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Uniform oscillatory behavior of spherical functions of GL_n/U_n at the identity and a central limit theorem

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Abstract

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $n \in \mathbb{N}$. Let $(S_k)_{k\geq 0}$ be a time-homogeneous random walk on $GL_n(\mathbb{F})$ associated with an $U_n(\mathbb{F})$-biinvariant measure $\nu \in M^1(GL_n(\mathbb{F}))$. We derive a central limit theorem for the ordered singular spectrum $\sigma_{sing}(S_k)$ with a normal distribution as limit with explicit analytic formulas for the drift vector and the covariance matrix. The main ingredient for the proof will be a oscillatory result for the spherical functions $\varphi_{i\rho+\lambda}$ of $(GL_n(\mathbb{F}), U_n(\mathbb{F}))$. More precisely, we present a necessarily unique mapping $m_{\mathbf{1}} : G \to \mathbb{R}^n$ such that for some constant C and all $g \in G$, $\lambda \in \mathbb{R}^n$,

$$|\varphi_{i\rho+\lambda}(g) - e^{i\lambda\cdot m_{\mathbf{1}}(g)}| \leq C\|\lambda\|^2.$$

KEYWORDS: Biinvariant random walks on $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$, asymptotics of spherical functions, central limit theorem for the singular spectrum, random walks on the positive definite matrices, dispersion.

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1 Introduction

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $n \geq 2$ an integer, and $G := GL(n,\mathbb{F})$ the general linear group with maximal compact subgroup $K := U_n(\mathbb{F})$. Consider i.i.d. G-valued random variables $(X_k)_{k\geq 1}$ with the common K-biinvariant distribution $\nu \in M^1(G)$ and the associated G-valued random walk $(S_k := X_1 \cdot X_2 \cdots X_k)_{k\geq 0}$ with the convention that S_0 is the identity I_n. Moreover, let

$$\sigma_{sing}(g) \in \{x = (x_1,\ldots,x_n) \in \mathbb{R}^n : \, x_1 \geq x_2 \geq \cdots \geq x_n > 0\}$$

denote the singular (or Lyapunov) spectrum of $g \in G$ where the singular values of g, i.e., square roots of the eigenvalues of gg^*, are ordered by size. Consider the mapping $\ln \sigma_{sing}$ from G onto the Weyl chamber

$$W_n := \{x = (x_1,\ldots,x_n) \in \mathbb{R}^n : \, x_1 \geq x_2 \geq \cdots \geq x_n\},$$

with the logarithm $\ln(x_1,\ldots,x_n) := (\ln x_1,\ldots,\ln x_n)$. We show that under a natural moment condition, the $\mathbb{R}^n$-valued random variables

$$\frac{1}{\sqrt{k}}(2\cdot \ln \sigma_{sing}(S_k) - k\cdot m_{\mathbf{1}}(\nu)) \tag{1.1}$$

tend for $k \to \infty$ to some n-dimensional normal distribution $N(0, \Sigma^2(\nu))$ where the drift vector $m_{\mathbf{1}}(\nu)$ and the covariance matrix $\Sigma^2(\nu)$ are given explicitely depending on ν.

This central limit theorem (CLT) can be also seen as follows: By polar decomposition of $g \in G$, the symmetric space G/K can be identified with the cone $P_n(\mathbb{F})$ of positive definite symmetric or hermitian $n \times n$ matrices via

$$gK \mapsto I(g) := gg^* \in P_n(\mathbb{F}) \qquad (g \in G),$$

where G acts on $P_n(\mathbb{F})$ via $a \mapsto gag^*$. In this way, the double coset space $G//K$ can be identified with the Weyl chamber W_n via

$$KgK \mapsto \ln \sigma_{sing}(g) = \frac{1}{2} \ln \sigma(gg^*)$$

where σ denotes the spectrum, i.e., the ordered eigenvalues, of a positive definite matrix. Therefore, the CLT above may be regarded as a CLT for the spectrum of K-invariant random walks on $P_n(\mathbb{F})$. Such CLTs have a long history. CLTs where ν is renormalized first into some measure $\nu_k \in M^1(G)$, and then the convergence of the convolution powers ν_k^k is studied, can be found e.g. in [KTS], [Tu], [FH], [Te1], [Te2], [Ri], [G1], and [G2]. In this case, so-called dispersions of ν appear as parameters of the limits, where these dispersions are defined in terms of derivatives of the spherical functions of (G, K). These dispersions will also appear in our CLT in order to describe $m_{\mathbf{1}}(\nu)$ and $\Sigma^2(\nu)$. Our CLT is in principle well-known; see Theorem 1 of [Vi], as well as the CLTs of Le Page [L] and the monograph [BL]. However, our approach, which directly leads to analytic formulas for drift and covariance, seems to be new for $n > 2$. For $n = 2$, our CLT can be splitted into two one-dimensional parts, namely a classical part for the sum $\ln \det S_k = \sum_{l=1}^{k} \ln \det X_l$ of i.i.d. random variables, and a CLT for $(SL_2(\mathbb{F}), SU(2, \mathbb{F}))$. The associated spherical functions are the Jacobi functions $\varphi_\lambda^{(0,0)}(t)$ and $\varphi_\lambda^{(1/2,1/2)}(t)$ depending on $\mathbb{F}$ (see [K] for details), and the CLT above for $(SL_2(\mathbb{F}), SU(2, \mathbb{F}))$ appears as a special case of a CLT of Zeuner [Z] for certain Sturm-Liouville hypergroups on $[0, \infty[$. The proof of Zeuner depends on some uniform estimate for the oscillatory behavior of the associated multiplicative functions, i.e., the Jacobi functions here. This idea was later on transfered to certain random walks on the nonnegative integers associated with orthogonal polynomials in [V1]. Moreover, the result of Zeuner [Z] was recently slightly improved for Jacobi functions in [V2] by using the well-known Harish-Chandra integral representation of the Jacobi functions from [K]. We here adopt this approach and use the Harish-Chandra integral representation of the spherical function of (G, K) to derive a uniform estimate for their oscillatory behavior. The CLT above then follows easily.

