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Abstract

We consider the problem of estimating the Pickands dependence function corresponding to a multivariate distribution. A minimum distance estimator is proposed which is based on a L^2-distance between the logarithms of the empirical and an extreme-value copula. The minimizer can be expressed explicitly as a linear functional of the logarithm of the empirical copula and weak convergence of the corresponding process on the simplex is proved. In contrast to other procedures which have recently been proposed in the literature for the nonparametric estimation of a multivariate Pickands dependence function [see Zhang et al. (2008) and Gudendorf and Segers (2011)], the estimators constructed in this paper do not require knowledge of the marginal distributions and are an alternative to the method which has recently been suggested by Gudendorf and Segers (2012). Moreover, the minimum distance approach allows the construction of a simple test for the hypothesis of a multivariate extreme-value copula, which is consistent against a broad class of alternatives. The finite-sample properties of the estimator and a multiplier bootstrap version of the test are investigated by means of a simulation study.

Keywords and Phrases: Extreme-value copula, minimum distance estimation, Pickands dependence function, weak convergence, copula process, test for extreme-value dependence
AMS Subject Classification: Primary 62G05, 62G32; secondary 62G20

1 Introduction

Consider a d-dimensional random variable $\mathbf{X} = (X_1, \ldots, X_d)$ with continuous marginal distribution functions $F_1, \ldots, F_d$. It is well known that the dependency between the different components of $\mathbf{X}$ can be described in a margin-free way by the copula C, which is based on the representation

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots F(x_d))$$

of the joint distribution function F of the random vector $\mathbf{X}$ [see Sklar (1959)]. A prominent class of copulas is the class of extreme-value copulas which arise naturally as the possible limits of copulas of component-wise maxima of independent, identically distributed or strongly mixing stationary sequences [see Deheuvels (1984) and Hsing (1989)]. For some applications of extreme-value copulas we refer to the work of Tawn (1988), Ghoudi et al. (1998), Coles et al. (1999) or Cebrian et al. (2003) among others.

A (d-dimensional) copula C is an extreme-value copula if and only if there exists a copula $\tilde{C}$ such that the relation

$$\lim_{n\to\infty} \tilde{C}(u_1^{1/n}, \ldots, u_d^{1/n})^n = C\,(u_1, \ldots, u_d) \tag{1.1}$$

holds for all $\mathbf{u} = (u_1, \ldots, u_d) \in [0,1]^d$. There exists an alternative description of multivariate extreme-value copulas, which is based on a function on the simplex

$$\Delta_{d-1} := \Big\{ \mathbf{t} \in [0,1]^{d-1} \,\Big|\, \sum_{j=1}^{d-1} t_j \le 1 \Big\}.$$

To be precise, a copula C is an extreme-value copula if and only if there exists a function $A : \Delta_{d-1} \to [1/d, 1]$ such that C has a representation of the form

$$C(u_1, \ldots, u_d) = \exp\Big\{ \Big(\sum_{j=1}^{d} \log u_j \Big) A\Big(\frac{\log u_2}{\sum_{j=1}^d \log u_j}, \ldots, \frac{\log u_d}{\sum_{i=1}^d \log u_j} \Big) \Big\}. \tag{1.2}$$

The function A is called Pickands dependence function [see Pickands (1981)]. If relation (1.2) holds true then the corresponding Pickands dependence function A is necessarily convex and satisfies the inequalities

$$\max\Big\{ 1 - \sum_{j=1}^{d-1} t_j, t_1, \ldots, t_{d-1} \Big\} \le A(\mathbf{t}) \le 1$$

for all $\mathbf{t} = (t_1, \ldots, t_{d-1}) \in \Delta_{d-1}$. In the case $d = 2$ these conditions are also sufficient for A to be a Pickands dependence function. By the representation (1.2) of the extreme-value copula C the problem of estimating C reduces to the estimation of the $(d-1)$-dimensional function A and statistical inference for an extreme-value copula C may now be reduced to inference for its corresponding Pickands dependence function.

The problem of estimating Pickands dependence function nonparametrically has a long history. Early work dates back to Pickands (1981) and Deheuvels (1991). Alternative estimators have been proposed and investigated in the papers by Capéraà et al. (1997), Jiménez et al. (2001), Hall and Tajvidi (2000), Segers (2007). These authors discuss the estimation of Pickands dependence function in the bivariate case and assume knowledge of the marginal distributions. Recently Genest and Segers (2009) and Bücher et al. (2011) proposed new estimators in the two-dimensional case which do not require this knowledge. While Genest and Segers (2009) considered rank-based versions of the estimators of Pickands (1981) and Capéraà et al. (1997), the approach of Bücher et al. (2011) is based on the minimum distance principle and yields an infinite class of estimators. The estimation problem of Pickands dependence function in the case $d > 2$ was studied by Zhang et al. (2008) and Gudendorf and Segers (2011) assuming knowledge of the marginal distributions. Their estimators are based on functionals of the transformed random variables $Y_{ij} = -\log F_j\left(X_{ij}\right)$ $(i = 1, \ldots, n\,, j = 1, \ldots, d)$, which were also the basis for the estimators proposed by Pickands (1981) and Capéraà et al. (1997) in the bivariate case. Zhang et al. (2008) considered the random variable

$$Z_{ij}\left(\mathbf{s}\right) = \frac{\bigwedge_{k:k\neq j} \frac{Y_{ik}}{s_k}}{\frac{Y_{ij}}{1-s_j} + \bigwedge_{k:k\neq j} \frac{Y_{ik}}{s_k}}$$

where $\mathbf{s} = (s_1, \ldots, s_d)$ such that $(s_2, \ldots, s_d) \in \Delta_{d-1}$ and $s_1 = 1 - \sum_{j=2}^{d} s_j$ and $\bigwedge_{j\in\mathcal{J}} a_j = \min\{a_j \mid j \in \mathcal{J}\}$. They showed that the corresponding distribution function depends in a simple way on a partial derivative of the logarithm of Pickands dependence function and proposed to estimate Pickands dependence function by using a functional of the empirical distribution function of the random variables $Z_{ij}(\mathbf{s})$. The obtained estimator is uniformly consistent and converges point-wise to a normal distribution.

Gudendorf and Segers (2011) discussed the random variable $\xi_i\left(\mathbf{s}\right) = \bigwedge_{j=1}^{d} \frac{Y_{ij}}{s_j}$ which is Gumbel-distributed with location parameter $\log A\left(s_2, \ldots, s_d\right)$. They suggested to estimate Pickands dependence function by the method-of-moments and also provided an endpoint correction to impose (some of) the properties of Pickands dependence function. They also discussed the asymptotic properties of the estimator and a way to get optimal weight functions needed in the endpoint corrections. It was shown that the least squares estimator leads to weight functions which minimize the asymptotic variance. Furthermore, Gudendorf and Segers (2011) showed that in some cases their estimator coincides with the one proposed by Zhang et al. (2008). Recently, Gudendorf and Segers (2012) extended this methodology of Gudendorf and Segers (2011) to the case of unknown marginals.

The present paper is devoted to the construction of an alternative class of estimators of Pickands dependence function in the general multivariate case $d \geq 2$, which also do not require knowledge of the marginal distribution. For this purpose we will use the minimum distance approach proposed in Bücher et al. (2011), which allows us to construct an infinite dimensional class of estimators, which depend in a linear way on the logarithm of the d-dimensional empirical copula. Because this statistic does not require knowledge of the marginals the resulting estimator of Pickands dependence function does automatically not depend on the marginal distributions of X. We also briefly discuss the properties of our methods in the case of dependent data.

