# Partial quasi-morphisms and symplectic quasi-integrals on cotangent bundles

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Alexandra Monzner

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Prüfungskommission:
Vorsitzender: Prof. Dr. H. Blum
Erster Gutachter: Prof. Dr. K. F. Siburg
Zweiter Gutachter: Prof. Dr. L. Schwachhöfer
Weiterer Prüfer: Prof. Dr. R. Scharlau
Wiss. Mitarbeiter: Dr. A. Schnurr

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# 1 Introduction

## 1.1 Overview

The construction of quasi-morphisms on the group of Hamiltonian diffeomorphisms using spectral invariants coming from Floer homology and the construction of symplectic quasi-integrals on the space of continuous functions with compact support has become an increasingly interesting subject in symplectic topology in the last years since the existence of these functions possesses various applications to a variety of topics.

In the first part of this work we construct a family of functions on the group of Hamiltonian diffeomorphisms of a cotangent bundle of a closed connected manifold, where each function possesses properties including those of a partial quasi-morphism. The family is obtained in terms of Lagrangian spectral invariants from Lagrangian Floer homology and gives rise to a family of functionals on the space of compactly supported smooth functions, where each functional has properties analogous to those of a partial symplectic quasi-integral. On cotangent bundles of tori we prove that the partial quasi-morphisms are equivalent to Viterbo's symplectic homogenization, and with this observation we define the latter for more general cotangent bundles. Moreover, we deduce various applications from the existence and properties of the partial quasi-morphisms and the partial symplectic quasi-integrals such as to Hofer geometry, Aubry-Mather theory, Banyaga's fragmentation norm, and symplectic rigidity.

In the second part of this work we compare two particular symplectic quasi-integrals in two dimensions. On the one hand, we prove the existence of a genuine symplectic quasi-integral on $T^*S^1$ which is uniquely characterized by its additional properties. On the other hand, there exists a quasi-state on $S^2$ due to Entov and Polterovich which is uniquely characterized by its additional properties as well. We compare the two symplectic quasi-integrals on an open neighborhood of the zero section in $T^*S^1$ and give a necessary and sufficient condition for them to be equal. The comparison has to do with the general question of uniqueness of symplectic quasi-integrals. Moreover, it will turn out that the unique symplectic quasi-integral on $T^*S^1$ is closely related to Viterbo's symplectic homogenization in two dimensions. In fact, using this quasi-integral, we can prove the existence and uniqueness of an operator on $T^*S^1$ which has the properties of symplectic homogenization by an axiomatic approach. To prove the existence of the symplectic quasi-integral on $T^*S^1$, its uniqueness, as well as the comparison theorem for the symplectic quasi-integrals, we introduce the notion of quasi-integrals and topological measures on locally compact Hausdorff spaces and develop a representation theory for them which is a generalization of Aarnes' representation theory for compact Hausdorff spaces.

## 1.2 Partial quasi-morphisms and partial symplectic quasi-integrals

In symplectic geometry, the notions of (partial) quasi-morphisms and (partial) symplectic quasi-states for closed symplectic manifolds were introduced and first studied by Entov and Polterovich [EP1], [EP2]. In general, the notion of homogeneous quasi-morphisms is a group-theoretic one; a homogeneous quasi-morphism on a group is a homomorphism up to a bounded error. If a group does not admit a nontrivial homomorphism to the reals, a homogeneous quasi-morphism is the best approximation to a homomorphism one can try to construct. We refer to [Ca] for an introduction to the theory of homogeneous quasi-morphisms. The notion of symplectic quasi-states[1)] and quasi-integrals is related to the one of quasi-states on compact Hausdorff spaces which was adapted from the theory of quantum mechanics, and introduced and first studied by Aarnes [Aa1]. In symplectic geometry, a symplectic quasi-integral is a certain real-valued functional on the set of all continuous functions with various algebraic properties involving, in particular, the structure which is given by the Poisson bracket. Thus, the theory of symplectic quasi-integrals can be interpreted as a connection between symplectic geometry and functional analysis.

Entov and Polterovich constructed the first homogeneous quasi-morphisms and symplectic quasi-states on closed symplectic manifolds. In fact, they proved the existence of homogeneous quasi-morphisms on the universal cover of the group of Hamiltonian diffeomorphisms $\widetilde{\mathrm{Ham}}$ for certain closed symplectic manifolds which descends to the group of Hamiltonian diffeomorphisms Ham for some particular manifolds. The homogeneous quasi-morphisms on $\widetilde{\mathrm{Ham}}$ yield the existence of symplectic quasi-states on these manifolds [EP1], [EP2]. The construction was thereby motivated by the fact that the (universal cover of the) group of Hamiltonian diffeomorphisms of a closed symplectic manifold is perfect according to Banyaga [Ba], and therefore does not admit a nontrivial homomorphism to the reals. In contrast, when the symplectic manifold is open and the symplectic form is exact, the (universal cover of the) group of Hamiltonian diffeomorphisms admits a homomorphism to the reals, the Calabi homomorphism. Now, if one covers a closed symplectic manifold by sufficiently small open disks, one can consider the collection of Calabi homomorphisms on these disks and ask whether it is possible to extend this collection to a global homomorphism. This is, of course, not possible according to Banyaga's result, but for certain closed symplectic manifolds it is possible to extend the Calabi homomorphisms to a homogeneous quasi-morphism on the (universal cover of the) group of Hamiltonian diffeomorphisms.

In particular, Entov and Polterovich constructed a homogeneous quasi-morphism $\mu_{EP}\colon \widetilde{\mathrm{Ham}} \to \mathbb{R}$ on the universal cover of the group of Hamiltonian diffeomorphisms for any closed spherically monotone symplectic manifold whose even-dimensional quantum homology (which is a commutative algebra with the quantum product) satisfies the algebraic condition of semi-simplicity. The homogeneous quasi-morphisms coincide with the Calabi homomorphism on any open and displaceable subset and are therefore known under the name Calabi quasi-morphisms. They yield the existence of a symplectic quasi-state $\zeta_{EP}\colon C(M) \to \mathbb{R}$, referred to as Calabi quasi-state, on any such manifold. In particular, the construction applies to the complex projective space $\mathbb{C}P^n$; there exists a Calabi quasi-morphism on $\widetilde{\mathrm{Ham}}(\mathbb{C}P^n)$ which yields a symplectic quasi-state on $\mathbb{C}P^n$. Moreover, the former descends to a homogeneous quasi-morphism on $\mathrm{Ham}(\mathbb{C}P^n)$. On more general closed symplectic manifolds (on any strongly semi-positive closed connected symplectic manifold) Entov and Polterovich

1) A symplectic quasi-state is a normalized symplectic quasi-integral on a compact symplectic manifold.

proved the existence of a partial quasi-morphism on $\widetilde{\mathrm{Ham}}$ which yields the existence of a partial symplectic quasi-state. Thereby, both notions were introduced by Entov and Polterovich and are weaker than the ones of genuine homogeneous quasi-morphisms and genuine symplectic quasi-states.

The (partial) quasi-morphisms are constructed by homogenizing a certain spectral invariant coming from Hamiltonian Floer homology which is, in this setting, isomorphic to the quantum homology of the symplectic manifold. It depends on the algebraic structure of the even-dimensional quantum homology whether one can extract genuine homogeneous or partial quasi-morphisms and genuine or partial symplectic quasi-states.

The existence and properties of the (partial) quasi-morphisms and (partial) symplectic quasi-states coming from Hamiltonian spectral invariants due to Entov and Polterovich on closed symplectic manifolds possess various applications, see [EP1], [EP2], [EP3], [EPZ], [BEP], [EP4], [EPP], [EPZ]. In particular, the quasi-morphisms lead to applications to Banyaga's fragmentation norm, the commutator norm, Poisson brackets, and restrictions on partitions of unity. The symplectic quasi-states yield applications to symplectic rigidity; the latter is a phenomenon in symplectic topology meaning that certain subsets of symplectic manifolds cannot be completely displaced from itself by a Hamiltonian diffeomorphism while they can be displaced by a genuine diffeomorphism.

Starting with Entov's and Polterovich's works, the construction of (partial) quasi-morphisms and (partial) symplectic quasi-integrals using Floer theory on symplectic manifolds has become an increasingly interesting subject. Several authors generalized and adapted the construction of quasi-morphisms on certain symplectic manifolds using Hamiltonian spectral invariants and deduced several applications. Usher, for instance, generalized Entov's and Polterovich's construction to more general closed symplectic manifolds under certain conditions on the homology using deformed spectral invariants [Us]. Lanzat generalized it to certain non-closed symplectic manifolds [La]. For certain convex strongly semi-positive compact symplectic manifolds, as well as for certain open convex symplectic manifolds, in particular, for cotangent bundles, Lanzat constructs partial quasi-morphisms on $\widetilde{\mathrm{Ham}}$ and partial symplectic quasi-integrals using a version of Hamiltonian Floer homology for compactly supported Hamiltonians and quantum homology. Under certain conditions on these manifolds he proves that the partial quasi-morphisms on $\widetilde{\mathrm{Ham}}$ are not homogeneous quasi-morphisms and that they descend to Ham. In particular, the above applies to cotangent bundles $T^*N$ of closed connected manifolds under the assumption that $N$ does admit a nowhere vanishing closed 1-form; in this case Lanzat constructs a partial quasi-morphism on $\widetilde{\mathrm{Ham}}$ which descends to Ham and gives rise to a partial symplectic quasi-integral.

In the context of symplectic geometry, quasi-morphisms and symplectic quasi-integrals can be viewed as an algebraic way of encoding certain information of Hamiltonian diffeomorphisms contained in spectral invariants respectively contained in Floer homology. In fact, spectral invariants coming from Floer homology are the only known way to construct quasi-morphisms on the (universal cover of the) group of Hamiltonian diffeomorphisms and symplectic quasi-integrals on symplectic manifolds with dimension higher than two.

In general, there are certain types of spectral invariants on symplectic manifolds which are interesting objects in themselves; they can be interpreted as "homologically visible" critical values of a certain functional associated to homology classes. More precisely, there are spectral invariants coming from filtered Floer homology and spectral invariants coming from generating function theory.

