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LIMIT TRANSITION BETWEEN HYPERGEOMETRIC FUNCTIONS OF TYPE BC AND TYPE A

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ABSTRACT. Let $F_{BC}(\lambda, k; t)$ be the Heckman-Opdam hypergeometric function of type BC with multiplicities $k = (k_1, k_2, k_3)$ and weighted half sum $\rho(k)$ of positive roots. We prove that $F_{BC}(\lambda + \rho(k), k; t)$ converges for $k_1 + k_2 \to \infty$ and $k_1/k_2 \to \infty$ to a function of type A for $t \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}^n$. This limit is obtained from a corresponding result for Jacobi polynomials of type BC, which is proven for a slightly more general limit behavior of the multiplicities, using an explicit representation of Jacobi polynomials in terms of Jack polynomials.

Our limits include limit transitions for the spherical functions of non-compact Grassmann manifolds over one of the fields $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ when the rank is fixed and the dimension tends to infinity. The limit functions turn out to be exactly the spherical functions of the corresponding infinite dimensional Grassmann manifold in the sense of Olshanski.

1. Introduction

Consider the Heckman-Opdam hypergeometric functions $F_R(\lambda, k; t)$ for the root systems $R = BC_n = \{\pm e_i, \pm 2e_i, \pm e_i \pm e_j,\ 1 \le i < j \le n\}$ and $A_{n-1} = \{\pm(e_i - e_j) : 1 \le i < j \le n\}$ with multiplicities $k = (k_1, k_2, k_3)$ and $k = \kappa$ respectively as studied e.g. in [BO], [H1], [H2], [H3], [HS], [O1], [O2]. Fix a positive subsystem R_+ in each case and denote by $\rho_R(k) = \frac{1}{2}\sum_{\alpha \in R_+} k_\alpha \alpha$ the weighted half-sum of positive roots. The Jacobi polynomials of type BC_n are indexed by the cone of dominant weights

$$P_+ = \{(\lambda_1, \dots \lambda_n) \in \mathbb{Z}_+^n : \lambda_1 \ge \dots \ge \lambda_n\}$$

and can be written as

$$P_\lambda^{BC}(k; t) = \frac{1}{c(\lambda + \rho_{BC}(k), k)} F_{BC}(\lambda + \rho_{BC}(k), k; t)$$

where c is the generalized c-function. The Jacobi polynomials of type A_{n-1} are indexed by the set $\pi(P_+)$, where π denotes the orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}_0^n$. They can be written as monic Jack polynomials,

$$P_{\pi(\lambda)}^A(\kappa; t) = j_\lambda^\kappa(e^t), \quad t \in \mathbb{R}_0^n;$$

see Section 4 for the precise notation.

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In this paper, we shall prove the following limit transition for the Jacobi polynomials of type BC_n:

$$\lim_{\substack{k_1+k_2\to\infty\\ k_1/k_2\to a}} P_\lambda^{BC}(k;t) \;=\; 4^{|\lambda|}\cdot j_\lambda^{k_3}(x(t)) \tag{1.1}$$

for $a\in[0,\infty]$, with the transform

$$\mathbb{R}^n\to\mathbb{R}^n_+,\quad t\mapsto x(t)\quad\text{with}\quad x_i(t)=\gamma_a+\sinh^2\big(\frac{t_i}{2}\big),\quad \gamma_a=\frac{a+1}{a+2}.$$

This result was already stated in [K2] without proof. A proof different from the one in the present paper was given by R. J. Beerends and the second author in an unpublished manuscript.

Restricting to the case $a=\infty$ we shall next extend the limit with respect to the spectral variable λ and prove that

$$\begin{aligned}\lim_{\substack{k_1+k_2\to\infty\\ k_1/k_2\to\infty}} & F_{BC}(\lambda+\rho_{BC}(k),k;t) \\ &= \prod_{i=1}^n\big(\cosh^2\frac{t_i}{2}\big)^{\sum_{i=1}^n\lambda_i/n}\cdot F_A\big(\pi(\lambda)+\rho_A(k_3),k_3;\pi\big(\log\cosh^2\frac{t}{2}\big)\big)\end{aligned} \tag{1.2}$$

for all $t\in\mathbb{R}^n$, locally uniformly in $\lambda\in\mathbb{C}^n$.

Let us briefly discuss the above limits for the rank one case $n=1$ where k_3 does not appear, the functions $F_{BC}(\lambda,k;t)$ are essentially Jacobi functions

$$\varphi_\lambda^{(\alpha,\beta)}(t)={}_2F_1(\tfrac12(\alpha+\beta+i\lambda),\tfrac12(\alpha+\beta+1-i\lambda);\alpha+1;-\sinh^2 t),$$

for which we refer to [K1], and where F_A reduces to the constant function 1. More precisely, comparing the examples on p. 89 of [O1] and [K1], we have

$$F_{BC_1}(\lambda,k;t)=\varphi_{-2i\lambda}^{(\alpha,\beta)}\big(\frac{t}{2}\big)\quad\text{with}\quad \alpha=k_1+k_2-1/2,\ \beta=k_2-1/2,$$

and (1.2) becomes the limit

$$\lim_{\alpha\to\infty,\alpha/\beta\to\infty}\varphi_{\lambda+i(\alpha+\beta+1)}^{(\alpha,\beta)}(t)=(\cosh t)^{-i\lambda}\quad(\lambda\in\mathbb{C}).$$

This limit is easily seen from

$$\begin{aligned}\varphi_{\lambda+i(\alpha+\beta+1)}^{(\alpha,\beta)}(t)&={}_2F_1(\tfrac12 i\lambda,\alpha+\beta+1-\tfrac12 i\lambda;\alpha+1;-\sinh^2 t)\\ &=(\cosh t)^{-i\lambda}{}_2F_1(\tfrac12 i\lambda,-\beta+\tfrac12 i\lambda;\alpha+1;\tanh^2 t).\end{aligned}$$

Moreover, the Heckman-Opdam polynomials in rank one are related to the monic Jacobi polynomials $p_n^{(\alpha,\beta)}$ by

$$P_n^{BC}(k;it)=2^n p_n^{(\alpha,\beta)}(\cos t),\ n\in\mathbb{Z}_+.$$

Limit (1.1) means that for $c=a+1\in[1,\infty)$ and $x=\cos t\in[-1,1]$,

$$\lim_{\alpha\to\infty,\alpha/\beta\to c}p_n^{(\alpha,\beta)}(x)=\left(x+\frac{c-1}{c+1}\right)^n.$$

This limit is easily seen from

$$p_n^{(\alpha,\beta)}(x)=\frac{2^n(\alpha+1)_n}{(n+\alpha+\beta+1)_n}\sum_{l=0}^n\frac{(n+\alpha+\beta+1)_l}{(\alpha+1)_l}\binom{n}{l}\left(\frac{x-1}{2}\right)^l.$$

We shall obtain (1.1) by means of an explicit representation of P_λ^{BC} in terms of Jack polynomials which goes back to ideas of [SK] and to [Ha]. The limit (1.2) for the hypergeometric function is then obtained from (1.1) by Phragmén-Lindelöf principles and sharp explicit estimates for general hypergeometric functions which slightly improve estimates by Opdam [O1] and Schapira [Sch]. Our limit transition (1.2) includes a limit result for the spherical functions of the Grassmannians $SO_0(p,n)/SO(p)\times SO(n)$, $SU(p,n)/S(U(p)\times U(n))$ and $Sp(p,n)/Sp(p)\times Sp(n)$, where $Sp(p,n)$ denotes the pseudo-unitary group of index (p,n) over $\mathbb{H}$. As $p\to\infty$ (and the rank n is fixed), the spherical functions of these Grassmannians converge to (restrictions of) the spherical functions of the reductive symmetric space $GL_+(n,\mathbb{R})/SO(n), GL(n,\mathbb{C})/U(n)$ and $GL(n,\mathbb{H})/Sp(n)$, respectively. We shall also show that the obtained limits are exactly the spherical functions of the corresponding infinite dimensional Grassmannians in the sense of Olshanski. Our results for infinite dimensional Grassmannians are also of interest in comparison with the recent results of [DOW]. There it is shown that under natural conditions on an infinite dimensional symmetric space $G_\infty/K_\infty = \varinjlim G_n/K_n$ where G_n/K_n are Riemannian symmetric of compact type, spherical functions of G_n/K_n can have a limit which is K_∞-spherical only if the G_n/K_n are Grassmannians.

