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Abstract

In this paper we consider a class of processes that can be represented in the form $Y_s = X_{\mathcal{T}(s)}$, where X is a Lévy process and $\mathcal{T}$ is a non-negative and non-decreasing stochastic process independent of X. The aim of this work is to infer on the Blumenthal-Getoor index of the process X from low-frequency observations of the time-changed Lévy process Y. We propose a consistent estimator for this index, derive the minimax rates of convergence and show that these rates can not be improved in general. The performance of the estimator is illustrated by numerical examples.

Keywords: time-changed Lévy processes, Abelian theorem, Blumenthal-Getoor index

1 Introduction

The problem of nonparametric statistical inference for jump processes or more generally for semimartingale models has long history and goes back to the works of Rubin an Tucker [16] and Basawa and Brockwell [4]. In the past decade one has witnessed the revival of interest in this topic which is mainly related to a wide availability of financial and economical time series data and new types of statistical issues that have not been addressed before. There are two major strands of recent literature dealing with statistical inference for semimartingale models. The first type of literature considers the so-called high-frequency setup, where the asymptotic properties of the corresponding estimates are studied under the assumption that the frequency of observations tends to infinity. In the second strand of literature, the frequency of observations is assumed to be fixed (the so-called low-frequency setup) and the asymptotic analysis is done under the premiss that the observational horizon tends to infinity. It is clear that none of the above asymptotic hypothesis can be perfectly realized on real data and they can only serve as a convenient approximation, as in practice both the frequency of observations and the horizon are always finite. The present paper studies the problem of statistical inference for a class of semimartingale models in low-frequency setup.

Consider two one-dimensional real-valued not necessarily independent stochastic processes - $X = (X_t)_{t\geq 0}$ and $\mathcal{T} = (\mathcal{T}(s))_{s\geq 0}$. Let X be a Lévy process and $\mathcal{T}$ be a non-negative, non-decreasing stochastic process with $\mathcal{T}(0) = 0$. Then the time-changed Lévy process is defined as

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$Y_s = X_{\mathcal{T}(s)}$. The change of time can be motivated by the fact that some economical effects (e.g., nervousness of the market which is indicated by volatility) can be better expressed in terms of "business" time, which may run faster than physical one in some periods. This resulting class of processes is very large; e.g. Monroe [14] shows that even in the case of the Brownian motion X, the class $\{Y\}$ basically coincides with the class of all semimartingales. For identifiability reasons, in this paper we restrict our attention to the case of independent processes X and $\mathcal{T}$. The corresponding class of processes remains rather large but its full characterization remains an open problem (see [3]).

Suppose now that the time-changed process Y is observable on the equidistant time grid $0 < \Delta < \ldots < n\Delta$ with some $n \in \mathbb{N}$ and $\Delta > 0$. A natural question is which parameters of the underlying Lévy process X can be identified from the observations $Y_\Delta, \ldots, Y_{n\Delta}$ as $n \to \infty$. This question has been recently addressed in the literature and the answer turns out to depend crucially on the asymptotic behavior of Δ and on the degree of our knowledge about $\mathcal{T}$. So in the case of high-frequency data with increasing time horizon, i.e., $\Delta_n \to 0$ with $n \cdot \Delta_n \to \infty$, one basically can, under some regularity conditions, identify X completely, provided $\mathbb{E}[\mathcal{T}]$ is known (see Figueroa-Lopez, [11]). If the time horizon remains fixed, only the diffusion part of X and the behavior of the Lévy measure of X at 0 can be identified. The latter behavior can be characterized in terms of the so-called Blumenthal-Getoor index or successive Blumenthal-Getoor indexes (see Aït-Sahalia and Jacod [1]). The Blumenthal-Getoor index is a characteristic of the activity of small jumps and for a one-dimensional Lévy process $Z = (Z_t)_{t\geq 0}$ with a Lévy measure ν can be defined via

$$\mathtt{BG}(Z) = \inf\left\{ r > 0 : \int_{|x|\leq 1} |x|^r \nu(dx) < \infty \right\}.$$

In the case of low-frequency data, i.e., if Δ is fixed and $n \to \infty$, one can not in general identify the corresponding Lévy measure as shown in Belomestny [6]. However, the question remains open whether the behavior of ν at 0 can be recovered. The aim of this paper is to answer this question and to propose a consistent estimate for the B-G index of the process X based on low-frequency observations of the process Y. It turns out that consistent estimation is basically possible if the process X has a nonzero diffusion part and $\mathcal{T}$ has stationary increments. It is worth pointing out that we do not assume any prior knowledge about the time change $\mathcal{T}$ or the Lévy process X, except the fact that X has a non-zero diffusion part.

The paper is organized as follows. In the next section, we present the main setup and give a short overview of the considered problem. Next, we introduce the main object of our study, the time-changed Lévy processes and formulate the main assumptions. Section 3 contains the so-called Abelian theorem describing the asymptotic behavior of the characteristic function of $Y_{t+\Delta} - Y_t$ for some $t > 0$. Estimation algorithm for the Blumenthal-Getoor index of X is given in Section 6. Theoretical results showing the consistency of the proposed estimator and the rates of convergence are presented in Section 7. A numerical example can be found in Section 8.

2 Main setup

2.1 Lévy process X

In this paper we assume that the process X_t is a one-dimensional Lévy process on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. This in particularly means that the characteristic function of X has the form:

$$\phi(u) := \mathbb{E}\left[\exp\left\{\mathrm{i}u^{\top}X_t\right\}\right] = \exp\left\{t\psi(u)\right\}, \quad t>0,$$

where the function $\psi(u)$ is called the characteristic exponent of X. The Lévy-Khintchine formula yields

$$\psi(u) = \mathrm{i}\mu u - \frac{1}{2}\sigma^2 u^2 + \mathcal{V}(u), \qquad \mathcal{V}(u) := \int_{\mathbb{R}\setminus\{0\}} \left(e^{\mathrm{i}ux} - 1 - \mathrm{i}ux\cdot \mathbf{1}_{\{|x|\le 1\}}\right)\nu(dx),$$

where $\mu \in \mathbb{R}$, σ is a non-negative number and ν is a Lévy measure on $\mathbb{R}\setminus\{0\}$, which satisfies

$$\int_{\mathbb{R}\setminus\{0\}} (|x|^2 \wedge 1)\,\nu(dx) < \infty.$$

A triplet (μ, σ^2, ν) is usually called the characteristic triplet of the Lévy process X_t. In this paper, we assume that

(AL) σ is a strictly positive and the function $\mathcal{V}(u)$ has the following representation:

$$\mathcal{V}(u) = -\lambda_1\,|u|^{\gamma}\,\Psi_1(u), \tag{1}$$

where $\lambda_1 > 0$, $\gamma \in (0,2)$, and moreover

$$|1-\Psi_1(u)| \le \vartheta_1 |u|^{-\chi_1}, \quad u \to +\infty \tag{2}$$

with $\chi_1 \in (0,\gamma)$, $\vartheta_1 > 0$.

The assumption (AL) is, for example, fulfilled in the case when there exist $\beta^{(0)} > 0$, $\beta^{(1)} \in I\!R$ such that

$$\int_{|x|>\varepsilon} \nu(dx) = \varepsilon^{-\gamma}(\beta^{(0)} + \beta^{(1)}\varepsilon^{\chi_1}(1+O(\varepsilon))), \quad \varepsilon \to +0, \tag{3}$$

see Lemma 7.2 for the proof. Note that the parameter γ in (3) is equal to the Blumenthal-Getoor index, because this index can be equivalently defined as the number $p \in [0,2]$ such that

$$\lim_{\varepsilon\to 0+} \varepsilon^p \int_{|x|>\varepsilon} \nu(dx) \in (0,+\infty). \tag{4}$$

We refer to [15] for a detailed discussion of the condition (3) and the property (4).

2.2 Time change

Let $\mathcal{T} = (\mathcal{T}(s))_{s\ge 0}$ be an increasing right-continuous process with left limits such that $\mathcal{T}(0) = 0$ and for each fixed s, the random variable $\mathcal{T}(s)$ is a stopping time with respect to the filtration $\mathcal{F}$. This setup is quite typical for the processes that are referred to as time change (see, e.g., the book by Barndorff-Nielsen and Shiryaev, [3]). In this paper, it is also assumed that

(AT1) processes X and $\mathcal{T}$ are independent;

(AT2) process $\mathcal{T}$ has stationary and ergodic increments;

(AT3) the Laplace transform of $\mathcal{T}(\Delta)$ has the following asymptotic behavior:

$$\mathcal{L}_\Delta(u) = \mathbb{E}[\exp(-u\mathcal{T}(\Delta))] \asymp A \exp\left\{-\lambda_2 u^\alpha \Psi_2(u)\right\}, \qquad u \to +\infty, \tag{5}$$

with $\lambda_2 > 0$, $A > 0$, $\alpha \in (0,1)$, and $\Psi_2(u)$ such that

$$|1 - \Psi_2(u)| \leq \vartheta_2 u^{-\chi_2}$$

with some $\chi_2, \vartheta_2 > 0$.

Typical examples of the processes that satisfy the assumptions (AT2) and (AT3) are the integrated CIR process ($\alpha = 1/2$) and the tempered stable process ($\alpha \in (0,1)$), see the next section for the detailed description. Note that A and λ_2 may depend on Δ. In the case, when $\mathcal{T}(s)$ is an increasing Lévy process, i.e., a subordinator, the parameter α coincides with the Blumenthal-Getoor index of $\mathcal{T}$ which is always smaller than or equal to 1.

2.3 Examples

Tempered stable process. The tempered stable distribution with parameters (a, b, α) can be defined via its Laplace transform

$$\mathcal{L}^{TS}(u) := \exp\left\{ab - a(b^{1/\alpha} + 2u)^\alpha\right\},$$

where $a > 0$, $b \geq 0$, $\alpha \in (0,1)$. The tempered stable process is a process Z_t, which has increments $Z_{t+s} - Z_t$ following a tempered stable law with parameters (sa, b, α). The Lévy measure of this process is of the form

$$\varrho(x) := \frac{c}{x^{\alpha+1}} \exp\{-\lambda x\} I\{x > 0\},$$

where $\lambda = b^{1/\alpha}/2$. Here the decay rate of big jumps, $c = -a * 2^\alpha/\Gamma(-\alpha)$ alters the intensity of all jumps simultaneously, and α is the Blumenthal-Getoor index of the process [9]. Note also that the parameter α from the assumption (AT3) coincides with the third parameter of the tempered stable process. The parameter χ_2 is equal to 1 if $b \neq 0$, and can be taken arbitrary otherwise. Other parameters that appear in (5) are $\lambda_2 = 2^\alpha a\Delta$, $\vartheta_2 = b^{1/\alpha}\alpha/2$, $A = \exp\{ab\Delta\}$. The tempered stable process is a subordinator [17] and therefore it can be used for a time change. Interestingly enough, the case $\alpha = 1/2$ coincides with the Inverse Gaussian process, which can be determined as the first time when a standard Brownian motion with drift b reaches the positive level a. Note also that the limiting case $\alpha \to 0$ gives the Gamma process.