Let us describe this uniform oscillatory result. Recapitulate that a K-biinvariant continuous function $\varphi \in C(G)$ on G is called spherical iff

$$\varphi(g_1)\varphi(g_2) = \int_K \varphi(g_1 k g_2)\, dk$$

for all $g_1, g_2 \in G$ where dk is the normalized Haar measure on K. It is well-known (see [H1] or [Te2]) that all spherical functions of (G, K) are given by the Harish-Chandra integral

$$\varphi_{i\rho+\lambda}(g) = \int_K \Delta_1^{i\lambda_1 - i\lambda_2}(k^* g g^* k) \cdots \Delta_{n-1}^{i\lambda_{n-1} - i\lambda_n}(k^* g g^* k) \Delta_n^{i\lambda_n}(k^* g g^* k)\, dk \tag{1.2}$$

where the Δ_r are the principal minors of order r, $\lambda \in \mathbb{C}^n$, and where $\rho = (\rho_1, \ldots, \rho_n)$ is the half sum of roots with $\rho_l = \frac{d}{2}(n + 1 - 2l)$ with the dimension $d = 1, 2$ of $\mathbb{F}$ over $\mathbb{R}$. Notice that by (1.2), $\varphi_{i\rho} \equiv 1$, and that for $\lambda \in \mathbb{R}^n$ and $g \in G$, $|\varphi_{i\rho+\lambda}(g)| \leq 1$.

We now follow the usual approach to the dispersion for (G, K) (see [FH],[Te1], [Te2], [Ri], [G1], [G2]) and to so-called moment functions on hypergroups in [Z], [V1], and Section 7.2.2 of [BH]: For multiindices $l = (l_1, \ldots, l_n) \in \mathbb{N}_0^n$ we define the so called moment functions

$$
\begin{aligned}
m_l(g) :=& \frac{\partial^{|l|}}{\partial \lambda^l} \varphi_{i\rho - i\lambda}(g)\Big|_{\lambda=0} := \frac{\partial^{|l|}}{(\partial\lambda_1)^{l_1} \cdots (\partial\lambda_n)^{l_n}} \varphi_{i\rho - i\lambda}(g)\Big|_{\lambda=0} \\
=& \int_K (\ln \Delta_1(k^*gg^*k))^{l_1} \cdot \left(\ln\left(\frac{\Delta_2(k^*gg^*k)}{\Delta_1(k^*gg^*k)}\right)\right)^{l_2} \cdots \left(\ln\left(\frac{\Delta_n(k^*gg^*k)}{\Delta_{n-1}(k^*gg^*k)}\right)\right)^{l_n} dk
\end{aligned}
\tag{1.3}
$$

of order $|l| := l_1 + \cdots + l_n$ for $g \in G$. Clearly, the last equality follows immediately from (1.2) by interchanging integration and derivatives. Using the n moment functions m_l of first order $|l| = 1$, we form the vector-valued moment function

$$
m_{\mathbf{1}}(g) := (m_{(1,0,\ldots,0)}(g), \ldots, m_{(0,\ldots,0,1)}(g)) \tag{1.4}
$$

of first order. Moreover, we use the usual scalar product $x \cdot y := \sum_{l=1}^n x_l y_l$ on $\mathbb{R}^n$. We can now formulate the following oscillatory result; it will be proved in Section 2.

1.1 Theorem. *There exists a constant $C = C(n)$ such that for all $g \in G$ and and $\lambda \in \mathbb{R}^n$,*

$$
|\varphi_{i\rho+\lambda}(g) - e^{i\lambda \cdot m_{\mathbf{1}}(g)}| \le C\|\lambda\|^2.
$$

The function $m_{\mathbf{1}}$ is obviously determined uniquely by the property of the theorem.

We return to the CLT. Similar to collecting the moment functions of first order in the vector $m_{\mathbf{1}}$, we group the moment functions of second order by

$$
\begin{aligned}
m_{\mathbf{2}}(g) :=& \begin{pmatrix} m_{1,1}(g) & \cdots & m_{1,n}(g) \\ \vdots & & \vdots \\ m_{n,1}(g) & \cdots & m_{n,n}(g) \end{pmatrix} \\
:=& \begin{pmatrix} m_{(2,0,\ldots,0)}(g) & m_{(1,1,0,\ldots,0)}(g) & \cdots & m_{(1,0,\ldots,0,1)}(g) \\ m_{(1,1,0,\ldots,0)}(g) & m_{(0,2,0,\ldots,0)}(g) & \cdots & m_{(0,1,0,\ldots,0,1)}(g) \\ \vdots & \vdots & & \vdots \\ m_{(1,0,\ldots,0,1)}(g) & m_{(0,1,0,\ldots,0,1)}(g) & \cdots & m_{(0,\ldots,0,2)}(g) \end{pmatrix}
\end{aligned}
\tag{1.5}
$$

for $g \in G$. We show in Section 3 as an easy consequence of (1.3) that the $n \times n$ matrices $m_{\mathbf{2}}(g) - m_{\mathbf{1}}(g)^t \cdot m_{\mathbf{1}}(g)$ are positive semidefinite.

Now consider $\nu \in M^1(G)$ such that the moment functions $m_{j,j} \ge 0$ $(j = 1, \ldots, n)$ are ν-integrable. We then say that ν admits finite second moments. In this case, (1.3) and the Cauchy-Schwarz inequality yield that all moments of order one and two are ν-integrable, and we form the modified expectation vector and covariance matrix

$$
m_{\mathbf{1}}(\nu) := \int_G m_{\mathbf{1}}(g)\, d\nu \in \mathbb{R}^n, \qquad \Sigma^2(\nu) := \int_G m_{\mathbf{2}}(g)\, d\nu \;-\; m_{\mathbf{1}}(\nu)^t \cdot m_{\mathbf{1}}(\nu)
$$

of ν. The precise statement of our CLT is now as follows:

1.2 Theorem. *If $\nu \in M^1(G)$ is K-biinvariant and admits finite second moments, then*

$$
\frac{1}{\sqrt{k}}(2 \cdot \ln \sigma_{sing}(S_k) - k \cdot m_{\mathbf{1}}(\nu)) \longrightarrow N(0, \Sigma^2(\nu)) \tag{1.6}
$$

for $k \to \infty$ in distribution.

