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Abstract

We propose a monitoring procedure to test for the constancy of the correlation coefficient of a sequence of random variables. The idea of the method is that a historical sample is available and the goal is to monitor for changes in the correlation as new data become available. We introduce a detector which is based on the first hitting time of a CUSUM-type statistic over a suitably constructed threshold function. We derive the asymptotic distribution of the detector and show that the procedure detects a change with probability approaching unity as the length of the historical period increases. The method is illustrated by Monte Carlo experiments and the analysis of a real application with the log-returns of the Standard & Poor's 500 (S&P 500) and IBM stock assets.

Keywords: Correlation changes; Gaussian process; Online detection; Threshold function.

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1. Introduction

The correlation coefficient is the most widely used method to measure dependence between a sequence of two random variables. In the particular case of financial time series, the analysis of the correlations between returns are very important in risk management. Indeed, there is compelling empirical evidence that the correlation structure of financial returns cannot be assumed to be constant over time, see e.g. Longin and Solnik (1995) and Krishan et al. (2009). Consequently, in periods of financial crisis, investors are extremely concerned about changes on correlations because in such periods, the correlation often increases, a phenomenon which is referred to as "Diversification Meltdown" (Campbell et al., 2008).

In order to construct an adequate model and to forecast future data, structural stability is a key point. Testing for structural stability has recently become one of the principal objectives of statistical analysis. There are two distinctly different approaches to tackle this problem. On the one hand, the main goal of retrospective procedures is to look for the presence of change points given an historical dataset of fixed size. On the other hand, the main goal of sequential detection procedures is to detect as soon as possible the presence of a change point once new data become available. This article is concerned with the latter kind of procedures. We adopt the framework in Chu et al. (1996) in which a historical sample is available and the goal is to monitor for a change point as new data become available. In particular, we analyze the case of changes in the correlation structure of a sequence of random variables. Other papers analyzing related problems under this framework are Chu et al. (1996), Horváth et al. (2004), Aue et al. (2006), Aue et al. (2009) and Aue et al. (2011), among others.

The paper is organized as follows. Section 2 proposes a monitoring procedure for detecting a correlation change and presents its asymptotic properties under the null and alternative hypothesis as well. Section 3 analyzes the finite sample properties of the proposed procedure via Monte Carlo experiments. Section 4 illustrates the procedure by analyzing log-returns of the S&P 500 and IBM stock assets. Finally, all proofs are given in an appendix.

2. The monitoring procedure

Let (X_t, Y_t), for $t \in \mathbb{Z}$, be a sequence of bivariate random variables with finite 4-th moments and correlation

$$\rho_t = \frac{\mathsf{Cov}(X_t, Y_t)}{\sqrt{\mathsf{Var}(X_t)\mathsf{Var}(Y_t)}}.$$

We are interested in the hypothesis of correlation stability of the sequence. For that, assume that we have observed a sequence of the bivariate random vector (X_t, Y_t) of size m. Since we are interested in sequentially monitoring whether or not the correlation coefficient remain stable over time, we require that the correlation is constant over the historical period of length m, i.e.:

Assumption 1. $\rho_1 = \ldots = \rho_m$, *where* m *is a positive integer.*

Although Assumption 1 may appear a strong assumption, in practice, if a sufficient amount of historical data is available, it can be analyzed with the retrospective change point method proposed by Galeano and Wied (2012). Given the results of this procedure, one can make necessary adjustments to ensure correlation stability. Now, we want to test the null hypothesis given by:

$$H_0 : \rho_1 = \ldots = \rho_m = \rho_{m+1} = \ldots.$$

versus the alternative H_1 that ρ_t changes at some $t \geq m + 1$, i.e.:

$$H_1 : \exists k^* \geq 1 : \rho_1 = \ldots = \rho_m = \ldots = \rho_{m+k^*-1} \neq \rho_{m+k^*} = \rho_{m+k^*+1} = \ldots,$$

where k^* is referred to as the change point and is assumed unknown.

Denote with $\hat{\rho}_k^l$ the empirical correlation coefficient calculated from the observations k to

l with $k < l$, given by:

$$\hat{\rho}_k^l = \frac{\sum_{t=k}^{l}(X_t - \overline{X}_{k,l})(Y_t - \overline{Y}_{k,l})}{\sqrt{\sum_{t=k}^{l}(X_t - \overline{X}_{k,l})^2}\sqrt{\sum_{t=k}^{l}(Y_t - \overline{Y}_{k,l})^2}}$$

where $\overline{X}_{k,l} = \frac{1}{l-k+1}\sum_{t=k}^{l} X_t$ and $\overline{Y}_{k,l} = \frac{1}{l-k+1}\sum_{t=k}^{l} Y_t$. The sequential procedure is based on the detector:

$$V_k = \hat{D}\frac{k}{\sqrt{m}}\left(\hat{\rho}_{m+1}^{m+k} - \hat{\rho}_1^m\right), \ k \in \mathbb{N}, \tag{1}$$

where $\hat{D}$ is an estimator which is calculated from the first m observations and is given in the appendix, see also Wied et al. (2011). We stop and declare H_0 to be invalid at the first time k such that the detector V_k exceeds the value of a scaled threshold function w, therefore yielding the stopping rule:

$$\tau_m = \min\left\{k \leq [mT] : |V_k| > c \cdot w\left(\frac{k}{m}\right)\right\}, \tag{2}$$

where T is a positive constant, c is a suitably chosen constant such that under H_0, $\lim_{m\to\infty} \mathsf{P}(\tau_m < \infty) = \alpha$, with $\alpha \in (0,1)$, and w is a positive and continuous function. Here, we write $\tau_m < \infty$ to indicate that the monitoring has been terminated during the testing period, i.e., the detector V_k has crossed the boundary $c \cdot w(k/m)$ for some $k \leq [mT]$. We write $\tau_m = \infty$ if the detector has not crossed the boundary during the testing period (compare Aue et al., 2011). Note that the stopping time τ_m need not be the change point; in fact the change point might be before τ_m. Some comments on the issue of estimating the change point once this has been detected will be given at the end of this section.

For deriving asymptotic results under H_0, some additional assumptions are necessary. The next three assumptions correspond to (A1), (A2) and (A3) in Wied et al. (2011).

