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Discussion Paper

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NONCENTRAL LIMIT THEOREM AND THE BOOTSTRAP FOR QUANTILES OF DEPENDENT DATA

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ABSTRACT. We will show under minimal conditions on differentiability and dependence that the central limit theorem for quantiles holds and that the block bootstrap is weakly consistent. Under slightly stronger conditions, the bootstrap is strongly consistent. Without the differentiability condition, quantiles might have a non-normal asymptotic distribution and the bootstrap might fail.

1. LIMIT BEHAVIOUR OF QUANTILES

Let $(X_n)_{n\in\mathbb{Z}}$ be a stationary sequence of real-valued random variables with distribution function F and $p \in (0,1)$. Then the p-quantile t_p of F is defined as

$$t_p := F^{-1}(p) := \inf\left\{t \in \mathbb{R} \middle| F(t) \geq p\right\}$$

and can be estimated by the empirical p-quantile, i.e. the $\lceil \frac{n}{p} \rceil$-th order statistic of the sample $X_1 \ldots, X_n$. This also can be expressed as the p-quantile $F_n^{-1}(p)$ of the empirical distribution function $F_n(t) := \frac{1}{n}\sum_{i=1}^n \mathbb{1}_{X_i \leq t}$. It is clear that $F_n^{-1}(p)$ is greater than t_p iff $F_n(t_p)$ is smaller than p. In the case of independent random variables, this converse behaviour was exploited by Bahadur [3] to show that the asymptotic behaviour of the quantile $F_n^{-1}(p)$ and the empirical distribution function F_n at the point t_p is the same under the condition that F is differentiable twice in a neighborhood of t_p. Ghosh [7] established a weak form of the Bahadur representation, only assuming that F is differentiable once in t_p. He showed that

$$F_n^{-1}(p) - t_p = \frac{p - F_n(t_p)}{f(t_p)} + R_n,$$

where $f = F'$ is the derivative of the distribution function and $R_n = o_P\left(n^{-\frac{1}{2}}\right)$. As noticed by Lahiri [12], the condition that F is differentiable is also necessary for the central limit theorem for $F_n^{-1}(p)$. Ghosh and Sukhatme [9] and de Haan and Taconis-Haantjes [10] investigated the noncentral limit theorem for $F_n^{-1}(p)$, if F is not differentiable, but regular varying. We will extend their results to strongly mixing random variables.

There is a broad literature on the Bahadur representation for strongly mixing data. Babu and Singh [2] proved such a representation under an exponentially fast decay of the strong mixing coefficients, this was weakened by Yoshihara [21], Sun [18] and Wendler [20] to a polynomial decay of the strong mixing coefficients. All these articles deal with the case that F is differentiable.

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Definition 1.1. *Let $(X_n)_{n\in\mathbb{Z}}$ be a stationary process. Then the strong mixing coefficients are defined as*

$$\alpha(k) := \sup\left\{|P[AB] - P[A]P[B]| : A \in \mathcal{F}_1^n, B \in \mathcal{F}_{n+k}^{\infty}, n \in \mathbb{Z}\right\} \tag{1}$$

where $\mathcal{F}_a^l$ is the σ-field generated by random variables $X_a, \ldots, X_l$. We say that $(X_n)_{n\in\mathbb{Z}}$ is strongly mixing if $\lim_{k\to\infty}\alpha(k) = 0$.

For further information on strong mixing and a detailed description of the other mixing assumptions, see Bradley [5].

Theorem 1. *Let $(X_n)_{n\in\mathbb{Z}}$ be a stationary, strongly mixing sequence of random variables with distribution function F, such that for a $\rho > 0$*

$$F(t_p + h) - F(t_p) = M|h|^{\rho}\operatorname{sgn}(h) + o(|h|^{\rho})$$

as $h \to 0$ and $\sum_{n=1}^{\infty}\alpha(n) < \infty$. Then

$$F_n^{-1}(p) - t_p = \left(\frac{|p - F_n(t_p)|}{M}\right)^{\frac{1}{\rho}} \operatorname{sgn}(p - F_n(t_p)) + R_n,$$

where $R_n = o_P(n^{-\frac{1}{2\rho}})$.