For instance, Schwarz defines Hamiltonian spectral invariants associated to homology classes for certain types of closed symplectic manifolds (for aspherically symplectic manifolds) using filtered Hamiltonian Floer homology [Sch]. The latter can be viewed as infinite-dimensional Morse theory of the classical action functional corresponding to a given Hamiltonian on a certain path space, and is, in this setting, isomorphic to the standard homology of the manifold. In the Floer theoretic context, Hamiltonian spectral invariants are critical values of the action functional. Using these spectral invariants Schwarz defines, for instance, a bi-invariant metric on the group of Hamiltonian diffeomorphisms and deduces several applications such as to Hofer geometry and Hofer-Zehnder capacity. On more general closed symplectic manifolds, Hamiltonian Floer theory involves quantum effects and uses a Novikov ring; it is isomorphic to the quantum homology of the manifold. In this context Oh defines and studies Hamiltonian spectral invariants which are associated to quantum homology classes [Oh3], [Oh4]; these are the ones which were used by Entov and Polterovich to construct the quasi-morphisms. Moreover, there is a version of Hamiltonian Floer homology for certain non-closed symplectic manifolds using a particular class of compactly supported Hamiltonians. Frauenfelder and Schlenk constructed Floer homology for weakly exact convex symplectic manifolds and deduced Hamiltonian spectral invariants which are associated to homology classes [FS]; in particular, their construction applies to cotangent bundles. Lanzat generalized the latter construction to strongly semi-positive compact convex symplectic manifolds; he obtains Hamiltonian spectral invariants associated to quantum homology classes which he uses to construct his quasi-morphisms [La].

In case of open symplectic manifolds given by cotangent bundles Viterbo defines Lagrangian spectral invariants using the theory of generating functions for Lagrangian submanifolds [Vi1]. He associates a critical value of a generating function of a Lagrangian submanifold which is Hamiltonian isotopic to the zero section to each homology class. On $\mathbb{R}^{2n}$ and $T^*\mathbb{T}^n$ Viterbo uses his spectral invariants to define a norm on the group of Hamiltonian diffeomorphisms and a norm on the set of Lagrangian submanifolds which are Hamiltonian isotopic to the zero section. Motivated by Weinstein's observation that the classical action functional corresponding to a given Hamiltonian $H$ is a generating function of the Lagrangian submanifold which is obtained by the image of the zero section under the time-1 map of $H$, Oh developed a Floer theoretic approach to Viterbo's construction [Oh1], [Oh2]. He constructs Lagrangian Floer homology for cotangent bundles which is isomorphic to the standard homology of the manifold and defines Lagrangian spectral invariants by replacing the generating function by the classical action functional. Milinković and Oh related Viterbo's finite-dimensional approach and Oh's infinite-dimensional approach to the construction of Lagrangian spectral invariants; they proved that both Lagrangian spectral invariants coincide under some natural assumptions [MO1], [MO2].

However, spectral invariants do not automatically give rise to the construction of quasi-morphisms and symplectic quasi-integrals; they need to possess additional algebraic properties in order to do so.

In this work we use several versions of spectral invariants. The family of functions on the group of Hamiltonian diffeomorphisms of cotangent bundles of closed connected manifolds is defined in terms of Lagrangian spectral invariants coming from Lagrangian Floer homology which were introduced by Oh [Oh1], [Oh2]. To be able to construct the functions we prove some additional properties of these Lagrangian spectral invariants. Moreover, we compare the Lagrangian spectral invariants with the Hamiltonian spectral invariants on cotangent bundles defined by Frauenfelder and Schlenk [FS]. We obtain an inequality between the Lagrangian and Hamiltonian spectral invariants which allows to prove that the functions we obtain are quasi-morphisms and which yields a vanishing property for the partial quasi-morphisms. To prove the equivalence between Viterbo's symplectic homogenization and the partial quasi-morphisms on cotangent bundles of tori, we make use of the equality between the Lagrangian spectral invariants coming from Lagrangian Floer homology due to Oh and the Lagrangian spectral invariants coming from generating function theory due to Viterbo [Vi1] established by Milinković and Oh [MO1], [MO2]. Moreover, we give a very short overview about the construction of Hamiltonian spectral invariants associated to quantum homology classes of certain closed symplectic manifolds according to Oh [Oh3], [Oh4] in order to introduce the Calabi quasi-morphism and the Calabi quasi-state due to Entov and Polterovich.

### 1.2.1 Construction of partial quasi-morphisms and partial symplectic quasi-integrals on $T^*N$

Let $N$ be a closed connected $n$-dimensional manifold and consider its cotangent bundle $(T^*N, \omega = d\lambda)$ as a symplectic manifold. Let $C(T^*N)$ be the space of continuous functions on $T^*N$ and $C_c(T^*N) \subset C(T^*N)$ the subspace of all continuous functions with compact support. On $C_c(T^*N)$ we use the $C^0$-norm given by $\|F\|_{C^0} = \sup_{T^*N} |F(x)|$. Denote by $C_c^\infty(T^*N)$ the space of all compactly supported smooth functions. Two such functions $F, G \in C_c^\infty(T^*N)$ are said to Poisson commute if their Poisson bracket vanishes, i.e. $\{F, G\} = 0$. Denote by $\mathrm{Ham}(T^*N)$ the group of compactly supported Hamiltonian diffeomorphisms on $T^*N$ and by $\rho$ Hofer's metric on $\mathrm{Ham}(T^*N)$. A subset $S \subset T^*N$ is called displaceable if there is a Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(T^*N)$ such that $\phi(S) \cap \overline{S} = \varnothing$, it is said to be dominated by an open subset $U \subset T^*N$ if there is a Hamiltonian diffeomorphism $\varphi \in \mathrm{Ham}(T^*N)$ such that $S \subset \varphi(U)$. We denote by $\|\phi\|_{\mathcal{U}}$ Banyaga's fragmentation norm for a Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(T^*N)$ relative to a family $\mathcal{U}$ of open subsets. We refer to Subsection 1.5 for details concerning the above definitions and preliminaries of symplectic geometry.

We are interested in (partial) quasi-morphisms on the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(T^*N)$ and formulate:

**Definition 1.1.** A quasi-morphism on $\mathrm{Ham}(T^*N)$ is a function $\mu\colon \mathrm{Ham}(T^*N) \to \mathbb{R}$ for which there is a constant $D$, called the defect of $\mu$, such that

$$|\mu(\phi\psi) - \mu(\phi) - \mu(\psi)| \leq D$$

for all $\phi, \psi \in \mathrm{Ham}(T^*N)$. A quasi-morphism $\mu$ is called homogeneous if $\mu(\phi^k) = k\mu(\phi)$ for all $\phi \in \mathrm{Ham}(T^*N)$ and $k \in \mathbb{Z}$.

**Definition 1.2.** A partial quasi-morphism on $\mathrm{Ham}(T^*N)$ is a function $\mu$: $\mathrm{Ham}(T^*N) \to \mathbb{R}$ such that the following properties hold:

(i) *Controlled quasi-additivity*[2)]: For any open and displaceable subset $U \subset T^*N$ there is a constant $R$ which only depends on $U$ such that for all $\phi, \psi \in \mathrm{Ham}(T^*N)$, where $\psi$ is generated by a Hamiltonian whose support is dominated by $U$, we have

$$|\mu(\phi\psi) - \mu(\phi) - \mu(\psi)| \leq R\,;$$

(ii) *Semi-homogeneity*: $\mu(\phi^k) = k\mu(\phi)$ for any $k \in \mathbb{Z}_{\geq 0}$.

We construct a family of functions

$$\mu_a\colon \mathrm{Ham}(T^*N) \to \mathbb{R}$$

parameterized by the first real cohomology $H^1(N;\mathbb{R})$, where any function $\mu_a$ with $a \in H^1(N;\mathbb{R})$ is a partial quasi-morphism and has various additional properties which are listed in Theorem 3.5 in Section 3. For instance, they include:

- for any collection $\mathcal{U}$ of open and displaceable subsets with finite spectral displacement energy[3)] $e(\mathcal{U}) < \infty$ we have

$$|\mu_0(\phi\psi) - \mu_0(\psi)| \leq e(\mathcal{U})\, \|\phi\|_{\mathcal{U}}\ ;$$

- $\mu_a$ is invariant under conjugation in $\mathrm{Ham}(T^*N)$;
- $\mu_a$ is Lipschitz continuous with respect to Hofer's metric;
- $\mu_a$ vanishes on Hamiltonian diffeomorphisms which are generated by Hamiltonians with displaceable support;
- $\mu_a(\phi) = c$ $(\geq c, \leq c)$ if $\phi$ is generated by a Hamiltonian whose restriction to the graph of a closed 1-form in the class $a$ is $= c$ $(\geq c, \leq c)$.

The partial quasi-morphisms $\mu_a$ are thereby obtained by homogenizing a certain Lagrangian spectral invariant coming from Lagrangian Floer homology. The Lagrangian spectral invariants were thereby defined by Oh. In [Oh1], [Oh2] Oh proves that one can extract spectral invariants $\ell(A, H)$ for any Hamiltonian $H$ and any $A \in H_*(N;\mathbb{Z}_2)$ via Lagrangian Floer homology. In this work we prove that Oh's Lagrangian spectral invariants satisfy some additional properties which allow to define the functions $\mu_a$ and which guarantee that the $\mu_a$ are partial quasi-morphisms with the properties listed above. For instance, we prove that the Lagrangian spectral invariants satisfy

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2) Our definition of controlled quasi-additivity is actually weaker than the original one introduced by Entov and Polterovich but it suffices for our purposes; in particular, it suffices to extract partial symplectic quasi-integrals from partial quasi-morphisms.

3) We refer to Remark 2.11 in Section 2 for the precise definition of the spectral displacement energy $e(\mathcal{U})$.

· a version of Poincaré duality for the top and the point spectral invariant;

· a sharp triangle inequality;

· the independence of isotopy.

In addition, we deduce an inequality between the Lagrangian spectral invariants and the Hamiltonian spectral invariants on cotangent bundles which were introduced and studied by Frauenfelder and Schlenk [FS]. It is needed to prove that the functions $\mu_a$ satisfy the controlled quasi-additivity property and the vanishing property. The precise statements and the precise properties of the function

$$\ell(A,\cdot)\colon\ \mathrm{Ham}(T^*N)\to\mathbb{R}\,,$$

where $A\in H_*(N;\mathbb{Z}_2)$, are summarized in Theorem 2.14 in Section 2. With the Lagrangian spectral invariants we define the partial quasi-morphism $\mu_0\colon\ \mathrm{Ham}(T^*N)\to\mathbb{R}$ by

$$\mu_0(\phi)=\lim_{k\to\infty}\frac{\ell_+(\phi^k)}{k}\,,$$

where $\ell_+(\cdot)=\ell([N],\cdot)$ with $[N]\in H_n(N;\mathbb{Z}_2)$ denotes the top Lagrangian spectral invariant. For any $a\in H^1(N;\mathbb{R})$ the partial quasi-morphisms $\mu_a$ are then obtained from $\mu_0$ via

$$\mu_a(\phi)=\mu_0(T_{-\alpha}\phi T_\alpha)\,,$$

where $T_\alpha\colon T^*N\to T^*N$ is the symplectomorphism given by $T_\alpha(q,p)=T(q,p+\alpha(q))$ for any $\alpha\in a$.