This paper is organized as follows: Section 2 is devoted exclusively to the proof of Theorem 1.1. In Section 3 we then shall present the proof of Theorem 1.2. There we also give a precise condition on ν when $\Sigma^2(\nu)$ is positive definite.

We finally remark that the results of our paper can be transfered to the Grassmann manifolds $(SO_0(p,q)/(SO(p) \times SO(q))$ and $(SU(p,q)/S(U(p) \times U(q))$. In this case, the spherical functions are certain Heckman-Opdam hypergeometric functions of type BC, for which a Harish-Chandra integral representation analog to (1.2) is available; see [Sa] and [RV]. We plan to carry out this in near future.

2 Proof of the oscillatory behavior of spherical functions

This section is devoted to the proof of Theorem 1.1 which depends on several facts which may be more or less well-known. As we could not find suitable published references, we include proofs for sake of completeness. We start with a result about the principal minors Δ_r:

2.1 Lemma. *Let $1 \le r \le n$ be integers, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $u \in U_n(\mathbb{F})$. Then*

$$\Delta_r(u^* \cdot diag(a_1, \ldots, a_n) \cdot u) = \sum_{1 \le i_1 < i_2 < \ldots < i_r \le n} c_{i_1,\ldots,i_r} a_{i_1} \cdot a_{i_2} \cdots a_{i_r}$$

for all $a_{i_1}, a_{i_2}, \ldots, a_{i_r} \in \mathbb{R}$ with coefficients $c_{i_1,\ldots,i_r} = c_{i_1,\ldots,i_r}(u)$ satisfying $c_{i_1,\ldots,i_r} \ge 0$ for $1 \le i_1 < i_2 < \ldots < i_r \le n$ and $\sum_{1 \le i_1 < i_2 < \ldots < i_r \le n} c_{i_1,\ldots,i_r} = 1$.

Proof. Clearly, $h_r(a_1, \ldots, a_n) := \Delta_r(u^* \cdot diag(a_1, \ldots, a_n) \cdot u)$ is a homogeneous polynomial of degree r, i.e.,

$$h_r(a_1, \ldots, a_n) = \sum_{1 \le i_1 \le i_2 \le \ldots \le i_r \le n} c_{i_1,\ldots,i_r} a_{i_1} \cdot a_{i_2} \cdots a_{i_r}.$$

We first check that $c_{i_1,\ldots,i_r} \neq 0$ is possible only for $1 \le i_1 < i_2 < \ldots < i_r \le n$. For this consider $i_1, \ldots, i_r$ with $|\{i_1, \ldots, i_r\}| =: q < r$. By changing the numbering of the variables $a_1, \ldots, a_n$ (and of rows and columns of u in an appropriate way), we may assume that $\{i_1, \ldots, i_r\} = \{1, \ldots, q\}$. In this case, $u^* \cdot diag(a_1, \ldots, a_q, 0, \ldots, 0) \cdot u$ has rank at most $q < r$. Thus

$$0 = h_r(a_1, \ldots, a_q, 0, \ldots, 0) = \sum_{1 \le i_1 \le i_2 \le \ldots \le i_r \le q} c_{i_1,\ldots,i_r} a_{i_1} \cdot a_{i_2} \cdots a_{i_r}$$

for all $a_1, \ldots, a_q$. This yields $c_{i_1,\ldots,i_r} = 0$ for $1 \le i_1 \le i_2 \le \ldots \le i_r \le q$ and proves that

$$h_r(a_1, \ldots, a_n) = \sum_{1 \le i_1 < i_2 < \ldots < i_r \le n} c_{i_1,\ldots,i_r} a_{i_1} \cdot a_{i_2} \cdots a_{i_r}.$$

For the nonnegativity we again may restrict our attention to $c_{1,\ldots,r}$. In this case,

$$0 \le \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \le I_n \qquad \text{and thus} \qquad 0 \le u^* \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} u \le I_n$$

w.r.t. the usual ordering of positive semidefinite matrices. As this inequality holds also for the upper left $r \times r$ block, we obtain

$$c_{1,\ldots,r} = h_r(1, \ldots, 1, 0, \ldots, 0) = \Delta_r \left(u^* \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} u \right) \ge 0.$$

Finally, as

$$\sum_{1\le i_1<i_2<\ldots<i_r\le n} c_{i_1,\ldots,i_r} = h_r(1,\ldots,1) = 1,$$

the proof is complete. $\square$

Let us keep the notation of Lemma 2.1. We now compare $h_r(a_1,\ldots,a_n)$ with the homogeneous polynomial

$$C_r(a_1,\ldots,a_n) := \frac{1}{\binom{n}{r}} \sum_{1\le i_1<i_2<\ldots<i_r\le n} a_{i_1}a_{i_2}\cdots a_{i_r} > 0 \qquad (r=1,\ldots,n). \tag{2.1}$$

2.2 Lemma. *For all $a_1,\ldots,a_n>0$,*

$$0 < \frac{C_r(a_1,\ldots,a_n)}{h_r(a_1,\ldots,a_n)} \le \frac{1}{\binom{n}{r}} \sum_{1\le i_1<i_2<\ldots<i_r\le n} c_{i_1,\ldots,i_r}(u)^{-1},$$

where, depending on u, on both sides the value ∞ is possible.