Moreover, the minimum distance approach also allows us to construct a simple test for the hypothesis that a given copula is an extreme-value copula. In this case the distance between the copula and its best approximation by an extreme-value copula is 0, and as a consequence a consistent estimator of the minimum distance should be small. Therefore the hypothesis of an extreme-value copula can be rejected for large values of this estimator. A multiplier bootstrap for the approximation of the critical values is proposed and its consistency proved. Moreover, we demonstrate that the new bootstrap is also applicable in the context of dependent data. Alternative tests for extreme-value dependence in dimension $d > 2$ in the case of independent data have recently been proposed by Kojadinovic et al. (2011) [for tests in dimension $d = 2$ for independent data see, e.g., Ghoudi et al. (1998); Ben Ghorbal et al. (2009); Kojadinovic and Yan (2010); Bücher et al. (2011); Genest et al. (2011); Quessy (2011)].

The remaining part of the paper is organized as follows. In Section 2 we present the necessary notation and define the class of minimum distance estimators. The main asymptotic properties are given in Section 3, while the corresponding test for the hypothesis of an extreme value copula is investigated in Section 4. Here we also establish consistency of the multiplier bootstrap such that critical values can easily be calculated by numerical simulation. The finite-sample properties of the new estimators and the test are investigated in Section 5, where we also present a brief comparison with the estimators proposed by Gudendorf and Segers (2012). Finally some technical details are deferred to an Appendix in Section 5.

2 Measuring deviations from an extreme-value copula

Throughout this paper we define $\mathcal{A}$ as the set of all functions $A : \Delta_{d-1} \to [1/d, 1]$ and Π is the independence copula, that is $\Pi(u_1, \ldots, u_d) = \prod_{j=1}^{d} u_j$. For most statements in this paper we will assume that the copula C satisfies $C \geq \Pi$ (or a slight modification of this statement). This assumption is equivalent to positive quadrant dependence of the random variables, that is for every $(x_1, \ldots, x_d) \in \mathbb{R}^d$ we have

$$\mathbb{P}(X_1 \leq x_1, \ldots, X_d \leq x_d) \geq \prod_{j=1}^{d} \mathbb{P}(X_j \leq x_j).$$

Obviously it holds for any extreme-value copula because of the lower bound of Pickands dependence function. Following Bücher et al. (2011) the construction of minimum distance estimators for

Pickands dependence function is based on a weighted L^2-distance

$$M_h(C,A) = \int_{(0,1)\times\Delta_{d-1}} \left\{\log C\left(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}}\right) - \log(y)\, A(\mathbf{t})\right\}^2 h(y)\, d(y,\mathbf{t}), \quad (2.1)$$

where $h : [0,1] \to \mathbb{R}^+$ is a continuous weight function and $\mathbf{t} = (t_1, \ldots, t_{d-1}) \in \Delta_{d-1}$. The result below gives an explicit expression of the best L^2-approximation of the logarithm of a copula satisfying this condition.

Theorem 2.1. *Assume that the copula C satisfies $C \geq \Pi^\kappa$ for some $\kappa \geq 1$ and that the weight function h satisfies $\int_0^1 (\log y)^2\, h(y)\, dy < \infty$. Then*

$$A^* = \operatorname{argmin}\{M_h(C,A) \mid A \in \mathcal{A}\}$$

is well-defined and given by

$$A^*(\mathbf{t}) = B_h^{-1} \int_0^1 \frac{\log C\left(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}}\right)}{\log y} h^*(y)\, dy, \quad (2.2)$$

where we use the notations

$$h^*(y) = \log^2(y)\, h(y) \quad (2.3)$$

and $B_h = \int_0^1 (\log y)^2\, h(y)\, dy = \int_0^1 h^*(y)\, dy$. Moreover, if $C \geq \Pi$, the function A^* defined in (2.2) satisfies

$$\max\left\{1 - \sum_{j=1}^{d-1} t_j, t_1, \ldots, t_{d-1}\right\} \leq A^*(\mathbf{t}) \leq 1.$$

Proof. Since $C \geq \Pi^\kappa$ we obtain

$$1 \geq C\left(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}}\right) \geq \Pi\left(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}}\right)^\kappa = y^\kappa$$

and thus

$$0 \geq \log\left(C\left(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}}\right)\right) \geq \kappa \log y.$$

This yields $|\log C\left(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}}\right)| \leq \kappa |\log y|$ and therefore the integral in (2.2) exists. By Fubini's theorem the weighted L^2-distance can be rewritten as

$$M_h(C,A) = \int_{\Delta_{d-1}} \int_0^1 \left(\frac{\log C\left(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}}\right)}{\log y} - A(\mathbf{t})\right)^2 \log^2(y)\, h(y)\; dy\, d\mathbf{t},$$

and now the first part of the assertion is obvious.
For a proof of the second part we make use of the upper Fréchet-Hoeffding-bound and obtain

$$A^*(\mathbf{t}) \geq B_h^{-1} \int_0^1 \frac{\log \min\{y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}}\}}{\log y} h^*(y)\, dy$$

$$= B_h^{-1} \int_0^1 \max\Big\{1 - \sum_{j=1}^{d-1} t_j, t_1, \ldots, t_{d-1}\Big\} h^*(y)\, dy = \max\Big\{1 - \sum_{j=1}^{d-1} t_j, t_1, \ldots, t_{d-1}\Big\}.$$

With a similar calculation and the assumption $C \geq \Pi$ we obtain the upper bound. □

A possible choice for the weight function is given by $h(y) = -y^k / \log y$, where $k \geq 0$, see Example 2.5 in Bücher et al. (2011). In Section 5 we consider this weight function with $k = 0.5$.
If the copula C is not an extreme-value copula the function A^* has not necessarily to be convex for any copula satisfying $C \geq \Pi$ [see Bücher et al. (2011)]. However, for every copula satisfying $C \geq \Pi^\kappa$ for some $\kappa \geq 1$ the equality $M_h(C, A^*) = 0$ holds if and only if the copula C is an extreme-value copula with Pickands dependence function A^*. This property will be useful for the construction of a test for the hypothesis that C is an extreme-value copula, which will be discussed in Section 4. For this purpose we will need an empirical analogue of the "best approximation" A^* which is constructed and investigated in the following section.