The family of partial quasi-morphisms $\mu_a$ gives rise to a family of functionals

$$\zeta_a\colon C_c^\infty(T^*N)\to\mathbb{R}$$

via $\zeta_a(F)=\mu_a(\phi_F)$, where $\phi_F$ denotes the time-1 map of $F$. Each functional $\zeta_a$ has properties analogous to those of a partial symplectic quasi-integral. Thereby, we formulate:

**Definition 1.3.** A functional $\zeta\colon C_c(T^*N)\to\mathbb{R}$ is called a symplectic quasi-integral if it satisfies:

(i) *Monotonicity*: $\zeta(F)\le\zeta(G)$ for all $F,G\in C_c(T^*N)$ with $F\le G$;

(ii) *Lipschitz continuity*: For every compact subset $K\subset T^*N$ there is a number $N_K\ge 0$ such that $|\zeta(F)-\zeta(G)|\le N_K\,\|F-G\|_{C^0}$ for all $F,G\in C_c(T^*N)$ with support contained in $K$;

(iii) *Strong quasi-additivity*: $\zeta$ is linear on Poisson commutative subspaces of $C_c^\infty(T^*N)$.

**Definition 1.4.** Let $\zeta\colon C_c(T^*N)\to\mathbb{R}$ be monotone and Lipschitz continuous as above. $\zeta$ is called a partial symplectic quasi-integral if it satisfies:

(i) *Partial quasi-additivity*: $\zeta(F+G)=\zeta(F)$ for all $F,G\in C_c^\infty(T^*N)$ such that the support of $G$ is displaceable and $\{F,G\}=0$;

(ii) *Semi-homogeneity*: $\zeta(\lambda F)=\lambda\zeta(F)$ for all $\lambda\in\mathbb{R}_{\ge 0}$.

If the underlying symplectic manifold is closed and the (partial) symplectic quasi-integral $\zeta$ is normalized such that $\zeta(1) = 1$, it is called a (partial) symplectic quasi-state.

The partial symplectic quasi-integrals $\zeta_a$ have some additional properties, which are listed in Theorem 3.10 in Section 3, which follow directly from the ones of the partial quasi-morphisms $\mu_a$; they include:

· $\zeta_a$ is invariant under the natural action of $\mathrm{Ham}(T^*N)$;

· $\zeta_a$ vanishes on Hamiltonians with displaceable support;

· $\zeta_a(F) = c$ $(\geq c, \leq c)$ if $F = c$ $(\geq c, \leq c)$ when restricted to the graph of a closed 1-form in the class $a$.

The existence and the properties of both, the family of partial quasi-morphisms $\mu_a$ and the family of partial symplectic quasi-integrals $\zeta_a$ on $T^*N$, lead to various applications; an overview is given in the following subsections.

### 1.2.2 Symplectic homogenization

Recently, Viterbo introduced the notion of symplectic homogenization for Hamiltonan diffeomorphisms on cotangent bundles of tori [Vi4]. Motivated by the classical homogenization his aim was to define a symplectic notion of homogenization for Hamiltonian diffeomorphisms on $T^*\mathbb{T}^n$. In fact, associated to a Hamiltonian $H \in C_c^\infty([0,1] \times T^*\mathbb{T}^n)$ he considers the "rescaled" Hamiltonian $H_k(t,q,p) = H(kt,kq,p)$ and asks whether $H_k$ converges to some Hamiltonian $\widehat{H}$ which only depends on the fiber variable $p$. The convergence is thereby understood as a convergence of the time-1 maps of $H_k$ in the sense that the limit of the time-1 maps of $H_k$ with respect to Viterbo's metric is, in a certain precise sense, generated by the $q$-independent Hamiltonian $\widehat{H}$. Viterbo's construction is based on the theory of Lagrangian spectral invariants coming from generating functions [Vi1], and he defines symplectic homogenization as an operator

$$\mathcal{H}\colon C_c^\infty([0,1] \times T^*\mathbb{T}^n) \to C_c(\mathbb{R}^n)$$

which indeed sends a Hamiltonian $H$ to a continuous functions $\widehat{H}$ which is related to the rescaled Hamiltonian $H_k$ and only depends on the fiber variable. In addition to its existence Viterbo claims in [Vi4] that the operator $\mathcal{H}$ has various properties which, for instance, include a notion of convergence of the time-1 maps of $H_k$ to the time-1 map of $\widehat{H}$ and Lipschitz continuity in Viterbo's metric. Moreover, he claims that the operator $\mathcal{H}$ gives rise to a notion of a symplectic homogenization operator for time-independent Hamiltonians which has various properties including, for instance, monotonicity and strong quasi-additivity.

However, in [Vi4] Viterbo uses his (still unproven) conjecture concerning a certain bound on Lagrangian spectral invariants on $T^*\mathbb{T}^n$ from time to time. Viterbo claims that the quantity $\ell_+(\phi)+\ell_+(\phi^{-1})$, where $\ell_+$ denotes the top Lagrangian spectral invariant coming from Lagrangian Floer homology, is bounded by a constant depending only on the Riemannian metric on $\mathbb{T}^n$ for any Hamiltonian diffeomorphism $\phi$ with support contained in the unit disk cotangent bundle [Vi3]. A careful consideration of Viterbo's constructions and proofs in [Vi4] shows that the Viterbo bound is not needed to give the definition of the symplectic homogenization operator but to prove some of its properties; we will explain this in more detail where appropriate.

In this work we prove that the family of partial quasi-morphisms $\mu_a$ is equivalent to Viterbo's symplectic homogenization. More precisely, we prove that, if we identify $H^1(\mathbb{T}^n;\mathbb{R}) = \mathbb{R}^n$, we have

$$\mathcal{H}(H)(a) = \mu_a(\phi_H)$$

for any $a \in \mathbb{R}^n$, see Theorem 4.13 in Section 4. The equivalence is thereby based on the assertion due to Milinković and Oh that the Lagrangian spectral invariants coming from Lagrangian Floer homology introduced by Oh, which we use to define the functions $\mu_a$, and the Lagrangian spectral invariants coming from generating functions introduced by Viterbo, which Viterbo uses to define $\mathcal{H}$, coincide under some natural assumptions.

The above equivalence on $T^*\mathbb{T}^n$ leads to a definition of symplectic homogenization for more general cotangent bundles $T^*N$, where $N$ is a closed connected manifold, as an operator

$$\mathcal{H}\colon C_c^\infty([0,1] \times T^*N) \to C_c(H^1(N;\mathbb{R}))\,.$$

The various properties of the symplectic homogenization operator can then be extracted from the properties of the partial quasi-morphisms $\mu_a$. Moreover, we can define a symplectic homogenization operator for time-independent Hamiltonians using the partial symplectic quasi-integrals $\zeta_a$, and the properties of the latter yield the ones of the symplectic homogenization operator. The precise statements are given in Section 4.

It will turn out that our observations concerning the Viterbo bound are consistent with the equivalence between symplectic homogenization and the partial quasi-morphisms in the following sense: The equivalence can be interpreted as a way to define symplectic homogenization, and the properties of symplectic homogenization, which are just the extracted ones of the partial quasi-morphisms, are precisely the properties in whose proofs Viterbo does not use his bound.

Moreover, in Section 6 we give an axiomatic proof of the existence and uniqueness of the above symplectic homogenization operator in two dimensions; we refer to the sequel for a more detailed description of the two-dimensional case.

In the proof of the equivalence between symplectic homogenization and the partial quasi-morphisms we use the equality between Oh's Lagrangian spectral invariants coming from Floer homology and Viterbo's spectral invariants coming from generating functions, and thus one could ask whether it is possible to define the partial quasi-morphisms $\mu_a$ in terms of Viterbo's Lagrangian spectral invariants. In this context one should note that we prove various properties of the functions $\mu_a$ which explicitly follow from the Floer theoretic approach to Lagrangian spectral invariants. For instance, the notions of displaceability, spectral displacement energy and Poisson commutativity, which are essentially contained in the definition and properties of the partial quasi-morphisms, are more naturally contained in a Floer theoretic approach rather than in an approach using generating functions.

### 1.2.3 Applications

In addition to the equivalence to symplectic homogenization there are several other applications (some of them appeared in the existing literature and we indicate the connections where appropriate) which stem from the existence and properties of the families $\mu_a$ and $\zeta_a$ such as to Banyaga's fragmentation norm, Hofer geometry, and symplectic rigidity:

- we get lower bounds on Banyaga's fragmentation norm for Hamiltonian diffeomorphisms relative to displaceable subsets, see Proposition 5.1;
- we are able to construct either an isometric embedding of $\mathbb{R}$ into $\mathrm{Ham}(T^*N)$ or an isometric embedding of $(C_c((0,1)), \mathrm{osc})$ into $\mathrm{Ham}(T^*N)$, depending on whether $N$ admits a nowhere vanishing closed 1-form or not, see Proposition 5.9;
- we can deduce inequalities like $\mathrm{osc}_{a\in H^1(N;\mathbb{R})} \le \rho(\phi)$, where $\rho(\phi)$ denotes the Hofer norm of $\phi$, see Proposition 5.8;
- following the ideas of Entov and Polterovich we can prove rigidity results using the partial symplectic quasi-integral $\zeta_0$ and, in particular, extract examples of non-displaceable subsets in $T^*N$, see Proposition 5.18.

Moreover, there is a relation between Aubry-Mather theory and the partial quasi-morphisms $\mu_a$. Aubry-Mather theory originally dealt with action minimizing orbits in Hamiltonian systems in two dimensions, and was generalized by Mather to a theory of action minimizing invariant measures, instead of orbits, for certain convex Hamiltonian systems on cotangent bundles $T^*N$ of higher dimension [Ma]. In the latter context Mather associates a function $\beta_H\colon H_1(N;\mathbb{R}) \to \mathbb{R}$, called beta function, to each Tonelli Hamiltonian, where its value can be interpreted to represent the minimal average Lagrangian action needed to carry out motions with a given rotation vector. The associated conjugate function $\alpha_H\colon H^1(N;\mathbb{R}) \to \mathbb{R}$ is known as Mather's alpha function.

We can define the functions $\mu_a$ for Hamiltonians with complete flow, and if such a Hamiltonian is Tonelli, we prove that Mather's alpha function is equivalent to the partial quasi-morphisms, that is, we prove

$$\alpha_H(a) = \mu_a(\phi_H)$$

for any $a \in H^1(N;\mathbb{R})$, see Proposition 5.3. This result was first established by Viterbo for $T^*\mathbb{T}^n$ in the language of symplectic homogenization [Vi4] and provides a relation between the dynamical view of Aubry-Mather theory and symplectic topology. Moreover, it gives a new proof of the invariance of the alpha function under Hamiltonian diffeomorphisms.