This paper is organized as follows: In Section 2 we recapitulate some basic notions and facts on the Cherednik kernel and Heckman-Opdam hypergeometric functions. We need the Cherednik kernel because we improve in Section 3 estimates of Opdam [O1] and Schapira [Sch] for this function. This results in an estimate for the Heckman-Opdam hypergeometric functions which is uniform in the multiplicity parameters. The Cherednik kernel will not be further used in the main part of the paper, starting in Section 4, where the limit (1.1) for Jacobi polynomials of type BC is proved. This result, the estimates of Section 3, and Phragmén-Lindelöf principles are combined in Section 5, leading to the limit (1.2). In Section 6 we briefly discuss this limit in terms of spherical functions for non-compact Grassmann manifolds of growing dimension and fixed rank. Finally, in Section 7 the Olshanski spherical functions of the associated infinite dimensional Grassmannians are characterized.

2. Notation and Preliminaries

Let $\mathfrak{a}$ be a finite-dimensional Euclidean space with inner product $\langle\,.\,,.\,\rangle$ which is extended to a complex bilinear form on the complexification $\mathfrak{a}_\mathbb{C}$ of $\mathfrak{a}$. We identify $\mathfrak{a}$ with its dual space $\mathfrak{a}^* = \mathrm{Hom}(\mathfrak{a},\mathbb{R})$ via the given inner product. Let $R\subset\mathfrak{a}$ be a (not necessarily reduced) crystallographic root system and let W be the Weyl group of R. For $\alpha\in R$ we write $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$ and denote by $\sigma_\alpha(t) = t - \langle t,\alpha^\vee\rangle\alpha$ the orthogonal reflection in the hyperplane perpendicular to α. We denote by $\mathcal{K}$ the vector space of multiplicity functions $k=(k_\alpha)_{\alpha\in R}$, satisfying $k_\alpha = k_\beta$ if α and β are in the same W-orbit. We shall write $k\geq 0$ $(k>0)$ if $k_\alpha\geq 0$ $(k_\alpha>0)$ for all $\alpha\in R$. For $k\in\mathcal{K}$ let

$$\rho = \rho(k) := \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha\alpha \tag{2.1}$$

be the weighted half-sum of positive roots, where R_+ is some fixed positive subsystem of R. Let

$$\mathfrak{a}_+ := \{t\in\mathfrak{a} : \langle t,\alpha\rangle > 0\ \forall\alpha\in R_+\}$$

be the positive Weyl chamber associated with R_+. If $k \geq 0$, then $\rho(k) \in \overline{\mathfrak{a}_+}$, and if $k > 0$, then $\rho(k) \in \mathfrak{a}_+$. This follows from the fact that for a simple system $\{\alpha_i\} \subset R_+$ (with indivisible roots α_i), the reflection σ_{α_i} leaves $R_+ \setminus \{\alpha_i\}$ invariant, and hence

$$\langle \rho(k), \alpha_i^\vee \rangle = k_{\alpha_i} + 2k_{2\alpha_i}$$

(with the understanding that $k_{2\alpha_i} = 0$ if $2\alpha_i \notin R$), c.f. [M], Section 11.
For fixed $k \in \mathcal{K}$, the Cherednik operator in direction $\xi \in \mathfrak{a}$ is defined by

$$T_\xi = T_\xi(k) := \partial_\xi + \sum_{\alpha \in R_+} k_\alpha \langle \alpha, \xi \rangle \frac{1}{1 - e^{-\alpha}} (1 - \sigma_\alpha) - \langle \rho(k), \xi \rangle$$

where ∂_ξ is the usual directional derivative and

$$e^\lambda(t) := e^{\langle \lambda, t \rangle} \quad \forall \lambda, t \in \mathfrak{a}_{\mathbb{C}}.$$

For fixed k, the operators $\{T_\xi(k),\ \xi \in \mathfrak{a}\}$ commute. According to Theorem 3.15 of [O1], there exist a W-invariant tubular neighborhood U of $\mathfrak{a}$ in $\mathfrak{a}_{\mathbb{C}}$ and a unique holomorphic function G on $\mathfrak{a}_{\mathbb{C}} \times K^{reg} \times U$ which satisfies

$$\begin{aligned} &\text{(i) } \forall\, \xi \in \mathfrak{a},\ \lambda \in \mathfrak{a}_{\mathbb{C}} :\ T_\xi(k) G(\lambda, k;\, .\,) = \langle \lambda, \xi \rangle G(\lambda, k;\, .\,); \\ &\text{(ii) } G(\lambda, k; 0) = 1. \end{aligned} \tag{2.2}$$

The function G is called the Cherednik-Opdam kernel. We shall mainly be concerned with the hypergeometric function associated with R, which is given by

$$F(\lambda, k; t) := \frac{1}{|W|} \sum_{w \in W} G(\lambda, k; w^{-1} t).$$

It is actually W-invariant both in λ and t. The functions $F(\lambda, k;\, .\,)$ generalize the spherical functions of Riemannian symmetric spaces of the non-compact type, which occur for specific values of the multiplicity parameter $k \geq 0$.

In order to interpret the main results below in the geometric context, we shall use the following scaling property:

Lemma 2.1. *Let R be a root system in a Euclidean space $\mathfrak{a}$ with multiplicity function k. For a constant $c > 0$ consider the rescaled root system $\widetilde{R} := cR := \{c\alpha, \alpha \in R\}$ and define $\widetilde{k}$ on $\widetilde{R}$ by $\widetilde{k}_{c\alpha} := k_\alpha$. Then the associated Cherednik kernels are related via*

$$G_\lambda(\widetilde{k}; t) \,=\, G_{\lambda/c}(k; ct).$$

A corresponding result holds also for the associated hypergeometric functions.

Proof. Write $\widetilde{f}(t) = f(ct)$ for functions f on $\mathfrak{a}$. Then

$$(T_\xi(\widetilde{k}) \widetilde{f}\,)(t) = (T_{c\xi}(k) f)(ct).$$

In view of characterization (2.2), this implies the assertion. □

In this paper, we shall always assume that $k \geq 0$ and we often write

$$G(\lambda, k; t) = G_\lambda(k; t), \quad F(\lambda, k; t) = F_\lambda(k; t).$$

For certain spectral variables λ, the hypergeometric functions F_λ are actually exponential polynomials, called Heckman-Opdam Jacobi polynomials. To introduce these, let $P = \{\lambda \in \mathfrak{a} : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\ \forall\, \alpha \in R\}$ denote the weight lattice of R and $P_+ = \{\lambda \in P : \langle \lambda, \alpha^\vee \rangle \geq 0\ \forall\, \alpha \in R_+\}$ the set of dominant weights associated

with R_+. We equip P_+ with the usual dominance order, that is, $\mu < \lambda$ iff $\lambda - \mu$ is a sum of positive roots. Let

$$\mathcal{T} := \mathrm{span}_{\mathbb{C}}\{e^{\lambda},\ \lambda \in P\}$$

denote the space of exponential polynomials associated with R. The monomial symmetric functions

$$M_\lambda = \sum_{\mu \in W\lambda} e^{\mu}, \quad \lambda \in P_+ \tag{2.3}$$

form a basis of the subspace $\mathcal{T}^W$ of W-invariant elements from $\mathcal{T}$.

The Jacobi polynomials $\{P_\lambda(k),\ \lambda \in P_+\}$ associated with R are uniquely characterized by the following two conditions:

(i) $P_\lambda(k) = M_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu}(k) M_\mu \quad (c_{\lambda\mu}(k) \in \mathbb{C})$;

(ii) $L_k P_\lambda(k) = \langle \lambda, \lambda + 2\rho(k)\rangle P_\lambda(k)$

with the operator

$$L_k = \Delta_{\mathfrak{a}} + \sum_{\alpha \in R_+} k_\alpha \coth \frac{\langle \alpha, t\rangle}{2}\, \partial_\alpha\,. \tag{2.4}$$

This is just the W-invariant part of the Heckman-Opdam Laplacian, which is given by restriction to W-invariant functions of

$$\sum_{i=1}^{n} T_{\xi_i}(k)^2 - |\rho(k)|^2,$$

with an arbitrary orthonormal basis $\{\xi_1, \ldots, \xi_n\}$ of $\mathfrak{a}$. The operator L_k generalizes the radial part of the Laplace-Beltrami operator on a Riemannian symmetric space of the non-compact type.

Let us point out that in the definition of the Jacobi polynomials, condition (ii) is frequently replaced by an orthogonality condition. As remarked in Proposition 8.1 of [H1], both sets of conditions are equivalent. Note also that in [H1], the Jacobi polynomials are indexed by $-P_+$ instead of P_+, which leads to a different sign in (ii).