For the special case $\rho = 1$ (differentiablilty), we get Lemma 5.1 of Sun and Lahiri [19] (central limit theorem for $F_n^{-1}(p)$), as $F_n(t_p)$ is asymptotically normal by Theorem 1.6 of Ibragimov [11].

Corollary 1. *If the assumptions of Theorem 1 hold with $\rho = 1$, then $\sqrt{n}(F_n^{-1}(p) - t_p)$ is asymptotically normal.*

In the other case, we get a noncentral limit theorem:

Corollary 2. *If the assumptions of Theorem 1 hold with $\rho \neq 1$, then $n^{\frac{1}{2\rho}}(F_n^{-1}(p) - t_p)$ converges in distribution to $C|W|^{\frac{1}{\rho}}\operatorname{sgn}(W)$, where C is a constant and W is a normal random variable.*

2. Block Bootstrap for Quantiles

The statistical inference for quantiles is a dificult task, many methods rely on estimates of the unknown density. An alternative method is the Bootstrap. Bickel and Freedman [4] established the consistitency of the Bootstrap for quantiles for independent data, more work on this topic was done by Ghosh et. al [8] and Babu [1].

For dependent data, normal approximation becomes even more difficult, but there is up to our knowledge only one article about the bootstrap for quantiles under dependence: Sun and Lahiri [19] have shown the strong consistency of the bootstrap under strong mixing. We will establish a weak Bahadur representation for the bootstrap version of the quantile and will conclude that the bootstrap is weakly consistent for $\rho = 1$ and inconsistent for $\rho \neq 1$.

There are different ways to resample blocks, for example the circular block bootstrap or the moving block bootstrap (for a detailed description of the different bootstrapping methods see Lahiri [13]). We consider the circular block bootstrap introduced by Politis and Romano [15]. Instead of the original sample of n observations with an unknown distribution, construct new samples $X_1^{\star}, \ldots, X_{bl}^{\star}$ as follows: Extend the sample $X_1, \ldots, X_n$ periodically by $X_{i+n} = X_i$, choose blocks of $l = l_n$

consecutive observations of the sample randomly and repeat that $b = \lfloor \frac{n}{l} \rfloor$ times independently: For $j = 1, \ldots, n$, $k = 0, \ldots, b-1$

$$P^\star\left(X^\star_{kl+1} = X_j, \ldots, X^\star_{(k+1)l} = X_{j+l-1}\right) = \frac{1}{n},$$

where $P^\star$ is the bootstrap distribution conditionally on $(X_n)_{n\in\mathbb{N}}$, $E^\star$ and $\mathrm{Var}^\star$ are the conditional expectation and variance. For the circular block bootstrap version of the sample mean, Radulović [16] has established weak consistency under very weak conditions.

$F^\star_n(t) = \frac{1}{bl}\sum_{i=1}^{bl} \mathbb{1}_{\{X^\star_i \le t\}}$ denotes the Bootstrap version of the empirical distribution function and $F^{\star-1}_n(p)$ the p-quantile of the bootstrap sample.

Theorem 2. *Let $(X_n)_{n\in\mathbb{Z}}$ be a stationary, strongly mixing sequence of random variables with distribution function F, such that for a $\rho > 0$ and $M \neq 0$*

$$F(t_p + h) - F(t_p) = M|h|^\rho \operatorname{sgn}(h) + o(h)$$

as $h \to 0$ and $\sum_{n=1}^\infty \alpha(n) < \infty$. Furthermore, choose the block length in such a way that $\frac{1}{l} + \frac{l}{n} \to 0$. Then

$$F^{\star-1}_n(p) - t_p = \left(\frac{|p - F^\star_n(t_p)|}{M}\right)^{\frac{1}{\rho}} \operatorname{sgn}(p - F^\star_n(t_p)) + R^\star_n,$$

where $R^\star_n = o_P(n^{-\frac{1}{2\rho}})$.