Integrated CIR process. Another candidate for the time change process is given by the integrated Cox-Ingersoll-Ross (CIR) process. The CIR process is defined as a solution of the following SDE:

$$dZ_t = (a - bZ_t)dt + \zeta\sqrt{Z_t}\, dW_t, \quad Z_0 = 1,$$

where a, b and ζ are positive, and W_t is a Wiener process. The time change process $\mathcal{T}(s)$ is then defined as

$$\mathcal{T}(s) = \int_0^s Z_t\, dt.$$

The Laplace transform of $\mathcal{T}(\Delta)$ is given by

$$\mathcal{L}_{\Delta}^{iCIR}(u) = \exp\left\{-a\varphi_{\Delta}(u) - u\psi_{\Delta}(u)\right\},$$

where

$$\varphi_{\Delta}(u) = -\frac{2}{\zeta^2}\log\left(\frac{2\gamma(u)e^{(\gamma(u)+b)\Delta/2}}{\gamma(u)-b+e^{\gamma(u)\Delta}(\gamma(u)+b)}\right), \qquad \psi_{\Delta}(u) = \frac{2\left(e^{\gamma(u)\Delta}-1\right)}{\gamma(u)-b+e^{\gamma(u)\Delta}(\gamma(u)+b)},$$

and $\gamma(u) = \sqrt{b^2+2\zeta^2 u}$, see [10] and [17]. Function $\mathcal{L}_{\Delta}^{iCIR}(u)$ has the following asymptotics:

$$\mathcal{L}_{\Delta}^{iCIR}(u) \asymp \exp\left\{\frac{a(b\Delta+2\log 2)}{\zeta^2} - \sqrt{2u}\,\frac{a\Delta+1}{\zeta}\Psi_2(u)\right\}, \qquad u\to\infty,$$

where

$$|1-\Psi_2(u)| \le \frac{2b}{\sqrt{2\sigma}\,(a\Delta+1)}\,u^{-1/2}. \tag{6}$$

This means that condition (AT3) is fulfilled with $\alpha = 1/2$ and $\chi_2 = 1/2$.

3 Abelian theorem

The first objective of this paper is to infer on the asymptotic behavior of the characteristic function of Y_Δ which is denoted by $\phi^\Delta(u)$. We have

$$\phi^{\Delta}(u) \quad = \quad \mathbb{E}_{\mathcal{T}(\Delta)}\left[\mathbb{E}_X\left[\exp\left\{\mathrm{i}uX_{\mathcal{T}(\Delta)}\right\}|\mathcal{T}(\Delta)\right]\right].$$

Since the inside (conditional) expectation is equal to $\exp\{\mathcal{T}(\Delta)\psi(u)\}$,

$$\phi^{\Delta}(u) = \mathbb{E}\exp\left\{\mathcal{T}(\Delta)\psi(u)\right\}.$$

The latter formula yields

$$\left|\phi^{\Delta}(u)\right| = \mathbb{E}\exp\left\{\mathcal{T}(\Delta)\operatorname{Re}(\psi(u))\right\} = \mathcal{L}_{\Delta}\left(-\operatorname{Re}(\psi(u))\right), \tag{7}$$

i.e., $\left|\phi^{\Delta}(u)\right|$ can be considered as the Laplace transform of $\mathcal{T}_\Delta$ computed at the point $-\operatorname{Re}(\psi(u))$. The first theorem can be viewed as the Abelian theorem [12] for time - changed Lévy processes.

Theorem 3.1. *Consider the process $Y_s := X_{\mathcal{T}(s)}$, where the processes X_t and $\mathcal{T}(s)$ satisfy the conditions (AL), (AT1)-(AT3) with*

$$\gamma - 2 > -2\chi_2. \tag{8}$$

Then the absolute value of the characteristic function of increments $Y_{s+\Delta} - Y_s$ has the following representation:

$$\left|\phi^{\Delta}(u)\right| = A\exp\left\{-\tau^{(1)}|u|^{2\alpha}\left(1+\tau^{(2)}|u|^{\gamma-2}+r(u)\right)\right\}, \tag{9}$$

where

$$\tau^{(1)} = \lambda_2\left(\sigma^2/2\right)^{\alpha}, \qquad \tau^{(2)} = 2\alpha\lambda_1/\sigma^2$$

and

$$|r(u)| \le \max\left\{\tau^{(2)}\vartheta_1\,|u|^{(\gamma-2)-\chi_1}, \vartheta_2\left(\sigma^2/2\right)^{-\chi_2}|u|^{-2\chi_2}\right\} \qquad \textit{for } |u| \textit{ large enough.}$$

Remark 3.2. The examples given in Section 2.3 show that the condition (8) is not restrictive. For example, if the tempered stable distribution serves as a time change then this condition holds for any Lévy process X since $\chi_2 = 1$, see Section 2.3.

Remark 3.3. Later on, we will use the notation

$$\tilde{\chi}_1 := \min\{\chi_1, 2\chi_2 + \gamma - 2\}$$

and

$$\tau^{(3)} := \tau^{(2)}\vartheta_1 I\Big\{\tilde{\chi}_1 = \chi_1\Big\} + \left(\sigma^2/2\right)^{-\chi_2} \vartheta_2 I\Big\{\tilde{\chi}_1 = 2\chi_2 + \gamma - 2\Big\} \le \max\Big\{\tau^{(2)}\vartheta_1, \left(\sigma^2/2\right)^{-\chi_2}\vartheta_2\Big\},$$

In this notation, $|r(u)| \le \tau^{(3)}|u|^{\gamma-2-\tilde{\chi}_1}$ for $|u|$ large enough.

4 Estimation of the Blumenthal-Getoor index

4.1 Main idea

Consider the processes X_t and $\mathcal{T}(s)$ satisfying the assumptions (AL), (AT1)-(AT3). First, assume that $A = 1$ and fix some $\theta > 2$. In this case, Theorem 3.1 yields

$$\begin{aligned} \mathcal{Y}_1(u) &:= \log\Big\{-\log\Big[|\phi^{\Delta}(u)|^{\theta^{2\alpha}} / \left|\phi^{\Delta}(\theta u)\right|\Big]\Big\} \\ &= \log(Q) + (2\alpha + \gamma - 2)\,\log|u| + \log(R_1(u)), \end{aligned} \tag{10}$$

where $Q = \tau^{(1)}\tau^{(2)}\theta^{2\alpha}(1 - \theta^{\gamma-2}) > 0$ and $R_1(u) \to 1$ as $u \to +\infty$. The representation (10) tells us that $\mathcal{Y}_1(u)$ is, up to a reminder term $\log(R_1(u))$, a linear function of $\log|u|$ with the slope $2\alpha + \gamma - 2$. If the parameter α is assumed to be known, one can view the estimation of γ as a linear regression problem (at least for large u) and apply the (weighted) least-squares approach. Otherwise, if α is unknown, one should first estimate α. This can be also done by the method of (weighted) least-squares. Indeed, define

$$\mathcal{Y}_2(u) := \log\left(-\log|\phi^{\Delta}(u)|\right) = \log(\tau^{(1)}) + 2\alpha\,\log|u| + \log(R_2(u)),$$

where $R_2(u) \to 1$ as $u \to +\infty$. So, $\mathcal{Y}_2(u)$ is (at least for large u) a linear function of $\log|u|$ with the slope proportional to α. If $A \ne 1$, then one can first apply the transformation:

$$\widetilde{\phi}^{\Delta}(u) := |\phi^{\Delta}(2u)|/|\phi^{\Delta}(u)| = \exp\Big\{-\bar{\tau}^{(1)}|u|^{2\alpha}\left(1 + \bar{\tau}^{(2)}|u|^{\gamma-2} + \bar{r}(u)\right)\Big\},$$

where

$$\bar{\tau}^{(1)} := \tau^{(1)}\left(2^{2\alpha} - 1\right), \quad \bar{\tau}^{(2)} := \tau^{(2)}\frac{2^{2\alpha+\gamma-2} - 1}{2^{2\alpha} - 1}, \quad \bar{r}(u) = \frac{2^{2\alpha}r(2u) - r(u)}{2^{2\alpha} - 1},$$

and then work with $\widetilde{\phi}^{\Delta}(u)$ instead of $\phi^{\Delta}(u)$. The above discussion shows that one can consistently estimate the parameters α and γ, provided a consistent estimate for the c.f. of Y_Δ is available.

4.2 Estimation of the characteristic function

Suppose that the discrete observations $Y_0, Y_\Delta, \ldots, Y_{n\Delta}$ of the state process Y are available for some fixed $\Delta > 0$. We estimate $\phi^\Delta(u)$ by its empirical counterpart $\phi_n^\Delta(u)$ defined as

$$\phi_n^\Delta(u) := \frac{1}{n}\sum_{k=1}^{n} e^{\mathrm{i}u(Y_{\Delta k}-Y_{\Delta(k-1)})}. \tag{11}$$

Note that due to the assumption (AT2) and by virtue of the Birkhoff ergodic theorem (see [2]),

$$\frac{1}{n}\sum_{k=1}^{n} e^{\mathrm{i}u(Y_{\Delta k}-Y_{\Delta(k-1)})} \longrightarrow \phi^\Delta(u), \quad n \to \infty,$$

almost surely and in $\mathcal{L}_1$.

4.3 The case of known α

Introduce a weighting function $w^{V_n}(u) = V_n^{-1} w^1(u/V_n)$, where V_n is a sequence of positive numbers tending to infinity, and the smooth function w^1 supported on $[\varepsilon, 1]$ for some $\varepsilon > 0$ and satisfying

$$\int_\varepsilon^1 w^1(u)\, du = 0, \qquad \int_\varepsilon^1 w^1(u) \log u \, du = 1. \tag{12}$$

Some examples of such weighting functions are given in [15]. If $2-\gamma < 2\alpha$, we define an estimator of γ by

$$\hat{\gamma}_n(\alpha) := 2(1-\alpha) + \int_0^\infty w^{V_n}(u) \log\left(-\log \frac{|\phi_n^\Delta(u)|^{\theta^{2\alpha}}}{|\phi_n^\Delta(\theta u)|}\right) du. \tag{13}$$

If otherwise $2-\gamma \geq 2\alpha$, consider an estimate

$$\hat{\gamma}_n^*(\alpha) := 2(1-\alpha) + \int_0^\infty w^{V_n}(u) \log\left(1 - \frac{|\phi_n^\Delta(u)|^{\theta^{2\alpha}}}{|\phi_n^\Delta(\theta u)|}\right) du. \tag{14}$$

In our theoretical study we mainly focus on the first case (some remarks about the second case can be found in Section 7.1). In fact, the estimate $\hat{\gamma}_n(\alpha)$ can be represented as the solution of some optimization problem. More precisely, $\hat{\gamma}_n(\alpha) = 2(1-\alpha) + m_{n,1}$, where

$$(m_{n,1}, m_{n,2}) := \operatorname*{arg\,min}_{\beta_1,\beta_2} \int_0^\infty \widetilde{w}^{V_n}(u) \left\{\log\left(-\log\frac{|\phi_n^\Delta(u)|^{\theta^{2\alpha}}}{|\phi_n^\Delta(\theta u)|}\right) - \beta_2 \log(u) - \beta_1\right\}^2 du. \tag{15}$$

Introduce the deterministic quantity

$$\bar{\gamma}_n(\alpha) := 2(1-\alpha) + \int_0^\infty w^{V_n}(u) \log\left(-\log \frac{|\phi^\Delta(u)|^{\theta^{2\alpha}}}{|\phi^\Delta(\theta u)|}\right) du. \tag{16}$$

The next lemma shows that $\bar{\gamma}_n(\alpha)$ is close to γ.

Lemma 4.1. *In the setup of Theorem 3.1, it holds for n large enough,*

$$|\gamma - \bar{\gamma}_n(\alpha)| \lesssim V_n^{-\tilde{\chi}_1}, \qquad n \to \infty, \tag{17}$$

where the notation introduced in Remark 3.3 is used. More precisely,

$$|\gamma - \bar{\gamma}_n(\alpha)| \leq C^{(1)} \frac{\tau^{(3)}}{\tau^{(2)}} \frac{1 + \theta^{\gamma - 2 - \tilde{\chi}_1}}{1 - \theta^{\gamma - 2}} (\varepsilon V_n)^{-\tilde{\chi}_1}, \qquad n \to \infty, \tag{18}$$

where $C^{(1)} > 0$ does not depend on the parameters of Y.