Assumption 2. *For*

$$U_t := \left(X_t^2 - \mathsf{E}(X_t^2), \quad Y_t^2 - \mathsf{E}(Y_t^2), \quad X_t - \mathsf{E}(X_t), \quad Y_t - \mathsf{E}(Y_t), \quad X_tY_t - \mathsf{E}(X_tY_t)\right)'$$

and $S_j := \sum_{t=1}^{j} U_t$, *we have*

$$\lim_{m\to\infty} \mathsf{E}\left(\frac{1}{m} S_m S_m'\right) =: D_1 \text{ (finite and positive definite).}$$

Assumption 3. *The* r*-th absolute moments of the components of* U_t *are uniformly bounded for some* $r > 2$.

Assumption 4. *The vector* (X_t, Y_t) *is* L_2*-NED (near-epoch dependent) with size* $-\frac{r-1}{r-2}$, *where* r *from 3, and constants* $(c_t), t \in \mathbb{Z}$, *on a sequence* $(V_t), t \in \mathbb{Z}$, *which is* α*-mixing of size* $\phi^* := -\frac{r}{r-2}$, *i.e.*

$$||(X_t, Y_t) - \mathsf{E}\left((X_t, Y_t)|\sigma(V_{t-l}, \ldots, V_{t+l})\right)||_2 \leq c_t v_l$$

with $\lim_{l\to\infty} v_l = 0$, *such that*

$$c_t \leq 2||U_t||_2$$

with U_t *from Assumption 3 and the* L_2*-norm* $||\cdot||_2$.

Furthermore, we impose a stationarity condition. This condition might be slightly relaxed to allow for some fluctuations in the first and second moments (see A4 and A5 in Wied et al., 2011), but for ease of exposition and because the procedure would keep exactly the same we stick to this notation.

Assumption 5. $(X_t, Y_t), t \in \mathbb{Z}$, *is weak-sense stationary.*

Our main result is then:

Theorem 1. *Under* H_0, *Assumptions 1, 2, 3, 4 and 5 and for any* $T > 0$,

$$\lim_{m\to\infty} \mathsf{P}(\tau_m < \infty) = \lim_{m\to\infty} \mathsf{P}\left(\sup_{0\leq b\leq T} \frac{|V_{[m\cdot b]+2}|}{w(b)} > c\right) = \mathsf{P}\left(\sup_{0\leq b\leq T} \frac{|G(b)|}{w(b)} > c\right), \quad (3)$$

where $G(\cdot)$ *is a mean zero Gaussian process with covariance* $\mathsf{E}(G(k)G(l)) = \min(k,l) + kl$.

Theorem 1 establishes the asymptotic behavior of the monitoring procedure based on the stopping rule τ_m in Eq. (2). Following Aue et al. (2011), the limiting probability in Eq. (3) can be written in an alternative way that allows for finite sample statistical inference. First, it is easy to see that $\{G(b) : b \in [0,T]\} =_d \{W(b) + b\xi : b \in [0,T]\}$, where $\{W(b) : b \geq 0\}$ is a standard Brownian Motion independent of the standard Gaussian random variable ξ. Then, it is also easy to see that $\{W(b) + b\xi : b \in [0,T]\} =_d \{(1+b)W(b/(1+b)) : b \in [0,T]\}$ just by comparing their covariance structures. Therefore,

$$\sup_{0 \leq b \leq T} \frac{|G(b)|}{w(b)} =_d \sup_{0 \leq b \leq T} \frac{1+b}{w(b)} W\left(\frac{b}{1+b}\right). \tag{4}$$

Eq. (4) leads to an obvious choice of the threshold function: take $w(b) = 1 + b$, because in this case:

$$\sup_{0 \leq b \leq T} \frac{|G(b)|}{w(b)} =_d \sup_{0 \leq b \leq T} \left| W\left(\frac{b}{1+b}\right) \right|.$$

With this expression, quantiles of interest can be easily simulated with Monte Carlo methods. However, once there occurs a change point, it is very important to quickly detect it. Therefore, we consider a kind of generalization of this threshold function, previously considered in Horváth et al. (2004), which is given by:

$$w(b) = (1+b)\left(\frac{b}{1+b}\right)^{\gamma}, \tag{5}$$

where $0 \leq \gamma < \frac{1}{2}$. Note that, if a correlation change occurs soon after the historical dataset, then, choosing γ as large as possible, the stopping rule τ_m will stop nearly instantaneously. Note that $\gamma = 1/2$ is excluded, since H_0 would else be rejected with probability one regardless whether it is true or not because of the law of the iterated logarithm for Brownian Motions at zero, see Aue et al. (2009). Using the threshold function in Eq. (5) and calling $u = b/(1+b)$, Eq. (4) leads to:

$$\sup_{0 \leq b \leq T} \frac{|G(b)|}{w(b)} =_d \sup_{0 \leq u \leq \frac{T}{T+1}} \frac{1}{u^{\gamma}} |W(u)|. \tag{6}$$

Finally, calling $s = u\left(T+1\right)/T$ and taking into account that $W\left(u\right)$ has the same covariance structure as $\sqrt{\frac{T}{1+T}}W\left(s\right)$, Eq. (6) transforms into:

$$\sup_{0\le b\le T}\frac{|G(b)|}{w(b)} =_d \left(\frac{T}{1+T}\right)^{\frac{1}{2}-\gamma}\sup_{0\le s\le 1}\frac{1}{s^{\gamma}}|W(s)|. \tag{7}$$

Therefore, under the conditions in Theorem 1:

$$\lim_{m\to\infty}\mathsf{P}(\tau_m<\infty)=\mathsf{P}\left(\left(\frac{T}{1+T}\right)^{\frac{1}{2}-\gamma}\sup_{0\le s\le 1}\frac{1}{s^{\gamma}}|W(s)|>c\right)$$

and Monte Carlo simulations can be used to obtain the constant $c\left(\alpha\right)$ such that:

$$\mathsf{P}\left(\left(\frac{T}{1+T}\right)^{\frac{1}{2}-\gamma}\sup_{0\le s\le 1}\frac{1}{s^{\gamma}}|W(s)|>c\left(\alpha\right)\right)=\alpha,$$

for any $\alpha\in(0,1)$. In this way, the probability of a false change point is approximately α if m is large enough.

For a local power analysis, we impose the assumption

Assumption 6. *$(X_t,Y_t), t\in\mathbb{Z}$, is weak-sense stationary with the difference that $\mathsf{Cov}(X_t,Y_t)=\frac{1}{\sqrt{m}}g\left(\frac{t}{m}\right)$ with a bounded function g which is can be approximated by step functions such that $g(z)=0, z\in[0,1]$, and $\int_1^{T+1}|g(z)|dz>0$.*

Theorem 2 yields consistency of the monitoring procedure. Therefore, a correlation change will be detected with high probability if the historical period is large enough.