Proof. Positivity is clear by Lemma 2.1. Moreover,

$$\begin{aligned} C_r(a_1,\ldots,a_n) =& \frac{1}{\binom{n}{r}} \sum_{1\le i_1<i_2<\ldots<i_r\le n} a_{i_1}a_{i_2}\cdots a_{i_r} \\ \le& \frac{\max_{1\le i_1<i_2<\ldots<i_r\le n} c_{i_1,\ldots,i_r}^{-1}}{\binom{n}{r}} \sum_{1\le i_1<i_2<\ldots<i_r\le n} c_{i_1,\ldots,i_r}a_{i_1}a_{i_2}\cdots a_{i_r} \end{aligned}$$

which immediately leads to the claim. $\square$

We also need the following observation from linear algebra:

2.3 Lemma. *Let $u\in U_n(\mathbb{C})$ have the block structure $u = \begin{pmatrix} u_1 & * \\ * & u_2 \end{pmatrix}$ with quadratic blocks $u_1\in M_r(\mathbb{C})$ and $u_2\in M_{n-r}(\mathbb{C})$ with $1\le r\le n$. Then $|\det u_1| = |\det u_2|$.*

Proof. W.l.o.g. we may assume $2r\le n$. By the KAK-decomposition of $U_n(\mathbb{C})$ with $K = U_r(\mathbb{C})\times U_{n-r}(\mathbb{C})$ (see e.g. Theorem VII.8.6 of [H2]), we write u as

$$u = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} \cdot \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & I_{q-2r} \end{pmatrix} \cdot \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}$$

with $a_1,a_2\in U_r(\mathbb{C})$, $b_1,b_2\in U_{n-r}(\mathbb{C})$ and with $c = diag(\cos\varphi_1,\ldots,\cos\varphi_r)$ and $s = diag(\sin\varphi_1,\ldots,\sin\varphi_r)$ for suitable $\varphi_1,\ldots,\varphi_r\in\mathbb{R}$. Therefore,

$$u_1 = a_1ca_2 \qquad \text{and} \qquad u_2 = b_1\begin{pmatrix} c & 0 \\ 0 & I_{q-2r} \end{pmatrix} b_2$$

which immediately implies the claim. $\square$

We shall also need the following elementary observation:

2.4 Lemma. *Let $\varepsilon \in]0,1]$, $M \geq 1$ and $m \in \mathbb{N}$. Then there exists a constant $C = C(\varepsilon, M, m) > 0$ such that for all $z \in]0, M]$,*

$$|\ln(z)|^m \leq C\left(1 + z^{-\varepsilon}\right).$$

Proof. Elementary calculus yields $|x^\varepsilon \cdot \ln x| \leq 1/(e\varepsilon)$ for $x \in]0,1]$ and the Euler number $e = 2,71....$ This leads to the estimate for $z \in]0,1]$. The estimate is trivial for $z \in]1, M]$. □

Proof of Theorem 1.1: As the spherical functions and the moment functions on G are constant on the double cosets w.r.t. K by definition, and as each double coset has a representative g such that $gg^* = diag(a_1, \ldots, a_n)$ is diagonal with $a_1 \geq \ldots \geq a_n > 0$, it suffices to consider these group elements $g \in G$. We thus fix $\lambda \in \mathbb{R}^n$ and $a_1 \geq \ldots \geq a_n > 0$ and put $a := diag(a_1, \ldots, a_n)$. According to (1.2), (1.3) and (1.4) we have to estimate

$$\begin{aligned} R :=& R(\lambda, a) := |\varphi_{i\rho+\lambda}(g) - e^{i\lambda\cdot m_{\mathbf{1}}(g)}| \\ =& \Bigg|\int_K exp\left(i\sum_{r=1}^{n}(\lambda_r - \lambda_{r+1}) \cdot \ln \Delta_r(k^*ak)\right) dk \\ &\quad - exp\left(i\int_K \sum_{r=1}^{n}(\lambda_r - \lambda_{r+1}) \cdot \ln \Delta_r(k^*ak)\, dk\right)\Bigg| \end{aligned} \tag{2.2}$$

with the convention $\lambda_{n+1} := 0$. For $r = 1, \ldots, n$, we now use the polynomial C_r from Eq. (2.1) and write the logarithms of the principal minors in (2.2) as

$$\ln \Delta_r(k^*ak) = \ln C_r(a_1, \ldots, a_r) + \ln(H_r(k, a)) \quad \text{with} \quad H_r(k, a) := \frac{\Delta_r(k^*ak)}{C_r(a_1, \ldots, a_n)}. \tag{2.3}$$

With this notation and with $|e^{ix}| = 1$ for $x \in \mathbb{R}$, we rewrite (2.2) as

$$\begin{aligned} R =& \Bigg|\int_K exp\left(i\sum_{r=1}^{n}(\lambda_r - \lambda_{r+1}) \cdot \ln(H_r(k, a))\right) dk \\ &\quad - exp\left(i\int_K \sum_{r=1}^{n}(\lambda_r - \lambda_{r+1}) \cdot \ln(H_r(k, a))\, dk\right)\Bigg|. \end{aligned} \tag{2.4}$$

We now use the power series for both exponential functions and observe that the terms of order 0 and 1 are equal in the difference above. Hence,

$$R \leq R_1 + R_2$$

for

$$R_1 := \int_K \left| exp\left(i\sum_{r=1}^{n}(\lambda_r - \lambda_{r+1}) \cdot \ln(H_r(k, a))\right) - \left(1 + i\sum_{r=1}^{n}(\lambda_r - \lambda_{r+1}) \cdot \ln(H_r(k, a))\right)\right| dk$$

and

$$R_2 := \left| exp\left(i\int_K \sum_{r=1}^{n}(\lambda_r - \lambda_{r+1}) \cdot \ln(H_r(k, a))\, dk\right) - 1 - i\int_K \sum_{r=1}^{n}(\lambda_r - \lambda_{r+1}) \cdot \ln(H_r(k, a))\, dk\right|.$$

Using the well-known elementary estimates $|\cos x - 1| \leq x^2/2$ and $|\sin x - x| \leq x^2/2$ for $x \in \mathbb{R}$, we obtain $|e^{ix} - (1+ix)| \leq x^2$ for $x \in \mathbb{R}$. Therefore, defining

$$A_m := \int_K \left| \sum_{r=1}^{n} (\lambda_r - \lambda_{r+1}) \cdot \ln(H_r(k,a)) \right|^m dk \qquad (m = 1, 2),$$

we conclude that

$$R \leq R_1 + R_2 \leq A_2 + A_1^2.$$

In the following, let $C_1, C_2, \ldots$ suitable constants. As $A_1^2 \leq A_2$ by Jensen's inequality, and as

$$A_2 \leq \|\lambda\|^2 \cdot C_1 \cdot \int_K \sum_{r=1}^{n} |\ln(H_r(k,a))|^2 \, dk =: \|\lambda\|^2 \cdot B_2,$$

we obtain $R \leq B_2 \cdot 2\|\lambda\|^2$. In order to complete the proof, we must check that B_2, i.e., that the integrals

$$L_r := \int_K |\ln(H_r(k,a))|^2 \, dk \tag{2.5}$$

remain bounded independent of $a_1, \ldots, a_n > 0$ for $r = 1, \ldots, n$.