The next theorem shows that $\hat{\gamma}_n(\alpha)$ converges to $\bar{\gamma}_n(\alpha)$ in probability.

Lemma 4.2. *Let the sequence V_n be such that*

$$\varepsilon_n := \frac{\log n}{\sqrt{n}} \exp\left\{ \tau^{(4)} (\theta V_n)^{2\alpha} \right\} = o(1), \quad n \to \infty, \tag{19}$$

where $\tau^{(4)} := \tau^{(1)} \left(1 + \tau^{(2)} + \tau^{(3)} \right)$. Assume that the conditions of Theorem 3.1 are fulfilled and moreover $2 - \gamma < 2\alpha$. Then there exist positive constants $C^{(2)}$, $\varkappa$ and δ such that

$$\mathbb{P}\left\{ |\bar{\gamma}_n(\alpha) - \hat{\gamma}_n(\alpha)| \leq C^{(2)} \varepsilon_n V_n^{(2-\gamma)-2\alpha} \right\} > 1 - \varkappa\, n^{-1-\delta}. \tag{20}$$

The last two lemmas can be combined into the following minimax convergence theorem.

Theorem 4.3 (minimax upper bounds for $\hat{\gamma}_n(\alpha)$)**.** *Fix some set of positive numbers*

$$\mathscr{P} = \left(\alpha_\circ, \alpha^\circ, \chi_{1\circ}, \lambda_{1\circ}, \lambda_1^\circ, \vartheta_1^\circ, \chi_{2\circ}, \lambda_{2\circ}, \lambda_2^\circ, \vartheta_2^\circ \right)$$

and consider a class of time-changed Lévy models $\mathscr{A} = \mathscr{A}(\mathscr{P})$ such that

- *Assumptions (AT1), (AT2) hold;*
- *Assumption (AT3) is fulfilled with $\alpha \in \left[\alpha_\circ, \alpha^\circ\right] \subset \left(0, 1\right)$, $\chi_2 \geq \chi_{2\circ} > 0$, $\lambda_2 \in \left[\lambda_{2\circ}, \lambda_2^\circ\right]$ and $\vartheta_2 \in (0, \vartheta_2^\circ)$;*
- *Assumption (AL) is fulfilled with $\gamma \in \left(\gamma_\circ, 2\right)$, where $\gamma_\circ := \max\{\chi_{1\circ}, 2(1-\alpha)\}$, $\chi_1 \in \left[\chi_{1\circ}, \gamma\right)$, $\lambda_1 \in \left[\lambda_{1\circ}, \lambda_1^\circ\right]$, and $\vartheta_1 \in \left(0, \vartheta_1^\circ\right)$.*

Take the sequence $V_n = (q \log n)^{1/(2\alpha^\circ)}$ with $q < \left(2\theta^{2\alpha^\circ} \min_{\mathscr{A}} \tau^{(4)}\right)^{-1}$, where $\tau^{(4)}$ is defined in Lemma 4.2. Then

$$\sup_{\mathscr{A}} \mathbb{P}\left\{ |\hat{\gamma}_n(\alpha) - \gamma| \leq \Xi_1 (\log n)^{-\tilde{\chi}_1/(2\alpha^\circ)} \right\} > 1 - \varkappa\, n^{-1-\delta}, \tag{21}$$

where $\tilde{\chi}_1 := \min\{\chi_1, 2\chi_2 + \gamma - 2\}$, the supremum is taken over the set of all models from $\mathscr{A}$, constants $\varkappa$ and δ do not depend on the parameters of the underlined models, and Ξ_1 depends on $\mathscr{P}$ only.

Remark 4.4. Introduce a constant

$$\tilde{\chi}_{1\circ} := \min\left\{\chi_{1\circ}, 2\chi_{2\circ} + \gamma_\circ + 2\right\}. \tag{22}$$

Obviously, $\tilde{\chi}_1$ can be changed to $\tilde{\chi}_{1\circ}$ in (55).

The next result shows that the rates obtained in Theorem 4.3 are optimal.

Theorem 4.5. ***Lower bounds.*** *It holds*

$$\liminf_{n\to\infty}\left\{(\log n)^{\tilde{\chi}_{1\circ}/\alpha^\circ} \inf_{\hat{\gamma}_n}\sup_{\mathscr{A}} \mathbb{E}_\gamma \left|\hat{\gamma}_n - \gamma\right|^2\right\} \geq \Xi_2, \tag{23}$$

where Ξ_2 is some positive constant, the infimum is taken over all possible estimates of the parameter γ, the supremum - over the set of all models from $\mathscr{A}$.

4.4 The case of unknown α

Estimation of α. Define an estimate for the parameter α via

$$\hat{\alpha}_n := \frac{1}{2}\int_0^\infty w_\alpha^{U_n}(u)\log\left(-\log\left|\phi_n^\Delta(u)\right|\right)\, du, \tag{24}$$

where U_n is a sequence of positive numbers tending to infinity, and a weighting function $w_\alpha^{V_n}$ satisfies the same properties as the function w^{U_n}, see (12).

This estimate can be alternatively defined as the solution $\hat{\alpha}_n = l_{n,1}$ of the following optimization problem:

$$(l_{n,1}, l_{n,2}) := \operatorname*{arg\,min}_{\beta_1,\beta_2} \int_0^\infty \widetilde{w}_\alpha^{U_n}(u)\left(\frac{1}{2}\log\left(-\log\left|\phi_n^\Delta(u)\right|\right) - \beta_2\log(u) - \beta_1\right)^2 du, \tag{25}$$

where $\widetilde{w}_\alpha^{U_n}(u)$ is a smooth positive function on $I\!\!R$ having the representation:

$$\widetilde{w}_\alpha^{U_n}(u) = \frac{1}{U_n}\,\widetilde{w}_\alpha^1\left(\frac{u}{U_n}\right)$$

with some function $\widetilde{w}_\alpha^1$ supported on the interval $[\varepsilon, 1]$. The upper bound for the estimate $\hat{\alpha}_n$ is given in the next theorem.

Theorem 4.6 (upper bound for $\hat{\alpha}_n$)**.** *Take the sequence*

$$U_n = (q\log n)^{1/(2\alpha^\circ)}, \quad \textit{where } q < q^\circ := \left(2^{1-\alpha_\circ}\lambda_2^\circ \max\left\{\sigma^{2\alpha_\circ}, \sigma^{2\alpha^\circ}\right\}\right)^{-1}.$$

There exists a positive constant Ξ_3 such that

$$\sup_{\mathscr{A}} \mathbb{P}\left\{\left|\hat{\alpha}_n - \alpha\right| \leq \Xi_3(\log n)^{(\gamma-2)/\alpha^\circ}\right\} > 1 - \varkappa\, n^{-1-\delta}, \tag{26}$$

where the supremum is taken over the set of all models from $\mathscr{A}$, and the constants $\varkappa$ and δ are defined in Theorem 4.3.

Estimation of γ in the case of unknown α. After estimating α, one can determine the estimate of γ as

$$\hat{\gamma}_n(\hat{\alpha}_n) := 2(1-\hat{\alpha}_n) + \int_0^\infty w^{V_n}(u) \log\left(-\log \frac{|\phi_n^\Delta(u)|^{\theta^{2\hat{\alpha}_n}}}{|\phi_n^\Delta(\theta u)|}\right) du, \tag{27}$$

where U_n and w^{U_n} are already defined in Section 4.3.

The next theorem shows that the upper bound for the estimate $\hat{\gamma}_n(\hat{\alpha}_n)$ is the same as in the case of known α as long as $\gamma < 4/3$.

Theorem 4.7 (upper bound for $\hat{\gamma}_n(\hat{\alpha}_n)$)**.** *Take the sequences*

$$\begin{aligned} U_n &= (q \log n)^{1/(2\alpha^\circ)}, \quad \textit{where } q < q^\circ := \left(2^{1-\alpha_\circ} \lambda_2^\circ \max\left\{\sigma^{2\alpha_\circ}, \sigma^{2\alpha^\circ}\right\}\right)^{-1}, \\ V_n &= (p \log n)^{1/(2\alpha^\circ)}, \quad \textit{where } p < p^\circ := q^\circ / \left(1 + \varepsilon^{2\alpha^\circ}\right). \end{aligned}$$

Then

$$\sup_{\mathscr{A}} \mathbb{P}\left\{|\hat{\gamma}_n(\hat{\alpha}_n) - \gamma| \le \Xi_4 (\log n)^{\max\{-\tilde{\chi}_1, 2(\gamma-2)\}/(2\alpha^\circ)}\right\} > 1 - \varkappa\, n^{-1-\delta}, \tag{28}$$

where the supremum is taken over the set of all models from $\mathscr{A}$, constants $\varkappa$ and δ do not depend on the parameters of Y, and Ξ_4 depends on $\mathscr{P}$ only. In particular, for the class of models $\mathscr{A}_1$ that consists of the models from $\mathscr{A}$ with $\gamma < 4/3$, we get

$$\sup_{\mathscr{A}_1} \mathbb{P}\left\{|\hat{\gamma}_n(\hat{\alpha}_n) - \gamma| \le \Xi_4 (\log n)^{-\tilde{\chi}_1/(2\alpha^\circ)}\right\} > 1 - \varkappa\, n^{-1-\delta}. \tag{29}$$

5 Numerical example

We consider the following time-changed Lévy model. The Lévy process X_t is taken as a sum of the Brownian motion and a γ-stable process S_t such that the characteristic exponent of the combined process is of the form

$$\psi(u) = -u^2/2 - \sigma_1 |u|^\gamma \Big(1 - \mathrm{i}\beta \operatorname{sign}(u) \tan(\pi\gamma/2)\Big),$$

with some $\beta \in [-1, 1]$, $\sigma_1 > 0$, and $\gamma \in [0, 2]$, $\gamma \neq 1$. Note that the assumption (AL) is fulfilled with $\lambda_1 = \sigma_1$. For our numerical study, we take $\gamma = 1.2$, $\sigma_1 = 0.25$ and $\beta = 0.3$. The time change $\mathcal{T}(s)$ is given by the integrated CIR process with parameters $a = 1.3$, $b = 0.01$, $\zeta = 1.6$ (see Section 2.3). Note that (AT3) holds with $\alpha = 1/2$ for this model. First, we generate a trajectory $Y_0, Y_\Delta, ..., Y_{n\Delta}$ with $\Delta = 1$. Next, we estimate the characteristic function by $\phi_n^\Delta(u)$, see (11), and consider the optimization problem (25):

$$(l_{n,1}, l_{n,2}) := \arg\min_{\beta_1,\beta_2} \int_{U_{low}}^{U_{up}} \left(\frac{1}{2}\log\left(-\log|\phi_n^\Delta(uU_n)|\right) - \beta_2 \log(uU_n) - \beta_1\right)^2 du.$$

where U_{low} and U_{up} are the truncation levels. The solution $l_{n,1}$ of this problem gives an estimate of α, which we denote by $\hat{\alpha}_n$. Figure 1 shows the box plots of $\hat{\alpha}_n$ as a function of n based on 25 simulation runs.