According to equation (4.4.10) of [HS], the $P_\lambda(k)$ can be expressed in terms of the hypergeometric function via

$$F_{\lambda+\rho}(k;t) = c(\lambda+\rho,k) P_\lambda(k;t), \tag{2.5}$$

where $c(\lambda,k)$ is the generalized c-function as defined in [HS], Definition 3.4.2. As the polynomial $P_0(k)$ is a constant, it follows that

$$F_\rho(k;t) = 1. \tag{2.6}$$

3. Some estimates for G and F

The growth behavior and asymptotic properties of the Cherednik kernel G and the hypergeometric function F have been studied in detail in [O1] as well as in [Sch], where the precise asymptotic behavior in the space variable was determined. We recall the following results:

Lemma 3.1. *([O1]) Let $k \geq 0$. Then for all $\lambda \in \mathfrak{a}_{\mathbb{C}}$ and all $t \in \mathfrak{a}$,*

$$|G_\lambda(k;t)| \leq \sqrt{|W|} \cdot e^{max_{w\in W} Re\langle w\lambda, t\rangle}. \tag{3.1}$$

Lemma 3.2. *([Sch]) Let $k \geq 0$. Then*

(1) *For $\lambda \in \mathfrak{a}$, the kernel $G_\lambda(k;\,.\,)$ is real and strictly positive on $\mathfrak{a}$.*
(2) *$|G_\lambda(k;t)| \leq G_{Re\lambda}(k;t)$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}$ and $t \in \mathfrak{a}$.*

By symmetrization over the Weyl group, one obtains the same properties and estimates for the hypergeometric function F.

In [Sch], Opdam's estimate (3.1) was substantially improved. In fact, it is shown there that for all $\lambda \in \mathfrak{a}$ and all $t \in \mathfrak{a}$,

$$G_\lambda(k;t) \leq G_0(k;t) \cdot e^{\max_{w\in W}\langle w\lambda,t\rangle} \tag{3.2}$$

and that for fixed $k > 0$, the kernel G_0 has the asymptotic behavior

$$G_0(k;t) \asymp \prod_{\alpha\in R_+^0\,|\langle\alpha,t\rangle\geq 0} \big(1+\langle\alpha,t\rangle\big) e^{-\langle\rho,t_+\rangle}$$

where R_+^0 denotes the set of indivisible positive roots and t_+ is the unique element from the orbit Wt which is contained in $\overline{\mathfrak{a}_+}$.

The following result generalizes Schapira's estimate (3.2).

Theorem 3.3. *Let $k \geq 0$. Then for all $\lambda \in \mathfrak{a}$, all $\mu \in \overline{\mathfrak{a}_+}$ and all $t \in \mathfrak{a}$,*

$$G_{\lambda+\mu}(k;t) \leq G_\mu(k,t) \cdot e^{\max_{w\in W}\langle w\lambda,t\rangle}.$$

The same estimate holds for the hypergeometric function F instead of G.

For $\mu = \rho \in \overline{\mathfrak{a}_+}$ we obtain, in view of identity (2.6) and of Lemma 3.2, the following

Corollary 3.4. *Let $k \geq 0$. Then for all $\lambda \in \mathfrak{a}_{\mathbb{C}}$ and all $t \in \mathfrak{a}$,*

$$\big|F_{\lambda+\rho}(k;t)\big| \leq e^{\max_{w\in W}\,\mathrm{Re}\langle w\lambda,t\rangle}. \tag{3.3}$$

Remarks.

(1) While the proof of (3.2) is by real-analytic methods and uses the Cherednik operators, we shall present a different approach, based on methods from complex analysis.
(2) Remark 3.1 of [Sch] implies the following asymptotics for $t \in \overline{\mathfrak{a}_+}$, when $k > 0$ and some real $\lambda \in \overline{\mathfrak{a}_+}$ are fixed:

$$F_{\lambda+\rho}(k;t) \asymp e^{\langle\lambda,t\rangle}.$$

For our purposes, it will however be important to have an estimate which is uniform in k.

For the proof of Theorem 3.3, we shall use the Phragmén-Lindelöf principle, see e.g. Theorem 5.61 of [T]:

Lemma 3.5. *(Phragmén-Lindelöf). Let f be holomorphic in an open neighborhood of the right half plane $H = \{z \in \mathbb{C} : Re\, z \geq 0\}$, and suppose that f satisfies*

$$|f(iy)| \leq M \quad \forall y \in \mathbb{R}$$

and, as $|z| = r \to \infty$,

$$f(z) = O(e^{r^\beta})$$

for some $\beta < 1$, uniformly in H. Then actually $|f(z)| \leq M$ for all $z \in H$.

Proof of Theorem 3.3. Fix $t \in \mathfrak{a}$ and denote again by t_+ the unique element from the orbit Wt which is contained in $\overline{\mathfrak{a}_+}$. Further, put

$$S := \{\lambda \in \mathfrak{a}_{\mathbb{C}} : \operatorname{Re}\lambda \in \mathfrak{a}_+\}.$$

The geometry of root systems implies that for $\lambda \in \overline{S}$ and all $w \in W$,

$$\langle w\operatorname{Re}\lambda, t\rangle \le \langle \operatorname{Re}\lambda, t_+\rangle.$$

Fix now $w \in W$ and consider the function

$$f(\lambda) := e^{-\langle \lambda, t_+\rangle} \cdot \frac{G_{w\lambda+\mu}(k;t)}{G_\mu(k;t)}$$

which is holomorphic on $\mathfrak{a}_{\mathbb{C}}$. We shall investigate f on the closure $\overline{S}$ of S. By part (2) of Lemma 3.2, we have

$$|f(\lambda)| \le f(\operatorname{Re}\lambda).$$

Hence for $\lambda \in \overline{S}$, Lemma 3.1 leads to the estimate

$$|f(\lambda)| \;\le\; e^{-\langle \operatorname{Re}\lambda, t_+\rangle} \cdot \frac{G_{w\operatorname{Re}\lambda+\mu}(k;t)}{G_\mu(k;t)} \;\le\; \sqrt{|W|}\cdot \frac{e^{\langle \mu, t_+\rangle}}{G_\mu(k;t)}. \tag{3.4}$$

Note that the right side is independent of λ. Again by Lemma 3.2, we further obtain for real $\lambda \in \mathfrak{a}$ the uniform estimate

$$|f(i\lambda)| = \frac{|G_{iw\lambda+\mu}(k;t)|}{G_\mu(k;t)} \;\le\; 1. \tag{3.5}$$

We claim that $|f| \le 1$ on $\overline{S}$. For this, fix a basis $\{\lambda_1, \ldots, \lambda_n\} \subseteq P_+$ of fundamental weights. Then each $\lambda \in \overline{S}$ has a unique expansion $\lambda = \sum_{i=1}^n z_i\lambda_i$ with $z_i \in H = \{z \in \mathbb{C} : \operatorname{Re} z \ge 0\}$. Consider first $\lambda = z_1\lambda_1$ with $z_1 \in H$. In view of estimates (3.5) and (3.4), we may apply Lemma 3.5 with $\beta = 0$, thus obtaining

$$|f(z_1\lambda_1)| \le 1 \quad \forall z_1 \in H.$$

We proceed by induction: Suppose, for $1 \le m < n$, that

$$|f(z_1\lambda_1 + \ldots + z_m\lambda_m)| \le 1 \quad \forall\, z_1, \ldots, z_m \in H.$$

Consider $h(z_{m+1}) := f(z_1\lambda_1 + \ldots + z_m\lambda_m + z_{m+1}\lambda_{m+1})$ for $z_{m+1} \in H$. This function is uniformly bounded on H according to (3.4), and for purely imaginary $z_{m+1} \in i\mathbb{R}$ we have

$$|h(z_{m+1})| \le \big|f\big(\operatorname{Re}(z_1\lambda_1 + \ldots + z_m\lambda_m + z_{m+1}\lambda_{m+1})\big)\big| = |f\big(\operatorname{Re}z_1\cdot\lambda_1 + \ldots + \operatorname{Re}z_m\cdot\lambda_m\big)|$$

which is less or equal to 1 by our induction hypothesis. By Lemma 3.5, we conclude that $|h(z)| \le 1$ for all $z \in H$. Thus, induction shows that $|f(\lambda)| \le 1$ for all $\lambda \in \overline{S}$, and in particular for all $\lambda \in \overline{\mathfrak{a}_+}$. If $\lambda \in \mathfrak{a}$ is arbitrary, just use the fact that $\lambda = w\lambda'$ with some $w \in W$ and $\lambda' \in \overline{\mathfrak{a}_+}$. This implies the assertion.