3 Weak convergence of minimal distance estimators

Throughout the remaining part of this paper let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ denote independent identically distributed $\mathbb{R}^d$-valued random variables. We define the components of each observation by $\mathbf{X}_i = (X_{i1}, \ldots, X_{id})$ $(i = 1, \ldots, n)$ and assume that all marginal distribution functions of $\mathbf{X}_i$ are continuous. The copula of $\mathbf{X}_i$ can easily be estimated in a nonparametric way by the empirical copula [see, e.g., Rüschendorf (1976)] which is defined for $\mathbf{u} = (u_1, \ldots, u_d)$ by

$$C_n(\mathbf{u}) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{\hat{U}_{i1} \leq u_1, \ldots \hat{U}_{id} \leq u_d\}, \tag{3.1}$$

where $\hat{U}_{ij} = \frac{1}{n+1} \sum_{k=1}^n \mathbb{I}\{X_{kj} \leq X_{ij}\}$ denote the normalized ranks of X_{ij} amongst $X_{1j}, \ldots, X_{nj}$. Following Bücher et al. (2011) we use Theorem 2.1 to construct an infinite class of estimators for Pickands dependence function by replacing the unknown copula with the empirical copula. To avoid zero in the logarithm we use a modification of the empirical copula. We set $\tilde{C}_n = C_n \vee n^{-\gamma}$ where $\gamma > \frac{1}{2}$ and obtain the estimator

$$\widehat{A}_{n,h}(\mathbf{t}) = B_h^{-1} \int_0^1 \frac{\log \tilde{C}_n\left(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}}\right)}{\log y} h^*(y)\, dy. \tag{3.2}$$

The weak convergence of the empirical process $\sqrt{n}(C_n - C)$ was investigated by Rüschendorf (1976) and Fermanian et al. (2004) among others under various assumptions on the partial derivatives of the copula C. Recently Segers (2012) proved the weak convergence of the empirical copula process

$$\mathbb{C}_n = \sqrt{n}\,(C_n - C) \rightsquigarrow \mathbb{G}_C \qquad \text{in } (\ell^\infty [0,1]^d, \|\cdot\|_\infty) \tag{3.3}$$

under a rather weak assumption, which is satisfied for many copulas, that is

$$\partial_j C(\mathbf{u}) \text{ exists and is continuous on } \{u \in [0,1]^d \mid u_j \in (0,1)\} \tag{3.4}$$

for every $j = 1, \ldots, d$. The limiting process $\mathbb{G}_C$ in (3.3) depends on the unknown copula and is given by

$$\mathbb{G}_C(\mathbf{u}) = \mathbb{B}_C(\mathbf{u}) - \sum_{j=1}^{d} \partial_j C(\mathbf{u}) \mathbb{B}_C(1, \ldots, 1, u_j, 1, \ldots, 1), \tag{3.5}$$

where we set $\partial_j C(\mathbf{u}) = 0$, $j = 1, \ldots, d$ for the boundary points $\{\mathbf{u} \in [0,1]^d \mid u_j \in \{0,1\}\}$. Here, $\mathbb{B}_C$ is a centered Gaussian field on $[0,1]^d$ with covariance structure

$$r(\mathbf{u}, \mathbf{v}) = \mathrm{Cov}(\mathbb{B}_C(\mathbf{u}), \mathbb{B}_C(\mathbf{v})) = C(\mathbf{u} \wedge \mathbf{v}) - C(\mathbf{u}) C(\mathbf{v}),$$

and the minimum is understood component-wise. Note that it can be shown that (3.4) holds for any extreme-value copula with continuously differentiable Pickands dependence function. The following result describes the asymptotic properties of the new estimators $\widehat{A}_{n,h}$ for Pickands dependence function. Weak convergence takes place in the space of all bounded functions on the unit simplex Δ_{d-1}, equipped with the topology induced by the sup-norm $\|\cdot\|_\infty$. The proof is given in the Appendix.

Theorem 3.1. *If the copula $C \geq \Pi$ satisfies condition (3.4), and the weight function h^* satisfies*

$$\left\| \frac{h^*}{\log} \right\|_\infty < \infty \quad \textit{and} \quad \int_0^1 h^*(y) (-\log y)^{-1} y^{-\lambda} dy < \infty \tag{3.6}$$

for some $\lambda > 1$, then we have for any $\gamma \in \left(\frac{1}{2}, \frac{\lambda}{2}\right)$ as $n \to \infty$

$$\sqrt{n}(\widehat{A}_{n,h} - A^*) \rightsquigarrow \mathbb{A}_{C,h} \tag{3.7}$$

in $\ell^\infty(\Delta_{d-1})$, where the limiting process is defined by

$$\mathbb{A}_{C,h} = B_h^{-1} \int_0^1 \frac{\mathbb{G}_C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})} \frac{h^*(y)}{\log y} dy.$$

Remark 3.2. A careful inspection of the proof of this result shows that weak convergence of the empirical copula process lies at the heart of its proof. Since the latter converges under fairly more general conditions on the serial dependence of a stationary time series, the i.i.d. assumption on the series $\mathbf{X}_1, \ldots, \mathbf{X}_n$ can be easily dropped. Exploiting the results in Bücher and Volgushev (2011), the assertion of Theorem 3.1 holds true for every stationary sequence of random vectors provided Condition 2.1 in that reference is met. This condition is so mild that all usual concepts of weak serial dependence are included, e.g., strong mixing or absolute regularity of a time series at a mild polynomial decay of the corresponding coefficients. For details we refer to Bücher and Volgushev (2011). The only difference to the i.i.d. case is reflected in a differing asymptotic covariance of the process $\mathbb{B}_C$ which is now given by

$$\mathrm{Cov}(\mathbb{B}_C(\mathbf{u}), \mathbb{B}_C(\mathbf{v})) = \sum_{j \in \mathbb{Z}} \mathrm{Cov}(\mathbb{I}\{\mathbf{U}_0 \leq \mathbf{u}\}, \mathbb{I}\{\mathbf{U}_j \leq \mathbf{v}\}),$$

and which, of course, reduces to $C(\mathbf{u} \wedge \mathbf{v}) - C(\mathbf{u}) C(\mathbf{v})$ in the i.i.d. setting.

Note that the result of Theorem 3.1 is correct even in the case where C is not an extreme-value copula because the centering in (3.7) uses the best approximation with respect to the L^2-distance. The discussed estimator $\widehat{A}_{n,h}$ in general will neither be convex nor will it necessarily satisfy the boundary conditions of a multivariate Pickands dependence function. To ensure the latter restriction, one can replace the estimator $\hat{A}_{n,h}$ by the statistic

$$\max\Big\{1 - \sum_{j=1}^{d-1} t_j, t_1, \ldots t_{d-1}, \min\{\hat{A}_{n,h}(\mathbf{t}), 1\}\Big\}.$$

Furthermore, to provide convexity, the greatest convex minorant of this statistics can be used. As a consequence the estimator $\widehat{A}_{n,h}$ is replaced by a convex estimator with a smaller sup-norm between the true Pickands dependence function and the corresponding estimator, see Wang (1986). An alternative way to achieve convexity and to correct for boundary properties of $\widehat{A}_{n,h}$ is the calculation of the L^2-projection on the space of partially linear functions satisfying these properties. This proposal was investigated by Fils-Villetard et al. (2008) and decreases the L^2-distance instead of the sup-norm.

Finally, we would like to point that none of these procedures guarantee that the modified estimator is in fact a Pickands dependence function, because these properties do not characterize Pickands dependence function in the case $d \geq 3$.

4 A test for extreme-value dependence

To construct a test for extreme-value dependence we reconsider the L^2-distance $M_h(C, A^*)$ defined in (2.1). The following result will motivate the choice of the test statistic.

Lemma 4.1. *If h is a strictly positive weight function with $h^* \in L^1(0,1)$, then $C \geq \Pi^\kappa$ for some $\kappa \geq 1$ is an extreme-value copula if and only if*

$$\min\{M_h(C, A) \mid A \in \mathcal{A}\} = M_h(C, A^*) = 0.$$

Proof. If C is an extreme-value copula then A^* is the Pickands dependence function of the copula C and the weighted L^2-distance is equal to 0.