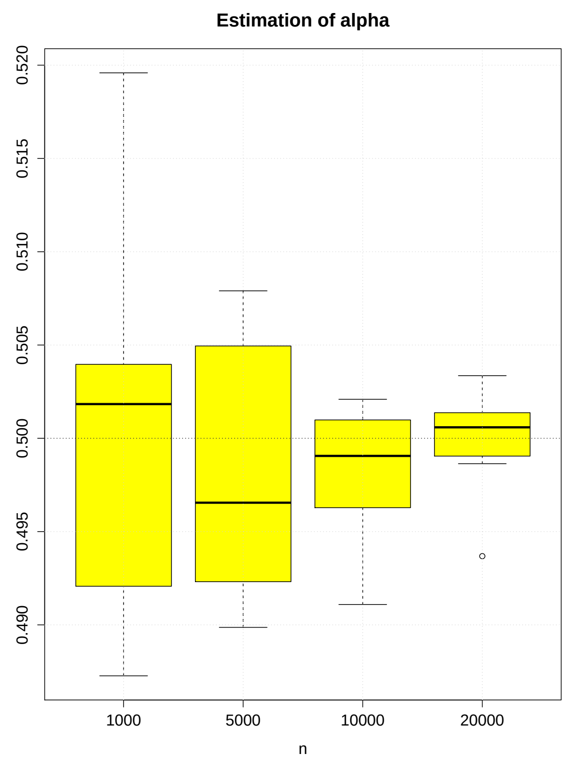

Figure 1: Boxplots of the estimate $\hat{\alpha}_n$ for different values of n

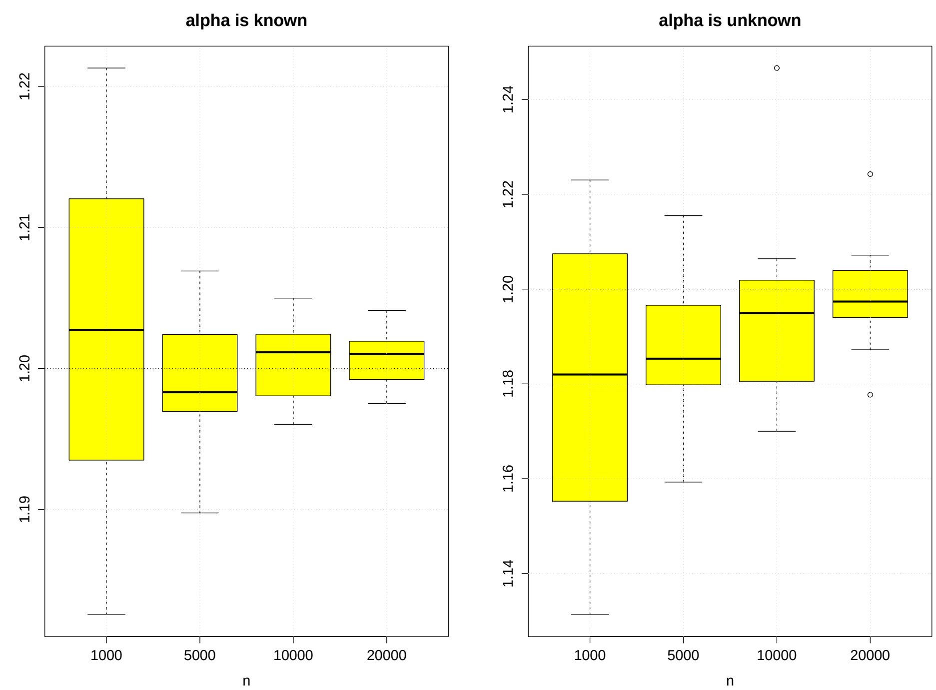

Figure 2: Boxplots of the estimates $\hat{\gamma}_n(\alpha)$ and $\hat{\gamma}_n(\hat{\alpha}_n)$ for different values of n.

Taking into account the fact that $2-\gamma < 2\alpha$ in our case, we proceed to the next step - estimation of γ - by considering the optimization problem (15) with $\alpha = \hat{\alpha}_n$:

$$(m_{n,1}, m_{n,2}) := \arg\min_{\beta_1,\beta_2} \int_{V_{low}}^{V_{up}} \left\{ \log\left(-\log \frac{|\phi_n^{\Delta}(uV_n)|^{\theta^{2\hat{\alpha}_n}}}{|\phi_n^{\Delta}(\theta u V_n)|} \right) - \beta_2 \log(u) - \beta_1 \right\}^2 du,$$

where V_{low} and V_{up} are the truncation levels, and $\theta = 2$. The estimate of γ is then defined as $\hat{\gamma}_n(\hat{\alpha}_n) = 2(1-\hat{\alpha}_n) + m_{n,1}$. The boxplots shown in Figure 2 indicate that the quality of the estimates $\hat{\gamma}_n(\alpha)$ and $\hat{\gamma}_n(\hat{\alpha}_n)$ is quite similar.

6 Proofs

In the sequel we use the simplified notation: $\phi(u) := \phi^{\Delta}(u)$ and $\phi_n(u) := \phi_n^{\Delta}(u)$.

6.1 Proof of Theorem 3.1

Substitung (5) into (7), we get

$$|\phi(u)| \asymp A \exp\left\{ -\lambda_2 \left(-\operatorname{Re}\psi(u) \right)^{\alpha} \Psi_2 \left(-\operatorname{Re}\psi(u) \right) \right\}.$$

Recall that by Assumption (AL),

$$-\operatorname{Re}\psi(u) = \frac{1}{2}\sigma^2 u^2 + \lambda_1 |u|^{\gamma} \operatorname{Re}(\Psi_1(u)).$$

Therefore

$$|\phi(u)| \quad \asymp \quad A\exp\left\{-\lambda_2\Big(\frac{1}{2}\sigma^2\Big)^{\alpha}|u|^{2\alpha}\Big(1+\frac{2\alpha\lambda_1}{\sigma^2}|u|^{\gamma-2}\operatorname{Re}(\Psi_1(u))+R(u)\Big)\right\}$$

with

$$|R(u)| \leq |1-\Psi_2\left(-\operatorname{Re}\psi(u)\right)| \leq \vartheta_2\left(-\operatorname{Re}\psi(u)\right)^{-\chi_2} \leq \vartheta_2\left(\sigma^2/2\right)^{-\chi_2}|u|^{-2\chi_2},$$

where the last two inequalities hold for u large enough. The inequality:

$$|1-\operatorname{Re}(\Psi_1(u))| \leq |1-\Psi_1(u)| \leq \vartheta_1|u|^{-\chi_1}$$

yields the form of $r(u)$ and completes the proof.

6.2 Estimation of γ when α is known

6.2.1 Upper bounds

Proof of Lemma 4.1. First note that

$$|\gamma-\bar{\gamma}_n(\alpha)| \quad = \quad \left|\int_{\varepsilon V_n}^{V_n} w^{V_n}(u)\log\left(1+\frac{r(u)-r(\theta u)}{u^{\gamma-2}\tau^{(2)}\left(1-\theta^{\gamma-2}\right)}\right)du\right|.$$

Since $|r(u)-r(\theta u)|/u^{\gamma-2} \leq \tau^{(3)}\left(1+\theta^{\gamma-2-\tilde{\chi}_1}\right)u^{-\tilde{\chi}_1}$ as $u\to\infty$, and $|\log(1+x)|\leq 2|x|$ for any $|x|\leq 1/2$, it follows that for u large enough,

$$|\gamma-\bar{\gamma}_n(\alpha)| \leq (\varepsilon V_n)^{-\tilde{\chi}_1}\ 2\frac{\tau^{(3)}(1+\theta^{\gamma-2-\tilde{\chi}_1})}{\tau^{(2)}\left(1-\theta^{\gamma-2}\right)}\int_{\varepsilon}^{1}|w^1(u)|du$$

with $C>0$. The statement of the lemma follows with $C^{(1)} := 2\int_{\varepsilon}^{1}|w^1(u)|du$.

Proof of Lemma 4.2. The proof of this theorem follows the same lines as the proof of its analogue for the case of affine stochastic volatility models [7], [15]. We begin the proof with the following lemma.

Lemma 6.1. *Suppose that*

$$\widetilde{\varepsilon}_n := \left[\inf_{u\in[\varepsilon V_n, V_n]}|\phi(u)|\right]^{-\theta^{2\alpha}}\frac{\log n}{\sqrt{n}} = o(1), \quad n\to\infty. \tag{30}$$

Then there exist positive constants B_1, $\varkappa$ and δ such that for any $n>1$

$$\mathbb{P}\left\{|\bar{\gamma}_n(\alpha)-\hat{\gamma}_n(\alpha)| \leq B_1\widetilde{\varepsilon}_n\int_{\varepsilon V_n}^{V_n}\left|w^{V_n}(u)\right|\left|\log^{-1}\left(\mathcal{G}(u)\right)\right|du\right\} > 1-\varkappa n^{-1-\delta}, \tag{31}$$

where $\mathcal{G}(u) = |\phi(u)|^{\theta^{2\alpha}}/|\phi(u\theta)|$.

Proof. We divide the proof of the lemma into several steps.

1. Denote $\mathcal{G}_n(u) = |\phi_n(u)|^{\theta^{2\alpha}}/|\phi_n(\theta u)|$. It holds

$$\begin{aligned}\mathcal{G}_n(u)-\mathcal{G}(u) &= \frac{|\phi_n(u)|^{\theta^{2\alpha}}-|\phi(u)|^{\theta^{2\alpha}}}{|\phi_n(u\theta)|}+\frac{|\phi(u)|^{\theta^{2\alpha}}}{|\phi(u\theta)|}\,\frac{|\phi(u\theta)|-|\phi_n(u\theta)|}{|\phi_n(u\theta)|}\\ &= \mathcal{G}(u)\left[\frac{\xi_{1,n}(u)+\xi_{2,n}(u)}{1-\xi_{2,n}(u)}\right] =: \mathcal{G}(u)\Lambda_n(u)\end{aligned} \tag{32}$$

with

$$\xi_{1,n}(u) = \frac{|\phi_n(u)|^{\theta^{2\alpha}} - |\phi(u)|^{\theta^{2\alpha}}}{|\phi(u)|^{\theta^{2\alpha}}} \qquad \text{and} \qquad \xi_{2,n}(u) = \frac{|\phi(u\theta)| - |\phi_n(u\theta)|}{|\phi(u\theta)|}.$$

2. Lemma 3.6.1 from [15] shows that the event

$$\mathcal{W}_n = \left\{ \sup_{u \in [\varepsilon V_n, V_n]} |\xi_{k,n}(u)| \le B_2\, \widetilde{\varepsilon}_n,\ k = 1,2 \right\}$$

has a probability that tends to 1 as n tends to infinity. More precisely, it holds

$$\mathbb{P}(\mathcal{W}_n) \ge \mathbb{P}\left\{ \sup_{u \in [0, V_n]} |\xi_{k,n}(u)| \le B_2 \widetilde{\varepsilon}_n \right\} \ge 1 - \varkappa n^{-1-\delta}, \quad k = 1,2 \tag{33}$$

for some positive constants B_2, $\varkappa$ and δ.

3. For any $u \in [\varepsilon V_n, V_n]$, the Taylor expansion for the function $f(x) = \log(-\log(x))$ in the vicinity of the point $x = \mathcal{G}(u)$ yields

$$\mathcal{Y}_n(u) - \mathcal{Y}(u) = K_1(u)(\mathcal{G}_n(u) - \mathcal{G}(u)) + K_2(u)(\mathcal{G}_n(u) - \mathcal{G}(u))^2 \tag{34}$$

with

$$K_1(u) = \mathcal{G}^{-1}(u) \log^{-1}(\mathcal{G}(u)) \qquad \text{and} \qquad |K_2(u)| \le 2^{-1} \max_{z \in I_n(u)} \left[\frac{1 + |\log(z)|}{z^2 \log^2(z)} \right], \tag{35}$$

where by $I_n(u)$ we denote the interval between $\mathcal{G}(u)$ and $\mathcal{G}_n(u)$. Due to Theorem 3.1,

$$\begin{aligned} \mathcal{G}(u) &= \exp\left\{ -\tau^{(1)} \theta^{2\alpha} u^{2\alpha} \left[1 + \tau^{(2)} |u|^{\gamma-2} + r(u) \right] + \tau^{(1)} (\theta u)^{2\alpha} \left[1 + \tau^{(2)} |\theta u|^{\gamma-2} + r(\theta u) \right] \right\} \\ &= \exp\left\{ -A_1 |u|^{2\alpha + (\gamma-2)} + R(u) \right\}, \end{aligned} \tag{36}$$

where $A_1 = \tau^{(1)}\tau^{(2)}\theta^{2\alpha}(1 - \theta^{\gamma-2}) > 0$ and $|R(u)| \le A_2 |u|^{2\alpha+(\gamma-2)-\tilde{\chi}_1}$ for u large enough with $A_2 = \tau^{(1)}\tau^{(3)}\theta^{2\alpha}\left[1 + \theta^{(\gamma-2)-\tilde{\chi}_1}\right] > 0$.