In addition, we are able to extract a connection between Aubry-Mather theory and Hofer geometry. As done in [Sib1] we can relate the minimum of the alpha function to the Hofer norm of the Hamiltonian in question, see Proposition 5.10. Similar to the above this result can be interpreted as to relate the dynamical and the geometrical approach to the study of Hamiltonian systems.

## 1.3 Comparison of symplectic quasi-integrals in two dimensions

The existence of the partial symplectic quasi-integrals $\zeta_a$ on cotangent bundles leads to a particular case of the general question concerning the uniqueness of quasi-integrals. Certain cotangent disk bundles, such as those of tori $\mathbb{T}^n$, admit symplectic embeddings into closed symplectic manifolds ($\mathbb{C}P^n$, for instance) which themselves admit (partial) symplectic quasi-states. These functionals can be pulled back to yield (partial) symplectic quasi-integrals on the disk bundle and one can ask whether this pull-back coincides with the restriction of the partial symplectic quasi-integral $\zeta_a$ on $T^*\mathbb{T}^n$.

In this work we present a result in this direction in two dimensions; we consider the case where a cotangent disk bundle of $T^*S^1$ is embedded into $S^2$. On $S^2$ there exists a symplectic quasi-state, the Calabi quasi-state $\zeta_{EP}$, due to Entov and Polterovich which is uniquely characterized by its additional properties [EP1], [EP2]. On $T^*S^1$ we can prove the existence of a genuine symplectic quasi-integral $\eta_0$ which is uniquely characterized by its additional properties as well, see Proposition 6.19. More precisely, we prove the existence of a unique symplectic quasi-integral

$$\eta_0\colon C_c(T^*S^1) \to \mathbb{R}$$

which has the following additional properties:

- it is invariant under Hamiltonian diffeomorphisms;
- it has the Lagrangian property, i.e. $\eta_0(F) = c$ if $F \in C_c(T^*S^1)$ is such that $F|_{S^1\times\{0\}} = c \in \mathbb{R}$.

We compare the two symplectic quasi-integrals $\eta_0$ and $\zeta_{EP}$, which both are universal in some sense, on an open neighborhood of the zero section in $T^*S^1$. More precisely, for $r \in (0, \frac{1}{2}]$ we consider a symplectic embedding

$$j_r\colon S^1 \times (-r, r) \to S^2$$

such that $j_r(S^1 \times \{0\})$ is the equator, where the symplectic forms are such that the area of $S^2$ is 1 and the area of $S^1 \times (-r, r) \subset T^*S^1$ is $2r$. By pulling $\zeta_{EP}$ back to $C_c(S^1 \times (-r, r))$ via this embedding

$$\zeta_r = j_r^*\zeta_{EP}\,,$$

we can compare it with the restriction of $\eta_0$ to $C_c(S^1 \times (-r, r))$. In fact, we provide a necessary and sufficient condition for the symplectic quasi-integrals to be equal; in Theorem 6.26 we prove

$$\eta_0|_{C_c(S^1\times(-r,r))} = \zeta_r \;\Leftrightarrow\; r \in (0, \frac{1}{4}]\,.$$

For the proof of the existence of the symplectic quasi-integral $\eta_0$ and the proof of the comparison of the symplectic quasi-integrals we introduce the notion of quasi-integrals and topological measures on locally compact Hausdorff spaces and develop a representation theory for quasi-integrals in terms of topological measures, see Theorem 6.6. It is a generalization of the representation theory for quasi-states and topological measures on compact Hausdorff spaces due to Aarnes [Aa1]. Since every symplectic quasi-integral is a quasi-integral, we can make use of this representation theory and prove the main statements in terms of topological measures. Moreover, we introduce a reduction argument for topological measures and prove a statement about the symplecticity of quasi-integrals on surfaces without boundary; both are needed in the proofs of the main statements.

In addition, it will turn out that the quasi-integral $\eta_0$ on $T^*S^1$ is closely related to Viterbo's symplectic homogenization. In fact, using the existence and the properties of $\eta_0$ we prove that there exists a unique operator

$$\mathcal{H}\colon C_c(T^*S^1) \to C_c(\mathbb{R})$$

which has the properties of symplectic homogenization, where partial quasi-additivity is replaced by strong quasi-additivity, by an axiomatic approach, see Theorem 6.20. This operator allows to define symplectic quasi-integrals $\eta_\sigma$ on $T^*S^1$ by integration against Radon measures $\sigma$. More precisely, we prove that the functionals given by

$$\eta_\sigma(F) = \int_{\mathbb{R}} \mathcal{H}(F)\, d\sigma$$

are symplectic quasi-integrals for any Radon measure $\sigma$, see Proposition 6.22. If we take the Radon measure $\sigma$ to be the Dirac measure centered at zero, we obtain a quasi-integral $\eta_0$ via $\eta_0(F) = \mathcal{H}(F)(0)$ which turns out to be the unique symplectic quasi-integral on $T^*S^1$.

## 1.4 Organization

The rest of this section is devoted to necessary preliminaries and definitions of symplectic geometry.

In Section 2 we review Lagrangian Floer homology, introduce the Lagrangian spectral invariants and prove their additional properties. A summary is given in Theorem 2.14 in Subsection 2.4.

In Section 3 we define the partial quasi-morphisms $\mu_a\colon \mathrm{Ham}(T^*N) \to \mathbb{R}$ and the partial symplectic quasi-integrals $\zeta_a\colon C_c^\infty(T^*N) \to \mathbb{R}$ and prove their properties, see Theorem 3.5 and Theorem 3.10.

In Section 4 we give an overview about Viterbo's symplectic homogenization and state the equivalence between symplectic homogenization and the partial quasi-morphisms on $T^*\mathbb{T}^n$ in Theorem 4.13. Moreover, we give a general definition of a symplectic homogenization operator and list its properties.

In Section 5 we formulate and prove the various applications following from the existence and properties of the partial quasi-morphisms $\mu_a$ and the partial symplectic quasi-integrals $\zeta_a$.

In Section 6 we develop a representation theory for quasi-integrals and topological measures on locally compact Hausdorff spaces, Theorem 6.6. Using the representation theory we prove the existence of a symplectic quasi-integral $\eta_0$ on $T^*S^1$ which is uniquely characterized by its additional properties, see Proposition 6.19, and relate it to Viterbo's symplectic homogenization in the sense that we prove the existence and uniqueness of an operator $\mathcal{H}\colon C_c(T^*S^1) \to C_c(\mathbb{R})$ which has the properties of symplectic homogenization, see Theorem 6.20. In Subsection 6.5 we give an overview about the definition and construction of the Calabi quasi-state on $S^2$ due to Entov and Polterovich. In Subsection 6.6 we compare the two symplectic quasi-integrals.

Parts of this work are based on the article [MZ] and results obtained in [MVZ].

## 1.5 Preliminaries of symplectic geometry

In this subsection we give a short overview about the basic concepts in symplectic geometry and introduce the relevant notation for the sequel of this work. We refer to [McS1] for a detailed introduction to symplectic geometry.

### Symplectic manifolds and Hamiltonian vector fields

A symplectic manifold is a pair $(M, \omega)$, where $M$ is a smooth manifold (throughout we assume that $M$ has no boundary) and $\omega$ is a non-degenerate closed 2-form on $M$, called the symplectic form. A symplectic manifold $(M, \omega)$ is necessarily of even dimension $2n$, and the form $\omega^n = \omega \wedge \cdots \wedge \omega$ defines a volume form on $M$. The standard example of a symplectic manifold is the $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$ with the so-called canonical symplectic form $\omega_0 = dp \wedge dq = \sum_{i=1}^n dp_i \wedge dq_i$. According to Darboux's theorem, any symplectic manifold $(M, \omega)$ looks locally like $(\mathbb{R}^{2n}, \omega_0)$, that is, there are always local coordinates in which the symplectic form is given by the canonical symplectic form $\omega_0$.

A diffeomorphism $\varphi\colon M \to M'$ on symplectic manifolds $(M, \omega)$ and $(M', \omega')$ is said to be symplectic (a symplectomorphism) if it preserves the symplectic forms in the sense that $\varphi^*\omega' = \omega$. The set $\mathrm{Symp}(M, \omega)$ of all symplectomorphisms with compact support on $(M, \omega)$ is a group with respect to composition.

A submanifold $L \subset M$ is said to be Lagrangian if its dimension is $n$ and $\omega|_L = 0$. An embedding $i\colon L^n \to M^{2n}$ is called Lagrangian if $i^*\omega = 0$. A Lagrangian submanifold in $(\mathbb{R}^{2n}, \omega_0)$ is, for example, given by $\{(q,p) \in \mathbb{R}^{2n} \,|\, p = 0\}$. Moreover, if $(M, \omega)$ is a symplectic manifold, then so is the product $(M \times \overline{M}, \omega \oplus -\omega)$, where the overline indicates that the sign of the symplectic form is the negative of the usual one. The graph $\Gamma_\varphi \subset M \times \overline{M}$ of a symplectomorphism $\varphi\colon M \to M$ is a Lagrangian submanifold in $(M \times \overline{M}, \omega \oplus -\omega)$. In particular, the diagonal $\Delta$ of $M \times \overline{M}$ is a Lagrangian submanifold.

If $(M, \omega)$ is a symplectic manifold, the symplectic form $\omega$ establishes an isomorphism between vector fields $X$ and 1-forms on $M$ given by $X \mapsto \iota_X\omega = \omega(X, \cdot)$. A vector field $X$ is called symplectic if the corresponding 1-form is closed, it is called Hamiltonian if the corresponding 1-form is exact. If $H\colon M \to \mathbb{R}$ is a smooth function on $M$, called a Hamiltonian, such that $\iota_X\omega = -dH$, the unique vector field $X = X_H$ is called the Hamiltonian vector field of $H$.

A time-dependent Hamiltonian is a smooth function $H\colon [0,1] \times M \to \mathbb{R}$. By $H_t$ we denote the function $H(t, \cdot) \in C^\infty(M)$. To a time-dependent compactly supported Hamiltonian $H \in C_c^\infty([0,1] \times M)$, that is, a Hamiltonian such that $H_t$ has compact support for any $t$, one can associate a time-dependent Hamiltonian vector field $X_H$

which is given by the equation

$$\iota_{X_{H_t}}\omega = -dH_t\,.$$

It gives rise to a flow on $M$, which is called the Hamiltonian flow generated by $H$, via

$$\frac{d}{dt}\phi_H^t = X_{H_t}\circ\phi_H^t, \quad \phi_H^0 = \mathrm{id}_M\ .$$

The time-t map of the flow of $H$ is denoted by $\phi_H^t$, while $\phi_H$ denotes its time-1 map; these maps are symplectic. Moreover, we define $\phi_H^t := \phi_H^{t-k}\phi_H^k$ for $t\in[k,k+1]$, where $k\in\mathbb{Z}$; here $\phi_H^k := (\phi_H)^k$. Whenever $H$ is defined for all $t\in\mathbb{R}$ and is 1-periodic in $t$, the time-$t$ flow of $H$ equals $\phi_H^t$.