□

4. Limit transition for Jacobi polynomials of type BC

Let $\mathfrak{a} = \mathbb{R}^n$ with the usual Euclidian scalar product and denote by $(e_i)_{i=1,\ldots,n}$ the standard basis of $\mathbb{R}^n$. We consider the root system BC_n in $\mathbb{R}^n$ with the positive subsystem

$$BC_n^+ = \{e_i, 2e_i,\, 1 \le i \le n\} \cup \{e_i \pm e_j,\, 1 \le i < j \le n\},$$

as well as the root system A_{n-1} in the linear subspace

$$\mathbb{R}_0^n := \{t \in \mathbb{R}^n : t_1 + \ldots + t_n = 0\}$$

with the positive subsystem

$$A_{n-1}^+ = \{e_i - e_j,\, 1 \le i < j \le n\}.$$

The Jacobi polynomials associated with these root systems as well as their relationship have been widely studied; see in particular [BO], [BF], [H1] and [H2]. We recall the fundamental facts: Let

$$\pi(t) := t - \frac{1}{n}\langle t, \omega_n\rangle \omega_n \tag{4.1}$$

with

$$\omega_n = e_1 + \ldots + e_n$$

denote the orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}_0^n$. The cone of dominant weights of BC_n is

$$P_+^{BC} = \{(\lambda_1, \ldots \lambda_n) \in \mathbb{Z}_+^n : \lambda_1 \ge \ldots \ge \lambda_n\},$$

and the dominant weights of A_{n-1} are given by

$$P_+^A = \pi\big(P_+^{BC}\big).$$

For abbreviation, we write $P_+ := P_+^{BC}$, which is just the set of partitions of length n. The dominance order on P_+ is given by

$$\lambda \le \mu \iff \sum_{j=1}^{i} \lambda_j \le \sum_{j=1}^{i} \mu_j\,, \quad i = 1, \ldots, n.$$

For the A_{n-1}-case, we take a real parameter $\kappa \ge 0$ and consider the *monic* Jack polynomials j_λ^κ in n variables which are indexed by partitions $\lambda \in P_+$ and are uniquely characterized by the following conditions:

(1) j_λ^κ is homogeneous of degree $|\lambda|$ and of the form

$$j_\lambda^\kappa = m_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu}(\kappa) m_\mu$$

where $\mu < \lambda$ refers to the dominance order on P_+ and the m_λ, $\lambda \in P_+$, are the monomial symmetric polynomials

$$m_\lambda(x) = \sum_{\mu \in S_n \lambda} x^\mu \quad (x \in \mathbb{R}^n).$$

(2) j_λ^κ is an eigenfunction of the operator

$$D_\kappa = \sum_{i=1}^n x_i^2 \frac{\partial^2}{\partial x_i^2} + 2\kappa \sum_{i \ne j} \frac{x_i^2}{x_i - x_j} \frac{\partial}{\partial x_i}.$$

In fact, the Jack polynomials satisfy

$$D_\kappa j_\lambda^\kappa = d_\lambda(\kappa) j_\lambda^\kappa \quad \text{with } d_\lambda(\kappa) = \sum_{i=1}^n \lambda_i\big(\lambda_i - 1 + 2\kappa(n-i)\big),$$

see [Ha] or [St]. For $\kappa = 0$, we have $j_\lambda^0 = m_\lambda$, while for $\kappa > 0$, the polynomial $j_\lambda^\kappa(x)$ coincides up to constant positive factor with the Jack polynomial $J_\lambda(x; 1/\kappa)$ in standard normalization as introduced in [St].

The Heckman-Opdam Jacobi polynomials of type A_{n-1} with multiplicity parameter $\kappa \geq 0$ are essentially Jack polynomials; according to Proposition 3.3. of [BO], they are given by

$$P^A_{\pi(\lambda)}(\kappa; t) = j_\lambda^\kappa(e^t) \quad \text{where } e^t = (e^{t_1}, \ldots, e^{t_n}),\ t = (t_1, \ldots, t_n) \in \mathbb{R}_0^n\,.$$

Notice that the homogeneity of the Jack polynomials implies that for arbitrary $t \in \mathbb{R}^n$,

$$j_\lambda^\kappa(e^t) = e^{|\lambda|(t_1+\ldots+t_n)/n} \cdot j_\lambda^\kappa(e^{\pi(t)}). \tag{4.2}$$

The Heckman-Opdam Jacobi polynomials of type BC_n are parameterized by a multiplicity function $k = (k_1, k_2, k_3) \geq 0$ on BC_n, where k_1 stands for the parameter on e_i, k_2 for the parameter on $2e_i$ and k_3 for the parameter on $e_i \pm e_j$. Let L_k^{BC} be the associated operator (2.4) of type BC_n. The corresponing eigenvalue (see [Ha]) is

$$e_\lambda(k) := d_\lambda(k_3) + (k_1 + 2k_2 + 1)|\lambda|,$$

with d_λ as above. We then obtain from [Ha] the following representation of the BC_n-type Jacobi polynomials $P_\lambda^{BC}(k)$ in terms of the Jack polynomials $j_\lambda^{k_3}$:

Proposition 4.1. *For all λ, k, t as above,*

$$P_\lambda^{BC}(k; t) = 4^{|\lambda|} \prod_{\mu \subset \lambda} \frac{L_k^{BC} - e_\mu(k)}{e_\lambda(k) - e_\mu(k)}\, j_\lambda^{k_3}\big(-\sinh^2\big(\frac{t}{2}\big)\big). \tag{4.3}$$

Here $\sinh^2\big(\frac{t}{2}\big)$ is understood component-wise, and $\mu \subset \lambda$ means that $\mu \neq \lambda$ and $\mu_i \leq \lambda_i$ for each i.

Proof. Denote the right hand side of (4.3) by $\tilde{P}_\lambda^{BC}(k; t)$. It follows from relation (13) of [Ha] that $\tilde{P}_\lambda^{BC}(k; t)$ is equal to $P_\lambda^{BC}(k; t)$ up to a multiplicative constant. In order to identify this constant, we compare the leading terms of both polynomials in the expansion with respect to the monomial symmetric functions M_μ^{BC} of type BC as defined in (2.3). In fact,

$$4^{|\lambda|} m_\lambda\big(\sinh^2\frac{t}{2}\big) = M_\lambda^{BC}(t) + \sum_{\mu<\lambda} d_{\lambda\mu} M_\mu^{BC}(t)$$

with certain constants $d_{\lambda\mu}$ (c.f. [SK], p. 383). Taking the characterization of the Jack polynomials j_λ^κ as well as part 1 of the characterization of the $\tilde{P}_\lambda^{BC}(k; t)$ and the definition of t_i on p. 1580 of [Ha] into account, we conclude that

$$\tilde{P}_\lambda^{BC}(k; t) = M_\lambda^{BC}(t) + \sum_{\mu<\lambda} e_{\lambda\mu} M_\mu^{BC}(t) \tag{4.4}$$

with certain coefficients $e_{\lambda\mu}$. Therefore $\tilde{P}_\lambda^{BC}(k; t) = P_\lambda^{BC}(k; t)$ as claimed. □

We notice that representations such as (4.3) were already observed by Macdonald [M] and were used in [SK] for limit transitions between different families of orthogonal polynomials.

From (4.3), we shall deduce the following limit result which was already stated as Theorem 1 in [K2] without proof.

Theorem 4.2. *Fix a parameter $0 \le a \le \infty$ and consider $k = (k_1, k_2, k_3)$ where $k_3 \ge 0$ is fixed. Then*

$$\lim_{\substack{k_1+k_2\to\infty\\ k_1/k_2\to a}} P_\lambda^{BC}(k;t) \;=\; 4^{|\lambda|}\cdot j_\lambda^{k_3}(x(t)), \tag{4.5}$$

where the transform $t \mapsto x(t), \mathbb{R}^n \to \mathbb{R}^n_+$ is given by

$$x_i(t) = \gamma_a + \sinh^2\big(\frac{t_i}{2}\big), \quad \gamma_a = \frac{a+1}{a+2}$$

with the understanding that $\gamma_\infty = 1$. The convergence in (4.5) is locally uniform in $t \in \mathbb{R}^n$. Especially if $a = \infty$, then

$$\begin{aligned}\lim_{\substack{k_1+k_2\to\infty\\ k_1/k_2\to\infty}} P_\lambda^{BC}(k;t) \;&= 4^{|\lambda|} j_\lambda^{k_3}\big(\cosh^2\frac{t}{2}\big)\\ &= \Big(\prod_{i=1}^n 4\cosh^2\frac{t_i}{2}\Big)^{|\lambda|/n}\cdot P^A_{\pi(\lambda)}\big(k_3;\pi\big(\log\cosh^2\frac{t}{2}\big)\big)\end{aligned} \tag{4.6}$$

The case $a = \infty$ occurs for instance if k_2, $k_3 \ge 0$ are fixed and $k_1 \to \infty$.