Theorem 2. *Under a sequence of local alternatives, Assumptions 1, 2, 3, 4, 5 and 6 and for any $T>0$,*

$$\lim_{m\to\infty}\mathsf{P}(\tau_m<\infty)=\lim_{m\to\infty}\mathsf{P}\left(\sup_{0\le b\le T}\frac{|V_{[m\cdot b]+2}|}{w(b)}>c\right)=\mathsf{P}\left(\sup_{0\le b\le T}\frac{|G(b)+h(b)|}{w(b)}>c\right),$$

where $G(\cdot)$ is a mean zero Gaussian process with covariance $\mathsf{E}(G(k)G(l))=\min(k,l)+kl$ and $h(b)=H\left(\int_1^{b+1}g(z)dz-b\cdot\int_0^1 g(z)dz\right)$ for a constant H depending on the data *generating process.*

Once the presence of a correlation change is detected, an estimate of its location is provided by using the statistic proposed in Wied et al. (2011). The estimate of the change point is $\widehat{k} = \underset{1\leq j\leq \tau_m-1}{\arg\max} D_{\tau_m}$ with

$$D_{\tau_m} := \hat{D}\frac{j}{\sqrt{\tau_m}}\left|\hat{\rho}_{m+1}^{m+j} - \hat{\rho}_{m+1}^{m+\tau_m-1}\right|. \tag{8}$$

Note that we do not use the historical period to compute the value of the statistic D_{τ_m} but only the observations from $m+1$ to $m+\tau_m-1$. Monte Carlo experiments have shown that the inclusion of the historical period severely distorts the estimates of the change point location. A theoretical analysis of this estimator is out of the scope of this paper.

3. Simulations

In this section, we report the results of the Monte Carlo experiments that we have performed to assess the finite sample performance of the proposed monitoring procedure. In all the experiments, we consider three different values of the parameter γ of the threshold function $w(t)$ in Eq. (5), $\gamma = 0$, 0.25 and 0.45. Figure 1 shows the plot of the three threshold functions considered. Note that the larger values of γ, the smaller the values of $w(t)$. The threshold function with $\gamma = 0.45$ is expected to allow for a quick detection of early change points. On the other hand, we consider four different values of the size of the historical sample, $m = 250$, 500 and 1000. Note that these values are specially designated for financial returns in which we can consider large historical samples. Finally, we consider four values of the parameter T, $T = 0.5$, 1, 2 and 4. Note that these values cover a large number of sample sizes of the generated bivariate series which is given by $n = m + [Tm]$. For instance, for $m = 500$, the sample sizes of the series generated are 750, 1000, 1500 and 2500, respectively.

Figure 1 goes around here

First, we obtain critical values to apply the monitoring procedure for the different values considered of γ and T. Table 1 shows the critical values at level $\alpha = 0.05$ based on 10000 standard Brownian Motion processes approximated on a grid of 10000 equispaced points in the interval $[0, 1]$. Note that the critical values increases with T and/or with γ as expected.

Table 1 goes around here

Second, in order to obtain empirical sizes of the monitoring procedure, we generate 1000 bivariate series (X_t, Y_t), for $t = 1, \ldots, n$, and any choice of γ, T and m, as follows. Initially, we generate two series $\left(\widetilde{X}_t, \widetilde{Y}_t\right)$, for $t = 1, \ldots, n$, independently, following the GARCH$(1, 1)$ models given by:

$$\widetilde{X}_t = \sqrt{h_{1,t}}\epsilon_{1,t}$$
$$h_{1,t} = 0.01 + 0.05\widetilde{X}_{t-1}^2 + 0.8h_{1,t-1}$$

and,

$$\widetilde{Y}_t = \sqrt{h_{2,t}}\epsilon_{2,t}$$
$$h_{2,t} = 0.01 + 0.1\widetilde{Y}_{t-1}^2 + 0.75h_{2,t-1}$$

respectively, where $\epsilon_{1,t}$ and $\epsilon_{2,t}$ are standard Gaussian distributed. Then, we transform the bivariate series $\left(\widetilde{X}_t, \widetilde{Y}_t\right)$ into (X_t, Y_t) by multiplying each value of the pair $\left(\widetilde{X}_t, \widetilde{Y}_t\right)$ with $\Sigma^{1/2}$ where Σ is a square symmetric matrix with ones in the main diagonal and with $\rho = 0.5$ outside the main diagonal. Then, the correlation between X_t and Y_t is $\rho = 0.5$. Afterwards, for each simulated dataset, we apply the monitoring procedure from time $m + 1$ until time n, with level $\alpha = 0.05$. Table 2 reports the simulated empirical sizes for the monitoring procedure based on the detector V_k. In most cases, the simulated empirical sizes slightly exceed the nominal sizes, specially for $\gamma = 0.45$. However, empirical and nominal sizes get closer as m increases which is reasonable based on the results in section 2. Also, larger empirical sizes are found as γ gets larger and m is small. Therefore, if a correlation change is expected to occur not shortly after the historical period and we want to minimize the type I error, the choice of the threshold function with $\gamma = 0$ appears to be appropriate. However, if a correlation change is expected to occur shortly after the historical period and we want to detect it as soon as possible even if false change point can happen, it is better the threshold function with $\gamma = 0.45$.

Table 2 goes around here

Third, in order to estimate the power of the monitoring procedure, the Monte Carlo setup is similar to the one described previously, but the series are generated with a single change point in the correlation at two different positions $k = [0.05mT]$ and $k = [0.5mT]$, in which $\rho = 0.5$ increases to $\rho = 0.75$. Therefore, the first m observations have the same correlation coefficient, that changes after k observations of the monitoring time. The first change point is at the initial 5% of the monitoring time, so that it is specially designated to estimate the power of the procedure in situations in which the change point occurs shortly after the historical period. The second change point is at the middle of the monitoring time, so that it is specially designated to estimate the power of the procedure in situations in which the change point does not occur shortly after the historical period. Tables 3, 4, 5, 6, 7 and 8 show the results for the three possible values of the γ parameter, $\gamma = 0$, 0.25 and 0.45, and the two possible change points, $k = [0.05mT]$ and $k = [0.5mT]$. These tables show the empirical power of the procedure and a summary of both, the empirical stopping time distribution and the estimated change points, including the quartiles, the mean, the standard deviation and the coefficient of variation. The tables show that the power increases with m and it can be large except in cases in which m and T are small. Besides, the power for early changepoints is larger than the power for changes at the middle of the monitoring period. Regarding the empirical stopping time distribution, if a change occurs shortly after the beginning of the monitoring period, then the threshold function with $\gamma = 0.45$ have the shortest detection delay time. However, for a change point at the middle of the monitoring period with $m = 250$ and $m = 500$, the first quartiles of the empirical stopping times with $\gamma = 0.45$ are very small indicating that is more likely to falsely detect a correlation change even before it occurred. On the other hand, regarding the change point estimates, we can observe that the estimates of the change point at the beginning of the monitoring period are upward biased, while the estimates of the change point at the middle of the monitoring period are downward biased. However, in both cases, the bias reduces substantially if m and/or T increases. In any case, the precision of the change point detection estimate is quite acceptable specially when the power is large.