Note that we do not center $F^{\star-1}_n(p)$ with respect to the bootstrapped expectation, but with respect to the true quantil t_p. With the help of this theorem, we get weak consistency respectively inconsistency of the bootstrap:

Corollary 3. *If the assumptions of Theorem 2 hold with $\rho = 1$ and additionally $\lim_{n\to\infty} \mathrm{Var}[\sqrt{n}F_n(t_p)] > 0$, then*

$$\sup_{t\in\mathbb{R}} \left|P^\star\left(F^{\star-1}_n(p) - F^{-1}_n(p) \le t\right) - P\left(F^{-1}_n(p) - t_p \le t\right)\right| \xrightarrow{n\to\infty} 0$$

in probability.

Corollary 4. *If the assumptions of Theorem 2 hold with $\rho \neq 1$ and additionally $\lim_{n\to\infty} \mathrm{Var}[\sqrt{n}F_n(t_p)] > 0$, then*

$$\sup_{t\in\mathbb{R}} \left|P^\star\left(F^{\star-1}_n(p) - F^{-1}_n(p) \le t\right) - P\left(F^{-1}_n(p) - t_p \le t\right)\right| \xrightarrow{n\to\infty} Z_\rho$$

in distribution, where Z_ρ is a non-degenerate (non-constant) random variable.

We also want to establish the almost sure consistency and we need slightly stronger conditions on the mixing coefficients and the block length:

Theorem 3. *Let $(X_n)_{n\in\mathbb{Z}}$ be a stationary, strongly mixing sequence of random variables with distribution function F which is differentiable in t_p. We assume that the mixing coefficients satisfy $\alpha(n) = O(n^{-1-\epsilon})$ for an $\epsilon > 0$. Furthermore, choose the block length in such a way that for some constants $C_1, C_2, \epsilon_1 > 0$*

$$C_1 n^{\epsilon_1} \le l_n \le C_2 n^{1-\epsilon_1}$$

and for all $k \in \mathbb{N}$

$$l_{2^k} = l_{2^k+1} = \ldots = l_{2^{k+1}-1}.$$

Then

$$F_n^{\star-1}(p) - F_n^{-1}(p) = \frac{F_n(t_p) - F_n^{\star}(t_p)}{f(t_p)} + R_n^{\star},$$

where $R_n^{\star} = o_P^{\star}(n^{-\frac{1}{2}})$ *almost surely.*

With this Bahadur-Ghosh representation and Theorem 2.4 of Shao and Yu [17], the strong consistency of the bootstap follows easily:

Corollary 5. *If the assumptions of Theorem 3 hold and* $\lim_{n\to\infty} \mathrm{Var}[\sqrt{n}F_n(t_p)] > 0$*, then*

$$\sup_{t\in\mathbb{R}} \left| P^{\star}\left(F_n^{\star-1}(p) - F_n^{-1}(p) \le t\right) - P\left(F_n^{-1}(p) - t_p \le t\right)\right| \xrightarrow{n\to\infty} 0$$

almost surely.

Compared to Theorem 3.1 of Sun and Lahiri [19], our assumptions on the mixing coefficient $\alpha(n)$, on the distribution function F and on the block length l are weaker.

3. Proofs

In the proofs, C denotes an arbitrary constant, which may have different values from line to line and may depend on several other values, but not on $n \in \mathbb{N}$. We use the following lemma proved by Ghosh [7]:

Lemma 3.1. *Let* $(V_n)_{n\in\mathbb{N}}$ *and* $(W_n)_{n\in\mathbb{N}}$ *be two sequences of random variables, such that*

1. *the sequence* $(W_n)_{n\in\mathbb{N}}$ *is tight,*
2. *For all* $k \in \mathbb{R}$ *and* $\epsilon > 0$

$$\lim_{n\to\infty} P\left(V_n \le k, W_n \ge k+\epsilon\right) = 0$$
$$\lim_{n\to\infty} P\left(V_n \ge k+\epsilon, W_n \le k\right) = 0.$$