For this fix r. Lemma 2.1 in particular implies that for all $a_1, \ldots, a_n > 0$,

$$\Delta_r(k^* a k) \leq \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n} a_{i_1} \cdot a_{i_2} \cdot \cdots \cdot a_{i_r} = \binom{n}{r} C_r(a_1, \ldots, a_n)$$

and $\Delta_r(k^* a k) > 0$. In other words,

$$0 < \frac{\Delta_r(k^* a k)}{C_r(a_1, \ldots, a_n)} = H_r(k,a) \leq \binom{n}{r}. \tag{2.6}$$

We conclude from (2.5), (2.6) and Lemma 2.4 that for any $\varepsilon \in]0,1[$ and suitable $C_2 = C_2(\varepsilon)$,

$$L_r \leq C_2 \int_K \left(1 + H_r(a_1, \ldots, a_n)^{-\varepsilon}\right) \, dk.$$

Therefore, by Lemma 2.2,

$$\begin{aligned} L_r \leq & C_2 + C_3 \int_K \left(\sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n} c_{i_1, \ldots, i_r}(k)^{-1} \right)^{\varepsilon} dk \\ \leq & C_2 + C_3 \cdot \binom{n}{r}^{\varepsilon} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n} \int_K c_{i_1, \ldots, i_r}(k)^{-\varepsilon} \, dk. \end{aligned} \tag{2.7}$$

The right hand side of (2.7) is independent of $a_1, \ldots, a_n$, and, by the definition of the $c_{i_1, \ldots, i_r}(k)$ in Lemma 2.1, $\int_K c_{i_1, \ldots, i_r}(k)^{-\varepsilon} \, dk$ is independent of $1 \leq i_1 < i_2 < \ldots < i_r \leq n$. Therefore, it suffices to check that

$$I_r := \int_K c_{1, \ldots, r}(k)^{-\varepsilon} \, dk = \int_K \Delta_r \left(k^* \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} k \right)^{-\varepsilon} dk < \infty.$$

For this, we write k as block matrix $k = \begin{pmatrix} k_r & * \\ * & k_{n-r} \end{pmatrix}$ with $k_r \in M_r(\mathbb{C})$ and $k_{n-r} \in M_{n-r}(\mathbb{C})$ and observe that

$$\Delta_r \left(k^* \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} k \right) = \Delta_r \begin{pmatrix} k_r^* k_r & * \\ * & * \end{pmatrix} = |\det k_r|^2.$$

We thus have to check that $\int_K |\det k_r|^{-2\varepsilon}\, dk < \infty$. As this is a consequence of the following lemma, the proof of the theorem is complete. □

2.5 Lemma. *Keep the block matrix notation above. For $\varepsilon < 1/2$,*

$$\int_K |\det k_r|^{-2\varepsilon}\, dk < \infty.$$

Proof. The statement is obvious for $r = n$. Moreover, by Lemma 2.3 we may assume that $1 \le r \le n/2$ which we shall assume now. In this case, we introduce the matrix ball

$$B_r := \{w \in M_r(\mathbb{F}) : \; w^* w \le I_r\}$$

as well as the ball $B := \{y \in M_{1,r}(\mathbb{F}) \equiv \mathbb{F}^n : \; \|y\|_2^2 \le 1\}$. We conclude from the truncation lemma 2.1 of [R2] that

$$\frac{1}{\kappa_r} \int_K |\det k_r|^{-2\varepsilon}\, dk = \int_{B_r} |\det w|^{-2\varepsilon} \Delta(I_r - w^* w)^{(n-2r+1)\cdot d/2-1}\, dw$$

where dw is the usual Lebesgue measure on the ball B_r and

$$\kappa_r := \left(\int_{B_r} \det(I_r - w^* w)^{(n-2r+1)\cdot d/2-1}\, dw \right)^{-1}.$$

Moreover, by Lemma 3.7 and Corollary 3.8 of [R1], the mapping $P : B^r \to B_r$ with

$$P(y_1, \ldots, y_r) := \begin{pmatrix} y_1 \\ y_2 (I_r - y_1^* y_1)^{1/2} \\ \vdots \\ y_r (I_r - y_{r-1}^* y_{r-1})^{1/2} \cdots (I_r - y_1^* y_1)^{1/2} \end{pmatrix} \tag{2.8}$$

establishes a diffeomorphism such that the image of the measure $\det(I_r - w^* w)^{(n-2r+1)\cdot d/2-1} dw$ under P^{-1} is $\prod_{j=1}^r (1 - \|y_j\|_2^2)^{(n-r-j+1)\cdot d/2-1} dy_1 \ldots dy_r$. Moreover, we show in Lemma 2.6 below that

$$\det P(y_1, \ldots, y_r) = \det \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix}.$$

We thus conclude that

$$\int_K |\det k_r|^{-2\varepsilon}\, dk = \frac{1}{\kappa_r} \int_B \cdots \int_B \left| \det \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} \right|^{-2\varepsilon} \prod_{j=1}^r (1 - \|y_j\|_2^2)^{(n-r-j+1)\cdot d/2-1} dy_1 \ldots dy_r. \tag{2.9}$$

This integral is finite for $\varepsilon < 1/2$, as one can use Fubini with an one-dimensional inner integral w.r.t. the (1,1)-variable. After this inner integration, no further singularities appear from the determinant-part in the remaining integral. □

2.6 Lemma. *Keep the notations of the preceding proof. For all $y_1, \ldots, y_n \in B$,*

$$\det P(y_1, \ldots, y_r) = \det \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix}.$$

