A Worst-Case Optimization Approach to Impulse Perturbed Stochastic Control with Application to Financial Risk Management

Dissertation

zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften
(Dr. rer. nat.)

Der Fakultät für Mathematik
der Technischen Universität Dortmund

vorgelegt von

Laurenz Mönnig

Dortmund, Juni 2012

Abstract

This work presents the main ideas, methods and results of the theory of impulse perturbed stochastic control as an extension of the classic stochastic control theory. Apart from the introduction and the motivation of the basic concept, two stochastic optimization problems are the focus of the investigations. On the one hand we consider a differential game as analogue of the expected utility maximization problem in the situation with impulse perturbation, and on the other hand we study an appropriate version of a target problem. By dynamic optimization principles we characterize the associated value functions by systems of partial differential equations (PDEs). More precisely, we deal with variational inequalities whose single inequalities comprise constrained optimization problems, where the corresponding admissibility sets again are given by the seeked value functions. Using the concept of viscosity solutions as weak solutions of PDEs, we avoid strong regularity assumptions on the value functions. To use this concept as sufficient verification method, we additionally have to prove the uniqueness of the solutions of the PDEs.

As a second major part of this work we apply the presented theory of impulse perturbed stochastic control in the field of financial risk management where extreme events have to be taken into account in order to control risks in a reasonable way. Such extreme scenarios are modelled by impulse controls and the financial decisions are made with respect to the worst-case scenario. In a first example we discuss portfolio problems as well as pricing problems on a capital market with crash risk. In particular, we consider the possibility of trading options and study their influence on the investor's performance measured by the expected utility of terminal wealth. This brings up the question of crash-adjusted option prices and leads to the introduction of crash insurance. The second application concerns an insurance company which faces potentially large losses from extreme damages. We propose a dynamic model where the insurance company controls its risk process by reinsurance in form of proportional reinsurance and catastrophe reinsurance. Optimal reinsurance strategies are obtained by maximizing expected utility of the terminal surplus value and by minimizing the required capital reserves associated to the risk process.

Contents

Chapter 1

Introduction

Extreme events like stock market crashes or large insurance claims can cause exceptionally large financial losses often resulting in the threat to existence for investors or insurers. To be prepared for such situations is a desirable goal. However, such rare events are difficult or actually impossible to predict in advance. In this thesis we present a type of stochastic control taking into account system crashes without modelling the crashes explicitly in a stochastic way. Under minimal assumptions concerning possible system crashes this approach aims at avoiding large losses in any possible situation.

Modelling of stock market crashes in finance or large claim sizes in insurance are actively researched mathematical topics (see, e.g., the pioneering work of Merton [37], or for a comprehensive survey Embrechts, Klüppelberg and Mikosch [14], Cont and Tankov [9] and the references therein). In the majority of cases those approaches rely on modelling the stock prices or the total claims as Lévy processes. Unfortunately, their analytical handling is not easy. Even more seriously, choosing the right distribution and fitting it to the market data is a challenging task, in particular because extreme jumps are rare events, so that there is no sufficient data available for an estimation. Another problem which arises from Lévy process models is that they neglect worst-case situations if decisions are based on expected value optimization. So the obtained optimal solutions do not necessarily provide sufficient protection in the course of an unlike movement in the system's state.

In contrast to modelling the extreme events in form of a stochastic jump process, we use a different control approach. The controlled state process considered in this setup is the same as if we would not take into account a possible system crash. The central idea is to amend a finite impulse control expressing system perturbation. An admissible impulse strategy represents a possible scenario with at most a fixed number of system perturbations up to the considered finite time horizon. The extent of these crashes is limited by the use of a compact set of admissible impulses. The impulse control is chosen by a virtual second agent acting as an opponent of the real decision maker, meaning that decisions are made from a worst-case point of view.

In summary, the main characteristics of our model are:

- Decisions are made in respect of the worst-case scenario. This guarantees best performance and the reachability of a given target relative to any possible situation.
- We focus on a finite time horizon. This parameter is of crucial importance in crash modelling because risk positions have to be adjusted to the time to maturity.
- We do not need any distribution assumption, neither for the time nor for the extent of the crash. Instead we only need the possible number of crashes within the time horizon and a range for the possible crash sizes.

Such a worst-case approach in the context of stock market crashes was firstly introduced by Hua and Wilmott [22]. They derived worst-case option prices in discrete time by replicating the option along the worst-case path by a portfolio consisting of shares of the underlying risky asset and the secure bond. In the same crash model Wilmott [58] proposed a static hedge of a portfolio against market crashes by the purchase of a fixed number of options. Korn and Wilmott [34] took up the crash model and formulated the so-called worst-case portfolio problem in continuous time. Their work represents the first given example of impulse perturbed stochastic control. Further studies on the worst-case portfolio problem are [29], [31], [32] and [33].

The type of control dealt with in this thesis is a combination of stochastic control and impulse control. Nevertheless, it varies from the type of control refered to as combined stochastic control and impulse control in the literature. In [2] Bensoussan and Lions developed a general methodology for solving impulse control problems based on the concept of quasi-variational inequalities (QVIs). A QVI is a non-linear partial differential equation (PDE) consisting of a differential part combined with an impulse intervention term. Approximating impulse control by iterated optimal stopping leads to variational inequalities (VIs) in which the intervention part is given in form of an explicit constraint. If the impulse controller additionally is free to choose a control process, we are concerned with so-called Hamilton-Jacobi-Bellman (HJB) (quasi-)variational inequalities, see Øksendal and Sulem [42]. In this case one considers in the differential part the usual partial differential operator occurring in the HJB equation of stochastic control. We also refer to Seydel [52] for a deep treatment of the relation between combined stochastic and impulse control and viscosity solutions of HJBQVIs. In spite of formal similarities, our situation is not covered by the above results. This is due to the contrary objectives of both controllers in our model. Anticipating the worst impulse strategy for any chosen stochastic control constrains the set of feasible stochastic control processes. Therefore impulse perturbed stochastic control is related to a VI, reflecting a simultaneous optimum with respect to both the continuation (without immediate perturbation) and the perturbation situation.

Via a dynamic programming principle (DPP) we reduce impulse perturbed stochastic control problems to problems of antagonistic control and stopping. By the choice of a stochastic control process the controller governs the state process, while the stopper can halt the controlled process at any time involving an additional cost for the controller. In particular, this cost depends on the current state of the controller's strategy what differentiates our control problem from other already studied mixed games of control and stopping. Here most of the authors emphasize on relating mixed games to backward stochastic differential equations which are intended to characterize the value of the game (see, e.g., [12], [19]). Other approaches are the reduction to a simple stopping problem [26] or a martingale approach [27]. These methods work when only the drift of the underlying state process can be controlled or under strong restrictions on the diffusion coefficients. Our approach is more direct which is possible because we work in a Markovian framework, allowing us to apply a DPP in order to link the problem to a PDE. Differential games in the context of controlled Markov processes are widely studied when both players control continuously, see for instance the monograph of Fleming and Soner [15]. To our knowledge, there are no references dealing with controller-and-stopper problems with simultaneous control of the state process and the costs due to stopping.

This work consists of two major parts, a rigorous development of the relevant theory and its applications to financial risk management. The setting of impulse perturbed stochastic control is introduced in Chapter 2. In this control framework we consider two types of problems. In Chapter 3 we analyze a differential game as analogue of the expected utility maximization problem in the situation with impulse perturbation. In this way we obtain a worst-case bound for the expected utility. In Chapter 4 we study an appropriate version of a target problem. Its objective is to determine the minimal initial value of some target process to reach a given stochastic target region at terminal time almost surely, in respect of any possible scenario. For both problems our guideline is to establish verification results under minimal assumptions. We prove that the value functions of the problems are unique viscosity solutions of a system of PDEs. Our results on the differential game are generalizations of the work of Korn and Steffensen [33] who presented a HJB-system approach (system of VIs) for solving the worst-case portfolio problem. Formulating the problem in very broad terms for controlled jump-diffusion processes and using the concept of viscosity solutions, we hope to allow for applications to a wide variety of problems. The analysis of the target problem, on the other hand, creates the theoretical foundation for the worst-case option pricing problem in [22] in a more general framework. From a mathematical point of view, we deal with a stochastic target problem for a system with jumps limited in number and size. For an introduction to stochastic target problems without jumps we refer to Soner and Touzi [54], [55]. Bouchard studied in [5] the problem when the controlled system follows a jump diffusion. So the stochastic target problem under impulse perturbation can be regarded as the analogue of this for a finite number of jumps.

The second part of the thesis is devoted to sample applications concerning stock market crashes and large insurance claims. We solve the respective impulse perturbed control problems by using the results of the first part of the thesis, and we present some examples illustrating the shape of the value functions and the corresponding optimal strategies. In Chapter 5 we take up the worst-case portfolio problem on an extended market on which the investor can trade in stocks and derivatives to discuss crash hedging possibilities and pricing techniques for the derivatives. Besides lower and upper price bounds, which can be seen as the continuous versions of the worst-case prices derived in [22] (with a correction with respect to the terminal condition), we establish a market completion approach. To this end we consider crash insurance and calculate the insurance premium. As an application of the obtained pricing rules we introduce a valuation method for defaultable bonds. In Chapter 6 we present a new model of an insurance company utilizing impulse perturbed stochastic control to dynamically reinsure its insurance risk. Here a main focus of attention is the investigation of minimal capital requirements for the insurer. Our thesis is complemented by a summary and conclusions at the end.

In sum, the most important aims of this thesis are

- the presentation of the concept of impulse perturbed stochastic control as an extension of the classic stochastic control theory,
- the generalization of the HJB-system approach presented by Korn and Steffensen [33] to differential games of stochastic control and finite impulse intervention in a very general framework with the analysis of the viscosity solution property of the value functions,
- the adaption of the stochastic target problem to impulse perturbed stochastic control,
- the application of the impulse perturbed stochastic control approach to finance (including the pricing of derivatives and crash insurance contracts in our crash model and the valuation of defaultable bonds) and insurance (computation of optimal reinsurance strategies and minimal capital requirements for a risk process under the threat of large insurance claims).

Acknowledgements

At this point, I want to thank Private Docent Flavius Guias for supervising my thesis and, in particular, for the trust he put in me. I appreciated his comments and his advice which improved my thesis considerably.

Moreover, I would like to express my thanks to Professor Ralf Korn for valuable comments and his patient explanations on stochastic control in finance. I also benefited from fruitful discussions with all the members of the Stochastic Control and Financial Mathematics Group at the University of Kaiserslautern, especially from Thomas Seifried who inspired me to new ideas on crash options and insurance.

Last, but not least, special thanks to my parents for their never-ending support.

Notation

Let us introduce some notation used throughout the thesis. Elements of $\mathbb{R}^d$, $d \geq 1$, are identified as column vectors, the superscript ^{T} stands for transposition and $|.|$ is the Euclidean norm. The i-th component of $x \in \mathbb{R}^d$ is specified by x_i, and $\operatorname{diag}(x)$ is the diagonal matrix whose i-th diagonal element is x_i. For a subset $\mathcal{O} \in \mathbb{R}^d$ we write $\overline{\mathcal{O}}$ for its closure, $\mathcal{O}^c$ for its complement and $\partial\mathcal{O}$ for its boundary. The open ball around $x \in \mathbb{R}^d$ with radius $\varepsilon > 0$ is refered to as $\mathcal{B}(x, \varepsilon)$. If A is a quadratic matrix, then $\operatorname{tr}(A)$ is its trace. By $\mathbb{S}^d$ we denote the set of all symmetric $(d \times d)$-matrices equipped with the spectral norm (as the induced matrix norm), I_d is the identity matrix. The set $\mathcal{C}^{1,2}$ contains all functions with values in $\mathbb{R}$ which are once continuously differentiable with respect to the time and twice with respect to the space variables. For $\varphi \in \mathcal{C}^{1,2}$ the notations $D_x\varphi$ and $D_x^2\varphi$ correspond to the gradient and the Hessian matrix, respectively, of φ with respect to the x variable. For $x \in \mathbb{R}$ we also write φ_x and φ_{xx}. The positive and negative parts of a function φ are denoted by $\varphi^+ = \max(\varphi, 0)$ and $\varphi^- = -\min(\varphi, 0)$, respectively, such that $\varphi = \varphi^+ - \varphi^-$.

For ease of reference, here is a partial index of notation introduced in the course of this work:

Chapter 2

The general setup

In this chapter we introduce the setting of impulse perturbed stochastic control. That is we set up the underlying state process which is controlled both continuously and by finite impulse control and which forms the basis on which we carry out our further analysis. Here the two types of control are chosen by different decision makers with contrary objectives. After giving the controlled process along with the specific notations of finite impulse control in Section 2.1, we show in Section 2.2 the strong Markov property as fundamental property of the state process. In Section 2.3 we outline briefly the proceeding in handling stochastic control problems in this special setting, and in Section 2.4 we conclude with a short discussion on the idea of the presented model.

2.1 Combined stochastic and impulse control

Let $T > 0$ be a fixed finite time horizon and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ be a filtered probability space. In the classical framework of stochastic control, the state $X(t) \in \mathbb{R}^d$ of a system at time t is given by a stochastic process of the form

$$dX(t) = \varphi(t, X(t-), \pi(t-))dL(t), \tag{2.1}$$

where L is a semimartingale and φ an appropriate function ensuring a unique solution of (2.1). The process π with values in a compact non-empty set $U \subset \mathbb{R}^p$ is the *control process*, $X = X^\pi$ is called the *(purely continuously) controlled process*. We refer to the responsible decision maker concerning π as *player A*.

Now suppose that there exists a second agent, *player B*, who is able to influence at any time the state of the system by giving impulses. An *impulse* ζ, exercised at time t when the system is in state x and controlled with π, lets X jump immediately to $\Gamma(t, x, \pi(t), \zeta)$, where Γ is some given function. Notice particularly that the jump height depends on the state of control chosen from agent A. As only restriction we assume that first the allowed impulses lie in a compact non-empty set $Z \subset \mathbb{R}^q$ and second the number of allowed interventions is limited by some integer n.

To cover situations in which player B does not intend to intervene, we assume that there exists an ineffective impulse ζ_0 such that $\Gamma(t,x,\pi,\zeta_0) = x$ for all $(t,x,\pi) \in [0,T] \times \mathbb{R}^d \times U$. Then we can specify the notion of an impulse control:

Definition 2.1.1. *An* (admissible) n-impulse control *is a double sequence*

$$\xi = (\tau_1, \tau_2, \ldots, \tau_n; \zeta_1, \zeta_2, \ldots \zeta_n),$$

where $0 =: \tau_0 \le \tau_1 \le \tau_2 \le \ldots \le \tau_n \le \tau_{n+1} := T$ *are stopping times and* ζ_i *are* $\mathcal{F}_{\tau_i}$*-measurable random variables with values in* $Z \cup \{\zeta_0\}$*. The stopping time* τ_i *is called* i-th intervention time *with associated* impulse ζ_i*. We denote by* $\mathcal{V}_n$ *the set of all admissible* n*-impulse controls.*

The interventions of player B are unknown to player A a priori. But of course he notices them, so that he is able to adapt his strategy to the new circumstances. So player A will specify his action completely by a sequence of strategies $\pi^{(n)}, \pi^{(n-1)}, \ldots, \pi^{(0)}$ depending on the number of intervention possibilities that are still left. At player B's i-th intervention time player A switches immediately from the pre-intervention strategy $\pi^{(n-i+1)}$ to the post-intervention strategy $\pi^{(n-i)}$. In this context a sequence $\pi = (\pi^{(n)}, \ldots, \pi^{(0)})$ of admissible processes $\pi^{(i)}$, $i = 0, 1, \ldots, n$, describes a control for player A.

Definition 2.1.2. *Denote by* $\mathcal{U}_0$ *the set of all progressively measurable processes* π *which have values in* U *and are square-integrable, i.e.* $\mathbb{E}[\int_0^T |\pi(t)|^2 dt] < 0$*. Then* $\mathcal{U}_n := \mathcal{U}_0^{n+1}$ *is the set of all* admissible (continuous) n-controls.

If we deal with a stochastic jump process, we have to distinguish at intervention time τ_i between a jump stemming from the semimartingale L and a jump caused by the impulse ζ_i. For this purpose we set

$$\check{X}(\tau_i-) := X(\tau_i-) + \Delta X(\tau_i),$$

where $\Delta X(\tau_i)$ denotes the jump of the stochastic process without the impulse.

With the above definitions the process $X = X^{\pi,\xi}$, controlled continuously by $\pi \in \mathcal{U}_n$ and with impulse control $\xi \in \mathcal{V}_n$, is given by

$$\begin{aligned} dX(t) &= \varphi(t, X(t-), \pi^{(n-i)}(t-))dL(t), \quad \tau_i < t < \tau_{i+1}, \quad i = 0, \ldots, n, \\ X(\tau_i) &= \Gamma(\tau_i, \check{X}(\tau_i-), \pi^{(n-i+1)}(\tau_i), \zeta_i), \quad i = 1, \ldots, n. \end{aligned} \tag{2.2}$$

If two or more impulses happen to be at the same time, i.e. $\tau_{i+1} = \tau_i$, we want to understand the jump condition in (2.2) as concatenation in form of

$$\Gamma(t, \Gamma(t, \check{X}(\tau_i-), \pi^{(n-i+1)}(t), \zeta_i), \pi^{(n-i)}(t), \zeta_{i+1}).$$

Let us conclude this section with some helpful notions: For $(t,x) \in [0,T] \times \mathbb{R}^d$, we denote by $X^{\pi,\xi}_{t,x}$ the solution of the controlled SDE (2.2) with initial condition $X(t) = x$. In the case without impulse interventions we use the notation $X^{\pi}_{t,x}$, $\pi \in \mathcal{U}_0$, for the solution of the purely continuously controlled SDE (2.1). For the sake of simplicity we will omit the indices t, x, π, ξ if the context is clear. To still illustrate the dependence of the expectation on the starting point (t, x), we write

$$\mathbb{E}_{t,x}[.] := \mathbb{E}[.|X(t) = x].$$

For a given impulse control $\xi \in \mathcal{V}_n$ with intervention times τ_i, $i = 1, \dots, n$, we introduce the counting process

$$N^{\xi}(t) := \max\{i = 0, 1, \dots, n : \tau_i \leq t\}.$$

Then we can indicate the current control process of a control $\pi \in \mathcal{U}_n$ by $\pi^{(n-N^{\xi})}$.

2.2 Markov property

To derive dynamic programming principles for the control problems analyzed in the following chapters, the controlled process

$$Y^{\pi,\xi}(t) := (t, X^{\pi,\xi}(t), N^{\xi}(t))$$

has to be a *strong Markov process*, i.e. for any bounded, Borel measurable function h, any stopping time $\tau < \infty$ and all $t \geq 0$ we have

$$\mathbb{E}[h(Y^{\pi,\xi}(\tau + t))|\mathcal{F}_{\tau}] = \mathbb{E}[h(Y^{\pi,\xi}(\tau + t))|Y^{\pi,\xi}(\tau)].$$

That means that at any arbitrary random time the process $Y^{\pi,\xi}$ "starts infresh" independently of the past. In the following we will often just say that $X^{\pi,\xi}$ is a strong Markov process if we refer to this affair.

It is a well known fact that the uncontrolled process $\tilde{Y}(t) := (t, \tilde{X}(t))$, with $\tilde{X}$ as the solution of SDE (2.1) without a control process π, has the strong Markov property, see for example Protter [46], Theorem V.32. Seydel [52] proved that the strong Markov property can be extended to controlled processes if we restrict ourselves to *Markov controls*. That means that the controls only depend on the current state of the system and do not use information of the past. This is the case for control processes $\pi^{(i)}$, $i = 0, 1 \dots, n$, given by some measurable feedback function $\bar{\pi}^{(i)}$ in form of $\pi^{(i)}(t) = \bar{\pi}^{(i)}(t, X(t-))$. For impulse controls the requirement of the irrelevance of the past leads to the consideration of exit times of $(t, \check{X}(t-))$ as intervention times τ_i and $\sigma(\tau_i, \check{X}(\tau_i-))$-measurable impulses ζ_i.

Proposition 2.2.1 (Proposition 2.3.1 in [52]). *For Markov controls π, ξ the controlled process $Y^{\pi,\xi}(t)$ is a strong Markov process.*

As a consequence of this proposition, from now on we consider exclusively Markov controls, i.e. we suppose that the admissibility sets $\mathcal{U}_n$, $\mathcal{V}_n$ only contain such controls.

2.3 Procedure

In this thesis we analyze two types of control problems. In Chapter 3 we study a differential game where player A tries to maximize some objective function $\mathcal{J}$, whereas player B intends to minimize $\mathcal{J}$. In Chapter 4 we deal with a target problem in the form that we want to determine the minimal initial value $Y(0) = y$ of some target process Y that allows player A to reach a given target $Y(T) \geq g(X(T))$ at final time almost surely, whatever player B attemps to prevent him from that. In both cases the solution is obtained by an iterative proceeding:

Step 0. We start with computing the solution of the control problem without impulse control.

Step n. Given the solution of the $(n-1)$-impulse control problem we can reduce the n-intervention problem to an impulse problem with only one intervention possibility.

2.4 Discussion of the model

The generated strategies secure best performance and the reachability of a given target, respectively, relative to any possible situation, in particular they are *crash-resistant*. Moreover, the basic assumptions of the model are very simple. We do not need any distribution assumptions for jumps due to impulses, neither for the jump time nor for its height. Of course the assumption of a limited finite number of possible perturbations is critical. This restriction is not urgently necessary. But for an unbounded intervention number there exists the risk of no end of interventions in an instant. To avoid these constellations we would have to weaken the effects of an impulse in a way that guarantees non-optimality of such degenerate intervention strategies. But this is inconsistent to the intention of modelling extreme situations where we want to understand the impulse interventions as rare events with grave consequences. So a small number of interventions reflects exactly the characteristic feature of such catastrophes.

Of course there are several extensions of the model which are possible. For example we could allow for time- and state-depending sets $U(t,x)$ and $Z(t,x)$. Else the assumption of a compact control set U could be dropped. Both changes would lead to other additional requirements on the admissible set $\mathcal{U}_0$ which provide for existence and uniqueness of a solution of the state equation (2.2). Moreover, we could include a possible regime switch as result of an intervention by using different coefficients $\varphi^{(n-i)}$, $i = 0, 1, \ldots, n$, in (2.2). For example, think of a capital market where a crash usually leads to an increase in volatility, see Korn and Menkens [31] for results of crash hedging with changing market parameters. However, to keep the notation as easy to get along with, we neglect these generalizations.

Chapter 3

Differential game with combined stochastic and impulse control

In this chapter we investigate a stochastic worst-case optimization problem consisting of stochastic control and impulse control. This problem corresponds to the maximization of the expected utility of a state process in consideration of system crashes, where crash scenarios are modelled by impulse interventions and we maximize with respect to the worst-case scenario.

The motivation of this problem comes from the portfolio problem under the threat of market crashes. Hua and Wilmott introduced in [22] a crash model where the sudden drops in market prices are not modelled stochastically. Based on very simple assumptions on possible crashes, they chose a worst-case approach to manage the crash risk in any possible situation. While [22] is focused on the pricing of options in discrete time, Korn and Wilmott [34] applied this approach in the context of continuous-time portfolio optimization. They formulated the portfolio problem as a differential game in which the investor meets the market as an opponent who systematically acts against the interests of the investor by causing crash scenarios. In this way they obtained portfolio strategies that maximize the worst-case bound of the investor's utility. In this chapter we basically refer to Korn and Steffensen [33] who presented a HJB-system approach (system of inequalities) for solving the worst-case portfolio problem. Our results are generalizations of their work, in particular we do not require strong regularity assumptions on the value function by using the concept of viscosity solutions. Examples for further studies of the worst-case portfolio problem with respect to market crashes are [29], [31] and [32]. Moreover, there are similar worst-case approaches in the context of portfolio optimization for considering model risk (see, e.g., [20], [38] and [56] and the references therein). A reference for deterministic worst-case design with numerous examples and applications to risk management is the book of Rustem and Howe [47].

To start with we present in Section 3.1 a detailed problem formulation including all assumptions we make. In the next section we state a PDE characterization of the associated value function in the viscosity sense as main result of this chapter, and we explain in more detail how we want to proceed in the following to prove this viscosity property. The main steps are the transformation of the problem and the derivation of a dynamic programming principle in Section 3.3, the proof of the continuity of the value function in Section 3.4 and its characterization as a unique viscosity solution in Section 3.5. An extension of the model and a relaxation of assumptions is discussed in the last section.

3.1 Problem formulation

Let $T > 0$ denote a fixed finite time horizon and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, i.e. the filtration $(\mathcal{F}_t)_t$ is complete and right-continuous. We consider further an adapted m-dimensional Brownian motion B and an adapted independent k-dimensional pure-jump Lévy process L. We denote by

$$\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$$

its compensated Poisson random measure with jump measure N and Lévy measure ν. Throughout this chapter we assume that

$$\int_{|z|\ge 1} |z|\nu(dz) < \infty. \tag{3.1}$$

This condition gives us the Lévy-Itô-decomposition in form of

$$L(t) = \int_0^t \int_{z\in\mathbb{R}^k} z\tilde{N}(ds, dz).$$

For more precise definitions and properties of random measures we refer to Jacod and Shiryaev [24] or Sato [49]. On this stochastic basis we consider the process $X = X^{\pi,\xi}$, controlled continuously by $\pi \in \mathcal{U}_n$ and with impulse control $\xi \in \mathcal{V}_n$, given by

$$\begin{aligned} dX(t) &= \mu(t, X(t), \pi^{(n-i)}(t))dt + \sigma(t, X(t), \pi^{(n-i)}(t))dB(t) \\ &\quad + \int_{\mathbb{R}^k} \gamma(t, X(t-), \pi^{(n-i)}(t-), z)d\tilde{N}(dt, dz), \\ &\qquad\qquad \tau_i < t < \tau_{i+1}, \quad i = 0, \dots, n, \\ X(\tau_i) &= \Gamma(\tau_i, \check{X}(\tau_i-), \pi^{(n-i+1)}(\tau_i), \zeta_i), \qquad i = 1, \dots, n. \end{aligned} \tag{3.2}$$

Here the functions $\mu : [0,T]\times\mathbb{R}^d\times U \to \mathbb{R}^d$, $\sigma : [0,T]\times\mathbb{R}^d\times U \to \mathbb{R}^{d\times m}$, $\gamma : [0,T]\times\mathbb{R}^d\times U\times\mathbb{R}^k \to \mathbb{R}^d$ and $\Gamma : [0,T]\times\mathbb{R}^d\times U\times Z \to \mathbb{R}^d$ satisfy conditions detailed in Assumption 3.1.1 below.

As performance criterion we consider a functional of the form

$$\begin{aligned} \mathcal{J}^{(\pi,\xi)}(t,x) = \mathbb{E}_{t,x}\Bigg[&\int_t^T f(s, X(s), \pi^{(n-N^\xi(s))}(s))ds + g(X(T)) \\ &+ \sum_{t\le\tau_i\le T} K(\tau_i, \check{X}(\tau_i-), \pi^{(n-i+1)}(\tau_i), \zeta_i)\Bigg], \end{aligned}$$

where $f : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}$, $g : \mathbb{R}^d \to \mathbb{R}$ and $K : [0,T] \times \mathbb{R}^d \times U \times Z \to \mathbb{R}$ are functions satisfying conditions detailed in Assumption 3.1.1 below. Here f can be interpreted as running profit, g as end profit and K stands for the additional profit resulting from an impulse intervention (or, from another point of view, as loss, respectively). Bearing in mind the role of ζ_0 as ineffective impulse, we assume $K(t,x,\pi,\zeta_0) = 0$ for all $(t,x,\pi) \in [0,T] \times \mathbb{R}^d \times U$.

In view of player B's role as creator of system perturbation we suppose that both players have contrary objectives. While player A tries to maximize $\mathcal{J}$, player B is intended to minimize $\mathcal{J}$. In order to reach their aims both players select their respective strategy from their admissibity sets $\mathcal{U}_n$ and $\mathcal{V}_n$, respectively. First agent A settles his control π. Depending on this choice his opponent B decides to intervene or not. So our stochastic optimization problem reads as follows: Find *optimal* controls $\hat{\pi} \in \mathcal{U}_n$, $\hat{\xi} \in \mathcal{V}_n$ with associated *value function* $v^{(n)}$ such that

$$v^{(n)}(t,x) = \sup_{\pi \in \mathcal{U}_n} \inf_{\xi \in \mathcal{V}_n} \mathcal{J}^{(\pi,\xi)}(t,x) = \mathcal{J}^{(\hat{\pi},\hat{\xi})}(t,x). \tag{3.3}$$

Such two-player-games with contrary objectives are called differential games. So the problem (3.3) represents a *stochastic differential game combining stochastic control and impulse control.*

Now let us summarize the conditions that form the basis for the investigation of the value function $v^{(n)}$ in the following assumption:

Assumption 3.1.1. *(G1) The control set $U \subset \mathbb{R}^p$ and the impulse set $Z \subset \mathbb{R}^q$ are compact and non-empty.*

(G2) The functions μ, σ, γ are continuous with respect to (t,x,π), $\gamma(t,x,\pi,.)$ is bounded for $|z| \le 1$, and there exist $C > 0$, $\delta : \mathbb{R}^k \to \mathbb{R}_+$ with $\int_{\mathbb{R}^k} \delta^2(z)\nu(dz) < \infty$ such that for all $t \in [0,T]$, $x, y \in \mathbb{R}^d$ and $\pi \in U$,

$$|\mu(t,x,\pi) - \mu(t,y,\pi)| + |\sigma(t,x,\pi) - \sigma(t,y,\pi)| \le C|x-y|,$$
$$|\gamma(t,x,\pi,z) - \gamma(t,y,\pi,z)| \le \delta(z)|x-y|,$$
$$|\gamma(t,x,\pi,z)| \le \delta(z)(1+|x|).$$

(G3) The transaction function Γ and the profit functions f, g, K are continuous and Lipschitz in x (uniformly in t, π and ζ), i.e. there is a constant $C > 0$ such that for all $t \in [0,T]$, $x, y \in \mathbb{R}^d$, $\pi \in U$ and $\zeta \in Z$,

$$|\Gamma(t,x,\pi,\zeta) - \Gamma(t,y,\pi,\zeta)| + |f(t,x,\pi) - f(t,y,\pi)|$$
$$+|g(x) - g(y)| + |K(t,x,\pi,\zeta) - K(t,y,\pi,\zeta)| \le C|x-y|.$$

The assumptions (G1) and (G2) guarantee the existence of a unique strong solution to (3.2) with initial condition $X(t) = x \in \mathbb{R}^d$ (for general existence and uniqueness results for SDEs with random coefficients see Gichman and Skorochod [17] or Protter [46]). The assumption (G3) implies that Γ, f, g, K satisfy a global linear growth condition with respect to x.

Therefore the objective function $\mathcal{J}$ is well-defined and, using the estimate (B.2) of Lemma B.1.1 together with the tower property of conditional expectation (see the argumentation in the appendix subsequent to Lemma B.1.1), we can conclude that

$$|v^{(n)}(t,x)| \leq C(1+|x|) \tag{3.4}$$

for some constant $C > 0$ independent of t, x.

Example 3.1.1. Consider the controlled deterministic process

$$\begin{aligned} dX(t) &= -\alpha\pi^{(1)}(t)dt, \qquad t \in (0,\tau_1), \\ X(\tau_1) &= X(\tau_1-) - (1-\pi^{(1)}(\tau))\zeta_1, \\ dX(t) &= -\alpha\pi^{(0)}(t)dt, \qquad t \in (\tau_1, T), \end{aligned}$$

with parameter $\alpha > 0$, 1-impulse perturbation $(\tau_1, \zeta_1) \in [0,T]\times(\{\zeta\}\cup\{0\})$, $\zeta > 0$, and control $\pi = (\pi^{(1)}, \pi^{(0)})$ whose processes have values in $[0,1]$, i.e. the setting is $n = 1$, $U = [0,1]$, $Z = \{\zeta\}$.

For an interpretation think of an insurer who is faced with one possible claim of size ζ up to time T. To limit the risk exposure the insurer can reinsure a proportion π of the claim by paying a continuous reinsurance premium α. If the claim is admitted at time τ, the insurer has to pay the sum $(1-\pi(\tau))\zeta$. The controlled process X then reflects the insurer's surplus.

We now want to maximize the expected value of $X(T)$ in the worst-case sense, i.e. we consider the differential game

$$v^{(1)}(t,x) = \sup_{\pi\in\mathcal{U}_1} \inf_{\xi\in\mathcal{V}_1} \mathbb{E}^{\pi,\xi}_{t,x}[X(T)],$$

where $\mathcal{U}_1, \mathcal{V}_1$ are the sets of admissible controls introduced in the preceding chapter.

It is easy to check that (G1)-(G3) from Assumption 3.1.1 are satisfied for this problem formulation. So the controlled process as well as the differential game are well-defined.

Step 0. If there is no perturbation to fear, it cannot be optimal to pay for protection against one. So we have $v^{(0)}(t,x) = x$ with $\hat{\pi}^{(0)} \equiv 0$.

Step 1. By intuition we try to determine the optimal strategy $\hat{\pi}^{(1)}$ and the value function $v^{(1)}$ via an indifference consideration. We demand as much protection as to be left with the same final expected value, regardless of the occurrence of a perturbation. So choose $\hat{\pi}^{(1)}$ such that

$$x - (1-\hat{\pi}^{(1)}(t))\zeta = x - \int_t^T \alpha\hat{\pi}^{(1)}(s)ds. \tag{3.5}$$

The left hand side of (3.5) is the expected value for an immediate impulse intervention and the right hand side is the expected value without impulse intervention. To be indifferent at final time we require $\hat{\pi}^{(1)}(T) = 1$. Subtracting x in (3.5), it seems that the optimal control process $\hat{\pi}^{(1)}$ is independent of the state of X, so that the process coincides with its feedback function. Differentiating on both sides of (3.5) we obtain a ODE for $\hat{\pi}^{(1)}$,

$$\frac{\partial}{\partial t}\hat{\pi}^{(1)}(t) = \frac{\alpha}{\zeta}\hat{\pi}^{(1)}(t), \quad \hat{\pi}^{(1)}(T) = 1.$$

Solving this ODE and calculating the corresponding value of one side of equation (3.5), we conclude

$$v^{(1)}(t,x) = x - \left(1 - \exp\left(-\frac{\alpha}{\zeta}(T-t)\right)\right)\zeta,$$
$$\pi^{(1)}(t) = \exp\left(-\frac{\alpha}{\zeta}(T-t)\right).$$

We will verify in the next subsection that this is indeed the solution to our problem. Just note that the strategy $\hat{\pi}$ is admissible since $\hat{\pi}^{(i)}$, $i = 1, 2$, has values in U, is Markovian (given in form of some feedback function) and progressively measurable (because it is deterministic).

3.2 Procedure and main result

In the rest of this chapter we devote ourselves to the characterization of the value function $v^{(n)}$ as the unique viscosity solution of a system of PDEs. To formulate the corresponding PDEs we introduce for a value $\pi \in U$ the *Dynkin second order integro-differential operator* $\mathcal{L}^\pi$ associated to the process X^π (with constant control $\pi(.) \equiv \pi$),

$$\begin{aligned}\mathcal{L}^\pi \varphi(t,x) = \frac{\partial \varphi}{\partial t}(t,x) + \mu(t,x,\pi)^T D_x\varphi(t,x) + \frac{1}{2}\mathrm{tr}\left((\sigma\sigma^T)(t,x,\pi)D_x^2\varphi(t,x)\right)\\ + \int_{\mathbb{R}^k}\{\varphi(t,x+\gamma(t,x,\pi,z)) - \varphi(t,x) - \gamma(t,x,\pi,z)^T D_x\varphi(t,x)\}\nu(dz),\end{aligned} \tag{3.6}$$

and the *intervention operator* $\mathcal{M}^\pi$ transacting the best immediate impulse for player B,

$$\mathcal{M}^\pi\varphi(t,x) = \inf_{\zeta \in Z}\{\varphi(t,\Gamma(t,x,\pi,\zeta)) + K(t,x,\pi,\zeta)\}. \tag{3.7}$$

Supposed n interventions might happen, it seems reasonable for player A to choose at the point (t,x) a control policy $\pi \in \mathcal{U}_0$ fulfilling $v^{(n)}(t,x) \le \mathcal{M}^{\pi(t)}v^{(n-1)}(t,x)$. Otherwise an immediate intervention would dissuade player A from achieving the optimal utility $v^{(n)}(t,x)$. To be able to preserve a similar optimality condition for approximately optimal controls also after starting time t, we need the following additional requirement:

Assumption 3.2.1. *For all $(t,x) \in [0,T) \times \mathbb{R}^d$ and $\varepsilon > 0$ there exists an ε-optimal control $\pi \in \mathcal{U}_n$ in the sense of*

$$\inf_{\xi \in \mathcal{V}_n} \mathcal{J}^{(\pi,\xi)}(t,x) \ge v^{(n)}(t,x) - \varepsilon,$$

whose feedback functions $\bar{\pi}^{(i)}$, i.e. $\pi^{(i)}(t) = \bar{\pi}^{(i)}(t, X(t-))$, are continuous.

In particular, this assumption is satisfied if the optimal control itself has continuous feedback functions. For one part of the characterization of the value function (see the subsolution property in Theorem 3.5.5 below) we will use the following consequence: For π as in Assumption 3.2.1 we can find for any $\delta > \varepsilon$ a neighborhood $\mathcal{B}$ of (t,x) such that $\mathcal{M}^{\pi(s)}v^{(n-1)}(s,y) \ge v^{(n)}(s,y) - \delta$ for all $(s,y) \in \mathcal{B}$.

In the following we show how the differential game with at most n intervention possibilities can be solved iteratively:

Step 0. We start with computing the value function $v^{(0)}$ of the classic pure continuous control problem. This problem is studied extensively in the existing literature, also for jump processes (see, e.g., Øksendal and Sulem [42]), so that we already know that $v^{(0)}$ is a viscosity solution of the Hamilton-Jacobi-Bellman equation (HJB-equation)

$$\begin{aligned} \inf_{\pi\in U}\{-\mathcal{L}^\pi v^{(0)} - f(.,.,\pi)\} = 0 \quad \text{on } [0,T)\times\mathbb{R}^d, \\ v^{(0)} - g = 0 \quad \text{on } \{T\}\times\mathbb{R}^d. \end{aligned} \tag{3.8}$$

Step n. Given the value function of the $(n-1)$-differential game we can transfer the n-intervention problem into a one time stopping problem with a final payoff depending on $v^{(n-1)}$. Then it turns out that the value function $v^{(n)}$ is a viscosity solution of the equation

$$\begin{aligned} \inf_{\pi\in U}\max\left(-\mathcal{L}^\pi v^{(n)} - f(.,.,\pi), v^{(n)} - \mathcal{M}^\pi v^{(n-1)}\right) = 0 \quad \text{on } [0,T)\times\mathbb{R}^d, \\ v^{(n)} - \min\left(g, \sup_{\pi\in U}\mathcal{M}^\pi v^{(n-1)}\right) = 0 \quad \text{on } \{T\}\times\mathbb{R}^d. \end{aligned} \tag{3.9}$$

Note that (3.8) and (3.9) are partial integro-differential equations. The precise definition of a viscosity solution in this case (with a non-local integral term) is given in Section 3.5. Further we will prove the uniqueness of a viscosity solution of (3.8) and (3.9) in a special class of functions, so that we can be sure that a solution of the respective equation is indeed the seeked value function.

Compared to existing results on combined stochastic and impulse control, the special quality of the equation (3.9) is the simultaneous minimization with respect to the control π of both the differential and the impulse part. This illustrates the balance problem of player A who has to find a strategy that is optimal with respect to the worse case of no intervention at the moment (first argument in the max-term) and an immediate impulse of the most horrendous proportion (second argument in the max-term). If the effects of player B's impulses did not depend on player A's strategy, the impulse component in (3.9) would not depend on π any more, so that we could move the infimum within the first argument of the max-term resulting in the so-called Hamilton-Jacobi-Bellman variational inequality (HJBVI). That means that player A would not be restricted any more in his decision and could concentrate on his performance in intervention-free times. On the other hand player A is at player B's mercy because A cannot influence the consequences resulting from impulses at the intervention times. But in view of applications to risk management we are of course especially interested in limiting the loss even in the extreme situations.

Following Korn and Steffensen in [33] we could also state the value function on $[0,T)\times\mathbb{R}^d$ as solution of a variational inequality of two constrained optimization problems:

$$\max\left(\inf_{\pi\in U_2}\{-\mathcal{L}^\pi v^{(n)} - f(.,.,\pi)\}, \inf_{\pi\in U_1}\{v^{(n)} - \mathcal{M}^\pi v^{(n-1)}\}\right) = 0,$$

where we minimize on the subsets

$$U_1 = \{\pi \in U : -\mathcal{L}^\pi v^{(n)} - f(.,.,\pi) \le 0\},$$
$$U_2 = \{\pi \in U : v^{(n)} - \mathcal{M}^\pi v^{(n-1)} \le 0\}.$$

Note that $U_i = U_i(t,x,n)$, $i = 1,2$, are not given explicitly because they depend on the studied value function.

Before going any further, we want to validate the above equations for the derived solution of the deterministic Example 3.1.1.

Example 3.2.1 (Continuation of Example 3.1.1)*.* We concentrate on the PDE characterization for $v^{(1)}$ and recall our guess from Example 3.1.1,

$$v^{(1)}(t,x) = x - \left(1 - \exp\left(-\frac{\alpha}{\zeta}(T-t)\right)\right)\zeta.$$

In view of the dynamics of the controlled process given in Example 3.1.1 we calculate

$$\mathcal{L}^\pi v^{(1)}(t,x) = \frac{\partial}{\partial t} v^{(1)}(t,x) - \alpha\pi \frac{\partial}{\partial x} v^{(1)}(t,x) = \alpha\left(\exp\left(-\frac{\alpha}{\zeta}(T-t)\right) - \pi\right),$$

and from the jump condition related to an impulse we get

$$\begin{aligned} v^{(1)}(t,x) - \mathcal{M}^\pi v^{(0)}(t,x) &= x - \left(1 - \exp\left(-\frac{\alpha}{\zeta}(T-t)\right)\right)\zeta - (x - (1-\pi)\zeta) \\ &= \zeta\left(\exp\left(-\frac{\alpha}{\zeta}(T-t)\right) - \pi\right). \end{aligned}$$

So the PDE in (3.9) reads

$$\inf_{\pi\in[0,1]} \max\left(\alpha\left(\pi - \exp\left(-\frac{\alpha}{\zeta}(T-t)\right)\right), \zeta\left(\exp\left(-\frac{\alpha}{\zeta}(T-t)\right) - \pi\right)\right) = 0. \tag{3.10}$$

We see that the max-term in (3.10) is zero for

$$\pi = \pi^{(1)}(t) = \exp\left(-\frac{\alpha}{\zeta}(T-t)\right) \in [0,1]$$

and strictly positive else. So we have found a minimizer for the left hand side of (3.10) with minimum zero. So $v^{(1)}$ fulfills equation (3.10). Furthermore, the terminal condition is satisfied because of

$$v^{(1)}(T,x) = x = \min\left(x, \sup_{\pi\in[0,1]}\{x - (1-\pi)\zeta\}\right),$$

where the above supremum is realized by $\pi = \pi^{(1)}(T) = 1$. This ends our validation which confirms the predications of Example 3.1.1.

The rest of this chapter is organized as follows: From now on we assume $n \geq 1$. Then we start in Section 3.3 with the derivation of a dynamic programming principle (DPP) which is associated to the stochastic differential game. More precisely, we will state two versions of a DPP. The first one establishes the connection of $v^{(n)}$ to the value function $v^{(n-1)}$ of the inferior problem. Here we point out that the impulse part of the control problem can be reduced to a problem of finding an optimal stopping time for the first intervention. The solution of the differential game with n intervention possibilities thus turns out to be an iteration of n differential games combining stochastic control and optimal stopping. Afterwards we give a second version of a DPP which is similar to the classic principles in which the controlled system is analyzed up to some stopping time τ while it is supposed to be controlled optimally in the sequel. The value function $v^{(n)}$ at initial state (t, x) is then refered to itself at the stochastic state $(\tau, X(\tau))$. After showing in Section 3.4 that $v^{(n)}$ is continuous, we use this representation of $v^{(n)}$ in Section 3.5 to point out the connection to the PDE (3.9). Since we cannot guarantee that $v^{(n)}$ is smooth enough, we use the concept of viscosity solutions as weak solutions of PDEs. This gives us a necessary condition for the value function in contrast to a traditional verification theorem (applicable only under strong regularity requirements) which presents a sufficient condition. However, the concept of viscosity solutions only makes sense if we can be sure that a viscosity solution is indeed the seeked value function. So we have to deal with the question of uniqueness of viscosity solutions of PDE (3.9) in the sequel. We conclude this chapter in Section 3.6 with a discussion on an extension of the model framework including relaxed assumptions.

In this chapter we follow Korn and Steffensen [33] where a special differential game of our type is analyzed, in particular for the derivation of the first DPP. For the handling of a differential game combining stochastic control and optimal stopping, we copy techniques of Pham [44] who studied optimal stopping of controlled jump diffusion processes.

3.3 Dynamic programming

In this section we provide a DPP for our differential game that we want to use in the following to characterize the value function as viscosity solution of a PDE.

We proceed in two steps: First we transform the original problem with impulse control into a problem associated with combined stochastic control and optimal stopping. Then we derive a DPP for optimal stopping of a controlled process.

Before doing so, we need to discuss some essential requirements. The basic idea of dynamic programming is to split the problem into subproblems, so that the optimal solution is composed of the solutions of the separate parts. For this method to work, two properties of the controlled process and the admissible controls are crucial. Firstly, the admissibility sets have to be stable under concatenation. This means that for any stopping time $\tau \in [0, T]$ a control switching at time τ from one admissible control to another admissible one is admissible, too. This condition guarantees the admissibility of the composition of solutions of subproblems. Secondly, splitting the relevant time interval $[t, T]$ into $[t, \tau)$ and $[\tau, T]$ and observing the system on the latter one separately, it is necessary that, given the state at time τ, the system is independent of its course on $[t, \tau)$. This requirement corresponds to the strong Markov property of the underlying process which actually is satisfied in our case according to Proposition 2.2.1.

3.3.1 From impulse control to optimal stopping

For a first version of a DPP we consider the problem up to the first intervention and the problem with $n-1$ remaining intervention possibilities as isolated cases. For this decomposition of the problem the stability assumption on $\mathcal{U}_n$, $\mathcal{V}_n$ is obviously satisfied in the following sense:

- If $\pi^{(n)} \in \mathcal{U}_0$ and $\pi' = (\pi^{(n-1)}, \ldots, \pi^{(0)}) \in \mathcal{U}_{n-1}$, then $\pi = (\pi^{(n)}, \pi^{(n-1)}, \ldots, \pi^{(0)}) \in \mathcal{U}_n$.
- If $(\tau_1; \zeta_1) \in \mathcal{V}_1$ and $\xi' = (\tau_2, \ldots, \tau_n; \zeta_2, \ldots, \zeta_n) \in \mathcal{V}_{n-1}$ such that $\tau_1 \leq \tau_2$, then $\xi = (\tau_1, \tau_2, \ldots, \tau_n; \zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathcal{V}_n$.

Recall that $X^\pi = X^\pi_{t,x}$ denotes the controlled process with start in $(t, x) \in [0, T] \times \mathbb{R}^d$ under control $\pi \in \mathcal{U}_0$ and no impulse intervention. We define by $\mathcal{T} = \mathcal{T}(t)$ the set of all stopping times $\tau \in [t, T]$ which represent admissible intervention times in the sense that they are independent of the past. Further we recall the definition of the intervention operator $\mathcal{M}^\pi$ from (3.7).

Now we can state the first important result of this section which is a generalization of Lemma 2 in Korn and Steffensen [33]. For the problem of optimization of the final utilility of a diffusion process with impulse perturbation (with fixed impulse ζ) they derived the relation

$$v^{(n)}(t, x) = \sup_{\pi \in \mathcal{U}_0} \inf_{\tau \in \mathcal{T}} \mathbb{E}_{t,x} \left[v^{(n-1)}(\tau, \Gamma(\tau, X^\pi(\tau-), \pi(\tau), \zeta)) \right].$$

We notice that this result implies that intervening is the optimal strategy for player B which is not true in general. Moreover, in view of our general choice of the performance criterion, we need to take into consideration the utility functions f, g and K as well as different admissible impulses. However, the correct version in our case is very intuitive and can be derived similarly to the lemma in [33].

Theorem 3.3.1 (Dynamic programming principle (version 1)). *The value function $v^{(n)}$ in* (3.3) *can be represented in form of*

$$\begin{aligned} v^{(n)}(t, x) = \sup_{\pi \in \mathcal{U}_0} \inf_{\tau \in \mathcal{T}} \mathbb{E}_{t,x} \Bigg[& \int_t^\tau f(s, X^\pi(s), \pi(s)) ds \\ & + \mathcal{M}^{\pi(\tau)} v^{(n-1)}(\tau, X^\pi(\tau)) \mathbb{1}_{\{\tau < T\}} \\ & + \min \left(g, \mathcal{M}^{\pi(\tau)} v^{(n-1)} \right) (\tau, X^\pi(\tau)) \mathbb{1}_{\{\tau = T\}} \Bigg]. \end{aligned} \tag{3.11}$$

Proof: Fix $(t,x) \in [0,T] \times \mathbb{R}^d$ and consider an impulse control $\xi \in \mathcal{V}_n$ which is composed of the first intervention $(\tau_1;\zeta_1)$ and $\xi' := (\tau_2,\dots,\tau_n;\zeta_2,\dots,\zeta_n)$. On the stochastic interval $[t,\tau_1)$ the continuous control given by $\pi = (\pi^{(n)},\dots,\pi^{(0)}) \in \mathcal{U}_n$ equals $\pi^{(n)}$. Afterwards it switches to the $(n-1)$-control $\pi' := (\pi^{(n-1)},\dots,\pi^{(0)})$. For each path with no intervention in the relevant time interval $[t,T]$, we are free to set $\tau_1 = T$ and $\zeta_1 = \zeta_0$. Noting that $\check{X}^{\pi,\xi}(\tau_1-) = X^{\pi^{(n)}}(\tau_1)$ the controlled process jumps at intervention time to $x' := \Gamma(\tau_1, X^{\pi^{(n)}}(\tau_1), \pi^{(n)}(\tau_1), \zeta_1)$. Since $E[.|\mathcal{F}_t] = E[E[.|\mathcal{F}_{\tau_1}]|\mathcal{F}_t]$ holds for all $\tau_1 \ge t$ and any controlled process has the strong Markov property, we can state the objective functional in form of

$$\mathcal{J}^{(\pi,\xi)}(t,x) = \mathbb{E}_{t,x}\Bigg[\int_t^{\tau_1} f(s,X^{\pi^{(n)}}(s),\pi^{(n)}(s))ds + K(\tau_1,X^{\pi^{(n)}}(\tau_1),\pi^{(n)}(\tau_1),\zeta_1)$$
$$+\mathbb{E}_{\tau_1,x'}\Bigg[\int_{\tau_1}^{T} f(s,X^{\pi',\xi'}(s),\pi^{(n-1-N^{\xi'}(s))}(s))ds + g(X^{\pi',\xi'}(T))$$
$$+\sum_{\substack{\tau_1\le\tau_i\le T\\ i\ge 2}} K(\tau_i,\check{X}^{\pi',\xi'}(\tau_i-),\pi^{(n-i+1)}(\tau_i),\zeta_i)\Bigg]\Bigg].$$

To make the notation more convenient we set

$$Y^{\pi^{(n)},(\tau_1;\zeta_1)} := \int_t^{\tau_1} f(s,X^{\pi^{(n)}}(s),\pi(s))ds + K(\tau_1,X^{\pi^{(n)}}(\tau_1),\pi^{(n)}(\tau_1),\zeta_1).$$

Then, using the definition of the objective functional $\mathcal{J}$ we obtain

$$\mathcal{J}^{(\pi,\xi)}(t,x) = \mathbb{E}_{t,x}\left[Y^{\pi^{(n)},(\tau_1;\zeta_1)} + \mathcal{J}^{(\pi',\xi')}(\tau_1,x')\right]. \tag{3.12}$$

Let $\varepsilon > 0$. For a given first intervention $(\tau_1;\zeta_1)$ choose a control $\hat{\pi} \in \mathcal{U}_n$ that is $\frac{\varepsilon}{2}$-optimal both up to the first intervention and later on, i.e.

$$\sup_{\pi^{(n)}\in\mathcal{U}_0} \mathbb{E}_{t,x}\left[Y^{\pi^{(n)},(\tau_1;\zeta_1)} + v^{(n-1)}(\tau_1,x')\right] \le \mathbb{E}_{t,x}\left[Y^{\hat{\pi}^{(n)},(\tau_1;\zeta_1)} + v^{(n-1)}(\tau_1,\hat{x}')\right] + \frac{\varepsilon}{2}, \tag{3.13}$$
$$\sup_{\pi'\in\mathcal{U}_{n-1}} \inf_{\xi'\in\mathcal{V}_{n-1}} \mathcal{J}^{(\pi',\xi')}(\tau_1,\hat{x}') \le \inf_{\xi'\in\mathcal{V}_{n-1}} \mathcal{J}^{(\hat{\pi}',\xi')}(\tau_1,\hat{x}') + \frac{\varepsilon}{2}, \tag{3.14}$$

where $\hat{x}' = \Gamma(\tau_1, X^{\hat{\pi}^{(n)}}(\tau_1), \hat{\pi}^{(n)}(\tau_1), \zeta_1)$ is the state after the first intervention $(\tau_1;\zeta_1)$ if the strategy $\hat{\pi}^{(n)}$ is applied until the first intervention. Furthermore, for a given control $\pi \in \mathcal{U}_n$ choose an impulse intervention $\hat{\xi} \in \mathcal{V}_n$ that is $\frac{\varepsilon}{2}$-optimal both for the first intervention and in the sequel, i.e.

$$\inf_{\substack{\tau_1\in\mathcal{T}\\ \zeta_1\in Z\cup\{\zeta_0\}}} \mathbb{E}_{t,x}\left[Y^{\pi^{(n)},(\tau_1;\zeta_1)} + v^{(n-1)}(\tau_1,x')\right] \ge \mathbb{E}_{t,x}\left[Y^{\pi^{(n)},(\hat{\tau}_1;\hat{\zeta}_1)} + v^{(n-1)}(\hat{\tau}_1,\hat{x}')\right] - \frac{\varepsilon}{2}, \tag{3.15}$$
$$\inf_{\xi'\in\mathcal{V}_{n-1}} \mathcal{J}^{(\pi',\xi')}(\tau_1,\hat{x}') \ge \mathcal{J}^{(\pi',\hat{\xi}')}(\tau_1,\hat{x}') - \frac{\varepsilon}{2}, \tag{3.16}$$

where in this context $\hat{x}' = \Gamma(\hat{\tau}_1, X^{\pi^{(n)}}(\hat{\tau}_1), \pi^{(n)}(\hat{\tau}_1), \hat{\zeta}_1)$. Then we have the following series of inequalities,

$$\begin{aligned}
v^{(n)}(t,x) &\geq \inf_{\xi \in \mathcal{V}_n} \mathbb{E}_{t,x}\left[Y^{\hat{\pi}^{(n)},(\tau_1;\zeta_1)} + \mathcal{J}^{(\hat{\pi}',\xi')}(\tau_1, \hat{x}')\right] \\
&\geq \inf_{\substack{\tau_1 \in \mathcal{T} \\ \zeta_1 \in Z \cup \{\zeta_0\}}} \mathbb{E}_{t,x}\left[Y^{\hat{\pi}^{(n)},(\tau_1;\zeta_1)} + \inf_{\xi' \in \mathcal{V}_{n-1}} \mathcal{J}^{(\hat{\pi}',\xi')}(\tau_1, \hat{x}')\right] \\
&\geq \inf_{\substack{\tau_1 \in \mathcal{T} \\ \zeta_1 \in Z \cup \{\zeta_0\}}} \mathbb{E}_{t,x}\left[Y^{\hat{\pi}^{(n)},(\tau_1;\zeta_1)} + \sup_{\pi' \in \mathcal{U}_{n-1}} \inf_{\xi' \in \mathcal{V}_{n-1}} \mathcal{J}^{(\pi',\xi')}(\tau_1, \hat{x}')\right] - \frac{\varepsilon}{2} \\
&\geq \inf_{\substack{\tau_1 \in \mathcal{T} \\ \zeta_1 \in Z \cup \{\zeta_0\}}} \sup_{\pi^{(n)} \in \mathcal{U}_0} \mathbb{E}_{t,x}\left[Y^{\pi^{(n)},(\tau_1;\zeta_1)} + v^{(n-1)}(\tau_1, x')\right] - \varepsilon \\
&\geq \sup_{\pi^{(n)} \in \mathcal{U}_0} \inf_{\substack{\tau_1 \in \mathcal{T} \\ \zeta_1 \in Z \cup \{\zeta_0\}}} \mathbb{E}_{t,x}\left[Y^{\pi^{(n)},(\tau_1;\zeta_1)} + v^{(n-1)}(\tau_1, x')\right] - \varepsilon.
\end{aligned}$$

Here the first inequality follows from the definition of $v^{(n)}$, (3.12) and plugging in strategy $\hat{\pi}$. The second inequality follows from interchanging the expectation and the infimum over intervention strategies after the first intervention. The third inequality follows from (3.14). The fourth inequality follows from the definition of $v^{(n-1)}$ and (3.13). The fifth inequality follows from the relation $\inf\sup \geq \sup\inf$.