In summary, if the bivariate series is going to be monitored for a long time and the type I error is to be avoided, or if a change in the correlation is expected to occur not shortly after the beginning of monitoring period, the threshold function with $\gamma = 0$ may be a good choice. However, if the focus is to detect a change point in the correlation as soon as possible, even if a false change point is accepted, and if the change point is expected to occur shortly after the beginning of monitoring period, then it is better to use $\gamma = 0.45$. Alternately, the threshold function with $\gamma = 0.25$ appears to be a good compromise between the previous frameworks.

Table 3 goes around here

Table 4 goes around here

Table 5 goes around here

Table 6 goes around here

Table 7 goes around here

Table 8 goes around here

4. Real data example

In this section, we apply the proposed monitoring procedure discussed in section 2 to a real data example. Galeano and Wied (2012) analyzed the log-return series of two U.S. assets: the Standard & Poors 500 Index and the IBM stock using a posteriori change point tests. In particular, Galeano and Wied (2012) considered the sample period starting from January 2, 1997 to December 31, 2010 consisting of $n = 3524$ observations, that are plotted in Figure 2. The binary segmentation procedure proposed in that paper detected a first change point at August 19, 1999 (observation number 664), that can be associated with the collapse of the dot-com bubble started at the end of the 1990s and the beginning of the 2000s, and a second change point at November 12, 2007 (observation number 2734), that can be associated with the beginning of the Global Financial Crisis around the end of 2007, which is considered by many economists the worst financial crisis since the Great Depression of the 1930s.

Figure 2 goes around here

Here, we apply the proposed monitoring procedure as follows. The analysis in Galeano and Wied (2012) indicated that the correlations between both log-returns remained constant for the period starting from January 2, 1997 to August 19, 1999. Then, we use the log-returns from January, 2, 1997 until May, 28, 1999, as the historical period, i.e., we take $m = 607$. If no correlation changes are found after $n-m = 2917$ observations (then, $T = 4.8056$) the procedure would be terminated. Otherwise, a change point is detected and a new historical period is defined with $m = 607$. Then, the monitoring procedure is applied again in a similar fashion. The results of our analysis are summarized in Table 9 for the three threshold functions with $\gamma = 0$, 0.25 and 0.45, for which the corresponding critical values at 5% level are 2.0510 for $\gamma = 0$, 2.2630 for $\gamma = 0.25$, and 2.7435 for $\gamma = 0.45$, respectively. The proposed procedure with the three values of the threshold functions detects four change points sequentially. Regarding the first hitting times, the procedure with $\gamma = 0.45$ has the shortest detection delay time whereas the procedure with $\gamma = 0$ the longest. This is in accordance with the Monte Carlo experiments in section 3. Regarding the estimated change points, the procedure with the three values gives very similar estimates. Indeed, the first and the last detected change points coincide with the ones given in Galeano and Wied (2012). Finally, Table 10 shows the empirical correlations between the Standard & Poors 500 and IBM log-returns in the periods given by the monitoring procedure. As it can be seen, there are substantial differences between correlations at different periods.

Table 9 goes around here

Table 10 goes around here

References

Aue, A., S. Hörmann, L. Horváth, M. Hušková, and J. Steinebach (2011): "Sequential testing for the stability of high frequency portfolio betas," *Econometric Theory, forthcoming, doi: 10.1017/S0266466611000673*.

Aue, A., L. Horváth, M. Hušková, and P. Kokoszka (2006): "Change-point monitoring in linear models," *Econometrics Journal*, 9, 373–403.

Aue, A., L. Horváth, and M. Reimherr (2009): "Delay times of sequential procedures for multiple time series regression models," *Journal of Econometrics*, 149(2), 174–190.

Campbell, R., C. Forbes, K. Koedijk, and P. Kofman (2008): "Increasing correlations or just fat tails?" *Journal of Empirical Finance*, 15, 287–309.

Chu, C.-S. J., M. Stinchcombe, and H. White (1996): "Monitoring structural change," *Econometrica*, 64(5), 1045–1065.

Galeano, P. and D. Wied (2012): "Multiple change point detection in the correlation structure of financial assets," *submitted for publication.*

Horváth, L., M. Hušková, P. Kokoszka, and J. Steinebach (2004): "Monitoring changes in linear models," *Journal of Statistical Planning and Inference*, 126, 225–251.

Krishan, C., R. Petkova, and P. Ritchken (2009): "Correlation risk," *Journal of Empirical Finance*, 16, 353–367.

Longin, F. and B. Solnik (1995): "Is the correlation in international equity returns constant: 1960-1990?" *International Money and Finance*, 14(1), 3–26.

Wied, D., W. Krämer, and H. Dehling (2011): "Testing for a change in correlation at an unknown point in time using an extended functional delta method," *Econometric Theory, forthcoming, doi: 10.1017/S0266466611000661.*

A. Appendix

A.1. The scalar $\hat{D}$ from the test statistic in Eq. (1)

The scalar $\hat{D}$ from our test statistic in Eq. (1) based on observations from $t = 1, \ldots, r$ can be written as

$$\hat{D} = (\hat{F}_1 \hat{D}_{3,1} + \hat{F}_2 \hat{D}_{3,2} + \hat{F}_3 \hat{D}_{3,3})^{-\frac{1}{2}}$$

where

$$\begin{pmatrix} \hat{F}_1 & \hat{F}_2 & \hat{F}_3 \end{pmatrix} = \begin{pmatrix} \hat{D}_{3,1}\hat{E}_{11} + \hat{D}_{3,2}\hat{E}_{21} + \hat{D}_{3,3}\hat{E}_{31} \\ \hat{D}_{3,1}\hat{E}_{12} + \hat{D}_{3,2}\hat{E}_{22} + \hat{D}_{3,3}\hat{E}_{32} \\ \hat{D}_{3,1}\hat{E}_{13} + \hat{D}_{3,2}\hat{E}_{23} + \hat{D}_{3,3}\hat{E}_{33} \end{pmatrix}',$$