Proof. We split the coordinate transform $t \to x(t)$ and consider first the transform

$$y_i = -\sinh^2\frac{t_i}{2}$$

which is frequently used in the BC-setting. In y-coordinates, the operator L_k^{BC} becomes

$$\widetilde{L}_k^{BC} \;=\; \sum_{i=1}^n y_i(y_i-1)\frac{\partial^2}{\partial y_i^2} - \sum_{i=1}^n\big(k_1+k_2+\frac{1}{2}-(k_1+2k_2+1)y_i\big)\frac{\partial}{\partial y_i} + 2k_3\sum_{i\ne j}\frac{y_i(y_i-1)}{y_i-y_j}\,\frac{\partial}{\partial y_i},$$

see Section 4 of [BO] (or also [Ha], Section 2.3). Next, we carry out the linear transform $x_i = \gamma_a - y_i$, under which $\widetilde{L}_k^{BC}$ becomes

$$\begin{aligned}\widehat{L}_k^{BC} \;&= \sum_{i=1}^n(\gamma_a - x_i)(\gamma_a - 1 - x_i)\frac{\partial^2}{\partial x_i^2} \;+\; 2k_3\sum_{i\ne j}\frac{(\gamma_a-x_i)(\gamma_a-1-x_i)}{x_i-x_j}\,\frac{\partial}{\partial x_i}\\ &\quad+\sum_{i=1}^n\big(k_1+k_2+\frac{1}{2}-(k_1+2k_2+1)(\gamma_a-x_i)\big)\frac{\partial}{\partial x_i}.\end{aligned}$$

Equation (4.3) thus writes

$$P_\lambda^{BC}(k;t) = 4^{|\lambda|}\Big(\prod_{\mu\subset\lambda}\frac{\widehat{L}_k^{BC}-e_\mu(k)}{e_\lambda(k)-e_\mu(k)}\,j_\lambda^{k_3}\Big)(x) \tag{4.7}$$

with $x = x(t)$. As $k_1 + k_2 \to \infty$, we have

$$e_\lambda(k) \;\sim\; |\lambda|(k_1+2k_2).$$

If in addition $k_1/k_2 \to a$, then

$$\frac{k_1+k_2}{k_1+2k_2} \to \gamma_a\,.$$

Now let $\mu \subset \lambda$. Then $|\mu| < |\lambda|$ and for $f \in C^\infty(\mathbb{R}^n)$ we obtain, as $(k_1,k_2) \to \infty$ in the required way,

$$\frac{\widehat{L}_k^{BC} - e_\mu(k)}{e_\lambda(k) - e_\mu(k)} f(x) \;\longrightarrow\; \frac{1}{|\lambda|-|\mu|}\Big(\sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \;- |\mu|\Big) f(x).$$

For f a symmetric polynomial, the convergence is locally uniform in $x \in \mathbb{R}^n$. In our case, $f = j_\lambda^{k_3}$ is homogeneous of degree $|\lambda|$. Thus

$$\sum_{i=1}^n x_i \frac{\partial}{\partial x_i} j_\lambda^{k_3}(x) = |\lambda| \cdot j_\lambda^{k_3}(x)$$

and therefore

$$\frac{\widehat{L}_k^{BC} - e_\mu(k)}{e_\lambda(k) - e_\mu(k)}\, j_\lambda^{k_3} \;\longrightarrow\; j_\lambda^{k_3}$$

locally uniformly, for each $\mu \subset \lambda$. Iteration according to formula (4.7) yields

$$P_\lambda^{BC}(k;t) \;\longrightarrow\; 4^{|\lambda|} \cdot j_\lambda^{k_3}(x(t)),$$

locally uniformly in t which completes the proof of relation (4.5). Finally, in the setting of relation (4.6) we have $\gamma_a = 1$, and the claimed limit result follows from formula (4.2).

□

5. Limit transition for hypergeometric functions of type BC

We now extend the above limit transition to the associated hypergeometric functions, where we restrict our attention to the case $a = \infty$.

For abbreviation, we write C_B for the closed Weyl chamber associated with the positive system BC_n^+, i.e.

$$C_B = \{t \in \mathbb{R}^n : t_1 \geq \ldots \geq t_n \geq 0\}.$$

Observe that under the projection $\pi : \mathbb{R}^n \to \mathbb{R}_0^n$, the chamber C_B is mapped onto the closed Weyl chamber associated with the positive subsystem A_{n-1}^+ of A_{n-1}.

Again, we consider $k = (k_1, k_2, k_3)$ where $k_3 \geq 0$ is fixed. We also recapitulate that the half-sums (2.1) of positive roots for BC_n and A_{n-1} are given by

$$\rho_{BC}(k) = \sum_{i=1}^n (k_1 + 2k_2 + 2k_3(n-i))e_i \quad \text{and} \quad \rho_A(k_3) = k_3 \sum_{i=1}^n (n+1-2i)e_i\,. \tag{5.1}$$

Theorem 5.1. *For each $t \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}^n$,*

$$\lim_{\substack{k_1+k_2 \to \infty \\ k_1/k_2 \to \infty}} F_{BC}(\lambda + \rho_{BC}(k), k; t)$$
$$= \prod_{i=1}^n \big(\cosh^2 \frac{t_i}{2}\big)^{\langle \lambda, \omega_n\rangle/n} \cdot F_A\big(\pi(\lambda) + \rho_A(k_3), k_3; \pi\big(\log \cosh^2 \frac{t}{2}\big)\big).$$

The convergence is locally uniform with respect to λ.

Notice that in this situation, $\rho_{BC}(k) \to \infty$. The proof of Theorem 5.1 will be based on Theorem 4.2 above and the following well-known theorem of Carlson (see e.g. [T], Theorem 5.81):

Theorem 5.2 (Carlson's Theorem)**.** *Let f be a function, which is holomorphic in a neighborhood of $\{z \in \mathbb{C} : Re\, z \geq 0\}$ and satisfies $f(z) = O(e^{c|z|})$ for some constant $c < \pi$. Suppose that $f(n) = 0$ for all $n \in \mathbb{N}_0$. Then f is identically zero.*

Proof of Theorem 5.1. Let $\mathcal{K}_+ := \{k = (k_1, k_2, k_3) \in \mathbb{R}^3 : k_i \geq 0\ \forall\, i\}$ and fix some $t \in \mathbb{R}^n$. By the BC-symmetry of both sides, we may assume that $t \in C_B$. For $k \in \mathcal{K}_+$ define

$$f_k(\lambda) := e^{-\langle \lambda, t\rangle} \cdot F_{BC}(\lambda + \rho_{BC}(k), k; t)$$

and

$$g(\lambda) := e^{-\langle \lambda, t\rangle} \cdot \prod_{i=1}^{n} \big(\cosh^2 \frac{t_i}{2}\big)^{\langle \lambda, \omega_n\rangle/n} \cdot F_A\big(\pi(\lambda) + \rho_A(k_3), k_3; \pi\big(\log \cosh^2 \frac{t}{2}\big)\big).$$

The functions f_k and g are holomorphic on $\mathbb{C}^q$. Corollary 3.4 readily implies that the family $\{f_k : k \in \mathcal{K}_+\}$ is locally bounded on $\mathbb{C}^q$ and uniformly bounded on the set $S := \{\lambda \in \mathbb{C}^q : \mathrm{Re}\, \lambda \in C_B\}$; indeed, as $t \in C_B$ we obtain

$$|f_k(\lambda)| \leq 1 \quad \forall\, \lambda \in S. \tag{5.2}$$

Now let $(k(j))_{j\in\mathbb{N}} \subset \mathcal{K}_+$ be a sequence of multiplicities such that $k(j)_3 = k_3$ with fixed $k_3 \geq 0$ and $k(j)_1 + k(j)_2 \to +\infty$, $k(j)_1/k(j)_2 \to +\infty$. For abbreviation, we write

$$f_j := f_{k(j)},\ j \in \mathbb{N}.$$

We have to show that $f_j \to g$ locally uniformly on $\mathbb{C}^q$. By Montel's theorem in several complex variables (see for instance [G]), each locally bounded sequence of holomorphic functions on $\mathbb{C}^q$ has a subsequence which converges locally uniformly to some limit function which is again holomorphic on $\mathbb{C}^q$. It therefore suffices to verify the following condition:

(M) If (f_{j_ν}) is a subsequence of (f_j) such that $f_{j_\nu} \to h$ locally uniformly on $\mathbb{C}^q$ for some h, then $h = g$ on $\mathbb{C}^q$.