On the other side we have

$$\begin{aligned}
v^{(n)}(t,x) &\leq \sup_{\pi \in \mathcal{U}_n} \mathbb{E}_{t,x}\left[Y^{\pi^{(n)},(\hat{\tau}_1;\hat{\zeta}_1)} + \mathcal{J}^{(\pi',\hat{\xi}')}(\hat{\tau}_1, \hat{x}')\right] \\
&\leq \sup_{\pi^{(n)} \in \mathcal{U}_0} \mathbb{E}_{t,x}\left[Y^{\pi^{(n)},(\hat{\tau}_1;\hat{\zeta}_1)} + \sup_{\pi' \in \mathcal{U}_{n-1}} \mathcal{J}^{(\pi',\hat{\xi}')}(\hat{\tau}_1, \hat{x}')\right] \\
&\leq \sup_{\pi^{(n)} \in \mathcal{U}_0} \mathbb{E}_{t,x}\left[Y^{\pi^{(n)},(\hat{\tau}_1;\hat{\zeta}_1)} + \sup_{\pi' \in \mathcal{U}_{n-1}} \inf_{\xi' \in \mathcal{V}_{n-1}} \mathcal{J}^{(\pi',\xi')}(\tau_1, \hat{x}')\right] + \frac{\varepsilon}{2} \\
&\leq \sup_{\pi^{(n)} \in \mathcal{U}_0} \inf_{\substack{\tau_1 \in \mathcal{T} \\ \zeta_1 \in Z \cup \{\zeta_0\}}} \mathbb{E}_{t,x}\left[Y^{\pi^{(n)},(\tau_1;\zeta_1)} + v^{(n-1)}(\tau_1, x')\right] + \varepsilon.
\end{aligned}$$

Here the first inequality follows from the definition of $v^{(n)}$, (3.12) and plugging in strategy $\hat{\xi}$. The second inequality follows from interchanging the expectation and the supremum over continuous $(n-1)$-controls used after the first intervention. The third inequality follows from (3.16). The fourth inequality follows from the definition of $v^{(n-1)}$ and (3.15).

Since $\varepsilon > 0$ has been chosen arbitrarily, it follows

$$v^{(n)}(t,x) = \sup_{\pi \in \mathcal{U}_0} \inf_{\substack{\tau \in \mathcal{T} \\ \zeta \in Z \cup \{\zeta_0\}}} \mathbb{E}_{t,x}\left[Y^{\pi,(\tau;\zeta)} + v^{(n-1)}(\tau, x')\right].$$

Recall that the impulse ζ is some $\mathcal{F}_\tau$-measurable random variable with values in $Z \cup \{\zeta_0\}$, so that it is deterministic at time τ. Considering the optimal choice of the first intervention as gradual process of first finding an optimal intervention time τ and then minimizing the objective function with respect to the associated impulse ζ, we may move the infimum over all impulses within the expectation. Obviously it is of no advantage for player B to exercise the ineffective impulse ζ_0 prematurely since by this strategy he only looses one possibility of intervening in the future. It is just an auxiliary expression to define the intervention strategy properly even in the no intervention case. So it is sufficient to consider ζ_0 as admissible impulse only at time T. Using the ineffective impulse ζ_0 at that time player B leaves the system as it is, so that the game is over with a final payment of $g(x)$. Then, by the definitions of $Y^{\pi,(\tau;\zeta)}$, x' and the impulse operator $\mathcal{M}^{\pi(\tau)}$, we finally arrive at the representation of $v^{(n)}$ in (3.11). $\square$

Theorem 3.3.1 motivates to introduce the notation

$$\mathcal{J}_n^{(\pi,\tau)}(t,x) := \mathbb{E}_{t,x}\Bigg[\int_t^\tau f(s,X^\pi(s),\pi(s))ds + \mathcal{M}^{\pi(\tau)}v^{(n-1)}(\tau,X^\pi(\tau))\mathbb{1}_{\{\tau<T\}} + \min\left(g,\mathcal{M}^{\pi(\tau)}v^{(n-1)}\right)(\tau,X^\pi(\tau))\mathbb{1}_{\{\tau=T\}}\Bigg].$$

Then it is our task to solve the following stochastic optimization (sub)problem: Find $v^{(n)}$, $\hat{\pi}\in\mathcal{U}_0$ and $\hat{\tau}\in\mathcal{T}$ such that

$$v^{(n)}(t,x) = \sup_{\pi\in\mathcal{U}_0}\inf_{\tau\in\mathcal{T}}\mathcal{J}_n^{(\pi,\tau)}(t,x) = \mathcal{J}_n^{(\hat{\pi},\hat{\tau})}(t,x). \tag{3.17}$$

Given the solution $v^{(n-1)}$ of the problem with $n-1$ intervention possibilities, this modified problem embodies a differential game combining stochastic control and optimal stopping. Going through the proof of Theorem 3.3.1 shows that the stopping time corresponds to the first intervention time τ_1 of player B and the continuous control is player A's strategy $\pi^{(n)}$ up to the first intervention. Here it is already assumed that both players act optimally in the sequel. In particular, player B's impulse ζ_1 excercised at τ_1 satisfies

$$\zeta_1 \in \arg\min_{\zeta\in Z\cup\{\zeta_0\}}\mathcal{M}^{\pi^{(n)}(\tau_1)}v^{(n-1)}(\tau_1,X^{\pi^{(n)}}(\tau_1))$$

if this expression is well-defined.

Moreover, note that the representation formula (3.11) gives us directly the final condition

$$v^{(n)} = \min\left(g,\sup_{\pi\in U}\mathcal{M}^\pi v^{(n-1)}\right) \quad \text{on } \{T\}\times\mathbb{R}^d. \tag{3.18}$$

3.3.2 DPP for optimal stopping

After we have translated our initial problem to a differential game combining stochastic control and optimal stopping, we now state a DPP for this modified problem. The right formulation of such a principle in our setting is the following:

Theorem 3.3.2 (Dynamic programming principle (version 2)). *The value function $v^{(n)}$ in (3.17) can be represented in form of*

$$v^{(n)}(t,x) = \sup_{\pi\in\mathcal{U}_0}\inf_{\tau\in\mathcal{T}}\mathbb{E}_{t,x}\Bigg[\int_t^{\tau\wedge\theta} f(s,X^\pi(s),\pi(s))ds + v^{(n)}(\theta,X^\pi(\theta))\mathbb{1}_{\{\theta<\tau\}} + \mathcal{M}^{\pi(\tau)}v^{(n-1)}(\tau,X^\pi(\tau))\mathbb{1}_{\{\theta\geq\tau,\tau<T\}} + \min\left(g,\mathcal{M}^{\pi(\tau)}v^{(n-1)}\right)(\tau,X^\pi(\tau))\mathbb{1}_{\{\theta=\tau=T\}}\Bigg] \tag{3.19}$$

for any stopping time $\theta\in[t,T]$.

Note that Theorem 3.3.2 can be reformulated equivalently in the following way:
Let $(t,x)\in[0,T]\times\mathbb{R}^d$. Then we have:

(DP1) For all $\varepsilon>0$ there exists $\pi\in\mathcal{U}_0$ such that for all $\tau\in\mathcal{T}$ and for each stopping time $\theta\in[t,\tau]$ we have

$$v^{(n)}(t,x) \leq \mathbb{E}_{t,x}\Bigg[\int_t^{\theta} f(s,X^\pi(s),\pi(s))ds + v^{(n)}(\theta,X^\pi(\theta))\mathbb{1}_{\{\theta<\tau\}} + \mathcal{M}^{\pi(\tau)}v^{(n-1)}(\tau,X^\pi(\tau))\mathbb{1}_{\{\theta=\tau<T\}} + \min\left(g,\mathcal{M}^{\pi(\tau)}v^{(n-1)}\right)(\tau,X^\pi(\tau))\mathbb{1}_{\{\theta=\tau=T\}}\Bigg] + \varepsilon.$$

(DP2) For all $\pi\in\mathcal{U}_0$ and $\varepsilon>0$ there exists $\tau\in\mathcal{T}$ such that for each stopping time $\theta\in[t,\tau]$ we have

$$v^{(n)}(t,x) \geq \mathbb{E}_{t,x}\Bigg[\int_t^{\theta} f(s,X^\pi(s),\pi(s))ds + v^{(n)}(\theta,X^\pi(\theta))\mathbb{1}_{\{\theta<\tau\}} + \mathcal{M}^{\pi(\tau)}v^{(n-1)}(\tau,X^\pi(\tau))\mathbb{1}_{\{\theta=\tau<T\}} + \min\left(g,\mathcal{M}^{\pi(\tau)}v^{(n-1)}\right)(\tau,X^\pi(\tau))\mathbb{1}_{\{\theta=\tau=T\}}\Bigg] - \varepsilon.$$

Proof of Theorem 3.3.2: We are going to prove the version (DP1)-(DP2). To start with let us remark that by the definition of the admissibility set $\mathcal{U}_0$ we can be sure that two admissible strategies applied sequentially in time form a new admissible strategy, see [55]. This allows us the following procedure: Fix $(t,x)\in[0,T]\times\mathbb{R}^d$ and consider $\pi\in\mathcal{U}_0$ and $\tau\in\mathcal{T}(t)$. For the sake of brevity of notation we write

$$g_n(\tau,X^\pi(\tau)) := \mathcal{M}^{\pi(\tau)}v^{(n-1)}(\tau,X^\pi(\tau))\mathbb{1}_{\{\tau<T\}} + \min\left(g,\mathcal{M}^{\pi(\tau)}v^{(n-1)}\right)(\tau,X^\pi(\tau))\mathbb{1}_{\{\tau=T\}},$$

so that the expected final value resulting from the strategies π and τ reads

$$\mathcal{J}_n^{(\pi,\tau)}(t,x) = \mathbb{E}_{t,x}\left[\int_t^{\tau} f(s,X^\pi(s),\pi(s))ds + g_n(\tau,X^\pi(\tau))\right].$$

Then, by the strong Markov property, for each stopping time $\theta \in [t, \tau]$ we have

$$\begin{aligned} \mathcal{J}_n^{(\pi,\tau)}(t,x) &= \mathbb{E}_{t,x}\bigg[\int_t^\theta f(s, X^\pi(s), \pi(s))ds \\ &\qquad + \mathbb{E}_{\theta, X^\pi(\theta)}\left[\int_\theta^\tau f(s, X^\pi(s), \pi(s))ds + g_n(\tau, X^\pi(\tau)) \right] \bigg] \\ &= \mathbb{E}_{t,x}\left[\int_t^\theta f(s, X^\pi(s), \pi(s))ds + \mathcal{J}_n^{(\pi,\tau)}(\theta, X^\pi(\theta)) \right]. \end{aligned} \tag{3.20}$$

Let $\varepsilon > 0$. By the characterization of the value function in (3.17) there exist $\hat{\pi} \in \mathcal{U}_0$ and $\hat{\tau} \in \mathcal{T}(\theta)$ such that

$$v^{(n)}(t,x) \le \mathcal{J}^{(\hat{\pi},\tau)}(t,x) + \frac{\varepsilon}{2} \quad \text{and} \quad \mathcal{J}^{(\hat{\pi},\hat{\tau})}(\theta, X^{\hat{\pi}}(\theta)) \le v^{(n)}(\theta, X^{\hat{\pi}}(\theta)) + \frac{\varepsilon}{2}. \tag{3.21}$$

Since (DP1) does not depend on the actual stopping time τ in the case of $\theta < \tau$, we may choose $\tau = \hat{\tau}$ in this case, i.e. we relabel τ by $\theta \mathbb{1}_{\{\theta = \tau\}} + \hat{\tau}\mathbb{1}_{\{\theta < \tau\}}$. Considering (3.20) and noting that $\mathcal{J}_n^{(\hat{\pi},\tau)}(\theta, X^{\hat{\pi}}(\theta)) = g_n(\tau, X^{\hat{\pi}}(\tau))$ for $\theta = \tau$, (3.21) proves (DP1).

In the same way there exist $\hat{\tau} \in \mathcal{T}(t)$ and $\hat{\pi} \in \mathcal{U}_0$ such that

$$v^{(n)}(t,x) \ge \mathcal{J}^{(\pi,\hat{\tau})}(t,x) - \frac{\varepsilon}{2} \quad \text{and} \quad \mathcal{J}^{(\hat{\pi},\hat{\tau})}(\theta, X^{\pi}(\theta)) \ge v^{(n)}(\theta, X^{\pi}(\theta)) - \frac{\varepsilon}{2}. \tag{3.22}$$

Since (DP2) depends on the strategy π only through its realizations in the stochastic interval $[t, \theta)$, we may pass to the control $\hat{\pi}$ on $[\theta, T]$. Therefore, combining (3.20) and (3.22) gives us (DP2). □

We will use the version (DP1) for the proof of the subsolution property of $v^{(n)}$ in Section 3.5. More precisely, we need the following consequence of (DP1): For all $(t,x) \in [0,T) \times \mathbb{R}^d$ and $\varepsilon > 0$, there exists $\pi \in \mathcal{U}_0$ such that for all $\theta \in \mathcal{T}$ we have

$$v^{(n)}(t,x) \le \mathbb{E}_{t,x}\left[\int_t^\theta f(s, X^\pi(s), \pi(s))ds + v^{(n)}(\theta, X^\pi(\theta)) \right] + \varepsilon$$

as well as

$$v^{(n)}(t,x) \le \mathcal{M}^{\pi(t)} v^{(n-1)}(t,x) + \varepsilon.$$

To see the first inequality, choose in (DP1) the specific stopping time $\tau = T$ and note the terminal condition stated in (3.18). The second inequality is (DP1) for $\tau = t$.

For the proof of the supersolution property of $v^{(n)}$ we will make use of the version (DP2) as follows: At a closer look on (DP2) we can even say more about the ε-optimal stopping time τ. For $(t,x) \in [0,T] \times \mathbb{R}^d$, $\pi \in \mathcal{U}_0$ and $\varepsilon > 0$ define the stopping time

$$\tau_\varepsilon := \inf\{s \ge t : v^{(n)}(s, X^\pi(s)) \ge \mathcal{M}^{\pi(s)} v^{(n-1)}(s, X^\pi(s)) - \varepsilon\}.$$

Suppose that $\tau < \tau_\varepsilon \wedge T$ with positive probability. Then, by the definition of τ_ε we conclude that $v^{(n)}(\tau, X^\pi(\tau)) < \mathcal{M}^{\pi(\tau)} v^{(n-1)}(\tau, X^\pi(\tau)) - \varepsilon$ with positive probability. But this represents a contradiction to (DP2) in the state $(\tau, X^\pi(\tau))$ for $\theta = \tau$. So we have shown that $\tau \geq \tau_\varepsilon \wedge T$ almost surely. Therefore we are allowed to consider in (DP2) the stopping time $\theta = \tau^\varepsilon \wedge (t+h)$ with $h \in (0, T-t)$. According to (DP1) we can modify each control $\pi \in \mathcal{U}_0$ on $[\theta, T]$ such that

$$v^{(n)}(\theta, X^\pi(\theta)) \leq \mathcal{M}^{\pi(\theta)} v^{(n-1)}(\theta, X^\pi(\theta)) + \varepsilon.$$

Then the inequality in (DP2) reads

$$v^{(n)}(t,x) \geq \mathbb{E}_{t,x}\left[\int_t^\theta f(s, X^\pi(s), \pi(s))ds + v^{(n)}(\theta, X^\pi(\theta))\right] - 2\varepsilon$$

which holds true for all $\pi \in \mathcal{U}_0$ because the expression on the right hand side does not depend on the course of π on $[\theta, T]$. In view of the arbitrary choice of $\varepsilon > 0$ and the fact that $\tau_{\varepsilon'} \geq \tau_\varepsilon$ for $\varepsilon' < \varepsilon$, it follows

$$v^{(n)}(t,x) \geq \sup_{\pi \in \mathcal{U}_0} \mathbb{E}_{t,x}\left[\int_t^\theta f(s, X^\pi(s), \pi(s))ds + v^{(n)}(\theta, X^\pi(\theta))\right]$$

for $\theta = \tau_\varepsilon \wedge (t+h)$ with arbitrary $\varepsilon > 0$ and $h \in (0, T-t)$.

3.4 Continuity of the value function

In this section we prove the continuity of the value function. The main result is the following:

Proposition 3.4.1. *Let Assumption 3.1.1 be satisfied. Then the value function $v^{(n)}$ in* (3.3) *(resp.* (3.17)*) is continuous and Lipschitz in x (uniformly in t). More precisely, there exists a constant $C > 0$ such that for all $s, t \in [0,T]$, $x, y \in \mathbb{R}^d$,*

$$|v^{(n)}(t,x) - v^{(n)}(s,y)| \leq C\{(1+|x|)|t-s|^{\frac{1}{2}} + |x-y|\}. \tag{3.23}$$

We adapt the proof to a proof presented in Pham [44] where the continuity of the value function of a similar stopping problem is shown. The proof essentially makes use of the estimates on the moments of the purely π-controlled process X^π, $\pi \in \mathcal{U}_0$, see Lemma B.1.1 in the Appendix. Furthermore, for the continuity with respect to the time variable t we apply the DPP in form of (3.19). The difference to [44] lies in necessary sup inf manipulations and in the extension of the mentioned estimates to impulse perturbed processes which is not a problem in view of the discussion following Lemma B.1.1.

Proof of Proposition 3.4.1: By the definition of $v^{(n)}$ and the relation $|\sup_{a\in A}\inf_{b\in B}\varphi(a,b) - \sup_{c\in A}\inf_{d\in B}\psi(c,d)| \leq \sup_{a\in A}\sup_{b\in B}|\varphi(a,b) - \psi(a,b)|$ we have for all $t \in [0,T]$, $x, y \in \mathbb{R}^d$,

$$\begin{aligned}|v^{(n)}(t,x) - v^{(n)}(t,y)| \leq \sup_{\pi\in\mathcal{U}_n}\sup_{\xi\in\mathcal{V}_n}\mathbb{E}\Bigg[&|g(X^{\pi,\xi}_{t,x}(T)) - g(X^{\pi,\xi}_{t,y}(T))| \\ &+ \int_t^T |f(s, X^{\pi,\xi}_{t,x}(s), \pi^{(n-N^\xi(s))}(s)) - f(s, X^{\pi,\xi}_{t,y}(s), \pi^{(n-N^\xi(s))}(s))|ds \\ &+ \sum_{t\leq\tau_i\leq T}|K(\tau_i, \check{X}^{\pi,\xi}_{t,x}(\tau_i-), \pi^{(n-i+1)}(\tau_i), \zeta_i) - K(\tau_i, \check{X}^{\pi,\xi}_{t,y}(\tau_i-), \pi^{(n-i+1)}(\tau_i), \zeta_i)|\Bigg].\end{aligned}$$

Using the Lipschitz conditions on Γ, f, g, K of the assumption (G3) and the estimate (B.5) of Lemma B.1.1 together with the tower property of conditional expectation, we conclude that

$$|v^{(n)}(t,x) - v^{(n)}(t,y)| \leq C|x-y|, \tag{3.24}$$

i.e. the value function is Lipschitz in x (uniformly in t).

To prove continuity of $v^{(n)}$ in time t, let $0 \leq t < s \leq T$, $x, y \in \mathbb{R}^d$ and use the DPP (3.19) with $\theta = s$ implying that

$$\begin{aligned} v^{(n)}(t,x) = \sup_{\pi \in \mathcal{U}_0} \inf_{\tau \in \mathcal{T}} \mathbb{E} \Bigg[\int_t^{\tau \wedge s} & f(r, X^\pi_{t,x}(r), \pi(r)) dr + v^{(n)}(s, X^\pi_{t,x}(s)) \mathbb{1}_{\{s<\tau\}} \\ & + \mathcal{M}^{\pi(\tau)} v^{(n-1)}(\tau, X^\pi_{t,x}(\tau)) \mathbb{1}_{\{s \geq \tau, \tau < T\}} \\ & + \min\left(g, \mathcal{M}^{\pi(\tau)} v^{(n-1)}\right)(\tau, X^\pi_{t,x}(\tau)) \mathbb{1}_{\{s \geq \tau = T\}} \Bigg]. \end{aligned}$$

Using the specific stopping time $\tau = T$ and noting that $v^{(n)} = \min(g, \sup_{\pi \in U} \mathcal{M}^\pi v^{(n-1)})$ on $\{T\} \times \mathbb{R}^d$, this leads to

$$|v^{(n)}(t,x) - v^{(n)}(s,x)| \leq \sup_{\pi \in \mathcal{U}_0} \mathbb{E} \left[\int_t^s |f(r, X^\pi_{t,x}(r), \pi(r))| dr + |v^{(n)}(s, X^\pi_{t,x}(s)) - v^{(n)}(s,x)| \right].$$

In view of the linear growth condition on f as consequence of the assumption (G3) and the Lipschitz relation (3.24) of $v^{(n)}$, this yields

$$|v^{(n)}(t,x) - v^{(n)}(s,x)| \leq \sup_{\pi \in \mathcal{U}_0} C \left\{ \int_t^s (1 + \mathbb{E}\left[|X^\pi_{t,x}(r)|\right]) dr + \mathbb{E}\left[|X^\pi_{t,x}(s) - x|\right] \right\}.$$

With the estimates (B.2) and (B.3) of Lemma B.1.1 we deduce

$$|v^{(n)}(t,x) - v^{(n)}(s,x)| \leq C(1+|x|)|s-t|^{\frac{1}{2}},$$

which ends the proof. □

That is $v^{(n)}$ is in $\mathcal{UC}_x([0,T] \times \mathbb{R}^d)$, the set of continuous functions on $[0,T] \times \mathbb{R}^d$, uniformly continuous in x (uniformly in t).

3.5 PDE characterization of the value function

We now come to the main result of this chapter showing that the value function of the differential game (3.3) can be represented as a viscosity solution of a PDE.

In this way we can apply the viscosity solution concept as a kind of verification method, allthough the value function does not need to satisfy strong regularity conditions: It is sufficient to verify that $v^{(n)}$ is a viscosity solution of the corresponding PDE. But for this method to work, it is necessary that we know that $v^{(n)}$ is the unique viscosity solution.

So we first give an existence theorem of the value function as viscosity solution and after that turn to the question of the uniqueness of these viscosity solutions. But to start with we need some preliminaries in order to extend the viscosity approach presented in the appendix to partial integro-differential equations with its non-local integral part.

3.5.1 Preliminaries

In this section we consider the partial integro-differential equation

$$\begin{aligned} \inf_{\pi\in U} \max\left(-\mathcal{L}^{\pi}v - f(.,.,\pi), v - \mathcal{M}^{\pi}v^{(n-1)}\right) &= 0 \quad \text{on } [0,T)\times\mathbb{R}^d, \\ v - \min\left(g, \sup_{\pi\in U}\mathcal{M}^{\pi}v^{(n-1)}\right) &= 0 \quad \text{on } \{T\}\times\mathbb{R}^d, \end{aligned} \tag{3.25}$$

where $\mathcal{L}^{\pi}$ is the integro-differential operator from (3.6).

Denote by $\mathcal{C}_1 = \mathcal{C}_1([0,T]\times\mathbb{R}^d)$ the space of functions $\varphi \in \mathcal{C}([0,T]\times\mathbb{R}^d)$ with linear growth rate, i.e. there exists a constant $C > 0$ such that

$$|\varphi(t,x)| \le C(1+|x|) \quad \text{for all } (t,x)\in[0,T]\times\mathbb{R}^d.$$

Note that by (3.4) we have $v^{(n)} \in \mathcal{C}_1([0,T]\times\mathbb{R}^d)$. As detailed below, $\mathcal{L}^{\pi}\varphi$ is well-defined for $\varphi \in \mathcal{C}_1 \cap \mathcal{C}^{1,2}([0,T)\times\mathbb{R}^d)$.

Now let us state more precisely what we mean by a viscosity solution of (3.25).

Definition 3.5.1. *Let $v \in \mathcal{C}_1([0,T]\times\mathbb{R}^d)$.*

(i) We say that v is a viscosity subsolution *of* (3.25) *if*

$$v = \min\left(g, \sup_{\pi\in U}\mathcal{M}^{\pi}v^{(n-1)}\right) \quad \text{on } \{T\}\times\mathbb{R}^d \tag{3.26}$$

and for any point $(t,x)\in[0,T)\times\mathbb{R}^d$ and all $\varphi \in \mathcal{C}_1\cap\mathcal{C}^{1,2}([0,T)\times\mathbb{R}^d)$ such that $(v-\varphi)$ has a global maximum in (t,x),

$$\inf_{\pi\in U}\max\left(-\mathcal{L}^{\pi}\varphi(t,x) - f(t,x,\pi), v(t,x) - \mathcal{M}^{\pi}v^{(n-1)}(t,x)\right) \le 0.$$

(ii) We say that v is a viscosity supersolution *of* (3.25) *if* (3.26) *holds and for any point $(t,x)\in[0,T)\times\mathbb{R}^d$ and all $\varphi\in\mathcal{C}_1\cap\mathcal{C}^{1,2}([0,T)\times\mathbb{R}^d)$ such that $(v-\varphi)$ has a global minimum in (t,x),*

$$\inf_{\pi\in U}\max\left(-\mathcal{L}^{\pi}\varphi(t,x) - f(t,x,\pi), v(t,x) - \mathcal{M}^{\pi}v^{(n-1)}(t,x)\right) \ge 0.$$

(iii) We say that v is a viscosity solution *of* (3.25) *if it is both a viscosity subsolution and a viscosity supersolution of* (3.25).

Remark 3.5.1. Without loss of generality we can replace the requirement of a "global maximum" of $(v-\varphi)$ in (t,x) in the definition of a viscosity subsolution by $\varphi \geq v$, $\varphi(t,x) = v(t,x)$ because adding constants to φ does not change the value of $\mathcal{L}^\pi\varphi$.

Except for the terminal condition and the non-local integral part of the PDE, the definition for a viscosity solution used here corresponds to the one given in Appendix A: Recall the form of the integro-differential operator

$$\begin{aligned}\mathcal{L}^\pi\varphi(t,x) = {}& \frac{\partial\varphi}{\partial t}(t,x) + \mu(t,x,\pi)^T D_x\varphi(t,x) + \frac{1}{2}\mathrm{tr}\left((\sigma\sigma^T)(t,x,\pi)D_x^2\varphi(t,x)\right)\\ &+ \int_{\mathbb{R}^k}\{\varphi(t,x+\gamma(t,x,\pi,z)) - \varphi(t,x) - \gamma(t,x,\pi,z)^T D_x\varphi(t,x)\}\nu(dz).\end{aligned}$$

To handle a possible singularity of ν in the origin, i.e. $\nu(\mathbb{R}^k) = \infty$, we split as in [44] the integral in $\mathcal{L}^\pi$ for $\eta \in (0,1)$ into two parts,

$$\int_{|z|<\eta} + \int_{|z|\geq\eta}.$$

To this end define for $t \in [0,T]$, $x,p,l \in \mathbb{R}^d$, $X \in \mathbb{S}^d$, $\pi \in U$ and $\varphi \in \mathcal{C}_1 \cap \mathcal{C}^{1,2}([0,T)\times\mathbb{R}^d)$,

$$\begin{aligned}F^\pi(t,x,p,X,l) &:= -\frac{1}{2}\mathrm{tr}\left((\sigma\sigma^T)(t,x,\pi)X\right) - \mu(t,x,\pi)^T p - f(t,x,\pi) - l,\\ \mathcal{I}_\pi^{1,\eta}[t,x,\varphi] &:= \int_{|z|<\eta}\{\varphi(t,x+\gamma(t,x,\pi,z)) - \varphi(t,x) - \gamma(t,x,\pi,z)^T D_x\varphi(t,x)\}\nu(dz),\\ \mathcal{I}_\pi^{2,\eta}[t,x,p,\varphi] &:= \int_{|z|\geq\eta}\{\varphi(t,x+\gamma(t,x,\pi,z)) - \varphi(t,x) - \gamma(t,x,\pi,z)^T p\}\nu(dz),\\ \mathcal{I}_\pi[t,x,\varphi] &:= \mathcal{I}_\pi^{1,\eta}[t,x,\varphi] + \mathcal{I}_\pi^{2,\eta}[t,x,D_x\varphi,\varphi].\end{aligned}$$

We remark that F is a continuous function satisfying the degenerate ellipticity condition

$$F(t,x,p,X,l_1) \leq F(t,x,p,Y,l_2) \quad \text{whenever} \quad X \geq Y, l_1 \geq l_2.$$

Here the fact that F is nonincreasing in l is indeed part of the needed requirements which make sure that any classical solution in $\mathcal{C}^{1,2}([0,T)\times\mathbb{R}^d)$ is also a viscosity solution. Since taking supremum or infimum preserves the ellipticity property, the left hand side of the PDE (3.25) is degenerate elliptic. Moreover, recalling the definition of the intervention operator,

$$\mathcal{M}^\pi\varphi(t,x) = \inf_{\zeta\in Z}\{\varphi(t,\Gamma(t,x,\pi,\zeta)) + K(t,x,\pi,\zeta)\},$$

and noting that Γ, K are continuous and Z is compact by the made assumptions and that $v^{(n-1)} \in \mathcal{UC}_x([0,T]\times\mathbb{R}^d)$, we conclude by Lemma A.1.1 in the appendix that $\mathcal{M}^\pi v^{(n-1)}$ is continuous on $[0,T]\times\mathbb{R}^d$. So, due to the compactness of U, the left hand side of the PDE (3.25) is continuous in $(t,x,v(t,x),\frac{\partial v}{\partial t}(t,x),D_x v(t,x),D_x^2 v(t,x))$.

Let us now turn to the integral part of the PDE in which the test function φ is not only evaluated in the point (t,x). By the Taylor expansion we have for the first integral term

$$\mathcal{I}_\pi^{1,\eta}[t,x,\varphi] \leq \frac{1}{2}\int_{|z|<\eta}|\gamma(t,x,\pi,z)|^2|D_x^2\varphi(t,\hat{x})|\nu(dz)$$

for some $\hat{x} \in \mathcal{B}(x, \sup_{|z|<\eta} |\gamma(t,x,\pi,z)|)$ (note that $\sup_{|z|<\eta} |\gamma(t,x,\pi,z)| < \infty$ according to the boundedness assumption of γ in the assumption (G2)). Using the growth condition on γ from the assumption (G2), the integral is well-defined. In view of the dominated convergence theorem it follows

$$\lim_{\eta \searrow 0} \mathcal{I}_\pi^{1,\eta}[t,x,\varphi] = 0. \tag{3.27}$$

For the second integral $\mathcal{I}_\pi^{2,\eta}[t,x,p,\varphi]$ we observe that, because of $\varphi \in \mathcal{C}_1([0,T] \times \mathbb{R}^d)$, the integrand is bounded by $C_{x,p}(1 + |\gamma(t,x,\pi,z)|^2)$ for a constant $C_{x,p} > 0$ depending on x, p. Since $\int_{\mathbb{R}^k} (|z|^2 \wedge 1)\nu(dz) < \infty$ holds for any Lévy measure ν (see, e.g., Protter [46], Theorem 42, or Jacod and Shiryaev [24], Corollary 4.19) and $|\gamma(t,x,\pi,z)|$ is bounded for $|z| \leq 1$ according to assumption (G2), the integral $\int_{|z|\geq\eta} \nu(dz)$ is finite. Again by the linear growth condition on γ from assumption (G2), this implies the well-definedness of $\mathcal{I}_\pi^{2,\eta}[t,x,p,\varphi]$. If ν has finite activity, i.e. $\nu(\mathbb{R}^k) < \infty$, then $\mathcal{I}_\pi^{2,\eta}[t,x,p,\varphi]$ is also well-defined for $\eta = 0$.

In view of the introduced notation, a function $v \in \mathcal{C}_1([0,T] \times \mathbb{R}^d)$ is a viscosity subsolution (resp. supersolution) of (3.25) if and only if (3.26) holds and for all $(t,x) \in [0,T) \times \mathbb{R}^d$ we have

$$\inf_{\pi \in U} \max\left(-\frac{\partial \varphi}{\partial t}(t,x) + F^\pi(t,x,D_x\varphi(t,x), D_x^2\varphi(t,x), \mathcal{I}_\pi[t,x,\varphi]), v(t,x) - \mathcal{M}^\pi v^{(n-1)}(t,x) \right) \leq 0$$

(resp. ≥ 0), whenever $\varphi \in \mathcal{C}_1 \cap \mathcal{C}^{1,2}([0,T) \times \mathbb{R}^d)$ such that $(v - \varphi)$ has a global maximum (resp. minimum) in (t,x).

We now want to replace in the characterization of a viscosity solution the test function φ in the integral $\mathcal{I}_\pi^{2,\eta}[t,x,p,\varphi]$ by the viscosity solution itself. To this end we need a further assumption:

Assumption 3.5.2. *If we have $\nu(\mathbb{R}^k) = \infty$, then the function γ does not depend on π and for all $(t,x) \in [0,T] \times \mathbb{R}^d$,*

$$\begin{aligned} |\gamma(t,x,z_1)| &< |\gamma(t,x,z_2)| \quad \textit{for } |z_1| < |z_2| \leq 1, \\ |\gamma(t,x,z)| &\geq \sup_{|z'|\leq 1} |\gamma(t,x,z')| \quad \textit{for } |z| \geq 1. \end{aligned}$$

Example 3.5.1. Assumption 3.5.2 is satisfied for $k = d$, $\gamma(t,x,z) = \tilde{\gamma}(t,x)z$ with $\tilde{\gamma} : [0,T] \times \mathbb{R}^d \to \mathbb{R} \setminus \{0\}$.

The following proposition states the intended equivalent formulation for viscosity solutions in the class $\mathcal{C}_1$.

Proposition 3.5.3. *Let Assumption 3.5.2 be satisfied and let $v \in \mathcal{C}_1([0,T] \times \mathbb{R}^d)$. If we have $\nu(\mathbb{R}^k) = \infty$, then v is a viscosity subsolution (resp. supersolution) of* (3.25) *if and only if* (3.26) *holds and for all $(t,x) \in [0,T) \times \mathbb{R}^d$ and $\eta \in (0,1)$,*

$$\inf_{\pi \in U} \max\Bigg(-\frac{\partial \varphi}{\partial t}(t,x) + F^\pi(t,x,D_x\varphi(t,x), D_x^2\varphi(t,x), \\ \mathcal{I}_\pi^{1,\eta}[t,x,\varphi] + \mathcal{I}_\pi^{2,\eta}[t,x,D_x\varphi(t,x),v]), v(t,x) - \mathcal{M}^\pi v^{(n-1)}(t,x) \Bigg) \leq 0$$

(resp. ≥ 0), whenever $\varphi \in \mathcal{C}^{1,2}([0,T) \times \mathbb{R}^d)$ such that $(v - \varphi)$ has a global maximum (resp. minimum) in (t,x). The same holds for $\nu(\mathbb{R}^k) < \infty$ with $\eta = 0$.

We remark that in the proposition's version of a viscosity solution it is not necessary to consider only test functions φ in $\mathcal{C}_1$ because the integral $\mathcal{I}^{1,\eta}_\pi[t,x,\varphi]$ is also well-defined without this requirement. Since any viscosity sub or supersolution v is stipulated to be in $\mathcal{C}_1$, the integral $\mathcal{I}^{2,\eta}_\pi[t,x,D_x\varphi(t,x),v]$ is well-defined, too.

Proof of Proposition 3.5.3: We prove the statement for subsolutions in the case $\nu(\mathbb{R}^k) = \infty$. The other statements are proved in quite the same way. So in the following we consider for some fixed point $(t_0,x_0) \in \mathcal{S} := [0,T) \times \mathbb{R}^d$ a test function $\varphi \in \mathcal{C}^{1,2}(\mathcal{S})$ with $\varphi \geq v$ and $\varphi(t_0,x_0) = v(t_0,x_0)$.

Sufficiency. Supposed in addition $\varphi \in \mathcal{C}_1(\mathcal{S})$, this assertion is a consequence of

$$\mathcal{I}^{2,\eta}_\pi[t_0,x_0,D_x\varphi(t_0,x_0),v] \leq \mathcal{I}^{2,\eta}_\pi[t_0,x_0,D_x\varphi(t_0,x_0),\varphi]$$

and the ellipticity of F.

Necessity. For this implication we have to construct a function $\varphi^\varepsilon \in \mathcal{C}^{1,2}(\mathcal{S})$ such that

$$\begin{aligned} \varphi^\varepsilon(t,x) &= \varphi(t,x) \quad && \text{for } x \in N_\eta(x_0), \\ \varphi^\varepsilon(t,x) &= \psi^\varepsilon(t,x) \quad && \text{for } x \in \mathbb{R}^d \setminus N_{\eta+\varepsilon}(x_0), \end{aligned}$$

where $\psi^\varepsilon \in \mathcal{C}^{1,2}(\mathcal{S})$ satisfies $||v - \psi^\varepsilon||_{L^\infty(\mathcal{S})} < \varepsilon$ with $\psi^\varepsilon \geq v$ and the subset $N_\eta(x_0)$ is defined by

$$N_\eta(x_0) := \{\lambda x_0 + (1-\lambda)\gamma(t_0,x_0,z) : |z| \leq \eta, \lambda \in [0,1]\}.$$

For the concrete construction we refer to Soner [53], Lemma 2.1, or Sayah [50], Proposition 2.1. Both references are concerned with elliptic PDEs involving a first order integro-differential operator with jump coefficient function $\gamma(x,z) = z$. But since the second order derivative D_x^2 only has impact on the local part of the PDE and the non-local integral part only takes into account values starting at time t_0, their procedure can be adapted to the parabolic case with second order differential operator. The different jump reaction is handled by the consideration of the set $N_\eta(x_0)$ instead of the ball $\mathcal{B}(x_0,\eta)$ which is possible because of Assumption 3.5.2. According to this construction, φ^ε inherits all local properties in (t_0,x_0) from φ as well as the linear growth property from v. So for each $\varepsilon > 0$ we can use φ^ε as a test function with respect to the original subsolution definition. Thus we deduce

$$\begin{aligned} \inf_{\pi \in U} \max\Bigg(&-\frac{\partial \varphi^\varepsilon}{\partial t}(t_0,x_0) + F^\pi(t_0,x_0,D_x\varphi^\varepsilon(t_0,x_0),D_x^2\varphi^\varepsilon(t_0,x_0), \\ &\mathcal{I}^{1,\eta}_\pi[t_0,x_0,\varphi^\varepsilon] + \mathcal{I}^{2,\eta}_\pi[t_0,x_0,D_x\varphi^\varepsilon(t_0,x_0),\varphi^\varepsilon]), v(t_0,x_0) - \mathcal{M}^\pi v^{(n-1)}(t_0,x_0)\Bigg) \leq 0. \end{aligned} \tag{3.28}$$

This infimum is attained in $\pi^\varepsilon \in U$ because of the continuity of the respective objective function and the compactness of the domain U. Choose a subsequence such that $\pi^\varepsilon \to \pi \in U$ for $\varepsilon \searrow 0$. Along this subsequence we have

$$\lim_{\varepsilon \searrow 0} \mathcal{I}^{2,\eta}_{\pi^\varepsilon}[t_0, x_0, D_x\varphi^\varepsilon(t_0, x_0), \varphi^\varepsilon] = \mathcal{I}^{2,\eta}_{\pi}[t_0, x_0, D_x\varphi(t_0, x_0), v],$$

so that we conclude our result by using the limit in (3.28). □

Remark 3.5.2. It is sufficient to require in the assumption (G2) that $\gamma(t, x, \pi, .)$ is bounded in a neighborhood of $z = 0$, say for $|z| \le \rho \in (0, 1]$, and in Assumption 3.5.2

$$\begin{aligned} |\gamma(t, x, z_1)| &< |\gamma(t, x, z_2)| \quad \text{for } |z_1| < |z_2| \le \rho, \\ |\gamma(t, x, z)| &\ge \sup_{|z'| \le \rho} |\gamma(t, x, z')| \quad \text{for } |z| \ge \rho. \end{aligned}$$

Then we only consider $\eta \in (0, \rho)$ for $\nu(\mathbb{R}^k) = \infty$.

For proving the uniqueness result we have to characterize a viscosity solution by the notion of semijets introduced by Crandall, Ishii and Lions [10]. We refer to the appendix for the definitions of the semijets $\mathcal{J}^{2,+}$, $\mathcal{J}^{2,-}$ and their limiting versions $\bar{\mathcal{J}}^{2,+}$, $\bar{\mathcal{J}}^{2,-}$. In the very same way as in the appendix, we obtain the following formulation of a viscosity solution in $\mathcal{C}_1$ in terms of the limiting semijets:

Corollary 3.5.4. *Let Assumption 3.5.2 be satisfied and let $v \in \mathcal{C}_1([0, T] \times \mathbb{R}^d)$. If we have $\nu(\mathbb{R}^k) = \infty$, then $v \in \mathcal{C}_1([0, T] \times \mathbb{R}^d)$ is a viscosity subsolution (resp. supersolution) of* (3.25) *if and only if* (3.26) *holds and for all $(t, x) \in [0, T) \times \mathbb{R}^d$, $(a, p, X) \in \bar{\mathcal{J}}^{2,+}v(t, x)$ (resp. $(a, p, X) \in \bar{\mathcal{J}}^{2,-}v(t, x)$) and $\eta \in (0, 1)$,*

$$\inf_{\pi \in U} \max\Big(-a + F^\pi(t, x, p, X, \mathcal{I}^{1,\eta}_\pi[t, x, \varphi] + \mathcal{I}^{2,\eta}_\pi[t, x, p, v]), v(t, x) - \mathcal{M}^\pi v^{(n-1)}(t, x)\Big) \le 0$$

(resp. ≥ 0), whenever $\varphi \in \mathcal{C}^{1,2}([0, T) \times \mathbb{R}^d)$ such that $(v - \varphi)$ has a global maximum (resp. minimum) in (t, x). The same holds for $\nu(\mathbb{R}^k) < \infty$ with $\eta = 0$.

3.5.2 Viscosity solution existence

Using the dynamic programming principle derived in Section 3.3 and the concept of viscosity solutions presented in the preceding section, we link the value function $v^{(n)}$ of our differential game with the PDE (3.25):

Theorem 3.5.5. *Let Assumptions 3.1.1 and 3.2.1 be satisfied. Then the value function $v^{(n)}$ in* (3.3) *(resp.* (3.17)*) is a viscosity solution of* (3.25).

Proof: We have already verified that $v^{(n)} = \min(g, \sup_{\pi \in U} \mathcal{M}^\pi v^{(n-1)})$ on $\{T\} \times \mathbb{R}^d$, see (3.18). So let us focus now on the viscosity property of $v^{(n)}$ on $\mathcal{S} := [0, T) \times \mathbb{R}^d$.

Subsolution property. Let $(t,x) \in \mathcal{S}$ and $\varphi \in \mathcal{C}_1 \cap \mathcal{C}^{1,2}(\mathcal{S})$ such that $\varphi(t,x) = v^{(n)}(t,x)$ and $\varphi \geq v^{(n)}$ on $\mathcal{S}$. In order to prove the subsolution inequality we argue by contradiction and assume that there exists a $\delta > 0$ such that

$$\inf_{\pi \in U} \max\Big(-\mathcal{L}^{\pi}\varphi(t,x) - f(t,x,\pi), v^{(n)}(t,x) - \mathcal{M}^{\pi}v^{(n-1)}(t,x)\Big) = \delta.$$

So for all $\pi \in U$ we have

$$-\mathcal{L}^{\pi}\varphi(t,x) - f(t,x,\pi) \geq \delta \quad \text{or} \quad v^{(n)}(t,x) - \mathcal{M}^{\pi}v^{(n-1)}(t,x) \geq \delta. \tag{3.29}$$

Let $\varepsilon \in (0,\delta)$. Then, as detailed subsequent to the proof of Theorem 3.3.2, the DPP (DP1) yields the existence of a control $\pi \in \mathcal{U}_0$ fulfilling

$$v^{(n)}(t,x) \leq \mathcal{M}^{\pi(t)}v^{(n-1)}(t,x) + \varepsilon \tag{3.30}$$

as well as, for each stopping time $\tau < T$,

$$v^{(n)}(t,x) \leq \mathbb{E}_{t,x}\left[\int_t^{\tau} f(s,X^{\pi}(s),\pi(s))ds + v^{(n)}(\tau,X^{\pi}(\tau))\right] + \varepsilon. \tag{3.31}$$

Thus, combining (3.29) and (3.30) and noting the continuity of the involved functions, in particular the continuity of the feedback function of π according to Assumption 3.2.1, we can find some $\rho > 0$ such that the parabolic ball $\mathcal{B} := [t,t+\rho) \times \mathcal{B}(x,\rho)$ surrounding (t,x) with radius ρ lies within $\mathcal{S}$ and

$$f(.,.,\pi) + \mathcal{L}^{\pi(.)}\varphi \leq -\delta \quad \text{on } \mathcal{B}. \tag{3.32}$$

Consider the stopping time

$$\tau_\rho := \inf\{s \geq t : |X^{\pi}(s) - x| \geq \rho\} \wedge (t+\rho).$$

Using the relation $v^{(n)} \leq \varphi$ and applying Dynkin's formula (see Theorem B.2.1 in the appendix) to $\varphi(\tau_\rho, X^{\pi}(\tau_\rho))$, we deduce from (3.31)

$$\begin{aligned}
v^{(n)}(t,x) &\leq \mathbb{E}_{t,x}\left[\int_t^{\tau_\rho} f(s,X^{\pi}(s),\pi(s))ds + \varphi(\tau_\rho,X^{\pi}(\tau_\rho))\right] + \varepsilon \\
&\leq \varphi(t,x) + \mathbb{E}_{t,x}\left[\int_t^{\tau_\rho} \{f(s,X^{\pi}(s),\pi(t)) + \mathcal{L}^{\pi(s)}\varphi(s,X^{\pi}(s))\}ds\right] + \varepsilon.
\end{aligned}$$

In view of τ_ρ as first exit time of the set $\mathcal{B}$, we know from (3.32) that $f(s,X^{\pi}(s),\pi(s)) + \mathcal{L}^{\pi(s)}\varphi(s,X^{\pi}(s)) \leq -\delta$ almost surely for all $s \in [t,\tau_\rho)$. Hence, by $v^{(n)}(t,x) = \varphi(t,x)$ we conclude

$$0 \leq \mathbb{E}_{t,x}\left[\int_t^{\tau_\rho} \{f(s,X^{\pi}(s),\pi(s)) + \mathcal{L}^{\pi(s)}\varphi(s,X^{\pi}(s))\}ds\right] + \varepsilon \leq -\delta\mathbb{E}_{t,x}[\tau_\rho - t] + \varepsilon.$$

Since this inequality holds for all $\varepsilon \in (0, \delta)$, it follows $\tau_\rho = t$ almost surely. Therefore, by Markov's inequality and the estimate (B.4) of Lemma B.1.1, we have for any $h \in (0, \rho)$,

$$\begin{aligned}1 = \mathbb{P}[\tau_\rho < t+h] &\leq \mathbb{P}\left[\sup_{t \leq s \leq t+h} |X^\pi(s) - x| \geq \rho\right] \\ &\leq \frac{1}{\rho^2}\mathbb{E}\left[\left(\sup_{t \leq s \leq t+h} |X^\pi(s) - x|\right)^2\right] \\ &\leq Ch,\end{aligned}$$

where $C > 0$ is independent of h. Consequently, sending $h \to 0$ leads to the desired contradiction.

Supersolution property. Let $(t, x) \in \mathcal{S}$ and $\varphi \in \mathcal{C}_1 \cap \mathcal{C}^{1,2}(\mathcal{S})$ such that $\varphi(t, x) = v^{(n)}(t, x)$ and $\varphi \leq v^{(n)}$ on $\mathcal{S}$. We must prove that

$$\inf_{\pi \in U} \max\left(-\mathcal{L}^\pi \varphi(t, x) - f(t, x, \pi), v^{(n)}(t, x) - \mathcal{M}^\pi v^{(n-1)}(t, x)\right) \geq 0.$$

If $v^{(n)}(t, x) \geq \mathcal{M}^\pi v^{(n-1)}(t, x)$ for all $\pi \in U$ with $f(t, x, \pi) + \mathcal{L}^\pi \varphi(t, x) > 0$, then the supersolution inequality holds trivially. So assume from now on the case that there exists a $\pi \in U$ such that

$$f(t, x, \pi) + \mathcal{L}^\pi \varphi(t, x) > 0 \quad \text{and} \quad v^{(n)}(t, x) < \mathcal{M}^\pi v^{(n-1)}(t, x). \tag{3.33}$$

For $\varepsilon \in (0, \mathcal{M}^\pi v^{(n-1)}(t, x) - v^{(n)}(t, x))$, $\rho > 0$ and the constant control $\pi(.) \equiv \pi$ define the stopping times

$$\begin{aligned}\tau_\varepsilon &:= \inf\{s \geq t : v^{(n)}(s, X^\pi(s)) \geq \mathcal{M}^\pi v^{(n-1)}(s, X^\pi(s)) - \varepsilon\}, \\ \tau_\rho &:= \inf\{s \geq t : |X^\pi(s) - x| \geq \rho\}.\end{aligned}$$

For $h \in (0, T - t)$ set $\tau := \tau_\varepsilon \wedge \tau_\rho \wedge (t + h)$. In view of the DPP (DP2) and the remarks after the proof of Theorem 3.3.2, we have

$$v^{(n)}(t, x) \geq \mathbb{E}_{t,x}\left[\int_t^\tau f(s, X^\pi(s), \pi)ds + v^{(n)}(\tau, X^\pi(\tau))\right].$$

Now we use the relation $v^{(n)} \geq \varphi$, so that we can deduce by applying Dynkin's formula to $\varphi(\tau, X^\pi(\tau))$,

$$\begin{aligned}v^{(n)}(t, x) &\geq \mathbb{E}_{t,x}\left[\int_t^\tau f(s, X^\pi(s), \pi)ds + \varphi(\tau, X^\pi(\tau))\right] \\ &\geq \varphi(t, x) + \mathbb{E}_{t,x}\left[\int_t^\tau \{f(s, X^\pi(s), \pi) + \mathcal{L}^\pi \varphi(s, X^\pi(s))\}ds\right].\end{aligned}$$

Then, using the equality $v^{(n)}(t, x) = \varphi(t, x)$ and dividing by h we obtain

$$\mathbb{E}_{t,x}\left[\frac{1}{h}\int_t^\tau \{f(s, X^\pi(s), \pi) + \mathcal{L}^\pi \varphi(s, X^\pi(s))\}ds\right] \leq 0. \tag{3.34}$$

We split the integral in the expectation term into two parts,

$$\left(\int_t^{t+h}\right)\mathbb{1}_{\{\tau=t+h\}}+\left(\int_t^{\tau}\right)\mathbb{1}_{\{\tau<t+h\}}.$$

As in the proof of the subsolution property, we deduce from Markov's inequality and Lemma B.1.1 that $\mathbb{P}[\tau < t+h] \leq Ch$ for some constant $C > 0$ which is independent of h, i.e. $\mathbb{1}_{\{\tau<t+h\}} \to 0$ almost surely for $h \searrow 0$. Since the integrand is bounded on $[t,\tau)$, the expectation of the second integral disappears for $h \searrow 0$. Thus, sending $h \searrow 0$ in (3.34), we conclude by the dominated convergence theorem that

$$f(t,x,\pi)+\mathcal{L}^{\pi}\varphi(t,x)\leq 0$$

which is a contradiction to (3.33). □

Remark 3.5.3. We only used Assumption 3.2.1 for the proof of the subsolution property.