Now assume $M_h(C, A^*) = 0$. With the definition of the L^2-distance we obtain

$$\log C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}}) = \log(y) A^*(\mathbf{t})$$

almost surely with respect to the Lebesgue measure on the set $(0,1) \times \Delta_{d-1}$. Since the functions $\log C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})$ and $(\log y)A^*(\mathbf{t})$ are continuous functions, the equality holds on the whole domain. This yields with a transformation for every $u_1, \ldots, u_d \in (0,1]$

$$C(u_1, \ldots, u_d) = \exp\Big\{\Big(\sum_{j=1}^d \log u_j\Big) A^*\Big(\frac{\log u_2}{\sum_{j=1}^d \log u_j}, \ldots \frac{\log u_d}{\sum_{i=1}^d \log u_j}\Big)\Big\}.$$

and it can easily be shown that this identity also holds on the boundary. As a consequence, C is max-stable and thus an extreme-value copula. $\square$

Lemma 4.1 suggests to use $M_h(\tilde{C}_n, \hat{A}_{n,h})$ as a test statistic for the hypothesis

$$H_0: \ C \text{ is an extreme-value copula} \tag{4.1}$$

and to reject the null hypothesis for large values of $M_h(\tilde{C}_n, \hat{A}_{n,h})$. We now will investigate the asymptotic distribution of the test statistic under the null hypothesis and the alternative.

Theorem 4.2. *Let C be an extreme-value copula satisfying condition (3.4) with Pickands dependence function A. If the weight function h is strictly positive, satisfies* (3.6) *and additionally the conditions*

$$\|h\|_\infty < \infty \quad \text{and} \quad \int_0^1 \frac{h(y)}{y^\lambda} \, dy < \infty \tag{4.2}$$

hold for some $\lambda > 2$, then we have for any $\gamma \in \left(\frac{1}{2}, \frac{\lambda}{4}\right)$ as $n \to \infty$

$$n M_h(\tilde{C}_n, \hat{A}_{n,h}) \rightsquigarrow Z_0,$$

where the random variable Z_0 is defined by

$$Z_0 := \int_{\Delta_{d-1}} \int_0^1 \left\{\frac{\mathbb{G}_C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}\right\}^2 h(y)\, dy\, d\mathbf{t} - B_h \int_{\Delta_{d-1}} \mathbb{A}_{C,h}^2(\mathbf{t})\, d\mathbf{t},$$

and the constant B_h and the process $\mathbb{A}_{C,h}$ are defined in Theorem 3.1.

The following result will give the asymptotic distribution of the test statistic under the alternative. Note that this is the case if and only if $M_h(C, A^*) > 0$.

Theorem 4.3. *Let $C \geq \Pi$ be a copula satisfying condition (3.4) such that $M_h(C, A^*) > 0$. If the strictly positive weight function h and the function h^* defined in* (2.3) *satisfy the conditions (3.6) and (4.2) for some $\lambda > 1$, then we have for any $\gamma \in (\frac{1}{2}, \frac{1+\lambda}{4} \wedge \frac{\lambda}{2})$ as $n \to \infty$*

$$\sqrt{n}(M_h(\tilde{C}_n, \hat{A}_{n,h}) - M_h(C, A^*)) \rightsquigarrow Z_1.$$

Here the random variable Z_1 is defined by

$$Z_1 := 2 \int_{\Delta_{d-1}} \int_0^1 \frac{\mathbb{G}_C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})} \nu(y, \mathbf{t}) \, dy \, d\mathbf{t}$$

with weight function

$$\nu(y, \mathbf{t}) = \Big\{ \log C(y^{1-\sum_{i=1}^{d-1} t_i}, y^{t_1}, \ldots, y^{t_{d-1}}) - \log(y) A^*(\mathbf{t}) \Big\} h(y).$$

Remark 4.4.
a) Again, the i.i.d. assumption on $\mathbf{X}_1, \ldots, \mathbf{X}_n$ in both preceding Theorems can be relaxed to weak serial dependence and strong stationarity, see Remark 3.2 above.
b) From Theorem 4.2 and 4.3 we obtain an asymptotic level α test for the hypothesis (4.1) by rejecting the null hypothesis H_0 if

$$n M_h(\tilde{C}_n, \hat{A}_{n,h}) > z_{1-\alpha},$$

where $z_{1-\alpha}$ denotes the $(1-\alpha)$-quantile of the distribution of the random variable Z_0. By Lemma 4.1 and Theorem 4.3 the test is (at least) consistent against all alternatives $C \geq \Pi$ satisfying assumption (3.4).
c) Note that the random variable Z_1 in Theorem 4.3 is normal distributed, with mean 0 and variance, say σ^2. Consequently the power of the test is approximately given by

$$P(n M_h(\tilde{C}_n, \hat{A}_{n,h}) > z_{1-\alpha}) \approx 1 - \Phi\Big(\frac{z_{1-\alpha}}{\sqrt{n}\sigma} - \sqrt{n}\frac{M_h(C, A^*)}{\sigma}\Big) \approx \Phi\Big(\sqrt{n}\frac{M_h(C, A^*)}{\sigma}\Big),$$

where A^* is defined in (2.2) and Φ denotes the standard normal distribution function. Thus the power of the test is an increasing function with respect to n depending on the quantity $\frac{M_h(C,A^*)}{\sigma}$, see Bücher et al. (2011).

For the construction of the test we need the $(1-\alpha)$-quantile of the distribution of the random variable Z_0. Unfortunately, this distribution depends on the unknown copula C and therefore it cannot be explicitly determined. However, we can easily construct a test if we approximate the distribution of the random variable Z_0 by the multiplier bootstrap [see Rémillard and Scaillet (2009), Bücher and Dette (2010) and Segers (2012)].
To this end, let $\widehat{\partial_j C_n}(\mathbf{u})$ be an estimator for $\partial_j C(\mathbf{u})$ which is uniformly bounded in n and $\mathbf{u}$ and for any $\delta \in (0, 1/2)$ satisfies the condition

$$\sup_{\mathbf{u} \in [0,1]^d : u_j \in [\delta, 1-\delta]} |\widehat{\partial_j C_n}(\mathbf{u}) - \partial_j C(\mathbf{u})| \xrightarrow{\mathbb{P}} 0,$$

as $n \to \infty$. It is easily seen, that for instance the following estimator based on finite differencing of the empirical copula satisfies these conditions:

$$\widehat{\partial_j C_n}(\mathbf{u}) = \begin{cases} \frac{C_n(\mathbf{u}+h_n\mathbf{e}_j)-C_n(\mathbf{u}-h_n\mathbf{e}_j)}{2h_n} & \text{if } u_j \in [h_n, 1-h_n] \\ \widehat{\partial_j C_n}(u_1,\ldots,u_{j-1},h_n,u_{j+1},\ldots,u_d) & \text{if } u_j \in [0,h_n) \\ \widehat{\partial_j C_n}(u_1,\ldots,u_{j-1},1-h_n,u_{j+1},\ldots,u_d) & \text{if } u_j \in (1-h_n,1], \end{cases}$$

where $h_n \to 0$ is a bandwidth such that $\inf_n h_n\sqrt{n} > 0$ and where $\mathbf{e}_j$ denotes the jth unit vector in $\mathbb{R}^d$.

