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# Remarks on pseudo stable laws on contractible groups

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# REMARKS ON PSEUDO STABLE LAWS ON CONTRACTIBLE GROUPS

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Abstract. In this note we discuss a class of probability distributions on homogeneous groups, called *pseudo stable laws* which were investigated in [5] for the real line. In particular it is shown that under mild assumptions these distributions belong to the domain of normal attraction of stable laws.

## Introduction

In [5] the authors investigate a class of distributions of real random variables, called $(c, p)$-pseudo stable laws, which are closely related to symmetric stable laws. These investigations are motivated by an earlier publication [8]. See also [3] and [7] for some generalizations (in the context of *weak stability*). A real random variable $X$ resp. its distribution $\mu = X(P)$ is called $(c, p)$-pseudo stable if for all real $a, b$ and iid $X, Y$, and independent $X, Z$, we have $a \cdot X + b \cdot Y \stackrel{d}{=} c \cdot X + d^{1/r} \cdot Z$, where $c = c(a, b) \cong (a^p + b^p)^{1/p}$ for some (fixed) $p > 0$, $d = d(a, b) \geq 0$, and $Z$ with distribution $Z(P) = \gamma$ denotes a symmetric stable random variable with characteristic function $\widehat{\gamma}(u) = \mathrm{e}^{-|u|^r}$, for some $0 < r \leq 2$. $(\gamma_t)_{t \geq 0}$ denotes the corresponding continuous convolution semigroup.

In order to investigate similar structures of distributions on more general state spaces, in particular on locally compact groups, we first re-write the defining equation in an equivalent form. Let, for $\alpha \in \mathbb{R}$, $H_\alpha$ denote the homothetical transformation $H_\alpha : x \mapsto \alpha \cdot x$. $\mu \in \mathcal{M}^1(\mathbb{R})$ is pseudo stable iff, for some $d(\cdot, \cdot) \geq 0$,

$$H_a(\mu) \star H_b(\mu) = H_{c(a,b)}(\mu) \star \gamma_{d(a,b)}$$

Now replace $\mathbb{R}$ by a locally compact group $\mathbb{G}$, replace $(H_t)_{t>0}$ by a contracting continuous group of automorphisms $(\tau_t)_{t>0} \subseteq \mathrm{Aut}(\mathbb{G})$ (with $\tau_t \tau_s = \tau_{ts}$). Fix $p, r > 0$ and let $(\gamma_u)_{u \geq 0}$, $\gamma_0 = \varepsilon_e$, be a continuous convolution semigroup which is (strictly) stable w.r.t. $(\tau_{t^{1/r}})$, i.e., $\tau_t(\gamma_s) = \gamma_{t^r \cdot s}$ (resp. $\tau_{t^{1/r}}(\gamma_s) = \gamma_{t \cdot s}$) for $s \geq 0, t > 0$. We fix the function $c : \mathbb{R}_+^2 \to \mathbb{R}_+$, $c(a, b) := (a^p + b^p)^{1/p}$. $\mu \in \mathcal{M}^1(\mathbb{G})$ is called pseudo stable w.r.t. $((\tau_t), p, r, (\gamma_s))$ if for all $a, b > 0$ we have

$$\tau_a(\mu) \star \tau_b(\mu) = \tau_{c(a,b)}(\mu) \star \gamma_{d(a,b)} \tag{0.1}$$

for some $d = d(a, b) \geq 0$, and furthermore, if there exists a commutative $\star$-sub-semigroup $\mathcal{S} \subseteq \mathcal{M}^1(\mathbb{G})$ such that $\{\tau_u(\mu) : u > 0, \gamma_v : v \geq 0\} \subseteq \mathcal{S}$.

Note that the existence of stable continuous convolution semigroups $(\gamma_u)$ with $\gamma_0 = \varepsilon_e$, equivalently, the existence of a contracting one parameter group $(\tau_t)$ implies that $\mathbb{G}$ is a homogeneous group, in particular a simply connected nilpotent Lie group.

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Recall that Abelian homogeneous groups are just vector spaces. It should be pointed out that the setup chosen here covers also (strictly) operator stable laws on $\mathbb{R}^d$. For the background on stable laws on groups see e.g. [1], for operator stable laws on vector spaces cf. e.g., [6], or [1], Ch. I, and the references mentioned there.

Our assumptions differ slightly from [5]: In (0.1) $a \leq 0$ or $b \leq 0$ are not admitted, and the distributions $\gamma_u$ are not supposed to be symmetric. Hence, in particular, one-sided stable distributions on the positive half-line $\mathbb{R}_+$ are not allowed in [5]. Furthermore, here we fixed the functions $c$ (with given $p$), whereas in [5] $c$ belongs a priori to a slightly larger class of functions.

We show in Section 2 under mild conditions (satisfied for all examples) that pseudo stable laws belong to the domain of normal attraction of stable laws. Hence in particular, this has a strong impact on the tail behaviour of these laws. (For the existence of moments see e.g., [1], 1.10.17 resp. 2.10.18.)