3.5.3 Viscosity solution uniqueness

We now conclude by establishing a comparison result for the equation (3.25) which implies that $v^{(n)}$ is the unique viscosity solution of (3.25) in the class $\mathcal{UC}_x([0,T]\times\mathbb{R}^d)$ of continuous functions on $[0,T]\times\mathbb{R}^d$, uniformly continuous in x (uniformly in t). For its proof we refer to a more general comparison result presented in the next section. Since the proof for the standard version considered here and for the generalization are very similar, we prove the generalization.

Theorem 3.5.6 (Comparison theorem)**.** *Let Assumptions 3.1.1 and 3.5.2 be satisfied. Further, let $u \in \mathcal{UC}_x([0,T]\times\mathbb{R}^d)$ be a viscosity subsolution and $v \in \mathcal{UC}_x([0,T]\times\mathbb{R}^d)$ a viscosity supersolution of* (3.25)*. Then we have*

$$u \leq v \;\; on \;\; [0,T]\times\mathbb{R}^d.$$

Proof: Supposed all the underlying assumptions hold, Theorem 3.6.6 below states that we have $u \leq v$ on $[0,T]\times\mathbb{R}^d$ if $u \leq v$ holds on $\{T\}\times\mathbb{R}^d$. Noting that u and v are viscosity sub and supersolutions and that viscosity sub and supersolutions are equal at terminal time T, it remains to check the assumptions of Theorem 3.6.6.

Assumptions 3.6.1 and 3.6.2 correspond to Assumption 3.1.1 and the fact that $v^{(n-1)}$ is continuous. So let us finally verify Assumption 3.6.5 for the growth rate $r = 1$. The Lipschitz condition (U1) is satisfied because of Assumption 3.1.1. The convergence result in (U2) is true due to (3.27), and the estimates of the integrals in (U2) are again a consequence of Assumption 3.1.1. The requirement (U4) is shown in the preliminaries of this section, supposed Assumption 3.5.2 holds. The single crucial point is assumption (U3), the existence of a positive function $w \in \mathcal{C}^{1,2}([0,T)\times\mathbb{R}^d)\cap\mathcal{C}([0,T]\times\mathbb{R}^d)$ and a constant $C > 0$ such that

$$\min\left(\inf_{\pi\in U}\{-\mathcal{L}^{\pi}w\},w\right)\geq C \quad \text{on } [0,T)\times\mathbb{R}^d$$

and

$$\lim_{|x|\to\infty} \frac{|x|}{w(t,x)} = 0 \quad \text{(uniformly in t)}.$$

We claim that for some large $\lambda > 0$ the function $w(t,x) = e^{\lambda(T-t)}|x|^2 + C(T-t+1)$ is the right choice. The strict positivity and the convergence property are evident. For the condition on the differential operator we calculate

$$\begin{aligned}-e^{-\lambda(T-t)}\mathcal{L}^\pi w(t,x) &= \lambda|x|^2 + C - 2\mu(t,x,\pi)^T x - \operatorname{tr}\left((\sigma\sigma^T)(t,x,\pi)I_d\right)\\ &\quad - \int_{\mathbb{R}^k} \left\{|x+\gamma(t,x,\pi,z)|^2 - |x|^2 - 2\gamma(t,x,\pi,z)^T x\right\}\nu(dz).\end{aligned}$$

Using the estimates in (G2) of Assumption 3.1.1 and the equality $|x+y|^2 = |x|^2 + 2x^T y + |y|^2$, we conclude

$$-e^{-\lambda(T-t)}\mathcal{L}^\pi w(t,x) \geq \lambda|x|^2 + C - \tilde{C}(1+|x|^2).$$

For $\lambda \geq \tilde{C}$ we obtain the desired inequality.

□

Theorem 3.5.6 directly leads to the following uniqueness result:

Corollary 3.5.7. *Let Assumption 3.1.1 be satisfied. Then the value function $v^{(n)}$ in* (3.3) *(resp.* (3.17)*) is the unique viscosity solution of* (3.25) *in $\mathcal{UC}_x([0,T]\times\mathbb{R}^d)$.*

Going through the proof of Theorem 3.6.6 below, it is apparent how one proves the uniqueness of a viscosity solution $v^{(0)}$ of the HJB-equation (3.8) for the classic control problem without impulse intervention.

3.6 Extension of the model

Since Assumption 3.1.1, in particular the global Lipschitz condition (G3) on the profit functions f, g, K, is quite restrictive, we want to discuss in this section how we can weaken the basic assumptions of our model. As minimal requirement we need a set of coefficient functions μ, σ, γ, a transaction function Γ, profit functions f, g, K and admissible control sets $\mathcal{U}_n = \mathcal{U}_0^{n+1}$, $\mathcal{V}_n$ such that the following holds:

Assumption 3.6.1. *(G1') The controlled process $X = X_{t,x}^{\pi;\xi}$ has a unique strong solution for any starting value $(t,x) \in [0,T]\times\mathbb{R}^d$ and any controls $\pi \in \mathcal{U}_n$, $\xi \in \mathcal{V}_n$.*

(G2') The objective function $\mathcal{J}^{(\pi,\xi)}$ is well-defined for any controls $\pi \in \mathcal{U}_n$, $\xi \in \mathcal{V}_n$.

(G3') The admissible control sets $\mathcal{U}_n, \mathcal{V}_n$ only contain Markov controls and are stable under concatenation.

In addition let us suppose that the process $X = X^{\pi,\xi}$ is controlled in the time interval $[0,T]$ only as long as it stays within an open set $\mathcal{O} \subset \mathbb{R}^d$. We set $\mathcal{S} := [0,T) \times \mathcal{O}$ and fix a starting point $(t,x) \in [0,T] \times \mathbb{R}^d$. Then the stopping time

$$\tau_{\mathcal{S}} := \inf\{s \in [t,T] : (s, X(s)) \notin \mathcal{S}\} \tag{3.35}$$

indicates the end of control. Note that $X(s)$ is the value at s after all impulses in s have been exercised, so that "intermediate values" between directly successive jumps are not taken into account by this stopping time. The performance criterion then reads

$$\begin{aligned}\mathcal{J}^{(\pi,\xi)}(t,x) = \mathbb{E}_{t,x}\Bigg[\int_t^{\tau_{\mathcal{S}}} f(s, X(s), \pi^{(n-N^\xi(s))}(s))ds + g(\tau_{\mathcal{S}}, X(\tau_{\mathcal{S}}))\\ + \sum_{t\le\tau_i\le\tau_{\mathcal{S}}} K(\tau_i, \check{X}(\tau_i-), \pi^{(n-i+1)}(\tau_i), \zeta_i)\Bigg].\end{aligned}$$

To specify the assumption (G2'), we require as in [42] the integrability condition on the negative parts of f, g, K,

$$\mathbb{E}_{t,x}\Bigg[\int_t^{\tau_{\mathcal{S}}} f^-(s,X(s),\pi^{(n-N^\xi(s))}(s))ds + g^-(\tau_{\mathcal{S}},X(\tau_{\mathcal{S}}) + \sum_{t\le\tau_i\le\tau_{\mathcal{S}}} K^-(\tau_i,\check{X}(\tau_i-),\pi^{(n-i+1)}(\tau_i),\zeta_i)\Bigg] < \infty.$$

Of course, we are mainly interested in the value function on the closure $\bar{\mathcal{S}}$ of the control domain. However, since a possible jump, driven stochastically or by intervention, may send the system outside $\bar{\mathcal{S}}$, we have to determine $v^{(n)}$ on the entire space $[0,T] \times \mathbb{R}^d$. In particular, by fixing $\tau_{\mathcal{S}}$ after all jumps, player B is given the chance to intervene even when the system is in $\mathcal{S}^c$. Consequently, the right boundary conditions reads

$$v^{(n)} = \min\left(g, \sup_{\pi\in U} \mathcal{M}^\pi v^{(n-1)}\right) \quad \text{on } \mathcal{S}^c.$$

We can furthermore drop the assumption (3.1) concerning the Lévy measure ν. Then we have to replace the compensated Poisson random measure $\tilde{N}$ by

$$\bar{N}(dt, dz) = N(dt, dz) - \mathbb{1}_{\{|z|<1\}}\nu(dz)dt.$$

Dynkin's operator now takes the form

$$\begin{aligned}\mathcal{L}^\pi\varphi(t,x) = \frac{\partial\varphi}{\partial t}(t,x) + \mu(t,x,\pi)^T D_x\varphi(t,x) + \frac{1}{2}\mathrm{tr}\left((\sigma\sigma^T)(t,x,\pi)D_x^2\varphi(t,x)\right)\\ + \int_{\mathbb{R}^k}\{\varphi(t,x+\gamma(t,x,\pi,z)) - \varphi(t,x) - \gamma(t,x,\pi,z)^T D_x\varphi(t,x)\mathbb{1}_{\{|z|<1\}}\}\nu(dz),\end{aligned}$$

and we redefine

$$\mathcal{I}_\pi^{2,\eta}[t,x,p,\varphi] = \int_{|z|\ge\eta}\{\varphi(t,x+\gamma(t,x,\pi,z)) - \varphi(t,x) - \gamma(t,x,\pi,z)^T p\mathbb{1}_{\{|z|<1\}}\}\nu(dz).$$

Having this modified setting in the back of our mind, let us now go through the preceding argumentation of this chapter. For the derivation of the DPPs we have only made use of the Markov property of the underlying state process and the stability of admissible controls under concatenation. So by the assumption (G3') the representation formulas (3.11) and (3.19) remain true for the set of admissible stopping times

$$\mathcal{T} := \{\tau : \tau \text{ stopping time}, t \leq \tau \leq \tau_{\mathcal{S}}\}$$

and with the time horizon T replaced by $\tau_{\mathcal{S}}$. Here the exit time $\tau_{\mathcal{S}}$ defined in (3.35) refers to the state process X^π without impulse intervention, i.e. $\mathcal{T} = \mathcal{T}^\pi(t,x)$.

Unfortunately, the proof of the continuity of the value function fails because we cannot reduce the distance $|v^{(n)}(t,x) - v^{(n)}(s,y)|$ to the estimates of the purely continuously controlled process due to the missing Lipschitz conditions and troubles at the parabolic boundary $\partial\mathcal{S} := ([0,T) \times \partial\mathcal{O}) \cup (\{T\} \times \overline{\mathcal{O}})$. Nevertheless, we can show the viscosity property by using the notion of discontinuous viscosity solutions which are locally bounded, see Appendix A.2. To this end, let v^* and v_* be the upper and lower semicontinuous envelopes of a locally bounded function v on $\mathcal{S}$, i.e.

$$v^*(t,x) = \limsup_{\mathcal{S} \ni (s,y) \to (t,x)} v(s,y), \quad v_*(t,x) = \liminf_{\mathcal{S} \ni (s,y) \to (t,x)} v(s,y).$$

For more details on v^*, v_* and on upper and lower semicontinuous functions, we refer to Appendix A.1.

Under the following conditions we can rebuild the proof of Theorem 3.5.5:

Assumption 3.6.2. *(V1) The control set U and the impulse set Z are compact and non-empty.*

(V2) The value function $v^{(n-1)}$, the SDE coefficients μ, σ, γ, the transaction function Γ and the profit functions f, g, K are continuous.

Assumption 3.6.3. *(E1) The purely continuously controlled process X^π satisfies the estimates of Lemma B.1.1 for all $\pi \in \mathcal{U}_0$.*

(E2) For all $\varphi \in \mathcal{C}^{1,2}(\mathcal{S})$, $(t,x) \in \mathcal{S}$ and $\pi \in \mathcal{U}_0$ there exists a $\rho > 0$ such that $\mathcal{L}^\pi\varphi$ is well-defined on $[t, t+\rho) \times \mathcal{B}(x,\rho)$, and for each stopping time $\tau \in \mathcal{T}$ satisfying

$$\tau \leq \tau_\rho := \inf\{s \geq t : |X^\pi(s) - x| \geq \rho\} \wedge (t+\rho),$$

we have

$$\mathbb{E}_{t,x}\left[\varphi(\tau, X^\pi(\tau))\right] = \varphi(t,x) + \mathbb{E}_{t,x}\left[\int_t^\tau \mathcal{L}^\pi(s, X^\pi(s))ds\right].$$

By the assumption (V1) we ensure continuity properties of the supremum and the infimum with respect to $\pi \in U$ and $\zeta \in Z$, repectively, while the requirement (V2) makes the differential operator $\mathcal{L}^\pi$ and the intervention operator $\mathcal{M}^\pi$ preserving continuity. Furthermore, the assumptions (E1) and (E2) justify the crucial steps used in the proof of Theorem 3.5.5.

Now we can formulate a viscosity solution existence result for the extended model:

Theorem 3.6.4. *Let Assumptions 3.6.1, 3.6.2 and 3.6.3 be satisfied. Assume further that $v^{(n)}$ is locally bounded on $\mathcal{S}$. Then, on $\mathcal{S}$ the value function $v^{(n)}$ in* (3.3) *(resp.* (3.17)*) is a viscosity supersolution of the equation*

$$\inf_{\pi \in U} \max\Big(-\mathcal{L}^\pi v - f(.,.,\pi), v - \mathcal{M}^\pi v^{(n-1)}\Big) = 0. \tag{3.36}$$

If in addition Assumption 3.2.1 is satisfied, then it is also a viscosity subsolution of (3.36).

Proof: (Sketch) We do not provide the entire proof because, thanks to the made assumptions, it follows exactly the line of arguments as in the regular model. We only explain the technical point of dealing with discontinuous viscosity solutions. We work on sequences $(t_k, x_k)_k \subset \mathcal{S}$ converging to $(t_0, x_0) \in \mathcal{S}$ such that $v^{(n)}(t_k, x_k) \to (v^{(n)})^*(t_0, x_0)$ for the proof of the subsolution inequality in (t_0, x_0) (resp. $v^{(n)}(t_k, x_k) \to (v^{(n)})_*(t_0, x_0)$ for the proof of the supersolution inequality). As a test function we consider some $\varphi \in \mathcal{C}^{1,2}(\mathcal{S})$ such that $\varphi \geq (v^{(n)})^*$ and $\varphi(t_0, x_0) = (v^{(n)})^*(t_0, x_0)$ (resp. $\varphi \leq (v^{(n)})_*$ and $\varphi(t_0, x_0) = (v^{(n)})_*(t_0, x_0)$). Using the relation $(v^{(n)})^* \geq v^{(n)} \geq (v^{(n)})_*$ we can approximate $(v^{(n)})^*(t_0, x_0)$ and $(v^{(n)})_*(t_0, x_0)$ by $v^{(n)}(t_k, x_k)$. Compared to the situation in the proof of Theorem 3.5.5, both the ε-optimal control $\pi \in \mathcal{U}_0$ in the subsolution proof and the control value $\pi \in U$ in the supersolution proof are substituted by sequences of ε_k-optimal controls $(\pi_k)_k \subset \mathcal{U}_0$ and analogous control values $(\pi_k)_k \subset U$. As well, instead of the considered stopping times $\tau_\rho, \tau_\varepsilon, t + h$ we use sequences $(\tau_k^\rho)_k, (\tau_k^\varepsilon)_k, (t_k + h_k)_k$ of stopping times which are first exit times of the relevant controlled processes $(t, X^{\pi_k}_{t_k, x_k}(t))$ of appropriate bounded subspaces of $\mathcal{S}$ and which converge to t_0. Similarly to the standard case, one verifies by using Lemma B.1.2 that for all $\tau_k = \tau_k^\rho, \tau_k^\varepsilon, t_k + h_k$ the probability of $\{\tau_k < \tau_{\mathcal{S}}\}$ goes to 1 as $k \to \infty$, so that one finally obtains the corresponding inequalities by taking the limit $k \to \infty$. □

Next we want to state a comparison result for viscosity subsolutions and supersolutions which are polynomially bounded. Denote by $\mathcal{P}_r = \mathcal{P}_r([0,T] \times \mathbb{R}^d)$ the space of functions v at most polynomially growing with exponent $r > 0$, i.e. there exists a constant $C > 0$ such that

$$|v(t,x)| \leq C(1 + |x|^r) \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}^d.$$

Let us fix some $r > 0$ for the rest of this section. Studying viscosity solutions of (3.36) in the class $\mathcal{P}_r$, we only have to consider test functions $\varphi \in \mathcal{P}_r \cap \mathcal{C}^{1,2}(\mathcal{S})$. Moreover, we will need the following assumptions for the proof of the comparison result:

Assumption 3.6.5. *(U1) Let $\mu(.,.,\pi)$, $\sigma(.,.,\pi)$, $f(.,.,\pi)$ be locally Lipschitz continuous on $\mathcal{S}$ (uniformly in $\pi \in U$), i.e. for all $(t_0, x_0) \in \mathcal{S}$ there exist an open neigbourhood $\mathcal{B} \subset \mathcal{S}$ of (t_0, x_0) and a constant $C > 0$ (independent of π) such that*

$$\begin{aligned}&|\mu(t,x,\pi) - \mu(t,y,\pi)| + |\sigma(t,x,\pi) - \sigma(t,y,\pi)| \\ &\qquad + |f(t,x,\pi) - f(t,y,\pi)| \leq C|x - y| \quad \textit{for all } (t,x),(t,y) \in \mathcal{B}.\end{aligned}$$

(U2) For all $(t,x) \in \mathcal{S}$*,* $\pi \in U$ *and* $\varphi \in \mathcal{P}_r \cap \mathcal{C}^{1,2}(\mathcal{S})$ *the integral* $\mathcal{I}_\pi[t,x,\varphi]$ *is well-defined with*

$$\lim_{\eta \searrow 0} \mathcal{I}_\pi^{1,\eta}[t,x,\varphi] = 0,$$

and there exist an open neigbourhood $\mathcal{B} \subset \mathcal{S}$ *of* (t,x) *and a constant* $C > 0$ *such that for all* $(t,x),(t,y) \in \mathcal{B}$*,*

$$\int_{\mathbb{R}^k} |\gamma(t,x,\pi,z) - \gamma(t,y,\pi,z)|^2 \nu(dz) \leq C|x-y|^2,$$
$$\int_{|z|\geq 1} |\gamma(t,x,\pi,z) - \gamma(t,y,\pi,z)| \nu(dz) \leq C|x-y|.$$

(U3) There exist a positive function $w \in \mathcal{C}^{1,2}(\mathcal{S}) \cap \mathcal{C}([0,T]\times\mathbb{R}^d)$ *and a constant* $C > 0$ *such that*

$$\min\left(\inf_{\pi\in U}\{-\mathcal{L}^\pi w\}, w\right) \geq C \quad \text{on } \mathcal{S}$$

and

$$\lim_{|x|\to\infty} \frac{|x|^r}{w(t,x)} = 0 \quad \text{(uniformly in } t\text{)}.$$

(U4) Suppose a viscosity solution in $\mathcal{P}_r([0,T]\times\mathbb{R}^d)$ *can be characterized by its limiting semijets as in Corollary 3.5.4.*

We are now ready for the crucial result with regard to the uniqueness problem of viscosity solutions of PDE (3.36), a comparison result. For the proof we need the assumptions (U1) and (U2) in order to estimate the local and integral part of the PDE, respectively. By the assumption (U3) we guarantee to attain a compact subset for desired maxima constructed with the doubling of variables approach introduced from Crandall, Ishii and Lions [10] which is a standard device used in typical uniqueness proofs for PDEs. A maximum principle for semicontinuous functions then gives us elements in their limiting semijets which can be connected to the definition of a viscosity solution due to the assumption (U4).

Theorem 3.6.6 (Comparison theorem)**.** *Let Assumptions 3.6.1, 3.6.2 and 3.6.5 be satisfied. Further, let* $u \in \mathcal{P}_r([0,T]\times\mathbb{R}^d)$ *be a viscosity subsolution and* $v \in \mathcal{P}_r([0,T]\times\mathbb{R}^d)$ *a viscosity supersolution of* (3.36) *on* $\mathcal{S}$*. If we have* $u^* \leq v_*$ *on* $\mathcal{S}^c$*, then it follows*

$$u^* \leq v_* \quad \text{on } [0,T]\times\mathbb{R}^d.$$

Proof: We argue by contradiction and suppose that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \{u^*(t,x) - v_*(t,x)\} > 0. \tag{3.37}$$

Consider the function w from the assumption (U3). In view of the growth condition of w for $|x|\to\infty$, the above supremum remains strictly positive if we substitute v by $v^\varepsilon := v + \varepsilon w$ for a sufficiently small $\varepsilon > 0$, and it is attained on some compact subset of $[0,T]\times\mathbb{R}^d$. (Note that

$u^* - (v^\varepsilon)_*$ is upper semicontinuous (usc) and that an usc function assumes its maximum on a compact set.) It is easy to check via the assumption (U3) that v^ε is a viscosity supersolution of

$$\inf_{\pi \in U} \max\left(-\mathcal{L}^\pi v - f(.,.,\pi), v - \mathcal{M}^\pi v^{(n-1)}\right) - \varepsilon C = 0 \quad \text{on } \mathcal{S} \tag{3.38}$$

with $(v^\varepsilon)_* \geq v_* \geq u^*$ on the parabolic boundary $\mathcal{S}^c$. Therefore we can cut back the relevant domain of the considered optimization problem on the set $\mathcal{S} = [0,T) \times \mathcal{O}$. Furthermore, we can find $\beta, \rho > 0$ such that

$$M := \sup_{(t,x) \in [0,T) \times \mathcal{O}} \left\{ u^*(t,x) - (v^\varepsilon)_*(t,x) - \frac{\beta}{t+\rho} \right\} > 0.$$

We now employ the doubling of variables approach from [10]. For $k \in \mathbb{N}$ we define

$$M_k := \sup_{(t,x,y) \in [0,T) \times \mathcal{O} \times \mathcal{O}} \left\{ u^*(t,x) - (v^\varepsilon)_*(t,y) - \frac{\beta}{t+\rho} - \frac{k}{2}|x-y|^2 \right\}.$$

Because of the foregoing argumentation this supremum is attained on some compact subset of $[0,T) \times \mathcal{O} \times \mathcal{O}$, independent of large k, and we have $0 < M_k < \infty$. Let (t_k, x_k, y_k) be a corresponding maximizer.

In the following we use the characterization of viscosity solutions by the limiting semijets in order to work towards a contradiction. At first we conclude from Lemma 3.1 in Crandall, Ishii and Lions [10] (see Lemma A.3.2 in the appendix) that

$$\lim_{k \to \infty} k|x_k - y_k|^2 = 0. \tag{3.39}$$

Thus we may pass to a subsequence of (t_k, x_k, y_k) converging to $(t_0, x_0, x_0) \in [0,T) \times \mathcal{O} \times \mathcal{O}$. Let us relabel this subsequence as the original one.

Next we apply Theorem 8.3 in [10] (see Theorem A.3.1 in the appendix) to the function $\varphi(t,x,y) = \frac{\beta}{t+\rho} + \frac{k}{2}|x-y|^2$ at point (t_k, x_k, y_k). To justify its applicability in the case $t_k = 0$, note that we can extend the functions $u^*, (v^\varepsilon)_*$ to the time periode $(-\rho, T)$ in a trivial way such that the maximum and the maximizer remain unchanged, but with the maximizer as inner point of the considered domain. Hence we get a number $a_k \in \mathbb{R}$ and matrices $X_k, Y_k \in \mathbb{S}^d$ such that

$$(a_k, k(x_k - y_k), X_k) \in \bar{\mathcal{J}}^{2,+} u^*(t_k, x_k), \quad (a_k + \frac{\beta}{(t_k+\rho)^2}, k(x_k - y_k), Y_k) \in \bar{\mathcal{J}}^{2,-} (v^\varepsilon)_*(t_k, y_k)$$

and

$$x^T X_k x - y^T Y_k y \leq 3k|x-y|^2 \quad \text{for all } x, y \in \mathbb{R}^d. \tag{3.40}$$

As in Section 3.5 we define for $\pi \in U$ the function $F^\pi : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ by

$$F^\pi(t,x,p,X,l) = -\frac{1}{2} \mathrm{tr}\left((\sigma\sigma^T)(t,x,\pi)X\right) - \mu(t,x,\pi)^T p - f(t,x,\pi) - l.$$

Suppose $\nu(\mathbb{R}^k) = \infty$ and use the characterization of viscosity solutions of Corollary 3.5.4. For $\nu(\mathbb{R}^k) < \infty$ the proof works in the same way with the choice $\eta = 0$ below. Since

$u, v \in \mathcal{P}_r([0,T] \times \mathbb{R}^d)$ are respectively a sub and a supersolution of (3.36) and $v^\varepsilon = v + \varepsilon w$ with $w \in \mathcal{C}^{1,2}(\mathcal{S})$ is a supersolution of (3.38), in view of the assumption (U4) we can find $\varphi_k, \psi_k \in \mathcal{C}^{1,2}(\mathcal{S})$ and $\pi_k \in U$ such that

$$\begin{aligned} \max\Bigg(&-a_k + F^{\pi_k}(t_k, x_k, k(x_k - y_k), X_k, \mathcal{I}^{1,\frac{1}{k}}_{\pi_k}[t_k, x_k, \varphi_k] \\ &+\mathcal{I}^{2,\frac{1}{k}}_{\pi_k}[t_k, x_k, k(x_k - y_k), u^*]), u^*(t_k, x_k) - \mathcal{M}^{\pi_k} v^{(n-1)}(t_k, x_k)\Bigg) < \frac{1}{k}, \end{aligned} \tag{3.41}$$

and

$$\begin{aligned} \max\Bigg(&- a_k - \frac{\beta}{(t_k+\rho)^2} + F^{\pi_k}(t_k, y_k, k(x_k - y_k), Y_k, \mathcal{I}^{1,\frac{1}{k}}_{\pi_k}[t_k, y_k, \psi_k] \\ &+\mathcal{I}^{2,\frac{1}{k}}_{\pi_k}[t_k, y_k, k(x_k - y_k), (v^\varepsilon)_*]), (v^\varepsilon)_*(t_k, y_k) - \mathcal{M}^{\pi_k} v^{(n-1)}(t_k, y_k)\Bigg) \geq \varepsilon C. \end{aligned} \tag{3.42}$$

Let us consider the difference Λ_k of the first components of the max-terms in (3.41)-(3.42), and let us split Λ_k in form of

$$\Lambda_k = \frac{\beta}{(t_k+\rho)^2} - A^1_k - A^2_k - I^1_k - I^2_k$$

with

$$\begin{aligned} A^1_k &:= \frac{1}{2}\left(\mathrm{tr}\left((\sigma\sigma^T)(t_k, x_k, \pi_k) X_k\right) - \mathrm{tr}\left((\sigma\sigma^T)(t_k, y_k, \pi_k) Y_k\right)\right) \\ &\qquad + k\left(\mu(t_k, x_k, \pi_k) - \mu(t_k, y_k, \pi_k)\right)^T (x_k - y_k), \\ A^2_k &:= f(t_k, x_k, \pi_k) - f(t_k, y_k, \pi_k), \\ I^1_k &:= \mathcal{I}^{1,\frac{1}{k}}_{\pi_k}[t_k, x_k, \varphi_k] - \mathcal{I}^{1,\frac{1}{k}}_{\pi_k}[t_k, y_k, \psi_k], \\ I^2_k &:= \mathcal{I}^{2,\frac{1}{k}}_{\pi_k}[t_k, x_k, k(x_k - y_k), u^*] - \mathcal{I}^{2,\frac{1}{k}}_{\pi_k}[t_k, y_k, k(x_k - y_k), (v^\varepsilon)_*]. \end{aligned}$$

Using the local Lipschitz condition on μ, σ, f from the assumption (U1) and the relation (3.40), we obtain the estimates of A^1_k, A^2_k,

$$A^1_k \leq Ck|x_k - y_k|^2, \qquad A^2_k \leq C|x_k - y_k|.$$

By the assumption (U2) we directly get

$$\lim_{k\to\infty} I^1_k = 0.$$

Furthermore, using the fact that (t_k, x_k, y_k) is a maximum point corresponding to M_k and the equality $|x+y|^2 = |x|^2 + 2x^T y + |y|^2$, we calculate for arbitrary vectors $d, \tilde{d}$,

$$\begin{aligned} &u^*(t_k, x_k + d) - u^*(t_k, x_k) - kd^T(x_k - y_k) \\ &\qquad \leq (v^\varepsilon)_*(t_k, y_k + \tilde{d}) - (v^\varepsilon)_*(t_k, y_k) - k\tilde{d}^T(x_k - y_k) + \frac{k}{2}|d - \tilde{d}|^2. \end{aligned}$$

Integrating on both sides with $d = \gamma(t_k, x_k, \pi_k, z)$ and $\tilde{d} = \gamma(t_k, y_k, \pi_k, z)$, we obtain

$$I_k^2 \le \frac{k}{2} \int_{|z| \ge \frac{1}{k}} |\gamma(t_k, x_k, \pi_k, z) - \gamma(t_k, y_k, \pi_k, z)|^2 \nu(dz) \\ + k|x_k - y_k| \int_{|z|\ge 1} |\gamma(t_k, x_k, \pi_k, z) - \gamma(t_k, y_k, \pi_k, z)| \nu(dz) \le Ck|x_k - y_k|^2,$$

where the last inequality is a consequence of (U2). So in view of (3.39) we arrive at

$$\limsup_{k\to\infty} \Lambda_k \ge \frac{\beta}{(t_0 + \rho)^2} > 0.$$

Hence, comparing the subsolution and supersolution inequalities (3.41) and (3.42) of u and v^ε, respectively, we conclude by using the continuity of $(t, x, \pi) \mapsto \mathcal{M}^\pi v^{(n-1)}(t, x)$ that for large k we have

$$u^*(t_k, x_k) < (v^\varepsilon)_*(t_k, y_k).$$

Recalling the definition of M_k and that (t_k, x_k, y_k) realizes the supremum in M_k, this inequality is a contradiction to $M_k > 0$. □

Recall that a necessary condition for the value function on the parabolic boundary is

$$v^{(n)} = \min\left(g, \sup_{\pi\in U} \mathcal{M}^\pi v^{(n-1)}\right) \quad \text{on } \mathcal{S}^c. \tag{3.43}$$

Combining this boundary condition with Theorem 3.6.6 leads to the desired uniqueness.

Corollary 3.6.7. *Let Assumptions 3.6.1, 3.6.2 and 3.6.5 be satisfied. Then there exists at most one viscosity solution $v \in \mathcal{P}_r([0,T]\times\mathbb{R}^d)$ of (3.36) on $\mathcal{S}$ which satisfies the boundary condition (3.43), and it is continuous on $[0,T]\times\mathbb{R}^d$.*

Proof: Let u, v be two viscosity solutions of (3.36). Then, by Theorem 3.6.6 and the definition of the upper and lower semicontinuous envelopes, we deduce

$$u^* \le v_* \le v^* \le u_* \le u^*.$$

Thus we have $u = v$ and the function is continuous. □

Of course some of the assumptions made in this section are very vague. We do not want to get involved in delicate technical detail problems. Rather we intend to make the reader sensible on the critical points where the presented theory might fail under changed model assumptions. For an example of a precise formulation of a very general setting we refer to Seydel [52] who investigated a control problem combining continuous and (infinite) impulse control.

Chapter 4

Stochastic target problem under impulse perturbation

In this chapter we consider the problem of finding the minimal initial data of a controlled process which guarantees to reach a controlled target. According to the setup presented in Chapter 2, we assume that there might be perturbations of the system in form of impulses resulting in jumps both in the target process and the stochastic target. The target condition then is intended to be satisfied for any impulse strategy.

Motivated by applications in finance, namely the super-replication problem, stochastic target problems were first considered by Soner and Touzi [54], [55] assuming the controlled process follows a diffusion. Bouchard, Elie and Touzi [7] further examined the case where the target has to be reached only with a given probability, which is called the stochastic target problem with controlled loss. As an additional extension their investigations on the target problem include controls with values in an unbounded set. Bouchard [5] and Moreau [41] extended these results on the classic target problem and the target problem with controlled loss, respectively, to the jump diffusion case. It will turn out that the stochastic target problem under impulse perturbation is the analogue of [5] for a finite number of jumps. In Chapter 6 we will also consider an application of a version of the target problem with controlled loss in the framework of impulse perturbation, but without a rigorous treatment of unbounded control sets which are needed in general in the context of this problem type. Besides the already note resources we refer to [3], [6] and [48] for examples of super-replication problems in the financial literature.

For the treatment of this new control problem we proceed as in Chapter 3 for the differential game: The definition of the stochastic target problem under impulse perturbation is formulated in Section 4.1, followed in the next section by the statement of a PDE characterization of the associated value function in the viscosity sense as main result of this chapter. Section 4.3 deals with the derivation of the dynamic programming principle which forms the basis for the proof of the viscosity property in Section 4.4. The last section is concerned with a special variant of the model. We do not derive a general uniqueness theorem for a viscosity solution of the associated PDE. Anyway, we will show in Chapter 5 and Chapter 6 through uniqueness results for the considered examples that the PDE characterization may be sufficient in financial applications.

4.1 Problem formulation

For a fixed finite time horizon $T > 0$ let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ be a filtered probability space which is equipped with an adapted d-dimensional Brownian motion B and satisfies the usual conditions. On this stochastic basis we consider a state process $(X, Y) \subseteq \mathbb{R}^d \times \mathbb{R}$ which is controlled continuously by $\pi \in \mathcal{U}_n$ and with impulse control $\xi \in \mathcal{V}_n$. Think of Y as the target process and X as the stochastic component of the target condition. Between intervention times $\tau_i < t < \tau_{i+1}$, $i = 0, \dots, n$, the processes evolve according to

$$\begin{aligned} dX(t) &= \mu_X(t, X(t), \pi^{(n-i)}(t))dt + \sigma_X(t, X(t), \pi^{(n-i)}(t))dB(t), \\ dY(t) &= \mu_Y(t, X(t), Y(t), \pi^{(n-i)}(t))dt + \sigma_Y(t, X(t), Y(t), \pi^{(n-i)}(t))dB(t), \end{aligned} \tag{4.1}$$

while at jump times τ_i, $i = 1, \dots, n$, the systems are sent to

$$\begin{aligned} X(\tau_i) &= \Gamma_X(\tau_i, X(\tau_i-), \pi^{(n-i+1)}(\tau_i), \zeta_i), \\ Y(\tau_i) &= \Gamma_Y(\tau_i, X(\tau_i-), Y(\tau_i-), \pi^{(n-i+1)}(\tau_i), \zeta_i). \end{aligned} \tag{4.2}$$

Here the functions $\mu_X : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}^d$, $\sigma_X : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}^{d\times m}$, $\mu_Y : [0, T] \times \mathbb{R}^{d+1} \times U \to \mathbb{R}$, $\sigma_Y : [0, T] \times \mathbb{R}^{d+1} \times U \to \mathbb{R}^{1\times m}$ and $\Gamma_X : [0, T] \times \mathbb{R}^d \times U \times Z \to \mathbb{R}^d$, $\Gamma_Y : [0, T] \times \mathbb{R}^{d+1} \times U \times Z \to \mathbb{R}$ satisfy conditions detailed in Assumption 4.1.1 below.

In this chapter we restrict ourselves to Itô processes instead of the more general Itô-Lévy processes. The reason for this is to be found in the fact that we cannot control in general the jump component of the state processes such that we can guarantee to reach the target zone almost surely. Bouchard [5] solved the classic stochastic target problem for jump diffusions, supposed it has a solution. The extension of the results presented in this chapter for the target problem under impulse perturbation to Bouchards jump setting is quite obvious. So for the sake of simplicity we do not take into consideration any stochastic jumps. As consequence of this we may suppose that the state process is continuous between intervention times.

By the stochastic target problem we understand the problem of finding the minimal initial value $y \in \mathbb{R}$ of Y, so that we are able to reach at time T the target $Y(T) \geq g(X(T))$ almost surely, where $g : \mathbb{R}^d \to \mathbb{R}$ is a given function. In consideration of possible system breakdowns we want to reach this target for any admissible impulse strategy. So the *stochastic target problem under impulse perturbation* reads as follows: Find an *optimal* control $\hat{\pi} \in \mathcal{U}_n$ with associated *value function* $v^{(n)}$ such that

$$\begin{aligned} v^{(n)}(t, x) = \inf\{y \in \mathbb{R} : &\text{ there exists } \pi \in \mathcal{U}_n \text{ such that} \\ & Y^{\pi,\xi}_{t,x,y}(T) \geq g(X^{\pi,\xi}_{t,x}(T)) \text{ a.s. for all } \xi \in \mathcal{V}_n\} \end{aligned} \tag{4.3}$$

and

$$Y^{\hat{\pi},\xi}_{t,x,v^{(n)}(t,x)}(T) \geq g(X^{\hat{\pi},\xi}_{t,x}(T)) \text{ a.s. for all } \xi \in \mathcal{V}_n.$$

Let us now fix the basic assumptions for the following analysis of the target problem:

Assumption 4.1.1. *(T1) The control set $U \subset \mathbb{R}^p$ and the impulse set $Z \subset \mathbb{R}^q$ are compact and non-empty.*

(T2) The transaction functions Γ_X, Γ_Y are continuous.

(T3) The functions $\mu_X, \sigma_X, \mu_Y, \sigma_Y$ are continuous and Lipschitz in the variables (x, y) (uniformly in the variables (t, π)).

(T4) The value functions $v^{(i)}$, $i = 0, \ldots, n$, are locally bounded on $[0, T] \times \mathbb{R}^d$ and g is locally bounded on $\mathbb{R}^d$.

The assumptions (T1)-(T3) guarantee the existence and uniqueness of a strong solution of $(X^{\pi,\xi}_{t,x}, Y^{\pi,\xi}_{t,x,y})$ to the stochastic differential system (4.1)-(4.2) for each starting point $(t, x, y) \in [0, T] \times \mathbb{R}^{d+1}$ and all controls $\pi \in \mathcal{U}_n$, $\xi \in \mathcal{V}_n$. By the requirement (T4) the semicontinuous envelopes

$$(v^{(i)})^*(t,x) = \limsup_{(0,T)\times\mathbb{R}^d \ni (s,x') \to (t,x)} v^{(i)}(s,x'), \quad (v^{(i)})_*(t,x) = \liminf_{(0,T)\times\mathbb{R}^d \ni (s,x') \to (t,x)} v^{(i)}(s,x'),$$

$$g^*(x) = \limsup_{\mathbb{R}^d \ni x' \to x} g(x'), \qquad g_*(x) = \liminf_{\mathbb{R}^d \ni x' \to x} g(x')$$

are finite.

Example 4.1.1. Consider the controlled system

$$\begin{aligned} dX(t) &= \mu_X dt + \sigma_X dB(t), & X(\tau_1) &= X(\tau_1-) - \zeta_1, \\ dY(t) &= \mu_Y \pi^{(i)} dt + \sigma_Y \pi^{(i)} dB(t), & Y(\tau_1) &= Y(\tau_1-) - \zeta_1 \pi^{(1)}, \end{aligned}$$

with parameters $\mu_X, \mu_Y, \sigma_X, \sigma_Y > 0$, 1-impulse perturbation $(\tau_1, \zeta_1) \in [0, T] \times (\{\zeta\} \cup \{0\})$, $\zeta > 0$, and control $\pi = (\pi^{(1)}, \pi^{(0)})$ whose processes have values in $[0, m]$ for some $m > 0$, i.e. the setting is $n = 1$, $U = [0, m]$, $Z = \{\zeta\}$. The dynamics of Y in the above representation must be understood in the sense that control process $\pi^{(1)}$ is used on $[0, \tau_1]$ and $\pi^{(0)}$ on $(\tau_1, T]$.

Given this system and the target function $g(x) = x$, we want to solve the stochastic target problem

$$v^{(1)}(t,x) = \inf\{y \in \mathbb{R} : \text{ there exists } \pi \in \mathcal{U}_1 \text{ such that } Y^{\pi,\xi}_{t,y}(T) \geq X_{t,x}(T) \text{ a.s. for all } \xi \in \mathcal{V}_1\},$$

where $\mathcal{U}_1, \mathcal{V}_1$ are the sets of admissible controls introduced in the Chapter 2.

It is easy to check that (T1)-(T3) from Assumption 3.1.1 are satisfied for this problem formulation. So the the controlled processes are well-defined. Ex post we will see that the derived solutions satisfy the assumption (T4), too, which allows us to use the PDE approach presented in the next section as a verification tool.

Step 0. Without impulse perturbation we have

$$Y_{t,y}^{\pi^{(0)}}(T) - X_{t,x}(T) = y - x + \int_t^T (\mu_Y \pi^{(0)}(s) - \mu_X) ds + \int_t^T (\sigma_Y \pi^{(0)}(s) - \sigma_X) dB(s).$$

Since we want to eliminate uncertainty of the target's reachability, we have to zero the integrand of the above stochastic integral. So we are forced to choose $\hat{\pi}^{(0)} \equiv \frac{\sigma_x}{\sigma_Y}$. Here we assume that m is large enough, so that $\hat{\pi}^{(0)}$ is admissible. Under this control we obtain

$$Y_{t,y}^{\hat{\pi}^{(0)}}(T) - X_{t,x}(T) = y - x + \left(\mu_Y \frac{\sigma_x}{\sigma_Y} - \mu_X\right)(T-t)$$

which is strictly increasing in y with root at

$$v^{(0)}(t,x) = x + \left(\mu_X - \mu_Y \frac{\sigma_x}{\sigma_Y}\right)(T-t).$$

Step 1. For the same reason as in step 0 we arrive at $\hat{\pi}^{(1)} \equiv \frac{\sigma_x}{\sigma_Y}$. In consideration of an impulse perturbation we have to distinguish two cases. For $\sigma_X \leq \sigma_Y$ we have $\hat{\pi}^{(1)} \leq 1$, so that an impulse sets back the target X more than Y. Consequently, in this case we have $v^{(1)} = v^{(0)}$. For $\sigma_X > \sigma_Y$ instead we have $\hat{\pi}^{(1)} > 1$, so that an impulse endamages X more than Y. In this case an additional buffer to the amount of the difference between both jump sizes is necessary. It follows

$$v^{(1)}(t,x) = v^{(0)}(t,x) + (\hat{\pi}^{(1)} - 1)\zeta = x + \left(\mu_X - \mu_Y \frac{\sigma_X}{\sigma_Y}\right)(T-t) + \left(\frac{\sigma_X}{\sigma_Y} - 1\right)\zeta.$$

4.2 Procedure and main result

Our main result consists in a PDE characterization of the value function $v^{(n)}$ in the weak sense of viscosity solutions (see the appendix for the notion of a discontinuous viscosity solution of a PDE). In order to formulate the corresponding system of PDEs, we need some preliminaries:

For $\pi \in U$ let $\mathcal{L}^\pi$ denote the differential operator given by

$$\mathcal{L}^\pi \varphi(t,x) := \frac{\partial \varphi}{\partial t}(t,x) + \mu_X(t,x,\pi)^T D_x \varphi(t,x) + \frac{1}{2} \mathrm{tr}\left((\sigma_X \sigma_X^T)(t,x,\pi) D_x^2 \varphi(t,x)\right),$$

and let $\mathcal{M}_y^\pi$ denote the intervention operator transacting the worst possible impulse in the sense of

$$\mathcal{M}_y^\pi \varphi(t,x) := \inf_{\zeta \in Z} \{\Gamma_Y(t,x,y,\pi,\zeta) - \varphi(t, \Gamma_X(t,x,\pi,\zeta))\}.$$

Further we introduce the set-valued function

$$N(t,x,y,p) := \{\pi \in U : \sigma_Y(t,x,y,\pi)^T - \sigma_X(t,x,\pi)^T p = 0\}$$

which allows us to control the Brownian part of the SDE (4.1). We make the following assumption on N:

Assumption 4.2.1 (Continuity of N)**.** *Let $(t_0, x_0, y_0, p_0) \in [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$. Then, for any $\pi_0 \in N(t_0, x_0, y_0, p_0)$ there exist an open neighborhood $\mathcal{B}$ of (t_0, x_0, y_0, p_0) in $[0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ and a continuous map $\hat{\pi}$ defined on $\mathcal{B}$ such that $\hat{\pi}(t_0, x_0, y_0, p_0) = \pi_0$ and $\hat{\pi}(t, x, y, p) \in N(t, x, y, p)$ on $\mathcal{B}$.*

For a differentiable function φ on $[0,T) \times \mathbb{R}^d \to \mathbb{R}$ we set

$$\mathcal{N}\varphi(t,x) := N(t, x, \varphi(t,x), D_x\varphi(t,x)).$$

Furthermore, let $\delta_N : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ be a function which indicates whether the set $N(t,x,y,p)$ is non-empty in form of

$$N(t,x,y,p) \neq \varnothing \quad \Leftrightarrow \quad \delta_N(t,x,y,p) \geq 0.$$

We will point out in Section 4.4 how such a function can be constructed and that it is continuous. For a differentiable function φ on $[0,T) \times \mathbb{R}^d \to \mathbb{R}$ we set

$$\delta_N\varphi(t,x) := \delta_N(t, x, \varphi(t,x), D_x\varphi(t,x)).$$

In the following we show how the target problem with at most n system breakdowns can be solved iteratively:

Step 0. We start with computing the value function $v^{(0)}$ of the classic stochastic target problem without impulse perturbation. From Soner and Touzi [54], see also Bouchard, Elie and Touzi [7], we know that on $[0,T) \times \mathbb{R}^d$ the function $v^{(0)}$ is a viscosity solution of

$$\min\left(\sup_{\pi \in \mathcal{N}v^{(0)}} \left\{ -\mathcal{L}^\pi v^{(0)} + \mu_Y(.,.,v^{(0)},\pi) \right\}, \delta_N v^{(0)} \right) = 0.$$

To specify the terminal values we have to consider the limits

$$\overline{G}^{(0)}(x) := \limsup_{t \nearrow T, x' \to x} v^{(0)}(t,x'), \quad \underline{G}^{(0)}(x) := \liminf_{t \nearrow T, x' \to x} v^{(0)}(t,x')$$

which possibly disagree with the "natural" terminal condition $v^{(0)}(t,.) = g$. According to [54] and [7], $\overline{G}^{(0)}$ is a viscosity subsolution of

$$\min\left(\overline{G}^{(0)} - g^*, \delta_N(T,.,\overline{G}^{(0)}, D_x\overline{G}^{(0)}) \right) = 0,$$

and $\overline{G}^{(0)}$ is a viscosity supersolution of

$$\min\left(\underline{G}^{(0)} - g_*, \delta_N(T,.,\underline{G}^{(0)}, D_x\underline{G}^{(0)}) \right) = 0.$$

Step n. Suppose that we are given the value function $v^{(n-1)}$ of the inferior target problem. Then we will show that on $[0,T) \times \mathbb{R}^d$ the value function $v^{(n)}$ is a viscosity subsolution of the equation

$$\min\left(\sup_{\pi \in \mathcal{N}v^{(n)}} \min\left(-\mathcal{L}^\pi v^{(n)} + \mu_Y(.,.,v^{(n)},\pi), \mathcal{M}^\pi_{v^{(n)}}(v^{(n-1)})^* \right), \delta_N v^{(n)} \right) = 0$$

and, under a local Lipschitz condition concerning the intervention operator applied to $v^{(n-1)}$ (detailed in Section 4.4), a viscosity supersolution of the equation

$$\min\left(\sup_{\pi\in\mathcal{N}v^{(n)}}\min\left(-\mathcal{L}^\pi v^{(n)}+\mu_Y(.,.,v^{(n)},\pi),\mathcal{M}^\pi_{v^{(n)}}(v^{(n-1)})_*\right),\delta_N v^{(n)}\right)=0.$$

The "relevant" terminal data is given as follows: The map $\overline{G}^{(n)}(x):=\limsup_{t\nearrow T,x'\to x}v^{(n)}(t,x')$ is a viscosity subsolution of

$$\min\left(\overline{G}^{(n)}-g^*,\sup_{\pi\in N(T,.,\overline{G}^{(n)},D_x\overline{G}^{(n)})}\mathcal{M}^\pi_{\overline{G}^{(n)}}(v^{(n-1)})^*(T,.),\delta_N(T,.,\overline{G}^{(n)},D_x\overline{G}^{(n)})\right)=0,$$

and the map $\underline{G}^{(n)}(x):=\liminf_{t\nearrow T,x'\to x}v^{(n)}(t,x')$ is a viscosity supersolution of

$$\min\left(\underline{G}^{(n)}-g_*,\sup_{\pi\in N(T,.,\underline{G}^{(n)},D_x\underline{G}^{(n)})}\mathcal{M}^\pi_{\underline{G}^{(n)}}(v^{(n-1)})_*(T,.),\delta_N(T,.,\underline{G}^{(n)},D_x\underline{G}^{(n)})\right)=0.$$

The additional condition in the PDEs in step n in comparison with the PDEs in step 0 shows that in the case of possible perturbation we always have to take care that we do not lose sight of the target due to an immediate impulse intervention. Because such a jump scenario transfers the system into the $(n-1)$-impulse setting, the intervention condition depends on the value function $v^{(n-1)}$ of the inferior problem. This result is very similar to the classic target problem in the presence of stochastically modelled bounded jumps as studied in Bouchard [5]. As only difference, in that case the aftermath of a jump relate to the value function itself. The fact that we consider different PDEs for sub and supersolution is just a consequence of possibly discontinuities of the related value function $v^{(n-1)}$ and the terminal condition g.

Example 4.2.1 (Continuation of Example 4.1.1). Let us test the solution of Example 4.1.1 by verifying the above equations. The continuity assumption on $N(t,x,y,p)=\{\frac{\sigma_X}{\sigma_Y}p\}$ is obviously satisfied. We recall the form of the derived value functions,

$$v^{(1)}(t,x)=v^{(0)}(t,x)+\left(\frac{\sigma_X}{\sigma_Y}-1\right)^+\zeta=x+\left(\mu_X-\mu_Y\frac{\sigma_X}{\sigma_Y}\right)(T-t)+\left(\frac{\sigma_X}{\sigma_Y}-1\right)^+\zeta.$$

Due to the regularity of the value functions, the characterizing PDEs for a viscosity sub and supersolution are reduced to a single PDE which is solved in the classical sense as follows: In view of the design of the state process (X,Y), we compute

$$\begin{aligned}\mathcal{L}^\pi v^{(1)}(t,x)&=\mu_Y\frac{\sigma_X}{\sigma_Y},\\ \mathcal{M}^\pi_y v^{(0)}(t,x)&=y-x+(1-\pi)\zeta-\left(\mu_X-\mu_Y\frac{\sigma_X}{\sigma_Y}\right)(T-t),\\ \mathcal{N}v^{(1)}(t,x)&=\left\{\frac{\sigma_X}{\sigma_Y}\right\}.\end{aligned}$$

Since the set $\mathcal{N}v^{(1)}(t,x)$ is non-empty for all $(t,x) \in [0,T] \times \mathbb{R}$, we may ignore the condition with the function δ_N in the characterizing PDE. Indeed, on $[0,T) \times \mathbb{R}$ our solution satisfies

$$\begin{aligned}\sup_{\pi \in \mathcal{N}v^{(1)}(t,x)} & \min\left(-\mathcal{L}^\pi v^{(1)}(t,x) + \mu_Y \pi, \mathcal{M}^\pi_{v^{(1)}(t,x)} v^{(0)}(t,x)\right) \\ &= \min\left(-\mu_Y \frac{\sigma_X}{\sigma_Y} + \mu_Y \pi, \left(\left(\frac{\sigma_X}{\sigma_Y} - 1\right)^+ + 1 - \pi\right)\zeta\right)\Bigg|_{\pi = \frac{\sigma_X}{\sigma_Y}} = 0.\end{aligned}$$

Moreover, by the continuity of $v^{(n)}$ we have

$$G^{(1)}(x) := \lim_{t \nearrow T, x' \to x} v^{(1)}(t,x') = x + \left(\frac{\sigma_X}{\sigma_Y} - 1\right)^+ \zeta$$

and further

$$\begin{aligned}&G^{(1)}(x) \geq x = g(x), \\ &\mathcal{M}^\pi_{G^{(1)}(x)} v^{(0)}(T,x)\Big|_{\pi = \frac{\sigma_X}{\sigma_Y}} = \left(\left(\frac{\sigma_X}{\sigma_Y} - 1\right)^+ + 1 - \pi\right)\zeta\Bigg|_{\pi = \frac{\sigma_X}{\sigma_Y}} \geq 0.\end{aligned}$$

Noting that $N(T,x,G^{(1)}(x), D_x G^{(1)}(x)) = \{\frac{\sigma_X}{\sigma_Y}\}$, we conclude that $v^{(1)}(T,x) = G^{(1)}(x)$ represents the right terminal data. It remains to verify the PDE characterization of $v^{(0)}$ which is done in the same way.

The plan for the rest of this chapter is as follows: From now on we assume $n \geq 1$. In Section 4.3 we state an appropriate dynamic programming principle (DPP). On this basis we prove in Section 4.4 that the value function $v^{(n)}$ of the stochastic target problem under impulse perturbation is a viscosity solution of a PDE. Here we first show the viscosity property on $[0,T) \times \mathbb{R}^d$. Then we turn to the problem of finding a suitable terminal condition. The very technical proofs of the PDE characterization are sent to Appendix C. The last section of this chapter is devoted to a special setting of the model which allows us to simplify the structure of the PDE characterizing the value function.

4.3 Dynamic programming

In this section we prove a DPP for the stochastic target problem under impulse perturbation as essential tool for the further characterization of the value function.

Before stating the main result of this section, let us make some preliminary remarks: Let $(t,x) \in [0,T) \times \mathbb{R}^d$, $y \in \mathbb{R}$ and $\pi \in \mathcal{U}_0$. For some stopping time $\tau \in [t,T]$ we set $x' := X^\pi_{t,x}(\tau)$, $y' := Y^\pi_{t,x,y}(\tau)$. Then we have $(X^\pi_{t,x}, Y^\pi_{t,x,y}) = (X^\pi_{\tau,x'}, Y^\pi_{\tau,x',y'})$ on $[\tau, T]$. This is due to the strong Markov property of the controlled processes (see Proposition 2.2.1), taking into account that two $\mathcal{F}_\tau$-measurable random variables X_1, X_2 are $\mathbb{P}$-almost equal if and only if $\int \mathbb{1}_A X_1 d\mathbb{P} = \int \mathbb{1}_A X_2 d\mathbb{P}$ holds for all $A \in \mathcal{F}_\tau$. In the same way, for $\pi \in \mathcal{U}_n$, $\pi = (\pi^{(n)}, \dots, \pi^{(0)})$, and $\xi \in \mathcal{V}_n$, $\xi = (\tau_1, \dots, \tau_n; \zeta_1, \dots, \zeta_n)$, we have $(X^{\pi,\xi}_{t,x}, Y^{\pi,\xi}_{t,x,y}) = (X^{\pi',\xi'}_{\tau_1,x'}, Y^{\pi',\xi'}_{\tau_1,x',y'})$ on $[\tau_1, T]$, where

$\pi' = (\pi^{(n-1)}, \dots, \pi^{(0)})$ and $\xi' = (\tau_1, \dots, \tau_n; \zeta_1, \dots, \zeta_n)$ are the continuations of π and ξ after the first intervention.