Condition $2\alpha + (\gamma - 2) > 0$ guarantees that $\mathcal{G}(u) \to 0$ as $u \to +\infty$. The length of the interval $I_n(u)$ is equal to $\mathcal{G}(u)|\Lambda_n(u)|$; therefore, the length of $I_n(u)$ tends to 0 on the event $\mathcal{W}_n$, uniformly in $u \in [\varepsilon V_n, V_n]$. Thus, $I_n(u) \subset (0,1)$ on $\mathcal{W}_n$ for n large enough and the maximum on the right hand side of the inequality in (35) is attained at one of the endpoints on interval $I_n(u)$.

4. Denote $Q(u) = K_2(u)(\mathcal{G}_n(u) - \mathcal{G}(u))^2$. Lemma 3.6.2 from [15] shows that there exists a positive constant B_3 such that for any $u \in [\varepsilon V_n, V_n]$ and for n large enough

$$\mathcal{W}_n \subset \left\{ |Q(u)| \le B_3(\xi_{1,n}^2(u) + \xi_{2,n}^2(u)) \left| \log^{-1}(\mathcal{G}(u)) \right| \right\}. \tag{37}$$

5. The Taylor expansion (34) and previous discussion yield that on the set $\mathcal{W}_n$,

$$\begin{aligned} |\bar{\gamma}_n(\alpha) - \hat{\gamma}_n(\alpha)| &= \left| \int_0^{V_n} w^{V_n}(u)(\mathcal{Y}_n(u) - \mathcal{Y}(u))\, du \right| \\ &\le \int_0^{V_n} |w^{V_n}(u)| \left(\frac{|\mathcal{G}_n(u) - \mathcal{G}(u)|}{|\mathcal{G}(u)|} \left| \log^{-1}(\mathcal{G}(u)) \right| + |Q(u)| \right) du \\ &\le \int_0^{V_n} |w^{V_n}(u)| \log^{-1}\left(\mathcal{G}^{-1}(u)\right) \left(\frac{|\mathcal{G}_n(u) - \mathcal{G}(u)|}{|\mathcal{G}(u)|} + B_3(\xi_{1,n}^2(u) + \xi_{2,n}^2(u)) \right) du. \end{aligned}$$

By (32), expression in the brackets is equal to

$$P := \frac{|\mathcal{G}_n(u) - \mathcal{G}(u)|}{|\mathcal{G}(u)|} + B_3(\xi_{1,n}^2(u) + \xi_{2,n}^2(u)) \quad = \quad \frac{|\xi_{1,n}(u) + \xi_{2,n}(u)|}{|1 - \xi_{2,n}(u)|} + B_3(\xi_{1,n}^2(u) + \xi_{2,n}^2(u)).$$

Taking into account that $\xi_{2,n} < 1$ on the set $\mathcal{W}_n$ by (33), we conclude that P can be upper bounded on $\mathcal{W}_n$ as follows (all supremums are taken over $[\varepsilon V_n, V_n]$):

$$\begin{aligned} P &\leq \frac{\sup|\xi_{1,n}(u)| + \sup|\xi_{2,n}(u)|}{1 - \sup|\xi_{2,n}(u)|} + B_3\left((\sup|\xi_{1,n}(u)|)^2 + (\sup|\xi_{2,n}(u)|)^2\right) \\ &\leq \frac{2B_2\widetilde{\varepsilon}_n}{1 - B_2\widetilde{\varepsilon}_n} + 2B_3B_2^2\widetilde{\varepsilon}_n^2 \leq B_1\widetilde{\varepsilon}_n \end{aligned}$$

with $B_1 > 0$. This completes the proof of the lemma. □

Next, we proceed with the proof of Theorem 4.2. First, we get a lower bound for the infimum of the function $|\phi(u)|$ over $[\varepsilon V_n, V_n]$:

$$\begin{aligned} \inf_{u\in[\varepsilon V_n, V_n]} |\phi(u)| &\geq \exp\left\{-\tau^{(1)} V_n^{2\alpha}\left(1 + \tau^{(2)}|V_n|^{\gamma-2} + \tau^{(2)}\vartheta_1 V_n^{(\gamma-2)-\tilde{\chi}_1}\right)\right\} \\ &\geq \exp\left\{-\tau^{(1)}\left(1 + \tau^{(2)} + \tau^{(3)}\right) V_n^{2\alpha}\right\}. \end{aligned}$$

Applying Lemma 6.1 and taking into account that by (36), $|\log(\mathcal{G}(u))| \gtrsim u^{2\alpha+(\gamma-2)}$, we arrive at the desired result.

Proof of Theorem 4.3. Next, we combine Lemma 4.1 with Lemma 4.2. We choose the sequence V_n in the form $V_n^{2\alpha^\circ} = q\log n$. The assumption

$$q < \min_{\mathscr{A}} \left(2\tau^{(4)}\theta^{2\alpha^\circ}\right)^{-1}$$

guarantees the condition (19) for any model from $\mathscr{A}$. With this V_n, $\varepsilon_n \leq (\log n)/n^{\varkappa_1}$ where $\varkappa_1 > 0$. Therefore, on the set of probability $1 - \varkappa n^{-\delta-1}$, for any models from $\mathscr{A}$, it holds

$$|\hat{\gamma}_n(\alpha) - \gamma| \leq C_1(\log n)^{-\tilde{\chi}_1/(2\alpha^\circ)} + C_2\frac{(\log n)^{\varkappa_2}}{n^{\varkappa_1}},$$

with some $\varkappa_2$, positive C_1, C_2, and n large enough. From here it follows that the estimate $\hat{\gamma}_n(\alpha)$ is of logarithmic order, i.e.,

$$\mathbb{P}\left\{|\hat{\gamma}_n(\alpha) - \gamma| \leq C(\log n)^{-\tilde{\chi}_1/(2\alpha^\circ)}\right\} \geq 1 - \varkappa n^{-\delta-1}, \tag{38}$$

where

$$C = C^{(1)}\,\tau^{(3)}\frac{1 + \theta^{\gamma-2-\tilde{\chi}_1}}{1 - \theta^{\gamma-2}}\varepsilon^{-\tilde{\chi}_1} q^{-\tilde{\chi}_1/(2\alpha^\circ)}.$$

Note that this constant can be uniformly upper bounded on the set of models from $\mathscr{A}$. This completes the proof.

6.2.2 Lower bounds

<u>*Proof of Theorem 4.5.*</u> The aim of this proof is to show that

$$\liminf_{n\to\infty} \psi_n^{-2} \inf_{\hat{\gamma}_n} \sup_{\mathscr{A}} \mathbb{E}_\gamma \left|\hat{\gamma}_n - \gamma\right|^2 \geq c, \qquad \psi_n := C(\log n)^{-\tilde{\chi}_{1\circ}/(2\alpha^\circ)}, \tag{39}$$

where infimum is taken over all possible estimates of the parameter γ, supremum - over all models from $\mathscr{A}$, and c is some positive constant not depending on the parameters of the distribution. The main ingredient of the proof is the following lemma, which directly follows from [18], Theorem 2.2.

Lemma 6.2. *Let $\mathcal{P} = \{\mathbb{P}_\gamma\}$ be a (nonparametric) family of models in $I\!R^n$. Assume that there exist two values of parameter γ, say γ_1 and γ_2, such that $|\gamma_1 - \gamma_2| > 2\psi_n$ and moreover the corresponding measures $\mathbb{P}_1 := \mathbb{P}_{\gamma_1}$ and $\mathbb{P}_2 := \mathbb{P}_{\gamma_2}$ satisfy the following properties:*

1. *there exists a measure μ such that $\mathbb{P}_1 << \mu$ and $\mathbb{P}_2 << \mu$;*
2. *the χ^2 - distance between $\mathbb{P}_1^{\otimes n}$ and $\mathbb{P}_2^{\otimes n}$ is bounded by some constant $\eta \in I\!R_+$, where the χ^2 - distance is defined for any two measures $\mathbb{P}$ and $\mathbb{Q}$ as*

$$\chi^2\left(\mathbb{P},\mathbb{Q}\right) := \begin{cases} \int \left(\frac{d\mathbb{P}}{d\mathbb{Q}} - 1\right)^2 d\mathbb{Q}, & \text{if } \mathbb{P} << \mathbb{Q}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then the condition (39) *is fulfilled.*

In our case, we tackle with the models on the samples from $I\!R^n$. It is a worth mentioning that

$$\chi^2\left(\mathbb{P}_1^{\otimes n}, \mathbb{P}_2^{\otimes n}\right) = \left(1 + \chi^2\left(\mathbb{P}_1,\mathbb{P}_2\right)\right)^n - 1,$$

see [18]. Therefore, the boundedness of the χ^2 divergence is equivalent to the condition that

$$\chi^2\left(\mathbb{P}_1,\mathbb{P}_2\right) \leq q^{1/n} - 1 \asymp \frac{\log q}{n}, \tag{40}$$

where $q := 1 + \eta > 1$.

Further properties of the χ_1^2-divergence are discussed in [13].

Lemma 6.2 motivates the following result.

Lemma 6.3. *There exist two values γ_1 and γ_2 such that $|\gamma_1 - \gamma_2| > 2\psi_n$, and the measures $\mathbb{P}_1 = \mathbb{P}_{\gamma_1}$ and $\mathbb{P}_2 = \mathbb{P}_{\gamma_2}$ belong to $\mathscr{A}$ and satisfy the condition* (40)*.*

Proof. **1.** *Presentation of the models.* Let us fix some set of parameters $\mathscr{P}$. Consider the class $\mathscr{A} = \mathscr{A}(\mathscr{P})$, which is described in the formulation of Theorem 4.3.