## 1. Examples of pseudo stable laws

In the following we briefly investigate some examples of pseudo stable laws on groups and indicate methods to construct new examples. Of course, these examples are well-known for $\mathbb{R}$, more generally, for vector spaces. Throughout, as before, let $\mathbb{G}$ be a homogeneous group, let $p, r > 0$ and let $(\tau_t)_{t>0}$ be a fixed contracting one-parameter group of automorphisms. Furthermore, let $(\gamma_u)_{u\geq 0}$ be a $(\tau_{t^{1/r}})$-stable continuous convolution semigroup, $\gamma_u \not\equiv \varepsilon_e$, and we fix $c(a,b) := (a^p + b^b)^{1/p}$.

**Example 1.1.** *Let $p = r > 0$. Then the stable law $\mu := \gamma_1$ is pseudo stable (with $d(\cdot,\cdot) \equiv 0$).*
$[\![$ *Indeed,* $\tau_a(\mu) \star \tau_b(\mu) = \gamma_{a^p} \star \gamma_{b^p} = \gamma_{a^p+b^p} = \tau_{c(a,b)}(\mu)$. $]\!]$

**Example 1.2.** *Let $p > r > 0$. Again $\mu := \gamma_1$. Then $\mu$ is pseudo stable w.r.t. $((\tau_t), p, r, (\gamma_u))$, where $d(a,b) = a^r + b^r - c(a,b)^r$.*
$[\![$ *Let $c := c(a,b)$ and w.l.g., $b \leq a$. Then $\tau_a(\mu) \star \tau_b(\mu) = \tau_a[\mu \star \tau_u(\mu)]$ with $0 < u := b/a \leq 1$. And $\mu \star \tau_u(\mu) = \gamma_1 \star \gamma_{u^r} = \gamma_{1+u^r} = \gamma_{c(1,u)^r} \star \gamma_{1+u^r-c(1,u)^r} = \tau_{c(1,u)}(\mu) \star \gamma_{d(1,u)}$ with $d(1,u) = 1 + u^r - c(1,u)^r \geq 0$. Thus we can choose $d(a,b) = a^r d(1,u) = a^r + b^r - c(a,b)^r$ and it follows $\tau_a(\mu) \star \tau_b(\mu) = \tau_{c(a,b)}(\mu) \star \gamma_{d(a,b)}$.* $]\!]$

**Example 1.3.** *Let $p \geq r > 0$, and let $(\gamma_u)$ as before. Let $(\delta_s)_{s>0}$ be a $(\tau_{t^{1/p}})$-stable continuous convolution semigroup and assume that all $\gamma_u$ and $\delta_v$ commute. Then $\mu := \gamma_1 \star \delta_1$ is pseudo stable (w.r.t. $((\tau_t), p, r, (\gamma_u))$), with $d$ as in Example 1.2.*
$[\![$ $\tau_a(\mu) \star \tau_b(\mu) = \tau_a(\gamma_1) \star \tau_b(\gamma_1) \star \tau_a(\delta_1) \star \tau_b(\delta_1) = \tau_{c(a,b)}(\gamma_1) \star \gamma_{d(a,b)} \star \delta_{c(a,b)} = \tau_{c(a,b)}(\mu) \star \gamma_{d(a,b)}$. $]\!]$

**Example 1.4.** *a) Let $p \geq r > 0$, and let, for $i = 1, 2$, $\mu^{(i)}$ be pseudo stable, $\tau_a(\mu^{(i)}) \star \tau_b(\mu^{(i)}) = \tau_{c(a,b)}(\mu^{(i)}) \star \gamma^{(i)}_{d^{(i)}(a,b)}$, where $\left(\gamma^{(i)}_u\right)$ are $(\tau_{t^{1/r}})$-stable. Assume moreover $d^{(i)}$ to be linearly dependent, $d^{(2)}(\cdot,\cdot) = \alpha \cdot d^{(1)}(\cdot,\cdot)$ for some $\alpha \geq 0$, and $\left\{\tau_u(\mu^{(i)}), \gamma^{(i)}_v : u > 0, v \geq 0, i = 1, 2\right\}$ commute. Then $\mu := \mu^{(1)} \star \mu^{(2)}$ is pseudo stable w.r.t. $((\tau_t), p, r, (\gamma_u))$, where $\gamma_u := \gamma^{(1)}_u \star \gamma^{(2)}_{\alpha u}$, and with $d(\cdot,\cdot) = d^{(1)}(\cdot,\cdot)$.*

⟦ $\tau_a(\mu) \star \tau_b(\mu) = \tau_a(\mu^{(1)}) \star \tau_b(\mu^{(1)}) \star \tau_a(\mu^{(2)}) \star \tau_b(\mu^{(2)}) = \tau_{c(a,b)}(\mu^{(1)} \star \mu^{(2)}) \star \gamma^{(1)}_{d^{(1)}(a,b)} \star \gamma^{(2)}_{d^{(2)}(a,b)} = \tau_{c(a,b)}(\mu) \star \gamma^{(1)}_{d^{(1)}} \star \gamma^{(2)}_{\alpha \cdot d^{(1)}} =: \tau_c(\mu) \star \gamma_{d^{(1)}}$, *with* $c = c(a,b), d^{(1)} = d^{(1)}(a,b)$. *And* $\left(\gamma_u := \gamma^{(1)}_u \star \gamma^{(2)}_{\alpha\cdot u}\right)$ *is* $(\tau_{t^{1/r}})$*-stable.* ⟧