The Poisson bracket of two Hamiltonians $F,G\in C_c^\infty(M)$ is given by $\{F,G\} = \omega(X_F,X_G) = -dF(X_G) = dG(X_F)\in C^\infty(M)$. If $\{F,G\}=0$, we say that $F$ and $G$ Poisson commute, and in this case the Hamiltonian flows $\phi_F^t$ and $\phi_G^t$ commute. The space $C_c^\infty(M)$ of compactly supported smooth Hamiltonians on $M$ forms a Lie algebra with respect to the Poisson bracket.

### The group of Hamiltonian diffeomorphisms and Hofer's metric

To define the group of Hamiltonian diffeomorphisms of a symplectic manifold $(M,\omega)$ we consider a special class $\mathcal{H}(M)$ of Hamiltonians: If the symplectic manifold $(M,\omega)$ is open, $\mathcal{H}(M)$ is defined to be the set of all compactly supported Hamiltonians $H\colon [0,1]\times M\to\mathbb{R}$ such that there exists a compact subset of $M$ which contains the supports of the functions $H_t$, $t\in[0,1]$, simultaneously. If $(M,\omega)$ is closed, the set $\mathcal{H}(M)$ consists of Hamiltonians $H\colon[0,1]\times M\to\mathbb{R}$ which are normalized in the sense that $\int_M H_t\,\omega^n = 0$ for all $t$. Both assumptions on the Hamiltonians require that the map which sends a Hamiltonian in $\mathcal{H}(M)$ to its Hamiltonian vector field is injective, that is, a Hamiltonian vector field determines the corresponding Hamiltonian uniquely. The set of Hamiltonian diffeomorphisms $\mathrm{Ham}(M)=\mathrm{Ham}(M,\omega)$ of $(M,\omega)$ is then defined to be the set of all diffeomorphisms which are generated by $H\in\mathcal{H}(M)$, i.e.

$$\mathrm{Ham}(M) = \{\phi\colon M\to M \mid \phi=\phi_H \text{ for some } H\in\mathcal{H}(M)\}\,.$$

The set $\mathrm{Ham}(M)$ forms a group with respect to composition. Moreover, $\mathrm{Ham}(M)$ is a Lie subgroup of the group of all diffeomorphisms of $M$, and the Lie algebra of $\mathrm{Ham}(M)$ can be identified with the set of all time-independent Hamiltonians in $\mathcal{H}(M)$.

For an open subset $U\subset M$ we let $\mathrm{Ham}(U)$ be the subgroup of $\mathrm{Ham}(M)$ where an element $\phi\in\mathrm{Ham}(M)$ lies in $\mathrm{Ham}(U)$ if and only if it is generated by a Hamiltonian $H\in\mathcal{H}(M)$ with compact support contained in $U$, i.e. $\operatorname{supp} H_t\subset U$ for all $t$. We let $\widetilde{\mathrm{Ham}}(M)$ be the universal cover of $\mathrm{Ham}(M)$; its elements are smooth paths in $\mathrm{Ham}(M)$ based at the identity, considered up to homotopy with fixed end points.

The group $\mathrm{Ham}(M)$ carries a non-degenerate bi-invariant metric, the Hofer metric. For $H\in C_c^\infty([0,1]\times M)$ let $\operatorname{osc} H_t = \max H_t - \min H_t$ be the oscillation of $H_t$. For $\phi\in\mathrm{Ham}(M)$ consider

$$\rho(\phi) = \inf_H \int_0^1 \operatorname{osc} H_t\,dt\,,$$

where the infimum goes over all Hamiltonians in $\mathcal{H}(M)$ generating $\phi$. Thus, $\rho(\phi)$ can be interpreted to describe the minimal amount of energy needed to generate a given $\phi \in \mathrm{Ham}(M)$. The latter defines a norm, and by

$$\rho(\phi, \psi) = \rho(\phi\psi^{-1})$$

one can extend it to a bi-invariant metric on $\mathrm{Ham}(M)$, the Hofer metric. Thereby, it is a highly nontrivial fact that $\rho$ is indeed non-degenerate. It was first established for $M = \mathbb{R}^n$ by Hofer [Ho] and then proved for all symplectic manifolds by Lalonde and McDuff [LaMc]. By the triangle inequality of the Hofer metric, the Hofer norm is subadditive, and the asymptotic Hofer norm on $\mathrm{Ham}(M)$ given by

$$\rho_\infty(\phi) = \lim_{k\to\infty} \frac{\rho(\phi^k)}{k}$$

is well-defined. We refer to [Po] for details about the group $\mathrm{Ham}(M)$ and Hofer geometry.

### Cotangent bundles

Important examples of symplectic manifolds are cotangent bundles. In this work we will mainly consider the following situation: Let $N$ be an $n$-dimensional closed connected manifold and $T^*N = \{(q,p) \,|\, p \in T_q^*N\}$ its cotangent bundle. There is a canonical 1-form $\lambda$ on $T^*N$, called the Liouville form, which induces a symplectic form $\omega = d\lambda$ on $T^*N$. It is given by the following construction: Denote by $\pi\colon T^*N \to N$ the natural projection $(q,p) \mapsto q$. For any $(q,p) \in T^*N$ the map $\pi$ induces a map

$$\pi_*(q,p)\colon T_{(q,p)}T^*N \to T_{\pi(q,p)}N = T_qN\,,$$

and for any $(q,p) \in T^*N$ and $\xi \in T_{(q,p)}T^*N$ the Liouville form $\lambda$ is given by

$$\lambda(q,p)(\xi) = \langle p, \pi_*(q,p)\xi\rangle\ ,$$

where $\langle \cdot,\cdot\rangle$ denotes the natural pairing between $T^*N$ and $TN$. In local coordinates $(q_1,\dots,q_n,p_1,\dots,p_n)$ on $T^*N$ the Liouville form is given by

$$\lambda = \sum_{i=1}^n p_i\, dq_i = p\, dq\,.$$

Therefore, $\omega = d\lambda = dp\wedge dq$ is a symplectic form and $(T^*N, \omega = d\lambda)$ is a symplectic manifold.

We set $T_r^*N = \{(q,p) \,|\, \|p\| \le r\}$ for $r > 0$. Thereby, we fix an auxiliary Riemannian metric on $N$ and measure the lengths of cotangent vectors relative to this metric.

There are several important examples of Lagrangian submanifolds in $(T^*N, \omega = d\lambda)$. The zero section $\mathcal{O}_N = \{(q,0) \in T^*N \,|\, q \in N\}$ of $T^*N$ is a Lagrangian submanifold; it is denoted by $\mathcal{O}_N$ or $N$. A Lagrangian submanifold $L \subset T^*N$ is said to be Hamiltonian isotopic to the zero section if there is $\phi \in \mathrm{Ham}(T^*N)$ such that $\phi(N) = L$. If $\alpha$ is a 1-form on $N$, its graph $\Gamma_\alpha = \{(q,\alpha(q)) \,|\, q \in N\} \subset T^*N$ is a Lagrangian submanifold if and only if $\alpha$ is closed. Moreover, let $M \subset N$ be a closed connected submanifold. Its conormal bundle $\nu^*M \subset T^*N$ is a Lagrangian submanifold in $T^*N$. It is given by $\nu^*M = \{(q,p) \in T^*N \,|\, q \in M, p \in \nu_q^*M\}$, where $\nu_q^*M = \{p \in T_q^*N \,|\, \langle p, v\rangle = 0 \text{ for all } v \in T_qM\}$.

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# 2 Lagrangian spectral invariants

Lagrangian spectral invariants arise from Lagrangian Floer homology of cotangent bundles relative to closed connected submanifolds; both were introduced and first studied by Oh [Oh1], [Oh2]. In Subsection 2.1 we give a very short overview about the construction of Lagrangian Floer homology and Lagrangian spectral invariants; the general reference for this is Oh's works [Oh1], [Oh2]. Here we use different sign conventions than Oh; we follow the philosophy that the Floer theory of the action functional is a perturbation of the Morse theory of a function on a closed manifold. The effect of the different sign conventions is that our invariants are "dual" to Oh's; this is discussed in Subsection 4.1.2 in detail.

In Subsection 2.2 we prove some properties of the Lagrangian spectral invariants arising from Floer homology with respect to the zero section. In particular, in Subsection 2.2.2 we prove a notion of Poincaré duality for the top and the point Lagrangian spectral invariants, in Subsection 2.2.3 we prove a sharp triangle inequality from which we conclude in Subsection 2.2.4 that the Lagrangian spectral invariants descend to functions on $\mathrm{Ham}(T^*N)$. Moreover, in Subsection 2.3 we compare the Lagrangian spectral invariants with the Hamiltonian spectral invariants in cotangent bundles which were introduced and studied by Frauenfelder and Schlenk [FS]. In Subsection 2.4 we summarize the properties of the functions on $\mathrm{Ham}(T^*N)$ given by the Lagrangian spectral invariants.

The various properties of the Lagrangian spectral invariants allow to define the family of partial quasi-morphisms $\mu_a\colon \mathrm{Ham}(T^*N) \to \mathbb{R}$ by homogenizing a certain Lagrangian spectral invariant as well as the family of partial symplectic quasi-integrals $\zeta_a\colon C_c^\infty(T^*N) \to \mathbb{R}$, and to prove some additional properties of these functions, see Section 3.

In Subsection 2.5 we extend the definition of the Lagrangian spectral invariants to Hamiltonians which are not compactly supported but have complete flow.

In Subsection 2.6 we prove a product formula for these extended spectral invariants which, in particular, gives a product formula for the partial quasi-morphisms $\mu_a$ and the partial symplectic quasi-integrals $\zeta_a$; the latter can be used to prove rigidity results, see Subsection 5.4.

In the sequel we fix a closed connected $n$-dimensional manifold $N$ and a Riemannian metric on $N$, and consider the symplectic manifold $(T^*N, \omega = d\lambda)$, where $\lambda = p\,dq$ is the Liouville form. We identify $N$ with the zero section via the embedding $N \to T^*N$. Unless otherwise mentioned, all homology is with $\mathbb{Z}_2$ coefficients, and all moduli spaces are counted modulo 2.