Then $V_n - W_n \to 0$ *in probabality as* $n \to \infty$*.*

Proof of Theorem 1. The proof follows the ideas of Ghosh [7]. We set $g(x) = |x|^{\rho}\,\mathrm{sgn}(x)$ and $W_n = g^{-1}(\sqrt{n}(p - F_n(t_p)))$. By Theorem 1.6 of Ibragimov [11], $\sqrt{n}(p - F_n(t_p))$ converges to a normal limit and thus $(W_n)_{n\in\mathbb{N}}$ is tight. We define $V_n = n^{\frac{1}{2\rho}}(F_n^{-1}(p) - t_p)$ and $Z_{t,n} = g^{-1}(\sqrt{n}(F(t_p + \frac{t}{n^{\frac{1}{2\rho}}}) - F_n(t_p + \frac{t}{n^{\frac{1}{2\rho}}})))$. We have that by the definition of the generalized inverse

$$\{V_n \le t\} = \left\{p \le F_n(t_p + \frac{t}{n^{\frac{1}{2\rho}}})\right\} = \{Z_{t,n} \le t_n\}$$

where $t_n := g^{-1}(\sqrt{n}(F(t_p + \frac{t}{n^{\frac{1}{2\rho}}}) - p))$. By our assumptions on F, for all $t \in \mathbb{R}$ we have that $t_n \to t$ as $n \to \infty$. We assumed that $\sum_{k=1}^{\infty} \alpha(k) \le \infty$. By a well-known covaraince inequality $\mathrm{Cov}\left(\mathbb{1}_{\{t_p < X_1 \le t_p + \frac{t}{n^{\frac{1}{2\rho}}}\}}, \mathbb{1}_{\{t_p < X_k \le t_p + \frac{t}{n^{\frac{1}{2\rho}}}\}}\right) \le 4\alpha(k-1)$, so we have that

$$E\left(\sqrt{n}\left(p - F_n(t_p) - F(t_p + \frac{t}{n^{\frac{1}{2\rho}}}) + F_n(t_p + \frac{t}{n^{\frac{1}{2\rho}}})\right)\right)^2$$

$$\le 2\sum_{k=1}^{\infty}\left|\operatorname{Cov}\left(\mathbb{1}_{\{t_p<X_1\le t_p+\frac{t}{n^{\frac{1}{2\rho}}}\}}, \mathbb{1}_{\{t_p<X_k\le t_p+\frac{t}{n^{\frac{1}{2\rho}}}\}}\right)\right|$$

$$\le 2\sum_{k=1}^{\lfloor n^{\frac{1}{4}}\rfloor}\operatorname{Var}\left(\mathbb{1}_{\{t_p<X_1\le t_p+\frac{t}{n^{\frac{1}{2\rho}}}\}}\right) + 8\sum_{k=\lfloor n^{\frac{1}{4}}\rfloor}^{\infty}\alpha(k)$$

$$\le 2n^{\frac{1}{4}}\left|F(t_p + \frac{t}{n^{\frac{1}{2\rho}}}) - p\right| + 8\sum_{k=\lfloor n^{\frac{1}{4}}\rfloor}^{\infty}\alpha(k) \xrightarrow{n\to\infty} 0,$$

so $\sqrt{n}(p - F_n(t_p) - F(t_p + \frac{t}{n^{\frac{1}{2\rho}}}) + F_n(t_p + \frac{t}{n^{\frac{1}{2\rho}}})) \to 0$ in probability and consequently $Z_{t,n} - W_n \to 0$ in probability as $n \to \infty$, Lemma 3.1 completes the proof. □

We omit the proof of the Corollaries 1 and 2, as we think they are obvious.

Proof of Theorem 2. We will use a similar method as in the proof of Theorem 1. We define

$$W_n^\star := g^{-1}\left(\sqrt{bl}(p - F_n^\star(t_p))\right),$$
$$V_n^\star := (bl)^{\frac{1}{2\rho}}(F_n^{\star -1}(p) - t_p),$$
$$Z_{t,n}^\star := g^{-1}\left(\sqrt{bl}(F(t_p + \frac{t}{n^{\frac{1}{2\rho}}}) - F_n^\star(t_p + \frac{t}{n^{\frac{1}{2\rho}}}))\right)$$