$$\begin{aligned}
\hat{E}_{11} &= \hat{D}_{1,11} - 4\hat{\mu}_x \hat{D}_{1,13} + 4\hat{\mu}_x^2 \hat{D}_{1,33}, \\
\hat{E}_{12} &= \hat{E}_{21} = \hat{D}_{1,12} - 2\hat{\mu}_x \hat{D}_{1,23} - 2\hat{\mu}_y \hat{D}_{1,14} + 4\hat{\mu}_x \hat{\mu}_y \hat{D}_{1,34}, \\
\hat{E}_{22} &= \hat{D}_{1,22} - 4\hat{\mu}_y \hat{D}_{1,24} + 4\hat{\mu}_y^2 \hat{D}_{1,44}, \\
\hat{E}_{13} &= \hat{E}_{31} = -\hat{\mu}_y \hat{D}_{1,13} + 2\hat{\mu}_x \hat{\mu}_y \hat{D}_{1,33} - \hat{\mu}_x \hat{D}_{1,14} + 2\hat{\mu}_x^2 \hat{D}_{1,34} + \hat{D}_{1,15} - 2\hat{\mu}_x \hat{D}_{1,35}, \\
\hat{E}_{23} &= \hat{E}_{32} = -\hat{\mu}_y \hat{D}_{1,23} + 2\hat{\mu}_x \hat{\mu}_y \hat{D}_{1,44} - \hat{\mu}_x \hat{D}_{1,24} + 2\hat{\mu}_y^2 \hat{D}_{1,34} + \hat{D}_{1,25} - 2\hat{\mu}_y \hat{D}_{1,45}, \\
\hat{E}_{33} &= \hat{\mu}_y^2 \hat{D}_{1,33} + 2\hat{\mu}_x \hat{\mu}_y \hat{D}_{1,34} - 2\hat{\mu}_y \hat{D}_{1,35} + \hat{\mu}_x^2 \hat{D}_{1,44} + \hat{D}_{1,55} - 2\hat{\mu}_x \hat{D}_{1,45},
\end{aligned}$$

$$\hat{D}_1 = \begin{pmatrix} \hat{D}_{1,11} & \hat{D}_{1,12} & \hat{D}_{1,13} & \hat{D}_{1,14} & \hat{D}_{1,15} \\ \hat{D}_{1,21} & \hat{D}_{1,22} & \hat{D}_{1,23} & \hat{D}_{1,24} & \hat{D}_{1,25} \\ \hat{D}_{1,31} & \hat{D}_{1,32} & \hat{D}_{1,33} & \hat{D}_{1,34} & \hat{D}_{1,35} \\ \hat{D}_{1,41} & \hat{D}_{1,42} & \hat{D}_{1,43} & \hat{D}_{1,44} & \hat{D}_{1,45} \\ \hat{D}_{1,51} & \hat{D}_{1,52} & \hat{D}_{1,53} & \hat{D}_{1,54} & \hat{D}_{1,55} \end{pmatrix} = \sum_{t=1}^{r} \sum_{u=1}^{r} k\left(\frac{t-u}{\delta_r}\right) V_t V_u{}',$$

$$V_t = \frac{1}{\sqrt{r}} U_t^{***}, \delta_r = [\log r],$$

$$U_t^{***} = \left(X_t^2 - \overline{(X^2)}_r \quad Y_t^2 - \overline{(Y^2)}_r \quad X_t - \bar{X}_r \quad Y_t - \bar{Y}_r \quad X_t Y_t - \overline{(XY)}_r \right)',$$

$$k(x) = \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & otherwise \end{cases},$$

$$\hat{\mu}_x = \bar{X}_r, \hat{\mu}_y = \bar{Y}_r, \hat{D}_{3,1} = -\frac{1}{2}\frac{\hat{\sigma}_{xy}}{\hat{\sigma}_y}\hat{\sigma}_x^{-3}, \hat{D}_{3,2} = -\frac{1}{2}\frac{\hat{\sigma}_{xy}}{\hat{\sigma}_x}\hat{\sigma}_y^{-3}, \hat{D}_{3,3} = \frac{1}{\hat{\sigma}_x \hat{\sigma}_y}$$

and

$$\hat{\sigma}_x^2 = \overline{(X^2)}_r - (\bar{X}_r)^2, \hat{\sigma}_y^2 = \overline{(Y^2)}_r - (\bar{Y}_r)^2, \hat{\sigma}_{xy} = \overline{(XY)}_r - \bar{X}_r \bar{Y}_r.$$

This is the same expression as in Appendix A.1 in Wied et al. (2011).

A.2. Proofs

Proof of Theorem 1

The proof mainly bases on the fact that for given $c < d$, constants e_1, e_2 not depending on m and $m \to \infty$ the process

$$\hat{D}\frac{[m \cdot d] - [m \cdot c]}{\sqrt{m}}(\hat{\rho}_{[m \cdot c]+e_1}^{[m \cdot d]+e_2} - \rho_1)$$

converges in distribution to the process $W(b) - W(a)$ with $W(\cdot)$ being a standard Brownian Motion. This result anon is a minor generalization of Lemma 3 in Wied et al. (2011).

With it, we obtain, for $0 \leq b \leq T$, that

$$\begin{aligned} V_{[m\cdot b]+2} &= \hat{D}\frac{[m\cdot b]+2}{\sqrt{m}}\left(\hat{\rho}_{m+1}^{m+[m\cdot b]+2} - \hat{\rho}_1^m\right) \\ &= \hat{D}\frac{[m\cdot b]+2}{\sqrt{m}}\left(\hat{\rho}_{m+1}^{m+[m\cdot b]+2} - \rho_1\right) - \hat{D}\frac{[m\cdot b]+2}{\sqrt{m}}\left(\hat{\rho}_1^m - \rho_1\right) \end{aligned}$$

converges to the process $(W(b+1) - W(1)) - b \cdot W(1) = W(b+1) - (b+1) \cdot W(1)$. Applying the continuous mapping theorem and calculating the covariance structure of the limit process proves the result. ■

Proof of Theorem 2

The proof uses the same arguments as Theorem 1 and mainly bases on the fact that for given $c < d$, constants e_1, e_2 not depending on m and $m \to \infty$ the process

$$\hat{D}\frac{[m\cdot d] - [m\cdot c]}{\sqrt{m}}(\hat{\rho}_{[m\cdot c]+e_1}^{[m\cdot d]+e_2} - \rho_1)$$

converges in distribution to the process $W(b) - W(a) + \int_a^b g(z)dz$ with $W(\cdot)$ being a standard Brownian Motion. This result anon is a minor generalization of arguments used in Theorem 2 in Wied et al. (2011). The constant H is then the limit of $\hat{D}$ under the null hypothesis, compare the proof of Theorem 2 in Wied et al. (2011). ■