## 2.1 Lagrangian Floer homology and Lagrangian spectral invariants

Let $H \in C_c^\infty([0,1] \times T^*N)$ be a compactly supported time-dependent Hamiltonian. Floer homology of the Hamiltonian $H$ can be viewed as Morse homology of the action functional $\mathcal{A}_H$ corresponding to $H$. Here, the classical action functional $\mathcal{A}_H$ corresponding to the Hamiltonian $H$ is a functional on the space of smooth paths

$$\Omega = \{\gamma\colon [0,1] \to T^*N \mid \gamma(0) \in N\}$$

given by

$$\mathcal{A}_H(\gamma) = \int_0^1 H_t(\gamma(t))\, dt - \int \gamma^* \lambda \, .$$

Let $M \subset N$ be a closed connected submanifold and denote by $\nu^* M \subset T^* N$ the conormal bundle of $M$ in $N$. Note that in case $M = N$, the conormal bundle $\nu^* N$ equals the zero section $N$. Consider the space of paths

$$\Omega(M) = \{\gamma \in \Omega \,|\, \gamma(1) \in \nu^* M\}$$

and let $\mathcal{A}_{H:M}$ be the restriction of $\mathcal{A}_H$ to $\Omega(M)$. The set $\mathrm{Crit}(H : M) = \mathrm{Crit}\, \mathcal{A}_{H:M}$ of critical points of the action functional on $\Omega(M)$ is precisely the set of solutions $\gamma$ of the Hamiltonian equation $\dot{\gamma} = X_H(\gamma)$ with boundary conditions dictated by $\Omega(M)$. Therefore, there is a one-to-one correspondence between the set of critical points $\mathrm{Crit}(H : M)$ and the intersection points of $\phi_H(N) \cap \nu^* M$, the map $\mathrm{Crit}(H : M) \to \phi_H(N) \cap \nu^* M$ given by $\gamma \mapsto \gamma(1)$ is a bijection. Let the action spectrum of $H$ relative to $M$ be the set

$$\mathrm{Spec}(H : M) = \{\mathcal{A}_{H:M}(\gamma) \,|\, \gamma \in \mathrm{Crit}(H : M)\} \subset \mathbb{R} \, .$$

It is a compact nowhere dense subset, and it only depends on the time-1 map $\phi_H$ of $H$. Therefore, for $\phi \in \mathrm{Ham}(T^* N)$ we denote $\mathrm{Spec}(\phi : M) = \mathrm{Spec}(H : M)$, where $H$ is any Hamiltonian generating $\phi$.

Consider the vector space $CF(H : M)$ spanned over $\mathbb{Z}_2$ by the set $\mathrm{Crit}(H : M)$, and for $a \notin \mathrm{Spec}(H : M)$, the subspace $CF^{<a}(H : M) \subset CF(H : M)$ spanned by critical points with action $< a$, and the quotient space $CF^{>a}(H : M) := CF(H : M)/CF^{<a}(H : M)$. For a generic choice of $H$, the intersection $\phi_H(N) \cap \nu^* M$ is transverse and so $\mathrm{Crit}(H : M)$ is finite, and the various spaces $CF$ are all finite-dimensional; we refer to such a Hamiltonian as *regular*.

For any critical point there is an integer-valued index $m_{H:M}$: $\mathrm{Crit}(H : M) \to \mathbb{Z}$, the Conley-Zehnder index. We normalize it as follows: Let $f_0$: $M \to \mathbb{R}$ be a Morse function. Denote by $\nu_N M$ the normal bundle of $M$ in $N$; it is a vector bundle over $M$. Identify a neighborhood of $M \subset N$ with a disk bundle $\pi_0$: $DM \to M$ in the normal bundle $\nu_N M$. Extend the function $\pi_0^* f_0$ to a smooth function $f$ on $N$ and let $H = \pi^* f : T^* N \to \mathbb{R}$. The elements of $\mathrm{Crit}(H : M)$ are in one-to-one correspondence to the critical points of $f|_M = f_0$. We normalize $m_{H:M}$ so that it coincides with the Morse index of $f_0$ under this correspondence. We let $CF_k(H : M)$ denote the subspace of $CF(H : M)$ spanned by elements of index $m_{H:M} = k$.

To study the negative gradient flow of $\mathcal{A}_{H:M}$ on $\Omega(M)$, define an $L^2$-type metric on $\Omega(M)$ as follows: Let $J$: $[0, 1] \to \mathrm{End}(TT^* N)$ be a path of almost complex structures on $T^* N$ compatible with the symplectic structure in the sense that $\omega(\cdot, J_t \cdot)$ is a path of Riemannian metrics on $T^* N$. For $\gamma \in \Omega(M)$ and $\xi, \eta \in T_\gamma \Omega(M)$ define $\langle \xi, \eta \rangle = \int_0^1 \omega(\xi(t), J_t \eta(t))\, dt$. Then the gradient of $\mathcal{A}_{H:M}$ relative to this metric reads

$$\nabla \mathcal{A}_{H:M}(\gamma(t)) = J_t(\gamma(t))\big(\dot{\gamma}(t) - X_H(\gamma(t))\big) \, .$$

The corresponding negative gradient flow equation for $u$: $\mathbb{R}(s) \to \Omega(M)$ is Floer's equation (which is a perturbed Cauchy-Riemann equation with Lagrangian boundary conditions) given by

$$\frac{\partial u}{\partial s} + J_t(u) \left( \frac{\partial u}{\partial t} - X_H(u) \right) = 0 \, .$$

For critical points $\gamma_\pm \in \mathrm{Crit}(H : M)$ we denote by $\mathcal{M}(\gamma_-, \gamma_+)$ the set of solutions $u$ of this equation such that $u(\pm\infty, \cdot) = \gamma_\pm$. This set admits a natural action of $\mathbb{R}$ by translation in the $s$ variable, and we let $\widehat{\mathcal{M}}(\gamma_-, \gamma_+) = \mathcal{M}(\gamma_-, \gamma_+)/\mathbb{R}$ be the quotient if $\gamma_+ \neq \gamma_-$, and $\widehat{\mathcal{M}}(\gamma_-, \gamma_-) = \varnothing$.

If the path of almost complex structures $J_t$ is chosen generically, the moduli spaces $\mathcal{M}(\gamma_-, \gamma_+)$ and $\widehat{\mathcal{M}}(\gamma_-, \gamma_+)$ are finite-dimensional smooth manifolds for any $\gamma_\pm \in \mathrm{Crit}(H : M)$; we call such a $J$ *regular* for $H$. Moreover, for regular $J$ and $H$ we have $\dim \mathcal{M}(\gamma_-, \gamma_+) = m_{H:M}(\gamma_-) - m_{H:M}(\gamma_+)$.

If the path of almost complex structures $J_t$ is chosen such that it coincides with the almost complex structure induced by the Riemannian metric on the base outside of a compact subset of $T^*N$, the zero-dimensional component of the moduli space $\widehat{\mathcal{M}}(\gamma_-, \gamma_+)$ becomes compact, while the one-dimensional component becomes compact up to breaking. Indeed, for $m_{H:M}(\gamma_-) = m_{H:M}(\gamma_+) + 1$ we have $\dim \widehat{\mathcal{M}}(\gamma_-, \gamma_+) = 0$ and $\widehat{\mathcal{M}}(\gamma_-, \gamma_+)$ is a compact smooth manifold. Therefore, define

$$\partial\colon CF_k(H : M) \to CF_{k-1}(H : M)$$

by the linear extension of

$$\partial\gamma_- = \sum_{m_{H:M}(\gamma_+)=k-1} \#\mathcal{M}(\gamma_-, \gamma_+)\, \gamma_+ \, .$$

According to the boundary conditions of the compactification of the one-dimensional components of $\widehat{\mathcal{M}}(\gamma_-, \gamma_+)$, the map $\partial$ is a differential, i.e. $\partial^2 = 0$. We denote the corresponding Floer homology groups by $HF_*(H : M)$.

Since elements of $\widehat{\mathcal{M}}(\gamma_-, \gamma_+)$ are negative gradient flow lines of the action functional, it decreases along any such element; hence $\partial$ induces a differential on the subspace $CF_*^{<a}(H : M)$, as well as on the quotient space $CF_*^{>a}(H : M)$. We denote the corresponding relative Floer homology groups by $HF_*^{<a}(H : M)$ and $HF_*^{>a}(H : M)$, and let

$$i_*^a\colon HF_*^{<a}(H : M) \to HF_*(H : M)$$

and

$$j_*^a\colon HF_*(H : M) \to HF_*^{>a}(H : M)$$

be the induced maps on homology.

**Remark 2.1.** The various Floer homology groups $HF$ are canonically isomorphic for different choices of the regular almost complex structure $J$, and therefore we suppressed $J$ from the notation. Moreover, these isomorphisms induce canonical commuting diagrams relating the (filtered) Floer homology groups and the maps $i_*^a, j_*^a$, and thus the maps $i_*^a, j_*^a$ are independent of $J$.

If $K$ is another regular Hamiltonian, there is a canonical continuation isomorphism $HF_*(H : M) = HF_*(K : M)$. Moreover, the Floer homology $HF_*(H : M)$ of any regular $H$ is canonically isomorphic to the singular homology $H_*(M)$. Indeed, let $f$ be a function on $N$ constructed as above and $H = \pi^* f$. Then the Floer complex of $H$ degenerates into the Morse complex of $f_0$, including grading. Therefore, for such $H$, the Floer homology $HF_*(H : M)$ is canonically isomorphic to the singular homology $H_*(M)$. In summary, using the above continuation isomorphism, the Floer homology $HF_*(H : M)$ is canonically isomorphic to the singular homology $H_*(M)$ for any regular Hamiltonian $H$, i.e.

$$HF_*(H : M) = H_*(M)\,.$$

Using the identification between $HF_*(H : M)$ and $H_*(M)$ one can define Lagrangian spectral invariants $\ell(A, H : M)$ for any regular $H$ and $A \in H_*(M)$ by

$$\ell(A, H : M) := \inf\{a \mid A \in \operatorname{im} i^a_*\}\,.$$

These have the following properties proved by Oh [Oh1], [Oh2]:

(i) $\ell(A, H : M) \in \operatorname{Spec}(H : M)$, in particular, any spectral invariant is a finite number;

(ii) if $H_k$ is a sequence of regular Hamiltonians which tends to 0 in the $C^1$-topology, then $\ell(A, H_k : M) \to 0$;

(iii) $\int_0^1 \min(H_t - K_t)\,dt \leq \ell(A, H : M) - \ell(A, K : M) \leq \int_0^1 \max(H_t - K_t)\,dt$; in particular, the spectral invariants are Lipschitz continuous with respect to the $C^0$-norm.