Proof. Fix $y_1 \in B$. The mapping $y \mapsto y(I_r - y_1^* y_1)^{1/2}$ on B has the following form: If y is written as $y = ay_1 + y^\perp$ in a unique way with $a \in \mathbb{F}$ and $y^\perp \perp y_1$, then $y(I_r - y_1^* y_1)^{1/2} = \sqrt{1 - \|y_1\|_2^2} \cdot ay_1 + y^\perp$ (write $I_r - y_1^* y_1$ in an orthonormal basis with $y_1/\|y_1\|_2$ as a member!). Using linearity of the determinant in all lines, we thus conclude that

$$\det \begin{pmatrix} y_1 \\ y_2(I_r - y_1^* y_1)^{1/2} \\ \vdots \\ y_r(I_r - y_{r-1}^* y_{r-1})^{1/2} \cdots (I_r - y_1^* y_1)^{1/2} \end{pmatrix} = \det \begin{pmatrix} y_1 \\ y_2 \\ y_3(I_r - y_2^* y_2)^{1/2} \\ \vdots \\ y_r(I_r - y_{r-1}^* y_{r-1})^{1/2} \cdots (I_r - y_2^* y_2)^{1/2} \end{pmatrix}.$$

The lemma now follows by an obvious induction. $\square$

3 Moments and the proof of a central limit theorem

In this section we prove Theorem 1.2 and related results. We start with some facts about the moment functions of Section 1. The first result concerns an estimate for $m_{\mathbf{1}}$.

3.1 Lemma. *For $r = 1, \ldots, n$ let*

$$s_r(g) := m_{(1,0,\ldots,0)}(g) + \cdots + m_{(0,\ldots,0,1,0,\ldots,0)}(g) \quad (g \in G)$$

be the sum of the first r moment functions of first order. Moreover, let $\sigma_1(a) \geq \ldots \geq \sigma_n(a) > 0$ be the ordered eigenvalues of a positive definite $n \times n$ matrix a. Then:

(1) $s_n(g) = \ln \det(gg^*)$.

(2) *There is a constant $C = C(n)$ such that for all $r = 1, \ldots, n$ and $g \in G$,*

$$0 \leq \ln \sigma_1(gg^*) + \ldots + \ln \sigma_r(gg^*) \; - \; s_r(g) \leq C.$$

(3) *There is a constant $C = C(n)$ such that for all $g \in G$*

$$\|2 \ln \sigma_{sing}(g) - m_{\mathbf{1}}(g)\| \leq C.$$

Proof. We may assume that $gg^* = diag(a_1, \ldots, a_n)$ with $a_l = \sigma_l(gg^*)$ $(l = 1 \ldots, n)$. The integral representation (1.3) implies that

$$s_r(g) = \int_K \ln \Delta_r(k^* gg^* k)\, dk.$$

This proves (1) and, in combination with Lemma 2.1, the first inequality in (2). For the second inequality of (2), we use the notations of Lemmas 2.1 and 2.2. By the proof of Lemma 2.2, we have for $k \in K$,

$$a_1 \cdot a_2 \ldots a_r \leq \binom{n}{r} C_r(a_1, \ldots, a_n) \leq \max_{1 \leq i_1 < \ldots < i_r \leq n} \frac{\ln \Delta_r(k^* gg^* k)}{c_{i_1,\ldots,i_r}(k)}.$$

Therefore,

$$\ln \sigma_1(gg^*) + \ldots + \ln \sigma_r(gg^*) = \int_K \ln(a_1 \cdot a_2 \ldots a_r)\, dk \le \int_K \ln \Delta_r(k^* g g^* k)\, dk + M$$

for

$$M := \int_K \max_{1\le i_1<\ldots<i_r\le n} \frac{1}{c_{i_1,\ldots,i_r}(k)}\, dk \le \sum_{1\le i_1<\ldots<i_r\le n} \int_K \ln(c_{i_1,\ldots,i_r}(k)^{-1})\, dk.$$

As by the definition of $c_{i_1,\ldots,i_r}(k)$ all integrals in the sum are obviously equal, it suffices to show that

$$\int_K \ln(c_{1,\ldots,r}(k)^{-1})\, dk = -\int_K \ln \Delta_r \left(k^* \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} k \right) dk$$

is finite. But this follows immediately from Lemma 2.5. This proves (2). Finally, (3) is a consequence of (2). □

Lemma 3.1(3) implies that there exists $C = C(n) > 0$ such that for all $g \in G$,

$$|e^{2i\lambda\cdot\ln \sigma_{sing}(g)} - e^{i\lambda\cdot m_{\mathbf{1}}(g)}| \le C \cdot \|\lambda\|. \tag{3.1}$$

Therefore, we conclude from Theorem 1.1:

3.2 Corollary. *There exists a constant $C = C(n) > 0$ such that for all $g \in G$,*

$$\|\varphi_{i\rho-\lambda}(g) - e^{2i\lambda\cdot\ln \sigma_{sing}(g)}\| \le C \cdot \|\lambda\|.$$

3.3 Remark. It can be easily checked (e.g. for $n = 2$ from explicit formulas in [K]) that the uniform orders $\|\lambda\|^2$ and $\|\lambda\|$ in Theorem 1.1 and Corollary 3.2 respectively are sharp. We note that Corollary 3.2 is closely related to the Harish-Chandra expansion of the spherical functions; see e.g. Opdam [O] and Lemma I.4.2.2 of [HS] in the context of Heckman-Opdam hypergeometric functions which includes our setting. We also remark that in the proof of the CLT 1.2 below Corollary 3.2 would be sufficient instead of the stronger Theorem 1.1. On the other hand, Theorem 1.1 leads generally to stronger rates of convergence in the CLT; see e.g. Theorem 4.2 of [V2] for the rank one case.