For the time change $\mathcal{T}(s)$ in both models, we take the tempered stable process with $b = 0$, which is in fact a stable process, see Section 2.3. The choice of parameters $\alpha \in (\alpha_\circ, \alpha^\circ)$ and $a \in [2^{-\alpha}\lambda_{2\circ}, 2^{-\alpha}\lambda_2^\circ]$ quarantees that the assumption (AT3) holds with $A = 1$, $\lambda_2 \in [\lambda_{2\circ}, \lambda_2^\circ]$ and any ϑ_2, γ_2. For the process X_t in the first model, we take the sum of two independent process: the Brownian motion W_t and γ-stable process $\tilde{X}_t$ with $\gamma \in \Big(\gamma_\circ, 2\Big)$ and $\lambda_1 \in [\lambda_{1\circ}, \lambda_1^\circ]$, such that the characteristic exponent is equal to $\psi(u) = -u^2/2 - \lambda_1|u|^\gamma$. Note that the condition (AL) holds for any values of ϑ_1 and χ_1. According to (7), the characteristic function of the increments in the first model has the following asymptotics:

$$\phi(u) \asymp \exp\left\{-\lambda_2 \left(\frac{1}{2}u^2 + \lambda_1|u|^\gamma\right)^\alpha\right\}, \qquad u \to \infty.$$

We define the Lévy process for the second model by the characteristic exponent

$$\breve{\psi}(u) = -\frac{1}{2}u^2 - \lambda_1|u|^\gamma I\{|u| \leq M\} - \lambda_1 b|u|^{\gamma-2\psi_n}\left(1 + c|u|^{-\breve{\chi}_1}\right) I\{|u| \geq M\},$$

where $M, c > 0$, $\breve{\chi}_1 \in [\chi_{1\circ}, \gamma)$, $b = M^{2\psi_n}/\left(1 + c|M|^{-\breve{\chi}_1}\right)$. As it is explained in [5], Appendix A.4, this function determines some Lévy process with the BG index equal to $\breve{\gamma} = \gamma - 2\psi_n$ for M and c large enough. Moreover, for $b = 1$, any fixed M and c, this process satisfies the assumption (AL), since

$$\Psi_1(u) = |u|^{2\psi_n} I\{|u| \leq M\} + \left(1 + c|u|^{-\breve{\chi}_1}\right) I\{|u| \geq M\}$$

lies between $1 - c|u|^{-\breve{\chi}_1}$ and $1 + c|u|^{-\breve{\chi}_1}$ for $|u|$ large enough. Assumption $b = 1$ yields the following relation between M and n:

$$\psi_n = \frac{\log\left(1 + M^{-\breve{\chi}_1}\right)}{2\log M} \asymp 1/\left(M^{\breve{\chi}_1}\log M\right), \qquad M \to \infty. \tag{41}$$

The characteristic function of the second compound process is equal to

$$\breve{\phi}^\Delta(u) = \exp\Big\{\lambda_2\left(-\breve{\psi}(u)\right)^\alpha\Big\}.$$

Note that both models have absolute continuous distributions. Denoting the corresponding densities in the moment Δ by $p_\Delta(x)$ and $\breve{p}_\Delta(x)$, we can express the χ^2 - divergence in the following way:

$$\chi^2(\mathbb{P}_1, \mathbb{P}_2) = \int_{I\!R} \frac{\left(p_\Delta(x) - \breve{p}_\Delta(x)\right)^2}{p_\Delta(x)} dx, \tag{42}$$

since $p_\Delta(x)\breve{p}_\Delta(x) > 0,\ \forall x \in I\!R$.

2. *Lower bound for* $p_\Delta(x)$. The density function $p_\Delta(x)$ can be expressed as follows:

$$p_\Delta(x) = \int_{I\!R+} q_t(x)\pi_\Delta(t)dt,$$

where $\pi_\Delta(t)$ is the density function of the tempered stable process at the time moment t, and $q_t(x)$ is the density function of the sum of processes $\tilde{X}_t$ and W_t. Since $q_t(x)$ is a convolution of two density functions, and the (strictly) γ-stable process $\tilde{X}_t$ posses the property $\tilde{X}_t \stackrel{d}{=} t^{1/\gamma}\tilde{X}_1$ (see [9]), we conclude that

$$q_t(x) = \frac{1}{(2\pi)^{1/2}t^{1/2+1/\gamma}} \int_{I\!R} \exp\left\{-\frac{(x-v)^2}{2t}\right\} p_{st}\left(\frac{v}{t^{1/\gamma}}\right) dv \gtrsim |x|^{-(\gamma+1)}, \qquad x \to \infty,$$

where p_{st} is the density of the distribution of $\tilde{X}_1$; the last inequality follows from [20]. Fixing some $0 < d_1 < d_2 < 1/2$, we arrive at

$$p_\Delta(x) \geq \int_{d_1}^{d_2} q_t(x)\pi_\Delta(t)dt \gtrsim |x|^{-(\gamma+1)}.$$

Returning now to (42). Taking into account that $p_\Delta(x)$ is bounded on any set of the type $\{|x| \leq \mathcal{C}\}$ by some constant $\mathcal{D}$, we get that with $\mathcal{C}$ large enough,

$$\chi^2(\mathbb{P}_1, \mathbb{P}_2) \leq \mathcal{D} \int_{|x|\leq \mathcal{C}} (p_\Delta(x) - \breve{p}_\Delta(x))^2\, dx + \int_{|x|>\mathcal{C}} |x|^{1+\gamma} (p_\Delta(x) - \breve{p}_\Delta(x))^2\, dx =: I_1 + I_2$$

3. *Upper bound for* I_1. By Parseval-Plancherel theorem [19],

$$I_1 \leq \mathcal{D} \int_{I\!R} (p_\Delta(x) - \breve{p}_\Delta(x))^2\, dx \quad = \quad \frac{\mathcal{D}}{2\pi} \int_{|x|>M} \left|\phi(x) - \breve{\phi}(x)\right|^2 dx$$

because $\phi(x)$ coincides with $\breve{\phi}(x)$ for $|x| \leq M$. Next, note that

$$\int_{|x|>M} \left|\phi(x) - \breve{\phi}(x)\right|^2 dx \quad = \quad \int_{|x|>M} e^{-2\lambda_2\left(x^2/2+\lambda_1|x|^{\gamma-\psi_n}\right)^\alpha} \left(e^{\varkappa_1} - e^{\varkappa_2}\right)^2 du,$$

where

$$\begin{aligned}
\varkappa_1 &= -\lambda_2\left(x^2/2 + \lambda_1|x|^\gamma\right)^\alpha + \lambda_2\left(x^2/2 + \lambda_1|x|^{\gamma-\psi_n}\right)^\alpha \\
&\asymp -2\alpha\lambda_1\lambda_2\left(x^2/2\right)^\alpha |x|^{\gamma-2} < 0; \\
\varkappa_2 &= -\lambda_2\left(x^2/2 + \lambda_1|x|^{\gamma-2\psi_n}\left(1 + c|x|^{-\breve{\chi}_1}\right)\right)^\alpha + \lambda_2\left(x^2/2 + \lambda_1|x|^{\gamma-\psi_n}\right)^\alpha \\
&\asymp -2\alpha\lambda_1\lambda_2 c\left(x^2/2\right)^\alpha |x|^{\gamma-\psi_n-2} < 0.
\end{aligned}$$

Therefore,

$$I_1 \lesssim \int_{|x|>M} e^{-2\lambda_2\left(x^2/2+\lambda_1|x|^{\gamma-\psi_n}\right)^\alpha} dx \lesssim e^{-2\lambda_2\left(M^2/2\right)^\alpha} \int_{|x|>M} e^{-4\lambda_1\lambda_2\alpha|x|^{\gamma-\psi_n-2}} du.$$

The asymptotical bound of the last integral can be found using the change of the variable and integration by parts. Denote $\xi_1 = 4\lambda_1\lambda_2\alpha$ and $\xi_2 = \gamma - \psi_n - 2$. Then

$$\begin{aligned}
\frac{1}{2}\int_{|x|>M} e^{-\xi_1|x|^{\xi_2}} dx \leq &\int_{v>M^{\xi_2}} e^{-\xi_1 v} d(v^{1/\xi_2}) \\
&\leq -e^{-\xi_1 M^{\xi_2}} M + \xi_1 \int_{v>M^{\xi_2}} e^{-\xi_1 v} v^{1/\xi_2} dv \\
&\leq -e^{-\xi_1 M^{\xi_2}} M + \xi_1 e^{-\xi_1 M^{\xi_2}} \int_{v>M^{\xi_2}} v^{1/\xi_2} dv \\
&\lesssim e^{-\xi_1 M^{\xi_2}} M^{1+\xi_2}.
\end{aligned} \tag{43}$$

This leads to the following upper bound for the integral I_1:

$$I_1 \lesssim e^{-2\lambda_2\left(M^2/2\right)^\alpha} e^{-4\lambda_1\lambda_2\alpha M^{\gamma-\psi_n-2}} M^{\gamma-\psi_n-1} \leq e^{-2\lambda_2\left(M^2/2\right)^\alpha} M^{\gamma-\psi_n-1}.$$

4. *Upper bound for I_2.* Note that

$$I_2 \leq \int_{|x|>M} \left[x^2 \left(p_\Delta(x) - \breve{p}_\Delta(x)\right)\right]^2 dx \leq \frac{1}{2\pi} \int_{|x|>M} \left|\widehat{x^2 p_\Delta(x)} - \widehat{x^2 \breve{p}_\Delta(x)}\right|^2 dx,$$

where by $\widehat{g(x)}$ we denote the Fourier transform of a function $g(x)$. Making use of the property $\widehat{x^2 g(x)} = \partial^2 \widehat{g(x)}/\partial x^2$, we conclude that

$$I_2 \lesssim \int_{|x|>M} x^4 \left|\phi(x) - \breve{\phi}^\Delta(x)\right|^2 dx \lesssim e^{-2\lambda_2 \left(M^2/2\right)^\alpha} M^{\gamma - \psi_n + 3},$$

because by the arguments similar to (43),

$$\frac{1}{2} \int_{|x|>M} |x|^n e^{-\xi_1 |x|^{\xi_2}} dx \lesssim e^{-\xi_1 M^{\xi_2}} M^{n+1+\xi_2}$$

for any $n > 0$.

5. *Choice of M.* Thus,

$$\chi^2(\mathbb{P}_1, \mathbb{P}_2) \lesssim e^{-\mu_1 M^{2\alpha}} M^{\mu_2}, \qquad M \to +\infty,$$

where $\mu_1 = \lambda_2\, 2^{1-\alpha} > 0$, $\mu_2 = \gamma - \psi_n + 3 > 0$. The aim now is to choose the parameter M such that the conditions (40) and (41) are fulfilled simultaneously. The choice of such M can be made in the form $M = (A_n/\mu_1)^{1/(2\alpha)}$. Substituting this M into (40) gives the following condition on A_n:

$$-A_n + \frac{\mu_2}{2\alpha} \log\left(\frac{A_n}{\mu_1}\right) \lesssim -\log n,$$

which suggests to choose $A_n = \log\left(n \log^\beta n\right)$, where $\beta > \mu_2/(2\alpha)$. Our considerations are summarized in the choice

$$M = \left(\frac{\log\left(n \log^\beta n\right)}{\lambda_2\, 2^{1-\alpha}}\right)^{1/(2\alpha)}, \qquad \beta > (\gamma - \psi_n + 3)/(2\alpha),$$

which satisfies (40) and (41). This completes the proof. □

6.3 Estimation of α

The empirical counterpart of the estimate $\hat{\alpha}_n$ is equal to

$$\bar{\alpha}_n := \frac{1}{2} \int_0^\infty w_\alpha^{U_n}(u) \log\left(-\log|\phi_n(u)|\right) du.$$

The closeness of $\bar{\alpha}_n$ and α is proven in the next lemma.

Lemma 6.4.

$$|\alpha - \bar{\alpha}_n| \lesssim U_n^{\gamma-2}, \qquad n \to \infty. \tag{44}$$

Proof. The proof follows the same lines as Lemma 4.1. The basic observation is that

$$
\begin{aligned}
|\alpha - \bar{\alpha}_n| &= \frac{1}{2}\int_{\varepsilon U_n}^{U_n} w_\alpha^{U_n}(u)\log\left(1+\tau^{(2)}u^{\gamma-2}+r(u)\right)du \\
&\le \int_{\varepsilon U_n}^{U_n} w_\alpha^{U_n}(u)\left(\tau^{(2)}u^{\gamma-2}+r(u)\right)du \\
&\le U_n^{\gamma-2}\left(\tau^{(2)}+\tau^{(3)}U_n^{-\tilde{\chi}_1}\right)\int_{\varepsilon}^{1} w_\alpha^1(u)du.
\end{aligned}
$$

This completes the proof. □

The next lemma is an analogue of Lemma 4.2 for the estimate $\hat{\alpha}_n$.