Figure 1: Threshold functions for different values of γ

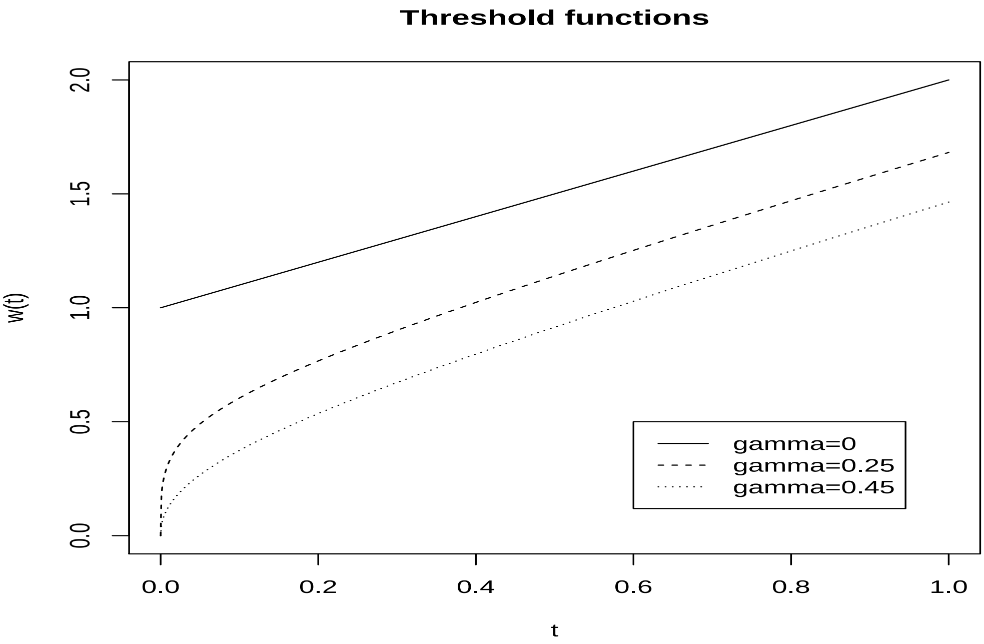

Table 1: Critical values.

T	$\gamma = 0$	$\gamma = 0.25$	$\gamma = 0.45$
0.5	1.2870	1.8001	2.6282
1	1.5578	1.9924	2.6844
2	1.8158	2.1684	2.7215
4	1.9980	2.2467	2.7660

Table 2: Empirical sizes.

	T	$m = 250$	$m = 500$	$m = 1000$
$\gamma = 0$	0.5	0.059	0.058	0.050
	1	0.077	0.069	0.061
	2	0.066	0.054	0.057
	4	0.063	0.071	0.060
$\gamma = 0.25$	0.5	0.075	0.079	0.047
	1	0.075	0.064	0.057
	2	0.087	0.063	0.052
	4	0.073	0.077	0.064
$\gamma = 0.45$	0.5	0.169	0.125	0.109
	1	0.174	0.136	0.116
	2	0.164	0.138	0.109
	4	0.161	0.128	0.106

Figure 2: Log-retuns of S&P 500 and IBM indexes

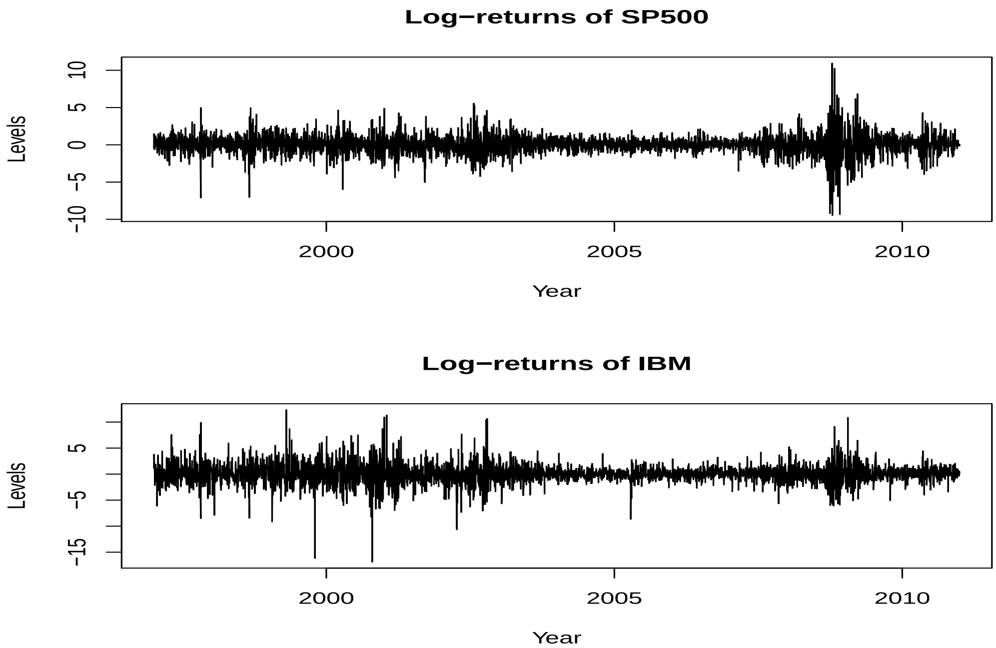

Table 3: Empirical power, stopping time and changepoint estimate for $\gamma = 0$ and $k = 5$.