We shall also need the following estimate which follows immediately from the integral representation (1.2):

3.4 Lemma. *For all $g \in G$ and $l \in \mathbb{N}_0^n$,*

$$\left| \frac{\partial^{|l|}}{\partial \lambda^l} \varphi_{i\rho-\lambda}(g) \right| \le m_l(g).$$

Let $m \in \mathbb{N}_0$ and $\nu \in M^1(G)$ a K-biinvariant probability measure. We say that ν admits finite m-th modified moments if in the notation of the introduction on the moment functions,

$$m_{(m,0,\ldots,0)}, m_{(0,m,0,\ldots,0)}, \ldots, m_{(0,\ldots,0,m)} \in L^1(G,\nu).$$

It follows immediately from (1.3) and Hölder's inequality that in this case all moment functions of order at most m are ν-integrable. Moreover, this moment condition implies a corresponding differentiability of the spherical Fourier transform of ν:

3.5 Lemma. *Let $m \in \mathbb{N}_0$ and $\nu \in M^1(G)$ a K-biinvariant probability measure with finite m-th moments. Then the spherical Fourier transform*

$$\tilde{\nu} : \mathbb{R}^n \to \mathbb{C}, \quad \lambda \mapsto \int_G \varphi_{i\rho-\lambda}(g)\, d\nu(g)$$

is m-times continuously partially differentiable, and for all $l \in \mathbb{N}_0^n$ with $|l| \leq m$,

$$\frac{\partial^{|l|}}{\partial \lambda^l} \tilde{\nu}(\lambda) = \int_G \frac{\partial^{|l|}}{\partial \lambda^l} \varphi_{i\rho-\lambda}(g)\, d\nu(g). \tag{3.2}$$

In particular,

$$\frac{\partial^{|l|}}{\partial \lambda^l} \tilde{\nu}(0) = (-i)^{|l|} \int_G m_l(g)\, d\nu(g). \tag{3.3}$$

Proof. We proceed by induction: The case $m = 0$ is trivial, and for $m \to m+1$ we observe that by our assumption all moments of lower order exist, i.e., (3.2) is available for all $|l| \leq m$. It now follows from Lemma 3.4 and the well-known result about parameter integrals that a further partial derivative and the integration can be interchanged. Finally, (3.3) follows from (3.2) and (1.3). Continuity of the derivatives is also clear by Lemma 3.4. □

We next turn to the positive (semi)definiteness of the modified covariance matrix $\sigma^2(\nu)$ for biinvariant measures with finite second modified moments. We start with measures concentrated on a double coset:

3.6 Lemma. *Let $n \geq 2$, $g \in G$, and $\Sigma^2(g) := m_{\mathbf{2}}(g) - m_{\mathbf{1}}(g)^t m_{\mathbf{1}}(g)$.*

(1) *$\Sigma^2(g)$ is positive semidefinite.*

(2) *If gg^* is not a multiple of the identity matrix, then $\Sigma^2(g)$ has rank $n-1$.*

(3) *If gg^* is a multiple of the identity matrix, then $\Sigma^2(g) = 0$.*

Proof. Let $a_1, \ldots, a_n \in \mathbb{R}$ with $a_1^2 + \ldots + a_n^2 > 0$ and the row vector $a = (a_1, \ldots, a_n)$. Put

$$f_1(k) := \ln \Delta_1(k^* gg^* k) \quad \text{and} \quad f_l(k) := \ln \Delta_l(k^* gg^* k) - \ln \Delta_{l-1}(k^* gg^* k) \quad (l = 2, \cdots, n).$$

Then, by (1.3), (1.4), (1.5), and the Cauchy-Schwarz inequality,

$$a\left(m_{\mathbf{2}}(g) - m_{\mathbf{1}}(g)^t m_{\mathbf{1}}(g)\right) a^t = \int_K \left(\sum_{l=1}^n a_l f_l(k)\right)^2 dk - \left(\int_K \sum_{l=1}^n a_l f_l(k)\, dk\right)^2 \geq 0.$$

Moreover, this expression is equal to 0 if and only if the function

$$k \mapsto \sum_{l=1}^n a_l f_l(k) = (a_1 - a_2) \ln \Delta_1(k^* gg^* k) + \cdots + (a_{n-1} - a_n) \ln \Delta_{n-1}(k^* gg^* k) + a_n \ln \Delta_n(k^* gg^* k)$$

is constant on K. As $k \mapsto \ln \Delta_n(k^* gg^* k)$ is constant on K, and as under the condition of (2), the functions $k \mapsto \ln \Delta_r(k^* gg^* k)$ $(r = 1, \ldots, n-1)$ and the constant function 1 are linearly independent on K by Corollary 4.2 in the appendix, the function $k \mapsto \sum_{l=1}^n a_l f_l(k)$ is constant on K precisely for $a_1 = a_2 = \ldots = a_n$. This proves (2). Part (3) is obvious. □

The arguments of the preceding proof lead to the following characterization of K-biinvariant measures with positive definite covariance matrices:

3.7 Lemma. *Let $\nu \in M^1(G)$ be a K-biinvariant probability measure having second modified moments. Then $\Sigma^2(\nu)$ is positive definite if and only if $supp \nu$ is not contained in the subgroups $\{cI_n : c \in \mathbb{F}, c \neq 0\}$ and $SL_n(\mathbb{F})$.*

We now turn to the proof of the CLT:

Proof of Theorem 1.2. Let $\nu \in M^1(G)$ be a K-biinvariant probability measure with finite second modified moments. Let $(X_k)_{k\geq 1}$ be i.i.d. G-valued random variables with distribution ν and $S_k := X_1 \cdot X_2 \cdots X_k$. Let $\lambda \in \mathbb{R}^n$. As the functions $\varphi_{i\rho-\lambda}$ are bounded on G (by the integral representation (1.2)) and multiplicative w.r.t. K-biinvariant measures, we have

$$E(\varphi_{i\rho-\lambda/\sqrt{k}}(S_k)) = \int_G \varphi_{i\rho-\lambda/\sqrt{k}}(g)\, d\nu^{(k)}(g) = \left(\int_G \varphi_{i\rho-\lambda/\sqrt{k}}(g)\, d\nu(g)\right)^k = \tilde{\nu}(\lambda/\sqrt{k})^k.$$