Suppose that (f_{j_ν}) is a subsequence with $f_{j_\nu} \to h$ locally uniformly on $\mathbb{C}^q$. According to Theorem 4.2 together with (2.5) and $F_{\lambda+\rho}(k; 0) = 1$, we have

$$f_{j_\nu}(\lambda) \to g(\lambda)$$

for all dominant weights $\lambda \in P_+$. Therefore $h(\lambda) = g(\lambda)$ for all $\lambda \in P_+$. Consider again the set S. We claim that

$$h(\lambda) = g(\lambda) \quad \forall \lambda \in S. \tag{5.3}$$

Once this is shown, the identity theorem will imply that $h = g$ on $\mathbb{C}^q$, and the verification of condition (M) will be accomplished. For the proof of (5.3) we shall apply Carlson's theorem to $g - h$ on S, which requires suitable growth bounds on the involved functions. First, h is the locally uniform limit of the sequence f_{j_ν} which is uniformly bounded on S according to (5.2). Hence

$$|h(\lambda)| \leq 1 \quad \forall \lambda \in S.$$

For an estimate of g on S, note that $\operatorname{Re}\pi(\lambda)$ is contained in the closed positive chamber associated with A_{n-1}^+ for each $\lambda \in S$. Application of Corollary 3.4 therefore yields

$$\Big|e^{-\langle \pi(\lambda), \pi(\log(\cosh^2 \frac{t}{2}))\rangle} \cdot F_A\big(\pi(\lambda)+\rho_A(k_3), k_3; \pi\big(\log\cosh^2\frac{t}{2}\big)\big)\Big| \;\le\; 1$$

for all $\lambda \in S$. Let us call the function on the left $E(\lambda)$ and write

$$|g(\lambda)| \;=\; \big|e^{-\langle \lambda, t\rangle} \cdot e^{\langle \pi(\lambda), \pi(\log(\cosh^2 \frac{t}{2}))\rangle} \cdot \prod_{i=1}^n \big(\cosh^2\frac{t_i}{2}\big)^{\langle \lambda, \omega_n\rangle/n}\big| \cdot E(\lambda).$$

As

$$\langle \pi(x), \pi(y)\rangle \;=\; \langle x, y\rangle - \frac{1}{n}\langle x, \omega_n\rangle\langle y, \omega_n\rangle \quad \forall x, y \in \mathbb{R}^n,$$

we obtain

$$|g(\lambda)| \;=\; \big|e^{-\langle \lambda, t\rangle} \cdot e^{\langle \lambda, \log(\cosh^2 \frac{t}{2})\rangle}\big| \cdot E(\lambda) \;\le\; \prod_{i=1}^n \big(e^{-t_i}\cosh^2\frac{t_i}{2}\big)^{\operatorname{Re}\lambda_i}$$

and therefore

$$|g(\lambda)| \le 1 \quad \forall \lambda \in S.$$

Summing up, we have

$$|g-h| \le 2 \;\text{ on }\; S \;\text{ and }\; (g-h)(\lambda) = 0 \;\; \forall\, \lambda \in P_+.$$

As in the proof of Theorem 3.3, we fix a set of fundamental weights $\{\lambda_1, \dots, \lambda_n\} \subset P_+$ and write $\lambda \in S$ as $\lambda = \sum_{i=1}^n z_i\lambda_i$ with coefficients $z_i \in \{z \in \mathbb{C} : \operatorname{Re} z \ge 0\}$. Then successive use of Carlson's Theorem with respect to the variables $z_1, \dots, z_n$ shows that actually $g - h = 0$ on S.

□

6. Limit transition for spherical functions of noncompact Grassmann manifolds

6.1. **Spherical functions of non-compact Grassmannians.** For specific multiplicities, hypergeometric functions of type BC occur as spherical functions of non-compact Grassmann manifolds. This was the starting point for the construction of hypergroup convolution algebras with hypergeometric functions as characters in [R]. Let us recall this connection. For each of the fields $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ we consider the Grassmann manifolds $\mathcal{G}_{p,q}(\mathbb{F}) = G/K$ where G is one of the groups $SO_0(p,q)$, $SU(p,q)$ or $Sp(p,q)$ with maximal compact subgroup $K = SO(p) \times SO(q)$, $S(U(p) \times U(q))$ or $Sp(p) \times Sp(q)$, where we assume that $p > q$. We regard G and K as subgroups of the indefinite unitary group $U(p,q;\mathbb{F})$ over $\mathbb{F}$. The Lie algebra $\mathfrak{g}$ of G has the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ consists of the $(p+q)$-block matrices

$$\begin{pmatrix} 0 & X \\ \overline{X}^t & 0 \end{pmatrix}, \quad X \in M_{p,q}(\mathbb{F}).$$

As a maximal abelian subspace of $\mathfrak{p}$ we choose

$$\mathfrak{a} = \left\{ H_t = \begin{pmatrix} & 0_{p\times p} & \\ \underline{t} & 0_{q\times(p-q)} & 0_{q\times q} \end{pmatrix}, \; t \in \mathbb{R}^q \right\}$$

where $\underline{t} := \operatorname{diag}(t_1, \dots, t_q)$ is the $q \times q$ diagonal matrix corresponding to t.

The restricted root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ is of type BC_q with the understanding that zero is allowed as a multiplicity on the long roots. We identify $\mathfrak{a}^*$ with $\mathfrak{a}$ via the Killing form and $\mathfrak{a}$ with $\mathbb{R}^q$ via the mapping $H_t \mapsto t$. Under this identification, the Killing form corresponds to a constant multiple of the Euclidean scalar product on $\mathbb{R}^q$, and

$$\Delta = BC_q = \{\pm e_i, \pm 2e_i, \pm e_i \pm e_j,\ 1 \le i < j \le q\} \subset \mathbb{R}^q.$$

The geometric multiplicities of the roots are given by

$$m_\alpha = \begin{cases} d(p-q) & \text{for } \alpha = \pm e_i \\ d-1 & \text{for } \alpha = \pm 2e_i \\ d & \text{for } \alpha = \pm e_i \pm e_j. \end{cases}$$

where $d = \dim_{\mathbb{R}} \mathbb{F}$. We consider the spherical functions of G/K as functions on $A = \exp \mathfrak{a}$. Let F_{BC} denote the hypergeometric function associated with $R = BC_q$ and multiplicity $k_\alpha = \frac{1}{2} m_\alpha$ (m_α as above), and denote by $\widetilde{F}_{BC}$ the hypergeometric function associated with the rescaled root system $\widetilde{R} = 2BC_q$ and multiplicity $\widetilde{k}_{2\alpha} = k_\alpha$. Then according to Remark 2.3 of [H3] and Lemma 2.1, the spherical functions of the Grassmannian $\mathcal{G}_{p,q}(\mathbb{F})$ are given by

$$\varphi_\lambda(a_t) \,=\, \widetilde{F}_{BC}(\lambda, \widetilde{k}; t) \,=\, F_{BC}(\lambda/2, k; 2t), \quad \lambda \in \mathbb{C}^q, \tag{6.1}$$

where

$$t \in \mathbb{R}^q \quad \text{and} \quad a_t = e^{H_t} = \begin{pmatrix} \cosh \underline{t} & 0 & \sinh \underline{t} \\ 0 & I_n & 0 \\ \sinh \underline{t} & 0 & \cosh \underline{t} \end{pmatrix}.$$

The limit $k_1 \to \infty$ in Theorem 5.1 here corresponds to $p \to \infty$. In order to identify the limit in this case, we recapitulate some facts on spherical functions of type A.