Lemma 6.5. *There exist positive constants D, $\varkappa$ and δ such that*

$$
\mathbb{P}\left\{|\bar{\alpha}_n - \hat{\alpha}_n| \le D\,\frac{\log n}{\sqrt{n}}\,\frac{\exp\{\tau^{(1)}U_n^{2\alpha}\}}{U_n^{2\alpha}}\right\} \ge 1-\varkappa n^{-1-\delta}.
$$

Proof. The main ingredient of the proof is that the difference between $\hat{\alpha}_n$ and $\bar{\alpha}_n$ allows the following representation:

$$
\begin{aligned}
|\hat{\alpha}_n - \bar{\alpha}_n| &= \frac{1}{2}\left|\int_{\varepsilon U_n}^{U_n} w_\alpha^{U_n}(u)\Big(\log\left(-\log|\phi(u)|\right)-\log\left(-\log|\phi_n(u)|\right)\Big)du\right| \\
&\le \frac{1}{2}\int_{\varepsilon U_n}^{U_n}\left|w_\alpha^{U_n}(u)\right|\cdot\left|\max_{\xi\in I_n(u)}\frac{1}{\xi\log\xi}\right|\cdot\big||\phi(u)|-|\phi_n(u)|\big|\,du,
\end{aligned}
\tag{45}
$$

where $I_n(u)$ is the interval between $\phi(u)$ and $\phi_n(u)$. Since $\phi_n(u)$ tends uniformly to $\phi(u)$ (see Section 4.2), we conclude that

$$
\left|\max_{\xi\in I_n(u)}\frac{1}{\xi\log\xi}\right| \le \max_{a\in(1/2,3/2)}\frac{1}{a|\phi(u)|\cdot|\log\left(a\,|\phi(u)|\right)|} = \frac{2}{|\phi(u)|\cdot\left(\log 2+|\log|\phi(u)||\right)}.
\tag{46}
$$

Next, note that Theorem 3.1 yields

$$
\begin{aligned}
\min_{u\in[\varepsilon U_n,U_n]}|\phi(u)| &= \left(\max_{u\in[\varepsilon U_n,U_n]}\exp\left\{\tau^{(1)}|u|^{2\alpha}\left(1+\tau^{(2)}|u|^{\gamma-2}+r(u)\right)\right\}\right)^{-1} \\
&= \exp\left\{-\tau^{(1)}U_n^{2\alpha}\left(1+\tau^{(2)}U_n^{\gamma-2}+r(U_n)\right)\right\} \\
&\asymp \exp\left\{-\tau^{(1)}U_n^{2\alpha}\right\}.
\end{aligned}
\tag{47}
$$

Similar to (33), there exist positive constants B, $\varkappa$ and δ such that

$$
\mathbb{P}\left\{\sup_{u\in[\varepsilon U_n,U_n]}\Big||\phi(u)|-|\phi_n(u)|\Big| \le B\frac{\log n}{\sqrt{n}}\right\} \ge 1-\varkappa n^{-1-\delta}.
\tag{48}
$$

Combining (46), (47), (48) with (45), we arrive at the desired result. □

Proof of Theorem 4.6. The choice $U_n^{2\alpha^\circ} = q\log n$ yields that on a set $\mathcal{W}_n$ of the probability measure larger than $1-\varkappa n^{-1-\delta}$, it holds for any model from $\mathscr{A}$

$$|\bar{\alpha}_n - \hat{\alpha}_n| \;\leq\; D\,\frac{\log n}{\sqrt{n}}\,\frac{\exp\{\tau^{(1)}U_n^{2\alpha^\circ}\}}{U_n^{2\alpha_\circ}} = D\,\frac{\log n}{\sqrt{n}}\,\frac{n^{q\tau^{(1)}}}{(q\log n)^{\alpha_\circ/\alpha^\circ}} \lesssim \frac{(\log n)^{\varkappa_2}}{n^{\varkappa_1}},$$

with $\varkappa_1 = 1/2 - q\tau^{(1)}$ and some $\varkappa_2$. Therefore, choosing $q < q^\circ = \min_{\mathscr{A}}\left\{1/(2\tau^{(1)}\right\}$ we get on $\mathcal{W}_n$

$$|\hat{\alpha}_n - \alpha| \leq |\bar{\alpha}_n - \hat{\alpha}_n| + |\hat{\alpha}_n - \alpha| \lesssim \Big(\log n\Big)^{(\gamma-2)/\alpha^\circ}.$$

This completes the proof.

6.4 Estimation of γ when α is unknown

Proof of Theorem 4.7.

1. *Preliminary remarks.* Note that

$$\begin{aligned}
|\gamma_n(\hat{\alpha}_n) - \gamma| &\leq |\gamma_n(\hat{\alpha}_n) - \bar{\gamma}_n(\alpha)| + |\bar{\gamma}_n(\alpha) - \gamma| \\
&\leq 2\,|\hat{\alpha}_n - \alpha| + |\bar{\gamma}_n(\alpha) - \gamma| + \left|\int_0^\infty w^{V_n}(u)\log\Big(\frac{-\theta^{2\hat{\alpha}_n}\log|\phi_n(u)| + \log|\phi_n(\theta u)|}{-\theta^{2\alpha}\log|\phi(u)| + \log|\phi(\theta u)|}\Big)\right| du.
\end{aligned}$$

The upper bound for the first two summands are given in Theorem 4.6 and Lemma 4.1 resp. So, the aim is to find the upper bound for the last summand, which we denote by I.

$$\begin{aligned}
I &\leq \left|\int_0^\infty w^{V_n}(u)\log\Big(\frac{-\theta^{2\hat{\alpha}_n}\log|\phi_n(u)| + \log|\phi_n(\theta u)|}{-\theta^{2\alpha}\log|\phi(u)| + \log|\phi(\theta u)|}\Big)\right| du \\
&\leq 2\left|\int_0^\infty w^{V_n}(u)\frac{-\theta^{2\alpha}\Upsilon_1 + \Upsilon_2}{-\theta^{2\alpha}\log|\phi(u)| + \log|\phi(\theta u)|}\right|,
\end{aligned}$$

where

$$\begin{aligned}
\Upsilon_1 &:= \theta^{2(\hat{\alpha}_n-\alpha)}\log|\phi_n(u)| - \log|\phi(u)|, \\
\Upsilon_2 &:= \log|\phi_n(\theta u)| - \log|\phi(\theta u)|.
\end{aligned}$$

2. *Upper bounds for Υ_1 and Υ_2.* Note that for any $u > 1$ and any $\beta \in I\!R$,

$$\Big|u^\beta - 1\Big| \leq u^{|\beta|} - 1.$$

Moreover, for β tending to zero,

$$u^{|\beta|} - 1 \leq C|\beta|\log(u),$$

where $C > 0$. This in particularly yields that for any small $\upsilon > 0$ and n large enough,

$$\Big|\theta^{2(\hat{\alpha}_n-\alpha)} - 1\Big| \leq 2C\log(\theta)|\hat{\alpha}_n - \alpha| \lesssim \Big(\log n\Big)^{(\gamma-2)/\alpha^\circ} := \upsilon_n,$$

where the last asymptotic inequality follows from Theorem 4.6; here we choose the sequence U_n as it is described in Theorem 4.6. Therefore, for n large enough,

$$|\Upsilon_1| \leq \left|\log\Big(\frac{|\phi_n(u)|}{|\phi(u)|}\Big)\right| + \upsilon_n \left|\log|\phi_n(u)|\right| \leq 2\frac{\left||\phi_n(u)| - |\phi(u)|\right|}{|\phi(u)|} + \upsilon_n \left|\log|\phi_n(u)|\right|. \tag{49}$$

On the other hand, (48) together with Theorem 3.1 yield that for n large enough,

$$\begin{aligned}
\left|\log|\phi_n(u)|\right| &= \left|\log|\phi(u)| + \log\Big(\frac{|\phi_n(u)| - |\phi(u)|}{|\phi(u)|} + 1\Big)\right| \\
&\lesssim \left|\log|\phi(u)|\right| + 2\frac{\left||\phi_n(u)| - |\phi(u)|\right|}{|\phi(u)|}.
\end{aligned}$$

Substituting this bound into (49), we conclude that

$$|\Upsilon_1| \leq 2\,(1+\upsilon_n)\,\frac{\left||\phi_n(u)| - |\phi(u)|\right|}{|\phi(u)|} + \upsilon_n \left|\log|\phi(u)|\right|. \tag{50}$$

Next, the arguments similar to given in the proof of Theorem 6.1, we get that for $u \in [\varepsilon V_n, V_n]$,

$$\begin{aligned}
\frac{\left||\phi_n(u)| - |\phi(u)|\right|}{|\phi(u)|} &\leq \frac{\log n}{\sqrt{n}} \left(\inf_{u\in[\varepsilon V_n, V_n]} |\phi(u)|\right)^{-1} \lesssim \frac{\log n}{\sqrt{n}} \exp\left\{\tau^{(1)} V_n^{2\alpha}\right\}, \\
\left|\log|\phi(u)|\right| &\lesssim \exp\left\{-\tau^{(1)}\left(\varepsilon V_n\right)^{2\alpha}\right\}.
\end{aligned}$$

Combining the last inequalities and the definition of υ_n, we conlcude that

$$|\Upsilon_1| \lesssim 3\frac{\log n}{\sqrt{n}} \exp\left\{\tau^{(1)} V_n^{2\alpha}\right\} + \Big(\log n\Big)^{(\gamma-2)/\alpha^\circ} \exp\left\{-\tau^{(1)}\left(\varepsilon V_n\right)^{2\alpha}\right\}.$$

As for Υ_2, it is bounded for large n up to a constant by the absolute value of $\xi_{2,n}$ from Lemma 6.1:

$$|\Upsilon_2| = \left|\log\Big(\frac{|\phi_n(\theta u)|}{|\phi(\theta u)|}\Big)\right| \leq 2\frac{\left||\phi_n(\theta u)| - |\phi(\theta u)|\right|}{|\phi(\theta u)|} := 2\left|\xi_{2,n}(u)\right| \lesssim \frac{\log n}{\sqrt{n}} \exp\left\{\tau^{(1)} V_n^{2\alpha}\right\}.$$

3. *Upper bound for I.* Taking into account that

$$-\theta^{2\alpha}\log|\phi(u)| + \log|\phi(\theta u)| \asymp -\tau^{(1)}\tau^{(2)}\theta^{2\alpha}\left(1-\theta^{\gamma-2}\right)|u|^{2\alpha+(\gamma-2)}$$

(see (36) for details), we conclude that

$$I \lesssim V_n^{-(2\alpha+(\gamma-2))}\left(\left(3\,\theta^{2\alpha}+1\right)\frac{\log n}{\sqrt{n}} \exp\left\{\tau^{(1)} V_n^{2\alpha}\right\} + \Big(\log n\Big)^{(\gamma-2)/\alpha^\circ} \exp\left\{-\tau^{(1)}\left(\varepsilon V_n\right)^{2\alpha}\right\}\right). \tag{51}$$

Denote

$$G_1(\alpha) := \left(3\,\theta^{2\alpha}+1\right)\frac{\log n}{\sqrt{n}} \exp\left\{\tau^{(1)} V_n^{2\alpha}\right\}, \qquad G_2(\alpha) := \Big(\log n\Big)^{(\gamma-2)/\alpha^\circ} \exp\left\{-\tau^{(1)}\left(\varepsilon V_n\right)^{2\alpha}\right\}.$$

It is a worth mentioning that for any model from $\mathscr{A}$ and n large enough,

$$G_1(\alpha) \leq G_1(\alpha^\circ) \lesssim G_2(\alpha^\circ) \leq G_2(\alpha)$$

provided that $V_n^{2\alpha^\circ} = p \log n$ with $p \le p^\circ$. This remark means that the asymptotical bound for I is given by the second summand in (51), i.e.,

$$I \lesssim V_n^{-(2\alpha+(\gamma-2))}\Big(\log n\Big)^{(\gamma-2)/\alpha^\circ} \exp\Big\{-\tau^{(1)}\left(\varepsilon V_n\right)^{2\alpha}\Big\} \lesssim \Big(\log n\Big)^{(-2\alpha+(\gamma-2))/(2\alpha^\circ)}. \quad (52)$$

4. *Upper bound for* $|\hat{\gamma}_n(\hat{\alpha}_n) - \gamma|$. To conclude the proof, we summarize the obtained upper bounds, see (52), Theorem 4.6 and Lemma 4.1:

$$\begin{aligned} I &\lesssim \Big(\log n\Big)^{(-2\alpha+(\gamma-2))/(2\alpha^\circ)}, \\ |\hat{\alpha}_n - \alpha| &\lesssim \Big(\log n\Big)^{(\gamma-2)/\alpha^\circ}, \\ |\bar{\gamma}_n(\alpha) - \gamma| &\lesssim \Big(\log n\Big)^{-\tilde{\chi}_1/(2\alpha^\circ)}. \end{aligned}$$

Since $2-\gamma < 2\alpha$ for class $\mathscr{A}$,

$$(\gamma-2)/\alpha^\circ > (-2\alpha + (\gamma-2))/(2\alpha^\circ),$$

and we arrive at (28). The remark that for $\gamma < 4/3$

$$-\tilde{\chi}_1/(2\alpha^\circ) > -\chi_1/(2\alpha^\circ) > -\gamma/(2\alpha^\circ) > (\gamma-2)/\alpha^\circ$$

gives (29) and completes the proof.