Now, let $\xi_1, \xi_2, ...$ denote independent identically distributed random variables with mean 0 and variance 1 independent from $\mathbf{X}_1, \mathbf{X}_2, ...$ satisfying $\int_0^\infty \sqrt{P(|\xi_1| > x)}dx < \infty$. Define

$$\alpha_n^\xi(\mathbf{u}) = \frac{1}{\sqrt{n}}\sum_{i=1}^n \xi_i \left\{\mathbb{I}\{\hat{U}_{i,1} \le u_1, \ldots, \hat{U}_{i,d} \le u_d\} - C_n(\mathbf{u})\right\}$$

and set

$$\mathbb{C}_n^\xi(\mathbf{u}) = \alpha_n^\xi(\mathbf{u}) - \sum_{j=1}^d \widehat{\partial_j C_n}(\mathbf{u})\, \alpha_n^\xi(1,\ldots,1,u_j,1,\ldots,1).$$

It follows from the results in Segers (2012) that if C satisfies condition (3.4), then

$$(\mathbb{C}_n, \mathbb{C}_n^\xi) \rightsquigarrow (\mathbb{G}_C, \mathbb{G}_C')$$

in $(\ell\,[0,1]^d, \|\cdot\|_\infty)^2$, where $\mathbb{G}_C$ denotes the process defined in (3.5) and $\mathbb{G}_C'$ is an independent copy of this process. By the results in Bücher and Ruppert (2012) we also obtain conditional weak convergence of $\mathbb{C}_n^\xi$ given the data in probability, which we denote by

$$\mathbb{C}_n^\xi \underset{\xi}{\overset{\mathbb{P}}{\rightsquigarrow}} \mathbb{G}_C.$$

For details on that type of convergence we refer to the monograph Kosorok (2008). Our final result now shows that the multiplier bootstrap procedure can be used to obtain a valid approximation for the distribution of the random variable Z_0.

Theorem 4.5. *Assume the Copula $C \ge \Pi$ satisfies condition (3.4). If the weight function h satisfies the conditions in Theorem 4.2 and the function $y \mapsto h^*(y)(y\log y)^{-2}$ is uniformly bounded, then we get for the random variable*

$$\hat{Z}_n = \int_{\Delta_{d-1}} \int_0^1 \left\{\frac{\mathbb{C}_n^\xi(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{\tilde{C}_n(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}\right\}^2 h(y)\,dy\,d\mathbf{t}$$
$$- B_h^{-1} \int_{\Delta_{d-1}} \left\{\int_0^1 \frac{\mathbb{C}_n^\xi(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{\tilde{C}_n(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})} \frac{h^*(y)}{\log y} dy\right\}^2 d\mathbf{t}$$

the weak conditional convergence $\hat{Z}_n \underset{\xi}{\overset{\mathbb{P}}{\rightsquigarrow}} Z_0$.

Proof. Due to the assumptions on the weight function all integrals in the definition of Z_0 are proper and therefore the mapping $(\mathbb{G}_C, C) \mapsto Z_0(\mathbb{G}_C, C)$ is continuous. Hence, the result follows from $\mathbb{C}_n^\xi \overset{\mathbb{P}}{\underset{\xi}{\rightsquigarrow}} \mathbb{G}_C$ and the continuous mapping theorem for the bootstrap, see, e.g., Theorem 10.8 in Kosorok (2008). $\square$

The bootstrap test is now obtained as follows. Repeating the procedure B times yields a sample $\hat{Z}_n(1), \ldots, \hat{Z}_n(B)$ that is approximately distributed according to Z_0. This suggests to reject the null hypothesis if

$$nM_h(\tilde{C}_n, \hat{A}_{n,h}) > \hat{z}_{1-\alpha},$$

where $\hat{z}_{1-\alpha}$ denotes the empirical $(1-\alpha)$-quantile of this sample. It follows from Theorem 4.5 that the test holds its level α asymptotically and that it is consistent. The finite-sample performance of the test is investigated in the following section.

Remark 4.6. By the results in Bücher and Ruppert (2012) a block multiplier bootstrap can be used to obtain a valid bootstrap approximation of Z_0 in the case of strongly mixing stationary time series. We omit the details for the sake of brevity.

5 Finite-sample properties

This section is devoted to a simulation study regarding the finite-sample properties of the proposed estimators and tests for extreme-value copulas. We begin our discussion with the performance of the estimators. For that purpose we consider the trivariate extreme-value copula of logistic type as presented in Tawn (1990) with Pickands dependence function defined for $\mathbf{t} = (t_1, t_2) \in \Delta_2$ by

$$\begin{aligned} A(\mathbf{t}) = &(\theta^{1/\alpha} s_1^{1/\alpha} + \phi^{1/\alpha} s_2^{1/\alpha})^\alpha + (\theta^{1/\alpha} s_2^{1/\alpha} + \phi^{1/\alpha} s_3^{1/\alpha})^\alpha + (\theta^{1/\alpha} s_3^{1/\alpha} + \phi^{1/\alpha} s_1^{1/\alpha})^\alpha \\ &+ \psi(s_1^{1/\alpha} + s_2^{1/\alpha} + s_3^{1/\alpha})^\alpha + 1 - \theta - \phi - \psi, \end{aligned}$$

where $\mathbf{s} = (s_1, s_2, s_3) := (1 - t_1 - t_2, t_1, t_2)$ and $(\alpha, \theta, \phi, \psi) \in (0, 1] \times [0, 1]^3$. For the sake of comparison with existing simulation studies in the literature [see Gudendorf and Segers (2012)] we considered the parameters $(\theta, \phi, \psi) = (0, 0, 1)$ corresponding to a symmetric copula model (also widely known as the Gumbel–Hougaard copula) and $(\theta, \phi, \psi) = (0.6, 0.3, 0)$ corresponding to an asymmetric logistic copula. The parameter α was chosen from the set $\{0.3, 0.5, 0.7, 0.9\}$.

Regarding the choice of the weight function we followed the proposal in Bücher et al. (2011) and considered the function $h(y) = -y^k / \log(y)$ with $k = 0.5$. This choice seems to be a good compromise between a possibly difficult data-adaptive way of choosing a weight function and anaytical tractability, see Section 3.7 in Bücher et al. (2011). Additional simulations in dimension $d = 3$ (which we do not state here for the sake of a clear exposition) revealed similar effects as

in the two-dimensional study in the last-named reference. Furthermore, we refer to Section 3.4 in Bücher et al. (2011) for a discussion of "optimal" weight functions.
In Tables 1 and 2 we report Monte-Carlo approximations for the mean integrated squared error (MISE) $\mathbb{E}[\int(\hat{A}-A)^2]$ for the multivariate CFG-estimator, Pickands estimator (see Gudendorf and Segers (2012)) and the estimator introduced in the present paper which we abbreviate by BDV according to Bücher et al. (2011). All estimators are corrected for the boundary conditions on Pickands dependence function. The BDV-estimator is replaced by the function

$$\max\Big\{1-\sum_{j=1}^{d-1} t_j, t_1, \dots t_{d-1}, \min\{\hat{A}_{n,h}(\mathbf{t}), 1\}\Big\}.$$

For the Pickands- and CFG-estimator we used the endpoint-corrections supposed in Gudendorf and Segers (2012). For each scenario we simulated 1.000 samples of size $n \in \{50, 100, 200\}$ using the simulation algorithms in Stephenson (2003) which are implemented in the R-package evd, Stephenson (2002). The main finings are as follows.

- The Pickands estimator is outperformed by the CFG and the BDV estimator. Regarding only the former two estimators this finding is in-line with the simulation study in Gudendorf and Segers (2012).
- The CFG and the BDV estimator yield comparable results with slight advantages for the CFG estimator for strong dependence, whereas weak dependence results in more efficiency for the BDV estimator.