$m = 250$													
		Empirical first hitting time						Changepoint estimate					
T (k)	Power	1st Q	Median	3rd Q	Mean	Std	CV	1st Q	Median	3rd Q	Mean	Std	CV
0.5 (6)	0.884	65	80	96	81.49	20.27	0.24	6	22	40.25	26.58	22.56	0.84
1 (12)	0.972	94	115	142.25	121.83	38.11	0.31	12	30	59	39.03	33.22	0.85
2 (25)	0.993	131	160	202	173.73	61.61	0.35	25	39	67	52.36	40.77	0.77
4 (50)	0.998	178	224	284.75	242.44	91.41	0.37	48	61	90	74.87	47.25	0.63
$m = 500$													
		Empirical first hitting time						Changepoint estimate					
T (k)	Power	1st Q	Median	3rd Q	Mean	Std	CV	1st Q	Median	3rd Q	Mean	Std	CV
0.5 (12)	0.997	95	112	137	118.50	31.78	0.26	12	30	58	37.99	31.67	0.83
1 (25)	1	136	160	188	166.34	43.53	0.26	24	39	69	50.61	37.24	0.73
2 (50)	1	191	224	264	231.52	55.53	0.23	47	58	83	69.03	36.81	0.53
4 (100)	1	268	319	376.25	328.61	81.78	0.24	95	105	126	113.91	37.11	0.32
$m = 1000$													
		Empirical first hitting time						Changepoint estimate					
T (k)	Power	1st Q	Median	3rd Q	Mean	Std	CV	1st Q	Median	3rd Q	Mean	Std	CV
0.5 (25)	1	142.75	165	188.25	168.74	35.13	0.20	23	38	69	49.42	36.17	0.73
1 (50)	1	198	228	259.25	232.47	46.74	0.20	48	61	90.25	72.77	39.66	0.54
2 (100)	1	285	318	364	326.92	62.46	0.19	96	104	124	111.45	33.47	0.30
4 (200)	1	422.75	475	537.25	483.86	85.81	0.17	191	200	214	203.85	37.22	0.18

Table 4: Empirical power, stopping time and changepoint estimate for $\gamma = 0$ and $k = 25$.

		Empirical first hitting time						Changepoint estimate					
							$m = 250$						
T (k)	Power	1st Q	Median	3rd Q	Mean	Std	CV	1st Q	Median	3rd Q	Mean	Std	CV
0.5 (62)	0.286	92	107	117	100.66	23.28	0.23	29	52	63	46.24	21.52	0.46
1 (125)	0.397	182	212	232	202.87	39.51	0.19	96	118	127	108.09	33.67	0.31
2 (250)	0.514	350.25	408	455.75	389.80	90.70	0.23	208.25	241.5	251	217.20	63.16	0.29
4 (500)	0.586	695.5	809.5	901.5	772.19	184.68	0.23	452.25	491	502	448.89	111.89	0.24
							$m = 500$						
T (k)	Power	1st Q	Median	3rd Q	Mean	Std	CV	1st Q	Median	3rd Q	Mean	Std	CV
0.5 (125)	0.505	188	208	230	205.53	31.18	0.15	95	119	127	108.33	31.80	0.29
1 (250)	0.740	357	404	448	396.81	70.90	0.17	211.75	243	250	223.17	51.46	0.23
2 (500)	0.856	696	774.5	850	769.60	125.25	0.16	457	490	501	466.11	71.96	0.15
4 (1000)	0.915	1323	1497	1679.5	1474.72	300.25	0.20	949	989	1000	930.13	166.93	0.17
							$m = 1000$						
T (k)	Power	1st Q	Median	3rd Q	Mean	Std	CV	1st Q	Median	3rd Q	Mean	Std	CV
0.5 (250)	0.833	363	403	438	398.32	56.30	0.14	215	242	251	227.90	41.92	0.18
1 (500)	0.958	676	744	823	745.32	116.43	0.15	446.25	489	501	457.01	83.14	0.18
2 (1000)	0.996	1271.5	1395	1542.5	1398.93	226.79	0.16	925	985.5	1000	926.62	155.20	0.16
4 (2000)	1	2438.5	2688	2916.5	2665.75	435.85	0.16	1905	1976	1998	1870.88	300.12	0.16

Table 5: Empirical power, stopping time and changepoint estimate for $\gamma = 0.25$ and $k = 5$.

$m = 250$													
		Empirical first hitting time						Changepoint estimate					
T (k)	Power	1st Q	Median	3rd Q	Mean	Std	CV	1st Q	Median	3rd Q	Mean	Std	CV
0.5 (6)	0.807	59.5	76	95	76.89	24.46	0.31	6	18	37	24.54	22.85	0.93
1 (12)	0.964	80.75	105	139.25	113.18	43.78	0.38	11	27	53	36.99	33.07	0.89
2 (25)	0.988	111	146	189	160.14	72.05	0.44	24	37	67	51.05	44.28	0.86
4 (50)	0.998	150.25	196	254	215.77	96.08	0.44	46	55	78	67.80	44.56	0.65
$m = 500$													
		Empirical first hitting time						Changepoint estimate					
T (k)	Power	1st Q	Median	3rd Q	Mean	Std	CV	1st Q	Median	3rd Q	Mean	Std	CV
0.5 (12)	0.993	80	101	128	107.05	36.75	0.34	11	24	52	34.05	30.50	0.89
1 (25)	1	111	135	172	144.28	49.03	0.33	24	36	61	45.88	32.29	0.70
2 (50)	1	159	190	233	202.50	70.92	0.35	47	55	76	65.50	38.04	0.58
4 (100)	1	227	275	332.25	285.95	83.72	0.29	92	102	117	106.95	36.24	0.33
$m = 1000$													
		Empirical first hitting time						Changepoint estimate					
T (k)	Power	1st Q	Median	3rd Q	Mean	Std	CV	1st Q	Median	3rd Q	Mean	Std	CV
0.5 (25)	1	113	135	163	140.62	38.53	0.27	23	34	58	42.99	30.75	0.71
1 (50)	1	156.75	187	219	192.01	49.18	0.25	46.75	56	77.25	64.14	30.84	0.48
2 (100)	1	235	275	319	281.93	64.80	0.22	92	101.5	117	106.19	31.44	0.29
4 (200)	1	361	412	472	421.46	86.12	0.20	187.75	199	212	197.38	33.11	0.16

Table 6: Empirical power, stopping time and changepoint estimate for $\gamma = 0.25$ and $k = 25$.