We now use Taylor's formula, Lemma 3.5, and

$$m_{\mathbf{2}}(\nu) := \int_G m_{\mathbf{2}}(g)\, d\nu(g) = \Sigma^2(\nu) + m_{\mathbf{1}}(\nu)^t m_{\mathbf{1}}(\nu)$$

and obtain

$$\begin{aligned}
exp(i\lambda \cdot m_{\mathbf{1}}(\nu)\sqrt{k}) \cdot E(\varphi_{i\rho-\lambda/\sqrt{k}}(S_k)) \quad &= \quad \left(exp(i\lambda \cdot m_{\mathbf{1}}(\nu)/\sqrt{k}) \cdot \tilde{\nu}(\lambda/\sqrt{k})\right)^k \qquad (3.4) \\
= \left(\left[1 + \frac{i\lambda \cdot m_{\mathbf{1}}(\nu)}{\sqrt{k}} - \frac{(\lambda \cdot m_{\mathbf{1}}(\nu))^2}{2k} + o(\frac{1}{k})\right] \cdot \left[1 - \frac{i\lambda \cdot m_{\mathbf{1}}(\nu)}{\sqrt{k}} - \frac{\lambda m_{\mathbf{2}}(\nu)\lambda^t}{2k} + o(\frac{1}{k})\right]\right)^k& \\
= \left(\left[1 + \frac{i\lambda \cdot m_{\mathbf{1}}(\nu)}{\sqrt{k}} - \frac{(\lambda \cdot m_{\mathbf{1}}(\nu))^2}{2k} + o(\frac{1}{k})\right] \cdot\right.& \\
\left. \times \left[1 - \frac{i\lambda \cdot m_{\mathbf{1}}(\nu)}{\sqrt{k}} - \frac{\lambda(\Sigma^2(\nu) + m_{\mathbf{1}}(\nu)^t m_{\mathbf{1}}(\nu))\lambda^t}{2k} + o(\frac{1}{k})\right]\right)^k& \\
= \left(1 - \frac{\lambda\Sigma^2(\nu)\lambda^t}{2k} + o(\frac{1}{k})\right)^k.&
\end{aligned}$$

Therefore,

$$\lim_{k\to\infty} exp(i\lambda \cdot m_{\mathbf{1}}(\nu)\sqrt{k}) \cdot E(\varphi_{i\rho-\lambda/\sqrt{k}}(S_k)) = exp(-\lambda\Sigma^2(\nu)\lambda^t/2).$$

Moreover, by Theorem 1.1,

$$\lim_{k\to\infty} E\left(\varphi_{i\rho-\lambda/\sqrt{k}}(S_k) - exp(-i\lambda \cdot m_{\mathbf{1}}(S_k)/\sqrt{k})\right) = 0.$$

We conclude that

$$\lim_{k\to\infty} exp(-i\lambda \cdot (m_{\mathbf{1}}(S_k) - k \cdot m_{\mathbf{1}}(\nu))/\sqrt{k}) = exp(-\lambda\Sigma^2(\nu)\lambda^t/2)$$

for all $\lambda \in \mathbb{R}^n$. Levy's continuity theorem for the classical n-dimensional Fourier transform now implies that $(m_{\mathbf{1}}(S_k) - k \cdot m_{\mathbf{1}}(\nu))/\sqrt{k}$ tends in distribution to $N(0, \Sigma^2(\nu))$. By the estimate of Lemma 3.1(1), this immediately implies Theorem 1.2. $\square$

On the basis of Theorem 1.1, also a Berry-Esseen-type estimate with the order $O(k^{-1/3})$ of convergence can be derived. As the details are technical, but quite similar to the proof of the corresponding rank-one-case in Theorem 4.2 of [V2], we here omit details. We also mention that Theorem 1.1 can be also used to derive further CLTs e.g. with stable distributions with domains of attraction or a Lindeberg-Feller CLT. The details of proof then would be also very similar to the classical cases for sums of iid random variables.

4 Appendix

Here we collect some results from linear algebra which are needed in Section 3.

4.1 Lemma. *Let* $x_1, \ldots, , x_n \in \mathbb{R}$. *Then*

$$\det \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & \cdots & x_n \\ x_1+x_2 & x_2+x_1 & x_3+x_2 & x_4+x_2 & \cdots & x_n+x_2 \\ x_1+x_2+x_3 & x_2+x_1+x_3 & x_3+x_2+x_1 & x_4+x_2+x_3 & \cdots & x_n+x_2+x_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{l=1}^n x_l & \sum_{l=1}^n x_l & \sum_{l=1}^n x_l & \sum_{l=1}^n x_l & \cdots & \sum_{l=1}^n x_l \end{pmatrix} =$$
$$= (x_1+x_2+\cdots+x_n)\cdot(x_1-x_2)\cdot(x_1-x_3)\cdots(x_1-x_n)$$

Proof. The determinant is a homogeneous polynomial in the the variables $x_1, \ldots, x_n$ of degree n. Moreover, the monomial x_1^n appears in this polynomial with coefficient 1, and for given $x_2, \ldots, x_n$, the determinant is a polynomial in the variable x_1 where $-(x_2+\cdots+x_n)$, x_2, x_3,..., x_n are the zeros of this polynomial. This leads readily to the claim. □

4.2 Corollary. *Let* $a_1, \ldots, a_n > 0$ *numbers such that at least two of them are different. Consider the diagonal matrix* $a = diag(a_1, \ldots, a_n)$. *Then the functions* $k \mapsto \ln \Delta_r(k^*ak)$ *with* $r = 1, \ldots, n-1$ *and the constant function 1 on* $K = U_n(\mathbb{F})$ *are linearly independent.*

Proof. Without loss of generality, a_1 is different from $a_2, \ldots, a_n$. Now consider the n permutation matrices k_l which permute the rows 1 and l and leave the other rows invariant for $l = 1, \cdots, n$. Then, using the notation $x_l := \ln a_l$, the number $\ln \Delta_r(k_j^* a k_j)$ is precisely the r, l-entry of the matrix in Lemma 4.1. Therefore, by Lemma 4.1, $\det((\ln \Delta_r(k_j^* a k_j))_{r,j=1,\ldots,n}) \neq 0$ for $x_1 + \ldots + x_n \neq 0$, i.e., a with $\det a \neq 1$. As $\ln \Delta_n(k^*ak)$ is constant, this proves the statement of the corollary for $\det a \neq 1$. The case $\det a = 1$ can be easily derived by considering $2a$ instead of a in the preceding argument. □

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