*b) In particular, let* $\mu^{(1)}$ *be as in a), let* $(\delta_t)$ *be a continuous convolution semigroup which is stable w.r.t.* $(\tau_{t^{1/r}})$*, and assume again that all measures belong to a commutative sub-semigroup* $\mathcal{S}$*. Assume* $d^{(1)}(a,b) = \alpha \cdot (a^r + b^r - c(a,b)^r)$ *for some constant* $\alpha \geq 0$*. Then* $\mu := \mu^{(1)} \star \delta_1$ *is pseudo stable.*

⟦ $\delta_1$ *is pseudo stable w.r.t.* $((\tau_t), p, r, (\delta_u))$ *(according to Example 1.2), i.e.,* $\mu^{(2)} := \delta_1$, $\gamma^{(2)}_u = \delta_u$ *satisfy the assumptions of a).* ⟧

**Example 1.5.** *A particular case: Let* $\mathbb{H}_i$ *be contractible groups with contracting automorphism groups* $\left(\tau^{(i)}_t\right)$*,* $i = 1, 2$*. Let* $\mu^{(i)}$ *be pseudo stable w.r.t.* $(\left(\tau^{(i)}_t\right), p, r, \left(\gamma^{(i)}_t\right))$*. Put* $\mathbb{G} := \mathbb{H}_1 \otimes \mathbb{H}_2$*,* $\tau_t := \tau^{(1)}_t \otimes \tau^{(2)}_t$ *and* $\mu := \mu^{(1)} \otimes \mu^{(2)}$*. Then* $\mu$ *is pseudo stable if, as in Example 1.4,* $d^{(2)} = \alpha \cdot d^{(1)}$.

⟦ *In fact,* $\gamma^{(i)}_u, \mu^{(i)}$ *may be considered as probabilities on* $\mathbb{G}$ *supported by the subgroups* $\mathbb{H}_i \subseteq \mathbb{G}$*, and hence the commutativity assumption is satisfied.* ⟧

Obviously we obtain

**Example 1.6.** *Let* $\mathbb{G}_1, \mathbb{G}_2$ *be homogeneous groups with contracting groups* $\left(\tau^{(i)}_t\right) \subseteq \mathrm{Aut}(\mathbb{G}_1)$*,* $i = 1, 2$*. Let* $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ *be a continuous homomorphism satisfying* $\varphi\tau^{(1)}_t = \tau^{(2)}_t\varphi$*. Let* $\mu \in \mathcal{M}^1(\mathbb{G}_1)$ *be pseudo stable w.r.t.* $((\tau^{(1)}_t), p, r, (\gamma^{(1)}_u))$*. Then* $\nu := \varphi(\mu)$ *is pseudo stable w.r.t.* $(\left(\tau^{(2)}_t\right), p, r, \left(\gamma^{(2)}_u := \varphi(\gamma^{(1)}_u)\right))$.

**Example 1.7.** *Subordination. Let* $\mathbb{G}_1$ *be a contractible group with contracting automorphism group* $(\tau_t)$*. Let* $(\lambda_t)_{t\geq 0}$ *be a* $(\tau_t)$*-stable continuous convolution semigroup. Let* $(\alpha_t) \subseteq \mathcal{M}^1(\mathbb{R}_+)$ *be (one-sided) stable w.r.t.* $(H_{t^{1/r}})$*. (Hence* $0 < r \leq 1$*.) Let furthermore,* $\xi \in \mathcal{M}^1(\mathbb{R}_+)$ *be pseudo stable w.r.t.* $((H_t), p, r, (\alpha_t))$*. Then* $\mu := \int_{\mathbb{R}_+} \lambda_t \mathrm{d}\xi(t) \in \mathcal{M}^1(\mathbb{G})$ *is pseudo stable w.r.t.* $((\tau_t), p, r, \left(\gamma_t := \int_{\mathbb{R}_+} \lambda_u \mathrm{d}\alpha_t(u)\right))$.

⟦ *We have* $H_a(\xi) * H_b(\xi) = H_{c(a,b)}(\xi) * \alpha_{d(a,b)}$*. Furthermore,* $\tau_a(\mu) = \int_{\mathbb{R}_+} \tau_a(\lambda_u)\mathrm{d}\xi(u) = \int_{\mathbb{R}_+} \lambda_{au}\mathrm{d}\xi(u) = \int_{\mathbb{R}_+} \lambda_u \mathrm{d}H_a(\xi)(u)$.

*Hence* $\tau_a(\mu) \star \tau_b(\mu) = \int_{\mathbb{R}_+} \lambda_u \mathrm{d}(H_a(\xi) * H_b(\xi))(u) = \int_{\mathbb{R}_+} \lambda_u \mathrm{d}(H_c(\xi) * \alpha_d)(u) = \left(\int_{\mathbb{R}_+} \lambda_u \mathrm{d}H_c(\xi)\right) \star \left(\int_{\mathbb{R}_+} \lambda_u \mathrm{d}\alpha_d\right) = \tau_c(\mu) \star \gamma_d$. ⟧

In fact, we do not know if on $\mathbb{R}_+$ there exist pseudo stable laws which are not convolution products of stable laws. (For $\mathbb{R}$ cf. Example 1.8 below.) If $\xi_t = \alpha_t * \beta_t \in \mathcal{M}^1(\mathbb{R}_+)$, where $(\alpha_t)$ and $(\beta_t)$ are $(H_{t^{1/p}})$- and $(H_{t^{1/r}})$-stable respectively, then $\mu = \left(\int_{\mathbb{R}_+} \lambda_u \mathrm{d}\alpha_1(u)\right) \star \left(\int_{\mathbb{R}_+} \lambda_u \mathrm{d}\beta_1(u)\right)$ is a convolution product of two $(\tau_{t^{1/p}})$- and $(\tau_{t^{1/r}})$-stable factors. So, we are in the situation of Example 1.3. In that case, no new examples are created via subordination.