6.2. **Spherical functions of type A.** Consider the symmetric spaces G/K where G is one of the connected reductive groups $GL_+(q, \mathbb{R})$, $GL(q, \mathbb{C})$, $GL(q, \mathbb{H})$ with maximal compact subgroup $K = SO(q)$, $U(q)$ and $Sp(q)$, respectively. We have the Cartan decomposition $G = KAK$ with

$$A = \exp \mathfrak{a}, \quad \mathfrak{a} = \{\, \underline{t} = \mathrm{diag}(t_1, \ldots, t_q),\ \ t = (t_1, \ldots, t_q) \in \mathbb{R}^q\}. \tag{6.2}$$

For the moment, we consider the spherical functions of G/K as functions on $\mathfrak{a}$, where we identify $\mathfrak{a} \cong \mathbb{R}^q$ via $\underline{t} \mapsto t$. The spherical functions of G/K are then characterized as the continuous functions on $\mathbb{R}^q$ which are symmetric and satisfy the product formula

$$\psi(t)\psi(s) = \int_K \psi\big(\log(\sigma_{sing}(e^{\underline{t}}\, k\, e^{\underline{s}}))\big) dk; \tag{6.3}$$

here $\sigma_{sing}(M) = (\sigma_1, \ldots, \sigma_q) \in \mathbb{R}^q$ denotes the singular values of $M \in M_q(\mathbb{F})$ ordered by size: $\sigma_1 \ge \ldots \ge \sigma_q$. The spherical functions of $G/K = GL(q, \mathbb{F})/U(q, \mathbb{F})$ are closely related to those of G_1/K_1 where G_1 is the corresponding semisimple group $SL(q, \mathbb{F})$ and $K_1 = SU(q, \mathbb{F})$. Indeed, consider the orthogonal projection $\pi : \mathbb{R}^q \to \mathbb{R}^q_0$ as in (4.1). In the same way as above, the spherical functions of G_1/K_1 may be characterized as the symmetric functions ψ on $\mathbb{R}^q_0$ which satisfy the same product formula (6.3). Now suppose that ψ is a spherical function of G/K. Then for $t \in \mathbb{R}^q$, we have

$$\psi(t) = \psi(t - \pi(t) + \pi(t)) = \,\psi(t - \pi(t)) \cdot \psi(\pi(t))$$

because $t-\pi(t)$ corresponds to the scalar matrix $\exp\left(\sum_{i=1}^q t_i/q\right)\cdot I_q$ which belongs to the subgroup $Z_{\mathbb{R}} := \{a\cdot I_q :\ a>0\}$ of the center of G. As the restriction of ψ to $Z_{\mathbb{R}}$ is multiplicative on $Z_{\mathbb{R}}$, we have $\psi(a\cdot I_q)=a^m$ with some exponent $m\in\mathbb{C}$. Therefore

$$\psi(t) = \exp\bigl(m\cdot\sum_{i=1}^q t_i/q\bigr)\cdot\psi(\pi(t)) \tag{6.4}$$

where the restriction $\psi|_{\mathbb{R}_0^q}$ corresponds to a spherical function of G_1/K_1. Conversely, it is easily checked that for a given spherical function ψ of G_1/K_1, formula (6.4) defines an extension to a spherical function ψ of G/K.

We now return to the usual convention and consider spherical functions as functions on the group. For G_1/K_1, the geometric multiplicity on the restricted root system $\Delta = A_{q-1}$ is given by $m=d$. Therefore, again according to Remark 2.3 of [H3] and Lemma 2.1, the spherical functions of G_1/K_1 can be identified as

$$\psi_\lambda(e^{\underline{t}}) \;=\; F_A(\lambda/2, d/2; 2t), \quad t\in\mathbb{R}_0^q, \tag{6.5}$$

with $\lambda\in\mathbb{C}_0^q := \{\lambda\in\mathbb{C}^q :\ \sum_{i=1}^q\lambda_i=0\}$. For $\lambda\in\mathbb{C}^q$, put $m=\sum_{i=1}^q\lambda_i$. Then

$$\langle t-\pi(t),\lambda\rangle = m\cdot\sum_{i=1}^q t_i/q\,. \tag{6.6}$$

This shows that we can parameterize the spherical functions of G/K according to

$$\psi_\lambda(e^{\underline{t}}) \;=\; e^{\langle t-\pi(t),\lambda\rangle}\cdot F_A\bigl(\pi(\lambda/2), d/2; \pi(2t)\bigr), \quad \lambda\in\mathbb{C}^q. \tag{6.7}$$

With the notions of (6.1) and (6.7), Theorem 5.1 now implies the following limit relation.

Corollary 6.1. *The spherical functions φ_λ of the Grassmannian $\mathcal{G}_{p,q}(\mathbb{F})$ and ψ_λ of $GL(q,\mathbb{F})/U(q,\mathbb{F})$ satisfy*

$$\lim_{p\to\infty}\varphi_{\lambda+\rho_{BC}^{geo}}(a_t) = \psi_{\lambda+\rho_A^{geo}}(\cosh\underline{t}\,)$$

for all $\lambda\in\mathbb{C}^q$ and $t\in\mathbb{R}^q$, with the "geometric" constants $\rho_R^{geo}=2\rho_R(k)$ given by

$$\rho_{BC}^{geo} = \sum_{i=1}^q (d(p+q+2-2i)-2)e_i \quad \textit{and} \quad \rho_A^{geo} = \sum_{i=1}^q d(q+1-2i)e_i.$$

Proof. From relation (6.1), Theorem 5.1 and identity (6.6) we obtain

$$\begin{aligned}\lim_{p\to\infty}\varphi_{\lambda+\rho_{BC}^{geo}}(a_t) &= \lim_{k_1\to\infty} F_{BC}(\lambda/2+\rho_{BC}(k), k; 2t)\\ &= \prod_{i=1}^q(\cosh^2 t_i)^{\langle\lambda,\omega_q\rangle/2q}\cdot F_A\bigl(\pi(\lambda/2)+\rho_A(k_3), d/2; \pi\bigl(\ln\cosh^2 t\bigr)\bigr)\\ &= e^{\langle\ln\cosh^2 t-\pi(\ln\cosh^2 t),\lambda/2\rangle}\cdot F_A\bigl(\pi(\lambda/2)+\rho_A(k_3), d/2; \pi\bigl(\ln\cosh^2 t\bigr)\bigr),\end{aligned}$$

with $k=(d(p-q)/2, (d-1)/2, d/2)$. Using $\rho_A(k_3)\in\mathbb{R}_0^q$ and (6.7), we conclude that this limit equals

$$\begin{aligned}&e^{\langle\ln\cosh^2 t-\pi(\ln\cosh^2 t),\,\lambda/2+\rho_A(k_3)\rangle}\cdot F_A\bigl(\pi(\lambda/2+\rho_A(k_3)), d/2; \pi\bigl(\ln\cosh^2 t\bigr)\bigr)\\ &= \psi_{\lambda+\rho_A^{geo}}(\cosh\underline{t}\,)\end{aligned}$$

as claimed. □

We finally mention that Corollary 6.1 can be also obtained with sharp estimates for the order of convergence by comparing explicit versions of the Harish-Chandra integral representations of the involved spherical functions; see [RV2]. We also remark that our limit transition for hypergeometric functions has a counterpart in the Euclidean case, namely the convergence of (suitably scaled) Dunkl-Bessel functions of type B to such of type A, which was obtained in [RV1] by completely different methods.

7. Spherical functions of infinite-dimensional Grassmannians

We now discuss an interpretation of the preceding limit results in the context of infinite dimensional symmetric spaces and Olshanski spherical pairs. For the general background on this subject we refer to Faraut [F] and Olshanski [Ol1], [Ol1] . In order to be in agreement with standard terminology, we slightly change our notation. We consider the Grassmann manifolds G_n/K_n with $G_n = SO_0(n+q,q)$, $SU(n+q,q)$ or $Sp(n+q,q)$ and maximal compact subgroup $K_n = SO(n+q)\times SO(q)$, $S(U(n+q)\times U(q))$ or $Sp(n+q)\times Sp(q)$. In all three cases, G_n is regarded as a closed subgroup of G_{n+1} with $K_n = G_n \cap K_{n+1}$. Consider the inductive limits $G_\infty = \lim_{\rightarrow} G_n$ and $K_\infty := \lim_{\rightarrow} K_n$. Then (G_∞, K_∞) is an Olshanski spherical pair, and G_∞/K_∞ is one of the infinite-dimensional Grassmannians $SO_0(\infty,q)/SO(\infty)\times SO(q)$, $SU(\infty,q)/S(U(\infty)\times U(q))$, $Sp(\infty,q)/Sp(\infty)\times Sp(q)$. A continuous function $\phi : G_\infty \to \mathbb{C}$ is called an Olshanski spherical function of (G_∞, K_∞) if ϕ is K_∞-biinvariant and satisfies the product formula

$$\phi(g)\cdot\phi(h) = \lim_{n\to\infty} \int_{K_n} \phi(gkh)\, dk \qquad \text{for} \quad g,h \in G_\infty.$$

We shall now classify the Olshanski spherical functions of (G_∞, K_∞) without representation theory.