As a consequence of the foregoing argumentation (and of results concerning the set $\mathcal{U}_0$ in [55]), the definition of $\mathcal{U}_n$ provides stability under concatenation of admissible controls, i.e. for all $\pi_1, \pi_2 \in \mathcal{U}_n$ and each stopping time τ the linked control $\pi_1 \mathbb{1}_{[0,\tau)} + \pi_2 \mathbb{1}_{[\tau,T)}$ is admissible, too. This is the crucial condition for the derivation of the following DPP:

Theorem 4.3.1 (Dynamic programming principle)**.** *Let $(t,x) \in [0,T] \times \mathbb{R}^d$. Then, for the value function $v^{(n)}$ in* (4.3) *we have the following:*

(DP1) If $y > v^{(n)}(t,x)$, then there exists $\pi \in \mathcal{U}_0$ such that for each stopping time $\tau \in [t,T]$,

$$Y^{\pi}_{t,x,y}(\tau) \geq v^{(n)}(\tau, X^{\pi}_{t,x}(\tau)) \text{ and } \mathcal{M}^{\pi(s)}_{Y^{\pi}_{t,x,y}(s)} v^{(n-1)}(s, X^{\pi}_{t,x}(s)) \geq 0 \text{ for all } s \in [t,\tau].$$

(DP2) For all $y < v^{(n)}(t,x)$, $\pi \in \mathcal{U}_0$ and each stopping time $\tau \in [t,T]$ we have

$$\mathbb{P}\left[Y^{\pi}_{t,x,y}(\tau) > v^{(n)}(\tau, X^{\pi}_{t,x}(\tau)) \text{ and } \mathcal{M}^{\pi(s)}_{Y^{\pi}_{t,x,y}(s)} v^{(n-1)}(s, X^{\pi}_{t,x}(s)) > 0 \text{ for all } s \in [t,\tau]\right] < 1.$$

Proof: Let $(t,x) \in [0,T] \times \mathbb{R}^d$.

In a first step we derive the part (DP1) of the DPP. To this end choose $y \in \mathbb{R}$ with $y > v^{(n)}(t,x)$. Then, by the definition of the target problem there exists a control $\pi \in \mathcal{U}_n$, $\pi = (\pi^{(n)}, \dots, \pi^{(0)})$, such that

$$Y^{\pi,\xi}_{t,x,y}(T) \geq g(X^{\pi,\xi}_{t,x}(T)) \quad \text{for all } \xi \in \mathcal{V}_n. \tag{4.4}$$

Choose an arbitrary impulse strategy $\xi \in \mathcal{V}_n$ with $\xi = (\tau_1, \dots, \tau_n; \zeta_1, \dots, \zeta_n)$. At the first intervention time τ_1 the controlled processes $X^{\pi,\xi}_{t,x}$ and $Y^{\pi,\xi}_{t,x,y}$ jump to the values $x' = \Gamma_X(\tau_1, X^{\pi^{(n)}}_{t,x}(\tau_1), \pi^{(n)}(\tau_1), \zeta_1)$ and $y' = \Gamma_Y(\tau_1, X^{\pi^{(n)}}_{t,x}(\tau_1), Y^{\pi^{(n)}}_{t,x,y}(\tau_1), \pi^{(n)}(\tau_1), \zeta_1)$, respectively. Let us denote by $\pi' = (\pi^{(n-1)}, \dots, \pi^{(0)}) \in \mathcal{U}_{n-1}$ and $\xi' = (\tau_2, \dots \tau_n; \zeta_2, \dots, \tau_n)$ the continuations of the strategies π and ξ after the first intervention. In view of this notation we have $(X^{\pi,\xi}_{t,x}, Y^{\pi,\xi}_{t,x,y}) = (X^{\pi',\xi'}_{\tau_1,x'}, Y^{\pi',\xi'}_{\tau_1,x',y'})$ on $[\tau_1, T]$. Thus the condition (4.4) implies

$$Y^{\pi',\xi'}_{\tau_1,x',y'}(T) \geq g(X^{\pi',\xi'}_{\tau_1,x'}(T)).$$

Since we have chosen $\xi \in \mathcal{V}_n$ arbitrarily, this inequality holds for all $\xi' \in \mathcal{V}_{n-1}$. So by the definition of the target problem it follows

$$y' \geq v^{(n-1)}(\tau_1, x').$$

Recall that $x' = \Gamma_X(\tau_1, X^{\pi^{(n)}}_{t,x}(\tau_1), \pi^{(n)}(\tau_1), \zeta_1)$ and $y' = \Gamma_Y(\tau_1, X^{\pi^{(n)}}_{t,x}(\tau_1), Y^{\pi^{(n)}}_{t,x,y}(\tau_1), \pi^{(n)}(\tau_1), \zeta_1)$ with $\tau_1 \in [t,T]$ and $\zeta_1 \in Z \cup \{\zeta_0\}$ chosen arbitrarily. Then we arrive at

$$\mathcal{M}^{\pi(\tau_1)}_{Y^{\pi^{(n)}}_{t,x,y}(\tau_1)} v^{(n-1)}(\tau_1, X^{\pi^{(n)}}_{t,x}(\tau_1)) \geq 0 \text{ for all } \tau_1 \in [t,T]. \tag{4.5}$$

Further, let us introduce another stopping time τ satisfying $\tau \leq \tau_1$. Set $x'' := X^{\pi^{(n)}}_{t,x}(\tau)$ and $y'' := Y^{\pi^{(n)}}_{t,x,y}(\tau)$ and observe that $(X^{\pi,\xi}_{t,x}, Y^{\pi,\xi}_{t,x,y}) = (X^{\pi,\xi}_{\tau,x''}, Y^{\pi,\xi}_{\tau,x'',y''})$ on $[\tau, T]$. In consideration of (4.4) we thus have

$$Y^{\pi,\xi}_{\tau,x'',y''}(T) \geq g(X^{\pi,\xi}_{\tau,x''}(T)).$$

Because this inequality holds for all $\xi \in \mathcal{V}_n$, it yields $y'' \geq v^{(n)}(\tau, x'')$, i.e. by the definition of x'' and y'' we have

$$Y^{\pi^{(n)}}_{t,x,y}(\tau) \geq v^{(n)}(\tau, X^{\pi^{(n)}}_{t,x}(\tau)).$$

This last assertion holds for each stopping time $\tau \in [t, T]$ as it is valid for all $\tau \in [t, \tau_1]$, where the stopping time $\tau_1 \in [t, T]$ is chosen arbitrarily. Hence, together with (4.5) this proves the statement (DP1).

Next we prove (DP2) indirectly. Let us fix some $y \in \mathbb{R}$ and suppose that there exists a control $\pi \in \mathcal{U}_0$ and a stopping time $\tau \in [t, T]$ such that we have almost surely

$$\begin{aligned}
&Y^{\pi}_{t,x,y}(\tau) > v^{(n)}(\tau, X^{\pi}_{t,x}(\tau)) \text{ and} \\
&\mathcal{M}^{\pi(s)}_{Y^{\pi}_{t,x,y}(s)} v^{(n-1)}(s, X^{\pi}_{t,x}(s)) > 0 \text{ for all } s \in [t, \tau].
\end{aligned}$$

Proceeding in this way we have to show that $y \geq v^{(n)}(t, x)$.

For that purpose consider an arbitrary impulse strategy $\xi \in \mathcal{V}_n$ with first intervention (τ_1, ζ_1) letting the controlled processes $X^{\pi}_{t,x}$ and $Y^{\pi}_{t,x,y}$ jump at time τ_1 to $x'' := \Gamma(\tau_1, X^{\pi}_{t,x}(\tau_1), \pi(\tau_1), \zeta_1)$ and $y'' := \Gamma_Y(\tau_1, X^{\pi}_{t,x}(\tau_1), Y^{\pi}_{t,x,y}(\tau_1), \pi(\tau_1), \zeta_1)$, respectively. Set $\theta := \tau \wedge \tau_1$ and $x' := X^{\pi}_{t,x}(\theta)$, $y' := Y^{\pi}_{t,x,y}(\theta)$. By assumption we then have

$$\begin{aligned}
y' &> v^{(n)}(\theta, x') \quad \text{on } \{\theta = \tau\}, \\
y'' &> v^{(n-1)}(\theta, x'') \quad \text{on } \{\theta < \tau\}.
\end{aligned}$$

So by the definition of the value functions $v^{(n)}$ and $v^{(n-1)}$ there exist controls $\pi' \in \mathcal{U}_n$ and $\pi'' \in \mathcal{U}_{n-1}$ such that

$$\begin{aligned}
Y^{\pi',\xi}_{\theta,x',y'}(T) &\geq g(X^{\pi',\xi}_{\theta,x',y'}(T)) \quad \text{on } \{\theta = \tau\}, \\
Y^{\pi'',\xi'}_{\theta,x'',y''}(T) &\geq g(X^{\pi'',\xi'}_{\theta,x'',y''}(T)) \quad \text{on } \{\theta < \tau\}.
\end{aligned}$$

If we define the strategy $\hat{\pi} = (\hat{\pi}^{(n)}, \ldots, \hat{\pi}^{(0)})$ according to

$$\hat{\pi}^{(n)} := \pi \mathbb{1}_{[t,\tau)} + \pi'^{(n)} \mathbb{1}_{[\tau,T]}, \quad (\hat{\pi}^{(n-1)}, \ldots, \hat{\pi}^{(0)}) := \pi''$$

and use the Markov property of the controlled processes, we deduce

$$Y^{\hat{\pi},\xi}_{t,x,y}(T) \geq g(X^{\hat{\pi},\xi}_{t,x,y}(T)) \text{ for all } \xi \in \mathcal{V}_n.$$

Consequently this implies $y \geq v^{(n)}(t, x)$ and therefore proves the assertion (DP2). $\square$

4.4 PDE characterization of the value function

With the DPP derived in the preceding section we are now prepared to formulate $v^{(n)}$ as viscosity solution of a PDE. Our notion of viscosity solutions used here agrees with the concept of discontinuous viscosity solutions as exposed in the appendix. For reasons of clearness let us recall the operators introduced in Section 4.2,

$$\begin{aligned}
\mathcal{L}^\pi\varphi(t,x) &= \frac{\partial\varphi}{\partial t}(t,x) + \mu_X(t,x,\pi)^T D_x\varphi(t,x) + \frac{1}{2}\mathrm{tr}\left((\sigma_X\sigma_X^T)(t,x,\pi)D_x^2\varphi(t,x)\right),\\
\mathcal{M}_y^\pi\varphi(t,x) &= \inf_{\zeta\in Z}\{\Gamma_Y(t,x,y,\pi,\zeta) - \varphi(t,\Gamma_X(t,x,\pi,\zeta))\},\\
\mathcal{N}\varphi(t,x) &= \{\pi\in U : \sigma_Y(t,x,\varphi(t,x),\pi)^T - \sigma_X(t,x,\pi)^T D_x\varphi(t,x) = 0\},\\
\delta_N\varphi(t,x) &= \delta_N(t,x,\varphi(t,x),D_x\varphi(t,x)).
\end{aligned}$$

Here we still have to define the function $\delta_N : [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ in a proper way. To this end consider the set-valued map Ψ on $[0,T]\times\mathbb{R}^d\times\mathbb{R}$ given by

$$\Psi(t,x,y,p) := \{r\in\mathbb{R}^d : r = \sigma_Y(t,x,y,\pi)^T - \sigma_X(t,x,\pi)^T p \text{ for some } \pi\in U\}.$$

We now identify δ_N as the signed distance function from its complement Ψ^c, i.e.

$$\delta_N := \mathrm{dist}(0,\Psi^c) - \mathrm{dist}(0,\Psi),$$

where dist stands for the Euclidean distance. In this way we obtain the desired relation

$$N(t,x,y,p) = \{\pi\in U : \sigma_Y(t,x,y,\pi)^T - \sigma_X(t,x,\pi)^T p = 0\} \neq \varnothing \quad\Leftrightarrow\quad \delta_N(t,x,y,p)\geq 0.$$

For the right treatment of δ_N in our PDE system approach the following result is helpful.

Lemma 4.4.1. *Let Assumption 4.1.1 be satisfied. Then the function δ_N is continuous.*
Proof: By the compactness of U, the continuity of σ_X, σ_Y and Lemma A.1.2 in the appendix, we deduce that $\mathrm{dist}(0,\Psi)$ and $\mathrm{dist}(0,\partial\Psi)$ are continuous in (t,x,y,p). Noting that

$$\delta_N = \begin{cases} \mathrm{dist}(0,\partial\Psi) & ,\ 0\in\Psi,\\ -\mathrm{dist}(0,\Psi) & ,\ 0\in\Psi^c,\end{cases}$$

and that $\delta_N(t_k,x_k,y_k,p_k)\to 0$ for any sequence (t_k,x_k,y_k,p_k) converging to (t_0,x_0,y_0,p_0) with $0\in\partial\Psi(t_0,x_0,y_0,p_0)$, proofs the statement. □

Before we can formulate the main results of this section, we need another lemma which states especially that the intervention operator $\mathcal{M}_y^\pi$ (for $y\in\mathbb{R},\pi\in U$) preserves continuity.

Lemma 4.4.2. *Let $v^{(n-1)}$ be locally bounded on $[0,T]\times\mathbb{R}^d$. Then we have:*

(i) $\mathcal{M}_y^\pi(v^{(n-1)})^*(t,x)$ is lower semicontinuous (lsc) in $(t,x,y,\pi)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}\times U$, and $\mathcal{M}_y^\pi v^{(n-1)}(t,x)\geq\mathcal{M}_y^\pi(v^{(n-1)})^*(t,x)$.

(ii) $\mathcal{M}^\pi((v^{(n-1)})_*(t,x),y)$ *is upper semicontinuous (usc) in* $(t,x,y,\pi) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}\times U$*, and* $\mathcal{M}^\pi_y v^{(n-1)}(t,x) \le \mathcal{M}^\pi_y(v^{(n-1)})_*(t,x)$.

Proof: In view of the properties of semicontinuous functions listed in Appendix A.1, the function to be minimized in the term $\mathcal{M}^\pi_y v(t,x)$ is lsc (resp. usc) if v is usc (resp. lsc). Thus we conclude the semicontinuity statements from Lemma A.1.1 in the appendix. The inequalities are direct consequences of the monotony of the operator $\mathcal{M}^\pi_y$. □

We are now ready for the PDE characterization of the value function.

4.4.1 In the interior of the domain

First we show the viscosity property on $[0,T)\times\mathbb{R}^d$. Theorem 4.4.3 states the subsolution property, and Theorem 4.4.4 states the supersolution property. For the proofs we use similar arguments as the ones in [54] and [7] which we adapt to our context. In particular, for the subsolution property we need the continuity of the set-valued function N (see Assumption 4.2.1), of the signed distance function δ_N (see Lemma 4.4.1), of the intervention operator $\mathcal{M}^\pi_y$ applied to semicontinuous functions (see Lemma 4.4.2) and of the differential operator $\mathcal{L}^\pi$ applied to sufficiently differentiable functions, in order to control the state processes such that the target conditions hold locally. The crucial point in the proof of the supersolution property is to derive that the optimal strategy has to take on values in the set N and, simultaneously, fulfills a jump constraint issued from a possible intervention. Therefore we consider the controlled processes here on time periods whose lengths converge to zero. Since we have no information about the continuity of the control processes, we are led to require that the functions depending on the control values $\pi \in U$ are locally Lipschitz. Thus we can use the Lipschitz condition as an estimate and then go to the limit along some almost surely converging subsequence. The Lipschitz condition for the differential part is included in the initial assumtion (T3). For the intervention part we have to require it explicitly for the negative part of $\mathcal{M}^\pi_y v^{(n-1)}(t,x)$.

The results then read as follows. For the detailed proofs we refer to the appendix.

Theorem 4.4.3. *Let Assumptions 4.1.1 and 4.2.1 be satisfied. Then, on* $[0,T)\times\mathbb{R}^d$ *the value function* $v^{(n)}$ *in* (4.3) *is a viscosity subsolution of the equation*

$$\min\left(\sup_{\pi\in\mathcal{N}v}\min\left(-\mathcal{L}^\pi v + \mu_Y(.,.,v,\pi), \mathcal{M}^\pi_v(v^{(n-1)})^*\right), \delta_N v\right) = 0. \tag{4.6}$$

Proof: See Appendix C.1. □

Theorem 4.4.4. *Let Assumptions 4.1.1 and 4.2.1 be satisfied. Further, let* $\mathcal{M}^\pi_y v^{(n-1)}(t,x)^-$ *be locally Lipschitz continuous in* $(t,x,y)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}$ *uniformly in* $\pi\in U$*. Then, on* $[0,T)\times\mathbb{R}^d$ *the value function* $v^{(n)}$ *in* (4.3) *is a viscosity supersolution of the equation*

$$\min\left(\sup_{\pi\in\mathcal{N}v}\min\left(-\mathcal{L}^\pi v + \mu_Y(.,.,v,\pi), \mathcal{M}^\pi_v(v^{(n-1)})_*\right), \delta_N v\right) = 0. \tag{4.7}$$

Proof: See Appendix C.2. □

Remark 4.4.1. Note that the term $-\mathcal{L}^\pi$ implies a degenerate ellipticity condition in the same way as in the situation of the differential game, see Subsection 3.5.1. Since there are no other second order derivatives included in the above PDEs, the left hand sides of (4.6) and (4.7) are degenerate elliptic. Furthermore, using the convention $\min(\sup \varnothing, a) = a$ for $a \in \mathbb{R}$, it is easy to check that the left hand sides of both PDEs are locally bounded. Finally, by the earlier consideration of the components of both PDEs, under Assumption 4.2.1 the left hand side of (4.6) is lsc and the left hand side of (4.7) is usc in $(t, x, v(t,x), \frac{\partial v}{\partial t}(t,x), D_x v(t,x), D_x^2 v(t,x))$. So the formulation of the PDEs agrees with the problem formulation presented in Appendix A.2.

Remark 4.4.2. The preliminary notes of this section heavily rely on the assumption of a compact control set U, giving us the semicontinuity of the considered PDEs. We further make use of the compactness of U for the convergence of a sequence of controls to an optimal one in the proof of the supersolution property. If we can exclude an exploding strategy as optimal solution, e.g. if $\mathcal{M}^\pi(v^{(n-1)}(t,x)), y) < 0$ for all $(t,x) \in [0,T) \times \mathbb{R}^d$, $y \geq v^{(n-1)}(t,x)$ and large $|\pi|$, then we may even allow for an unrestricted set of control values. Alternatively, we could capture an unbounded set U by passing over to the technique of relaxed semilimits presented by Bouchard, Elie and Touzi [7] resulting in a slightly different, not necessarily continuous PDE for $v^{(n)}$. On the other hand the compactness of U may reflect a constraint imposed by a regulator. So from a practical point of view this requirement is quite reasonable.

4.4.2 Terminal condition

In order to provide a complete characterization of the value function, we need to specify the terminal condition of $v^{(n)}$. According to the definition of $v^{(n)}$, we obviously have

$$v^{(n)}(T,x) = \max\Big(g(x), \inf\{y \in \mathbb{R} : \mathcal{M}_y^\pi v^{(n-1)}(T,x) \geq 0 \text{ for some } \pi \in U\}\Big).$$

However, $v^{(n)}$ may be discontinuous at time T, so that we are led to introduce the functions

$$\overline{G}^{(n)}(x) := \limsup_{t \nearrow T, x' \to x} v^{(n)}(t,x'), \quad \underline{G}^{(n)}(x) := \liminf_{t \nearrow T, x' \to x} v^{(n)}(t,x').$$

Following the argumentation in Bouchard [5] and Saintier [48], it is straightforward to derive the terminal data in our case of a bounded number of jumps. Approximating $\overline{G}^{(n)}, \overline{G}_{(n)}$ from inside the domain and using the viscosity property of the value function there, we extend the stochastic controllability and the intervention condition to the time horizon T. For the sake of completeness, we complemented the appendix by their proofs.

Theorem 4.4.5. *Under the assumptions of Theorem 4.4.3, $\overline{G}^{(n)}$ is a viscosity subsolution of*

$$\min\left(v - g^*, \sup_{\pi \in N(T,.,v,D_x v)} \mathcal{M}_v^\pi (v^{(n-1)})^*(T,.), \delta_N(T,.,v,D_x v)\right) = 0.$$

Proof: See Appendix C.4. □

Theorem 4.4.6. *Under the assumptions of Theorem 4.4.4, $\underline{G}^{(n)}$ is a viscosity supersolution of*

$$\min\left(v - g_*, \sup_{\pi \in N(T,.,v,D_x v)} \mathcal{M}_v^\pi(v^{(n-1)})_*(T,.), \delta_N(T,.,v,D_x v)\right) = 0.$$

Proof: See Appendix C.4. □

4.5 Variant of the model

In this section we consider a special variant of the target problem. It is the model treated in [54] in the pure diffusion case and in [5] in the jump-diffusion case. By a slight change of the model setup we do not need to consider a supremum in the derived PDEs any longer. We put particular emphasize on this modified setting because it represents the foundation for the analysis of the super-hedging problem in finance, see Section 5.3. The design of the model is detailed in the following list of assumptions.

Assumption 4.5.1. *(H1) We have $d = m = p$ (where m is the dimension of the Brownian motion and p the dimension of the control set U).*

(H2) The control set U is convex and has a non-empty interior.

(H3) The matrix $\sigma_X(t,x,\pi)$ is invertible and the map

$$\pi \mapsto \sigma_X^{-1}(t,x,\pi)^T \sigma_Y(t,x,y,\pi)^T$$

is one to one for all $(t,x,y) \in [0,T] \times \mathbb{R}^{d+1}$.

Here, (H3) is the crucial assumption which enables us to match the stochastic parts of the processes X and Y by a definite control $\hat{\pi} \in \mathcal{U}_n$. To state this control explicitly in form of a feedback function, let us denote by ψ the inverse of the map in (H3), i.e.

$$\sigma_Y(t,x,y,\pi)^T - \sigma_X(t,x,\pi)^T p = 0 \quad \Leftrightarrow \quad \pi = \psi(t,x,y,p). \tag{4.8}$$

Notice that due to the continuity of σ_X and σ_Y with respect to π and the compactness of U the function ψ is continuous in (t,x,y,p): Let $(t_k,x_k,y_k,p_k)_k$ be a sequence converging to (t,x,y,p) and $\pi_k = \psi(t_k,x_k,y_k,p_k)$. Then, for any subsequence of $(\pi_k)_k$ there exists a convergent subsequence $(\pi_{kj})_j$ with limit π fulfilling $\pi = \psi(t,x,y,p)$. Since π is uniquely determined, it follows $\pi_k \to \pi$.

Using the notion of this chapter we have $N(t,x,y,p) = \{\psi(t,x,y,p)\}$ if $N(t,x,y,p)$ is non-empty. So, as a consequence of the last fact, (H3) corresponds to Assumption 4.2.1 in Section 4.2. We note further that $\psi(t,x,y,p)$ is only well-defined if

$$p \in D(t,x,y) := \{\sigma_X^{-1}(t,x,\pi)^T \sigma_Y(t,x,y,\pi)^T : \pi \in U\} \subset \mathbb{R}^d.$$

In the following we want to use a continuous continuation of ψ to the domain $[0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ such that $\psi(t,x,y,p) \notin U$ if $p \notin D(t,x,y)$. Thus we conclude

$$\delta_N(t,x,y,p) \geq 0 \quad \Leftrightarrow \quad \psi(t,x,y,p) \in U.$$

We can therefore replace the left inequality as follows: We introduce the support function δ_U of the closed convex set U,

$$\delta_U(x) := \sup_{\pi \in U} x^T \pi, \quad x \in \mathbb{R}^d.$$

Then, using the definition

$$\chi_U(\pi) := \inf_{|x|=1} \{\delta_U(x) - x^T \pi\},$$

we obtain the following characterization of U and ∂U (see, e.g., [54] and the reference therein):

$$\pi \in U \;\Leftrightarrow\; \chi_U(\pi) \geq 0 \quad \text{and} \quad \pi \in \partial U \;\Leftrightarrow\; \chi_U(\pi) = 0.$$

Notice that δ_U as well as χ_U are continuous. This follows from the compactness of U and of the unit sphere of $\mathbb{R}^d$, respectively, and from Lemma A.1.1 in the appendix.

Remark 4.5.1. Example 4.1.1 fits into this framework. The associated feedback function reads $\psi(p) = \frac{\sigma_X}{\sigma_Y} p$.

Combining the considerations in this section with the results of the previous section, we directly obtain the right PDE characterization for the value function.

Corollary 4.5.2. *Let Assumptions 4.1.1 and 4.5.1 be satisfied. Then we have:*

(i) In the interior of the domain. On $[0,T) \times \mathbb{R}^d$ the value function $v^{(n)}$ in (4.3) is a viscosity subsolution of the equation

$$\min\left(-\mathcal{L}^{\hat\pi} v + \mu_Y(.,.,v,\hat\pi), \mathcal{M}_v^{\hat\pi}(v^{(n-1)})^*, \chi_U(\hat\pi)\right) = 0$$

with

$$\hat\pi(t,x) = \psi(t,x,v(t,x),D_x v(t,x)).$$

If in addition $\mathcal{M}_y^{\pi} v^{(n-1)}(t,x)^-$ is locally Lipschitz continuous in $(t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}$ uniformly in $\pi \in U$, then it is also a viscosity supersolution of the equation

$$\min\left(-\mathcal{L}^{\hat\pi} v + \mu_Y(.,.,v,\hat\pi), \mathcal{M}_v^{\hat\pi}(v^{(n-1)})_*, \chi_U(\hat\pi)\right) = 0.$$

(ii) Terminal condition. $\overline{G}^{(n)}(x) := \limsup_{t \nearrow T, x' \to x} v^{(n)}(t,x')$ is a viscosity subsolution of

$$\min\left(v - g^*, \mathcal{M}_v^{\hat\pi}(v^{(n-1)})^*(T,.), \chi_U(\hat\pi)\right) = 0$$

and $\underline{G}^{(n)}(x) := \liminf_{t \nearrow T, x' \to x} v^{(n)}(t,x')$ is a viscosity supersolution of

$$\min\left(v - g_*, \mathcal{M}_v^{\hat\pi}(v^{(n-1)})_*(T,.), \chi_U(\hat\pi)\right) = 0$$

with

$$\hat\pi(x) := \psi(T,x,v(x),D_x v(x)).$$

Chapter 5

Portfolio optimization and option pricing under the threat of a crash

This chapter deals with portfolio problems of an investor taking into consideration the possibility of market crashes. Here the investor distinguishes between "normal times" where the prices of risky assets are supposed to follow a geometric Brownian motion and "crash times" of large price movements. However, he does not know when these crashes take place and how strong they are. So it is his objective to be well prepared for any possible scenario - even the worst one. Therefore the portfolio problems can be represented as stochastic differential games where the investor meets the market as an opponent. The investor tries to maximize the utility of his terminal wealth by choosing a favourable portfolio strategy, while the "unruly" market is able to intervene by creating crash shocks. Thus the portfolio problems can be analyzed using the techniques derived in chapter 3.

As pointed out in the introduction of the impulse perturbed stochastic control framework, the attraction of this approach is manifold. First of all it pays attention to worst-case scenarios. A cautious investor following the advice presented in the following can limit his maximal potential loss. While portfolio optimization in a jump-diffusion model only provides strategies that perform best in the mean, but permit of extreme losses in unlikely market turmoil, the worst-case modelling gives the investor protection in any possible situation. Moreover, the approach comes up with very simple assumptions. In contrast to a jump-diffusion model, probability assumptions concerning the occurence and the extent of crashes are not necessary. Instead, only a maximal number of crashes that might happen up to a given time horizon as well as a range for possible crash sizes are fixed. Trading in addition derivatives on the stocks held in the portfolio, it turns out that it even makes sense to allow for an unrestricted number of crashes.

This worst-case approach to portfolio optimization is already studied in depth in the financial literature. Korn and Wilmott derived in [34] worst-case optimal strategies for an investor with a logarithmic utiliy function by choosing an indifference approach. These results were extended to a more general market setting by Korn and Menkens in [31]. Moreover, they took into consideration changing market coefficients after the occurence of a crash and introduced the notion of crash-hedging strategies which make the investor indifferent to sudden jumps in equity prices. See also Menkens [39] for a detailed analysis of crash-hedging strategies. In [33] Korn and Steffensen used a control approach fitting to the impulse perturbed stochastic control concept as presented in this work. Similar to our proceeding in Chapter 3, they stated a system of inequalities for verifying the value function of the worst-case portfolio problem. This so-called HJB-system corresponds to the PDE characterization given in Section 3.5 for the concerned general differential game. In [29] Korn solved the portfolio problem for an insurer who faces a claim process correlated to the developments on the capital market. Mönnig dealt in [40] with more concrete insurance contracts trying to calculate prices for equity-linked insurances capturing crash risk. A survey of the status quo of the portfolio problem under the threat of market crashes can be found in Korn and Seifried [32], including a recent martingale approach for solving the worst-case portfolio problem.

Here we want to turn to some new aspects of the worst-case approach for modelling market crashes. Firstly we observe the portfolio problem considering in addition options as investment class. Assuming that the option prices do not bear the crash risk in mind, we will show that the crash risk can be reduced or even be eliminated by the employment of suitable options. In a next step the question is tackled whether the price for such derivatives has to be adapted to the additional crash risk. To this end we propose two methods, the concept of super-hedging in the sense of the target problem analyzed in Chapter 4 and a market completion approach. The latter one makes use of a crash insurance which allows to replicate the option exactly. For the calculation of the corresponding insurance premium we argue by utility indifference. Therefore we optimize a portfolio including insurance contracts in the sense of the differential game of Chapter 3 such that the same utility is achieved as without insurance.

This chapter is organized as follows: To start with we repeat in the first section the results from Korn and Steffensen [33] on the portfolio problem under the threat of crashes for a simple market setting where the investor can trade only a risky asset and a secure bond. In Section 5.2 we investigate the extended model that gives the investor the possibility to hedge the crash risk by trading options. Section 5.3 deals with the pricing of derivatives under the threat of crashes and Section 5.4 with the premium calculation for a crash insurance. We conclude in Section 5.5 with an application of the crash-adjusted option pricing technique to defaultable bonds.

5.1 Portfolio with 1 risky asset and n possible crashes

This subsection emphasizes on the introduction of the basic model and sums up the existing solutions for simple worst-case portfolio problems.

Consider a capital market with two tradeable assets, a risk-free bond with constant interest rate $r > 0$ and a risky asset, say a stock. Suppose that in "normal times" the price of the stock is given by

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t) \tag{5.1}$$

with constant market coefficients $\mu > r$ and $\sigma > 0$ and with B as a Brownian motion on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume further that there exists the possibility of a jump in the stock price due to a market crash. A crash of height ζ then leads to a sudden decline in the equity price from S to $(1-\zeta)S$. No probabilistic assumptions are made about the distribution neither of the crash time nor of the crash size. Only suppose that up to the end of the time horizon $T > 0$ at most $n \in \mathbb{N}$ crashes might occur and that the crash heights lie in the interval $Z = [\underline{\zeta}, \bar{\zeta}]$ with $\underline{\zeta} \leq \bar{\zeta}$ and $0 < \bar{\zeta} < 1$.

The investor is allowed to trade in the assets continuously in $[0, T]$. Here every denomination of the assets is possible which particularly permits unrestricted borrowing and even short-selling of the stock. Moreover, transaction costs are neglected. At any time $t \in [0, T]$ the investor has to decide how to distribute his current wealth $X(t) = x \geq 0$ on both assets. Let $\pi^{(i)}(t)$ denote the fraction of his wealth invested in the stock at time t when there are still $i \in \{0, \ldots, n\}$ remaining crash possibilities. Since consumption and liabilities to pay are not considered, this automatically determines the bond investment as the remaining wealth $X(t)(1 - \pi^{(i)}(t))$. So the investor's wealth process is of the form

$$\begin{aligned} dX(t) &= (r + (\mu - r)\pi^{(n-i)}(t))X(t)dt + \sigma\pi^{(n-i)}(t)X(t)dB(t), \\ &\qquad\qquad \tau_i < t < \tau_{i+1}, \quad i = 0, \ldots, n, \\ X(\tau_i) &= X(\tau_i -)(1 - \zeta_i \pi^{(n-i+1)}(\tau_i)), \quad i = 1, \ldots, n, \end{aligned}$$

where τ_i is the i-th crash time with associated crash size ζ_i (where we make the conventions $\tau_0 := 0$ and $\tau_{n+1} := T$). Note that $X = X^{\pi,\xi}$ depends on the portfolio strategy $\pi = (\pi^{(n)}, \ldots, \pi^{(0)})$ and the crash parameters $\xi = (\tau_1, \ldots, \tau_n; \zeta_1, \ldots, \zeta_n)$. Nevertheless, in the following X is used most of the time as abbreviation for $X^{\pi,\xi}$ if the connections are obvious.

The separate portfolio strategies $\pi^{(i)}$ depend on the $\mathbb{P}$-augmentation $(\mathcal{F}_t)_t$ of the filtration generated by the Brownian motion B. According to our control framework of Chapter 2, we require that the process $\pi^{(i)}$ has values in a compact set

$$U \subset \begin{cases} (-\infty, \bar{\zeta}^{-1}], & \underline{\zeta} \geq 0, \\ [\underline{\zeta}^{-1}, \bar{\zeta}^{-1}], & \underline{\zeta} < 0 \end{cases}$$

and is progressively measurable with respect to $(\mathcal{F}_t)_t$ and Markovian. The choice of the control set U ensures that the wealth process stays non-negative. In practice U can be fixed afterwards, so that the optimal strategies are contained. The double sequence ξ of crash parameters can be interpreted as n-impulse control with $Z \cup \{0\}$-valued impulses which is assumed to be Markovian, too. We denote the corresponding admissibility sets by $\mathcal{U}_n$

and $\mathcal{V}_n$. Under the above assumptions on the crash model, decisions are made in order to perform best in a worst-case scenario. As performance measure consider the expected utility of the investor's wealth at time T using a general utility function $g : \mathbb{R} \to \mathbb{R}$, i.e. g is once continuously differentiable, monotonously increasing and strictly concave. Then the *worst-case scenario portfolio problem* can be stated as

$$\sup_{\pi \in \mathcal{U}_n} \inf_{\xi \in \mathcal{V}_n} \mathbb{E}_{t,x} \left[g\left(X^{\pi,\xi}(\tau_{\mathcal{S}}) \right) \right],$$

where the stopping time $\tau_{\mathcal{S}} = \inf\{s \geq t : (s, X^{\pi,\xi}(s)) \notin \mathcal{S} = [0,T) \times (0,\infty)\}$ takes into account an early ruin of the investor. It is our objective to determine the corresponding value function $v^{(n)}$ and to find optimal controls $\hat{\pi} \in \mathcal{U}_n$ and $\hat{\xi} \in \mathcal{V}_n$ such that

$$v^{(n)}(t,x) = \mathbb{E}_{t,x} \left[g\left(X^{\hat{\pi},\hat{\xi}}(\tau_{\mathcal{S}}) \right) \right].$$

Here we do not have to consider $v^{(n)}$ outside $\bar{\mathcal{S}} = [0,T] \times [0,\infty)$ because we a priori excluded jumps outside this domain by the made restriction on the control set U. We solve the problem by using the results of Section 3.6. To this end let us briefly check the necessary assumptions: In view of the simple stochastic model with constant SDE coefficients, the condition (G1') of Assumption 3.6.1 is satisfied. If we assume that the utility function g is bounded below on $[0,\infty)$, the assumption (G2') is true, too. Furthermore, (G3') holds by the definition of $\mathcal{U}_n$ and $\mathcal{V}_n$. The validity of Assumption 3.6.2 is obvious except for the continuity of $v^{(n-1)}$, and the statements of Assumption 3.6.3 follow from the argumentations in Appendix B. Supposed $v^{(n-1)}$ is continuous, $v^{(n)}$ is locally bounded and there exist ε-optimal controls with continuous feedback function (see Assumption 3.2.1), Theorem 3.6.4 says that on $\mathcal{S}$ the seeked function $v^{(n)}$ is a viscosity solution of the non-linear PDE

$$\inf_{\pi \in U} \max\left(-\mathcal{L}^{\pi} v^{(n)}, v^{(n)} - \mathcal{M}^{\pi} v^{(n-1)} \right) = 0, \tag{5.2}$$

where the differential operator $\mathcal{L}^{\pi}$ and the impulse operator $\mathcal{M}^{\pi}$ are given by

$$\begin{aligned} \mathcal{L}^{\pi}\varphi(t,x) &= \varphi_t(t,x) + (r + (\mu - r)\pi)x\varphi_x(t,x) + \frac{1}{2}\sigma^2\pi^2x^2\varphi_{xx}(t,x), \\ \mathcal{M}^{\pi}\varphi(t,x) &= \inf_{\zeta \in [\underline{\zeta},\bar{\zeta}]} \varphi(t, x(1-\zeta\pi)). \end{aligned}$$

To define $v^{(n)}$ as the unique solution of 5.2, we consider the boundary conditions

$$\begin{aligned} v^{(n)}(T,x) &= \min\left(g(x), \sup_{\pi \in U} \mathcal{M}^{\pi} v^{(n-1)}(T,x) \right), \quad x \in [0,\infty), \\ v^{(n)}(t,0) &= g(0), \quad t \in [0,T]. \end{aligned} \tag{5.3}$$

Noting that we do not allow for stochastic jumps in the stock price process, the only critical point with respect to the applicability of Corollary 3.6.7 is (U3) in Assumption 3.6.5. Let us suppose that $v^{(n)}$ is at most polynomially growing with exponent $m > 0$. Then it is not very difficult to compute that the auxiliary function $w(t,x) = e^{\lambda(T-t)}x^{m+1} + C(1+T-t)$, with an arbitrary constant $C > 0$ and λ chosen large enough, has all properties we need. Consequently, if there exists a polynomially bounded viscosity solution of 5.2 satisfying 5.3, then it is the seeked value function $v^{(n)}$.

Of course, in the simple market setting dealt with in this subsection the worst crash is always of maximal size as long as the investor holds a long position in the stock. On the other hand it is never optimal to invest a negative sum in the stock. Obviously the worst-case scenario would then be a sudden rise in the equity price or the no crash case. But in both scenarios the pure bond strategy $\pi \equiv 0$ which is resistant to any market movements performs better. Even if no crash happens, the pure bond strategy yields a higher worst-case bound because of $\mu > r$. Therefore, for $\pi \geq 0$, the worst first crash is given by

$$\hat{\tau}_1 = \inf\left\{ s \geq t : v^{(n-1)}\left(s, X^\pi(s)(1-\bar{\zeta}\pi(s))\right) \leq v^{(n)}\left(s, X^\pi(s)\right)\right\} \wedge T,$$

$$\hat{\zeta}_1 = \begin{cases} \bar{\zeta}\ , & \text{if } v^{(n-1)}\left(\hat{\tau}_1, X^\pi(\hat{\tau}_1)(1-\bar{\zeta}\pi(\hat{\tau}_1))\right) \leq v^{(n)}\left(\hat{\tau}_1, X^\pi(\hat{\tau}_1)\right), \\ 0\ , & \text{else.} \end{cases}$$

Here X^π denotes the wealth process under the current portfolio strategy $\pi = \pi^{(n)}$ and without any crash, and it is assumed that the investor trades optimally after the first crash. An optimal portfolio strategy can be obtained in form of (assuming this expression is well-defined)

$$\hat{\pi}^{(n)}(t,x) \in \arg \max_{\pi \in U^{(n)}(t,x)} \mathcal{L}^\pi v^{(n)}(t,x),$$

where

$$U^{(n)}(t,x) := \{\pi \in U : v^{(n-1)}(t, x - \bar{\zeta}\pi) \geq v^{(n)}(t,x)\}.$$

The following example summarizes concrete results on the value functions and the optimal portfolio strategies for various utility functions which are adopted from Korn and Steffensen [33].

Example 5.1.1. (a) Power utility: Consider a utility function of the form

$$g(x) = \frac{1}{\gamma}x^\gamma \quad \text{with } \gamma < 1, \gamma \neq 0.$$

Then the optimal portfolio strategy $\hat{\pi} = (\hat{\pi}^{(n)}, \ldots, \hat{\pi}^{(0)})$ only depends on the time variable t and can be calculated iteratively by solving the non-linear ordinary differential equations (ODEs)

$$\hat{\pi}_t^{(i)} = \frac{1}{\bar{\zeta}}\left(1 - \bar{\zeta}\hat{\pi}^{(i)}\right)\left((\mu - r)\left(\hat{\pi}^{(i)} - \hat{\pi}^{(i-1)}\right) - \frac{1}{2}(1-\gamma)\sigma^2\left(\left(\hat{\pi}^{(i)}\right)^2 - \left(\hat{\pi}^{(i-1)}\right)^2\right)\right),$$
$$\hat{\pi}^{(i)}(T) = 0, \quad i = 1, \ldots, n,$$

where the optimal strategy in the crash-free setting reads

$$\hat{\pi}^{(0)} = \frac{1}{1-\gamma}\frac{\mu - r}{\sigma^2}.$$

The corresponding value function is given by

$$v^{(n)}(t,x) = v^{(0)}\left(t, x\prod_{i=1}^{n}\left(1-\bar{\zeta}\hat{\pi}^{(i)}(t)\right)\right),$$

where

$$v^{(0)}(t,x) = \frac{1}{\gamma}x^{\gamma}\exp\left(\gamma\left(r+\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2\frac{1}{1-\gamma}\right)(T-t)\right)$$

is the value function in the crash-free setting.

(b) Log utility: Consider the logarithmic utility function

$$g(x) = \ln(1+x).$$

As in the case of power utility the optimal portfolio strategy only depends on the time variable t and is given by the system of non-linear ODEs

$$\hat{\pi}_t^{(i)} = \frac{1}{\bar{\zeta}}\left(1-\bar{\zeta}\hat{\pi}^{(i)}\right)\left((\mu-r)\left(\hat{\pi}^{(i)}-\hat{\pi}^{(i-1)}\right)-\frac{1}{2}\sigma^2\left(\left(\hat{\pi}^{(i)}\right)^2-\left(\hat{\pi}^{(i-1)}\right)^2\right)\right),$$
$$\hat{\pi}^{(i)}(T) = 0, \quad i=1,\dots,n,$$

where the optimal strategy in the crash-free setting reads

$$\hat{\pi}^{(0)} = \frac{\mu-r}{\sigma^2}.$$

The corresponding value function is given by

$$v^{(n)}(t,x) = v^{(0)}\left(t, x\prod_{i=1}^{n}(1-\bar{\zeta}\hat{\pi}^{(i)}(t))\right),$$

where

$$v^{(0)}(t,x) = \ln(1+x)+\left(r+\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2\right)(T-t)$$

is the value function in the crash-free setting.

(c) Exponential utility: Consider the exponential utility function

$$g(x) = -e^{-\alpha x} \quad \text{for some } \alpha > 0.$$

An investor with this utility function has a constant absolute risk aversion which means that the total amount of his wealth invested risky is independent of his wealth. Therefore it looks attractive to interprete the portfolio strategy π as the total sum of the stock investment. Then the optimal portfolio strategy can be stated as solution of the system of non-linear ODEs

$$\hat{\pi}_t^{(i)} = r\hat{\pi}^{(i)}+\frac{\mu-r}{\bar{\zeta}}\left(\hat{\pi}^{(i)}-\hat{\pi}^{(i-1)}\right)-\frac{\alpha e^{r(T-t)}\sigma^2}{2\bar{\zeta}}\left(\left(\hat{\pi}^{(i)}\right)^2-\left(\hat{\pi}^{(i-1)}\right)^2\right),$$
$$\hat{\pi}^{(i)}(T) = 0, \quad i=1,\dots,n,$$

where the optimal strategy in the crash-free setting reads

$$\hat{\pi}^{(0)}(t) = \frac{e^{-r(T-t)}}{\alpha} \frac{\mu - r}{\sigma^2}.$$

The corresponding value function is given by

$$v^{(n)}(t,x) = v^{(0)}\left(t, x - \sum_{i=1}^{n} \bar{\zeta}\hat{\pi}^{(i)}(t)\right),$$

where

$$v^{(0)}(t,x) = -\exp\left(-\alpha e^{r(T-t)}x - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 (T-t)\right)$$

is the value function in the crash-free setting. For $n = 1$ there exists an explicit solution which has the form

$$\hat{\pi}^{(1)}(t) = \frac{e^{-r(T-t)}}{\alpha} \frac{\mu - r}{\sigma^2} \frac{(\mu - r)(T-t)}{(\mu - r)(T-t) + 2\bar{\zeta}},$$

$$v^{(1)}(t,x) = -\exp\left(-\alpha e^{r(T-t)}x - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{(\mu - r)(T-t)^2}{(\mu - r)(T-t) + 2\bar{\zeta}}\right).$$

Remark 5.1.1. (a) The forms of the value function $v^{(n)}$ in the above examples show that the worst-case scenario is always a sudden crash of maximal size $\bar{\zeta}$.

(b) The explicit formula $\hat{\pi}^{(1)} = \frac{(\mu-r)(T-t)}{(\mu-r)(T-t)+2\bar{\zeta}}\hat{\pi}^{(0)}$ for the optimal strategy in the case of exponential utility reveils $\hat{\pi}^{(1)} < \hat{\pi}^{(0)}$. Indeed, one can show via induction that the solutions of the ODEs characterizing the optimal strategy in the case of power utility, log utility and exponential utility satisfy the relation

$$0 \leq \hat{\pi}^{(n)} \leq \hat{\pi}^{(n-1)} \leq \ldots \leq \hat{\pi}^{(0)}.$$

This is a very intuitive result since the more crashes might happen, the less money should be invested in the stock. By the way it shows that the optimal control processes take on values in a compact subset U. Moreover, in contrast to the constant (resp. time-discounted) portfolio processes $\hat{\pi}^{(0)}$ in the no-crash model, the crash strategies $\hat{\pi}^{(i)}$, $i = 1, \ldots, n$, turn out to be strictly decreasing until no stocks are held in the portfolio at closure time T. The reason can be found in the fact at time T there is no time left to compensate for losses. Since these properties of the optimal portfolio strategy $\hat{\pi}$ are only stated in [33] without a proof, we provide the proof in Appendix B.3.

(c) The semi-explicit formulas for the value functions show that their strict increase and their strict concavity with respect to the wealth variable are inherited in the crash setting from their counterparts in the crash-free setting.

(d) In consideration of the continuity of the portfolio processes as well as the continuity and the polynomial growth of the value functions, the assumptions made beforehand are indeed satisfied.

Note that without great difficulty the results of this section can be extended to a market with several risky assets. In this case we deal with a price process modelled by a multi-dimensional geometric Brownian motion. To keep the notation as clear as possible, we want to stick to the one-stock-model. However, in the following investigations we consider a derivative of the stock as additional risky asset.

5.2 Crash-hedging via options

Now suppose that the investor additionally has the possibility to trade with options on the stock as underlying in order to limit his exposure in the course of a crash. For example he might invest in a put option which is negatively correlated with the stock. Therefore he will be compensated if the stock price declines. On the other hand a rise in the equity price might then reduce his investment gains. So once again he is faced with a kind of balance problem.

Let $P = P(t, S)$ denote the price of such an option depending on the current state $S(t) = S$ of the price process (5.1). Throughout this section we assume that the price function P is once continuously differentiable with respect to the time variable and twice with respect to the stock price variable. Applying Itô's formula yields

$$\begin{aligned} dP(t) = &\left(P_t(t, S(t)) + \mu S(t) P_S(t, S(t)) + \frac{1}{2}\sigma^2 S^2(t) P_{SS}(t, S(t))\right) dt \\ &+ \sigma S(t) P_S(t, S(t)) dB(t). \end{aligned}$$

Furthermore, it is a well known fact that in the crash-free setting the option price P solves the Black-Scholes differential equation

$$-rP + P_t + rSP_S + \frac{1}{2}\sigma^2 S^2 P_{SS} = 0.$$

Designating $\delta^{(i)}(t)$ as the ratio of the number of options that are held in the portfolio and the present wealth at time t when there are still $i \in \{0, \dots, n\}$ remaining crash possibilities, the corresponding wealth process of the investor can thus be stated by

$$\begin{aligned} dX(t) &= \left(1 - \pi^{(n-i)}(t) - \delta^{(n-i)} P(t)\right) rX(t)dt + \frac{\pi^{(n-i)}(t)X(t)}{S(t)} dS(t) + \delta^{(n-i)} X(t) dP(t) \\ &= \left(r + (\mu - r)\left(\pi^{(n-i)}(t) + \delta^{(n-i)}(t) S(t) P_S(t, S(t))\right)\right) X(t) dt \\ &\quad + \sigma\left(\pi^{(n-i)}(t) + \delta^{(n-i)}(t) S(t) P_S(t, S(t))\right) X(t) dB(t) \end{aligned}$$

for $\tau_i < t < \tau_{i+1}$, $i = 0, \dots, n$, and at the i-th crash time, $i = 1, \dots, n$,

$$X(\tau_i) = X(\tau_i -)\left(1 - \zeta_i \pi^{(n-i+1)}(\tau_i) - \delta^{(n-i+1)}(P(\tau_i, S(\tau_i)) - P(\tau_i, (1 - \zeta_i) S(\tau_i)))\right).$$

We are now interested in solving the portfolio problem

$$\sup_{(\pi,\delta)\in\mathcal{U}_n} \inf_{\xi\in\mathcal{V}_n} \mathbb{E}_{t,x}\left[g\left(X^{(\pi,\delta),\xi}(\tau_S)\right)\right],$$

where g stands for one of the utility functions considered in Example 5.1.1, and $\mathcal{U}_n$ is the modified admissibility set of vectors of control processes $(\pi^{(i)}, \delta^{(i)})$, $i = 0, \ldots, n$, which have values in a compact subset $U \subset \mathbb{R}^2$ such that $(\pi, \delta) \in U$ satisfies

$$\zeta\pi - \delta(P(t,S) - P(t,(1-\zeta)S)) \leq 1 \quad \text{for all } (t,S) \in [0,T] \times [0,\infty), \zeta \in [\underline{\zeta}, \bar{\zeta}].$$

The differential operator $\mathcal{L}^{\pi,\delta}$ and the impulse operator $\mathcal{M}^{\pi,\delta}$ associated to the state process $(X(t), S(t))$ are defined by

$$\begin{aligned}
\mathcal{L}^{\pi,\delta}\varphi(t,x,S) &= \varphi_t(t,x,S) + (r + (\mu - r)(\pi + \delta S P_S(t,S)))\, x\varphi_x(t,x,S) \\
&\quad + \mu S\varphi_S(t,x,S) + \frac{1}{2}\sigma^2 (\pi + \delta S P_S(t,S))^2 x^2 \varphi_{xx}(t,x,S) \\
&\quad + \sigma^2 (\pi + \delta S P_S(t,S))\, xS\varphi_{xS}(t,x,S) + \frac{1}{2}\sigma^2 S^2 \varphi_{SS}(t,x,S), \\
\mathcal{M}^{\pi,\delta}\varphi(t,x,S) &= \inf_{\zeta\in[\underline{\zeta},\bar{\zeta}]} \varphi(t, x(1 - \zeta\pi - \delta(P(t,S) - P(t,(1-\zeta)S))), (1-\zeta)S).
\end{aligned}$$

Then, supposed $v^{(n)}$ is polynomially bounded, it is the unique polynomially bounded and continuous viscosity solution of the PDE (5.2) satisfying the boundary conditions (5.3), but with control variable $(\pi, \delta) \in U$ and the operators $\mathcal{L}^{\pi,\delta}$, $\mathcal{M}^{\pi,\delta}$.