[Insert Tables 1 and 2 about here]

Finally, we conducted Monte Carlo experiments to investigate the level and the power of the test for extreme-value dependence introduced in Section 4. We fixed the dimension to $d = 3$ and considered samples of size $n = 200$ and $n = 400$ where the level of the test is $\alpha = 5\%$. Under the null hypothesis we simulated from the symmetric logistic type model as defined in (6.1) with parameters $(\theta, \phi, \psi) = (0, 0, 1)$ (i.e., the Gumbel–Hougaard copula). For the sake of a comparison with the two-dimensional version of the test in Bücher et al. (2011) and with the extensive simulation study in Kojadinovic et al. (2011) we chose the remaining parameter α in such a way that Kendall's tau varies in the set $\{0.25, 0.5, 0.75\}$. Under the alternative we considered the Clayton, Frank, Normal and t-copula with four degrees of freedom and the same values for Kendall's tau. For the multiplier method we chose $B = 100$ Bootstrap-replicates. The test was carried out at the 5% significance level and empirical rejection rates were computed from 1.000 random samples in each scenario. The results are stated in Table 3. The main findings are as follows.

- The test seems to be globally too conservative, although the observed level improves with increasing sample size. This effect is observed to be stronger for increasing level of dependence (measured by Kendall's tau) and is in-line with other simulation studies on the multiplier method for copulas with strong dependence.

- In terms of power the test detects all alternatives with reasonable rejection rates. Clayton and Frank's copula are detected more often, as it was supposed to under consideration of the findings in Bücher et al. (2011).

[Insert Table 3 about here]

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6 Proofs

6.1 Proof of Theorem 3.1

The proof follows from a slightly more general result, which establishes weak convergence for the weighted process

$$W_{n,\omega}(\mathbf{t}) = \int_0^1 \log \frac{\tilde{C}_n(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})} \omega(y, \mathbf{t})\, dy, \tag{6.1}$$

where the weight function $\omega : [0,1] \times \Delta_{d-1}$ may depend on y and $\mathbf{t}$. Theorem 3.1 is a simple consequence of the following result using the weight function $\omega(y, \mathbf{t}) = B_h^{-1} \frac{h^*(y)}{\log y}$ which does not depend on $\mathbf{t}$.

Theorem 6.1. *Assume that for the weight function $\omega : [0,1] \times \Delta_{d-1} \to \overline{\mathbb{R}}$ there exists a bounded function $\overline{\omega} : [0,1] \to \mathbb{R}_0^+$ such that $|\omega(y, \mathbf{t})| \le \overline{\omega}(y)$ for all $y \in [0,1]$ and all $\mathbf{t} \in \Delta_{d-1}$ and such that*

$$\int_0^1 \overline{\omega}(y)\, y^{-\lambda} dy < \infty \text{ for some } \lambda > 1. \tag{6.2}$$

If the copula $C \ge \Pi$ satisfies (3.4) then we have for every $\gamma \in \left(\frac{1}{2}, \frac{\lambda}{2}\right)$ as $n \to \infty$

$$\sqrt{n} W_{n,\omega}(\mathbf{t}) \rightsquigarrow \mathbb{W}_{C,\omega}(\mathbf{t}) \quad \text{in} \quad \ell^\infty(\Delta_{d-1}),$$

where the limiting process is given by

$$\mathbb{W}_{C,\omega}(\mathbf{t}) = \int_0^1 \frac{\mathbb{G}_C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})} \omega(y, \mathbf{t})\, dy.$$

Proof of Theorem 6.1. Fix $\lambda > 1$ and $\gamma \in \left(\frac{1}{2}, \frac{\lambda}{2}\right)$. Due to Lemma 1.10.2 in Van der Vaart and Wellner (1996), the processes $\sqrt{n}(\tilde{C}_n - C)$ and $\sqrt{n}(C_n - C)$ will have the same weak limit. For $i = 1, 2, \ldots$ we consider the following random functions in $\ell^\infty(\Delta_{d-1})$:

$$\begin{aligned} W_n(\mathbf{t}) &:= \int_0^1 \sqrt{n}\big\{ \log \tilde{C}_n(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}}) \\ &\qquad - \log C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})\big\} \omega(y, \mathbf{t})\, dy \\ W_{i,n}(\mathbf{t}) &:= \int_{1/i}^1 \sqrt{n}\big\{ \log \tilde{C}_n(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}}) \\ &\qquad - \log C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})\big\} \omega(y, \mathbf{t})\, dy \\ W(\mathbf{t}) &:= \int_0^1 \frac{\mathbb{G}_C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})} \omega(y, \mathbf{t})\, dy \\ W_i(\mathbf{t}) &:= \int_{1/i}^1 \frac{\mathbb{G}_C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})} \omega(y, \mathbf{t})\, dy \end{aligned}$$

With this notation we have to show the following three assertions :

(i) $W_{i,n} \rightsquigarrow W_i$ in $\ell^\infty(\Delta_{d-1})$ for $n \to \infty$,

(ii) $W_i \rightsquigarrow W$ in $\ell^\infty(\Delta_{d-1)}$ for $i \to \infty$,

(iii) for every $\varepsilon > 0$: $\lim_{i\to\infty} \overline{\lim}_{n\to\infty} \mathbb{P}^*\left(\sup_{t\in\Delta_{d-1}} |W_{i,n}(t) - W_n(t)| > \varepsilon\right) = 0$,

then Lemma B.1 in Bücher et al. (2011) yields the convergence $W_n \rightsquigarrow W$ in $\ell^\infty(\Delta_{d-1})$.
We begin with the proof of assertion (i). For this purpose we set $T_i = [1/i, 1]^d$ for $i \in \mathbb{N}$ and consider the mapping

$$\Phi_1 : \begin{cases} \mathbb{D}_{\Phi_1} \to \ell^\infty(T_i) \\ f \mapsto \log \circ f, \end{cases}$$

where the domain is defined by $\mathbb{D}_{\Phi_1} := \{f \in \ell^\infty(T_i) \mid \inf_{\mathbf{x}\in T_i} |f(\mathbf{x})| > 0\}$. Due to Lemma 12.2 in Kosorok (2008), it follows that Φ_1 is Hadamard-differentiable at C tangentially to $\ell^\infty(T_i)$ with derivative $\Phi'_{1,C}(f) = \frac{f}{C}$. Since $\tilde{C}_n \geq n^{-\gamma}$ and $C \geq \Pi$, we have $\tilde{C}_n, C \in \mathbb{D}_{\Phi_1}$ and with the functional delta method we obtain

$$\sqrt{n}(\log \tilde{C}_n - \log C) \rightsquigarrow \frac{\mathbb{G}_C}{C}$$

for $n \to \infty$ in $\ell^\infty(T_i)$. Now we consider the mapping

$$\Phi_2 : \begin{cases} \ell^\infty(T_i) \to \ell^\infty([1/i,1] \times \Delta_{d-1}) \\ f \mapsto f \circ \phi \end{cases},$$

where the mapping $\phi : [1/i,1] \times \Delta_{d-1} \to T_i$ is defined by

$$\phi(y, \mathbf{t}) = (y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}}).$$

For Φ_2 the following inequality holds:

$$\begin{aligned}\|\Phi_2(f) - \Phi_2(g)\|_\infty &= \sup_{y\in[1/i,1],\mathbf{t}\in\Delta_{d-1}} |f\circ\phi(y,\mathbf{t}) - g\circ\phi(y,\mathbf{t})| \\ &\le \sup_{\mathbf{x}\in T_i} |f(\mathbf{x}) - g(\mathbf{x})| = \|f-g\|_\infty.\end{aligned}$$