The next class of examples is found in [5], cf. also [3, 8]: there exist pseudo stable laws which are *not representable as convolution products* *of stable laws.* We mention these examples in our list since they seem to be the only known concrete examples with that property:

**Example 1.8.** *For* $\mathbb{G} = \mathbb{R}$*, it is shown in* [5], [8]*, that for* $0 < r \leq 2$ *there exist* $A, B > 0$*,* $p > 2$ *such that* $f : \mathbb{R} \ni t \mapsto \exp(-A|t|^r - B|t|^p)$ *is the characteristic function of a probability measure* $\mu$*, i.e.* $f(t) = \widehat{\mu}(t)$*. In fact, for* $r \in (0, 1] \cup \{2\}$*,* $p > 2$ *and sufficiently large* $A, B$*,* $f$ *is a characteristic function, but for* $r \leq 2$*,* $A, B > 0$*, the set of* $p > 2$*, such that* $f$ *is not a characteristic function, is not empty. It is easily shown that such a probability is pseudo stable w.r.t.* $((H_u), p, r, (\gamma_u))$*, where* $(\gamma_u)$ *is the stable continuous convolution semigroup with characteristic function* $t \mapsto \exp(-A|t|^r)$*. As* $p > 2$*,* $\mu$ *cannot be represented as convolution product of two stable laws.*

*Almost verbatim as in the preceding Examples 1.2, 1.3 it follows that again* $d(a, b) = a^r + b^r - c(a, b)^r$ *(though* $t \mapsto \exp(-B|t|^p)$*) is not the Fourier transform of a probability).*

**Remarks 1.9.** *a) In Examples 1.4, 1.5 we had to suppose that the functions* $d^{(i)}$ *are linearly dependent. In all concrete examples in our list we obtained – for fixed* $p \geq r$ *–* $d(a, b) = a^r + b^r - c(a, b)^r$*, hence in that cases linear dependence is trivially satisfied. (The case* $p < r$ *turns out to be trivial, cf. Theorem 2.1 a).)*

*b) In* [5, 3, 8] *the authors show furthermore that for* $\mathbb{G} = \mathbb{R}$ *the characteristic functions of (symmetric) pseudo stable laws are always representable as* $t \mapsto \exp(-A|t|^r - B|t|^p)$ *for constants* $A, B \geq 0$*,* $0 < r \leq 2$ *and* $p > 0$*. Hence a convolution product of two stable laws is pseudo stable, a product of* $n \geq 3$ *stable laws (with different stability indices) is not pseudo stable. The last result holds true in general:*

*Let* $\left(\gamma_t^{(i)}\right)$ *be* $(\tau_{t^{1/r_i}})$*-stable,* $i = 1, 2, 3$ *with* $0 < r := r_1 < r_2 < r_3 := p$*. Then* $\mu := \gamma_1^{(1)} \star \gamma_1^{(2)} \star \gamma_1^{(3)}$ *is not pseudo-stable (though, according to our examples,* $\gamma_1^{(i)}$ *and the products of two factors are pseudo stable).* $[\![$ *In fact, for* $a, b > 0, c = c(a, b)$ *we have* $\tau_a(\mu) \star \tau_b(\mu) = \gamma_{c^r}^{(1)} \star \gamma_{c^{r_2}}^{(2)} \star \gamma_{c^p}^{(1)} \star \gamma_{(a^r+b^r-c^r)}^{(1)} \star \gamma_{(a^{r_2}+b^{r_2}-c^{r_2})}^{(2)} = \tau_c(\mu) \star \left[\gamma_{(a^r+b^r-c^r)}^{(1)} \star \gamma_{(a^{r_2}+b^{r_2}-c^{r_2})}^{(2)}\right]$*. If* $\mu$ *is pseudo stable, we have* $\tau_a(\mu) \star \tau_b(\mu) = \tau_c(\mu) \star \lambda_{d(a,b)}$ *for some* $(\tau_{t^{1/r}})$*-stable continuous convolution semigroup* $(\lambda_t)$*. By the injectivity of the convolution operator of the embeddable law* $\tau_c(\mu)$ *(cf.* [4]*) it follows that* $\lambda_{d(a,b)} = \gamma_{(a^r+b^r-c^r)}^{(1)} \star \gamma_{(a^{r_2}+b^{r_2}-c^{r_2})}^{(2)}$*. But the right side is not* $(\tau_{t^{1/r}})$*-stable, a contradiction.* $]\!]$

As mentioned afore, our examples enable us to construct new non-trivial pseudo stable measures on homogeneous groups. We show e.g., for $\mathbb{G} = \mathbb{H}_1$, the 3-dimensional Heisenberg group, that there exist pseudo stable laws with full support which are not representable as convolution products of stable laws. The reader will easily see how this result may be generalized to arbitrary homogeneous groups.