Note that $\sqrt{bl}(F_n(t_p) - F_n^\star(t_p))$ converges to a normal limit by Theorem 2 of Radulović [16], so the sequence $(\sqrt{bl}((p - F_n(t_p)) + (F_n(t_p) - F_n^\star(t_p)))_{n\in\mathbb{N}}$ is tight and consequently the sequence $(W_n^\star)_{n\in\mathbb{N}}$ is also tight. It remains to show for any $t \in \mathbb{R}$ that $Z_{t,n}^\star - W_n^\star \to 0$ in probability as $n \to \infty$. By the construction of the circular block bootstrap $E^\star F_n^\star(t) = F_n(t)$, so

$$EE^\star\left(\sqrt{bl}\left(F(t_p + \frac{t}{n^{\frac{1}{2\rho}}}) - F_n^\star(t_p + \frac{t}{n^{\frac{1}{2\rho}}}) - p + F_n^\star(t_p)\right)\right)^2$$
$$= EE^\star\left(\sqrt{bl}\left(F_n(t_p + \frac{t}{n^{\frac{1}{2\rho}}}) - F_n^\star(t_p + \frac{t}{n^{\frac{1}{2\rho}}}) - F_n(t_p) + F_n^\star(t_p)\right)\right)^2$$
$$+ E\left(\sqrt{bl}\left(F(t_p + \frac{t}{n^{\frac{1}{2\rho}}}) - F_n(t_p + \frac{t}{n^{\frac{1}{2\rho}}}) - p + F_n(t_p)\right)\right)^2.$$

In the proof of Theorem 1, we have already shown that the second summand converges to zero. For the first summand, we conclude from the conditional independence of the resampled blocks and the definition of empirical distribution function

$$EE^\star\left(\sqrt{bl}\left(F_n(t_p + \frac{t}{n^{\frac{1}{2\rho}}}) - F_n^\star(t_p + \frac{t}{n^{\frac{1}{2\rho}}}) - F_n(t_p) + F_n^\star(t_p)\right)\right)^2$$
$$= lEE^\star\left(F_n(t_p + \frac{t}{n^{\frac{1}{2\rho}}}) - \frac{1}{l}\sum_{i=1}^{l}\mathbb{1}_{\{X_i^\star\le t_p+\frac{t}{n^{\frac{1}{2\rho}}}\}} - F_n(t_p) + \frac{1}{l}\sum_{i=1}^{l}\mathbb{1}_{\{X_i^\star\le t_p\}}\right)^2.$$

With probability $\frac{1}{n}$, we have $(X_1^\star,\ldots,X_l^\star)=(X_{j+1},\ldots,X_{j+l})$ (with $X_i=X_{i-n}$ for $i>n$), so

$$\begin{aligned}
&lEE^\star\left(F_n(t_p+\frac{t}{n^{\frac{1}{2\rho}}})-\frac{1}{l}\sum_{i=1}^{l}\mathbb{1}_{\{X_i^\star\leq t_p+\frac{t}{n^{\frac{1}{2\rho}}}\}}-F_n(t_p)+\frac{1}{l}\sum_{i=1}^{l}\mathbb{1}_{\{X_i^\star\leq t_p\}}\right)^2\\
&=\frac{l}{n}\sum_{j=1}^{n}E\left(F_n(t_p+\frac{t}{n^{\frac{1}{2\rho}}})-\frac{1}{l}\sum_{i=1}^{l}\mathbb{1}_{\{X_{j+1}\leq t_p+\frac{t}{n^{\frac{1}{2\rho}}}\}}-F_n(t_p)+\frac{1}{l}\sum_{i=1}^{l}\mathbb{1}_{\{X_{j+i}\leq t_p\}}\right)^2\\
&\leq 2lE\left(F_n(t_p+\frac{t}{n^{\frac{1}{2\rho}}})-F(t_p+\frac{t}{n^{\frac{1}{2\rho}}})-F_n(t_p)+F(t_p)\right)^2\\
&\qquad+2lE\left(F_l(t_p+\frac{t}{n^{\frac{1}{2\rho}}})-F(t_p+\frac{t}{n^{\frac{1}{2\rho}}})-F_l(t_p)+F(t_p)\right)^2.
\end{aligned}$$

These two summands converge to 0 as in the proof of Theorem 1, which completes the proof. □