This implies that Φ_2 is Lipschitz-continuous. By the continuous mapping theorem and the boundedness of the weight function ω we obtain

$$\begin{aligned}&\sqrt{n}\left\{\log\tilde{C}_n(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}}) - \log C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})\right\}\omega(y,\mathbf{t}) \\ &\qquad\qquad \rightsquigarrow \frac{\mathbb{G}_C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}\omega(y,\mathbf{t})\end{aligned}$$

in $\ell^\infty\left([\frac{1}{i},1]\times\Delta_{d-1}\right)$. By integration with respect to $y \in [1/i, 1]$ assertion (i) follows. Assertion (ii) follows directly from the observation, that the process $\mathbb{G}_C$ is bounded on $[0,1]^d$ and from the fact, that the function

$$\mathbf{t} \mapsto \frac{\omega(y,\mathbf{t})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}$$

can be bounded by the integrable function $\overline{\omega}(y)\, y^{-1}$. The proof of (iii) is obtained by the same arguments as given in Bücher et al. (2011) in the case $d = 2$ and is therefore omitted. $\square$

6.2 Proof of Theorem 4.2

Since integration is continuous, it suffices to show the weak convergence $\bar{W}_n(t) \rightsquigarrow \bar{W}(t)$ in $\ell^\infty(\Delta_{d-1})$, where we define

$$\begin{aligned}\bar{W}_n(\mathbf{t}) &= \int_0^1 n\left(\log\frac{\tilde{C}_n(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}\right)^2 h(y)\,dy - nB_h(\hat{A}_{n,h}(\mathbf{t}) - A(\mathbf{t}))^2 \\ \bar{W}(\mathbf{t}) &= \int_0^1 \left(\frac{\mathbb{G}_C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}\right)^2 h(y)\,dy - B_h\mathbb{A}^2_{C,h}(\mathbf{t}).\end{aligned}$$

Now we will proceed similar to the proof of Theorem 6.1 and consider

$$\begin{aligned}
\bar W_{i,n}(\mathbf{t}) &= \int_{1/i}^{1} n\left(\log \frac{\tilde C_n(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}\right)^2 h(y)\, dy \\
&\qquad - B_h^{-1}\left(\int_{1/i}^{1} \sqrt{n} \log \frac{\tilde C_n(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})} \frac{h^*(y)}{\log y}\, dy \right)^2 \\
\bar W_i(\mathbf{t}) &= \int_{1/i}^{1} \left(\frac{\mathbb{G}_C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}\right)^2 h(y)\, dy \\
&\qquad - B_h^{-1}\left(\int_{1/i}^{1} \frac{\mathbb{G}_C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})}{C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}})} \frac{h^*(y)}{\log y}\, dy \right)^2.
\end{aligned}$$

Due to Lemma B.1 in Bücher et al. (2011) it suffices to show

(i) $\bar W_{i,n} \rightsquigarrow \bar W_i$ in $\ell^\infty(\Delta_{d-1})$ for $n \to \infty$,

(ii) $\bar W_i \rightsquigarrow \bar W$ in $\ell^\infty(\Delta_{d-1})$ for $i \to \infty$,

(iii) for every $\varepsilon > 0$: $\lim_{i\to\infty} \overline{\lim}_{n\to\infty} \mathbb{P}^* \left(\sup_{t\in\Delta_{d-1}} |\bar W_{i,n}(t) - \bar W_n(t)| > \varepsilon \right) = 0$.

The proof of these assertions follows by similar arguments as in Bücher et al. (2011) and is omitted for the sake of brevity. $\square$

6.3 Proof of Theorem 4.3

We use the decomposition

$$M_h(\tilde C_n, \hat A_{n,h}) - M_h(C, A^*) = S_1 + S_2 + S_3, \tag{6.3}$$

where

$$\begin{aligned}
S_1 &= 2 \int_{\Delta_{d-1}} \int_0^1 \left\{ \bar C_n(y,\mathbf{t}) - \bar C(y,\mathbf{t}) \right\} \left\{ \bar C(y,\mathbf{t}) - A^*(\mathbf{t})(-\log y) \right\} h(y)\, dy\, d\mathbf{t}, \\
S_2 &= \int_{\Delta_{d-1}} \int_0^1 \left\{ \bar C_n(y,\mathbf{t}) - \bar C(y,\mathbf{t}) \right\}^2 h(y)\, dy\, d\mathbf{t} \\
S_3 &= -B_h \int_{\Delta_{d-1}} \left\{ \hat A_{n,h}(\mathbf{t}) - A^*(\mathbf{t}) \right\}^2 d\mathbf{t}
\end{aligned}$$

and we used the notations

$$\begin{aligned}
\bar C(y,\mathbf{t}) &= -\log C(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}}), \\
\bar C_n(y,\mathbf{t}) &= -\log \tilde C_n(y^{1-t_1-\ldots-t_{d-1}}, y^{t_1}, \ldots, y^{t_{d-1}}).
\end{aligned}$$

To investigate the convergence of the first term in (6.3) we first notice that $|\nu(y, \mathbf{t})| \leq \bar{\nu}(y)$, with $\bar{\nu}(y) := \frac{2h^*(y)}{-\log y}$. The assumptions of the Theorem on the weight function h imply that we van invoke Theorem 6.1. With the continuous mapping theorem this yields $\sqrt{n}S_1 \rightsquigarrow Z_1$ and it remains to show that the remaining two terms S_2 and S_3 can be neglected. By Theorem 3.1 and the continuous mapping theorem we have $S_3 = O_P\left(\frac{1}{n}\right)$ and finally S_2 can be estimated along similar lines as in the proof of Theorem 4.2 in Bücher et al. (2011). $\square$

References

Ben Ghorbal, N., Genest, C., and Nešlehová, J. (2009). On the Ghoudi, Khoudraji, and Rivest test for extreme-value dependence. *Canadian Journal of Statistics*, 37:534–552.

Bücher, A. and Dette, H. (2010). A note on bootstrap approximations for the empirical copula process. *Statist. Probab. Lett.*, 80:1925–1932.

Bücher, A., Dette, H., and Volgushev, S. (2011). New estimators of the Pickands dependence function and a test for extreme-value dependence. *Ann. Statist.*, 39(4):1963–2006.

Bücher, A. and Ruppert, M. (2012). Consistent testing for a constant copula under strong mixing based on the tapered block multiplier technique. *arXiv:1206.1675*.

Bücher, A. and Volgushev, S. (2011). Empirical and sequential empirical copula processes under serial dependence. *arXiv:1111.2778*.

Capéraà, P., Fougères, A. L., and Genest, C. (1997). A nonparametric estimation procedure for bivariate extreme value copulas. *Biometrika*, 84:567–577.

Cebrian, A., Denuit, M., and Lambert, P. (2003). Analysis of bivariate tail dependence using extreme values copulas: An application to the SOA medical large claims database. *Belgian Actuarial Journal*, 3:33–41.

Coles, S., Heffernan, J., and Tawn, J. (1999). Dependence measures for extreme value analyses. *Extremes*, 2:339–365.

Deheuvels, P. (1984). Probabilistic aspects of multivariate extremes. In de Oliveira, J. T., editor, *Statistical Extremes and Applications*. Reidel, Dordrecht.

Deheuvels, P. (1991). On the limiting behavior of the pickands estimator for bivariate extreme-value distributions. *Statistics and Probability Letters*, 12:429–439.

Fermanian, J., Radulović, D., and Wegkamp, M. J. (2004). Weak convergence of empirical copula processes. *Bernoulli*, 10:847–860.