**Example 1.10.** *Let* $(\tau_t)$ *denote the group of dilations on* $\mathbb{H}_1$ *acting on the Lie algebra* $\mathfrak{g} \equiv \mathbb{R}^3$ *as* $t^E$ *for the diagonal exponent* $E = \mathrm{diag}(1, 1, 2)$*. Let* $\mathcal{Z} \equiv \mathbb{R}$ *denote the center and* $i : \mathbb{R} \to \mathcal{Z} \subseteq \mathbb{H}_1$ *the canonical injection.*

*Let* $\left(\gamma_t^{(1)}\right) \subseteq \mathcal{M}^1(\mathbb{H}_1)$ *be* $(\tau_{t^{1/r}})$*-stable with full support, and let* $\mu^{(1)} \in \mathcal{M}^1(\mathbb{H}_1)$ *be pseudo stable w.r.t.* $((\tau_t), p, r, \left(\gamma_t^{(1)}\right))$ *(e.g., choose* $\mu^{(1)} = \gamma_1^{(1)}$*), such that* $d(a,b) = a^r + b^r - c(a,b)^r$.

*Let* $\left(\gamma_t^{(2)}\right) \subseteq \mathcal{M}^1(\mathbb{R})$ *be* $(H_{t^{1/2r}})$*-stable and let* $\nu \in \mathcal{M}^1(\mathbb{R})$ *be pseudo stable w.r.t.* $((H_t), 2p, 2r, \left(\gamma_t^{(2)}\right))$ *as in Example 1.8. (Therefore,* $2r \leq 1$*.) Put* $\mu^{(2)} := i(\nu) \in \mathcal{M}^1(\mathbb{H}_1)$. $\tau_{t^{1/r}}|_{\mathcal{Z}} = H_{t^{1/2r}}$ *implies that* $\left(i(\gamma_t^{(2)})\right)$ *is* $(\tau_{t^{1/r}})$*-stable, furthermore,* $i(\nu)$ *and* $i(\gamma_t^{(2)})$ *are central measures in* $\mathcal{M}^1(\mathbb{G})$*. Hence, by Example 1.4,* $\left(\gamma_t := \gamma_t^{(1)} \star \gamma_t^{(2)}\right)$ *is* $(\tau_{t^{1/r}})$*-stable and* $\mu := \mu^{(1)} \star \mu^{(2)}$ *is pseudo stable w.r.t.* $((\tau_t), p, r, (\gamma_t))$*, has full support and is not representable as convolution of stable laws.*

## 2. Pseudo stable laws and domains of attraction

In the following we point out that pseudo stable laws are closely related to limit laws. Let, as before, $\mu \neq \varepsilon_e$ be pseudo stable w.r.t. $((\tau_t), p, r, (\gamma_t))$. To avoid trivialities, assume again $\gamma_s \neq \varepsilon_e$ for $s \neq 0$. Then, putting $a = b = 1$ in (0.1), we obtain $\mu^2 = \tau_{2^{1/p}}(\mu) \star \gamma_{d_2}$ with $c(1,1) = 2^{1/p}$ and $d(1,1) =: d_2 \geq 0$. By induction we obtain (observing $c(k^{1/p}, 1) = (k+1)^{1/p}$) :

$$\mu^n = \tau_{n^{1/p}}(\mu) \star \gamma_{d_n} \tag{2.2}$$

with $d_1 := 0, d_2 = d(1,1), d_n = \sum_2^n d((k-1)^{1/p}, 1)$.

**Theorem 2.1.** *Assume that* $\{d_n/n\}$ *is bounded above. Then we have:*

*a) If* $p < r$ *then* $\tau_{n^{-1/p}}(\mu)^n \to \mu$*, hence* $\lambda_1 := \mu$ *is embeddable into a* $(\tau_{t^{1/p}})$*-stable continuous convolution semigroup* $(\mu_t)$*, and* $\gamma_{d(\cdot,\cdot)} \equiv \varepsilon_e$.

*b) If* $p > r$ *then* $\{\tau_{n^{-1/r}}(\mu)^n\}$ *is relatively compact with accumulation points* $\mathrm{LIM}\{\tau_{n^{-1/p}}(\mu)^n\} \subseteq \{\gamma_s : 0 \leq s \leq K\}$ *for some* $K > 0$*. In particular, any accumulation point is* $(\tau_{t^{1/r}})$*-stable.*

*b1) In particular, if* $d_n/n \to \alpha$ *for some* $\alpha > 0$*, then* $\mu$ *belongs to the domain of normal attraction* $\mathrm{DNA}(\gamma_\alpha)$ *(w.r.t.* $(\tau_{t^{1/r}})$*).*

*c) If* $p = r$ *then again* $\{\tau_{n^{-1/p}}(\mu)^n\}$ *is relatively compact, with accumulation points* $\mathrm{LIM}\{\tau_{n^{-1/p}}(\mu)^n\} \subseteq \{\mu \star \gamma_s : 0 \leq s \leq K\}$ *for some* $K > 0$*. In that case all accumulation points are embeddable into continuous convolution semigroups* $(\lambda_t)$ *with* $\lambda_1 = \mu \star \gamma_s$.