Proof of Corollary 3. By Theorem 2 of Radulović [16]

$$\sup_{t\in\mathbb{R}}\left|P^\star\left(\sqrt{n}(F_n(t_p)-p)\leq t\right)-P\left(Y\leq t\right)\right|\xrightarrow{n\to\infty}0$$

and

$$\sup_{t\in\mathbb{R}}\left|P^\star\left(\sqrt{bl}(F_n^\star(t_p)-F_n(t_p))\leq t\right)-P\left(Y\leq t\right)\right|\xrightarrow{n\to\infty}0$$

in probability for some normal random variable Y. Furthermore by Theorems 1 and 2

$$F_n^{-1}(p)-t_p=\frac{p-F_n(t_p)}{M}+R_n$$

and

$$\begin{aligned}
F_n^{\star-1}(p)-F_n^{-1}(p)&=\left(F_n^{\star-1}(p)-t_p\right)-\left(F_n^{-1}(p)-t_p\right)\\
&=\frac{p-F_n^\star(t_p)}{M}-\frac{p-F_n(t_p)}{M}+R_n^\star-R_n=\frac{F_n(t_p)-F_n^\star(t_p)}{M}+R_n^\star-R_n,
\end{aligned}$$

where $R_n=o_P(n^{-\frac{1}{2}})$ and $R_n^\star=o_P(n^{-\frac{1}{2}})$. So we can conclude that

$$\begin{aligned}
&\sup_{t\in\mathbb{R}}\left|P^\star\left(F_n^{\star-1}(p)-F_n^{-1}(p)\leq t\right)-P\left(F_n^{-1}(p)-t_p\leq t\right)\right|\\
&\leq\sup_{t\in\mathbb{R}}\left|P^\star\left(\sqrt{bl}(F_n^{\star-1}(p)-F_n^{-1}(p))\leq t\right)-P\left(\frac{-Y}{M}\leq t\right)\right|\\
&\qquad+\sup_{t\in\mathbb{R}}\left|P^\star\left(\sqrt{n}(F_n(p)-t_p)\leq t\right)-P\left(-\frac{Y}{M}\leq t\right)\right|\xrightarrow{n\to\infty}0
\end{aligned}$$

in probability. □

Proof of Corollary 4. By Theorems 1 and 2, we have that

$$F_n^{\star-1}(p)-F_n^{-1}(p)=g^{-1}(p-F_n^\star(t_p))-g^{-1}(p-F_n(t_p))+R_n+R_n^\star$$

with $R_n + R_n^\star = o_P(n^{-\frac{1}{2\rho}})$, so

$$\sup_{t\in\mathbb{R}} \Big| P^\star \left(n^{\frac{1}{2\rho}} (F_n^{\star -1}(p) - F_n^{-1}(p)) \le t \right) \\ - P^\star \left(g^{-1}(\sqrt{n}(p - F_n^\star(t_p))) - g^{-1}(\sqrt{n}(p - F_n(t_p))) \le t \right) \Big| \xrightarrow{n\to\infty} 0$$

in probability. Furthermore

$$\sup_{t\in\mathbb{R}} \Big| P \left(n^{\frac{1}{2\rho}} (F_n^{-1}(p) - t_p) \le t \right) - P \left(g^{-1}(\sqrt{n}(p - F_n(t_p))) \le t \right) \Big| \xrightarrow{n\to\infty} 0.$$

So we have to investigate

$$\sup_{t\in\mathbb{R}} \big| P^\star \left(g^{-1}(\sqrt{n}(p - F_n^\star(t_p))) - g^{-1}(\sqrt{n}(p - F_n(t_p))) \le t \right) \\ - P \left(g^{-1}(\sqrt{n}(p - F_n(t_p))) \le t \right) \big| \\ \xrightarrow{n\to\infty} \sup_{t\in\mathbb{R}} \big| P \left(g^{-1}(W_1 + W_2) - g^{-1}(W_2) \le t | W_2 \right) - P \left(g^{-1}(W_1) \le t \right) \big| =: Z_\rho$$

in distribution, where W_1 and W_2 are two independent normal random variables. As the functions $x \to g^{-1}(x+y) - g^{-1}(y)$ and $x \to g^{-1}(x)$ are not identical for $y \neq 0$, this random variables is not 0. □

Proof of Theorem 3. We define $a_n = 2^k$ for the $k \in \mathbb{N}$ such that $2^k \le n < 2^{k+1}$.