Assuming sufficient regularity and strict concavity of the value function $v^{(n)}$, the first-order condition for a maximum $(\hat{\pi}^{(n)}, \hat{\delta}^{(n)})$ of $\mathcal{L}^{\pi,\delta}v^{(n)}(t,x,S)$ reads as

$$\begin{aligned}
&\sigma^2 x^2 v_{xx}^{(n)} \hat{\pi}^{(n)} + (\mu - r) x v_x^{(n)} + \sigma^2 x S v_{xS}^{(n)} + \sigma^2 x^2 S P_S v_{xx}^{(n)} \hat{\delta}^{(n)} = 0 \\
\wedge \quad &\sigma^2 x^2 S^2 P_S^2 v_{xx}^{(n)} \hat{\delta}^{(n)} + (\mu - r) x S P_S v_x^{(n)} + \sigma^2 x S^2 P_S v_{xS}^{(n)} + \sigma^2 x^2 S P_S v_{xx}^{(n)} \hat{\pi}^{(n)} = 0,
\end{aligned}$$

where the partial derivatives $v_x^{(n)}$, $v_{xx}^{(n)}$ and $v_{xS}^{(n)}$ are evaluated in (t,x,S) and P_S in (t,S). Resolving the first equation for $\hat{\pi}^{(n)}$ gives

$$\hat{\pi}^{(n)} = -\frac{\mu - r}{\sigma^2} \frac{v_x^{(n)}}{x v_{xx}^{(n)}} - S\frac{v_{xS}^{(n)}}{x v_{xx}^{(n)}} - \hat{\delta}^{(n)} S P_S, \tag{5.4}$$

while the left hand side of the second equation disappears by inserting (5.4). So $\mathcal{L}^{\pi,\delta}v^{(n)}(t,x,S)$ is maximal for $\hat{\pi}^{(n)}$ as in (5.4), where $\hat{\delta}^{(n)}$ can be chosen arbitrarily. Thus the question arises how many options the investor is intended to hold. In a crash-free setting it is no matter whether to invest in the stock or in the option as long as the proper proportion between stocks and options, expressed in (5.4), is maintained. So for $n = 0$ the value function $v^{(0)}$ is independent of the stock price S and equals the value function $v^{(0)}$ from the previous section where no options are traded. But taking into consideration the threat of a crash, not any delta hedge is sufficient any more. To obtain an optimal mixture of $\hat{\pi}^{(n)}$ and $\hat{\delta}^{(n)}$ in the crash model, we have to consider the constraint

$$\mathcal{M}^{\hat{\pi}^{(n)},\hat{\delta}^{(n)}} v^{(n-1)}(t,x,S) \geq v^{(n)}(t,x,S).$$

In view of (5.4) it follows

$$v^{(n-1)}(t, x(1-\zeta\bar{\pi} - \hat{\delta}^{(n)}\Delta), (1-\zeta)S) \geq v^{(n)}(t,x,S) \quad \text{for all } \zeta \in [\underline{\zeta}, \bar{\zeta}],$$

where we have set

$$\begin{aligned} \bar{\pi} = \bar{\pi}(t,x,S) &:= -\frac{\mu - r}{\sigma^2}\frac{v_x^{(n)}}{xv_{xx}^{(n)}} - S\frac{v_{xS}^{(n)}}{xv_{xx}^{(n)}}, \\ \Delta = \Delta(t,S,\zeta) &:= P(t,S) - P(t,(1-\zeta)S) - \zeta S P_S(t,S). \end{aligned}$$

If we can find a $\hat{\delta}^{(n)}$ such that

$$\zeta\bar{\pi} + \hat{\delta}^{(n)}\Delta(t,S,\zeta) \leq 0 \quad \text{for all } \zeta \in [\underline{\zeta}, \bar{\zeta}], \tag{5.5}$$

the investor's wealth cannot be affected by large price movements. Since the objective function only depends on the wealth process, this would mean that the investor can realize the same utility as in the crash-free world, i.e. $v^{(n)}(t,x,S) = v^{(0)}(t,x)$. In particular, our initial assumptions on the regularity and concavity of the value function would be true, and $\bar{\pi}$ would correspond to the optimal strategy without crash risk and option trading, $\hat{\pi}^{(0)} = -\frac{\mu-r}{\sigma^2}\frac{v_x^{(0)}}{xv_{xx}^{(0)}} > 0$. In order to observe the hedging condition (5.5) let us suppose that the price function P is strictly convex which is the case if we deal with standard call or put options. Therefore we know that $\Delta(t,S,\zeta) < 0$ for all $\zeta \neq 0$. Then (5.5) is equivalent to

$$\hat{\delta}^{(n)} \geq -\frac{\zeta}{\Delta(t,S,\zeta)}\bar{\pi} \quad \text{for all } \zeta \neq 0. \tag{5.6}$$

For $\bar{\pi} > 0$ the right hand side of the above inequality is negative for $\zeta < 0$, and it is strictly decreasing in $\zeta > 0$ because of

$$\frac{d}{d\zeta}\frac{\zeta}{\Delta(t,S,\zeta)} = \frac{P(t,S) - P(t,(1-\zeta)S) - \zeta S P_S(t,(1-\zeta)S)}{\Delta^2(t,S,\zeta)} > 0.$$

So on the one hand we do not have to regard upward jumps by choosing $\hat{\delta}^{(n)} \geq 0$. On the other hand, if we neglect infinitesimal jump sizes, i.e. $\underline{\zeta} > 0$, then the maximum for the right hand side in (5.6) is attained for $\zeta = \underline{\zeta}$. So we can conclude that by trading according to

$$\begin{aligned} \hat{\delta} &= \frac{\underline{\zeta}}{\underline{P} - P + \underline{\zeta}SP_S}\hat{\pi}^{(0)}, \\ \hat{\pi} &= \hat{\pi}^{(0)} - \hat{\delta}SP_S = \frac{\underline{P} - P}{\underline{P} - P + \underline{\zeta}SP_S}\hat{\pi}^{(0)}, \end{aligned}$$

where $\underline{P}$ stands for $P(t,(1-\underline{\zeta})S)$, the investor can eliminate any crash risk without additional costs. Note in particular that this strategy is independent of the number of possible crashes.

But in contrast to the classic portfolio strategy $\hat{\pi}^{(0)}$, the hedging strategy $(\hat{\pi}, \hat{\delta})$ depends on S, although the value function does not. For $\underline{\zeta} \leq 0$ a full hedge is not possible any more since we have

$$\lim_{\zeta \searrow 0} -\frac{\zeta}{\Delta(t, S, \zeta)}\bar{\pi} = \infty.$$

But choosing $\hat{\delta}$ as large as possible (in the set of constrained admissible trading strategies, e.g. $\delta \in [\underline{\delta}, \bar{\delta}]$), we can reduce the negative effects of crashes.

So the message of this section reads as follows: While in the classic Black-Scholes model it is of no additional benefit to trade with options, we have seen that under the threat of crashes this extension of the market gives the investor the chance to compensate losses due to a crash scenario, so that he is even able to perform as well as in the classic Black-Scholes model. That is he can hedge any crash risk. Moreover, the limiting number n of possible crashes does not play any role, neither on the value function nor on the optimal investment strategies.

5.3 Option pricing

In the previous section we have seen that under Black-Scholes option prices the crash risk can be hedged completely. That might raise the question if Black-Scholes prices really fit in our crash framework. Black-Scholes prices are obtained by reconstruction of the option's payoff in the complete crash-free market. But the crash feature leads to an incompletion of the market, so that the replication technique does not work any more. In this section we want to discuss two different approaches leading to upper (or lower) price boundaries and to indifference prices. We only consider European options, i.e. the option's buyer is not allowed to execute the option prematurely.

5.3.1 Super-hedging

For the super-hedging approach we consider a portfolio including a position (long or short) in an option. The central idea is to find a stock trading strategy and an option pricing rule that offer at least the risk-free return almost surely for any market scenario. To start with we derive the PDEs for these super-hedging option prices in an heuristic way. Afterwards we prove the obtained statements with the theoretical tools provided in Chapter 4 for stochastic target problems.

Heuristic derivation of the super-hedging prices

We take a portfolio consisting of positions in the bond, the stock and a fixed number δ of options as basis. Let us denote by $P^{(n)}$ the crash-adjusted option price and by $\pi^{(n)}$ the total stock investment sum in the setting with n crashes. To state the corresponding wealth process X we suppose enough regularity of $P^{(n)}$. In view of Itô's formula the dynamic of the wealth process with n remaining crash possibilities then reads

$$\begin{aligned} dX(t) &= \left(X(t) - \pi^{(n)}(t) - \delta P^{(n)}\right) r dt + \frac{\pi^{(n)}(t)}{S(t)} dS(t) + \delta dP^{(n)}(t) \\ &= \left(rX(t) + (\mu - r)\pi^{(n)}(t) + \delta\left(-rP^{(n)} + P_t^{(n)} + \mu S P_S^{(n)} + \frac{1}{2}\sigma^2 S^2 P_{SS}^{(n)}\right)\right) dt \\ &\quad + \sigma\left(\pi^{(n)}(t) + \delta S P_S^{(n)}\right) dB(t), \end{aligned}$$

where $P^{(n)}$ and its partial derivatives are evaluated in $(t, S(t))$. Given a crash of size ζ at time τ there is a jump in form of

$$X(\tau) = X(\tau-) - \zeta\pi^{(n)}(\tau) - \delta\left(P^{(n)}(\tau, S(\tau)) - P^{(n-1)}(\tau, (1-\zeta)S(\tau))\right).$$

Now we want to adapt the replication method for complete markets to our crash setting. It is our purpose to construct a portfolio that is risk-free along the worst-case path. For such a risk-free portfolio we have to take into consideration the evolution of the wealth process if no crash occurs as well as the jump condition for a crash scenario. In order to exclude stochastic fluctuation driven by the Brownian motion, we have to choose

$$\pi^{(n)}(t) = -\delta S P_S^{(n)}. \tag{5.7}$$

To avoid losses from possible jumps we require

$$\sup_{\zeta \in [\underline{\zeta}, \bar{\zeta}]} \left\{\zeta\pi^{(n)}(t) + \delta\left(P^{(n)} - P^{(n-1)}(t, (1-\zeta)S(t))\right)\right\} \leq 0, \tag{5.8}$$

where equality holds if a crash embodies the worst-case scenario. To achieve the risk-free yield, the drift term in the wealth's SDE must be greater than $rX(t)$ almost surely at any time t because otherwise there are scenarios with a less return. This gives us the next condition,

$$(\mu - r)\pi^{(n)}(t) + \delta\left(-rP^{(n)} + P_t^{(n)} + \mu S P_S^{(n)} + \frac{1}{2}\sigma^2 S^2 P_{SS}^{(n)}\right) \geq 0, \tag{5.9}$$

where equality holds if the worst case is no immediate crash. Assembling the conditions (5.7)-(5.9) we conclude

$$\min\left(rP^{(n)} - P_t^{(n)} - rSP_S^{(n)} - \frac{1}{2}\sigma^2 S^2 P_{SS}^{(n)}, P^{(n)} - \sup_{\zeta \in [\underline{\zeta}, \bar{\zeta}]}\left\{\zeta S P_S^{(n)} + P^{(n-1)}(t, (1-\zeta)S)\right\}\right) = 0$$

for $\delta < 0$ (short position in option) and

$$\max\left(rP^{(n)} - P_t^{(n)} - rSP_S^{(n)} - \frac{1}{2}\sigma^2 S^2 P_{SS}^{(n)}, P^{(n)} - \inf_{\zeta \in [\underline{\zeta}, \bar{\zeta}]}\left\{\zeta S P_S^{(n)} + P^{(n-1)}(t, (1-\zeta)S)\right\}\right) = 0$$

for $\delta > 0$ (long position). Note that apart from the sign of δ the pricing equations are independent of the portfolio's position in options. The first equation presents the seller's price $P_{short}^{(n)}$ for the option and therefore an upper bound. The second one can be regarded as the buyer's price $P_{long}^{(n)}$ for the option and therefore as a lower bound.

Remark 5.3.1. Using the price rules of this subsection, crash-hedging is not possible any more: The hedging condition (5.5) cannot be satisfied since

$$\delta\Delta^{(n)}(t,S,\zeta) = \delta\left(P^{(n)}(t,S) - \zeta S P_S^{(n)}(t,S) - P^{(n-1)}(t,(1-\zeta)S)\right) \geq 0 \text{ for all } \delta.$$

Here we have to use the seller's option price for $\delta > 0$ and the buyer's option price for $\delta < 0$ because these are the prices for which the counterparty accepts the trade.

Stochastic target approach

If we choose $\delta = -1$ (short position in one option) and the initial capital $X(t) = 0$ and split the replicating portfolio into the option position $-P_{short}^{(n)}$ and the super-hedging portfolio Y, then the seller's option price at time t corresponds to the initial value of the process Y. Since we force $X(T) = 0$ along the worst-case path, $P_{short}^{(n)}$ is the minimal amount of money with which the option's seller can set up a hedging portfolio Y that exceeds the option's payoff at maturity almost surely. That is the option price is the value function of a stochastic target problem. The controlled target process is the seller's hedging process Y and the target is given by the payoff function depending on the stock price process S. Remember that the dynamics of (S,Y) in "calm" times $t \in (\tau_i, \tau_{i+1})$, $i = 0,\dots,n$, are given by

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S(t)dB(t), \\ dY(t) &= (rY(t) + (\mu - r)\pi^{(n-i)}(t))dt + \sigma\pi^{(n-i)}(t)dB(t), \end{aligned}$$

whereas at crash times τ_i, $i = 1,\dots,n$, we have jumps in form of

$$S(\tau_i) = S(\tau_i-)(1-\zeta_i), \qquad Y(\tau_i) = Y(\tau_i-) - \zeta_i \pi^{(n-i+1)}(\tau_i).$$

Once more is to point out that in contrast to the initially introduced portfolio problem we want to understand by the control $\pi^{(i)}$ the absolute investment sum. Moreover, let us consider here the unconstrained control set $U = \mathbb{R}$. Therefore we impose in addition that an admissible strategy $\pi \in \mathcal{U}_n$ provides further on a unique well-defined hedging process $Y_{t,S}^{\pi,\xi}$ for all $(t,S) \in [0,T] \times [0,\infty)$ and $\xi \in \mathcal{V}_n$. Then the crash-adjusted seller's option price can be formulated by

$$P_{short}^{(n)}(t,S) = \inf\{y \in \mathbb{R} : \text{ there exists } \pi \in \mathcal{U}_n \text{ s.t. } Y^{\pi,\xi}(T) \geq g(S^\xi(T)) \text{ for all } \xi \in \mathcal{V}_n\},$$

where g is the option's payoff function, assumed to be continuous in the following. To keep the notation short we just write $P^{(n)}$ instead of $P_{short}^{(n)}$ from now on.

Let us now use the theory of stochastic target problems from Chapter 4. The situation here corresponds for the most part to the model discussed in Section 4.5. The hedging strategy $\hat{\pi}$, which is essential in order to control the stochastic price movements driven by the Brownian motion, is uniquely identified by the function $\psi(S,p) = Sp$ in the sense of (supposed the derivatives exist)

$$\hat{\pi}^{(i)}(t) = \psi(S(t), P_S^{(i)}(t,S(t))) = S(t)P_S^{(i)}(t,S(t)), \quad i = 0,\dots,n.$$

However, we have to elaborate on two differences to the target problem of Section 4.5:

- *U is not compact.* We needed the compactness of U in Chapter 4 for three aspects. Firstly, for the existence of a solution of the controlled SDEs which is guaranteed in the situation here by the definition of the admissibility set $\mathcal{U}_n$. Secondly, for the preservation of continuity properties of the terms building the characterizing PDEs which applies because of the continuity of the function ψ. And thirdly, for the existence of a convergent subsequence in the proof of the supersolution property. This is satisfied if a "nearly optimal" hedging strategy π has values in a compact subset in a local sense, i.e. as long as $(s, S_{t,S}(s), Y^{\pi}_{t,y}(s))$ stays within some open neighborhood of (t, S, y). This condition is true if we deal with payoff functions $g \in \mathcal{P}_1([0,\infty))$, i.e. there exists $C > 0$ such that $|g(S)| \leq C(1+S)$. In this case we a-priori can restrict the hedge positions to compacts depending on the current underlying price S because it does not make sense to trade more than C shares, i.e. $\pi \in [-CS, CS]$. In particular, this gives us $|P^{(i)}(t,S)| \leq C(1+S)$, i.e. $P^{(i)} \in \mathcal{P}_1([0,T]\times[0,\infty))$. In sum, the viscosity property of Corollary 4.5.2 still holds, but we do not have to pay attention to the condition $\pi \in U$.

- *The domain of control is restricted to $\mathcal{S} = [0,T) \times [0,\infty)$.* The viscosity properties of the price functions in the domain $\mathcal{S}$ as well as on the boundary part $\{T\}\times(0,\infty)$ remain true. In addition we need boundary conditions for $S = 0$ in order to characterize the price functions as the unique solutions of the associated PDEs. Since the stock would remain worthless if the price once dropped to zero, the natural boundary condition in the no-crash setting is given by $P^{(0)}(t,0) = e^{-r(T-t)}g(0)$. This is also the natural boundary condition if a crash is possible because at the boundary $[0,T]\times\{0\}$ an impulse ζ has no impact. It remains to clarify if the natural boundary condition represents the correct data for a precise characterization of the price functions. This is the case if the price functions are continuous on the boundary part $[0,T]\times\{0\}$. To avoid a tedious proof of a general statement, let us be content with a simple justification for the use of the natural boundary condition in the case of a call and a put option. For both option types we know a-priori estimates. If $P^{(i)}$ denotes the price of a call with strike K, then we have $P^{(i)}(t,S) \in [0,S]$ and therefore $\lim_{\mathcal{S}\ni(t',S')\to(t,0)} P^{(i)}(t',S') = 0$. If $P^{(i)}$ denotes the price of a put with strike K, then we have $P^{(i)}(t,S) \in [Ke^{-r(T-t)} - S, Ke^{-r(T-t)}]$ and therefore $\lim_{\mathcal{S}\ni(t',S')\to(t,0)} P^{(i)}(t',S') = Ke^{-r(T-t)}$.

As it is known, in the no crash case the stochastic target approach leads to Black-Scholes prices, i.e. $P^{(0)}$ is linked to the system

$$
\begin{aligned}
rP^{(0)} - P^{(0)}_t - rSP^{(0)}_S - \frac{1}{2}\sigma^2 S^2 P^{(0)}_{SS} &= 0 \quad \text{on } \mathcal{S},\\
P^{(0)}(T,S) - g(S) &= 0 \quad \text{for } S \in [0,\infty),\\
P^{(0)}(t,0) - e^{-r(T-t)}g(0) &= 0 \quad \text{for } t \in [0,T).
\end{aligned}
$$

From the Feynman-Kac theorem we know that under weak assumptions on g there exists a unique classical solution $P^{(0)} \in \mathcal{C}^{1,2}(\mathcal{S}) \cap \mathcal{C}(\bar{\mathcal{S}})$ which is polynomially bounded, for example if g is continuous, non-negative and satisfies a polynomial growth condition (see, e.g., Karatzas and Shreve [25], Theorem 5.7.6 and Remark 5.7.8, or Korn and Korn [30], Chapter III, Theorem 18).

Under crash risk the result is the following:

Theorem 5.3.1. *Let $g \in \mathcal{P}_1([0,\infty))$ and assume that $P^{(n-1)}$ is continuous and locally Lipschitz in (t,x). Then the seller's option price $P^{(n)}$ is a viscosity solution on $\mathcal{S}$ of the PDE*

$$\min\Bigg(rP^{(n)} - P_t^{(n)} - rSP_S^{(n)} - \frac{1}{2}\sigma^2 S^2 P_{SS}^{(n)}, \\ P^{(n)} - \sup_{\zeta\in[\underline{\zeta},\bar{\zeta}]}\left\{\zeta S P_S^{(n)} + P^{(n-1)}(t,(1-\zeta)S)\right\}\Bigg) = 0. \tag{5.10}$$

The terminal data given in form of $\overline{G}^{(n)}(S) := \limsup_{t\nearrow T, S'\to S} P^{(n)}(t,S')$ and $\underline{G}^{(n)}(S) := \liminf_{t\nearrow T, S'\to S} P^{(n)}(t,S')$ is a the viscosity sub and supersolution, respectively, on $(0,\infty)$ of

$$\min\left(G^{(n)} - g, G^{(n)} - \sup_{\zeta\in[\underline{\zeta},\bar{\zeta}]}\left\{\zeta S G_S^{(n)} + P^{(n-1)}(T,(1-\zeta)S)\right\}\right) = 0. \tag{5.11}$$

Proof: In view of the argumentation preceding this theorem, the statement is a consequence of Corollary 4.5.2. We remark that the assumptions underlying Corollary 4.5.2 are evidently satisfied, in particular the required local Lipschitz condition on the map $(t,S,y) \mapsto \sup_{\zeta\in[\underline{\zeta},\bar{\zeta}]}\left\{y - \zeta\pi + P^{(n-1)}(t,(1-\zeta)S)\right\}^-$ follows from the local Lipschitz continuity of $P^{(n-1)}$. □

So Theorem 5.3.1 confirms the heuristic PDE characterization. In the same way we can formulate for $\delta = 1$ the super-hedging problem from the point of view of the option's buyer. The result is the target problem

$$P_{long}^{(n)}(t,S) = \sup\{y \in \mathbb{R} : \text{ there exists } \pi \in \mathcal{U}_n \text{ s.t. } -Y^{\pi,\xi}(T) \geq -g(S^\xi(T)) \text{ for all } \xi \in \mathcal{V}_n\}.$$

Writing this in form of

$$P_{long}^{(n)}(t,S) = -\inf\{y \in \mathbb{R} : \text{ there exists } \pi \in \mathcal{U}_n \text{ s.t. } Y^{\pi,\xi}(T) \geq -g(S^\xi(T)) \text{ for all } \xi \in \mathcal{V}_n\},$$

we come to the PDE for the buyer's option price as in the heuristic derivation. The proper terminal data is given by

$$\max\left(G^{(n)} - g, G^{(n)} - \inf_{\zeta\in[\underline{\zeta},\bar{\zeta}]}\left\{\zeta S g_S^{(n)} + P^{(n-1)}(T,(1-\zeta)S)\right\}\right) = 0.$$

Due to the similar structure of both pricing rules, we only analyze the seller's option price in detail.

Remark 5.3.2 (The worst-crash size). If the price function $P^{(n-1)}$ is strictly convex, the worst downward jump is related to the maximal crash size $\bar{\zeta} > 0$ and the worst upward jump to the minimal crash size $\underline{\zeta} < 0$. So in the case where both downward and upward jumps are possible (i.e. $\underline{\zeta} < 0$, $\bar{\zeta} > 0$) we have to answer the question when $\underline{\zeta}$ embodies the worst crash size and when $\bar{\zeta}$ does so, i.e. the PDE (5.10) simplifies to

$$\min\left(rP^{(n)}-P_t^{(n)}-rSP_S^{(n)}-\frac{1}{2}\sigma^2S^2P_{SS}^{(n)},P^{(n)}-\max_{\zeta\in\{\underline{\zeta},\bar{\zeta}\}}\left\{\zeta SP_S^{(n)}+P^{(n-1)}(t,(1-\zeta)S)\right\}\right)=0.$$

If either negative or positive jump sizes are excluded, the above max-term can be replaced by the relevant argument. For a price function $P^{(n-1)}$ which is not strictly convex, the worst-crash size might also be an inner point.

To prove uniqueness of a viscosity solution to the PDE (5.10), we need the following comparison result. As in the proof of the comparison theorem for the PDE considering in the context of the differential game in Chapter 3, we make use of the doubling of variables approach from Crandall, Ishii and Lions [10], so that most parts of the proof are equal to the proof of Theorem 3.6.6. We therefore only sketch the steps which do not require new arguments. Because of $P^{(n)} \in \mathcal{P}_1(\bar{\mathcal{S}})$ it is sufficient to show uniqueness in the class $\mathcal{P}_1(\bar{\mathcal{S}})$.

Theorem 5.3.2 (Comparison theorem 1)**.** *Let $u \in \mathcal{P}_1(\bar{\mathcal{S}})$ be a viscosity subsolution and $v \in \mathcal{P}_1(\bar{\mathcal{S}})$ a viscosity supersolution of* (5.10) *on $\mathcal{S}$. If we have $u^* \le v_*$ on the parabolic boundary $\partial\mathcal{S} = ([0,T) \times \{0\}) \cup (\{T\} \times [0,\infty))$, then it follows $u^* \le v_*$ on $\bar{\mathcal{S}}$.*

Proof: We argue by contradiction and suppose that

$$\sup_{(t,x)\in[0,T]\times[0,\infty)} \{u^*(t,x) - v_*(t,x)\} > 0.$$

In view of the growth conditions on u and v, we can find some $\beta, \delta, \lambda, \rho > 0$ such that

$$0 < M := \sup_{(t,x)\in[0,T]\times[0,\infty)} \{u^*(t,x) - v_*(t,x) - \frac{\beta}{t+\rho} - 2\delta e^{\lambda(T-t)}x^2\} < \infty.$$

For $k \in \mathbb{N}$ we set

$$\varphi_k(t,x,y) := \frac{\gamma}{t+\rho} + \frac{k}{2}|x-y|^2 + \delta e^{\lambda(T-t)}(x^2+y^2),$$

and we consider

$$M_k := \sup_{(t,x,y)\in[0,T]\times[0,\infty)^2} \{u^*(t,x) - v_*(t,y) - \varphi_k(t,x,y)\}$$

with $0 < M_k < \infty$. This supremum is attained on some compact subset of $[0,T] \times [0,\infty)^2$, independent of large k. We denote by (t_k, x_k, y_k) a corresponding maximizer. According to the relation of u^* and v_* on the parabolic boundary, we have $(t_k, x_k, y_k) \in [0,T) \times (0,\infty)^2$. From Lemma 3.1 in [10] (see Lemma A.3.2 in the appendix) we know that

$$\lim_{k\to\infty} k|x_k - y_k|^2 = 0. \tag{5.12}$$

Thus we may pass to a subsequence (t_k, x_k, y_k) converging to $(t_0, x_0, x_0) \in [0, T) \times (0, \infty)^2$.

Next we apply Theorem 8.3 in [10] (see Theorem A.3.1 in the appendix) to the function $w(t, x, y) = u^*(t, x) - v_*(t, y) - \varphi_k(t, x, y)$ at point (t_k, x_k, y_k) to get $a_k \in \mathbb{R}$ and $X_k, Y_k \in \mathbb{R}$ such that

$$(a_k, p_k, \tilde{X}_k) \in \bar{\mathcal{J}}^{2,+} u^*(t_k, x_k), \qquad (b_k, q_k, \tilde{Y}_k) \in \bar{\mathcal{J}}^{2,-} v_*(t_k, y_k),$$

where

$$\begin{aligned} b_k &:= a_k + \frac{\beta}{(t_k + \rho)^2} + \lambda \delta e^{\lambda(T - t_k)}(x_k^2 + y_k^2), \\ p_k &:= k(x_k - y_k) + 2\delta e^{\lambda(T - t_k)} x_k, \quad \tilde{X}_k := X_k + 2\lambda\delta e^{(T - t_k)}, \\ q_k &:= k(x_k - y_k) - 2\delta e^{\lambda(T - t_k)} y_k, \quad \tilde{Y}_k := Y_k - 2\lambda\delta e^{(T - t_k)} \end{aligned}$$

and X_k, Y_k satisfy

$$X_k x^2 - Y_k y^2 \le k(x - y)^2 \quad \text{for all } x, y \ge 0. \tag{5.13}$$

The sub and supersolution properties of u and v yield

$$\begin{aligned} \min\Bigg(& r u^*(t_k, x_k) - a_k - r x_k p_k - \frac{1}{2}\sigma^2 x_k^2 \tilde{X}_k, \\ & u^*(t_k, x_k) - \sup_{\zeta \in [\underline{\zeta}, \bar{\zeta}]} \Big\{ \zeta x_k p_k + P^{(n-1)}(t_k, (1 - \zeta) x_k) \Big\} \Bigg) \le 0, \end{aligned} \tag{5.14}$$

$$\begin{aligned} \min\Bigg(& r v_*(t_k, y_k) - b_k - r y_k q_k - \frac{1}{2}\sigma^2 y_k^2 \tilde{Y}_k, \\ & v_*(t_k, y_k) - \sup_{\zeta \in [\underline{\zeta}, \bar{\zeta}]} \Big\{ \zeta y_k q_k + P^{(n-1)}(t_k, (1 - \zeta) y_k) \Big\} \Bigg) \ge 0. \end{aligned} \tag{5.15}$$

Consider the difference Λ_k of the first components of the above min-terms and let us split Λ_k in form of $\Lambda_k = A_k^1 + A_k^2 - B_k$ with

$$\begin{aligned} A_k^1 &:= r(u^*(t_k, x_k) - v_*(t_k, y_k)) + \frac{\beta}{(t_k + \rho)^2}, \\ A_k^2 &:= \lambda \delta e^{\lambda(T - t_k)}(x_k^2 + y_k^2) - (2r + \sigma^2)\delta e^{\lambda(T - t_k)}(x_k^2 + y_k^2), \\ B_k &:= rk(x_k - y_k)^2 + \frac{1}{2}\sigma^2(X_k x_k^2 - Y_k y_k^2). \end{aligned}$$

Using the fact that (t_k, x_k, y_k) is a maximizer corresponding to $M_k > 0$, we have $A_k^1 > 0$ for all k. Choosing λ sufficiently large, i.e. $\lambda \ge 2r + \sigma^2$, we have $A_k^2 \ge 0$ for all k. From (5.13) we obtain the estimate $B_k \le (r + \frac{1}{2}\sigma^2)k(x_k - y_k)^2$. Hence, by (5.13) we conclude $\Lambda_k > 0$ for k chosen large enough. Thus we deduce from the viscosity inequalities (5.14)-(5.15) that

$$u^*(t_k, x_k) - \sup_{\zeta \in [\underline{\zeta}, \bar{\zeta}]} \Big\{ \zeta x_k p_k + P^{(n-1)}(t_k, (1 - \zeta) x_k) \Big\} \le 0. \tag{5.16}$$

On the other hand, using the fact $M_k > 0$ and afterwards the inequality (5.15), we deduce

$$\begin{aligned} u^*(t_k, x_k) &- \sup_{\zeta\in[\underline{\zeta},\bar{\zeta}]} \left\{\zeta x_k p_k + P^{(n-1)}(t_k,(1-\zeta)x_k)\right\} \\ &\geq v_*(t_k,y_k) + \varphi_k(t_k,x_k,y_k) - \sup_{\zeta\in[\underline{\zeta},\bar{\zeta}]} \left\{\zeta x_k p_k + P^{(n-1)}(t_k,(1-\zeta)x_k)\right\} + \frac{M}{2} \\ &\geq \varphi_k(t_k,x_k,y_k) - \sup_{\zeta\in[\underline{\zeta},\bar{\zeta}]} \{\zeta(x_k p_k - y_k q_k)\} \\ &\quad - \sup_{\zeta\in[\underline{\zeta},\bar{\zeta}]} \left\{P^{(n-1)}(t_k,(1-\zeta)x_k) - P^{(n-1)}(T,(1-\zeta)y_k)\right\} + \frac{M}{2}. \end{aligned}$$

In view of (5.12) and the continuity of $P^{(n-1)}$, the second of the last two suprema can be neglected. For the first one we have

$$\sup_{\zeta\in[\underline{\zeta},\bar{\zeta}]} \{\zeta(x_k p_k - y_k q_k)\} \leq \bar{\zeta}\left(k(x_k-y_k)^2 + 2\delta e^{\lambda(T-t_k)}(x_k^2+y_k^2)\right).$$

Consequently, choosing k large enough leads to a contradiction to (5.16). □

Theorem 5.3.2 says that if we have continuous boundary conditions for $P^{(n)}$ on $\partial\mathcal{S}$, then $P^{(n)}$ is the unique continuous viscosity solution of (5.10) in $\mathcal{P}_1(\bar{\mathcal{S}})$. So let us continue with a uniqueness result for the terminal data which implicitly gives us continuity of the terminal data if $P^{(n)}$ is continuous in $(T, 0)$.

Theorem 5.3.3 (Comparison theorem 2). *Let $u \in \mathcal{P}_1([0,\infty))$ be a viscosity subsolution and $v \in \mathcal{P}_1([0,\infty))$ a viscosity supersolution of* (5.11) *on* $(0,\infty)$. *If we have $u^*(0) \leq v_*(0)$, then it follows $u^* \leq v_*$ on* $[0,\infty)$.

Proof: Analogous to the proof of Theorem 5.3.2. □

Remark 5.3.3. We can write equation (5.11) in form of $G^{(n)} - H(S, G^{(n)}) = 0$. Comparison results for such elliptic first-order equations already exist, even for an unbounded domain, see for example Section 1.2 in Ishii [23]. But these results only compare continuous viscosity solutions. So if we are not sure that $G^{(n)}$ is continuous, we cannot apply them for a uniqueness statement. Instead, the continuity of $G^{(n)}$ is just an implication of the comparison theorem.

To simplify matters, from now on let us concentrate on the special case $n = 1$ and on options which are linear combinations of standard calls and puts. Then the regularity of $P^{(0)}$ gives us the Lipschitz condition required in Theorem 5.3.1 and, as shown above, we are allowed to use the natural boundary condition for $S = 0$. In sum, the price function $P^{(1)}$ is the unique viscosity solution in $\mathcal{P}_1(\bar{\mathcal{S}})$ of (5.10) which satisfies the boundary conditions

$$\begin{aligned} P^{(1)}(t,0) - e^{-r(T-t)}g(0) &= 0, \quad t \in [0,T], \\ P^{(1)}(T,S) - G^{(1)}(S) &= 0, \quad S \in (0,\infty), \end{aligned}$$

where the terminal data $G^{(1)}$ is the unique viscosity solution on $(0,\infty)$ of (5.11) with $G^{(1)}(0) = g(0)$, and $G^{(1)}$ and $P^{(1)}$ are continuous.

For a sample of plain vanilla options here are the explicit terminal data functions (with $g^{(0)} := g$, $g^{(1)} := G^{(1)}$):

(a) Call option with strike K (i.e. $g^{(0)}(S) = (S-K)^+$):

crash parameters	terminal data function
$\bar\zeta \geq \underline\zeta \geq 0$, $\bar\zeta > 0$	$g^{(1)}(S) = \begin{cases} \bar\alpha S^{\frac{1}{\bar\zeta}} & , S \leq \frac{K}{1-\bar\zeta} \\ S-K & , S > \frac{K}{1-\bar\zeta} \end{cases}$, $\bar\alpha := \bar\zeta \left(\frac{1-\bar\zeta}{K}\right)^{\frac{1-\bar\zeta}{\bar\zeta}}$
$\underline\zeta \leq \bar\zeta \leq 0$, $\underline\zeta < 0$	$g^{(1)}(S) = \begin{cases} 0 & , S \leq \frac{K}{1-\underline\zeta} \\ \underline\alpha S^{\frac{1}{\underline\zeta}} + S - K & , S > \frac{K}{1-\underline\zeta} \end{cases}$, $\underline\alpha := -\underline\zeta \left(\frac{1-\underline\zeta}{K}\right)^{\frac{1-\underline\zeta}{\underline\zeta}}$
$\underline\zeta < 0$, $\bar\zeta > 0$	$g^{(1)}(S) = \begin{cases} \bar\alpha S^{\frac{1}{\bar\zeta}} & , S \leq K \\ \underline\alpha S^{\frac{1}{\underline\zeta}} + S - K & , S > K, \end{cases}$, $\bar\alpha := -\frac{\bar\zeta\underline\zeta}{\bar\zeta-\underline\zeta} K^{-\frac{1-\bar\zeta}{\bar\zeta}}$, $\underline\alpha := -\frac{\bar\zeta\underline\zeta}{\bar\zeta-\underline\zeta} K^{-\frac{1-\underline\zeta}{\underline\zeta}}$

(b) Put option with strike K (i.e. $g^{(0)}(S) = (K-S)^+$):

crash parameters	terminal data function
$\bar\zeta \geq \underline\zeta \geq 0$, $\bar\zeta > 0$	$g^{(1)}(S) = \begin{cases} \bar\alpha S^{\frac{1}{\bar\zeta}} - S + K & , S \leq \frac{K}{1-\bar\zeta} \\ 0 & , S > \frac{K}{1-\bar\zeta} \end{cases}$, $\bar\alpha := \bar\zeta \left(\frac{1-\bar\zeta}{K}\right)^{\frac{1-\bar\zeta}{\bar\zeta}}$
$\underline\zeta \leq \bar\zeta \leq 0$, $\underline\zeta < 0$	$g^{(1)}(S) = \begin{cases} K-S & , S \leq \frac{K}{1-\underline\zeta} \\ \underline\alpha S^{\frac{1}{\underline\zeta}} & , S > \frac{K}{1-\underline\zeta} \end{cases}$, $\underline\alpha := -\underline\zeta \left(\frac{1-\underline\zeta}{K}\right)^{\frac{1-\underline\zeta}{\underline\zeta}}$
$\underline\zeta < 0$, $\bar\zeta > 0$	$g^{(1)}(S) = \begin{cases} \bar\alpha S^{\frac{1}{\bar\zeta}} - S + K & , S \leq K \\ \underline\alpha S^{\frac{1}{\underline\zeta}} & , S > K \end{cases}$, $\bar\alpha := -\frac{\bar\zeta\underline\zeta}{\bar\zeta-\underline\zeta} K^{-\frac{1-\bar\zeta}{\bar\zeta}}$, $\underline\alpha := -\frac{\bar\zeta\underline\zeta}{\bar\zeta-\underline\zeta} K^{-\frac{1-\underline\zeta}{\underline\zeta}}$

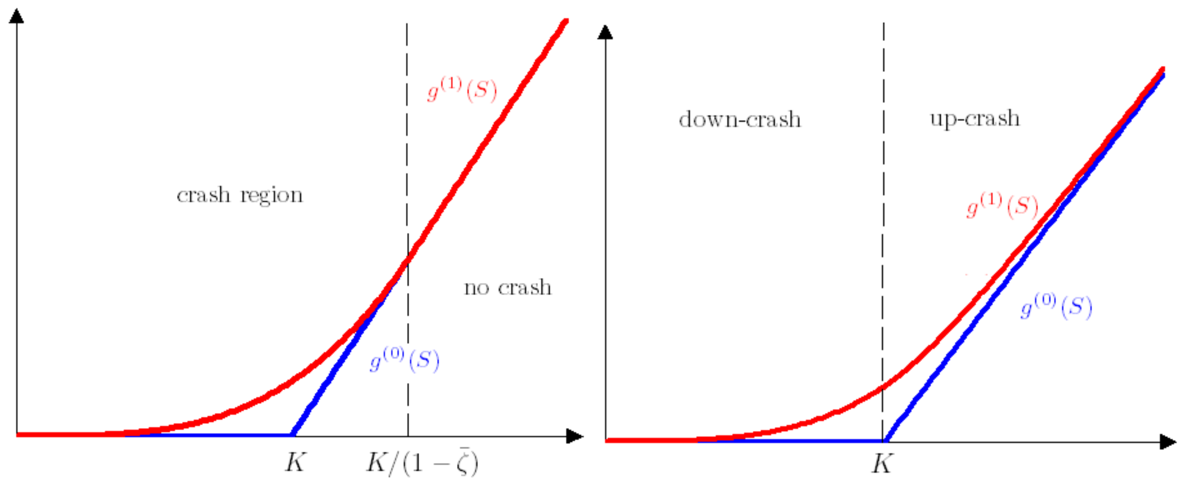

Figure 5.1: Terminal data for a call. On the left: Only downward jumps possible. On the right: Downward and upward jumps possible.

(c) Bull spread with strikes $K_1 < K_2$ (i.e. $g^{(0)}(S) = (S - K_1)^+ - (S - K_2)^+$):

crash parameters	terminal data function
$\bar\zeta \ge \underline{\zeta} \ge 0$, $\bar\zeta > 0$, $K_1 \le (1-\bar\zeta)K_2$	$g^{(1)}(S) = \begin{cases} \bar\alpha S^{\frac{1}{\bar\zeta}} & , S \le \frac{K_1}{1-\bar\zeta} \\ S - K_1 & , \frac{K_1}{1-\bar\zeta} < S \le K_2 \\ K_2 - K_1 & , S > K_2 \end{cases}, \ \bar\alpha := \bar\zeta \left(\frac{1-\bar\zeta}{K_1}\right)^{\frac{1-\bar\zeta}{\bar\zeta}}$
$\bar\zeta \ge \underline{\zeta} \ge 0$, $\bar\zeta > 0$, $K_1 > (1-\bar\zeta)K_2$	$g^{(1)}(S) = \begin{cases} \bar\alpha S^{\frac{1}{\bar\zeta}} & , S \le K_2 \\ K_2 - K_1 & , S > K_2 \end{cases}, \ \bar\alpha := K_2^{-\frac{1}{\bar\zeta}}(K_2 - K_1)$
$\underline{\zeta} \le \bar\zeta \le 0$, $\underline{\zeta} < 0$	$g^{(1)}(S) = \begin{cases} 0 & , S \le \frac{K_1}{1-\underline{\zeta}} \\ \underline{\alpha} S^{\frac{1}{\underline{\zeta}}} + S - K_1 & , \frac{K_1}{1-\underline{\zeta}} < S \le \frac{K_2}{1-\underline{\zeta}} \\ \underline{\tilde\alpha}(S - K_2) + K_2 - K_1 & , \frac{K_2}{1-\underline{\zeta}} < S \le \frac{K_2}{1-\bar\zeta} \\ \max(\tilde{\bar\alpha} S^{\frac{1}{\bar\zeta}} + K_2, S) - K_1 & , \frac{K_2}{1-\bar\zeta} < S \le K_2 \\ K_2 - K_1 & , S > K_2 \end{cases},$ $\underline{\alpha} := -\underline{\zeta}\left(\frac{1-\underline{\zeta}}{K_1}\right)^{\frac{1-\underline{\zeta}}{\underline{\zeta}}}, \quad \underline{\tilde\alpha} := 1 - \left(\frac{K_2}{K_1}\right)^{\frac{1-\underline{\zeta}}{\underline{\zeta}}}, \quad \tilde{\bar\alpha} := \bar\zeta\underline{\tilde\alpha}\left(\frac{1-\bar\zeta}{K_2}\right)^{\frac{1-\bar\zeta}{\bar\zeta}}$
$\underline{\zeta} < 0$, $\bar\zeta > 0$, $(1-\underline{\zeta})K_1 \le K_2$	$g^{(1)}(S) = \begin{cases} \bar\alpha S^{\frac{1}{\bar\zeta}} & , S \le K_1 \\ \underline{\alpha} S^{\frac{1}{\underline{\zeta}}} + S - K_1 & , K_1 < S \le \frac{K_2}{1-\underline{\zeta}} \\ \underline{\tilde\alpha}(S - K_2) + K_2 - K_1 & , \frac{K_2}{1-\underline{\zeta}} < S \le K_2 \\ K_2 - K_1 & , S > K_2 \end{cases},$ $\bar\alpha := -\frac{\bar\zeta\underline{\zeta}}{\bar\zeta-\underline{\zeta}} K_1^{-\frac{1-\bar\zeta}{\bar\zeta}}, \quad \underline{\alpha} := -\frac{\bar\zeta\underline{\zeta}}{\bar\zeta-\underline{\zeta}} K_1^{-\frac{1-\underline{\zeta}}{\underline{\zeta}}}, \quad \underline{\tilde\alpha} := 1 + \frac{1}{\underline{\zeta}}\underline{\alpha}\left(\frac{K_2}{1-\underline{\zeta}}\right)^{\frac{1-\underline{\zeta}}{\underline{\zeta}}}$
$\underline{\zeta} < 0$, $\bar\zeta > 0$, $(1-\underline{\zeta})K_1 > K_2$	$g^{(1)}(S) = \begin{cases} \bar\alpha S^{\frac{1}{\bar\zeta}} & , S \le \frac{K_2}{1-\underline{\zeta}} \\ \underline{\tilde\alpha}(S - K_2) + K_2 - K_1 & , \frac{K_2}{1-\underline{\zeta}} < S \le K_2 \\ K_2 - K_1 & , S > K_2 \end{cases},$ $\bar\alpha := -\frac{\bar\zeta}{\bar\zeta-\underline{\zeta}}(K_2 - K_1)\left(\frac{1-\underline{\zeta}}{K_2}\right)^{\frac{1}{\bar\zeta}}, \quad \underline{\tilde\alpha} := \frac{1}{\bar\zeta-\underline{\zeta}}(K_2 - K_1)\frac{1-\underline{\zeta}}{K_2}$

Numerical example. Consider the following options:

- Call with strike $K = 100$.
- Bull spread with strikes $K_1 = 80$ and $K_2 = 150$.
- Time to maturity in each case is $T = 1$.
- The parameters for the bond and the risky asset are $r = 0.05$, $\sigma = 0.3$, $\underline{\zeta} = -0.3$ and $\bar\zeta = 0.3$.

For the numerical treatment of the variational inequality we use a finite difference scheme and the PSOR (projected successive over-relaxation) method. The boundary conditions for

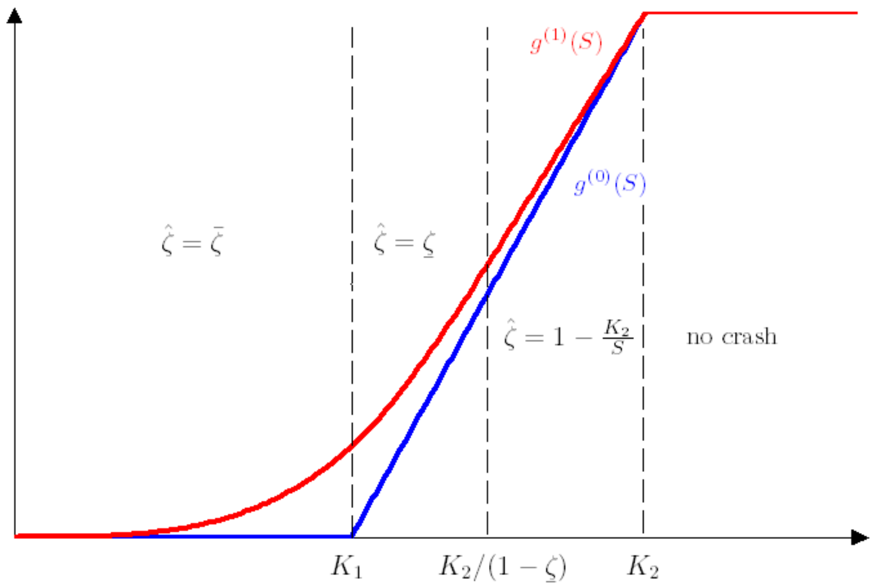

Figure 5.2: Terminal data for a bull spread: Downward and upward jumps possible (with $(1-\underline{\zeta})K_1 \leq K_2$).

$S \searrow 0$ and $S \to \infty$ can be chosen as in the crash-free case because the respective exercise decisions are certain, independent of crash scenarios,

$$\begin{array}{llll} \text{call:} & P^{(1)}(t,0) = 0 & \text{and} & P^{(1)}(t,S) \sim e^{-r(T-t)}(S-K) \text{ for } S \to \infty, \\ \text{bull spread:} & P^{(1)}(t,0) = 0 & \text{and} & P^{(1)}(t,S) \to e^{-r(T-t)}(K_2-K_1) \text{ for } S \to \infty. \end{array}$$

We make the transformation

$$\tau = \frac{1}{2}\sigma^2(T-t), \quad x = \ln(S), \quad k = \frac{2r}{\sigma^2}, \quad \alpha = \frac{1}{4}(k+1)^2, \quad \beta = \frac{1}{2}(k-1),$$
$$u^{(i)}(\tau,x) = e^{\alpha\tau+\beta x}P^{(i)}(t,S), \quad i = 0,1.$$

Then we have to solve the variational inequality

$$\min\left(u_\tau^{(1)} - u_{xx}^{(1)}, u^{(1)} - \sup_{\zeta\in[\underline{\zeta},\bar{\zeta}]}\left\{\zeta\left(-\beta u^{(1)} + u_x^{(1)}\right) - u^{(0)}(\tau, x+\ln(1-\zeta))\right\}\right) = 0 \qquad (5.17)$$

for $\tau \in (0,T_0]$, $T_0 := \sigma^2 T/2$, and $x \in \mathbb{R}$. For a discretization we restrict the space domain to $x \in [\underline{x},\bar{x}]$ for some adequate $\underline{x} < \bar{x}$, e.g. $\underline{x} = \ln(1)$ and $\bar{x} = \ln(300)$, and introduce an equidistant grid with points

$$(\tau_j, x_i) = (js, \underline{x} + ih), \quad j = 0,\dots,M, \; i = 0,\dots,N,$$

where $s = T_0/M$ and $h = (\bar{x}-\underline{x})/N$ denote respectively the time and the space mesh size. We further introduce the notation $w_i^j := u^{(1)}(\tau_j, x_i)$. Then the initial conditions in the discretized model read

$$w_i^0 = e^{\beta x_i} g^{(1)}\left(e^{x_i}\right), \quad i = 0,\dots,N,$$

where the respective function $g^{(1)}$ can be looked up in the above tables. The boundary conditions read

$$w_0^j = 0, \quad w_N^j = \begin{cases} e^{(\alpha-k)\tau_j+\beta\bar{x}}(e^{\bar{x}} - K) & \text{, call} \\ e^{(\alpha-k)\tau_j+\beta\bar{x}}(K_2 - K_1) & \text{, bull spread} \end{cases}, \quad j = 0, \ldots, M.$$

For the PDE approximation scheme we use the standard mixed explicit-implicit timestepping (θ-method with $\theta \in [0,1]$) for the first inequality,

$$\frac{1}{s}\left(w_i^j - w_i^{j-1}\right) - \frac{\theta}{h^2}\left(w_{i+1}^j - 2w_i^j + w_{i-1}^j\right) - \frac{1-\theta}{h^2}\left(w_{i+1}^{j-1} - 2w_i^{j-1} + w_{i-1}^{j-1}\right),$$

and a pure explicit scheme for the second inequality,

$$w_i^j - \sup_{\zeta\in[\underline{\zeta},\bar{\zeta}]}\left\{\zeta\left(-\beta w_i^j + \mathbb{1}_{\{\zeta>0\}}\frac{w_{i+1}^{j-1} - w_i^j}{h} + \mathbb{1}_{\{\zeta<0\}}\frac{w_i^j - w_{i-1}^{j-1}}{h}\right) - u^{(0)}(\tau_j, x_i + \ln(1-\zeta))\right\},$$

for $j = 1, \ldots, M$, $i = 1, \ldots, N-1$. The split difference quotient in the latter approximation ensures a monotony condition which is needed for the convergence result presented below. Writing everything in matrix notation we obtain in each time step $j = 1, \ldots, M$ a vector $w^j = (w_i^j)_{i=1,\ldots,N-1}$ as solution of a linear complementarity problem in form of

$$Aw^j - b^{j-1} \geq 0, \quad w^j - f_{\hat{\zeta}}^{j-1} \geq 0, \quad (Aw^j - b^{j-1})^T(w^j - f_{\hat{\zeta}}^{j-1}) = 0,$$

where the matrix A is fixed and the vectors b^{j-1} and $f_{\hat{\zeta}}^{j-1}$ are computed explicitly from the data of the last time step. Here the subscript $\hat{\zeta}$ suggests the dependence of $f_{\hat{\zeta}}^{j-1}$ from the respective worst-case jump size. Then we apply the SOR algorithm, an iterative method to solve linear equations, to $Aw^j = b^{j-1}$. But after each iteration we project the current value back to the admissible cone $w^j \geq f_{\hat{\zeta}}^{j-1}$. Since the matrix A is symmetric and positive definite, the resulting sequence converges to the unique solution of the above linear complementarity problem, see Cryer [11].

It remains to show that the solution of the discretized problem converges to the solution of the transformed original problem (5.17) for grid spacing $s, h \searrow 0$. For a detailed treatment of the convergence of approximation schemes in a viscosity solution framework we refer to Barles and Souganidis [1]. To prove convergence in our situation let us consider for simplicity a pure implicit timestepping ($\theta = 1$) with respect to the first part of the variational inequality. So we investigate the discretized PDE

$$\min\left(D_\tau^s u - D_x^{2,h}u, \mathcal{M}^{s,h}u\right) = 0,$$

where the operator $\mathcal{M}^{s,h}$ is defined by

$$\mathcal{M}^{s,h}u(\tau,x) = u(\tau,x) - \sup_{\zeta\in[\underline{\zeta},\bar{\zeta}]}\Big\{\zeta\left(-\beta u(\tau,x) + \left[\mathbb{1}_{\{\zeta>0\}}D_x^{s,h+} + \mathbb{1}_{\{\zeta<0\}}D_x^{s,h-}\right]u(\tau,x)\right) - u^{(0)}(\tau, x + \ln(1-\zeta))\Big\}$$

and the finite difference operators are given by

$$D_\tau^s u(\tau,x) = \frac{u(\tau,x)-u(\tau-s,x)}{s}, \quad D_x^{2,h}u(\tau,x) = \frac{u(\tau,x+h)-2u(\tau,x)-u(\tau,x-h)}{h^2},$$
$$D_x^{s,h+}u(\tau,x) = \frac{u(\tau-s,x+h)-u(\tau,x)}{h}, \quad D_x^{s,h-}u(\tau,x) = \frac{u(\tau,x)-u(\tau-s,x-h)}{h}.$$

We denote by $u_{s,h}$ a solution of the discretized equation and we define

$$\overline{u}(\tau,x) = \limsup_{\substack{s,h\searrow 0,\\ (js,\underline{x}+ih)\to(\tau,x)}} u_{s,h}(js,\underline{x}+ih), \quad \underline{u}(\tau,x) = \liminf_{\substack{s,h\searrow 0,\\ (js,\underline{x}+ih)\to(\tau,x)}} u_{s,h}(js,\underline{x}+ih).$$

If we are able to prove that $\overline{u}$ is a subsolution and $\underline{u}$ a supersolution of the nondiscretized PDE (5.17) on $\mathcal{S} := (0,T_0]\times(\underline{x},\bar{x})$, then, supposed $\overline{u}$ and $\underline{u}$ match on the parabolic boundary, by the comparison principle (which obviously holds for the transformed PDE, too) it follows that $u=\overline{u}=\underline{u}$ is the unique solution of (5.17), and $u_{s,h}$ converges locally uniformly to u for $s,h\searrow 0$.

We only sketch the proof that $\overline{u}$ is a subsolution, assuming that $u_{s,h}$ is uniformly bounded: Let $(\tau_0,x_0)\in\mathcal{S}$ and $\varphi\in\mathcal{C}^{1,2}(\mathcal{S})$ such that $\overline{u}-\varphi$ has a strict global maximum in (τ_0,x_0). Then there exist sequences $(s_k)_{k\in\mathbb{N}},(h_k)_{k\in\mathbb{N}}$ such that, using the abbreviation $u_k := u_{s_k,h_k}$, for $k\to\infty$ we have

$$s_k,h_k\searrow 0, \quad (\tau_k,x_k) := (j_k s_k, \underline{x}+i_k h_k)\to(\tau_0,x_0), \quad u_k(\tau_k,x_k)\to\overline{u}(\tau_0,x_0),$$
$$(\tau_k,x_k) \text{ is a global maximum point of } u_k-\varphi.$$

Set $\varphi_k := \varphi + u_k(\tau_k,x_k)-\varphi(\tau_k,x_k)$, so that we have $\varphi_k\ge u_k$ and $\varphi_k(\tau_k,x_k)=u_k(\tau_k,x_k)$. Then it follows

$$\begin{aligned} 0 &= \min\Big(D_\tau^{s_k}u(\tau_k,x_k)-D_x^{2,h_k}u(\tau_k,x_k), \mathcal{M}^{s_k,h_k}u(\tau_k,x_k)\Big)\\ &\ge \min\Big(D_\tau^{s_k}\varphi_k(\tau_k,x_k)-D_x^{2,h_k}\varphi_k(\tau_k,x_k), \mathcal{M}^{s_k,h_k}\varphi_k(\tau_k,x_k)\Big). \end{aligned} \tag{5.18}$$

If we only use mesh sizes s,k with $s\le h^2$ in our approximation scheme, then we get as $s,k\searrow 0$,

$$\begin{aligned} D_x^{s,h+}\varphi(\tau,x) &= \frac{\varphi(\tau-s,x+h)-\varphi(\tau,x)}{h}\\ &= \frac{\varphi(\tau,x+h)-\varphi(\tau,x)}{h} - \frac{\varphi(\tau,x+h)-\varphi(\tau-s,x+h)}{s}\cdot\frac{s}{h}\\ &\longrightarrow \varphi_x(\tau,x)+\varphi_\tau(\tau,x)\cdot 0 = \varphi_x(\tau,x) \end{aligned}$$

and the same for $D_x^{s,h-}\varphi(\tau,x)$. Hence, in view of the consistency of the standard finite difference approximations for $\varphi\in\mathcal{C}^{1,2}(\mathcal{S})$, the fact $\varphi_k(\tau_k,x_k)=u_k(\tau_k,x_k)\to\overline{u}(\tau_0,x_0)$ for $k\to\infty$ and the continuity of $u^{(0)}$, sending $k\to\infty$ in (5.18) yields

$$0\ge\min\left(\varphi_\tau-\varphi_{xx}, \overline{u}-\sup_{\zeta\in[\underline{\zeta},\bar{\zeta}]}\left\{\zeta\left(-\beta\overline{u}+\varphi_x\right)-u^{(0)}(\tau_0,x_0+\ln(1-\zeta))\right\}\right),$$

where the functions on the right hand side are evaluated in (τ_0, x_0). Therefore $\overline{u}$ is a subsolution of (5.17).