7 Appendix

7.1 The case $2-\gamma > 2\alpha$

Assume $A = 1$ and introduce an estimate

$$\hat{\gamma}_n^*(\alpha) := 2(1-\alpha) + \int_0^\infty w^{V_n}(u) \log\left(1 - \frac{|\phi_n(u)|^{\theta^{2\alpha}}}{|\phi_n(\theta u)|}\right) du. \quad (53)$$

The main idea behind this estimator is that

$$\frac{|\phi(u)|^{\theta^{2\alpha}}}{|\phi(\theta u)|} = \exp\Big\{-A_1|u|^{2\alpha+(\gamma-2)}R(u)\Big\}, \quad |R(u)| \lesssim 1 + A_2|u|^{-\tilde{\chi}_1}$$

with $A_1 = \tau^{(1)}\tau^{(2)}\theta^{2\alpha}(1-\theta^{\gamma-2}) > 0$ and $A_1A_2 = \tau^{(1)}\tau^{(3)}\theta^{2\alpha}(1+\theta^{(\gamma-2)-\chi_1})$, and therefore

$$\mathcal{Y}_3(u) = 1 - \frac{|\phi(u)|^{\theta^{2\alpha}}}{|\phi(\theta u)|} \asymp A_1|u|^{2\alpha+(\gamma-2)}R(u)$$

Taking logarithms of both parts, we conclude that $\mathcal{Y}_3(u)$ is linear in $\log|u|$ (at least for large $|u|$) with slope $2\alpha+\gamma-2$. Therefore, the weighted least squares approach leads to the estimate of α also in this case.

The study of $\hat{\gamma}_n^*(\alpha)$ does not differ principally from the study of $\hat{\gamma}_n(\alpha)$. In this article, we give only the formulation of the theorem that shows the upper bound for $\hat{\gamma}_n(\alpha)$.

Theorem 7.1. *Introduce a deterministic counterpart for $\hat{\gamma}_n^*(\alpha)$:*

$$\bar{\gamma}_n^*(\alpha) := 2(1-\alpha) + \int_0^\infty w^{V_n}(u) \log\left(1 - \frac{|\phi(u)|^{\theta^{2\alpha}}}{|\phi(\theta u)|}\right) du.$$

(i) In the setup of Theorem 3.1, it holds for n large enough,

$$|\gamma - \bar{\gamma}_n^*(\alpha)| \lesssim V_n^{-\tilde{\chi}_1}, \qquad n \to \infty.$$

(ii) Let the sequence V_n be such that

$$\varepsilon_n := \frac{\log n}{\sqrt{n}} \exp\left\{\tau^{(4)} \left(\theta V_n\right)^{2\alpha}\right\} = o(1), \quad n \to \infty.$$

Then there exist positive constants $C^{(3)}$, $\varkappa$ and δ such that

$$\mathbb{P}\left\{|\bar{\gamma}_n^*(\alpha) - \hat{\gamma}_n^*(\alpha)| \le C^{(3)} \varepsilon_n V_n^{(2-\gamma)-2\alpha}\right\} > 1 - \varkappa\, n^{-1-\delta}. \tag{54}$$

(iii) Take the sequence $V_n = (q \log n)^{1/(2\alpha^\circ)}$ with $q < \left(2\theta^{2\alpha^\circ} \min_{\mathscr{A}} \tau^{(4)}\right)^{-1}$. Then

$$\sup_{\mathscr{A}} \mathbb{P}\left\{|\hat{\gamma}_n(\alpha) - \gamma| \le \Xi_5 (\log n)^{-\tilde{\chi}_1/(2\alpha^\circ)}\right\} > 1 - \varkappa\, n^{-1-\delta}, \tag{55}$$

where $\tilde{\chi}_1 := \min\{\chi_1, 2\chi_2 + \gamma - 2\}$, the supremum is taken over the set of all models from $\mathscr{A}$, constants $\varkappa$ and δ do not depend on the parameters of the underlined models, and Ξ_5 depends on $\mathscr{P}$ only.

7.2 Asymptotic behavior of the characteristic exponent

Lemma 7.2. *Consider a Lévy measure ν on $\mathbb{R} \setminus \{0\}$ that fulfilles*

$$G(\varepsilon) := \int_{|x|>\varepsilon} \nu(dx) = \varepsilon^{-\gamma}(\beta^{(0)} + \beta^{(1)} \varepsilon^{\chi_1}(1 + O(\varepsilon))), \quad \varepsilon \to +0 \tag{56}$$

with $0 < \chi_1 < \gamma < 2$, and $\beta^{(0)} > 0$. Denote

$$\mathcal{V}(u) \;=\; \mathrm{Re}(\psi(u)) + \frac{1}{2}\sigma^2 u^2 = \int_{\mathbb{R}} \left(\cos(ux) - 1\right) d\nu(x).$$

Then as $u \to +\infty$,

$$\mathcal{V}(u) \;=\; -u^\gamma \left(\beta^{(0)} d_\gamma + \beta^{(1)} d_{\gamma-\chi_1} u^{-\chi_1}\right) + O(1).$$

where $d_\gamma = \Gamma(1-\gamma) \sin((1-\gamma)\pi/2)$.

Proof. We divide the proof into 3 steps.

1. First, apply integration by parts to get

$$\begin{aligned}\mathcal{V}(u) &= -\int_0^{+\infty}\big(\cos(ux)-1\big)dG(x)\\ &= -\big(\cos(ux)-1\big)G(x)\big|_0^{+\infty} - u\int_0^{+\infty}\sin(ux)G(x)dx\\ &= -\int_0^{+\infty}\sin(x)G(x/u)dx.\end{aligned}$$

2. Take $H=u^p$ with $0<p<1$ such that $p\,\gamma>\chi_1$, and represent the last integral as a sum of tho integrals:

$$\begin{aligned}\int_0^{+\infty}\sin(x)G(x/u)dx &= \int_0^{H}\sin(x)G(x/u)dx+\int_H^{+\infty}\sin(x)G(x/u)dx\\ &= I_1+I_2.\end{aligned}$$

The integral I_2 is bounded, because $G(x/u)$ is monotone, converges to 0 as $x\to\infty$ and the antiderivative of $\sin(x)$ is bounded.

3. Next, we apply (56) to I_1:

$$\begin{aligned}I_1 &= \int_0^{H}\sin(x)\,(x/u)^{-\gamma}\left(\beta^{(0)}+\beta^{(1)}\,(x/u)^{\chi_1}\,(1+O\,(x/u))\right)dx\\ &= \beta^{(0)}u^{\gamma}\int_0^{H}\frac{\sin(x)}{x^{\gamma}}dx+\beta^{(1)}u^{\gamma-\chi_1}\int_0^{H}\frac{\sin(x)}{x^{\gamma-\chi_1}}dx+\beta^{(1)}u^{\gamma-\chi_1-1}\int_0^{H}\frac{\sin(x)}{x^{\gamma-\chi_1-1}}dx.\end{aligned}$$

Note that the integral $\int_0^H\sin(x)x^{-\gamma}dx$ can be represented in the following way:

$$\int_0^{H}\frac{\sin(x)}{x^{\gamma}}dx = \int_0^{\infty}\frac{\sin(x)}{x^{\gamma}}dx-\int_H^{\infty}\frac{\sin(x)}{x^{\gamma}}dx=d_\gamma+O(H^{-\gamma}).$$

Analogously,

$$\int_0^{H}\frac{\sin(x)}{x^{\gamma-\chi_1}}dx = d_{\gamma-\chi_1}+O(H^{-(\gamma-\chi_1)}).$$

Finally, we arrive at

$$I_1 = \beta^{(0)}d_\gamma u^{\gamma}+\beta^{(1)}d_{\gamma-\chi_1}u^{\gamma-\chi_1}+T_1,$$

where

$$T_1 = O(u^{(1-p)\gamma})+O(u^{(1-p)(\gamma-\chi_1)})+O(u^{(1-p)(\gamma-\chi_1-1)})=O(u^{(1-p)\gamma}).$$

□

References

[1] Aït-Sahalia, Y. and Jacod, J. Estimating the degree of activity of jumps in high frequency financial data. *Ann.Stat.*, 37(5A):2202–2244, 2009.

[2] Athreya, K. and Lahiri, S. *Measure theory and probability theory.* Springer US, 2010.

[3] Barndorff-Nielsen, Ole E. and Shiryaev, A.N. *Change of Time and Change of Measure.* World Scientific, 2010.

[4] Basawa, I. and Brockwell, P. Nonparametric estimation for nondecreasing Lévy processes. *J. Roy. Statist. Soc. Ser. B*, 44:262–269, 1982.

[5] Belomestny, D. Spectral estimation of the fractional order of a Lévy process. *Ann.Stat.*, 38(1):317–351, 2010.

[6] Belomestny, D. Statistical inference for time-changed Lévy processes via composite characteristic function estimation. *Ann. Stat.*, 39(4):2205–2242, 2011.

[7] Belomestny, D. and Panov, V. Abelian theorems for stochastic volatility models with application to the estimation of jump activity of volatility. SFB 823 Preprint, 45/2011, 2011.

[8] Carr, P. and Wu, L. Time-changed Lévy processes and option pricing. *J.Fin.Ec.*, 71:113–141, 2004.

[9] Cont, R. and Tankov, P. *Financial modelling with jump process.* CRC Press UK, 2004.

[10] Elliott, R.J. and Kopp, P.E. *Mathematics of financial markets.* Springer, 1999.

[11] Figueroa-Lopez, J. E. Advances in Applied Probability. *Advances in Applied Probability*, 41:1161–1188, 2009.

[12] Korevaar, J. *Tauberian theory: a century of developments.* Springer, 2004.

[13] Liese, F. and Vajda, I. *Convex statistical distance.* BSB Teubner, 1987.

[14] Monroe, I. Processes that can be embedded in Brownian motion. *Ann. Prob.*, 6:42–56, 1978.

[15] Panov, V. *Abelian theorems for stochastic volatility models and semiparametric estimation of the signal space.* PhD thesis, Humboldt University, 2012.

[16] Rubin, H. and Tucker, H. Estimating the parameters of a differential process. *Ann. Math. Statist.*, 30:641–658, 1959.

[17] Schoutens, W. *Lévy processes in finance.* John Wiley and Sons, 2003.

[18] Tsybakov, A. *Introduction to nonparametric estimation.* Springer, New York, 2009.

[19] Ushakov, N.G. *Selected topics in characteristic functions.* VSP, Utrecht, 1999.

[20] Zolotarev, V.M. *One-dimensional stable distributions.* Providence, RI : American Mathematical Soc., 1986.