$m = 250$													
		Empirical first hitting time						Changepoint estimate					
T (k)	Power	1st Q	Median	3rd Q	Mean	Std	CV	1st Q	Median	3rd Q	Mean	Std	CV
0.5 (62)	0.241	84	102	115	91.17	33.62	0.36	18	50	61	42.75	24.16	0.56
1 (125)	0.350	167	205	225	185.14	60.89	0.32	77.25	114	126	99.30	42.03	0.42
2 (250)	0.472	350	409	455	384.27	103.04	0.26	210.75	241.5	252	217.57	63.89	0.29
4 (500)	0.556	682.75	804	880.25	739.39	230.86	0.31	432.75	488	501	430.50	136.65	0.31
$m = 500$													
		Empirical first hitting time						Changepoint estimate					
T (k)	Power	1st Q	Median	3rd Q	Mean	Std	CV	1st Q	Median	3rd Q	Mean	Std	CV
0.5 (125)	0.497	180	209	230	198.31	46.30	0.23	92	119	128	106.29	36.84	0.34
1 (250)	0.643	363	411	446	393.39	82.88	0.21	213	243	252	223.33	53.54	0.23
2 (500)	0.803	697.5	789	876	771.40	157.22	0.20	453.5	490	501	457.12	92.34	0.20
4 (1000)	0.897	1320	1503	1693	1459.73	362.23	0.24	936	988	1000	911.61	212.27	0.23
$m = 1000$													
		Empirical first hitting time						Changepoint estimate					
T (k)	Power	1st Q	Median	3rd Q	Mean	Std	CV	1st Q	Median	3rd Q	Mean	Std	CV
0.5 (250)	0.804	360	406	447	393.48	77.74	0.19	212	241.5	251	221.35	59.97	0.25
1 (500)	0.951	670.5	750	833.5	744.29	133.66	0.17	453	489	500	456.21	89.41	0.19
2 (1000)	0.993	1297	1427	1576	1422.85	256.28	0.18	943	986	999	935.72	152.72	0.16
4 (2000)	1	2484.5	2710	2962.5	2698.17	458.50	0.16	1914	1979	1999	1881.67	298.38	0.15

Table 7: Empirical power, stopping time and changepoint estimate for $\gamma = 0.45$ and $k = 5$.

		Empirical first hitting time						Changepoint estimate					
$m = 250$													
T (k)	Power	1st Q	Median	3rd Q	Mean	Std	CV	1st Q	Median	3rd Q	Mean	Std	CV
0.5 (6)	0.729	43	68	90	64.28	35.12	0.54	4	17	38.25	24.64	23.78	0.96
1 (12)	0.922	67	101.5	143	105.05	61.50	0.58	10	27	55	38.11	36.54	0.95
2 (25)	0.979	91	137	197	146.50	94.33	0.64	21	35	65	50.61	51.01	1.00
4 (50)	0.993	130	194	284	218.15	153.63	0.70	43	56	86	73.00	62.28	0.85
$m = 500$													
		Empirical first hitting time						Changepoint estimate					
T (k)	Power	1st Q	Median	3rd Q	Mean	Std	CV	1st Q	Median	3rd Q	Mean	Std	CV
0.5 (12)	0.980	63	92	127	95.96	54.13	0.56	10	24	51	34.87	33.70	0.96
1 (25)	0.998	92	124	166	129.43	66.50	0.51	21	33	57	43.29	38.22	0.88
2 (50)	1	127	169	219	171.02	86.40	0.50	41	53	72	59.20	39.16	0.66
4 (100)	1	203	260	325	259.13	120.38	0.46	86	100	114	98.22	46.35	0.47
$m = 1000$													
		Empirical first hitting time						Changepoint estimate					
T (k)	Power	1st Q	Median	3rd Q	Mean	Std	CV	1st Q	Median	3rd Q	Mean	Std	CV
0.5 (25)	1	87	116	150	117.69	55.93	0.47	19.75	30	53	40.03	33.73	0.84
1 (50)	1	129	167	205	162.49	69.44	0.42	41.75	53	70	56.14	31.80	0.56
2 (100)	1	200	242.5	291	238.43	95.57	0.40	86	99	113	97.02	40.77	0.42
4 (200)	1	330	394	454	379.52	138.11	0.36	173	197	208	179.24	61.41	0.34

Table 8: Empirical power, stopping time and changepoint estimate for $\gamma = 0.45$ and $k = 25$.

$m = 250$													
		Empirical first hitting time						Changepoint estimate					
T (k)	Power	1st Q	Median	3rd Q	Mean	Std	CV	1st Q	Median	3rd Q	Mean	Std	CV
0.5 (62)	0.231	3	8	96.5	40.12	47.99	1.19	2	4	51	22.11	26.54	1.20
1 (125)	0.310	5	157	216	119.25	100.07	0.83	3	81.5	122	68.57	55.37	0.80
2 (250)	0.416	6	371.5	440	264.38	202.17	0.76	6	217	248	156.63	112.02	0.71
4 (500)	0.427	21.5	748	884	557.69	391.11	0.70	45.5	474	498.5	343.19	213.24	0.62
$m = 500$													
		Empirical first hitting time						Changepoint estimate					
T (k)	Power	1st Q	Median	3rd Q	Mean	Std	CV	1st Q	Median	3rd Q	Mean	Std	CV
0.5 (125)	0.403	7	189	224	140.70	97.82	0.69	4	95	124	76.57	56.52	0.73
1 (250)	0.524	269.75	400	447.25	320.99	174.81	0.54	121	235	251	182.22	99.82	0.54
2 (500)	0.703	646	775	891.5	679.03	305.66	0.45	422.5	486	501	406.55	172.94	0.42
4 (1000)	0.820	1294	1544	1727.5	1333.82	613.14	0.45	891.5	983	1000	814.08	355.37	0.43
$m = 1000$													
		Empirical first hitting time						Changepoint estimate					
T (k)	Power	1st Q	Median	3rd Q	Mean	Std	CV	1st Q	Median	3rd Q	Mean	Std	CV
0.5 (250)	0.718	336	406	448	347.23	155.39	0.44	171.5	234	249	192.20	90.57	0.47
1 (500)	0.895	685.5	780	862.5	703.95	266.51	0.37	437.5	490	500	418.28	159.63	0.38
2 (1000)	0.980	1287.25	1444	1631.25	1345.95	477.67	0.35	910.75	980	998	860.08	293.36	0.34
4 (2000)	0.993	2570	2833	3114	2683.97	812.59	0.30	1898	1981	1999	1792.81	513.98	0.28

Table 9: Results of the monitoring procedure for three values of γ (EFHT stands for empirical first hitting times).

$\gamma = 0$		$\gamma = 0.25$		$\gamma = 0.45$	
EFHT	Est. changepoints	EFHT	Est. changepoints	EFHT	Est. changepoints
984	665 (1999/08/20)	808	682 (1999/09/15)	772	682 (1999/09/15)
1580	1399 (2002/07/25)	1554	1399 (2002/07/25)	1529	1399 (2002/07/25)
2222	2196 (2005/09/22)	2209	2053 (2005/03/01)	2208	2053 (2005/03/01)
3014	2936 (2008/09/02)	2945	2733 (2007/11/09)	2890	2733 (2007/11/09)

Table 10: Empirical correlations at different periods.

Period	$\gamma = 0$	$\gamma = 0.25$	$\gamma = 0.45$
1	0.6274	0.6237	0.6237
2	0.5245	0.5264	0.5264
3	0.7249	0.7410	0.7410
4	0.6033	0.5364	0.5364
5	0.8021	0.7800	0.7800