Figures 5.3 and 5.4 show the upper price bounds of the considered options in the incomplete market involving crash risk as well as the invested sums in the stock in order to super-hedge the options. For low stock prices these sums are greater than in the crash-free setting because by an upward price jump both options can be pushed into the money. In the case of the bull spread also a downward jump can send the option back to the stock price sensitive zone for high stock prices. But then the option's seller would lose money from the hedge and therefore the stock exposure is lower. If the options are in the money, the corresponding hedging strategies tend to be reduced compared to the situation without crash risk because a large jump might push the options out of the money. As an interesting numerical result it turns out that for both options a stock price jump only represents the worst-case scenario at maturity.

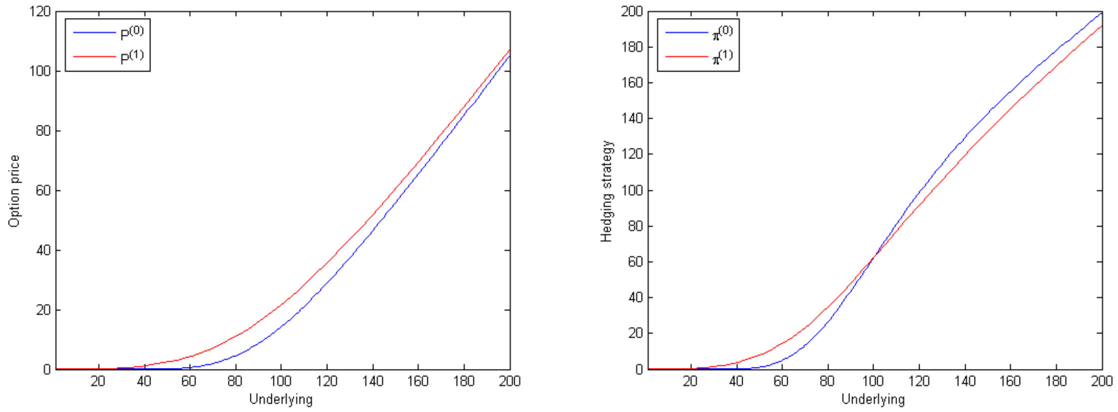

Figure 5.3: Super-hedging price of a call under crash risk and associated hedging strategy.

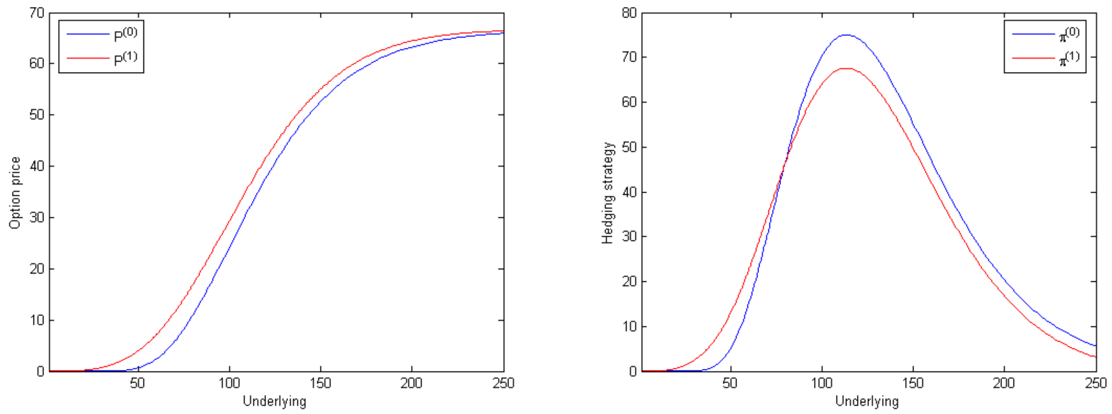

Figure 5.4: Super-hedging price of a bull spread under crash risk and associated hedging strategy.

5.3.2 Market completion

In this subsection we want to introduce a new approach to the option pricing problem under crash risk. Our procedure here will be rather heuristic, in particular it relies on a regularity assumption for the option price function. The central idea is to complete the market with crash risk such that the standard replication approach is applicable in some sense. This is done by introducing crash insurance as auxiliary asset class. Suppose that for insuring the sum $I \in \mathbb{R}$ one has to pay a continuous premium with rate $C(t, I)$. In the case of a crash of height ζ the insured is compensated by a payment of ζI. Here we only consider downward jumps, i.e. $\underline{\zeta} \geq 0$. For $\underline{\zeta} < 0$ we would have to include insurance against upward jumps as a second insurance type.

Now we are able to replicate a derivative by trading with the underlying stock and the insurance contract. More precisely, we replicate it along the worst-case path. To this end consider a portfolio with the amount π invested in the stock, the sum I insured against market crashes and a short position in one option. If the crash-adjusted option prices $P^{(n)}$ are sufficiently regular, we obtain by Itô's formula for the dynamic of the wealth process in the n-crash setting

$$dX(t) = \left(rX(t) + (\mu - r)\pi(t) - C(t, I(t)) - \left(-rP^{(n)} + P_t^{(n)} + \mu S P_S^{(n)} + \frac{1}{2}\sigma^2 S^2 P_{SS}^{(n)}\right)\right) dt$$
$$+ \sigma \left(\pi(t) - S P_S^{(n)}\right) dB(t).$$

Given a crash of size ζ at time τ there is a jump in form of

$$X(\tau) = X(\tau-) - \zeta\left(\pi(\tau) - I(\tau)\right) + P^{(n)}(\tau, S(\tau)) - P^{(n-1)}(\tau, (1-\zeta)S(\tau)).$$

To make the portfolio risk-free we choose π and I such that

$$\pi(t) = S P_S^{(n)} \quad \text{and} \quad \sup_{\zeta \in [\underline{\zeta}, \bar{\zeta}]} \left\{ \zeta(\pi(t) - I(t)) + P^{(n-1)}(t, (1-\zeta)S) \right\} = P^{(n)}.$$

By arbitrage arguments we then have $dX(t) = rX(t)$. It follows

$$rP^{(n)} - P_t^{(n)} - rSP_S^{(n)} - \frac{1}{2}\sigma^2 S^2 P_{SS}^{(n)} - C(t, I(t)) = 0, \tag{5.19}$$

$$P^{(n)} - \sup_{\zeta \in [\underline{\zeta}, \bar{\zeta}]} \left\{ \zeta(SP_S^{(n)} - I(t)) + P^{(n-1)}(t, (1-\zeta)S) \right\} = 0. \tag{5.20}$$

Resolving (5.19) for $I(t)$ and inserting this in (5.20), we get

$$P^{(n)} = \sup_{\zeta \in [\underline{\zeta}, \bar{\zeta}]} \left\{ \zeta\left(SP_S^{(n)} - J\left(t, rP^{(n)} - P_t^{(n)} - rSP_S^{(n)} - \frac{1}{2}\sigma^2 S^2 P_{SS}^{(n)}\right)\right) + P^{(n-1)}(t, (1-\zeta)S) \right\},$$

where $q \mapsto J(t, q)$ denotes the inverse of $I \mapsto C(t, I)$. If $C(t, I)$ is proportional to I, i.e. $C(t, I) = c(t)I$, then the above PDE reads

$$P^{(n)} = \sup_{\zeta \in [\underline{\zeta}, \bar{\zeta}]} \left\{ \frac{\zeta}{c(t)}\left(-rP^{(n)} + P_t^{(n)} + (r + c(t))SP_S^{(n)} + \frac{1}{2}\sigma^2 S^2 P_{SS}^{(n)}\right) + P^{(n-1)}(t, (1-\zeta)S) \right\},$$

As terminal value we can arrange the natural condition $P^{(n)}(T,S) = g(S)$ because with a final insurance sum of

$$I(T) = \pi(T) + \sup_{\zeta\in[\underline{\zeta},\bar{\zeta}]} \frac{g((1-\zeta)S(T)) - g(S(T))}{\zeta}$$

the portfolio maintains risk-free. With the same arguments as in the previous subsection, the boundary condition for $S = 0$ can be formulated as $P^{(n)}(t,0) = e^{-r(T-t)}g(0)$. For both conditions we have implicitly used a continuity assumption of the price functions on the parabolic boundary.

Remark that we have replicated the option's payoff only along the worst-case path for the seller of the option. So the above pricing rule states the seller's price of the option. The worst-case crash regarded from the buyer's point of view may differ. Going through the above argumentation for a portfolio with a long position in one option, we obtain the above pricing rule with an inf-term instead of the sup-term. Thus, in the case of a single crash size $\underline{\zeta} = \bar{\zeta}$ the seller's and the buyer's price coincide. Given this special case we obtain the linear PDE

$$\left(r + \frac{c(t)}{\bar{\zeta}}\right) P^{(n)} - P_t^{(n)} - (r + c(t))SP_S^{(n)} - \frac{1}{2}\sigma^2 S^2 P_{SS}^{(n)} = \frac{c(t)}{\bar{\zeta}} P^{(n-1)}(t,(1-\bar{\zeta})S). \quad (5.21)$$

A main advantage of this formula is that it generates option prices that are additive. For example, the value of a bull spread is the difference of the related call prices. Also the put-call parity holds. For computing the price $P^{(n)}$ in this setting we have to tackle the problem of calculating the insurance premium c. This task will be addressed in the next section.

Remark 5.3.4. The derivation of the pricing formula (5.21) goes smoothly even for a crash size of $\bar{\zeta} = 1$. This is of importance with regard to the pricing of defaultable bonds in Section 5.5.

Numerical example. Using the risk-neutral insurance premium formulas derived in Section 5.4 and a finite difference scheme, we can compute the option price under crash risk numerically. Let us concentrate on the case $n = 1$. Then we can apply the explicit premium rule

$$c(t) = \frac{2(\mu - r)\bar{\zeta}}{(\mu - r)(T-t) + 2\bar{\zeta}}.$$

Consequently the crash-adjusted option price is a solution of the PDE

$$\left(r + \frac{2(\mu - r)}{(\mu - r)(T-t) + 2\bar{\zeta}}\right)P^{(1)} - P_t^{(1)} - \left(r + \frac{2(\mu - r)\bar{\zeta}}{(\mu - r)(T-t) + 2\bar{\zeta}}\right) SP_S^{(1)} - \frac{1}{2}\sigma^2 S^2 P_{SS}^{(1)}$$
$$= \frac{2(\mu - r)}{(\mu - r)(T-t) + 2\bar{\zeta}} P^{(0)}(t,(1-\bar{\zeta})S).$$

In this example we consider the following option:

- Bull spread with strikes 80 and 150.
- Time to maturity is $T = 1$.

- The parameters for the bond and the risky asset are $r = 0.05$, $\mu = 0.1$, $\sigma = 0.3$ and $\bar{\zeta} = 0.3$.

For the difference scheme we use the standard boundary conditions for a bull spread,

$$P^{(1)}(T,S) = (S-80)^+ - (S-150)^+,$$
$$P^{(1)}(t,0) = 0 \quad \text{and} \quad P^{(1)}(t,S) \to (150-80)e^{-r(T-t)} \text{ for } S \to \infty.$$

The further proceeding is then very similar to the one presented in the numerical example of Subsection 5.3.1.

Figure 5.5 compares the option prices and the hedging strategies with and without crash risk. It promises a low impact of crash risk on the option price. Also the hedging positions in the risky asset do not differ dramatically if we consider crash risk or not. But under the threat of a crash the additional risk exposure has to be covered by crash insurance. At the upper part of the exercise region it is even recommended to sell crash insurance because the holder of the option is provided with a leeway of risk with regard to a crash scenario. So he can earn the insurance premium without being threatened by a crash.

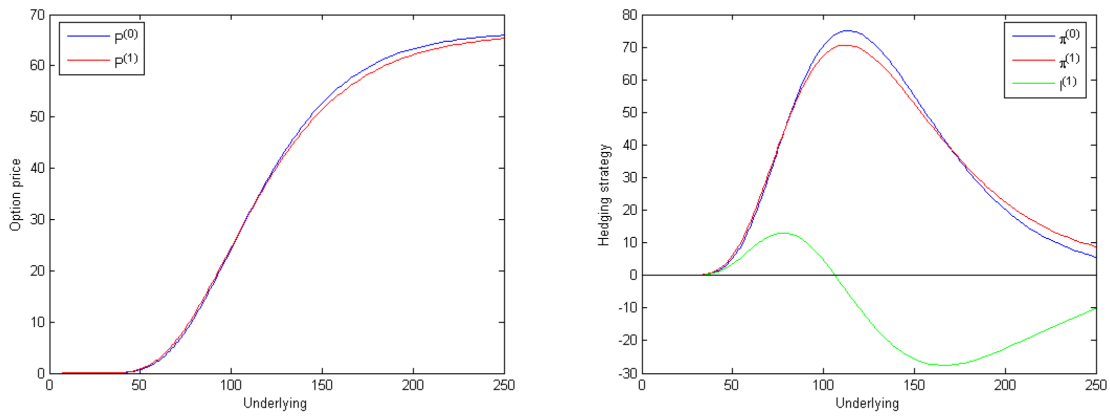

Figure 5.5: Price of a bull spread under crash risk and associated hedging strategy.

5.4 Crash insurance

In this section we calculate the price for crash insurance based on the equivalent utility principle. Here we consider the case of a continuous premium $C^{(n)}(t,I)$ payed by the investor for insuring the investment sum I in a market with n crash possibilities. If a crash of height ζ occurs, the investor will be compensated by a payment of ζI. Here we only consider downward jumps, i.e. $\zeta \geq 0$. Upward jumps can be treated as independent risk for which a separate insurance policy is necessary. The calculation of the premium is done in the same way for both cases. We exclusively use the exponential utility function $g(x) = -e^{-\alpha x}$ because this is a utility function with constant absolute risk aversion and therefore leads to premiums which are independent of the investment sum to be insured.

We consider a portfolio with the investment strategy $\pi = (\pi^{(n)}, \ldots, \pi^{(0)})$ and the insurance sum $I \geq 0$, where $\pi^{(i)}$ denotes the total sum invested in the risky asset for i left crash possibilities. Then the investor's wealth is given by

$$\begin{aligned} dX(t) &= \left(rX(t) + (\mu - r)\pi^{(n-i)}(t) - C^{(n-i)}(t, I)\right) dt + \sigma\pi^{(n-i)}(t)dB(t), \\ &\qquad\qquad \tau_i < t < \tau_{i+1}, \quad i = 0, \ldots, n, \\ X(\tau_i) &= X(\tau_i-) - \zeta_i \left(\pi^{(n-i+1)}(\tau_i) - I\right), \quad i = 1, \ldots, n. \end{aligned}$$

The expected utility maximization problem

$$\sup_{\pi \in \mathcal{U}_n} \inf_{\xi \in \mathcal{V}_n} \mathbb{E}_{t,x} \left[- \exp(-\alpha X^{\pi,I,\xi}(\tau_{\mathcal{S}}))\right] \tag{5.22}$$

was analyzed in Section 5.1 for the case without crash insurance, i.e. $I = 0$. We know the associated value function $v^{(n)}$ and the optimal investment strategy $\hat{\pi}$ in form of a system of ODEs, see Example 5.1.1. The threat of market crashes forces the investor to reduce his risk capital compared to the no crash model, i.e. $\hat{\pi}^{(n)} \leq \hat{\pi}^{(0)}$. By insuring an investment sum I the investor is able to increase his exposure in the risky asset by the amount I without loosing more money due to a crash as before. Since insurance yields costs, an investor does not demand an insurance sum which allows him to trade more risky than in the no crash case, i.e. $I \leq \hat{\pi}^{(0)} - \hat{\pi}^{(n)}$. We now want to calculate $C(t, I)$ for $I \in [0, \hat{\pi}^{(0)} - \hat{\pi}^{(n)}]$ such that the stochastic control problem (5.22) reveals the same optimal utility as without insurance. So for all $I \in [0, \hat{\pi}^{(0)} - \hat{\pi}^{(n)}]$ it has the value function $v^{(n)}$. Hence we have

$$\inf_{\pi \in [0, \hat{\pi}^{(0)}]} \max \left(-\mathcal{L}^{\pi,I} v^{(n)}(t, x), v^{(n)}(t, x) - \mathcal{M}^{\pi,I} v^{(n-1)}(t, x)\right) = 0, \tag{5.23}$$

where the differential operator $\mathcal{L}^{\pi,I}$ and the impulse operator $\mathcal{M}^{\pi,I}$ are given by

$$\begin{aligned} \mathcal{L}^{\pi,I} \varphi(t, x) &= \varphi_t(t, x) + (rx + (\mu - r)\pi - C^{(n)}(t, I))\varphi_x(t, x) + \frac{1}{2}\sigma^2\pi^2 x^2 \varphi_{xx}(t, x), \\ \mathcal{M}^{\pi,I} \varphi(t, x) &= \inf_{\zeta \in [\underline{\zeta}, \bar{\zeta}]} \varphi(t, x - \zeta(\pi - I)). \end{aligned}$$

Since $\pi \mapsto \mathcal{L}^{\pi,I} v^{(n)}(t, x)$ is strictly monotonously increasing up to

$$-\frac{\mu - r}{\sigma^2} \frac{v_x^{(n)}}{v_{xx}^{(n)}} = -\frac{\mu - r}{\sigma^2} \frac{v_x^{(0)}}{v_{xx}^{(0)}} = \hat{\pi}^{(0)}$$

and $\pi \mapsto \mathcal{M}^{\pi,I} v^{(n-1)}(t, x)$ is strictly mononously decreasing with $\mathcal{M}^{\bar{\pi},I} v^{(n-1)}(t, x) = v^{(n)}(t, x)$ for

$$\bar{\pi} = \hat{\pi}^{(n)} + I,$$

the infimum in (5.23) is attained for $\bar{\pi}$ and we have

$$\mathcal{L}^{\bar{\pi},I} v^{(n)}(t, x) = 0.$$

In view of

$$\mathcal{L}^{\hat{\pi}^{(n)},0}v^{(n)}(t,x) = 0$$

this yields the following equation for the insurance premium,

$$\left(C^{(n)}(t,I) - (\mu - r)I\right) v_x^{(n)} - \frac{1}{2}\sigma^2 \left(I^2 + 2\hat{\pi}^{(n)}(t)I\right) = 0.$$

Using the semi-explicit formula for $v^{(n)}$ from Example 5.1.1, it follows

$$\begin{aligned} C^{(n)}(t,I) &= (\mu - r)I - \frac{1}{2}\sigma^2\alpha e^{r(T-t)} \left(I^2 + 2\hat{\pi}^{(n)}(t)I\right) \\ &= (\mu - r)\left(1 - f^{(n)}(t)\right) I - \frac{1}{2}\sigma^2\alpha e^{r(T-t)} I^2, \end{aligned} \tag{5.24}$$

where $f^{(n)}(t) = \alpha e^{r(T-t)}\frac{\sigma^2}{\mu-r}\hat{\pi}^{(n)}(t)$ solves the system of ODEs

$$\begin{aligned} f_t^{(n)} &= \frac{\mu - r}{\bar{\zeta}}\left(f^{(n)} - f^{(n-1)}\right) - \frac{1}{2}\frac{\mu - r}{\bar{\zeta}}\left(\left(f^{(n)}\right)^2 - \left(f^{(n-1)}\right)^2\right), \\ f^{(n)}(T) &= 0, \quad i = 1, \dots, n, \\ f^{(0)} &\equiv 1. \end{aligned}$$

Since we have considered the portfolio decision for $I \geq 0$, the formula (5.24) states the indifference premium for the insurance holder. Therefore let us call the derived premium $C_{holder}^{(n)}$. Analogously we obtain the indifference premium for the insurer,

$$C_{insurer}^{(n)}(t,I) = (\mu - r)\left(1 - f^{(n)}\right) I + \frac{1}{2}\sigma^2\alpha e^{r(T-t)} I^2.$$

Of course we have $C_{holder}^{(n)} \leq C_{insurer}^{(n)}$, but for $\alpha \searrow 0$ both converge to the risk-neutral premium

$$C_{\alpha=0}^{(n)}(t,I) = (\mu - r)\left(1 - f^{(n)}\right) I. \tag{5.25}$$

So the risk-neutral insurance premium is proportional to I with premium rate $c_{\alpha=0}^{(n)}(t) = (\mu - r)\left(1 - f^{(n)}\right)$. Using the properties of the optimal portfolio strategy $\hat{\pi}$ stated in the remark following Example 5.1.1, we deduce that $c_{\alpha=0}^{(n)}$ is monotonously increasing with $c_{\alpha=0}^{(n)}(T) = \mu - r$ and we have the relation $0 \leq c_{\alpha=0}^{(1)} \leq \ldots \leq c_{\alpha=0}^{(n)} \leq \mu - r$. Further we note that

$$\hat{\pi}^{(0)}(t) - \hat{\pi}^{(n)}(t) = \frac{e^{-r(T-t)}}{\alpha}\frac{\mu - r}{\sigma^2}\left(1 - f^{(n)}(t)\right) \longrightarrow \infty \quad \text{for } \alpha \searrow 0,$$

so that the formula (5.25) holds for all $I \geq 0$. For $I < 0$ we set $C_{\alpha=0}^{(n)}(t,I) = -C_{\alpha=0}^{(n)}(t,-I)$. In the special case $n = 0$ we have $c_{\alpha=0}^{(0)}(t) = 0$ which goes with our intuition that a crash insurance in the no-crash model is worthless. For $n = 1$ we can state the explicit formula

$$c_{\alpha=0}^{(1)}(t) = \frac{2(\mu - r)\bar{\zeta}}{(\mu - r)(T - t) + 2\bar{\zeta}}.$$

A remarkable feature of the risk-neutral insurance premium is that it only depends on the drift coefficients r, μ and the jump size $\bar{\zeta}$, but not on the volatility σ.

Remark 5.4.1. Using the market completion approach discussed in the foregoing section with the indifference crash insurance premium of this section, crash-hedging via options is not possible for an investor with an exponential utility function. To see this, remark that for the option there exists a perfect hedging strategy in bond, stock and crash insurance. But the insurance premium is just calculated such that the investor cannot increase his utility by trading additionally insurance contracts.

Numerical example. We demonstrate the results of this section with a common set of parameters for the financial market:

- Terminal time is $T = 30$.
- The parameters for the bond and the risky asset are $r = 0.05$, $\mu = 0.1$ and $\bar{\zeta} = 0.3$.

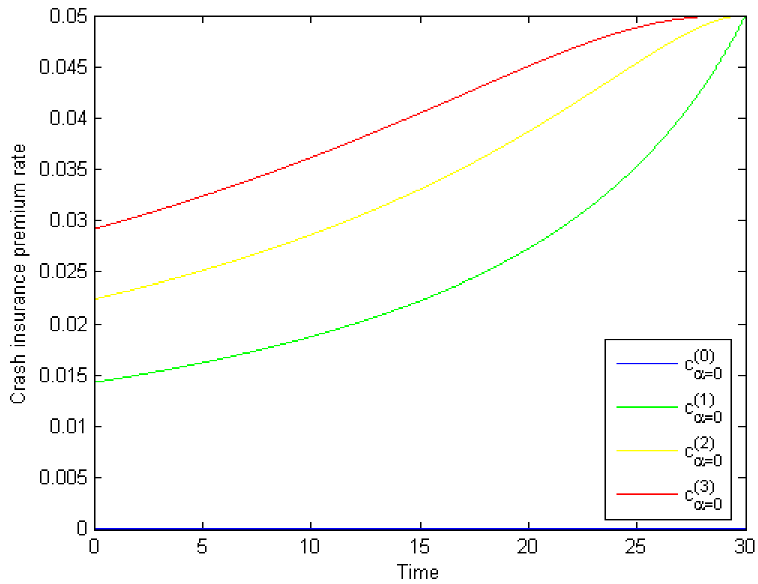

Figure 5.6: Crash insurance premium rate.

5.5 Defaultable bonds

For an application of the crash-adjusted option pricing rules from Section 5.3.2 in a different context we consider corporate bonds as an alternative risky asset class. The riskiness of the bonds refers to the possibility of default of the respective issuer which leads to large losses for the investor. The default event is an extreme scenario which can be modelled as an impulse perturbation. If the market allows for credit insurance, e.g. in form of credit default swaps, we can use the option pricing rules derived via market completion to derive a value of a defaultable bond.

We are geared to the firm value model in Merton [36]. The model is based on the assumption that the value process of a firm follows a geometric Brownian motion

$$dV(t) = \mu V(t)dt + \sigma V(t)dB(t)$$

for some constants μ, σ. The company has further issued one share of stock and a zero coupon bond with notional K and maturity T. If at time T the firm's asset value exceeds the promised payment K, the full notional of the bond can be repaid to the bondholders and the shareholders receive the residual asset value. If, however, the asset value is less than the promised payment, the firm defaults, the bondholders receive a payment equal to the asset value, and the share value drops to zero. Consequently, at the maturity T of the bond we have the following relations for the share value $S(T)$ and the bond value $D(T)$,

$$S(T) = (V(T) - K)^+, \quad D(T) = K - (K - V(T))^+.$$

Hence the value of the share can be interpreted as the price of a call option on the firm's asset value with strike K, and the corporate bond can be priced like a portfolio of a risk-free zero bond with notional K and a short position in a put option on V with strike K.

In this model default can only be detected at maturity. We now want to include the possibility of an early default at time $\tau \in [0, T]$. In this case the firm's asset value drops to zero at τ which corresponds to a jump of size $\bar{\zeta} = 1$ in the firm value. To make use of option pricing techniques let us assume that the value V is a tradeable asset. Furthermore, there exists a credit default insurance such that the default event triggers a payment of some insurance sum I. The associated premium is payed continuously with rate cI for some constant $c > 0$. The consideration of such an insurance secures a complete capital market. So we can apply the pricing rule (5.21) giving us the following PDEs for the call price C and the put price P (with V as underlying and strike K),

$$\begin{aligned} (r+c)C - C_t - (r+c)VC_V - \frac{1}{2}\sigma^2 V^2 C_{VV} &= 0, \\ (r+c)P - P_t - (r+c)VP_V - \frac{1}{2}\sigma^2 V^2 P_{VV} &= cKe^{-r(T-t)} \end{aligned}$$

with boundary conditions

$$\begin{aligned} C(T,V) &= (V-K)^+, \qquad & C(t,0) &= 0, \\ P(T,V) &= (K-V)^+, \qquad & P(t,0) &= Ke^{-r(T-t)}. \end{aligned}$$

Observe that the first PDE is the Black-Scholes PDE for a call on a market with interest $r + c$. It follows

$$C(t,V) = V\Phi(d_1(t,V)) - Ke^{-(r+c)(T-t)}\Phi(d_2(t,V))$$

with Φ denoting the cumulative distribution function of the standard normal distribution and

$$d_1(t,V) = \frac{\ln\left(\frac{V}{K}\right) + (r + c + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2(t,V) = d_1(t,V) - \sigma\sqrt{T-t}.$$

From the put-call parity we infer that

$$P(t,V) = C(t,V) - V + Ke^{-r(T-t)}.$$

Thus, in view of the relations $S(t) = C(t, V(t))$ and $D(t) = Ke^{-r(T-t)} - P(t, V(t))$, we obtain

$$S(t) = V(t)\Phi(d_1(t,V(t))) - Ke^{-(r+c)(T-t)}\Phi(d_2(t,V(t))),$$
$$D(t) = Ke^{-(r+c)(T-t)}\Phi(d_2(t,V(t))) + V(t)\Phi(-d_1(t,V(t))).$$

Numerical example. We illustrate the results for the following parameters:

- The issued bond has notional $K = 100$ and terminal time $T = 5$.
- The parameters for the (hypothetical) financial market are $r = 0.05$, $\mu = 0.1$, $\sigma = 0.3$ and $c = 0.02$.

Figure 5.7 compares the values of the share and the defaultable bond with a possible early default to the corresponding Merton prices (dotted lines). As expected the new bond price is lower than in the Merton model, especially for high firm values. Instead, the change in the stock price needs an explanation. An interpretation for the observed phenomenon of a higher stock price in spite of an additional default risk is that the weak development of the bond security leads to an increasing demand on the more profitable stock.

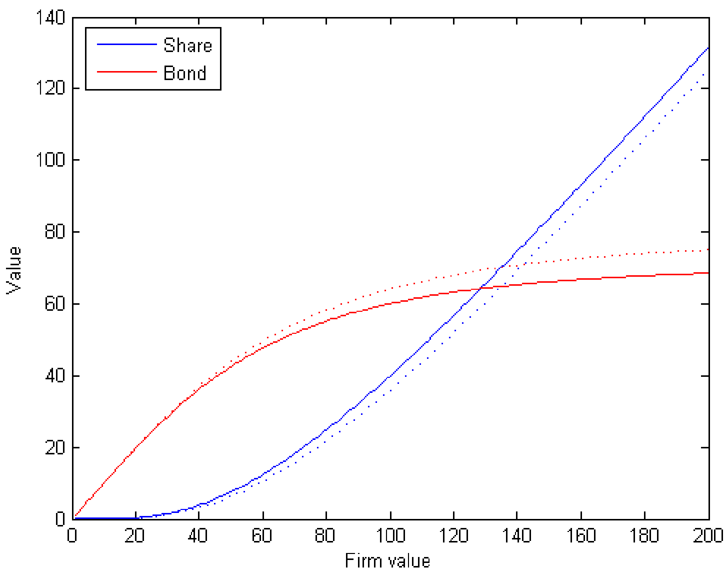

Figure 5.7: Share and bond value.

Without much difficulty one can use the derived pricing formula to make portfolio decisions concerning corporate bonds and credit default swaps. Considering that after an early default the secure bond is the only investment possibility, it cannot be optimal to enter a position which would lead to a loss in a default scenario. On the other hand, if one trades such that a premature default results in a positive profit, then the investor gives away possible insurance premium revenues. Therefore the relation between the number $\hat{\delta}$ of defaultable bonds and the insurance sum $\hat{I}$ in an optimal portfolio must be $\hat{I} = \hat{\delta}D$. Then, using the dynamics of the process $D(t, V(t))$, one can verify that, supposed there is no early default, the investor's wealth process evolves according to

$$dX(t) = \Big(rX(t) + (\mu - (r + c))\hat{\delta}V(t)D_V(t, V(t))\Big)\, dt + \sigma\hat{\delta}V(t)D_V(t, V(t))dB(t).$$

Remembering the known fact $D_V(t, V) = \Phi(-d_1(t, V)) > 0$, it is worthwile to enter a long position in the defaultable bond if and only if we have $c < \mu - r$. In the case $c < \mu - r$ it is optimal to sell short defaultable bonds.

Chapter 6

Optimal reinsurance and minimal capital requirement

In consideration of catastrophes and extreme damage events the worst-case control approach is very suitable to the insurance business. Our task in this chapter is to find worst-case optimal reinsurance strategies for an insurer who is faced with a given risk process. Moreover, we want to calculate the appropriate capital reserves that allow the insurer to bear all the future damages with a given probability.

The problem of optimal reinsurance strategies has been an area of active research in recent years. There is a large number of works which address several different kinds of reinsurance strategies in diverse insurance models. Most of the authors study strategies which minimize the insurer's ruin probability in a continuous-time framework where they use the classical Cramér-Lundberg model or a diffusion approximation of the surplus process as risk model (see, e.g., [21], [51] and the references therein). For variations of the problem we refer to Browne [8] and Promislow and Young [45] where the insurer can invest in risky and riskless assets. We take up their diffusion approximation approach complemented by impulse perturbation for large insurance losses for our investigations. In almost any cases the time period being considered is infinite which leads to static reinsurance strategies. In contrast, we concentrate on a finite time horizon which is conform to our control framework of finite impulse perturbation. So we obtain dynamic reinsurance strategies, so that the risk position is adjusted to its duration. However, since the problem of minimizing the ruin probability in finite time is very difficult to solve even numerically, we have to apply other decision rules such as utility optimization or minimization of the required capital reserves. To our knowledge, there are no references dealing with the calculation of capitalization requirements in this form.

The first section of this chapter introduces the insurance model in detail, and it is followed by the analysis of some reinsurance problems.

6.1 The insurance model

Let us consider a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For the risk process we distinguish between a multitude of small losses and a small number of imaginable large losses. The small claims can be approximated by a diffusion process as follows: Let $\tilde{C}(t) = \sum_{i=1}^{N(t)} Y_i$ be the claim process where $N(t)$ denotes the number of claims up to time t and Y_i is the size of the i-th claim. As usual in the classic theory of risk for small claims, N is assumed to be a Poisson process with rate $\lambda > 0$ and Y_i are independent and identically distributed random variables. Then we know from the well studied field of diffusion approximation (see, e.g., [18] and the references in [8]) that we have $\tilde{C} \approx C$ with

$$dC(t) = \alpha dt - \beta dB(t),$$

where B denotes a Brownian motion and the drift and the volatility of C are given by $\alpha = \lambda \mathbb{E}[Y_1]$ and $\beta^2 = \lambda \mathbb{E}[Y_1^2]$. The large claims are modelled by an impulse control. Therefore we fix a maximal claim size $\zeta > 0$ and a maximal number n of such catastrophes up to the time horizon $T > 0$. As filtration we use the $\mathbb{P}$-augmentation $(\mathcal{F}_t)_t$ of the filtration generated by the Brownian motion B.

We want to understand these large losses as exceptional events that are necessary to be taken into consideration for a robust risk control. But the insurer cannot refer to such worst-case scenarios for his premium calculation since no insurance holder would comply with the resulting premium rates. So we adopt a premium principle of the classic risk theory defining the premium rate by $c = (1+\theta)\alpha$ for some loading factor $\theta > 0$. Alternatively, one may adjust the insurance premium after a large loss. One could argue that in the aftermath of a catastrophe the insurance company has a better bargaining position to enforce higher premiums. So we could use different loading factors $\theta^{(i)}$ for the setting with $i \in \{0, \dots, n\}$ possible catastrophes left such that $\theta^{(n)} \leq \dots \leq \theta^{(0)}$. To keep the notation as short as possible, we want to abandon this technical generalization of the model. Then, ignoring interest payments on the insurer's capital, the surplus process of the insurer is given by

$$\begin{aligned} dX(t) &= c dt - dC(t) = \theta \alpha dt + \beta dB(t), \\ X(\tau_i) &= X(\tau_i -) - \zeta, \end{aligned}$$

where τ_i denotes the time of the i-th large loss. In order to lower the undertaken risk the insurer can reinsure a part of his risk portfolio. On the one hand we want to consider *proportional reinsurance*. Suppose the insurer cedes a fraction of his risk to a reinsurer. Let $\pi^{(i)}(t) \in [0,1]$ denote the reinsurance proportion at time t and for $i \in \{0, \dots, n\}$ possible catastrophes left and let $\eta > \theta$ be the reinsurance loading factor. Then we deal with the controlled surplus process

$$\begin{aligned} dX(t) &= (\theta - \eta \pi^{(n-i)}(t))\alpha dt + (1 - \pi^{(n-i)}(t))\beta dB(t), \\ X(\tau_i) &= X(\tau_i -) - (1 - \pi^{(n-i+1)}(\tau_i))\zeta. \end{aligned} \tag{6.1}$$

On the other hand a catastrophe linked reinsurance seems advisable in our situation. A type of reinsurance which is very common in the insurance of natural catastrophes are *CAT-bonds*. By issuing bonds that are linked to special catastrophe scenarios the insurance company can securitize a part of its catastrophe insurance risk. If such an extreme scenario takes place, the bond expires without continuing interest and redemption. As compensation for the risk taken by the investor of a CAT-bond, the insurer pays a risk premium $\mu > 0$. Let us assume that the insurance company can issue bonds continuously and let us denote by $\pi^{(i)}(t) \geq 0$ the nominal amount of all bonds issued until time t (in the situation with $i \in \{0, \dots, n\}$ possible catastrophes left). If we do not allow for a premature cancellation of the bond, $\pi^{(i)}$ has to be monotonously increasing. Then we can write the controlled surplus process in form of

$$\begin{aligned} dX(t) &= (\theta\alpha - \mu\pi^{(n-i)}(t))dt + \beta dB(t), \\ X(\tau_i) &= X(\tau_i -) - \zeta + \pi^{(n-i+1)}(\tau_i). \end{aligned} \tag{6.2}$$

Actually, there is no payment of $\pi^{(n-i+1)}(\tau_i)$ in the case of the i-th catastrophe scenario. But in the above representation we have settled the initial incoming payments and the redemption (at time T) which is omitted by the occurrence of a catastrophe.

Besides an optimal reinsurance strategy we strive for another goal. An important task in insurance is to determine the reserves of the insurance company that are required in order to fulfill all liabilities. We present two approaches for these connected problems which differ both in the problem formulation and in the type of reinsurance. In Section 6.2 we maximize the exponential utility of the insurer's surplus at final time T by proportional reinsuring. For the surplus process under the resulting optimal reinsurance policy we then determine the *minimal capital requirements (MCR)* of the insurer such that the surplus is positive at time T with some given probability. So the method used in this section is stepwise. In Section 6.3 we rather obtain the reinsurance strategy and the MCR simultaneously by solving a stochastic target problem. Here we use CAT-bonds to lower the risk and therefore the needed capital reserves, and the MCR is computed as the minimal initial value for the surplus process such that the surplus is positive at time T with some given probability. We transform this problem into a target problem whose optimal control is the optimal reinsurance strategy and the value function is the resulting MCR.

6.2 Exponential utility of terminal surplus

Our first objective is to maximize the exponential utility of terminal surplus of the insurer by choosing an optimal proportional reinsurance strategy. The analysis of this section is therefore based on the surplus process given by (6.1). By $\mathcal{U}_n$ and $\mathcal{V}_n$ we denote the adequate admissibility sets with the underlying control sets $U = [0, 1]$ and $Z = \{\zeta\}$ (see Chapter 2 for the requirements of admissiblity of the respective controls). Furthermore, by the stopping time $\tau_{\mathcal{S}} = \inf\{s \geq t : (s, X^{\pi,\xi}(s)) \notin \mathcal{S} = [0, T) \times (0, \infty)\}$ we indicate the end of control actions due to an early ruin of the insurer or the arrival at the time horizon. Then we want to study the problem

$$\sup_{\pi\in\mathcal{U}_n}\inf_{\xi\in\mathcal{V}_n}\mathbb{E}_{t,x}\left[g(X(\tau_{\mathcal{S}}))\right] \qquad \text{for} \qquad g(x)=-e^{-\gamma x}, \quad \gamma>0.$$

Note that the exponential utility function has a constant absolute risk aversion, $-\frac{g_{xx}}{g_x}\equiv\gamma$. As we will see in the following, this is exactly the property that yields an optimal reinsurance strategy which is independent of the level of reserves of the insurance company.

Let $v^{(n)}$ be the associated value function. The differential operator $\mathcal{L}^\pi$ and the impulse operator $\mathcal{M}^\pi$ corresponding to the surplus process (6.1) are given by

$$\begin{aligned}\mathcal{L}^\pi\varphi &= \varphi_t+(\theta-\eta\pi)\alpha\varphi_x+\frac{1}{2}(1-\pi)^2\beta^2\varphi_{xx},\\ \mathcal{M}^\pi\varphi &= \varphi(t,x-(1-\pi)\zeta).\end{aligned}$$

To simplify the calculation let us suppose that $\gamma\geq\frac{\eta\alpha}{\beta^2}$. By this assumption we avoid delicate boundary problems and are able to state explicit solutions, at least for $n=0$ and $n=1$. For $n=0$ we can derive the value function as solution of the HJB-equation

$$\begin{aligned}\sup_{\pi\in[0,1]}\mathcal{L}^\pi v^{(0)}(t,x)&=0, \quad (t,x)\in\mathcal{S}\ =[0,T)\times(0,\infty),\\ v^{(0)}(t,x)+\exp(-\gamma x)&=0, \quad (t,x)\in\mathcal{S}^c=([0,T]\times\mathbb{R})\setminus([0,T)\times(0,\infty)).\end{aligned}$$

As result we obtain the constant optimal reinsurance rate

$$\hat{\pi}^{(0)}(t)=1-\frac{\eta\alpha}{\gamma\beta^2},$$

and the value function is

$$v^{(0)}(t,x)=-\exp\left(-\gamma x+f^{(0)}(t)\right)$$

with

$$f^{(0)}(t)=\left(\gamma(\eta-\theta)\alpha-\frac{1}{2}\frac{\eta^2\alpha^2}{\beta^2}\right)(T-t).$$

Note that by the made assumptions the optimal control is admissible.

According to Chapter 3, taking into consideration extreme claim sizes we have to solve the system

$$\begin{aligned}\inf_{\pi\in[0,1]}\max\left(-\mathcal{L}^\pi v^{(n)}(t,x),v^{(n)}(t,x)-\mathcal{M}^\pi v^{(n-1)}(t,x)\right)&=0, \quad (t,x)\in\mathcal{S},\\ v^{(n)}(t,x)+\exp(-\gamma x)&=0, \quad (t,x)\in\mathcal{S}^c.\end{aligned} \tag{6.3}$$

Here we note that the problem formulation corresponds to the extended differential game studied in Section 3.6. The underlying assumptions are very easy to check, in particular we see that $v^{(n)}$ is polynomially bounded on $\mathcal{S}$. Consequently, if the PDE in (6.3) has a viscosity solution which is continuous at terminal time T with final data as in (6.3) and with a continuous associated feedback control function, then we are sure that it is the seeked value function.

In order to solve the problem (6.3) we state the following lemma:

Lemma 6.2.1. *Suppose that for $i = 1, \dots, n$ the value functions $v^{(i)}$ are in $\mathcal{C}^{1,2}([0,T] \times \mathbb{R})$ and are strictly concave in x. Then the characterizing PDE in* (6.3) *simplifies to*

$$\max_{\pi \in U^{(n)}(t,x)} \mathcal{L}^\pi v^{(n)}(t,x) = 0, \tag{6.4}$$

where we maximize on the subset

$$U^{(n)}(t,x) := \{\pi \in [0,1] : \mathcal{M}^\pi v^{(n-1)}(t,x) \geq v^{(n)}(t,x)\}.$$

Furthermore, the unique solution of this restricted maximization problem is

$$\hat{\pi}^{(n)}(t,x) = \max\left(\pi^{(n)*}(t,x), \bar{\pi}^{(n)}(t,x)\right)$$

with

$$\begin{aligned} \pi^{(n)*}(t,x) &= 1 + \frac{\eta\alpha}{\beta^2} \frac{v_x^{(n)}(t,x)}{v_{xx}^{(n)}(t,x)}, \\ \bar{\pi}^{(n)}(t,x) &= \min U^{(n)}(t,x). \end{aligned}$$

Proof: In view of the strict concavity of $v^{(n)}$, the map $\pi \mapsto -\mathcal{L}^\pi v^{(n)}$ is a convex parabola with minimum at $\pi^{(n)*}(t,x)$. Moreover, it is easy to verify that $v^{(n-1)}$ is strictly monotonuously increasing in x. Therefore the map $\pi \mapsto v^{(n)} - \mathcal{M}^\pi v^{(n-1)}$ is strictly monotonously decreasing. Because of the continuity of this map and the obvious relation $v^{(n)} \leq v^{(n-1)}$, the set $U^{(n)}(t,x)$ is a closed intervall of the form $[\bar{\pi}^{(n)}(t,x), 1]$ for some $\bar{\pi}^{(n)}(t,x) \in [0,1]$. Hence, using continuity arguments it follows that the infimum in the PDE characterization (6.3) is realized by $\hat{\pi}^{(n)}(t,x) = \max(\pi^{(n)*}(t,x), \bar{\pi}^{(n)}(t,x))$ which is the unique solution of the restricted maximization problem $\max_{\pi \in U^{(n)}(t,x)} \mathcal{L}^\pi(t,x)$, and the equation reads $\mathcal{L}^{\hat{\pi}^{(n)}(t,x)} v^{(n)}(t,x) = 0$. □

Lemma 6.2.1 yields the optimal reinsurance strategy $\hat{\pi}^{(n)}$ in the setting with n outstanding possible catastrophes. Here, the case $\bar{\pi}^{(n)}(t,x) \geq \pi^{(n)*}(t,x)$ reflects the necessity of a higher reinsurance rate in order to compensate large claims due to a catastrophe. It seems to be reasonable that this situation prevails at all time. So let us assume that $\hat{\pi}^{(n)}(t,x) = \bar{\pi}^{(n)}(t,x)$ (which has to be verified later on). Note that for $\bar{\pi}^{(n)}(t,x) \geq \pi^{(n)*}(t,x)$ the feedback function $\hat{\pi}^{(n)}$ satisfies

$$\mathcal{M}^{\hat{\pi}^{(n)}(t,x)} v^{(n-1)}(t,x) = v^{(n)}(t,x). \tag{6.5}$$

In view of the form of $v^{(0)}$, we make the ansatz

$$v^{(i)}(t,x) = -\exp\left(-\gamma x + f^{(i)}(t)\right), \quad i = 1, \dots, n.$$

Then we deduce from (6.5)

$$\hat{\pi}^{(n)}(t) = 1 - \frac{f^{(n)}(t) - f^{(n-1)}(t)}{\gamma\zeta} \tag{6.6}$$

and from (6.4)

$$f_t^{(n)} = (\theta - \eta\hat{\pi}^{(n)})\alpha\gamma - \frac{1}{2}(1-\hat{\pi}^{(n)})^2\beta^2\gamma^2. \tag{6.7}$$

Combining (6.6)-(6.7) we can conclude

$$\hat{\pi}_t^{(n)} = \frac{\eta\alpha}{\zeta}(\hat{\pi}^{(n)} - \hat{\pi}^{(n-1)}) + \frac{\gamma\beta^2}{2\zeta}\left((1-\hat{\pi}^{(n)})^2 - (1-\hat{\pi}^{(n-1)})^2\right).$$

To characterize the control process $\hat{\pi}^{(n)}$ uniquely we need the correct terminal condition. The optimal utility $-\exp(-\gamma x)$ at time $t = T$ is only attained if the risk process is fully reinsured at that time, i.e. $\hat{\pi}^{(n)}(T) = 1$.

It remains to check the made assumptions. The value functions are indeed sufficiently regular and strictly concave in x. The solution of the derived system of ODEs is a unique differentiable function. Using results from the theory of differential inequalities, we will show in Lemma 6.2.3 below the relation

$$1 \geq \hat{\pi}^{(n)} \geq \hat{\pi}^{(n-1)} \geq \ldots \geq \hat{\pi}^{(0)} \geq 0.$$

Therefore the assumption $\bar{\pi}^{(n)}(t,x) \geq \pi^{(n)*}(t,x) = \hat{\pi}^{(0)}(t,x)$ is true and the strategy $\hat{\pi} = (\hat{\pi}^{(n)}, \ldots, \hat{\pi}^{(0)})$ is admissible.

So we have proved the following result:

Theorem 6.2.2. *The optimal proportional reinsurance strategy $\hat{\pi} = (\hat{\pi}^{(n)}, \ldots, \hat{\pi}^{(0)})$ to maximize the expected utility of the surplus at a terminal time T is given by the system of ODEs*

$$\begin{aligned} \hat{\pi}_t^{(i)} &= \frac{\eta\alpha}{\zeta}(\hat{\pi}^{(i)} - \hat{\pi}^{(i-1)}) + \frac{\gamma\beta^2}{2\zeta}\left((1-\hat{\pi}^{(i)})^2 - (1-\hat{\pi}^{(i-1)})^2\right), \\ \hat{\pi}^{(i)}(T) &= 1, \quad i = 1, \ldots, n, \end{aligned} \tag{6.8}$$

where the optimal strategy in the catastrophe-free setting reads

$$\hat{\pi}^{(0)}(t) = 1 - \frac{\eta\alpha}{\gamma\beta^2}.$$

The corresponding value function is given by

$$v^{(n)}(t,x) = v^{(0)}\left(t, x - \sum_{i=1}^{n}\left(1 - \hat{\pi}^{(i)}(t)\right)\zeta\right),$$

where

$$v^{(0)}(t,x) = -\exp\left(-\gamma x + \left(\gamma(\eta-\theta)\alpha - \frac{1}{2}\frac{\eta^2\alpha^2}{\beta^2}\right)(T-t)\right)$$

is the value function in the catastrophe-free setting.

Remark 6.2.1. For $n = 1$ we can solve the ODE (6.8) explicitly, giving us

$$\hat{\pi}^{(1)}(t) = 1 - \frac{\eta\alpha}{\gamma\beta^2}\frac{\eta\alpha(T-t)}{\eta\alpha(T-t)+2\zeta}$$

and consequently

$$v^{(1)}(t,x) = -\exp\left(-\lambda e^{r(T-t)}x + \left(\gamma(\eta-\theta)\alpha - \frac{1}{2}\frac{\eta^2\alpha^2}{\beta^2}\frac{\eta\alpha(T-t)}{\eta\alpha(T-t)+2\zeta}\right)(T-t)\right).$$

The following lemma states the essential properties of the optimal proportional reinsurance strategy:

Lemma 6.2.3. *The solution* $\hat{\pi} = (\hat{\pi}^{(n)}, \ldots, \hat{\pi}^{(0)})$ *of* (6.8) *satisfies*

(i) $1 \geq \hat{\pi}^{(n)} \geq \hat{\pi}^{(n-1)} \geq \ldots \geq \hat{\pi}^{(0)} \geq 0,$

(ii) $\hat{\pi}_t^{(i)} \geq 0$ *for all* $i = 1, \ldots, n$.

Proof: The inequality $\hat{\pi}^{(0)} \geq 0$ is already verified. We prove the remaining statements via induction.

For $i = 1$ we can directly compute

$$\hat{\pi}^{(1)}(t) = 1 - \frac{\eta\alpha}{\gamma\beta^2}\frac{\eta\alpha(T-t)}{\eta\alpha(T-t)+2\zeta} \geq 1 - \frac{\eta\alpha}{\gamma\beta^2} = \hat{\pi}^{(0)}(t).$$

In particular we have $\hat{\pi}_t^{(1)} \geq 0$.

For the induction step we want to consider the function

$$h(z) := -\frac{\eta\alpha}{\zeta}z - \frac{\gamma\beta^2}{2\zeta}(1-z)^2.$$

For $z > \tilde{z} > 0$ we verify the one-sided Lipschitz condition

$$h(z) - h(\tilde{z}) \leq -\frac{\gamma\beta^2}{2\zeta}\left((1-z)^2 - (1-\tilde{z})^2\right) = \frac{\gamma\beta^2}{2\zeta}(2-z-\tilde{z})(z-\tilde{z}) \leq \frac{\gamma\beta^2}{\zeta}(z-\tilde{z}).$$

If we suppose $\hat{\pi}_t^{(i)} \geq 0$ for $i = 1, \ldots, n-1$, then we know from the induction hypothesis and the ODE characterization of $\hat{\pi}^{(i)}$ that

$$\hat{\pi}_t^{(i)} = -h(\hat{\pi}^{(i)}) + h(\hat{\pi}^{(i-1)}) \geq 0 \quad \text{for } i = 1, \ldots, n-1.$$

Using the ODE characterization of $\hat{\pi}^{(n)}$ it follows

$$\hat{\pi}_t^{(n)} = -h(\hat{\pi}^{(n)}) + h(\hat{\pi}^{(n-1)}) \geq -h(\hat{\pi}^{(n)}) + h(\hat{\pi}^{(0)}).$$

Since the map $z \mapsto -h(z) + h(\hat{\pi}^{(0)})$ is a convex parabola with minimum at $\hat{\pi}^{(0)}$, we deduce $\hat{\pi}_t^{(n)} \geq 0$. Furthermore, we calculate

$$-\hat{\pi}_t^{(n)} - h(\hat{\pi}^{(n)}) = -h(\hat{\pi}^{(n-1)}) \geq -\hat{\pi}_t^{(n-1)} - h(\hat{\pi}^{(n-1)}).$$

Thus, in view of the terminal condition $\hat{\pi}^{(n)}(T) = \hat{\pi}^{(n-1)}(T)$, the one-sided Lipschitz condition of h and Lemma B.3.1 in the appendix, we conclude the desired relation $\hat{\pi}^{(n)} \geq \hat{\pi}^{(n-1)}$. Because of $\hat{\pi}_t^{(n)} \geq 0$ and $\hat{\pi}^{(n)}(T) = 1$, it finally follows $\hat{\pi}^{(n)} \leq 1$. □

So under the threat of large losses the optimal reinsurance rate is not constant any more. Instead, it is increasing such that at final time the risk portfolio is reinsured completely. And, very intuitively, the more catastrophes the insurer expects the more he has to reinsure. Furthermore, note that the derived strategy is independent of the insurer's capitalization which allows us to split the decision on reinsurance and MCR calculation into two independent operations.