$$\tilde{W}_n := \frac{1}{\sqrt{a_n}} \sum_{i=1}^{n} \left(F_n(t_p) - \mathbb{1}_{\{X_i^\star \le t_p\}} \right),$$
$$\tilde{Z}_{t,n} := \frac{1}{\sqrt{a_n}} \sum_{i=1}^{n} \left(F_n(t_p + \frac{t}{\sqrt{a_n}}) - \mathbb{1}_{\{X_i^\star \le t_p + \frac{t}{\sqrt{a_n}}\}} - F_n(t_p) + \mathbb{1}_{\{X_i^\star \le t_p\}} \right).$$

Following the arguments of the proof of Theorem 1, we only have to show that the sequence $(\tilde{W}_n)_{n\in\mathbb{N}}$ is tight and that $\tilde{Z}_{t,n} \to 0$ in bootstrap probability for all $t \in \mathbb{R}$ almost surely. Note that by Theorem 2.4 of Shao and Yu [17], $(\sqrt{n}(F_n(t_p) - F_n^\star(t_p)))_{n\in\mathbb{N}}$ is almost surely asymptotically normal and thus $(\tilde{W}_n)_{n\in\mathbb{N}}$ is tight.

First note that by the construction of the bootstrap random variables, the summands of $\tilde{Z}_{t,n}$ are independent conditional on $X_1, \ldots, X_n$ when the indices i lie in different blocks. Additionally, the random variables are centered in bootstrap probability and the sequence of blocks are stationary for fixed n. So

$$E^\star \left(\tilde{Z}_{t,n} \right)^2 \\ = \lfloor \frac{n}{l_n} \rfloor \frac{1}{a_n} \operatorname{Var}^\star \left[\sum_{i=1}^{l_n} \left(F_n(t_p + \frac{t}{\sqrt{a_n}}) - \mathbb{1}_{\{X_i^\star \le t_p + \frac{t}{\sqrt{a_n}}\}} - F_n(t_p) + \mathbb{1}_{\{X_i^\star \le t_p\}} \right) \right].$$

Recall that the bootstrap random variables $X_1^\star, \ldots, X_l^\star$ take the values $X_j, \ldots, X_{j+l-1}$ for $j = 1, \ldots, n$ with probability $\frac{1}{n}$ and that we have to set $X_j = X_{j-n}$ for $j > n$.

So we have the following upper bound for the bootstrap variance:

$$\begin{aligned}
&\mathrm{Var}^\star\left[\sum_{i=1}^{l_n}\left(F_n(t_p+\frac{t}{\sqrt{a_n}})-\mathbb{1}_{\{X_i^\star\leq t_p+\frac{t}{\sqrt{a_n}}\}}-F_n(t_p)+\mathbb{1}_{\{X_i^\star\leq t_p\}}\right)\right]\\
&\leq\frac{1}{n}\sum_{j=1}^{n}\left(\max_{m=1,\ldots,l}\sum_{i=j}^{j+m-1}\left(F_n(t_p+\frac{t}{\sqrt{a_n}})-\mathbb{1}_{\{X_i\leq t_p+\frac{t}{\sqrt{a_n}}\}}-F_n(t_p)+\mathbb{1}_{\{X_i\leq t_p\}}\right)\right)^2\\
&\leq 2\frac{1}{n}\sum_{j=1}^{n}\left(\max_{m=1,\ldots,l}\sum_{i=j}^{j+m-1}\left(F(t_p+\frac{t}{\sqrt{a_n}})-\mathbb{1}_{\{X_i\leq t_p+\frac{t}{\sqrt{a_n}}\}}-F(t_p)+\mathbb{1}_{\{X_i\leq t_p\}}\right)\right)^2\\
&\qquad\qquad+2l_n^2\left(F_n(t_p+\frac{t}{\sqrt{a_n}})-F(t_p+\frac{t}{\sqrt{a_n}})-F_n(t_p)+F(t_p)\right)^2.
\end{aligned}$$