*c1) And again, if* $d_n/n \to \alpha > 0$*, then* $\mu$ *is embeddable into a (uniquely determined)* $(\tau_{t^{1/r}})$*-stable continuous convolution semigroup* $(\mu_t)$ *and, as in Example 1.2,* $\gamma_{d(a,b)} = \mu_{a^r + b^r - c(a,b)^r}$.

*Proof.* Equation (2.2) is equivalent with

$$\tau_{n^{-1/p}}(\mu)^n = \mu \star \gamma_{d_n \cdot n^{-r/p}} \tag{2.3}$$

(since $(\gamma_u)$ is $(\tau_{n^{1/r}})$-stable), resp.

$$\tau_{n^{-1/r}}(\mu)^n = \tau_{n^{1/p - 1/r}}(\mu) \star \gamma_{d_n \cdot n^{-1}} \tag{2.4}$$

In case a) we use equation (2.3), in case b) (2.4), to see that $\{\tau_{n^{-1/p}}(\mu)^n\}$ resp. $\{\tau_{n^{-1/r}}(\mu)^n\}$ are relatively compact: In case a), $r/p > 1$, we have $d_n \cdot n^{-r/p} \to 0$, thus $\tau_{n^{-1/p}}(\mu)^n \to \mu$.

In case b), $1/p - 1/r < 0$, we have $\tau_{n^{1/p - 1/r}}(\mu) \to \varepsilon_e$, thus (for some upper bound $K$ of $\{d_n/n\}$), $\mathrm{LIM}\{\tau_{n^{-1/r}}(\mu)^n\} \subseteq \{\gamma_s : 0 \leq s \leq K\}$.

In case c), $p = r$, it follows $\tau_{n^{-1/p}}(\mu)^n = \mu \star \gamma_{d_n/n}$ and therefore, $\mathrm{LIM}\{\tau_{n^{-1/p}}(\mu)^n\} \subseteq \{\mu \star \gamma_s : 0 \le s \le K\}$.

In either case, it follows embeddability of $\mu$ in a) resp. of any accumulation point $\gamma_s$ resp. $\mu\star\gamma_s$ in b) resp c), (cf. e.g., [1], 2.6.4). In case b1) or c1) we have in addition $b_n/n \to \alpha$. Then $\tau_{n^{-1/p}}(\mu)^n \to \lambda_1 := \gamma_\alpha$ resp. $\to \lambda_1 := \mu \star \gamma_\alpha$ yields $\mu \in \mathrm{DNA}(\lambda_1)$ w.r.t. $(\tau_{t^{1/r}})$; in particular, $\lambda_1$ is embeddable into a (uniquely determined) $(\tau_{t^{1/r}})$-stable continuous convolution semigroup $(\lambda_t)$ (cf. e.g., [1], 2.6.10 b)). If $\lambda_1 = \mu\star\gamma_\alpha$, stability of $(\lambda_t)$ implies for all $n \in \mathbb{N}$: $\lambda_1^n = \tau_{n^{1/r}} \star \gamma_{n\alpha}$, and on the other hand we have $\lambda_1^n = \mu^n \star \gamma_{n\alpha}$. Thus, again by the injectivity of the convolution operators of $\gamma_u$ ([4]) we obtain $\mu^n = \tau_{n^{1/r}}(\mu), n \in \mathbb{N}$. I.e., $\mu$ is B-stable (cf. [1], 2.6.13), and therefore $\mu$ is embeddable into a stable continuous convolution semigroup $(\mu_t)$ ([1], 2.6.14 b) and 2.6.11* (p. 256)). The last assertion follows as in Example 1.2: $\tau_a(\mu)\star\tau_b(\mu) = \tau_{(a^r+b^r)^{1/r}}(\mu)$ by stability, thus $= \mu_{a^r+b^r} = \mu_{c(a,b)^r} \star \mu_{a^r+b^r-c(a,b)^r}$. Hence in view of the defining equation (0.1), $\tau_{c(a,b)}(\mu) \star \gamma_{d(a,b)} = \tau_{c(a,b)}(\mu) \star \mu_{a^r+b^r-c(a,b)^r}$. Injectivity of the convolution operators of (the embeddable law) $\tau_{c(a,b)}(\mu)$ yields $\gamma_{d(a,b)} = \mu_{a^r+b^r-c(a,b)^r}$, as asserted.

Note that we assumed $\mu \ne \varepsilon_e$ and $\gamma_\alpha \ne \varepsilon_e$ and furthermore, that $(\tau_t)$ is contractive. Hence in either case a), b1), c1), the limit $\lambda_1 \ne \varepsilon_e$ is a full measure w.r.t. $\mathcal{B} := \{\tau_t\}_{t>0} \subseteq \mathrm{Aut}(\mathbb{G})$. And therefore, the convergence of types theorem holds true (e.g., verbatim as in [1], 1.13.33.) Hence, as in the proof of [1], 2.6.10 b), $\mu$ resp. $\lambda_1$ is embeddable into a stable continuous convolution semigroup. □

For $\alpha = 0$ we obtain $\gamma_\alpha = \varepsilon_e$. Then the assertions hold trivially.