Fils-Villetard, A., Guillou, A., and Segers, J. (2008). Projection estimators of Pickands dependence functions. *Canad. J. Statist.*, 36(3):369–382.

Genest, C., Kojadinovic, I., Nešlehová, J., and Yan, J. (2011). A goodness-of-fit test for bivariate extreme-value copulas. *Bernoulli*, 17(1):253–275.

Genest, C. and Segers, J. (2009). Rank-based inference for bivariate extreme-value copulas. *Annals of Statistics*, 37(5B):2990–3022.

Ghoudi, K., Khoudraji, A., and Rivest, L. (1998). Propriétés statistiques des copules de valeurs extrêmes bidimensionnelles. *Canadian Journal of Statistics*, 26:187–197.

Gudendorf, G. and Segers, J. (2011). Nonparametric estimation of an extreme-value copula in arbitrary dimensions. *Journal of Multivariate Analysis*, 102:37–47.

Gudendorf, G. and Segers, J. (2012). Nonparametric estimation of multivariate extreme-value copulas. *Journal of Statistical Planning and Inference, to appear.*

Hall, P. and Tajvidi, N. (2000). Distribution and dependence-function estimation for bivariate extreme-value distributions. *Bernoulli*, 6:835–844.

Hsing, T. (1989). Extreme value theory for multivariate stationary sequences. *Journal of Multivariate Analysis*, 29:274–291.

Jiménez, J. R., Villa-Diharce, E., and Flores, M. (2001). Nonparametric estimation of the dependence function in bivariate extreme value distributions. *Journal of Multivariate Analysis*, 76:159–191.

Kojadinovic, I., Segers, J., and Yan, J. (2011). Large-sample tests of extreme-value dependence for multivariate copulas. *Canad. J. Statist.*, 39(4):703–720.

Kojadinovic, I. and Yan, J. (2010). Nonparametric rank-based tests of bivariate extreme-value dependence. *J. Multivariate Anal.*, 101(9):2234–2249.

Kosorok, M. R. (2008). *Introduction to Empirical Processes and Semiparametric Inference*. Springer Series in Statistics, New York.

Pickands, J. (1981). Multivariate extreme value distributions (with a discussion). *Proceedings of the 43rd Session of the International Statistical Institute. Bull. Inst. Internat. Statist.*, 49:859–878,894–902.

Quessy, J.-F. (2011). Testing for bivariate extreme dependence using kendall's process. *Scandinavian Journal of Statistics, to appear.*

Rémillard, B. and Scaillet, O. (2009). Testing for equality between two copulas. *Journal of Multivariate Analysis*, 100:377–386.

Rüschendorf, L. (1976). Asymptotic distributions of multivariate rank order statistics. *Annals of Statistics*, 4:912–923.

Segers, J. (2007). Nonparametric inference for bivariate extreme-value copulas. In Ahsanullah, M. and Kirmani, S. N. U. A., editors, *Topics in Extreme Values*. Nova Science Publishers, New York.

Segers, J. (2012). Asymptotics of empirical copula processes under non-restrictive smoothness assumptions. *Bernoulli*, 18(3):764–782.

Sklar, M. (1959). Fonctions de répartition à n dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris*, 8:229–231.

Stephenson, A. (2003). Simulating multivariate extreme value distributions of logistic type. *Extremes*, 6(1):49–59.

Stephenson, A. G. (2002). evd: Extreme value distributions. *R News*, 2(2).

Tawn, J. (1990). Modelling multivariate extreme value distributions. *Biometrika*, 77:245–253.

Tawn, J. A. (1988). Bivariate extreme value theory: Models and estimation. *Biometrika*, 75:397–415.

Van der Vaart, A. W. and Wellner, J. A. (1996). *Weak Convergence and Empirical Processes*. Springer Verlag, New York.

Wang, J. L. (1986). Asymptotically minimax estimators for distributions with increasing failure rate. *Annals of Statistics*, 44:1113–1131.

Zhang, D., Wells, M. T., and Peng, L. (2008). Nonparametric estimation of the dependence function for a multivariate extreme value distribution. *J. Multivariate Anal.*, 99(4):577–588.

Sample size	Estimator	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
$n = 50$	P	2.37×10^{-4}	6.91×10^{-4}	1.70×10^{-3}	2.91×10^{-3}
	CFG	9.94×10^{-5}	4.09×10^{-4}	1.16×10^{-3}	2.26×10^{-3}
	BDV	1.24×10^{-4}	5.07×10^{-4}	1.27×10^{-3}	2.04×10^{-3}
$n = 100$	P	1.01×10^{-4}	3.31×10^{-4}	7.59×10^{-4}	1.43×10^{-3}
	CFG	4.12×10^{-5}	2.28×10^{-4}	6.04×10^{-4}	1.17×10^{-3}
	BDV	5.46×10^{-5}	2.69×10^{-4}	6.23×10^{-4}	1.01×10^{-3}
$n = 200$	P	4.69×10^{-5}	1.59×10^{-4}	3.92×10^{-4}	7.15×10^{-4}
	CFG	2.34×10^{-5}	1.07×10^{-4}	3.02×10^{-4}	5.21×10^{-4}
	BDV	2.84×10^{-5}	1.20×10^{-4}	2.93×10^{-4}	4.77×10^{-4}

Table 1: Symmetric logistic dependence function, $(\theta, \phi, \psi) = (0, 0, 1)$: Simulated MISE for the Pickands, CFG- and BDV-estimator.

Sample size	Estimator	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
$n = 50$	P	1.65×10^{-3}	1.98×10^{-3}	2.49×10^{-3}	3.13×10^{-3}
	CFG	1.10×10^{-3}	1.32×10^{-3}	1.77×10^{-3}	2.51×10^{-3}
	BDV	1.19×10^{-3}	1.34×10^{-3}	1.67×10^{-3}	2.16×10^{-3}
$n = 100$	P	8.55×10^{-4}	9.48×10^{-4}	1.23×10^{-3}	1.53×10^{-3}
	CFG	5.42×10^{-4}	6.56×10^{-4}	8.32×10^{-4}	1.19×10^{-3}
	BDV	5.69×10^{-4}	6.61×10^{-4}	8.04×10^{-4}	9.86×10^{-4}
$n = 200$	P	4.05×10^{-4}	4.59×10^{-4}	5.99×10^{-4}	7.45×10^{-4}
	CFG	2.85×10^{-4}	3.20×10^{-4}	4.13×10^{-4}	5.28×10^{-4}
	BDV	2.91×10^{-4}	3.34×10^{-4}	3.90×10^{-4}	4.67×10^{-4}

Table 2: Asymmetric logistic dependence function, $(\theta, \phi, \psi) = (0.6, 0.3, 0)$: Simulated MISE for the Pickands, CFG- and BDV-estimator.

Copula	τ	$n = 200$	$n = 400$
Gumbel	0.25	0.051	0.038
	0.5	0.026	0.049
	0.75	0.011	0.023
Clayton	0.25	1	1
	0.5	1	1
	0.75	1	1
Frank	0.25	0.668	0.917
	0.5	0.9111	1
	0.75	0.916	1
normal	0.25	0.591	0.831
	0.50	0.639	0.932
	0.75	0.306	0.820
t-copula	0.25	0.358	0.570
	0.5	0.470	0.783
	0.75	0.222	0.688

Table 3: Simulated rejection probabilities of the test for the null hypothesis of an extreme-value copula where the level is 5%.