MCR. After we have found an optimal reinsurance strategy which is independent of x we next want to calculate the required reserves of an insurer with an exponential utility function. The final surplus under some strategy $\pi \in \mathcal{U}_0$ and without large claims reads in integral form

$$X^\pi(T) = x + \theta\alpha(T-t) - \eta\alpha \int_t^T \pi(s)ds + \beta \int_t^T (1-\pi(s))dB(s).$$

If π is a deterministic strategy, we conclude that $X^\pi(T)$ is normally distributed with mean $m(t,x) = \mathbb{E}_{t,x}[X^\pi(T)]$ and variance $\nu(t,x) = \mathbb{E}_{t,x}[(X^\pi(T) - m(t,x))^2]$ given by

$$m(t,x) = x + \theta\alpha(T-t) - \eta\alpha \int_t^T \pi(s)ds,$$
$$\nu(t,x) = \beta^2 \int_t^T (1-\pi(s))^2 ds.$$

Thus, in the castastrophe-free setting the ruin probability reads

$$\mathbb{P}_{t,x}[X^\pi(T) \le 0] = 1 - \Phi\left(-\frac{m(t,x)}{\sqrt{\nu(t,x)}}\right),$$

where Φ is the cumulative distribution function of the standard normal distribution. So given some ruin probability $p > 0$ we can compute the MCR under strategy π by solving the equation

$$1 - \Phi\left(-\frac{m(t,x)}{\sqrt{\nu(t,x)}}\right) = p$$

for x. Let us denote by $R^{(0)}_\pi(t,p)$ the solution of this equation. Some elementary calculation gives us

$$R^{(0)}_\pi(t,p) = \eta\alpha \int_t^T \pi(s)ds - \theta\alpha(T-t) - \beta\Phi^{-1}(1-p)\sqrt{\int_t^T (1-\pi(s))^2 ds}.$$

Under catastrophe risk the insurer needs an additional cushion. If his reinsurance strategy $\pi \in \mathcal{U}_n$ only consists of deterministic control processes $\pi^{(i)}$, $i = 0, \dots, n$, then it is reasonable to maintain the reserve

$$R^{(n)}_\pi(t,p) = \max_{t =: \tau^{(0)} \le \tau^{(1)} \le \dots \le \tau^{(n+1)} := T} \Bigg\{ \eta\alpha \sum_{i=0}^n \int_{\tau^{(i)}}^{\tau^{(i+1)}} \hat{\pi}^{(n-i)}(s)ds - \theta\alpha(T-t)$$
$$-\beta\Phi^{-1}(1-p)\sqrt{\sum_{i=0}^n \int_{\tau^{(i)}}^{\tau^{(i+1)}} (1-\hat{\pi}^{(n-i)}(s))^2 ds} + \zeta \sum_{i=1}^n (1-\hat{\pi}^{(n-i+1)}(\tau^{(i)})) \Bigg\}.$$

Since the reinsurance strategy $\hat{\pi}$ of an insurer with exponential utility is deterministic, we can calculate the associated MCR in this way.

Numerical example. We demonstrate the results of this section with a common set of parameters for the insurance market:

- Terminal time is $T = 30$.
- The parameters for the claim process C are $\alpha = 1$, $\beta = 0.5$ and $\zeta = 1$.
- The loading factors are $\theta = 0.1$ and $\eta = 0.3$.
- The risk aversion parameter γ is chosen such the optimal reinsurance fraction in the catastrophe-free setting is 0.1, i.e. $\gamma = \frac{\eta\alpha}{0.9\beta^2} = \frac{4}{3}$.
- The confidence level for the MCR calculation is $p = 0.95$.

Figure 6.1 shows the optimal reinsurance strategies and the associated MCR for $n \leq 3$. It points out the trade-off between large insurance losses and the costs for reinsurance. The latter one increases with the period of reinsurance and becomes negligible for a short time to maturity, so that the optimal reinsurance proportion increases in time while the MCR decreases when we near the time horizon.

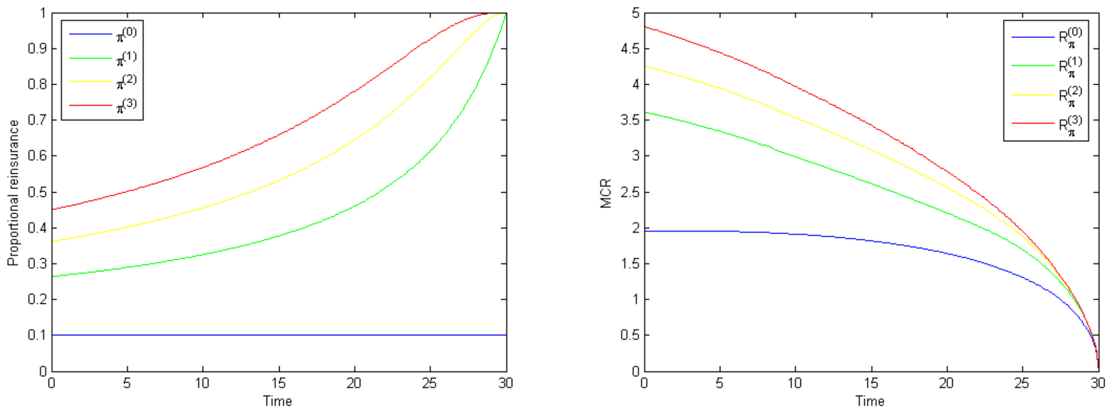

Figure 6.1: Optimal proportional reinsurance strategy and associated MCR.

6.3 Stochastic target approach

The focus of the last section was the performance of the insurance portfolio measured by exponential utility. Nevertheless, this approach may lead to a large MCR which is not in the insurer's interest because of opportunity costs. This fact shows the disadvantage of the exponential utility function with its constant absolute risk aversion. Therefore we now look for reinsurance strategies that minimize the MCR. To this end consider an (arbitrary) risk process $X^{\pi,\xi}$ (with $\pi \in \mathcal{U}_n, \xi \in \mathcal{V}_n$). Let us suppose that the company is required to hold capital as reserves such that loss can be avoided for a given probability. For $t \in [0,T]$, $p \in [0,1]$ we are interested in

$$v^{(n)}(t,p) = \inf\left\{x \in \mathbb{R} : \text{there exists } \pi \in \mathcal{U}_n \text{ s.t. } \mathbb{P}[X^{\pi,\xi}_{t,x}(T) \geq 0] \geq p \text{ for all } \xi \in \mathcal{V}_n\right\}. \quad (6.9)$$

In order to write $v^{(n)}$ as a value function of a target problem, we follow [7] and introduce an additional controlled process defined by

$$dP(t) = \phi(t)dB(t),$$

where the additional control ϕ is a progressively measurable $\mathbb{R}$-valued square-integrable process. By $\mathcal{A}$ we denote the set of all progressively measurable $\mathbb{R}$-valued square-integrable processes and we set $\tilde{\mathcal{U}}_n = \mathcal{U}_n \times \mathcal{A}^{n+1}$. Then we can transform the initial problem of reaching the target with a given probability of success into a standard control problem as follows.

Proposition 6.3.1. *For all $(t,p) \in [0,T) \times (0,1)$ we have*

$$v^{(n)}(t,p) = \inf\left\{x \in \mathbb{R} : \textit{there exists } (\pi,\phi) \in \tilde{\mathcal{U}}_n \textit{ s.t. } \mathbb{1}_{\{X^{\pi,\xi}_{t,x}(T)\geq 0\}} \geq P^{\phi}_{t,p}(T) \textit{ for all } \xi \in \mathcal{V}_n\right\}. \quad (6.10)$$

Proof: We denote by $\tilde{v}(t,p)$ the right-hand side of (6.10). For $x > v^{(n)}(t,p)$ we can find π such that $\mathbb{P}[X^{\pi,\xi}_{t,x}(T) \geq 0] \geq p$ for all ξ. By the martingale representation theorem there exists a progressively measurable square-integrable process ϕ such that

$$p + \mathbb{1}_{\{X^{\pi,\xi}_{t,x}(T)\geq 0\}} - \mathbb{P}[X^{\pi,\xi}_{t,x}(T) \geq 0] = p + \int_t^T \phi(s)dB(s) = P^{\phi}_{t,p}(T).$$

From this equality we deduce

$$\mathbb{1}_{\{X^{\pi,\xi}_{t,x}(T)\geq 0\}} - P^{\phi}_{t,p}(T) = \mathbb{P}[X^{\pi,\xi}_{t,x}(T) \geq 0] - p \geq 0,$$

and therefore $x \geq \tilde{v}(t,p)$ from the definition of $\tilde{v}$.

Conversely, for $x > \tilde{v}(t,p)$ there exist π, ϕ such that $\mathbb{1}_{\{X^{\pi,\xi}_{t,x}(T)\geq 0\}} \geq P^{\phi}_{t,p}(T)$ for all ξ. Since $P^{\phi}_{t,p}$ is a martingale, it follows that

$$\mathbb{P}[X^{\pi,\xi}_{t,x}(T) \geq 0] = \mathbb{E}[\mathbb{1}_{\{X^{\pi,\xi}_{t,x}(T)\geq 0\}}] \geq \mathbb{E}[P^{\phi}_{t,p}(T)] = p,$$

so that $x \geq v^{(n)}(t,p)$ by the definition of $v^{(n)}$. □

Note that with

$$g(p) := \begin{cases} -\infty\ , & p \leq 0, \\ 0\ , & p \in (0,1], \\ \infty\ , & p > 1 \end{cases}$$

the representation (6.10) yields a target problem of the form

$$v^{(n)}(t,p) = \inf\left\{x \in \mathbb{R} : \text{there exist } (\pi,\phi) \in \tilde{\mathcal{U}}_n \text{ s.t. } X^{\pi,\xi}_{t,x}(T) \geq g(P^{\phi}_{t,p}(T)) \text{ for all } \xi \in \mathcal{V}_n\right\}. \quad (6.11)$$

Obviously there are some inconsistencies with the target problem considered in Chapter 4. First of all, g is not locally bounded on $\mathbb{R}$. However, in view of the initial problem formulation, it only makes sense to consider $v^{(n)}$ on $[0,T]\times[0,1]$. If we assume that $v^{(n)}$ is locally bounded in the interior of this domain, the viscosity property still holds on $[0,T)\times(0,1)$. As well the viscosity property of the terminal condition holds on $(0,1)$ because of the local boundedness of $v^{(n)}$ in the interior and of g on $(0,1)$. So it remains to specify the boundary conditions on $[0,T]\times\{0,1\}$. Furthermore, the introduced stochastic control ϕ has an unbounded control set. To cope with this problem we have to make sure that, firstly, the considered SDEs always have a unique strong solution. Secondly, that the PDE for the subsolution (resp. supersolution) derived in Chapter 4 is given in form of $F(t,p,v(t,p),v_t(t,p),v_p(t,p),v_{pp}(t,x)) = 0$ for a locally bounded lsc (resp. usc) function F. And, thirdly, that the control ϕ has values in a compact subset in a local sense (i.e. as long as $(s,P^{\phi}_{t,p}(s),X^{\pi}_{t,x}(s))$ stays within some open neighborhood of (t,p,x)), allowing us to pass to a converging subsequence in the proof of the supersolution property (see Appendix C.2, steps 3 and 4). Here the first point is not critical because ϕ is square-integrable. The other points will be addressed in relation to the application considered below. Finally, let us note that the process P^{ϕ} inherits the strong Markov property from $X^{\pi,\xi}$. This fact follows directly from the construction of P^{ϕ} in the proof of Proposition 6.3.1. Moreover, since ϕ is progressively measurable, the set of admissible stochastic controls is stable under concatenation (see the corresponding introductory notes in Section 4.3). So the dynamic programming principle of Chapter 4 is valid for the target problem (6.11).

Application to proportional reinsurance

Let us now apply this reserve calculation technique in the context of proportional reinsurance, i.e. we consider the controlled system

$$\begin{aligned}
dX(t) &= (\theta-\eta\pi^{(n-i)}(t))\alpha dt + (1-\pi^{(n-i)}(t))\beta dB(t),\\
dP(t) &= \phi^{(n-i)}(t)dB(t), \quad i=0,\dots,n,\\
X(\tau_i) &= X(\tau_i-) - (1-\pi^{(n-i+1)}(\tau_i))\zeta,\\
P(\tau_i) &= P(\tau_i-), \qquad\qquad i=1,\dots,n.
\end{aligned}$$

According to Chapter 4, the reinsurance policy π and the associated strategy ϕ have to fulfill the relation (supposed the derivatives exist)

$$(1-\pi^{(i)}(t))\beta = \phi^{(i)}(t)v^{(i)}_p(t,p), \quad i=0,\dots,n \tag{6.12}$$

to match the stochastic integrals of X and P. From the definition of $v^{(n)}$ in (6.9) and the relation $X^{\pi,\xi}_{t,x} = x + X^{\pi,\xi}_{t,0}$ we deduce that $v^{(n)}$ is monotonously increasing in p. We may therefore assume a gradient constraint of the form $v^{(n)}_p \geq 0$, meaning that in the definition of a viscosity solution we only consider test functions φ with $\varphi_p(t,p)\geq 0$ at the relevant point

(t,p). If we have $\varphi_p(t,p) > 0$, then a control $\phi^{(n)}$ fulfilling (6.12) takes on values in a compact set locally around (t,p). So the proof of the supersolution property can be reproduced without difficulty in this case. For $\varphi_p(t,p) = 0$ we have to force the supersolution property explicitly. The resulting supersolution equation characterizing $(v^{(n)})_*$ on $[0,T) \times (0,1)$ is then given by

$$\sup_{\pi\in[0,1]} \min\left(-v_t^{(n)} - \frac{1}{2}(1-\pi)^2\beta^2 \frac{v_{pp}^{(n)}}{(v_p^{(n)})^2} + (\theta-\eta\pi)\alpha, v^{(n)} - (1-\pi)\zeta - (v^{(n-1)})_*, v_p^{(n)} \right) = 0.$$

For the subsolution property no changes are necessary. However, since an authority might require a non-negative MCR, it seems to be reasonable to reformulate the problem, so that the initial capital should be non-negative. Noting that in the proof of the subsolution property of $(v^{(n)})^*$ at point (t,p) a sequence of initial values is constructed which are non-negative if and only if $(v^{(n)})^*(t,p) > 0$ (see Appendix C.1, step 2), we have to exclude the case $(v^{(n)})^* \leq 0$. Then $(v^{(n)})^*$ is a viscosity subsolution on $[0,T) \times (0,1)$ of

$$\sup_{\pi\in[0,1]} \min\left(-v_t^{(n)} - \frac{1}{2}(1-\pi)^2\beta^2 \frac{v_{pp}^{(n)}}{(v_p^{(n)})^2} + (\theta-\eta\pi)\alpha, v^{(n)} - (1-\pi)\zeta - (v^{(n-1)})^*, -v^{(n)} \right) = 0.$$

Both equations satisfy the semicontinuity requirements (see above) which can be verified by a distinction of cases. Assuming continuity of $v^{(n)}$ at the parabolic boundary, we can set up the conditions

$$v^{(0)}(T,p) = 0, \quad p \in (0,1),$$
$$v^{(0)}(t,0) = 0 \quad \text{and} \quad v^{(0)}(t,1) = (\eta-\theta)\alpha(T-t) + n\zeta, \quad t \in [0,T),$$

where for the last condition we used the MCR for the completely reinsured risk process.

Alas, the non-linear term $v_{pp}^{(n)}/(v_p^{(n)})^2$ together with the fact that $v_p^{(n)}$ is not bounded away from zero makes standard numerical solution methods impracticable. So the stochastic target approach seems not very suitable for proportional reinsurance.

Application to hedging via CAT-bonds

For dealing with special catastrophe reinsurance in form of CAT-bonds as described in Section 6.1, the approach turns out to be more attractive. So let us consider from now on the controlled surplus process X given by (6.2), but here for simplicity we want to concentrate first on the risk of large claims and neglect small claim sizes modelled by the Brownian motion. So we study the process

$$dX(t) = (\theta\alpha - \mu\pi^{(n-i)}(t))dt, \quad i = 0, \ldots, n,$$
$$X(\tau_i) = X(\tau_i-) - \zeta + \pi^{(n-i+1)}(\tau_i) \quad i = 1, \ldots, n,$$

and the auxiliary controlled process P from above. Since X is deterministic up to the jumps through impulses, we can choose the control $\phi \equiv 0$ (see the construction of P in the proof of Proposition 6.3.1) and the value function is independent of p. The CAT-bond policy π is not affected by a restriction on the corresponding admissibility set $U := [0, \zeta]$. Of course, for $n = 0$ issuing CAT-bonds makes no sense and the MCR is $v^{(0)}(t) = -\theta\alpha(T-t)$. For $n \geq 1$ we have to solve the ODE

$$\sup_{\pi\in[0,\zeta]} \min\left(-v_t^{(n)} + \theta\alpha - \mu\pi, v^{(n)} - \zeta + \pi - v^{(n-1)}\right) = 0. \tag{6.13}$$

At time T we can avoid a loss for free by the choice $\hat{\pi}^{(i)}(T) = \zeta$ for $i = 1, \dots, n$, so that the right terminal condition is $v^{(n)}(T) = 0$. The supremum in the above equation is obviously obtained when both components of the min-term equal zero. This implies in particular that the worst-case scenario leading to the maximal MCR is an immediate occurrence of all damages which gives us the MCR in form of

$$v^{(n)}(t) = -\theta\alpha(T-t) + n\zeta - \sum_{i=1}^{n} \hat{\pi}^{(i)}. \tag{6.14}$$

Using this representation, the optimal CAT-bond strategy $\hat{\pi} = (\hat{\pi}^{(n)}, \dots, \hat{\pi}^{(0)})$ can be computed as the solution of the system of ODEs

$$\hat{\pi}_t^{(i)} = \mu\hat{\pi}^{(i)} - \sum_{j=1}^{i-1} \hat{\pi}_t^{(j)},$$
$$\hat{\pi}^{(i)}(T) = \zeta, \quad i = 1, \dots, n,$$

with $\hat{\pi}^{(0)} \equiv 0$. The solution of this ODE system is a unique differentiable function and the value function given in (6.14) is the unique viscosity solution of (6.13) with linear growth. Here the latter statement can be shown by a simple comparison result for an ODE.

The construction of the optimal control as solution of both equations for the differential and the impulse part shows that the optimal reinsurance strategy makes the insurer indifferent between catastrophe scenarios and "normal times". Furthermore, via induction one shows that it is increasing such that at final time the risk portfolio is hedged completely against large losses. For $n = 1$ we can explicitly solve the stated ODE giving us

$$\hat{\pi}^{(1)}(t) = \zeta e^{-\mu(T-t)}$$

and consequently

$$v^{(1)}(t,p) = -\theta\alpha(T-t) + \zeta\left(1 - e^{-\mu(T-t)}\right).$$

It is now very easy to incorporate small claim sizes in the MCR calculation by adding the stochastic integral $\int_0^t \beta dB(s)$ to the surplus $X(t)$. Since the risk of small and of large claims is independent and a CAT-bond strategy has no impact on the exposure to small claims, we can just add the MCRs for both sources of uncertainty. Proceeding as in the second part of the previous section, the resulting MCR is

$$v^{(n)}(t,p) = -\beta\Phi^{-1}(1-p)\sqrt{T-t} - \theta\alpha(T-t) + n\zeta - \sum_{i=1}^{n} \hat{\pi}^{(i)}.$$

Numerical example. For this example we take the following values for the parameters:

- Terminal time is $T = 15$.
- The parameters for the claim process C are $\alpha = 1$, $\beta = 0.5$ and $\zeta = 3$.
- The loading factor is $\theta = 0.1$ and the CAT-bond yield is $\mu = 0.3$.
- The confidence level for the MCR calculation is $p = 0.95$.

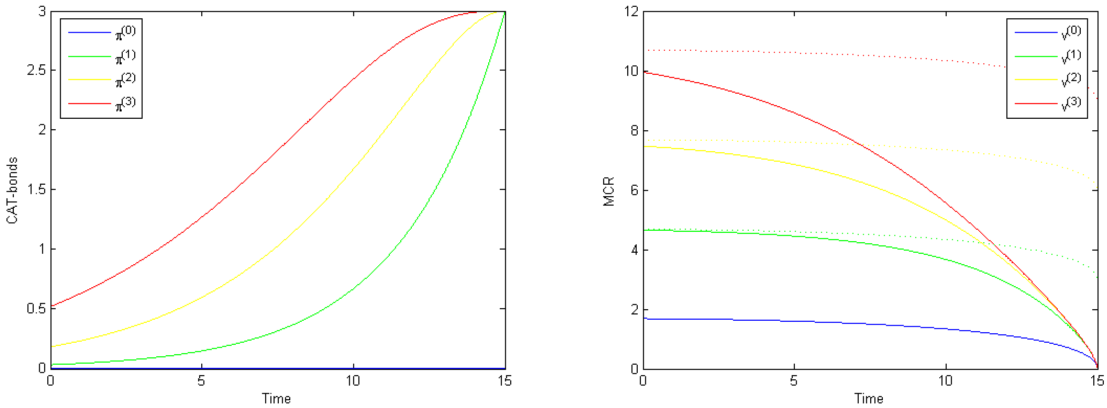

Figure 6.2: Optimal CAT-bond strategies and associated MCR.

Figure 6.2 shows the optimal reinsurance strategies and the associated MCR for $n \leq 3$. The dotted lines in the graphic on the right hand side refer to the MCR in the respective setting without issuing CAT-bonds. In contrast to the numerical example given in the previous section, we have chosen here a severer catastrophe scenario. Reinsuring such scenarios with a proportional reinsurance would have led to an almost complete risk transfer. That shows the suitability of CAT-bonds for our specific concern.

Chapter 7

Summary and conclusions

In this thesis we introduced a model for stochastic control taking into account system crashes. Decisions are made in respect of the worst-case scenario which can be an immediate crash or no crash relative to each instant. So the decision maker is faced with a balance problem of controlling satisfactorily in the no crash case and cutting down on the negative consequences of a crash.

We derived a PDE characterization of the value function for the associated control problem which is a differential game with combined stochastic and impulse control. As well we derived a PDE characterization of the value function for the appropriate target problem. In both cases we proved that the value function solves the respective PDE in the weak sense of a viscosity solution. For the differential game we even showed uniqueness of the solution which is necessary in order to use the PDE characterization as a verification result. For the target problem we formulated a uniqueness statement for a special setting. To allow access to a wide range of applications we tried to introduce the theory as general as possible. So in the case of the differential game we studied as underlying control process a jump-diffusion. To obtain suitable properties we first made some restrictive assumptions concerning the control sets, the coefficient functions of the jump-diffusion, the transaction function for impulse-driven jumps and the profit functions. Afterwards we attenuated them to a minimum of requirements. For the stochastic target problem we limited ourselves to diffusion processes. However, the results can be extended to underlying controlled jump-diffusion processes under suitable assumptions on the pure jump measure. The proof is carried out as presented in this thesis, only with some additional constraint associated to possibly jumps stemming from the Lévy process (see [5] for this extension in the classic stochastic control framework).

Since we cannot assume the value functions to be differentiable, we chose the viscosity solution approach. So the results may justify numerical PDE solution techniques for their computation. However, we hope that analytical presentations can be found in many applications. Admittedly, the charaterizing PDEs are often either trivial or very difficult to solve.

In particular, the force to control the stochastic part in the target problem might induce an uncomfortable non-linearity which complicates even a numerical solution.

As illustrative examples of impulse pertubed stochastic control we studied some problems in the field of financial risk management. We continued investigations on the worst-case portfolio problem under crash risk. Here we put our emphasis on the extension of the capital market by derivatives, crash insurance and defaultable bonds as alternative investment class. Above all we were concerned with the pricing of these new products. To this end we presented different approaches. For the pricing of derivatives we considered stochastic target problems leading to lower and upper price bounds, the buyer's and seller's price, in form of variational inequalities. As an alternative pricing method we introduced the concept of market completion. By the market completion approach we obtained a unique price function given by a linear PDE. We introduced the notion of crash insurance and priced such contracts by an equivalent utility principle. This gave us a time-depending ODE characterization for the premium rate. Finally we used option prices derived by the market completion approach to price defaultable bonds on a market with credit default insurance.

Furthermore, we used the impulse pertubed stochastic control framework to handle insurance risk. Modelling small claims by a diffusion and large claims in form of impulse perturbations, we strived for optimal reinsurance strategies. Since we focused on a finite time horizon, we derived dynamic reinsurance strategies, meaning that the risk position has to be adjusted to the time to maturity. By maximizing the expected exponential utility of the insurer's terminal surplus we first derived an ODE system representation for optimal proportional reinsurance. As well we determined the associated MCR in terms of the optimal strategy. To avoid large capital reserves to be held for the insurer, we considered in addition the MCR minimization problem. We transformed the origin problem to a stochastic target problem which allows to find MCR minimizing reinsurance strategies. For this method to work, we had to choose CAT-bonds as reinsurance type. As result we again obtained a system of ODEs for the optimal CAT-bond strategy and MCR formulas depending on its solution. Similar to the investing problem under crash risk, the main feature of the optimal reinsurance strategies is that they schedule a progressive reduction of the exposed risk until there is no residual risk any more at the end.

Looking at the worst that can happen is a quite reasonable approach. Nevertheless, allthough our strategies allow for risky asset investments and uncovered insurance risks, they can be regarded as too conservative. An interesting extension of our model would be to use further information on the probability of crash scenarios. For example, one can suppose that the probability of $i \in \{0, \dots, n\}$ crashes occuring is $p_i \in [0, 1]$ with $\sum_{i=0}^{n} p_i = 1$. Then the utility maximization problem, if we only consider final utility for simplicity, can be modified in form of

$$\sup_{\pi \in \mathcal{U}_n} \sum_{i=0}^{n} p_i \inf_{\xi \in \tilde{\mathcal{V}}_i} \mathbb{E}\left[g(X_{t,x}^{\pi_i,\xi}(T)) \right], \tag{7.1}$$

where $\tilde{\mathcal{V}}_i \subset \mathcal{V}_i$ denotes the set of all impulse controls representing i real perturbations, i.e. $\zeta \in Z$ for all impulses ζ, and with $\pi_i = (\pi^{(n)}, \ldots, \pi^{(n-i)})$. We assume that, via a dynamic programming principle, there is a connection to a control problem of the form

$$\sup_{\pi \in \mathcal{U}_0} \mathbb{E}\left[f(t, x, \pi(t)) + g(X^{\pi}_{t,x}(T))\right]. \tag{7.2}$$

To account for this, consider for example the worst-case portfolio problem in the case $n = 1$. Obviously, the optimal control process for $\pi^{(0)}$ agrees with the optimal strategy of the portfolio problem without crash risk. Further, we know that the worst crash scenario is always an immediate crash, so that (7.1) simplifies to

$$\sup_{\pi^{(1)} \in \mathcal{U}_0} \left\{ p_1 \mathcal{M}^{\pi^{(1)}(t)} v^{(0)}(t, x) + (1 - p_1) \mathbb{E}\left[g(X^{\pi^{(1)}}_{t,x}(T))\right] \right\},$$

where $v^{(0)}$ stands for the value function of the classic portfolio problem. By the appropriate (re)definition of f and g, it follows (7.2). To our knowledge, there are no studies on control problems of this form. An adequate formulation for the target problem, using a q-quantil approach as detailed in Section 6.3, $q \in [0, 1]$, reads

$$\inf \left\{ y \in \mathbb{R} : \text{ there exists } \pi \in \mathcal{U}_n \text{ s.t. } \sum_{i=0}^{n} p_i \mathbb{P}\left[Y^{\pi_i, \xi_i}_{t,x,y}(T) \geq g(X^{\pi_i, \xi_i}_{t,x}(T))\right] \geq q \text{ for all } \xi_i \in \tilde{\mathcal{V}}_i \right\}.$$

Moreover, by the use of information on the probability of crash scenarios, a Value at Risk approach seems reasonable. In the context of portfolio optimization, Menkens [39] implemented such a procedure exclusively for the “crash times” while the portfolio’s performance in “normal times” were averaged out. Continuing this approach to a universal Value at Risk consideration would be a challenging task.

Appendix A

Viscosity solutions of PDEs

A.1 Semicontinuous functions

Let $\mathcal{S}$ be an arbitrary set. A function $u : \mathcal{S} \to \mathbb{R}$ is upper semicontinuous (usc) if

$$\limsup_{k\to\infty} u(x_k) \le u(x) \quad \text{for any sequence } (x_k)_{k\in\mathbb{N}} \subset \mathcal{S} \text{ such that } x_k \to x \in \mathcal{S}.$$

A function $u : \mathcal{S} \to \mathbb{R}$ is lower semicontinuous (lsc) if

$$\liminf_{k\to\infty} u(x_k) \ge u(x) \quad \text{for any sequence } (x_k)_{k\in\mathbb{N}} \subset \mathcal{S} \text{ such that } x_k \to x \in \mathcal{S}.$$

We denote by USC($\mathcal{S}$) the set of all usc functions and by LSC($\mathcal{S}$) is the set of all lsc functions. Further, for a locally bounded function $u : \mathcal{S} \to \mathbb{R}$ we define the usc envelope $u^* : \mathcal{S} \to \mathbb{R}$ and the lsc envelope $u_* : \mathcal{S} \to \mathbb{R}$ of u by

$$u^*(x) = \limsup_{\substack{y\to x\\ y\in\mathcal{S}}} u(y), \quad u_*(x) = \liminf_{\substack{y\to x\\ y\in\mathcal{S}}} u(y).$$

Note that in general we have

$$u_* \le u \le u^*$$

and that u is usc if and only if $u = u^*$ and u is lsc if and only if $u = u_*$. In particular, u is continuous if and only if $u_* = u = u^*$.

In the following we want to list some properties of semicontinuous functions which are used in handling possibly discontinuities of viscosity solutions:

(i) u is usc if and only if $-u$ is lsc.

(ii) If u, v are usc with $u, v \ge 0$, then uv is usc; if u, v are lsc with $u, v \ge 0$, then uv is lsc.

(iii) If u is usc and v is continuous, then $u \circ v$ is usc; if u is lsc and v is continuous, then $u \circ v$ is lsc.

(iv) If u, v are usc, then $\max(u, v)$ and $\min(u, v)$ are usc; if u, v are lsc, then $\max(u, v)$ and $\min(u, v)$ are lsc.

(v) A usc function attains its maximum on any compact set; a lsc function attains its minimum on each compact set.

To derive continuity statements for the intervention operators $\mathcal{M}^\pi$ in Chapter 3 and $\mathcal{M}^\pi_y$ in Chapter 4, we need the following lemma which is a generalization of the property (iv).

Lemma A.1.1. *Let X be a subset of $\mathbb{R}^{N_1}$ and K a non-empty compact subset of $\mathbb{R}^{N_2}$. Given a function $f : X \times K \to \mathbb{R}$, we consider the function $F : X \to \mathbb{R}$,*

$$F(x) := \sup_{y \in K} f(x, y).$$

If f is usc (resp. lsc), then F is usc (resp. lsc).

Proof: Consider a sequence $(x_k)_{k\in\mathbb{N}} \subset X$ converging to $x \in X$.

(i) Let f be usc. Then, in view of the property (v) stated prior to this lemma, the upper semicontinuity of f and the compactness of K, there exist $(y_k)_{k\in\mathbb{N}} \subset K$ and $\hat{y} \in K$ such that

$$F(x_k) = f(x_k, y_k) \text{ for all } k \in \mathbb{N} \quad \text{and} \quad F(x) = f(x, \hat{y}).$$

Hence, due to the compactness of K and the upper semicontinuity of f, we can find some $\bar{y} \in K$ such that

$$\limsup_{k\to\infty} F(x_k) = \limsup_{k\to\infty} f(x_k, y_k) \le f(x, \bar{y}) \le f(x, \hat{y}) = F(x),$$

which proofs the upper semicontinuity of F.

(ii) Let f be lsc. For $\varepsilon > 0$ there exists some $y^\varepsilon \in K$ such that

$$f(x, y^\varepsilon) + \varepsilon \ge F(x).$$

Consider a sequence $(y_k)_{k\in\mathbb{N}} \subset K$ with $y_k \to y^\varepsilon$ as $k \to \infty$. Then, by the lower semicontinuity of f, it follows

$$\liminf_{k\to\infty} F(x_k) \ge \liminf_{k\to\infty} f(x_k, y_k) \ge f(x, y^\varepsilon) \ge F(x) - \varepsilon.$$

Since $\varepsilon > 0$ is chosen arbitrarily, we conclude that F is lsc. □

To proof the continuity of the signed distance function δ introduced in Chapter 4, the following lemma is crucial.

Lemma A.1.2. *Let X, Y be subsets of $\mathbb{R}^{N_1}$ and $\mathbb{R}^{N_2}$, respectively, and let K be a non-empty compact subset of $\mathbb{R}^{N_3}$. Given functions $f : X \to \mathbb{R}$ and $g : Y \times K \to \mathbb{R}^{N_1}$, we consider the function $F : Y \to \mathbb{R}$,*

$$F(y) := \sup_{x \in g(y,K)} f(x).$$

Let g be continuous. If f is usc (resp. lsc), then F is usc (resp. lsc).

Proof: Consider a sequence $(y_k)_{k\in\mathbb{N}} \subset Y$ converging to $\hat{y} \in Y$.

(i) Let f be usc. By the continuity of g and the compactness of U, the set $g(y, K)$ is compact for all $y \in Y$. Thus, in view of the upper semicontinuity of f and the property (v) from above, we can find $x_k \in g(y_k, K)$, $z_k \in K$ and $\hat{x} \in g(\hat{y}, K)$ such that

$$F(y_k) = f(x_k) \text{ with } x_k = g(y_k, z_k) \text{ for all } k \in \mathbb{N} \quad \text{and} \quad F(\hat{y}) = f(\hat{x}).$$

Since K is compact, we may pass to a subsequence of (z_k) converging to $\hat{z} \in K$. Using the continuity of g, we obtain $x_k \to \bar{x} := g(\hat{y}, \hat{z})$ for $k \to \infty$ along this subsequence. Hence we conclude

$$\limsup_{k\to\infty} F(y_k) = \limsup_{k\to\infty} f(x_k) \leq f(\bar{x}) \leq f(\hat{x}) = F(\hat{y}),$$

which proofs the upper semicontinuity of F.

(ii) Let f be lsc. For $\varepsilon > 0$ there exists some $x^\varepsilon \in g(\hat{y}, K)$ such that

$$f(x^\varepsilon) + \varepsilon \geq F(\hat{y}).$$

Consider $x_k \in g(y, K)$ such that $x_k \to x^\varepsilon$ as $k \to \infty$. Then, by the lower semicontinuity of f, it follows

$$\liminf_{k\to\infty} F(y_k) \geq \liminf_{k\to\infty} f(x_k) \geq f(x^\varepsilon) \geq F(y) - \varepsilon.$$

Since $\varepsilon > 0$ is chosen arbitrarily, we conclude that F is lsc. □

Recalling the relation in terms of semicontinuity between u and $-u$ from the property (i) in the list stated above, Lemma A.1.1 and Lemma A.1.2 also hold if we consider an infimum in the respective function F.

A.2 Definition of viscosity solutions

We consider a non-linear second order parabolic PDE of the form

$$F(t, x, u(t,x), \frac{\partial u}{\partial t}(t,x), D_x u(t,x), D_x^2 u(t,x)) = 0, \quad (t,x) \in \mathcal{S} := [0,T) \times \mathcal{O}, \tag{A.1}$$

where $T > 0$ is a fixed time horizon, $\mathcal{O}$ is an open subset of $\mathbb{R}^d$ and $F : \mathcal{S} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \to \mathbb{R}$ is a continuous map ($\mathbb{S}^d$ is the set of all symmetric $(d \times d)$-matrices). As we will see below, it is of crucial importance for the consistency of the definition of viscosity solutions of (A.1) that F satisfies the degenerate ellipticity condition

$$F(t, x, r, a, p, X) \leq F(t, x, r, a, p, Y) \quad \text{whenever} \quad X \geq Y. \tag{A.2}$$

Here $X \geq Y$ refers to the usual order in $\mathbb{S}^d$ in the sense of $x^T X x \geq x^T Y x$ for all $x \in \mathbb{R}^d$.

The following definition specifies a weak notion of a solution of equation (A.1):

Definition A.2.1. *Let $u : \mathcal{S} \to \mathbb{R}$ be a locally bounded function.*

(i) *We say that u is a* (discontinuous) viscosity subsolution *of* (A.1) *if for any point* $(t,x) \in \mathcal{S}$ *and all* $\varphi \in \mathcal{C}^{1,2}(\mathcal{S})$ *such that* $(u^* - \varphi)$ *has a local maximum in* (t,x),

$$F(t,x,u^*(t,x),\frac{\partial \varphi}{\partial t}(t,x),D_x\varphi(t,x),D_x^2\varphi(t,x)) \le 0.$$

(ii) *We say that u is a* (discontinuous) viscosity supersolution *of* (A.1) *if for any point* $(t,x) \in \mathcal{S}$ *and all* $\varphi \in \mathcal{C}^{1,2}(\mathcal{S})$ *such that* $(u_* - \varphi)$ *has a local minimum in* (t,x),

$$F(t,x,u_*(t,x),\frac{\partial \varphi}{\partial t}(t,x),D_x\varphi(t,x),D_x^2\varphi(t,x)) \ge 0.$$

(iii) *We say that u is a* (discontinuous) viscosity solution *of* (A.1) *if it is both a viscosity subsolution and a viscosity supersolution of* (A.1).

In consideration of the first and second order condition of a maximum and a minimum, the requirement (A.2) secures that any classical solution $u \in \mathcal{C}^{1,2}(\mathcal{S})$ of (A.1) is also a viscosity solution.

Remark A.2.1. (a) Without loss of generality we can replace the requirement of a "local maximum" of $(u^* - \varphi)$ in (t,x) in the definition of a viscosity subsolution by a "strict local maximum", "global maximum" or "strict global maximum". To see this, just add to φ the function $h(y) = (y-x)^4$ which satisfies $Dh(x) = 0$ and $D^2h(x) = 0$. Furthermore, if $(u^* - \varphi)$ has a local maximum in (t,x), by the local boundedness of u there exists a function $\tilde{\varphi} \in \mathcal{C}^{1,2}(\mathcal{S})$ such that $\tilde{\varphi} = \varphi$ locally around (t,x) and $(u^* - \tilde{\varphi})$ has a global maximum in (t,x). By adding constants to φ we may even assume $\varphi \ge u^*$, $\varphi(t,x) = u^*(t,x)$. Apparently, similar modifications for the minimum of $(u_* - \varphi)$ in the supersolution case hold.

(b) If F is not continuous, but locally bounded and fulfills the degenerate ellipticity condition A.2, then the definition for viscosity solutions persists, when we consider the lsc envelope F_* in the subsolution inequality and the usc envelope F^* in the supersolution inequality. For example, u is a viscosity subsolution if for any point $(t,x) \in \mathcal{S}$ and all $\varphi \in \mathcal{C}^{1,2}(\mathcal{S})$ such that $(u^* - \varphi)$ has a local maximum in (t,x),

$$F_*(t,x,u^*(t,x),\frac{\partial \varphi}{\partial t}(t,x),D_x\varphi(t,x),D_x^2\varphi(t,x)) \le 0.$$

Here the envelopes applied to F are taken with respect to all arguments.

It will be useful to give another definition of viscosity solutions which is equivalent to Definition A.2.1. To this end let us introduce as in Crandall, Ishii and Lions [10] the second order semijets: We define the superjet $\mathcal{J}^{2,+}v(t,x)$ of a function $v \in \mathrm{USC}(\mathcal{S})$ at the point $(t,x) \in \mathcal{S}$ by

$$\begin{aligned}\mathcal{J}^{2,+}v(t,x) := \{(a,p,X) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d : v(s,y) \le v(t,x) + a(s-t) + p^T(y-x) \\ + \frac{1}{2}(y-x)^T X(y-x) + o(|s-t| + |y-x|^2) \text{ as } \mathcal{S} \ni (s,y) \to (t,x)\}.\end{aligned}$$

Similarly, we define the subjet $\mathcal{J}^{2,-}w(t,x)$ of a function $w \in \mathrm{LSC}(\mathcal{S})$ at the point $(t,x) \in \mathcal{S}$ by

$$\mathcal{J}^{2,-}w(t,x) := \{(a,p,X) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d : w(s,y) \geq w(t,x) + a(s-t) + p^T(y-x) + \frac{1}{2}(y-x)^T X(y-x) + o(|s-t| + |y-x|^2) \text{ as } \mathcal{S} \ni (s,y) \to (t,x)\}.$$

When we want stress the dependence on the set $\mathcal{S} = [0,T) \times \mathcal{O}$, we write the semijets with subscript $\mathcal{S}$ or $\mathcal{O}$, e.g. $\mathcal{J}^{2,+}_{\mathcal{O}}$.

It is proved in Fleming and Soner [15] (see Chapter V, Lemma 4.1) that

$$\mathcal{J}^{2,+}v(t,x) = \left\{ \left(\frac{\partial \varphi}{\partial t}(t,x), D_x\varphi(t,x), D_x^2\varphi(t,x) \right) : \varphi \in \mathcal{C}^{1,2}(\mathcal{S}) \text{ and } (v-\varphi) \text{ has a local maximum in } (t,x) \right\}$$

and the same for $\mathcal{J}^{2,-}w(t,x)$ with a local minimum of $(w-\varphi)$ in (t,x) instead of a maximum. Consequently we can connect Definition A.2.1 with the concept of semijets: A function $u : \mathcal{S} \to \mathbb{R}$ is a viscosity subsolution of (A.1) if and only if

$$F(t,x,u^*(t,x),a,p,X) \leq 0 \quad \text{for all} \quad (t,x) \in \mathcal{S}, (a,p,X) \in \mathcal{J}^{2,+}u^*(t,x), \tag{A.3}$$

and it is a viscosity supersolution of (A.1) if and only if

$$F(t,x,u_*(t,x),a,p,X) \geq 0 \quad \text{for all} \quad (t,x) \in \mathcal{S}, (a,p,X) \in \mathcal{J}^{2,-}u_*(t,x). \tag{A.4}$$

For technical reasons we want to extend these viscosity solution qualifications to the following limiting semijets: For $v \in \mathrm{USC}(\mathcal{S})$, $w \in \mathrm{LSC}(\mathcal{S})$ and $(t,x) \in \mathcal{S}$ we set

$$\begin{aligned}\bar{\mathcal{J}}^{2,+}v(t,x) = \{&(a,p,X) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d : \text{ for all } k \in \mathbb{N} \text{ there exists} \\ &(t_k,x_k,a_k,p_k,X_k) \in \mathcal{S} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \text{ svch that } (a_k,p_k,X_k) \in \mathcal{J}^{2,+}v(t_k,x_k) \\ &\text{and } (t_k,x_k,v(t_k,x_k),a_k,p_k,X_k) \to (t,x,v(t,x),a,p,X)\}\end{aligned}$$

and similarly for $\bar{\mathcal{J}}^{2,-}w(t,x)$. Let us also remark the obvious relations $\mathcal{J}^{2,-}v = -\mathcal{J}^{2,+}(-v)$ and $\bar{\mathcal{J}}^{2,-}w = -\bar{\mathcal{J}}^{2,+}(-w)$.

For reasons of continuity of F, the characterizations of viscosity subsolutions and supersolutions by the inequalities (A.3) and (A.4), respectively, remain true if $\mathcal{J}^{2,+}$ and $\mathcal{J}^{2,-}$ are replaced by $\bar{\mathcal{J}}^{2,+}$ and $\bar{\mathcal{J}}^{2,-}$, respectively. So we arrive at the following result:

Proposition A.2.2. *A locally bounded function $u : \mathcal{S} \to \mathbb{R}$ is a viscosity subsolution of* (A.1) *if and only if*

$$F(t,x,u^*(t,x),a,p,X) \leq 0 \quad \textit{for all} \quad (t,x) \in \mathcal{S}, (a,p,X) \in \bar{\mathcal{J}}^{2,+}u^*(t,x).$$

The function u is a viscosity supersolution of (A.1) *if and only if*

$$F(t,x,u_*(t,x),a,p,X) \geq 0 \quad \textit{for all} \quad (t,x) \in \mathcal{S}, (a,p,X) \in \bar{\mathcal{J}}^{2,-}u_*(t,x).$$

A.3 Tools for uniqueness proof

Let us recall two results from Crandall, Ishii and Lions [10] which are essential for proving the uniqueness of viscosity solutions. The first one is a maximum principle for semicontinuous functions which is a generalization of the standard maximum principle for $\mathcal{C}^{1,2}$-functions:

Theorem A.3.1 (Theorem 8.3 in [10]). *Let* $u_i \in USC((0,T)\times\mathcal{O}_i)$, $i = 1,\dots,k$, *where* $\mathcal{O}_i$ *is a locally compact subset of* $\mathbb{R}^{N_i}$. *Let* φ *be defined on an open neighborhood of* $(0,T)\times\mathcal{O}_1\times\dots\times\mathcal{O}_k$ *and such that* $(t,x_1,\dots,x_k)\mapsto\varphi(t,x_1,\dots,x_k)$ *is once continuously differentiable in* t *and twice continuously differentiable in* $(x_1,\dots,x_k)\in\mathcal{O}_1\times\dots\times\mathcal{O}_k$. *Suppose that* $\hat{t}\in(0,T)$, $\hat{x}_i\in\mathcal{O}_i$, $i=1,\dots,k$, *and*

$$w(t,x_1,\dots,x_k) := u_1(t,x_1)+\dots+u_k(t,x_k)-\varphi(t,x_1,\dots,x_k)\le w(\hat{t},\hat{x}_1,\dots,\hat{x}_k)$$

for $0<t<T$ *and* $x_i\in\mathcal{O}_i$. *Assume, moreover, that there is an* $r>0$ *such that for every* $M>0$ *there is a* C *such that for* $i=1,\dots,k$,

$$b_i\le C \text{ whenever } (b_i,q_i,X_i)\in\mathcal{J}^{2,+}_{\mathcal{O}_i}u_i(t,x_i),$$
$$|x_i-\hat{x}_i|+|t-\hat{t}|\le r \text{ and } |u_i(t,x_i)|+|q_i|+|X_i|\le M.$$

Then, for each $\varepsilon>0$ *there are* $X_i\in\mathbb{S}(N_i)$ *such that*

$$(i)\ (b_i, D_{x_i}\varphi(\hat{t},\hat{x}_1,\dots,\hat{x}_k),X_i)\in\bar{\mathcal{J}}^{2,+}_{\mathcal{O}_i}u_i(\hat{t},\hat{x}_i) \text{ for } i=1,\dots,k,$$

$$(ii)\ -\left(\frac{1}{\varepsilon}+|A|\right)I\le\begin{pmatrix}X_1&\dots&0\\ \vdots&\ddots&\vdots\\ 0&\dots&X_k\end{pmatrix}\le A+\varepsilon A^2,$$

$$(iii)\ b_1+\dots+b_k=\varphi_t(\hat{t},\hat{x}_1,\dots,\hat{x}_k),$$

where $A=(D_x^2\varphi)(\hat{t},\hat{x}_1,\dots,\hat{x}_k)$.

Moreover, a general uniqueness proof makes use of the following Lemma:

Lemma A.3.2 (Lemma 3.1 in [10]). *Let* $\mathcal{O}$ *be a subset of* $\mathbb{R}^N$, $u\in USC(\mathcal{O})$, $v\in LSC(\mathcal{O})$ *and*

$$M_\alpha=\sup_{\mathcal{O}\times\mathcal{O}}\left\{u(x)-v(y)-\frac{\alpha}{2}|x-y|^2\right\}$$

for $\alpha>0$. *Let* $M_\alpha<\infty$ *for large* α *and* (x_α,y_α) *be such that*

$$\lim_{\alpha\to\infty}\left\{M_\alpha-\left(u(x_\alpha)-v(y_\alpha)-\frac{\alpha}{2}|x_\alpha-y_\alpha|^2\right)\right\}=0.$$

Then, $\lim_{\alpha\to\infty}\alpha|x_\alpha-y_\alpha|^2=0$ *and*

$$\lim_{\alpha\to\infty}M_\alpha=u(\hat{x})-v(\hat{y})=\sup_{\mathcal{O}}\{u(x)-v(x)\}$$

whenever $\hat{x}\in\mathcal{O}$ *is a limit point of* x_α *as* $\alpha\to\infty$.

Appendix B

Auxiliary tools

B.1 Estimates of the distribution of a jump diffusion process

Given the stochastic basis of Chapter 3 we consider the stochastic process $X = X^{\pi}_{t,x}$ started in $(t,x) \in [0,T) \times \mathbb{R}^d$ and continuously controlled with $\pi \in \mathcal{U}_0$,

$$\begin{aligned} dX(s) &= \mu(s, X(s), \pi(s))dt + \sigma(s, X(s), \pi(s))dB(s) \\ &\quad + \int_{\mathbb{R}^k} \gamma(s, X(s-), \pi(s-), z))d\tilde{N}(ds, dz), \quad X(t) = x, \end{aligned} \tag{B.1}$$

where the coefficients satisfy the usual growth and Lipschitz conditions, i.e. there exist $C > 0$, $\delta : \mathbb{R}^k \to \mathbb{R}_+$ with $\int_{\mathbb{R}^k} \delta^2(z)\nu(dz) < \infty$ such that for all $t \in [0,T]$, $x, y \in \mathbb{R}^d$ and $\pi \in U$,

$$\begin{aligned} |\mu(t,x,\pi) - \mu(t,y,\pi)| + |\sigma(t,x,\pi) - \sigma(t,y,\pi)| &\leq C|x-y|, \\ |\gamma(t,x,\pi,z) - \gamma(t,y,\pi,z)| &\leq \delta(z)|x-y|, \\ |\gamma(t,x,\pi,z)| &\leq \delta(z)(1+|x|). \end{aligned}$$

For the moments of the purely π-controlled jump diffusion (B.1), we get the following estimates:

Lemma B.1.1. *For any $k \in [0,2]$ there exists a constant $C > 0$ such that for all $t \in [0,T]$, $h \in [t,T]$, $x, y \in \mathbb{R}^d$, $\pi \in \mathcal{U}_0$ and any stopping time $\tau \in [t, t+h]$ we have*

$$\mathbb{E}\left[|X^{\pi}_{t,x}(\tau)|^k\right] \leq C(1+|x|^k), \tag{B.2}$$

$$\mathbb{E}\left[|X^{\pi}_{t,x}(\tau) - x|^k\right] \leq C(1+|x|^k)h^{\frac{k}{2}}, \tag{B.3}$$

$$\mathbb{E}\left[\left(\sup_{t \leq s \leq t+h} |X^{\pi}_{t,x}(s) - x|\right)^k\right] \leq C(1+|x|^k)h^{\frac{k}{2}}, \tag{B.4}$$

$$\mathbb{E}\left[|X^{\pi}_{t,x}(\tau) - X^{\pi}_{t,y}(\tau)|^k\right] \leq C|x-y|^k. \tag{B.5}$$

Proof: See Lemma 3.1 in [44]. □

We use the estimates (B.3) and (B.4) only in the pure continuous control setting without any impulses. If in the context of impulse perturbation the transaction function Γ is Lipschitz in x (uniformly in t, π and ζ), then, by the repeated use of the tower property of conditional expectation, (B.2) and (B.5) hold true also for impulse perturbed controlled processes. For example we obtain (B.5) for a 1-impulse control $\xi = (\tau_1; \zeta_1) \in \mathcal{V}_1$ and $\pi = (\pi^{(1)}, \pi^{(0)}) \in \mathcal{U}_1$ by

$$\begin{aligned}
\mathbb{E}\left[\left|X_{t,x}^{\pi,\xi}(\tau) - X_{t,y}^{\pi,\xi}(\tau)\right|^k\right] &= \mathbb{E}\left[\mathbb{E}\left[\left|X^{\pi^{(0)}}_{\tau_1, X_{t,x}^{\pi,\xi}(\tau_1)}(\tau) - X^{\pi^{(0)}}_{\tau_1, X_{t,y}^{\pi,\xi}(\tau_1)}(\tau)\right|^k \middle| \mathcal{F}_{\tau_1}\right]\right] \\
&\leq \mathbb{E}\left[C\left|\Gamma(\tau_1, X_{t,x}^{\pi^{(1)}}(\tau_1), \pi^{(1)}(\tau_1), \zeta_1) - \Gamma(\tau_1, X_{t,y}^{\pi^{(1)}}(\tau_1), \pi^{(1)}(\tau_1), \zeta_1)\right|^k\right] \\
&\leq \mathbb{E}\left[C'\left|X_{t,x}^{\pi^{(1)}}(\tau_1) - X_{t,y}^{\pi^{(1)}}(\tau_1)\right|^k\right] \\
&\leq C''|x-y|^k,
\end{aligned}$$

where $C, C', C'' > 0$ are constants and we have set $\xi = (\tau, \zeta_0)$ in the case $\tau_1 > \tau$.

Using the estimates of Lemma B.1.1 one can show the following two lemmas which are needed for the proof of the supersolution property of the value function of the target problem. To this end, it is not necessary to prove the assertions for an underlying jump diffusion process. But we also use the following lemma for the existence proof of a viscosity solution for the differential game in the extended model.

Lemma B.1.2. *Let $(t_k, x_k)_k$ be a sequence in $[0,T] \times \mathbb{R}^d$ converging to $(t_0, x_0) \in [0,T] \times \mathbb{R}^d$ and let $(\pi_k)_k$ be a sequence in $\mathcal{U}_0$. Then, for any sequence $(t'_k)_k$ in $[0,T]$ such that $t'_k \geq t_k$ and $t'_k \to t_0$, we have*

$$\sup_{t_k \leq s \leq t'_k} |X_{t_k,x_k}^{\pi_k}(s) - x_0| \to 0 \quad \text{in } L^2.$$

Proof: By Doob's inequality for martingales we have

$$\begin{aligned}
\mathbb{E}\left[\left(\sup_{t_n \leq s \leq t'_n} |X_{t_n,x_n}^{\pi_n}(s) - x_0|\right)^2\right] \leq C\Bigg\{ & |x_n - x_0|^2 + \mathbb{E}\left[\int_{t_n}^{t'_n} |\mu(s, X_{t_n,x_n}^{\pi_n}(s), \pi_n(s))|^2 ds\right] \\
& + \mathbb{E}\left[\int_{t_n}^{t'_n} |\sigma(s, X_{t_n,x_n}^{\pi_n}(s), \pi_n(s))|^2 ds\right] \\
& + \mathbb{E}\left[\int_{t_n}^{t'_n}\int_{\mathbb{R}^k} |\gamma(s, X_{t_n,x_n}^{\pi_n}(s), \pi_n(s), z)|^2 \nu(dz) ds\right]\Bigg\}.
\end{aligned}$$

Using the growth conditions on the coefficients, Fubini's theorem and the estimate (B.2), we obtain

$$\mathbb{E}\left[\left(\sup_{t_n \leq s \leq t'_n} |X_{t_n,x_n}^{\pi_n}(s) - x_0|\right)^2\right] \leq C\left\{|x_n - x_0|^2 + (t'_n - t_n)(1 + |x_n|^2)\right\}.$$

The proof is concluded by sending n to infinity. □

Lemma B.1.3. *Let $\psi : [0,T]\times\mathbb{R}^d\times U \to \mathbb{R}$ be locally Lipschitz in $(t,x) \in [0,T]\times\mathbb{R}^d$ uniformly in $\pi \in U$. Further, let $(h_k)_k$ be a strictly positive sequence converging to 0, $(t_k, x_k)_k$ a sequence in $[0,T] \times \mathbb{R}^d$ converging to $(t_0, x_0) \in [0,T] \times \mathbb{R}^d$ and let $(\pi_k)_k$ be a sequence in $\mathcal{U}_0$. Then we have*

$$\frac{1}{h_k}\int_{t_k}^{t_k+h_k}\{\psi(s, X^{\pi_k}_{t_k,x_k}(s), \pi_k(s)) - \psi(t_0, x_0, \pi_k(s))\}ds \to 0 \quad \text{in } L^2.$$

Proof: See Lemma 4.1 in [54]. □

B.2 Dynkin's formula

Given the stochastic basis of Chapter 3 we consider the stochastic process X evolving according to the SDE

$$dX(s) = \mu(s, X(s))dt + \sigma(s, X(s))dB(s) + \int_{\mathbb{R}^k}\gamma(s, X(s-), z))d\tilde{N}(ds, dz), \tag{B.6}$$

where μ, σ, γ are appropriate (possibly random) functions such that there is a unique strong solution to (B.6).