In [5, 3, 8] for $\mathbb{G} = \mathbb{R}$ the functions $d(\cdot,\cdot)$ are explicitly calculated solving functional equations of characteristic functions: If $c(a,b) = (a^p + b^p)^{1/p}$ then $d(\cdot,\cdot)$ has the form we obtained in the examples in Section 1. In general we can only show that $d$ is homogeneous, $d(ta,tb) = t^r d(a,b)$ for $a,b,t > 0$. We indicate a proof to point out which tools are needed if characteristic functions are not available:
$[\![$ $\tau_{ta}(\mu)\star\tau_{tb}(\mu) = \tau_{c(ta,tb)}(\mu)\star\gamma_{d(ta,tb)}$, with $c(ta,tb) = t\cdot c(a,b)$. On the other hand, this equals $\tau_t\left[\tau_a(\mu)\star\tau_b(\mu)\right]$, thus we obtain: $\tau_{tc(a,b)}(\mu)\star\gamma_{t^r d(a,b)} = \tau_{tc(a,b)}\star\gamma_{d(ta,tb)}$. Assume e.g., $t^r d(a,b) \ge d(ta,tb)$, then again injectivity of the convolution operators of $\gamma_u$ yields $\tau_{tc(a,b)}(\mu)\star\gamma_{t^r d(a,b)-d(ta,tb)} = \tau_{tc(a,b)}(\mu)$, i.e., $\gamma_{t^r d(a,b)-d(ta,tb)} = \varepsilon_e$ since $\mathbb{G}$ is aperiodic. Whence the assertion. $]\!]$

In our examples 1.2, 1.3, 1.8 we obtained for $r \le p$: $d(k^{1/p},1) = k^{r/p} + 1 - (k+1)^{r/p}$, thus $d_n = \sum_2^n d((k-1)^{1/p},1) = n - n^{r/p}$. Hence, $d_n/n \equiv 0$ for $r = p$ and $d_n/n \to 1$ for $r < p$. Thus these examples are covered by Theorem 2.1.

To show that the assumption $d_n/n \le K$ in Theorem 2.1 is quite natural we consider briefly the case of infinitely divisible $\mu$:

**Proposition 2.2.** *Let, with the above notations, $\mu$ be pseudo stable w.r.t. $((\tau_t), p, r, (\gamma_t))$, $\gamma_t \not\equiv \varepsilon_e$. Assume in addition that $\mu$ is infinitely divisible, hence embeddable into a continuous convolution semigroup $(\mu_t)$, $\mu_1 = \mu$. Furthermore, assume either (1) $\mathbb{G} = \mathbb{R}^d$, or (2) $\mu_t = \widetilde{\mu}_t$, $t \ge 0$. Then $\{d_n/n\}$ is bounded.*

*Proof.* Inserting $\mu = \mu_{1/n}^n$, $n \in \mathbb{N}$, in the defining equation, we obtain $\tau_a(\mu_{1/n})^n \star \tau_b(\mu_{1/n})^n = \tau_{c(a,b)}(\mu_{1/n})^n \star \gamma_{d_n/n}^n$. In case (1), $\mathcal{M}^1(\mathbb{G})$ is commutative and the roots $\mu_{1/n}, \gamma_{d_n/n}$ of infinitely divisible laws are uniquely determined. In case (2), the convolution operators $T_{\mu_t}$ on $L^2(\mathbb{G})$ are positive semi-definite functions of $T_{\mu_1}$, hence $\mu_t$ are uniquely determined by $\mu_1$ and commute with all measures commuting with $\mu_1$. Furthermore, the stable continuous convolution semigroup $(\gamma_t)$ is uniquely determined by $\gamma_1$.

Hence we obtain $\tau_a(\mu_{1/n}) \star \tau_b(\mu_{1/n}) = \tau_{c(a,b)}(\mu_{1/n}) \star \gamma_{d_n/n}$ for all $n$ and therefore it follows for all $t \geq 0$,

$$\tau_a(\mu_t) \star \tau_b(\mu_t) = \tau_{c(a,b)}(\mu_t) \star \gamma_{d_n \cdot t} \tag{2.5}$$

As immediately seen, symmetry of $(\mu_t)$ yields $\gamma_{d_n t} = \widetilde{\gamma}_{d_n t}$ for all $t, n$, this shows that $(\gamma_t)$ is symmetric too. Let $A, G$ denote the generating functionals of $(\mu_t)$ and $(\gamma_t)$ respectively, and denote the Lévy measures and Gaussian terms by $\eta_A, \eta_G$ and $\Gamma_A, \Gamma_G$ respectively. (2.5) easily yields $\tau_a(A) + \tau_b(A) = \tau_{c(a,b)}(A) + d_n \cdot G$, in particular, $2A = \tau_{2^{1/p}}(A) + d_2 \cdot G$, and by induction, for $n \in \mathbb{N}$, $n \cdot A = \tau_{n^{1/p}}(A) + d_n \cdot G$. Equivalently,

$$A = (1/n) \cdot \tau_{n^{1/p}}(A) + (d_n/n) \cdot G$$

Consequently,

$$\eta_A = (1/n) \cdot \tau_{n^{1/p}}(\eta_A) + (d_n/n) \cdot \eta_G \tag{2.6}$$

$$\Gamma_A = (1/n) \cdot \tau_{n^{1/p}}(\Gamma_A) + (d_n/n) \cdot \Gamma_G \tag{2.7}$$