To show the convergence of the bootstrap variance, we now need moment bounds for the maximum of the partial sums. By Davydov's inequality [6], we have that

$$\begin{aligned}
&\mathrm{Cov}\left(\mathbb{1}_{\{X_1\leq t_p+\frac{t}{\sqrt{a_n}}\}}-\mathbb{1}_{\{X_1\leq t_p\}},\mathbb{1}_{\{X_{1+k}\leq t_p+\frac{t}{\sqrt{a_n}}\}}-\mathbb{1}_{\{X_{1+k}\leq t_p\}}\right)\\
&\qquad\qquad\leq C\alpha^{\frac{2}{2+\epsilon}}(k)(\frac{t}{\sqrt{a_n}})^{\frac{\epsilon}{2+\epsilon}},
\end{aligned}$$

as $|F(t_p)-F(t_p+h)|\leq C|h|$. By standard calculations

$$\begin{aligned}
&E\left(\sum_{i=1}^{m}\left(F(t_p+\frac{t}{\sqrt{a_n}})-\mathbb{1}_{\{X_i\leq t_p+\frac{t}{\sqrt{a_n}}\}}-F(t_p)+\mathbb{1}_{\{X_i\leq t_p\}}\right)\right)^2\\
&\qquad\qquad\leq 2m\sum_{k=1}^{\infty}C\alpha^{\frac{2}{2+\epsilon}}(k)(\frac{t}{\sqrt{a_n}})^{\frac{\epsilon}{2+\epsilon}}\leq Cma_n^{-\frac{\epsilon}{4+\epsilon}}.
\end{aligned}$$

We obtain the following maximal inequality by Theorem 3 of Móricz [14]

$$\begin{aligned}
&E\left(\max_{m=1,\ldots,l}\sum_{i=1}^{m}\left(F(t_p+\frac{t}{\sqrt{a_n}})-\mathbb{1}_{\{X_i\leq t_p+\frac{t}{\sqrt{a_n}}\}}-F(t_p)+\mathbb{1}_{\{X_i\leq t_p\}}\right)\right)^2\\
&\qquad\qquad\leq Cl\log^2 la_n^{-\frac{\epsilon}{4+\epsilon}}.
\end{aligned}$$

To simplify the notation, we set

$$Y_n(i):=F(t_p+\frac{t}{\sqrt{a_n}})-\mathbb{1}_{\{X_i\leq t_p+\frac{t}{\sqrt{a_n}}\}}-F(t_p)+\mathbb{1}_{\{X_i\leq t_p\}}$$

By the Chebyshev inequality

$$
\begin{aligned}
&\sum_{k=1}^{\infty} P\big(\max_{n=2^k,\ldots,2^{k+1}-1} \tilde{Z}_{t,n} \geq \delta\big) \\
&\leq \frac{1}{\delta^2}\sum_{k=1}^{\infty} E\left[\max_{n=2^k,\ldots,2^{k+1}-1} \lfloor \frac{n}{l_n} \rfloor \frac{1}{a_n}\frac{2}{n}\sum_{j=1}^{n}\left(\max_{m=1,\ldots,l}\sum_{i=j}^{j+m-1} Y_n(i)\right)^2\right] \\
&\qquad + \frac{1}{\delta^2}\sum_{k=1}^{\infty} E\left[\max_{n=2^k,\ldots,2^{k+1}-1} \lfloor \frac{n}{l_n} \rfloor \frac{2}{a_n} l_n^2 \left(\frac{1}{n}\sum_{i=1}^{n} Y_n(i)\right)^2\right] \\
&\leq \frac{1}{\delta^2}\sum_{k=1}^{\infty} 8\frac{1}{l_{2^k}} E\left(\max_{m=1,\ldots,l}\sum_{i=1}^{m} Y_n(i)\right)^2 \\
&\qquad + \frac{1}{\delta^2}\sum_{k=1}^{\infty} 4\frac{l_{2^k}}{a_{2^k}^2} E\left(\max_{m=1,\ldots,2^{k+1}-1}\sum_{i=1}^{m} Y_n(i)\right)^2 \\
&\qquad \leq C\sum_{k=1}^{\infty} \log^2(l_{2^k}) a_{2^k}^{-\frac{\epsilon}{4+\epsilon}} + C\sum_{k=1}^{\infty} \log^2(a_{2^k}) a_{2^k}^{-\frac{\epsilon}{4+\epsilon}} < \infty.
\end{aligned}
$$

With the Borel-Cantelli-lemma, we have that $\tilde{Z}_{t,n}$ converges to 0 almost surely for all $t \in \mathbb{R}$ and the proof is complete. □

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