We refer to property (iii) as the *continuity* of the spectral invariants.

Similarly to the above one can define spectral invariants associated to cohomology classes of $M$. Consider the dual Floer complex $CF^*(H : M) = \operatorname{Hom}(CF_*(H : M), \mathbb{Z}_2) \equiv (CF^*(H : M))^*$. The universal coefficient theorem implies that the cohomology of this complex taken with the dual differential $\partial^*$ is canonically isomorphic to $(H_*(M))^*$. The latter with coefficients in a field is the same as the singular cohomology $H^*(M)$.

The dual complex is similarly filtered by action; the action increases along the differential. Consider the subcomplex $CF^*_{>a}(H : M)$ generated by orbits of action $>$ $a$ and the quotient complex $CF^*_{<a}(H : M) = CF^*(H : M)/CF^*_{>a}(H : M)$. Here we identify the basis of $CF_*$ with the dual basis of $CF^*$, and as a result we have canonical identifications $CF^*_{>a}(H : M) = (CF^{>a}_*(H : M))^*$ and $CF^*_{<a}(H : M) = (CF^{<a}_*(H : M))^*$, and the same for (co)homology. Let $j^*_a\colon HF^*_{>a}(H : M) \to HF^*(H : M)$ and $i^*_a\colon HF^*(H : M) \to HF^*_{<a}(H : M)$ be the maps induced on cohomology by the inclusion and projection maps. The short exact sequence of cochain complexes

$$0 \to CF^*_{>a}(H : M) \to CF^*(H : M) \to CF^*_{<a}(H : M) \to 0$$

is dual to the short exact sequence of chain complexes

$$0 \to CF^{<a}_*(H : M) \to CF_*(H : M) \to CF^{>a}_*(H : M) \to 0\,,$$

and the induced long exact sequence of cohomologies

$$\cdots \to HF^{k-1}_{<a}(H : M) \to HF^k_{>a}(H : M) \xrightarrow{j^k_a} HF^k(H : M) \xrightarrow{i^k_a} HF^k_{<a}(H : M) \to \ldots$$

is dual to the long exact sequence of homologies

$$\cdots \to HF^{>a}_{k+1}(H:M) \to HF^{<a}_k(H:M) \xrightarrow{i^a_k} HF_k(H:M) \xrightarrow{j^a_k} HF^{>a}_k(H:M) \to \dots .$$

The spectral invariant corresponding to $v \in H^*(M)$ is

$$\ell(v, H:M) = \sup\{a \,|\, i^*_a(v) = 0\}.$$

If $H \in C^\infty_c([0,1] \times T^*N)$ is an arbitrary compactly supported Hamiltonian, it can be approximated by regular Hamiltonians $H_k$ in the $C^\infty$-sense. From the continuity of the spectral invariants it follows that $\ell(A, H_k : M)$ is a convergent sequence and that its limit only depends on $H$. Therefore, spectral invariants can be uniquely extended to the set of all compactly supported Hamiltonians. These extended invariants satisfy the spectrality axiom (see [Oh4])

$$\ell(A, H:M) \in \mathrm{Spec}(H:M),$$

and they are continuous in the sense of property (iii) above, and so they are Lipschitz continuous with respect to the $C^0$-norm.

**Remark 2.2.** In the sequel we need to use Hamiltonians defined on $[0,\tau] \times T^*N$ with $\tau$ different from 1 from time to time. All the preceding constructions are modified in the obvious way, for example, the action functional is now defined on paths $\gamma$: $[0,\tau] \to T^*N$ by $\mathcal{A}(\gamma) = \int_0^\tau H_t(\gamma(t))\,dt - \int \gamma^*\lambda$, and so on. We will not mention this modification explicitly, and the context will always make clear the domain of definition of Hamiltonians, paths, and action functionals.

## 2.2 Properties of Lagrangian spectral invariants

In this subsection we prove some additional properties of the Lagrangian spectral invariants arising from Floer homology relative to the zero section $N$ of $T^*N$ which include a notion of Poincaré duality for the top and the point spectral invariants, a sharp triangle inequality, and the independence of isotopy. These properties are needed in order to define the family of functions $\mu_a$ and to extract the properties of the functions $\mu_a$ in Section 3.

We assume $M = N$ in the rest of this subsection and denote the corresponding Lagrangian spectral invariants by $\ell(A, H)$ for any $A \in H_*(N)$ and $\ell(v, H)$ for any $v \in H^*(N)$. Moreover, we set

$$\ell_+(H) = \ell([N], H) \text{ and } \ell_-(H) = \ell(\mathrm{pt}, H),$$

where $[N] \in H_n(N)$ and $\mathrm{pt} \in H_0(M)$ are generators.

### 2.2.1 Lagrangian property

We have the following observation which turns out to be crucial for many applications of Lagrangian spectral invariants. We refer to it as *Lagrangian* property.

**Lemma 2.3.** *Let $H \in C_c^\infty([0,1]\times T^*N)$ be such that $H|_N = c$ (respectively $H|_N \geq c$, $H|_N \leq c$) for some $c \in \mathbb{R}$. Then $\ell(A,H) = c$ (respectively $\ell(A,H) \geq c$, $\ell(A,H) \leq c$), for any $A \in H_*(N) \setminus \{0\}$.*

**Proof.** Assume $H|_N = c$. According to the spectrality property, $\ell(A,H)$ is a critical value of the action functional which means that it equals $\mathcal{A}_H(\gamma)$ for some $\gamma \in \mathrm{Crit}(H : N)$. The latter set consists of Hamiltonian trajectories beginning and ending on $N$. Since $N \subset T^*N$ is Lagrangian and $H$ is constant on it, $X_H$ is tangent to it, and so any element of $\mathrm{Crit}(H : N)$ is contained in $N$ which means that its action equals $c$. This shows that $\ell(A,H) = c$, as claimed. If $H|_N \geq c$, there is another time-dependent Hamiltonian $K$ with compact support which satisfies $H \geq K$ and $K|_N = c$. The claim then follows from the above consideration and the continuity of the spectral invariants. The other inequality is proved similarly. □

### 2.2.2 Poincaré duality

**Proposition 2.4.** *For any Hamiltonian $H \in C_c^\infty([0,1] \times T^*N)$ we have*

$$\ell_\pm(H) = -\ell_\mp(\overline{H})\,,$$

*where $\overline{H}$ is defined by $\overline{H}(t,x) = -H(1-t,x)$.*

**Proof.** Assume that the Hamiltonian $H$ is regular, that is, $\phi_H(N)$ intersects $N$ transversely. By standard duality considerations (see [Sch] for example) we obtain

$$\ell(\mathrm{pt},H) = \ell(1,H) \quad \text{and} \quad \ell([N],H) = \ell(\mu_N,H)\,,$$

where $\mathrm{pt} \in H_0(N)$, $[N] \in H_n(N)$, $1 \in H^0(N)$, $\mu_N \in H^n(N)$ are generators. In order to prove the claim we make use of the above duality by comparing the filtered Floer cohomology of $H$ with the filtered Floer homology of $\overline{H}$. The Hamiltonian $\overline{H}$ generates the isotopy $\phi^t_{\overline{H}}$ which is obtained from the one generated by $H$ by retracing it backwards, i.e.

$$\phi^t_{\overline{H}} = \phi_H^{1-t}\phi_H^{-1}\,.$$

The sets of critical points of $\mathcal{A}_H$ and $\mathcal{A}_{\overline{H}}$ are in one-to-one correspondence, the bijection is given by the involution $\Omega(N) \to \Omega(N)$, $\gamma \mapsto \overline{\gamma} = \gamma(1-\cdot)$. Moreover, we have $m_{H:N}(\gamma) = n - m_{\overline{H}:N}(\overline{\gamma})$. In summary, there is a canonical isomorphism

$$CF^*(H:N) = CF_{n-*}(\overline{H}:N)\,.$$

Since we have $\mathcal{A}_H(\gamma) = -\mathcal{A}_{\overline{H}}(\overline{\gamma})$, the filtrations are reversed for every $a \notin \mathrm{Spec}(H : N)$, i.e.

$$CF^*_{\mathcal{A}_H > a}(H:N) = CF_{n-*}^{\mathcal{A}_{\overline{H}} < -a}(\overline{H}:N)\,.$$

Moreover, if $J$ is a compatible almost complex structure which is regular for $H$, $\overline{J}(t,\cdot) = J(1-t,\cdot)$ is a compatible almost complex structure which is regular for $\overline{H}$, and there is a natural identification of the moduli spaces $\widehat{\mathcal{M}}(\gamma_-,\gamma_+,H,J)$ and $\widehat{\mathcal{M}}(\overline{\gamma}_+,\overline{\gamma}_-,\overline{H},\overline{J})$ given by $u \mapsto \overline{u}$, $\overline{u}(s,t) = u(-s,1-t)$. In summary, we conclude

$$\ell(\mathrm{pt},H) = \ell(1,H) = -\ell([N],\overline{H})\,,$$

and similarly we have

$$\ell([N],H)=\ell(\mu_N,H)=-\ell(\mathrm{pt},\overline{H})\,.$$

Due to the continuity of spectral invariants both equalities continue to hold if we replace $H$ by an arbitrary smooth Hamiltonian, and the claim is proved. □

### 2.2.3 Triangle inequality

Let $H,H'\colon [0,1]\times T^*N\to\mathbb{R}$ be such that $H(1,\cdot)=H'(0,\cdot)$. The concatenation $H\sharp H'\colon [0,2]\times T^*N\to\mathbb{R}$ of the two Hamiltonians is defined to be

$$H\sharp H'(t,x)=\begin{cases} H(t,x)\,, & \text{if } t\le 1\\ H'(t-1,x)\,, & \text{if } t\ge 1\end{cases}\,.$$

Note that if $H,H'$ are smooth and $H(1,\cdot)=H'(0,\cdot)$ with all the time derivatives, then $H\sharp H'$ is smooth as well.

**Proposition 2.5.** *Let $H,H'\in C_c^\infty([0,1]\times T^*N)$ be such that $H(1,\cdot)=H'(0,\cdot)$ with all the time derivatives. Then*

$$\ell(A\cap B,H\sharp H')\le\ell(A,H)+\ell(B,H')$$

*for all $A,B\in H_*(N)$ with $A\cap B\neq 0$, where $\cap\colon H_j(N)\times H_k(N)\to H_{j+k-n}(N)$ denotes the intersection product in homology.*

In the proof of the above proposition we will use a certain procedure, which we call smoothing, which allows to replace any given time-dependent Hamiltonian by a Hamiltonian that vanishes for values of time close to 0 and 1 (see [Po] for instance). We will see that this procedure leaves intact all the spectral invariants of the original Hamiltonian. Moreover, if $H$ and $K$ are such that $H\sharp K$ is smooth, then the spectral invariants of $H\sharp K$ are precisely the spectral invariants of the concatenation of any two smoothed versions of the Hamiltonians $H$ and $K$.