Hence, if $\eta_G \neq 0$ or $\Gamma_G \neq 0$, boundedness of $\{d_n/n\}$ follows by (2.6) resp. (2.7). Since, as mentioned above, $(\gamma_u)$ is symmetric, $\not\equiv \varepsilon_e$, the proof is complete. □

However, we do not know if in the group case there exist examples $\mu$ with different growth behaviour of $\{d_n\}$. Hence we mention

**Theorem 2.3.** *Let $(\gamma_t)$ be a $(\tau_{t^{1/r}})$-continuous convolution semigroup of full measures. (For the definition of full measures cf. e.g.,* [1], *§ 2.2. I.) Assume for a subsequence $(n') \subseteq \mathbb{N}$ that $d_n/n \overset{(n')}{\to} \infty$. Then we have:*

*a) If $p < r$ then $\tau_{d_n^{-1/p}}(\mu)^n \to \varepsilon_e$ along the subsequence $(n')$.*

*b) If $p \geq r$ then $\tau_{d_n^{-1/r}}(\mu)^n \to \gamma_1$ along $(n')$. Thus, if $(n') = \mathbb{N}$, $\mu$ belongs to the (general) domain of attraction $\mu \in \mathrm{DA}(\gamma_1)$ (with norming automorphisms belonging to $\mathcal{B} = (\tau_t)$).*

*Proof.* a) $r > p$, hence $1/r < 1/p$ and thus $\tau_{d_n^{-1/p}}(\mu)^n = \tau_{(d_n^{-1} \cdot n)^{1/p}}(\mu) \star \gamma_{d_n^{-r/p} \cdot d_n} \to \varepsilon_e$ (along $(n')$), since $d_n^{-1} \cdot n \to 0$ and $d_n^{1-r/p} \to 0$.

b) $r \leq p$. Then $\tau_{d_n^{-1/r}}(\mu)^n = \tau_{(d_n^{-1} \cdot n)^{1/p}} \tau_{d_n^{1/p-1/r}}(\mu) \star \gamma_1 \to \gamma_1$ (along $(n')$) as $d_n^{-1} \cdot n \to 0$, $1/p - 1/r \leq 0$, thus $\tau_{(d_n^{-1} \cdot n)^{1/p}} \tau_{d_n^{1/p-1/r}}(\mu) \to \varepsilon_e$. □

**Possible Generalizations.** a) As $\mathrm{Aut}(\mathbb{R}) = \{H_u : u \in \mathbb{R} \setminus \{0\}\}$, the definition of pseudo stability (on $\mathbb{R}$) could be generalized as follows:

$\mu \in \mathcal{M}^1(\mathbb{G})$ is called completely pseudo stable if for all $a, b \in \mathrm{Aut}(\mathbb{G})$ we have

$$a(\mu) \star b(\mu) = c(\mu) \star \gamma_d$$

where $c : \mathrm{Aut}(\mathbb{G}) \times \mathrm{Aut}(\mathbb{G}) \to \mathrm{Aut}(\mathbb{G})$ and $d : \mathrm{Aut}(\mathbb{G}) \times \mathrm{Aut}(\mathbb{G}) \to \mathbb{R}_+$ are (suitable) functions. However, for $d(\cdot, \cdot) \equiv 0$, this defines 'complete stability' (cf. e.g., [1], 1.14.28, 2.11.22). It is known (cf. [9]) that for $\mathbb{G} = \mathbb{R}^n$ and $n \geq 2$, completely stable measures are Gaussian, and vice versa. Thus the definition of *complete* pseudo stability turns out to be to restrictive.

b) At the first glance it seems natural to investigate pseudo stable laws on more general convolution structures admitting contracting groups $(\tau_t)$ of automorphisms (to guarantee the existence of stable laws). E. g. *generalized convolutions* for the state space $\mathbb{R}_+$ (cf. e.g., [3, 7] and the literature mentioned there). In the case of hypergroups, as in the group case, the existence of contracting $(\tau_t)$ has a strong impact on the underlying structure. E.g., on $\mathbb{R}_+$, only Bessel-Kingman hypergroups admit contracting automorphisms. More general examples are hypergroup structures on matrix cones (cf. [10]), which have a considerably rich structure of automorphisms, and hence there exist stable laws in abundance. (Cf. also [2]). Our Examples 1.1–1.7 and Theorem 2.1 hold true in this situation. However, the existence of nontrivial pseudo stable laws $\mu$, i.e., laws which are not products of stable laws, could not be proved. Therefore, we omit further details.

c) The existence of stable laws and hence of continuous contracting automorphism groups restricts the investigations to homogeneous groups. To get rid of that restrictions one can replace $\mathbb{G}$ by a contractible locally compact group, $(\tau_t)$ by a discrete contracting group $\left(\tau^k\right)_{k\in\mathbb{Z}}$ and assume $(\gamma_t)$ to be semistable. Under additional conditions on $\mathbb{G}$ which guarantee a convergence of types theorem and continuous embedding of (certain) infinitely divisible laws, a part of the afore mentioned results holds in this general situation. Natural candidates for investigations of examples of such 'pseudo semistable' laws are the real line on the one hand and certain contractible totally disconnected groups on the other, e.g., $p$-adic groups. Details will appear elsewhere.

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