For some sufficiently regular map φ and some finite stopping time τ, Dynkin's formula constitutes the expected value of $\varphi(\tau, X(\tau))$ in dependence of the Dynkin second order integro-differential operator $\mathcal{L}$ associated to the process X,

$$\begin{aligned}\mathcal{L}\varphi(t,x) = {}& \frac{\partial\varphi}{\partial t}(t,x) + \mu(t,x)^T D_x\varphi(t,x) + \frac{1}{2}\mathrm{tr}\left((\sigma\sigma^T)(t,x)D_x^2\varphi(t,x)\right)\\ &+ \int_{\mathbb{R}^k}\{\varphi(t, x+\gamma(t,x,z)) - \varphi(t,x) - \gamma(t,x,z)^T D_x\varphi(t,x)\}\nu(dz).\end{aligned}$$

For jump diffusion processes we need the following localized version of Dynkin's formula:

Theorem B.2.1 (Dynkin's formula)**.** *Let $\mathcal{O} \subset \mathbb{R}^d$ be an open set and define $\mathcal{S} := [0,T) \times \mathcal{O}$. Let $(t,x) \in \mathcal{S}$, $\varphi \in \mathcal{C}^{1,2}(\mathcal{S}) \cap \mathcal{C}(\overline{\mathcal{S}})$ and let $\tau \in [t,T]$ be a stopping time such that*

$$\begin{aligned}&\tau \le \tau_{\mathcal{S}} := \inf\{s \ge t : (s, X(s)) \notin \mathcal{S}\},\\ &(\tau, X(\tau)) \in \overline{\mathcal{S}} \text{ a.s.},\\ &\mathcal{L}\varphi(s, X(s)) \text{ is well-defined for } s \in [t,\tau]\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}_{t,x}\Bigg[&|\varphi(\tau, X(\tau))| + \int_t^\tau\{|\sigma(s,X(s))^T D_x\varphi(s,X(s))|^2\\ &+ \int_{\mathbb{R}^k}|\varphi(s, X(s-) + \gamma(s, X(s-), z)) - \varphi(s, X(s-))|^2\nu(dz)\}ds\Bigg] < \infty.\end{aligned} \tag{B.7}$$

Then we have

$$\mathbb{E}_{t,x}\left[\varphi(\tau, X(\tau))\right] = \varphi(t,x) + \mathbb{E}_{t,x}\left[\int_t^\tau \mathcal{L}(s, X(s))ds\right].$$

Proof: The proof of Dynkin's formula consists of applying Itô's formula to $\varphi(\tau, X(\tau))$ and taking expectation, so that the integrals with respect to the Brownian motion and the Poisson martingale measure disappear. □

Let us briefly verify that this version is strong enough for our applications:

Assume the growth and Lipschitz conditions corresponding to the setting in Chapter 3, i.e. there exist $C > 0$, $\delta : \mathbb{R}^k \to \mathbb{R}_+$ with $\int_{\mathbb{R}^k} \delta^2(z)\nu(dz) < \infty$ such that for all $t \in [0,T]$, $x, y \in \mathbb{R}^d$,

$$
\begin{aligned}
|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| &\leq C|x-y|, \\
|\gamma(t,x,z) - \gamma(t,y,z)| &\leq \delta(z)|x-y|, \\
|\gamma(t,x,z)| &\leq \delta(z)(1+|x|).
\end{aligned}
$$

We claim that Dynkin's formula is applicable for all $(t,x) \in [0,T) \times \mathbb{R}^d$, $\varphi \in \mathcal{C}_1 \cap \mathcal{C}^{1,2}([0,T) \times \mathbb{R}^d)$ and each stopping time

$$\tau \leq \tau_\rho := \inf\{s \geq t : |X(s) - x| \geq \rho\} \wedge (t+\rho), \quad \rho \in (0, T-t).$$

In Section 3.5 we have shown that $\mathcal{L}\varphi$ is well-defined on $[0,T) \times \mathbb{R}^d$. Moreover, $(\tau, X(\tau)) \in [0,T] \times \mathbb{R}^d$ holds trivially. It remains to check condition (B.7) which follows directly from the growth conditions on φ and on the coefficients σ, γ and from the boundedness of $D_x\varphi$ on the compact set $[t, t+\rho] \times \overline{\mathcal{B}(x,\rho)}$.

For the setting in Chapter 4 it is sufficient to consider the case when there are no jumps, i.e. $N = 0$. Apparently, Dynkin's formula is then applicable to $\varphi(\tau, X(\tau))$ for all $\varphi \in \mathcal{C}^{1,2}([0,T) \times \mathbb{R}^d)$ and $\tau \leq \tau_\rho$ with τ_ρ from above.

B.3 Comparison theorem for ODEs

For the discussion of qualitative properties of the optimal portfolio and reinsurance processes in Chapter 5 and Chapter 6, respectively, we make use of the following lemma, a comparison theorem for ODEs.

Lemma B.3.1. *Let $u, v \in \mathcal{C}^1((0,T)) \cap \mathcal{C}([0,T])$ with $v \geq 0$ and suppose that $h : \mathbb{R} \to \mathbb{R}$ satisfies for some $C > 0$ the one-sided Lipschitz condition*

$$h(z) - h(\tilde{z}) \leq C(z - \tilde{z}) \quad \textit{for } z \geq \tilde{z} \geq 0.$$

Then

$$
\begin{aligned}
-u_t - h(u) &\geq -v_t - h(v) \quad \textit{on } [0,T), \\
u(T) &\geq v(T)
\end{aligned}
$$

implies $u \geq v$ on $[0,T]$.

Proof: The statement is a special case of Lemma XV in Walter [57], Section 28 in Chapter IV. □

We continue with an application of Lemma B.3.1 in the context of portfolio optimization under the threat of market crashes.

Lemma B.3.2. *The optimal portfolio strategies* $\hat{\pi} = (\hat{\pi}^{(n)}, \dots, \hat{\pi}^{(0)})$ *derived in Example 5.1.1 to maximize expected power utility, log utility and exponential utility at a terminal time* T *each satisfy*

(i) $0 \le \hat{\pi}^{(n)} \le \hat{\pi}^{(n-1)} \le \dots \le \hat{\pi}^{(0)}$,

(ii) $\hat{\pi}^{(i)}_t \le 0$ *for all* $i = 1, \dots, n$.

Proof: We carry out the proof in the case of power utility. First we have to transform the ODE characterizing $\hat{\pi}^{(i)}$ to separate the variables $\hat{\pi}^{(i)}$ and $\hat{\pi}^{(i-1)}$ in the equation. We substitute $1 - \bar{\zeta}\hat{\pi}^{(i)} = e^{-f^{(i)}(t)}$ giving us the corresponding ODE in form of

$$
\begin{aligned}
f^{(i)}_t &= -\frac{\mu - r}{\bar{\zeta}} \left(e^{-f^{(i)}} - e^{-f^{(i-1)}} \right) - \frac{(1-\gamma)\sigma^2}{2\bar{\zeta}^2} \left(\left(1 - e^{-f^{(i)}}\right)^2 - \left(1 - e^{-f^{(i-1)}}\right)^2 \right), \\
f^{(i)}(T) &= 0, \quad i = 1, \dots, n.
\end{aligned}
$$

We set

$$h(z) := \frac{\mu - r}{\bar{\zeta}} e^{-z} + \frac{(1-\gamma)\sigma^2}{2\bar{\zeta}^2} \left(1 - e^{-z}\right)^2.$$

For $z \ge \tilde{z} \ge 0$ one derives the one-sided Lipschitz condition

$$
\begin{aligned}
h(z) - h(\tilde{z}) &\le \frac{(1-\gamma)\sigma^2}{2\bar{\zeta}^2} \left(\left(1 - e^{-z}\right)^2 - \left(1 - e^{-\tilde{z}}\right)^2 \right) \\
&= \frac{(1-\gamma)\sigma^2}{2\bar{\zeta}^2} \left(2 - e^{-z} - e^{-\tilde{z}}\right) \left(e^{-\tilde{z}} - e^{-z}\right) \le \frac{(1-\gamma)\sigma^2}{\bar{\zeta}^2} \left(e^{-\tilde{z}} - e^{-z}\right) \\
&\le \frac{(1-\gamma)\sigma^2}{\bar{\zeta}^2} \max_{\theta \ge 0} e^{-\theta} (z - \tilde{z}) = \frac{(1-\gamma)\sigma^2}{\bar{\zeta}^2} (z - \tilde{z}).
\end{aligned}
$$

For $i = 1$ we note that $\hat{\pi}^{(0)} = \frac{1}{1-\gamma}\frac{\mu - r}{\sigma^2} \ge 0$ is constant and therefore $f^{(0)} \ge 0$ and $f^{(0)}_t = 0$. Thus, in view of the ODE characterization of $f^{(1)}$ we have $-f^{(1)}_t - h(f^{(1)}) = -f^{(0)}_t - h(f^{(0)})$. Then, supposed $f^{(1)} \ge 0$, we can argue by the use of Lemma B.3.1 that the terminal condition $f^{(1)}(T) = 0$ implies $f^{(1)} \le f^{(0)}$, i.e. $\hat{\pi}^{(1)} \le \hat{\pi}^{(0)}$. Moreover, using the equation $f^{(1)}_t = -h(f^{(1)}) + h(f^{(0)})$ and noting that $f^{(0)}$ maximizes the map $z \mapsto -h(z) + h(f^{(0)})$, we conclude $f^{(1)}_t \le 0$ and equivalently $\hat{\pi}^{(1)}_t \le 0$. Combining the latter result with the terminal condition for $f^{(1)}$ and $\hat{\pi}^{(1)}$ proves the positivity of both processes.

For the induction step we suppose $f^{(i)}_t \le 0$ for $i = 1, \dots, n-1$. Then the ODE system tells us $f^{(i)}_t = -h(f^{(i)}) + h(f^{(i-1)}) \le 0$ for all $i = 1, \dots, n-1$, so that we can estimate $f^{(n)}_t = -h(f^{(n)}) + h(f^{(0)})$. Then the monotony and the positivity of $f^{(n)}$ follow as in the initial step. From the ODE for $f^{(n)}$ we deduce $-f^{(n)}_t - h(f^{(n)}) \le -f^{(n-1)}_t - h(f^{(n-1)})$. Noting the terminal condition $f^{(n)}(T) = f^{(n-1)}(T) = 0$, by Lemma B.3.1 it follows $f^{(n)} \le f^{(n-1)}$. Translating the results for the process $\hat{\pi}^{(n)}$ ends the proof.

The proof of the properties (i) and (ii) in the log utility case is completely analogous, the one in the exponential utility case is similar to the proof of Lemma 6.2.3. □

Appendix C

Proof of the PDE characterization for the target problem

C.1 Subsolution property on $[0,T) \times \mathbb{R}^d$

Proof of Theorem 4.4.3: To make the notion more convenient we write v instead of $v^{(n)}$ in the following. Let $(t_0, x_0) \in \mathcal{S} := [0,T) \times \mathbb{R}^d$ and $\varphi \in \mathcal{C}^{1,2}(\mathcal{S})$ such that $\varphi(t_0,x_0) = v^*(t_0,x_0)$ and $\varphi > v^*$ on $\mathcal{S} \setminus \{(t_0,x_0)\}$. If the set $\mathcal{N}\varphi(t_0,x_0)$ is empty, the subsolution inequality holds trivially by the convention $\min(\sup\varnothing, a) = a$ with $a = \delta_N\varphi(t_0,x_0) < 0$. So let us suppose from now on that $\mathcal{N}\varphi(t_0,x_0)$ is non-empty. We have to show that for all $\pi \in \mathcal{N}\varphi(t_0,x_0)$ we have

$$\min\Big(-\mathcal{L}^{\pi}\varphi(t_0,x_0) + \mu_Y(t_0,x_0,\varphi(t_0,x_0),\pi), \mathcal{M}^{\pi}_{\varphi(t_0,x_0)}(v^{(n-1)})^*(t_0,x_0)\Big) \leq 0.$$

Let us assume to the contrary that there exist $\pi_0 \in \mathcal{N}\varphi(t_0,x_0)$ and $\eta > 0$ such that

$$-\mathcal{L}^{\pi_0}\varphi(t_0,x_0) + \mu_Y(t_0,x_0,\varphi(t_0,x_0),\pi_0) \geq 2\eta \text{ and } \mathcal{M}^{\pi_0}_{\varphi(t_0,x_0)}(v^{(n-1)})^*(t_0,x_0) \geq 2\eta, \quad \text{(C.1)}$$

and let us work towards a contradiction to (DP2). We do so in several steps.

Step 1. According to Assumption 4.2.1 there exists a continuous map $\hat{\pi}$ on some open neighborhood of (t_0,x_0) in $\mathcal{S}$ satisfying $\hat{\pi}(t,x) \in \mathcal{N}\varphi(t,x)$ and $\hat{\pi}(t_0,x_0) = \pi_0$. Then, in view of the lower semicontinuity of $\mathcal{M}^{\pi}_y(v^{(n-1)})^*$ in (t,x,y,π), the relation $\mathcal{M}^{\pi}_y v^{(n-1)} \geq \mathcal{M}^{\pi}_y(v^{(n-1)})^*$ and the continuity of the remaining functions which are involved in (C.1), we can find some $\rho > 0$ such that

$$\begin{gathered} -\mathcal{L}^{\hat{\pi}(t,x)}\varphi(t,x) + \mu_Y(t,x,y,\hat{\pi}(t,x)) \geq \eta \text{ and } \mathcal{M}^{\hat{\pi}(t,x)}_y v^{(n-1)}(t,x) \geq \eta \\ \text{for all } (t,x,y) \in \mathcal{S} \times \mathbb{R} \text{ such that } (t,x) \in \overline{\mathcal{B}}_\rho \text{ and } |y - \varphi(t,x)| \leq \rho \end{gathered} \quad \text{(C.2)}$$

and

$$\hat{\pi}(t,x) \in \mathcal{N}\varphi(t,x) \quad \text{for all } (t,x) \in \overline{\mathcal{B}}_\rho. \quad \text{(C.3)}$$

Here $\mathcal{B}_\rho := \mathcal{B}((t_0, x_0), \rho) := (t_0 - \rho, t_0 + \rho) \times \mathcal{B}(x_0, \rho)$ denotes the open parabolic ball surrounding (t_0, x_0) with radius ρ (For $t_0 = 0$ we set $\mathcal{B}_\rho := [0, \rho) \times \mathcal{B}(x, \rho)$.).

Step 2. From the definition of v^* there exists a sequence $(t_k, x_k)_{k\in\mathbb{N}} \subset \mathcal{S}$ such that $(t_k, x_k) \to (t_0, x_0)$ and $v(t_k, x_k) \to v^*(t_0, x_0)$ as $k \to \infty$. Choose a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ with $\varepsilon_k \searrow 0$ and set $y_k := v(t_k, x_k) - \varepsilon_k$ and $\delta_k := y_k - \varphi(t_k, x_k)$. Then, $\delta_k \to 0$ for $k \to \infty$.

Let (X_k, Y_k) be the solution of the state equation (4.1) with start in $X_k(t_k) = x_k$, $Y_k(t_k) = y_k$ and controlled via $\hat{\pi}_k(t) := \hat{\pi}(t, X_k(t))$, i.e. $(X_k, Y_k) = (X^{\hat{\pi}_k}_{t_k,x_k}, Y^{\hat{\pi}_k}_{t_k,x_k,y_k})$. To be more precise, the feedback control function $\hat{\pi}_k(.,.)$ is only well-defined on $\overline{\mathcal{B}}_\rho$. But for our purpose it is sufficient to consider the controlled system up to the first exit time of $(t, X_k(t))$ of $\mathcal{B}_\rho$. Remark further that due to the continuity of the map $\hat{\pi}_k$ and the boundedness of $\overline{\mathcal{B}}_\rho$ we can ensure to find a continuation of $\hat{\pi}_k$ outside $\overline{\mathcal{B}}_\rho$ such that the associated control $\hat{\pi}_k$ is admissible. In addition we define the process

$$\hat{Y}_k(t) := \varphi(t, X_k(t)) - \varepsilon_k$$

and compare it to the process Y_k. Applying Itô's formula we obtain

$$d\hat{Y}_k(t) = \mathcal{L}^{\hat{\pi}_k(t)}\varphi(t, X_k(t))dt + D_x\varphi(t, X_k(t))^T \sigma_X(t, X_k(t), \hat{\pi}_k(t))dB(t).$$

Taking into account (C.3) and using the definition of $\mathcal{N}$, we can conclude that

$$D_x\varphi(t, X_k(t))^T \sigma_X(t, X_k(t), \hat{\pi}_k(t)) = \sigma_Y(t, X_k(t), Y_k(t), \hat{\pi}_k(t)) \text{ as long as } (t, X_k(t)) \in \overline{\mathcal{B}}_\rho.$$

So in this situation we have

$$d\hat{Y}_k(t) = \mathcal{L}^{\hat{\pi}_k(t)}\varphi(t, X_k(t))dt + \sigma_Y(t, X_k(t), Y_k(t), \hat{\pi}_k(t))dB(t). \tag{C.4}$$

Recall that Y_k solves the same SDE with a different drift term,

$$dY_k(t) = \mu_Y(t, X_k(t), Y_k(t), \hat{\pi}_k(t))dt + \sigma_Y(t, X_k(t), Y_k(t), \hat{\pi}_k(t))dB(t). \tag{C.5}$$

Step 3. We define the stopping times

$$\tau_k^{\mathcal{B}} := \inf\{t \ge t_k : (t, X_k(t)) \notin \mathcal{B}_\rho\} \qquad \text{and}$$
$$\tau_k := \inf\{t \ge t_k : |Y_k(t) - \varphi(t, X_k(t))| \ge \rho\} \wedge \tau_k^{\mathcal{B}}.$$

Because of $(t_k, x_k) \to (t_0, x_0) \in \mathcal{B}_\rho$ and $Y_k(t_k) - \varphi(t_k, X_k(t_k)) = \delta_k \to 0$ as $k \to \infty$, we have $\tau_k > t_k$ for large k. Further, the definition of τ_k guarantees that both $(t, X_k(t)) \in \overline{\mathcal{B}}_\rho$ and $|Y_k(t) - \varphi(t, x)| \le \rho$ hold for all $t \in [t_k, \tau_k]$. Therefore (C.2) yields

$$\mathcal{M}^{\hat{\pi}_k(t)}_{Y_k(t)} v^{(n-1)}(t, X_k(t)) \ge \eta \quad \text{for all } t \in [t_k, \tau_k] \tag{C.6}$$

and

$$-\mathcal{L}^{\hat{\pi}_k(t)}\varphi(t, X_k(t)) + \mu_Y(t, X_k(t), Y_k(t), \hat{\pi}_k(t)) \ge \eta \quad \text{for all } t \in [t_k, \tau_k].$$

In view of the dynamics of Y_k and $\hat{Y}_k$ given in (C.5) and (C.4), respectively, we can deduce from the last inequality the relation

$$Y_k(t) \geq \hat{Y}_k(t) \quad \text{for all } t \in [t_k, \tau_k]. \tag{C.7}$$

This leads to the conclusion

$$Y_k(\tau_k) - \varphi(\tau_k, X_k(\tau_k)) = \rho \quad \text{for } \tau_k < \tau_k^{\mathcal{B}}. \tag{C.8}$$

To see this, just note that in the case of $\tau_k < \tau_k^{\mathcal{B}}$ we have $|Y_k(\tau_k) - \varphi(\tau_k, X_k(\tau_k))| = \rho > 0$ with $Y_k(\tau_k) - \varphi(\tau_k, X_k(\tau_k)) = Y_k(\tau_k) - \hat{Y}_k(\tau_k) - \varepsilon_k$ and choose a suitably large k.

Step 4. Now we are in the position to induce a contradiction to (DP2). From the definition of $\tau_k^{\mathcal{B}}$ we know that $(\tau_k, X_k(\tau_k)) \in \partial\mathcal{B}_\rho$ holds for $\tau_k = \tau_k^{\mathcal{B}}$. Since the function $\varphi - v^*$ has a strict minimum in $(t_0, x_0) \in \mathcal{B}_\rho$ with minimum zero, its values on the boundary $\partial\mathcal{B}_\rho$ are limited from above by

$$\beta := \min_{\partial\mathcal{B}_\rho}\{\varphi - v^*\} > 0.$$

Using the inequalities $\varphi - \beta \geq v^* \geq v$ on $\partial\mathcal{B}_\rho$ and $\varphi \geq v^* \geq v$ on $\mathcal{B}_\rho$, we can deduce

$$\begin{aligned} Y_k(\tau_k) - v(\tau_k, X_k(\tau_k)) &\geq \mathbb{1}_{\{\tau_k = \tau_k^{\mathcal{B}}\}}\{Y_k(\tau_k) - \varphi(\tau_k, X_k(\tau_k)) + \beta\} \\ &\quad + \mathbb{1}_{\{\tau_k < \tau_k^{\mathcal{B}}\}}\{Y_k(\tau_k) - \varphi(\tau_k, X_k(\tau_k))\} \geq (\beta - \varepsilon_k) \wedge \rho, \end{aligned} \tag{C.9}$$

where the last inequality arises from (C.7), (C.8) and the definition of $\hat{Y}_k$. Recall that by construction we have $(X_k, Y_k) = (X^{\hat{\pi}_k}_{t_k,x_k}, Y^{\hat{\pi}_k}_{t_k,x_k,y_k})$ with $y_k = v(t_k, x_k) - \varepsilon_k < v(t_k, x_k)$. Hence, for a sufficiently large k the inequalities (C.6) and (C.9) represent a contradiction to (DP2).

□

C.2 Supersolution property on $[0, T) \times \mathbb{R}^d$

Proof of Theorem 4.4.4: To make the notion more convenient we write v instead of $v^{(n)}$ in the following. Let $(t_0, x_0) \in \mathcal{S}$ and $\varphi \in \mathcal{C}^{1,2}(\mathcal{S})$ such that $\varphi(t_0, x_0) = v_*(t_0, x_0)$ and $\varphi < v_*$ on $\mathcal{S} \setminus \{(t_0, x_0)\}$. In order to prove the supersolution inequality we have to show that for each $\eta > 0$ there exists $\pi_0 \in \mathcal{N}\varphi(t_0, x_0)$ such that

$$-\mathcal{L}^{\pi_0}\varphi(t_0, x_0) + \mu_Y(t_0, x_0, \varphi(t_0, x_0), \pi_0) \geq -\eta \text{ and } \mathcal{M}^{\pi_0}_{\varphi(t_0,x_0)}(v^{(n-1)})_*(t_0, x_0) \geq -\eta.$$

We split the proof into several steps.

Step 1. From the definition of v_* there exists a sequence $(t_k, x_k)_{k\in\mathbb{N}} \subset \mathcal{S}$ such that $(t_k, x_k) \to (t_0, x_0)$ and $v(t_k, x_k) \to v_*(t_0, x_0)$ for $k \to \infty$. Set $y_k := v(t_k, x_k) + \frac{1}{k}$ and $\delta_k := y_k - \varphi(t_k, x_k)$, and observe that $\delta_k \to 0$ as k tends to infinity. Then, according to (DP1) there exists a control process $\pi_k \in \mathcal{U}_0$ such that for each stopping time $\tau_k \in [t_k, T]$ we have

$$Y^{\pi_k}_{t_k,x_k,y_k}(\tau_k) \geq v(\tau_k, X^{\pi_k}_{t_k,x_k}(\tau_k)) \tag{C.10}$$

and

$$\mathcal{M}^{\pi_k(t)}_{Y^{\pi_k}_{t_k,x_k,y_k}(t)} v^{(n-1)}(t, X^{\pi_k}_{t_k,x_k}(t)) \geq 0 \quad \text{for all } t \in [t_k, \tau_k]. \tag{C.11}$$

We set $X_k := X^{\pi_k}_{t_k,x_k}$ and $Y_k := Y^{\pi_k}_{t_k,x_k,y_k}$. Because of $v \geq v_* \geq \varphi$ and the definition of y_k and δ_k, it follows from (C.10) that

$$\delta_k + (Y_k(\tau_k) - y_k) - (\varphi(\tau_k, X_k(\tau_k)) - \varphi(t_k, x_k)) \geq 0.$$

By Itô's formula this can be read as

$$\begin{aligned} 0 \leq \delta_k &+ \int_{t_k}^{\tau_k} \{\mu_Y(t, X_k(t), Y_k(t), \pi_k(t)) - \mathcal{L}^{\pi_k(t)}\varphi(t, X_k(t))\} dt \\ &+ \int_{t_k}^{\tau_k} \{\sigma_Y(t, X_k(t), Y_k(t), \pi_k(t)) - D_x\varphi(t, X_k(t))^T \sigma_X(t, X_k(t), \pi_k(t))\} dB(t). \end{aligned} \tag{C.12}$$

Step 2. For ease of notation let us use the abbreviation

$$\varsigma_k(t) := \sigma_Y(t, X_k(t), Y_k(t), \pi_k(t)) - D_x\varphi(t, X_k(t))^T \sigma_X(t, X_k(t), \pi_k(t)).$$

For $\lambda \in \mathbb{R}$ we introduce the probability measure $\mathbb{P}^\lambda_k$ equivalent to $\mathbb{P}$ defined by the density process

$$\left.\frac{d\mathbb{P}^\lambda_k}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L^\lambda_k(t) := \mathcal{E}\left(\lambda \int_{t_k}^t \varsigma_k(s) dB(s)\right).$$

Here $\mathcal{E}$ denotes the Doléans-Dade exponential operator, i.e. the process L^λ_k is given as the solution of the SDE

$$\begin{aligned} L^\lambda_k(t_k) &= 1, \\ dL^\lambda_k(t) &= L^\lambda_k(t)\lambda\varsigma_k(t) dB(t) \qquad \text{for } t \geq t_k. \end{aligned}$$

Set $M_k(t) := \int_{t_k}^t \varsigma_k(s) dB(s)$. According to Itô's product law the process $L^\lambda_k M_k$ obeys the dynamics

$$d(L^\lambda_k M_k)(t) = L^\lambda_k(t)\lambda|\varsigma_k(t)|^2 dt + (L^\lambda_k(t)\varsigma_k(t) + L^\lambda_k(t)\lambda\varsigma_k(t)M_k(t)) dB(t).$$

Let us call the volatility coefficient in the above representation $\tilde{\varsigma}_k(t)$. For some $\rho > 0$ we choose as stopping time

$$\tau_k := \inf\{t \geq t_k : |X_k(t) - x_k| \geq \rho\} \wedge \inf\{t \geq t_k : |Y_k(t) - y_k| \geq \rho\} \wedge T.$$

Then the process $\int_{t_k}^{\tau_k} \tilde{\varsigma}_k(s) dB(s)$ is a $\mathbb{P}$-martingale. Denoting by $\mathbb{E}^\lambda_k$ the expectation operator under $\mathbb{P}^\lambda_k$, we can conclude

$$\begin{aligned} \mathbb{E}^\lambda_k[M_k(\tau_k)] &= \mathbb{E}[L^\lambda_k(\tau_k)M_k(\tau_k)] = \mathbb{E}[\int_{t_k}^{\tau_k} L^\lambda_k(t)\lambda|\varsigma_k(t)|^2 dt] = \mathbb{E}[\int_{t_k}^{T} L^\lambda_k(t)\lambda|\varsigma_k(t)|^2 \mathbb{1}_{\{t\leq\tau_k\}} dt] \\ &= \int_{t_k}^{T} \mathbb{E}[L^\lambda_k(t)\lambda|\varsigma_k(t)|^2 \mathbb{1}_{\{t\leq\tau_k\}}] dt = \int_{t_k}^{T} \mathbb{E}^\lambda_k[\lambda|\varsigma_k(t)|^2 \mathbb{1}_{\{t\leq\tau_k\}}] dt \\ &= \mathbb{E}^\lambda_k[\int_{t_k}^{T} \lambda|\varsigma_k(t)|^2 \mathbb{1}_{\{t\leq\tau_k\}} dt] = \mathbb{E}^\lambda_k[\int_{t_k}^{\tau_k} \lambda|\varsigma_k(t)|^2 dt], \end{aligned}$$

where we are allowed to change the order of integrating by Fubini's theorem. Hence, taking expectation under $\mathbb{P}_k^\lambda$ in (C.12) leads to

$$0 \le \delta_k + \mathbb{E}_k^\lambda \left[\int_{t_k}^{\tau_k} \{\mu_Y(t, X_k(t), Y_k(t), \pi_k(t)) - \mathcal{L}^{\pi_k(t)}\varphi(t, X_k(t)) + \lambda|\sigma_Y(t, X_k(t), Y_k(t), \pi_k(t)) - \sigma_X(t, X_k(t), \pi_k(t))^T D_x\varphi(t, X_k(t))|^2\}dt \right].$$

In consideration of (C.11) we can write the above inequality as

$$0 \le \delta_k + \mathbb{E}_k^\lambda \left[\int_{t_k}^{\tau_k} \{\mu_Y(t, X_k(t), Y_k(t), \pi_k(t)) - \mathcal{L}^{\pi_k(t)}\varphi(t, X_k(t)) + \lambda|\sigma_Y(t, X_k(t), Y_k(t), \pi_k(t)) - \sigma_X(t, X_k(t), \pi_k(t))^T D_x\varphi(t, X_k(t))|^2 + \lambda\mathcal{M}_{Y_k(t)}^{\pi_k(t)} v^{(n-1)}(t, X_k(t))^-\}dt \right]. \tag{C.13}$$

Step 3. Choose a strictly positive sequence $(h_k)_{k\in\mathbb{N}}$ such that $h_k \to 0$ and $\frac{\delta_k}{h_k} \to 0$ as $k \to 0$. Since the inequality (C.13) holds for any stopping time smaller than τ_k, we may replace τ_k in (C.13) by $\theta_k = \tau_k \wedge (t_k + h_k)$. Now we divide this inequality by h_k and want to let converge k to infinity. The integrand H_λ of the right hand side in (C.13) satisfies

$$\frac{1}{h_k}\int_{t_k}^{\theta_k} H_\lambda(t, X_k(t), Y_k(t), \pi_k(t))dt \le |H_\lambda(\hat{t}_k, \hat{x}_k, \hat{y}_k, \hat{\pi}_k)|,$$

where $(\hat{t}_k, \hat{x}_k, \hat{y}_k, \hat{\pi}_k)$ is a maximizer of the continuous map $|H_\lambda|$ on the compact subset $[t_k, t_k + h_k] \times \overline{\mathcal{B}}(x_k, \rho) \times \overline{\mathcal{B}}(y_k, \rho) \times U$. Since $h_k \to 0$ and $(t_k, x_k, y_k) \to (t_0, x_0, y_0)$, $y_0 := v_*(t_0, x_0)$, for $k \to \infty$, the right hand side of the last inequality converges to $|H_\lambda(t_0, \hat{x}, \hat{y}, \hat{\pi})| < \infty$ for some suitable $(\hat{x}, \hat{y}, \hat{\pi}) \in \overline{\mathcal{B}}(x_0, \rho) \times \overline{\mathcal{B}}(y_0, \rho) \times U$. Moreover, using estimate (B.5) from the appendix, it can be checked easily that $\mathbb{1}_{\{\theta_k = t_k + h_k\}} \to 1$ almost surely for $k \to \infty$. Then, by the dominated convergence theorem and the right continuity of the filtration, we obtain

$$\liminf_{k\to\infty} \frac{1}{h_k}\int_{t_k}^{t_k+h_k} H_\lambda(t, X_k(t), Y_k(t), \pi_k(t))dt \ge 0.$$

By Lemma B.1.3 in the appendix and the Lipschitz conditions assumed in (T3) and in the statement, it follows

$$\liminf_{k\to\infty} \frac{1}{h_k}\int_{t_k}^{t_k+h_k} H_\lambda(t_0, x_0, y_0, \pi_k(t))dt \ge 0. \tag{C.14}$$

Furthermore, because of $\frac{1}{h_k}\int_{t_k}^{t_k+h_k} dt = 1$ we have

$$\frac{1}{h_k}\int_{t_k}^{t_k+h_k} H_\lambda(t_0, x_0, y_0, \pi_k(t))dt \in \overline{\text{conv}}\mathcal{H}_\lambda(t_0, x_0, y_0), \tag{C.15}$$

where $\overline{\text{conv}}\mathcal{H}_\lambda(t, x, y)$ is the closed convex hull of the set $\mathcal{H}_\lambda(t, x, y)$ given by

$$\mathcal{H}_\lambda(t, x, y) := \{H_\lambda(t, x, y, \pi) : \pi \in U\}.$$

Therefore it follows from (C.14) and (C.15), together with the convexity of U, that

$$\sup_{\pi\in U} H_\lambda(t_0,x_0,y_0,\pi) = \sup_{\psi\in\overline{\operatorname{conv}}\mathcal{H}_\lambda(t_0,x_0,y_0)} \psi \geq 0. \tag{C.16}$$

Step 4. Note that (C.16) holds for all $\lambda\in\mathbb{R}$, so that we can choose $\lambda=-k$ for $k\in\mathbb{N}$. By the compactness of U the supremum in (C.16) is attained at some $\hat\pi_k\in U$ and reads as

$$\mu_Y(t_0,x_0,y_0,\hat\pi_k) - \mathcal{L}^{\hat\pi_k}\varphi(t_0,x_0) - k\mathcal{M}^{\hat\pi_k}_{y_0}v^{(n-1)}(t_0,x_0)^- \\ -k|\sigma_Y(t_0,x_0,y_0,\hat\pi_k) - D_x\varphi(t_0,x_0)^T\sigma_X(t_0,x_0,\hat\pi_k)|^2 \geq 0.$$

Taking into account that $(\hat\pi_k)_{k\in\mathbb{N}}\subset U$ is a bounded sequence, we may pass to a subsequence converging to some $\pi_0\in U$. Then the last inequality gives us

$$\mu_Y(t_0,x_0,y_0,\pi_0) - \mathcal{L}^{\pi_0}\varphi(t_0,x_0) \geq 0, \tag{C.17}$$

$$\mathcal{M}^{\pi_0}_{y_0}v^{(n-1)}(t_0,x_0) \geq 0, \tag{C.18}$$

$$|\sigma_Y(t_0,x_0,y_0,\pi_0) - D_x\varphi(t_0,x_0)^T\sigma_X(t_0,x_0,\pi_0)|^2 = 0. \tag{C.19}$$

From (C.19) we conclude that $\pi_0\in N(t_0,x_0,y_0,D_x\varphi(t_0,x_0))$. Recalling that $y_0=v_*(t_0,x_0)$, the supersolution property thus follows from (C.17), (C.18) and the relation $\mathcal{M}^{\pi_0}_{y_0}(v^{(n-1)})_*\geq\mathcal{M}^{\pi_0}_{y_0}v^{(n-1)}$. □

C.3 Subsolution property on $\{T\}\times\mathbb{R}^d$

Proof of Theorem 4.4.5: For ease of simplicity we write v instead of $v^{(n)}$ and $\overline{G}$ instead of $\overline{G}^{(n)}$. Choose $x_0\in\mathbb{R}^d$ and $\varphi\in\mathcal{C}^2(\mathbb{R}^d)$ such that $\overline{G}(x_0)=\varphi(x_0)$ and $\overline{G}\leq\varphi$ on $\mathbb{R}^d$. Suppose to the contrary that there exists $\eta>0$ satisfying

$$\min\left(\overline{G}(x_0)-g^*(x_0), \sup_{\pi\in N(T,x_0,\overline{G}(x_0),D_x\varphi(x_0))} \mathcal{M}^{\pi}_{\overline{G}(x_0)}(v^{(n-1)})^*(T,x_0)\right) \geq 2\eta. \tag{C.20}$$

Consider the auxiliary test function $\tilde\varphi:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ given by

$$\tilde\varphi(t,x) := \varphi(x) + |x-x_0|^2 + \sqrt{T-t}.$$

Then, $\lim_{t\to T}\frac{\partial\tilde\varphi}{\partial t}(t,x) = -\infty$ uniformly in x. Therefore, by the compactness of U, the continuity of the coefficients μ_X, σ_X, μ_Y and the upper semicontinuity of v^*, we can find $\rho>0$ such that

$$-\mathcal{L}^\pi\tilde\varphi(t,x) + \mu_Y(t,x,v^*(t,x),\pi) > 0 \tag{C.21}$$

for all $(t,x)\in[T-\rho,T]\times\mathcal{B}(x_0,\rho)$ and $\pi\in U$.

Noting that we may substitute v by v^* in the definition of $\overline{G}$, there exists a sequence (s_k,χ_k) in $[T-\rho,T]\times\mathcal{B}(x_0,\rho)$ satisfying $(s_k,\chi_k)\to(T,x_0)$ and $v^*(s_k,\chi_k)\to\overline{G}(x_0)$. Let

(t_k, x_k) denote a maximum point of $v^* - \tilde{\varphi}$ on $[s_k, T] \times \overline{\mathcal{B}}(x_0, \rho)$. Arguing as in Section 5.2 of [5], we can show that, up to a subsequence,

$$\lim_{k\to\infty} (t_k, x_k) = (T, x_0) \text{ with } t_k < T \text{ for sufficiently large } k, \\ \text{and } \lim_{k\to\infty} v^*(t_k, t_k) = \overline{G}(x_0). \tag{C.22}$$

Passing to this subsequence we calculate

$$D_x\tilde{\varphi}(t_k, x_k) = D_x\varphi(x_k) + 2(x_k - x_0) \to D_x\varphi(x_0) \text{ for } k \to \infty. \tag{C.23}$$

The subsolution property of v on $[0, T) \times \mathbb{R}^d$ (see Theorem 4.4.3) gives us

$$\min\Big(-\mathcal{L}^\pi\tilde{\varphi}(t_k, x_k) + \mu_Y(t_k, x_k, v^*(t_k, x_k), \pi), \mathcal{M}^\pi_{v^*(t_k,x_k)}(v^{(n-1)})^*(t_k, x_k)\Big) \leq 0$$

for all $\pi \in N(t_k, x_k, v^*(t_k, x_k), D_x\tilde{\varphi}(t_k, x_k))$ and k large enough. In consideration of (C.21) this implies

$$\sup_{\pi\in N(t_k,x_k,v^*(t_k,x_k),D_x\tilde{\varphi}(t_k,x_k))} \mathcal{M}^\pi_{v^*(t_k,x_k)}(v^{(n-1)})^*(t_k, x_k) \leq 0. \tag{C.24}$$

On the other hand, we deduce from (C.20) and the lower semicontinuity of the map $(t, x, y, \pi) \mapsto \mathcal{M}^\pi_y(v^{(n-1)})^*(t, x)$ that

$$\mathcal{M}^\pi_y(v^{(n-1)})^*(t, x) \geq \eta \quad \text{for all } (t, x, y, \pi) \in \mathcal{B}_\delta,$$

where $\mathcal{B}_\delta := [T - \delta, T] \times \mathcal{B}(x_0, \delta) \times \mathcal{B}(\overline{G}(x_0), \delta) \times \mathcal{B}(\pi_0, \delta)$ for some $\delta > 0$ chosen small enough and $\pi_0 \in N(T, x_0, \overline{G}(x_0), D_x\varphi(x_0))$. In view of the convergence results (C.22)-(C.23) and the fact

$$N(t_k, x_k, v^*(t_k, x_k), D_x\tilde{\varphi}(t_k, x_k)) \cap \mathcal{B}(\pi_0, \delta) \neq \varnothing \quad \text{for large } k,$$

which is a consequence of Assumption 4.2.1, this represents a contradiction to (C.24). $\square$

C.4 Supersolution property on $\{T\} \times \mathbb{R}^d$

Proof of Theorem 4.4.6: For ease of simplicity we write v instead of $v^{(n)}$ and $\underline{G}$ instead of $\underline{G}^{(n)}$. First we show that $\underline{G} \geq g_*$. Let $x_0 \in \mathbb{R}^d$ and $(t_k, x_k) \in [0, T) \times \mathbb{R}^d$ such that $(t_k, x_k) \to (T, x_0)$ and $v(t_k, x_k) \to \underline{G}(x_0)$ for $k \to \infty$. Set $y_k := v(t_k, x_k) + \frac{1}{k}$. Then, according to (DP1) there exists $\pi_k \in \mathcal{U}$ such that

$$Y^{\pi_k}_{t_k,x_k,y_k}(T) \geq g(X^{\pi_k}_{t_k,x_k}(T)).$$

Lemma B.1.2 in the appendix implies

$$(X^{\pi_k}_{t_k,x_k}(T), Y^{\pi_k}_{t_k,x_k,y_k}(T)) \to (x_0, \underline{G}(x_0)) \quad \text{for } k \to \infty$$

after possibly passing to a subsequence. Thus, sending $k \to \infty$ in the last inequality gives us

$$\underline{G}(x_0) \geq \liminf_{x'\to x} g(x') \geq g_*(x_0).$$

Let $\varphi \in \mathcal{C}^2(\mathbb{R}^d)$ satisfying $\underline{G} \geq \varphi$ on $\mathbb{R}^d$ and $\underline{G}(x_0) = \varphi(x_0)$ for some $x_0 \in \mathbb{R}^d$. Since we may replace v by v_* in the definition of $\underline{G}$, there exists a sequence (s_k, χ_k) converging to (T, x_0) such that $s_k < T$ and $v_*(s_k, \chi_k) \to \underline{G}(x_0)$. For $k \in \mathbb{N}$ and $\alpha > 0$ consider the auxiliary test function

$$\varphi_k^\alpha(t,x) := \varphi(x) - \frac{\alpha}{2}|x - x_0|^2 + \alpha\frac{T-t}{T-s_k}.$$

Let $\mathcal{B} := \mathcal{B}(x_0, 1)$ be the unit ball in $\mathbb{R}^d$ centered at x_0. Choose $(t_k^\alpha, x_k^\alpha) \in [s_k, T] \times \overline{\mathcal{B}}$ which minimizes the difference $v_* - \varphi_k^\alpha$ on $[s_k, T] \times \overline{\mathcal{B}}$. Following line by line the arguments of the proof of Lemma 20 in [5], we can show that

$$\lim_{k\to\infty}(t_k^\alpha, x_k^\alpha) = (T, x_0) \text{ with } t_k^\alpha < T \text{ for sufficiently large } k, \quad \text{for all } \alpha > 0, \tag{C.25}$$

and

$$\lim_{\alpha\searrow 0}\lim_{k\to\infty} v_*(t_k^\alpha, t_k^\alpha) = \underline{G}(x_0). \tag{C.26}$$

Furthermore, we have

$$\lim_{\alpha\searrow 0}\limsup_{k\to\infty}\sup_{(t,x)\in[s_k,T]\times\overline{\mathcal{B}}}|D_x\varphi_k^\alpha(t,x) - D_x\varphi(x)| = 0. \tag{C.27}$$

From the supersolution property of v_* on $[0,T) \times \mathbb{R}^d$, we know that for any $\varepsilon > 0$ there exists $\pi_k^\alpha(\varepsilon) \in N(t_k^\alpha, x_k^\alpha, v_*(t_k^\alpha, x_k^\alpha), D_x\varphi_k^\alpha(t_k^\alpha, x_k^\alpha))$ such that

$$\mathcal{M}^{\pi_k^\alpha(\varepsilon)}_{v_*(t_k^\alpha, x_k^\alpha)}(v^{(n-1)})_*(t_k^\alpha, x_k^\alpha) \geq -\varepsilon. \tag{C.28}$$

Noting that U is compact, after possibly passing to a subsequence we get

$$\pi(\varepsilon) := \lim_{\alpha\searrow 0}\lim_{k\to\infty}\pi_k^\alpha(\varepsilon) \in U.$$

Then, by (C.25)-(C.27) and the remarks on the continuity of $(t,x,y,p) \mapsto N(t,x,y,p)$ in Section 4.4, it follows

$$\pi(\varepsilon) \in N(T, x_0, \underline{G}(x_0), D_x\varphi(x_0)).$$

In view of (C.25)-(C.26) and the upper semicontinuity of $(t,x,y,\pi) \mapsto \mathcal{M}_y^\pi(v^{(n-1)})_*(t,x)$, we deduce from (C.28) that

$$\mathcal{M}^{\pi(\varepsilon)}_{\underline{G}(x_0)}(v^{(n-1)})_*(T, x_0) \geq \limsup_{k\to\infty}\mathcal{M}^{\pi_k^\alpha(\varepsilon)}_{v_*(t_k^\alpha, x_k^\alpha)}(v^{(n-1)})_*(t_k^\alpha, x_k^\alpha) \geq -\varepsilon.$$

Since $\varepsilon > 0$ was chosen arbitrarily, this yields

$$\sup_{\pi\in N(T,x_0,\underline{G}(x_0),D_x\varphi(x_0))}\mathcal{M}^\pi_{\underline{G}(x_0)}(v^{(n-1)})_*(T, x_0) \geq 0$$

which ends the proof. □

Bibliography

[1] G. Barles, P. E. Souganidis: Convergence of approximation schemes for fully nonlinear equations. *Asymptotic Analysis*, **4**(3): 271–283, 1991.

[2] A. Bensoussan, J. L. Lions: *Impulse control and quasi-variational inequalities.* Gauthier-Villars, Paris, 1984.

[3] I. Bentahar, B. Bouchard: Barrier option hedging under constraints: a viscosity approach. *SIAM Journal on Control and Optimization*, **45**(5):1846-1874, 2006.

[4] T. Björk: *Arbitrage Theory in Continuous Time.* Oxford University Press, Oxford, New York, 1998.

[5] B. Bouchard: Stochastic targets with mixed diffusion processes and viscosity solutions. *Stochastic Processes and their Applications*, **101**:273-302, 2002.

[6] B. Bouchard, V. T. Nam: The obstacle version of the Geometric Dynamic Programming Principle: Application to the pricing of American options under constraints. Technical report, CEREMADE, 2009.

[7] B. Bouchard, R. Elie, N. Touzi: Stochastic target problems with controlled loss. Technical report, CEREMADE, 2008.

[8] S. Browne: Optimal Investment Policies for a Firm with a Random Risk Process: Exponential Utility and Minimizing the Probability of Ruin. *Mathematics of Operations Research*, **20**:937-958, 1995.

[9] R. Cont, P. Tankov: *Financial Modelling with Jump Processes.* Chapman & Hall/CRC Financial Mathematics Series, Boca Raton, 2004.

[10] M. G. Crandall, H. Ishii, P.-L. Lions: User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American Mathematical Society*, **27**(1):1-67, 1992.

[11] C. W. Cryer: The solution of a quadratic programming problem using systematic overrelaxation. *SIAM Journal on Control and Optimization*, **9**:385-392, 1971.

[12] J. Cvitanic, I. Karatzas: Backward SDEs with reflection and Dynkin games. *Annals of Probability*, **24**:2024-2056, 1996.

[13] E. B. Dynkin: *Markov Processes.* Volume I/II. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 121/122. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1965.

[14] P. Embrechts, C. Klüppelberg, T. Mikosch: *Modelling Extremal Events.* Springer-Verlag, Berlin, Heidelberg, New York, 2003.

[15] W. H. Fleming, M. H. Soner: *Controlled Markov Processes and Viscosity Solutions.* Springer-Verlag, Berlin, 1993.

[16] W. H. Fleming, P. E. Souganidis: On the Existence of Value Functions of Two-Player, Zero-Sum Stochastic Differential Games. *Indiana University Mathematics Journal*, **38**(2): 293-314, 1989.

[17] I. I. Gichman, A. W. Skorochod: *Stochastische Differentialgleichungen.* Akademie-Verlag, Berlin, 1971.

[18] J. Grandell: *Aspects of Risk Theory.* Springer-Verlag, New York, 1991.

[19] S. Hamadene: Mixed zero-sum stochastic differential game and american game options. *SIAM Journal on Control and Optimization*, **45**(2):496-518, 2006.

[20] D. Hernández-Hernández and A. Schied: Robust utility maximization in a stochastic factor model. *Statistics and Decisions*, **27**:109-125, 2006.

[21] C. Hipp: Stochastic Control with Application in Insurance. *Stochastic Methods in Finance*, Lecture Notes in Mathematics, **1856**: 127-164, Springer-Verlag, Berlin, 2004.

[22] P. Hua, P. Wilmott: Modelling Market Crashes: The Worst-Case Scenario. Working paper, OFRC, Oxford, 1999.

[23] H. Ishii: Uniqueness of Unbounded Viscosity Solutions of Hamilton-Jacobi Equations. *Indiana University Mathematics Journal*, **33**:721-748, 1984.

[24] J. Jacod, A. N. Shiryaev: *Limit Theorems for Stochastic Processes.* Springer-Verlag, Berlin, Heidelberg, New York, 2003.

[25] I. Karatzas, S. E. Shreve: *Brownian motion and stochastic calculus.* Springer-Verlag, New York, Berlin, 1991.

[26] I. Karatzas, W. D. Sudderth: The controller-and-stopper game for a linear diffusion. *Annals of Probability*, **29**(3):1111-1127, 2001.

[27] I. Karatzas, M. Zamfirescu: Martingale approach to stochastic differential games of control and stopping. *Annals of Probability*, **36**(4): 1495-1527, 2008.

[28] R. Korn: Optimal portfolios: new variations of an old theme. *Computational Management Science*, **5**:289–304, 2008.

[29] R. Korn: Worst-Case Scenario Investment for Insurers. *Insurance: Mathematics and Economics*, **36**:1-11, 2005.

[30] R. Korn, E. Korn: *Optionsbewertung und Portfolio-Optimierung.* Vieweg, Wiesbaden, 2009.

[31] R. Korn, O. Menkens: Worst-Case Scenario Portfolio Optimization: A New Stochastic Control Approach. *Mathematical Methods of Operations Research*, **62**(1):123–140, 2005.

[32] R. Korn, F. T. Seifried: A Worst-Case Approach to Continuous-time Portfolio Optimization. *Radon Series on Computational and Applied Mathematics*, **8**:327-345, 2009.

[33] R. Korn, M. Steffensen: Worst-Case Portfolio Optimization and HJB-Systems. Working paper, University of Copenhagen, Department of Economics, Kopenhagen, 2006.

[34] R. Korn, P. Wilmott: Optimal Portfolios under the Threat of a Crash. *International Journal of Theoretical and Applied Finance*, **5**(2):171-187, 2002.

[35] N. V. Krylov: *Controlled Diffusion Processes.* Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[36] R. Merton: On the pricing of corporate dept: The risk structure of interest rates. *Journal of Finance*, **29**:449-479, 1974.

[37] R. Merton: Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, **3**:125-144, 1976.

[38] S. Mataramvura, B. Øksendal: Risk minimizing portfolios and HJBI equations for stochastic differential games. *Stochastics*, **80**(4):317-337, 2008.

[39] O. Menkens: *Crash Hedging Strategies and Optimal Portfolios.* PhD thesis, University of Kaiserslautern, 2004.

[40] L. Mönnig: Aktiengebundene Versicherungen bei Crashgefahr: Ein Worst-Case Optimierungsansatz. Diploma thesis, TU Dortmund, Department of Mathematics, 2009.

[41] L. Moreau: Stochastic Target Problems with Controlled Loss in Jump Diffusion Models. Technical report, CEREMADE, 2010.

[42] B. Øksendal, A. Sulem: *Applied Stochastic Control of Jump Diffusions.* Springer-Verlag, Berlin, Heidelberg, 2005.

[43] B. Øksendal: *Stochastic Differential Equations.* Springer-Verlag, Berlin, Heidelberg, 2003.

[44] H. Pham: Optimal Stopping of Controlled Jump Diffusion Processes: A Viscosity Solutions Approach. *Journal of Mathematical Systems, Estimation, and Control*, **8**(1):1-27, 1998.

[45] S. D. Promislow, V. R. Young: Minimizing the probability of ruin when claims follow Brownian motion with drift. *North American Actuarial Journal*, **9**(3): 109-128, 2005.

[46] P. Protter: *Stochastic Integration and Differential Equations.* Springer-Verlag, Berlin, Heidelberg, New York, 1992.

[47] B. Rustem, M. Howe: *Algorithms for Worst-case Design and Applications to Risk Management.* Princeton University Press, Princeton, New Jersey, 2002.

[48] N. Saintier: A general stochastic target problem with jump diffusion and an aplication to a hedging problem for large investors. *Electronic Communications in Probability*, **12**:106-119, 2007.

[49] K. Sato: *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, Cambridge, 1999.

[50] A. Sayah: Equations d'Hamilton-Jacobi du premier ordre avec termes intégro-différentiels. *Communications in Partial Differential Equations*, **16**(6):1057-1074, 1991.

[51] H. Schmidli: *Stochastic Control in Insurance.* Springer-Verlag, London, 2008.

[52] R. C. Seydel: *Impulse Control for Jump-Diffusions: Viscosity Solutions of Quasi-Variational Inequalities and Applications in Bank Risk Management.* PhD thesis, University of Leipzig, 2009.

[53] H. M. Soner: Optimal control with state-space constraint II. *SIAM Journal on Control and Optimization*, **24**:1110-1122, 1986.

[54] H. M. Soner, N. Touzi: Stochastic target problems, dynamic programming and viscosity solutions. *SIAM Journal on Control and Optimization*, **41**:404-424, 2002.

[55] H. M. Soner, N. Touzi: Dynamic programming for stochastic target problems and geometric flows. *Journal of the European Mathematical Society*, **4**:201-236, 2002.

[56] D. Talay, Z. Zheng: Worst Case Model Risk Management. *Finance and Stochastics*, **6**:517-537, 2002.

[57] W. Walter: *Differential and Integral Inequalities.* Springer-Verlag, Berlin, Heidelberg, New York, 1970.

[58] P. Wilmott: *Paul Wilmott on Quantitative Finance.* John Wiley & Sons, Chichester, New York, 2006.