For this we use the decomposition $G_n = K_n A_n^+ K_n$

$$A_n^+ := \left\{ \begin{pmatrix} \cosh \underline{t} & 0 & \sinh \underline{t} \\ 0 & I_n & 0 \\ \sinh \underline{t} & 0 & \cosh \underline{t} \end{pmatrix} : \ t \in C_B \right\}$$

of representatives of the K_n-double cosets in G_n, where again

$$C_B := \{t = (t_1, \ldots, t_q) \in \mathbb{R}^q : \ t_1 \geq t_2 \geq \ldots \geq t_q \geq 0\}$$

denotes the closed Weyl chamber of type BC. Therefore, independently of n, we identify A_n^+ with the set of diagonal matrices

$$D := \{\cosh \underline{t} := \mathrm{diag}(\cosh t_1, \ldots, \cosh t_q) : \ t \in C_B\}.$$

This gives the topological identification $G_n//K_n \simeq A_n^+ \simeq D$. Notice that the elements of D are just the lower right $q\times q$-blocks of the matrices from A_n^+. In the same way,

$$G_\infty//K_\infty \simeq A_\infty^+ := \left\{ a_t^\infty := \begin{pmatrix} \cosh \underline{t} & 0 & \sinh \underline{t} \\ 0 & I_\infty & 0 \\ \sinh \underline{t} & 0 & \cosh \underline{t} \end{pmatrix} : \ t \in C_B \right\} \simeq D.$$

By definition of the inductive limit topology, a function $\phi : G_\infty \to \mathbb{C}$ is continuous and K_∞-biinvariant iff for all $n \in \mathbb{N}$, $\phi|_{G_n}$ is continuous and K_n-biinvariant. The space of all continuous, K_∞-biinvariant functions on G_∞ may thus be identified with

the space of all continuous functions on D. Using this convention, the Olshanski spherical functions of (G_∞, K_∞) can be characterized as follows:

Lemma 7.1. *A continuous K_∞-biinvariant function $\phi : G_\infty \to \mathbb{C}$ is an Olshanski spherical function if and only if there is a continuous function $\tilde\phi : D \to \mathbb{C}$ with $\phi(a_t^\infty) = \tilde\phi(\cosh \underline{t})$ for $t \in C_B$ such that $\tilde\phi$ satisfies the product formula*

$$\tilde\phi(a) \cdot \tilde\phi(b) = \int_{U(q,\mathbb{F})} \tilde\phi(\sigma_{sing}(akb))dk, \quad a, b \in D. \tag{7.1}$$

Here the vector $\sigma_{sing}(...) \in \mathbb{R}^q$ is identified with the corresponding diagonal matrix.

Proof. Let ϕ be a continuous K_∞-biinvariant function on G_∞. By the preceding discussion, ϕ is Olshanski spherical if and only if there is a continuous function $\tilde\phi : D \to \mathbb{C}$ with $\phi(a_t^\infty) = \tilde\phi(\cosh t)$ for $t \in C_B$ such that $\tilde\phi$ satisfies

$$\tilde\phi(\cosh \underline{t}\,) \cdot \tilde\phi(\cosh \underline{s}\,) \;=\; \lim_{n\to\infty} \int_{K_n} \phi(a_t^\infty \, k \, a_s^\infty)\, dk = \lim_{n\to\infty} \int_{K_n} \phi(a_t^n \, k \, a_s^n)\, dk \tag{7.2}$$

for $s, t \in C_B$. We shall use Proposition 2.2 of [R] to rewrite the integrals on the right hand side. Let $B_q := \{w \in M_q(\mathbb{F}) : \; w^*w < I\}$ and

$$c_n := \int_{B_q} \Delta(I - w^*w)^{(n+q)d/2-\gamma}\, dw \quad \text{with } \gamma := d(q-1/2)+1,$$

where Δ denotes the determinant and dw means integration with respect to Lebesgue measure. Then

$$\int_{K_n} \phi(a_t^n \, k \, a_s^n)\, dk = \, c_n^{-1} \int_{B_q} \int_{U_0(q,\mathbb{F})} \tilde\phi(\sigma_{sing}(\sinh \underline{t}\, w \sinh \underline{s} \; + \; \cosh \underline{t}\, k \cosh \underline{s})) \cdot \Delta(I - w^*w)^{(n+q)d/2-\gamma}\, dk\, dw \tag{7.3}$$

where $U_0(q,\mathbb{F})$ is the connected component of $U(q,\mathbb{F})$. The probability measures $c_n^{-1} \cdot \Delta(I - w^*w)^{(n+q)d/2-\gamma}\, dw$ are compactly supported in B_q and tend weakly to the point measure δ_0 for $n \to \infty$. Therefore (7.2) is equivalent to

$$\tilde\phi(\cosh \underline{t}\,) \cdot \tilde\phi(\cosh \underline{s}\,) \;=\; \int_{U_0(q,\mathbb{F})} \tilde\phi(\sigma_{sing}(\cosh \underline{t}\, k \cosh \underline{s}))\, dk. \tag{7.4}$$

Finally, is easily checked that the group $U_0(q,\mathbb{F})$ may be replaced by $U(q,\mathbb{F})$ in the integral, which completes the proof. □

We consider the reductive symmetric spaces $G/K = GL(q,\mathbb{F})/U(q,\mathbb{F})$ of subsection 6.2 and resume the notation from there. We introduce the set of diagonal matrices

$$D_0 := \{e^{\underline{t}} \in M_q(\mathbb{R}) : t = (t_1, \dots t_q) \in \mathbb{R}^q \text{ with } \; t_1 \geq \dots \geq t_q\}.$$

Then $G//K \cong D_0$, and a spherical function ψ of G/K may be characterized as a continuous function on D_0 satisfying the product formula

$$\psi(a) \cdot \psi(b) = \int_{U(q,\mathbb{F})} \psi(\sigma_{sing}(akb))dk, \quad a, b \in D_0. \tag{7.5}$$

Comparison with Lemma 7.1 gives

Theorem 7.2. *A continuous K_∞-biinvariant function $\phi : G_\infty \to \mathbb{C}$ is an Olshanski spherical function if and only if the function $\tilde{\phi} : D \to \mathbb{C}$ with $\phi(a_t^\infty) = \tilde{\phi}(\cosh t)$ for $t \in C_B$ is the restriction to $\mathcal{D}$ of a spherical function ψ of G/K. Each spherical function ψ of G/K is uniquely determined by its restriction to D, and the Olshanski spherical functions therefore correspond in a bijective way to the spherical functions of G/K.*

Proof. The if-part is clear from Lemma 7.1. The converse direction follows from Lemma 7.1 together with the following lemma. □

Lemma 7.3. *Each continuous function φ on D which satisfies product formula (7.1) admits a unique extension to a continuous function ψ on D_0 satisfying product formula (7.5).*

Proof. Assume first that $\psi : D_0 \to \mathbb{C}$ is such an extension of φ. Consider first a scalar matrix $a = rI_q$ with $r \geq 1$. Then $a \in D$ and hence $\psi(a) = \varphi(a)$. Moreover, as $\psi(a^{-1}) = 1/\psi(a)$ for a as above, the function ψ is uniquely determined by φ on the set of scalar matrices $Z = \{rI_q,\, r > 0\}$. Now let $a \in D_0$. We then find $r > 0$ and a matrix $b \in D$ such that $a = rb$. Using Product formula (7.5) we obtain

$$\psi(rI_q)\psi(b) \;=\; \int_{U_q(\mathbb{F})} \psi(\sigma_{sing}(rkb))\, dk \;=\; \psi(rb) = \psi(a). \tag{7.6}$$

Therefore, ψ is determined uniquely by φ.

Conversely, it is easily checked that for given φ, the definition of ψ first on Z as above and then on D_0 via (7.6) leads to a well-defined continuous function ψ on D_0 which satisfies the product formula. □

We notice at this point that our proof of Theorem 7.2 relies only on the explicit product formula (7.3) and does not require the results of the preceding sections. On the other hand, Corollary 6.1 and Theorem 7.2 imply the following

Corollary 7.4. *All Olshanski spherical functions of the infinite-dimensional Grassmannians G_∞/K_∞ appear as limits of the spherical functions of the Grassmannians G_n/K_n.*

Let us finally remark that the arguments above may also be applied to further Olshanski spherical pairs with fixed rank, for example to pairs related to the Cartan motion groups of Grassmann manifolds with growing dimension.

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