**Remark 2.6** (Smoothing)**.** Let $H\in C_c^\infty([0,1]\times T^*N)$ be a time-dependent Hamiltonian with compact support. Let $f\colon[0,1]\to[0,1]$ be a smooth function with $f'(t)\ge 0$ for any $t\in[0,1]$ and $f(t)\equiv 0$ for $t$ near 0 and $f(t)\equiv 1$ for $t$ near 1. Then the function $H^f$ defined by

$$H^f(t,x)=f'(t)H(f(t),x)$$

is a compactly supported smooth Hamiltonian which equals 0 for $t$ near 0 and 1. The flows of $H$ and $H^f$ satisfy $\phi^t_{H^f}=\phi^{f(t)}_H$.

Moreover, if $H$ is a regular Hamiltonian, then so is $H^f$. If $J$ is an almost complex structure regular for $H$, then $J^f=J(f'(\cdot),\cdot)$ is for $H^f$, with an obvious identification between the various moduli spaces relative to $H,J$ and $H^f,J^f$. For the various spectral invariants we have the following:

(i) The above procedure of smoothing leaves intact all the spectral invariants. Indeed, for $H\in C_c^\infty([0,1]\times T^*N)$ and a smooth function $f\colon[0,1]\to[0,1]$

with $f'(t) \geq 0$ for any $t \in [0,1]$ and $f(0) = 0$, $f(1) = 1$ consider the function $H^f$. Since the flows of $H$ and $H^f$ satisfy $\phi^t_{H^f} = \phi^{f(t)}_H$, there is a bijection between the sets of solutions of the corresponding Hamiltonian equations with boundary conditions dictated by $\Omega(N)$. The bijection is given by $\mathrm{Crit}(H : N) \to \mathrm{Crit}(H^f : N)$, $\gamma \mapsto \gamma^f$, where $\gamma^f(t) = \gamma(f(t))$, and it preserves the corresponding actions, i.e. $\mathcal{A}_H(\gamma) = \mathcal{A}_{H^f}(\gamma^f)$. Therefore, if $f_\tau\colon [0,1] \to [0,1]$, $\tau \in [0,1]$, is a continuous family of smooth functions with $f_0 = \mathrm{id}_{[0,1]}$, $f_1 = f$ and $f_\tau(0) = 0$, $f_\tau(1) = 1$, $f'_\tau \geq 0$, then the action spectrum $\mathrm{Spec}(H^{f_\tau} : N)$ is independent of $\tau$, and consequently, by spectrality, so is any spectral invariant. Thus, the spectral invariants of $H$ and $H^f$ coincide.

(ii) Consider two Hamiltonians $H$ and $K$ as in the above proposition and note that their concatenation $H \sharp K$ is smooth. Denote by $H^f$ and $K^g$ the corresponding smoothed Hamiltonians; the concatenation $H^f \sharp K^g$ is smooth as well. The spectral invariants of $H^f \sharp K^g$ are independent of the functions $f, g$ used for smoothing, and the spectral invariants of $H \sharp K$ coincide with those of $H^f \sharp K^g$.

**Proof** (of Proposition 2.5)**.** Let $H, H' \in C^\infty_c([0,1] \times T^*N)$ be two Hamiltonians as in the proposition; in particular, the concatenation $H \sharp H'$ is smooth. According to the continuity of the spectral invariants we can assume that $H$ and $H'$ are regular. Using the above smoothing procedure we can replace $H$ and $H'$ by smoothed Hamiltonians which are regular as well without altering the corresponding spectral invariants of $H, H'$ and $H \sharp H'$. Therefore, we can assume that $H, H'$ are regular and smoothed, i.e. $H = H' = 0$ for times $t$ near $0, 1$. Let $\varepsilon > 0$. Consider the concatenation $H''_0 = H \sharp H'$. It may not be regular anymore, so we perturb it to a regular Hamiltonian $H''$ such that $\|H'' - H''_0\|_{C^0} < \varepsilon$. We choose an additional smooth function $K\colon \mathbb{R} \times [0,2] \times T^*N \to \mathbb{R}$ such that

$$K(s,t,\cdot) = \begin{cases} H(t,\cdot)\,, & s \leq 1\,, t \in [0,1] \\ H'(t-1,\cdot)\,, & s \leq 1\,, t \in [1,2] \\ H''(t,\cdot)\,, & s \geq 2\,, t \in [0,2] \end{cases}$$

and for $s \in [1,2]$ we have $\left|\frac{\partial K}{\partial s}\right| < \varepsilon$ for all $t$. Moreover, we fix a $t$-dependent almost complex structure $J$, defined for $t \in [0,2]$, which coincides with the almost complex structure induced by the Riemannian metric outside of a compact subset in $T^*N$.

Denote by $\Upsilon$ the strip with a slit appearing in [AS]. This $\Upsilon$ is a Riemann surface with boundary which is conformally equivalent to a closed disk with three boundary punctures which can be described as a strip with a slit: Take the disjoint union $\mathbb{R} \times [0,1] \cup \mathbb{R} \times [1,2]$ and identify $(s, 1^-)$ with $(s, 1^+)$ for every $s \geq 0$. The resulting object is a Riemann surface with interior $\mathbb{R} \times (0,2) \setminus (-\infty, 0] \times \{1\}$ endowed with the complex structure of a subset of $\mathbb{R}^2 = \mathbb{C}$. The complex structure at each boundary point of the components $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{2\}$ is induced by the inclusion in $\mathbb{C}$. The conformal coordinate near the point $(0,1)$ is given by the square root. Thus, $\Upsilon$ carries a global coordinate $z = s + it$ which is holomorphic everywhere with exception of the point $(0,1)$; see [AS] for details.

Let $\gamma, \gamma', \gamma''$ be critical points of $\mathcal{A}_H$, $\mathcal{A}_{H'}$, $\mathcal{A}_{H''}$, respectively. We consider the moduli space $\mathcal{M}(\gamma, \gamma'; \gamma'')$ of solutions $u\colon \Upsilon \to T^*N$ with coordinates $(s,t)$, where

$t \in [0,2]$, of the equation

$$\frac{\partial u}{\partial s}(s,t) + J_t(u)\left(\frac{\partial u}{\partial t}(s,t) - X_K(s,t)\right) = 0$$

with boundary conditions $u(\partial\Upsilon) \subset N$ and asymptotic conditions $u(-\infty,\cdot) = \gamma$, $u(-\infty, \cdot - 1) = \gamma'$, $u(\infty,\cdot) = \gamma''$. For a generic choice of $J$, the moduli space $\mathcal{M}(\gamma,\gamma';\gamma'')$ is a smooth manifold of dimension $m_{H:N}(\gamma)+m_{H':N}(\gamma')-m_{H'':N}(\gamma'')-n$ which is compact in dimension 0 [AS]. This allows to define a bilinear map

$$CF_j(H:N) \times CF_k(H':N) \to CF_{j+k-n}(H'':N)$$

by the linear extension of

$$(\gamma,\gamma') \mapsto \sum_{\gamma''} \#\mathcal{M}(\gamma,\gamma';\gamma'')\,\gamma''\,.$$

This map is a chain map (according to the boundary conditions of the compactification of the one-dimensional components of $\mathcal{M}(\gamma,\gamma';\gamma'')$) and hence descends to homology,

$$HF_j(H:N) \times HF_k(H':N) \to HF_{j+k-n}(H'':N)\,.$$

We claim that, under the natural identification $HF_* = H_*(N)$, this map corresponds to the intersection product in homology. Postponing the proof of this claim for a moment, we conclude: A computation shows (see [AS]) that if $u \in \mathcal{M}(\gamma,\gamma';\gamma'')$, then

$$\mathcal{A}_H(\gamma) + \mathcal{A}_{H'}(\gamma') - \mathcal{A}_{H''}(\gamma'') \geq E(u) - \varepsilon\,,$$

where $E(u) = \int_\Upsilon |\partial_s u|^2 \,ds\,dt \geq 0$ is the energy of $u$. It follows that the above chain map restricts to a map on filtered subcomplexes,

$$CF_j^{<a}(H:N) \times CF_k^{<b}(H':N) \to CF_{j+k-n}^{a+b+\varepsilon'}(H'':N)$$

for any $a, b, \varepsilon'$ such that $a \notin \mathrm{Spec}(H:N)$, $b \notin \mathrm{Spec}(H':N)$, $\varepsilon' > \varepsilon$, and $a+b+\varepsilon' \notin \mathrm{Spec}(H'':N)$. This implies

$$\ell(A\cap B, H'') \leq \ell(A,H) + \ell(B,H') + \varepsilon\,.$$

Since $H''$ was chosen $\varepsilon$-close to the concatenation $H\sharp H'$, passing to the limit as $\varepsilon \to 0$, we obtain the desired triangle inequality

$$\ell(A\cap B, H\sharp H') \leq \ell(A,H) + \ell(B,H')\,.$$

To prove the remaining claim about the correspondence between the above map and the intersection product in homology we note the following: In [Oh2] Oh proved that a different version of the above $\Upsilon$-product corresponds to the cup product in singular cohomology. In his version, the Hamiltonian $K$ on the strip with a slit vanishes for $s$ near 0. If we use such a Hamiltonian in the definition of our moduli space, we will obtain the same map on homology. Indeed, one can define the corresponding moduli space of paths of solutions to the above equation, where the Hamiltonian depends on the variable of the path, say $K^\tau$. Studying the boundary of the one-dimensional component of the moduli spaces, one can see that counting the zero-dimensional moduli spaces amounts to a chain homotopy between the chain maps constructed from Hamiltonians $K^0$ and $K^1$. This implies that they define the same map in homology. Thus, it is immaterial whether to use our Hamiltonian $K$, "glued" from $H, H', H''$, or Oh's Hamiltonian which vanishes for $s$ near 0. In Oh's sign conventions his Floer homologies are isomorphic to $H^*(N)$ (see Subsection 4.1.2). Passing to our sign conventions amounts to applying the Poincaré duality in each variable. This transforms the cup product on cohomology into the intersection product on homology and therefore, in summary, our map corresponds to the intersection product in homology, proving the claim. $\square$

### 2.2.4 Independence of isotopy

As a consequence of the triangle inequality we can prove that the spectral invariants are independent of the Hamiltonian isotopy generated by $H$; in particular, they descend to the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(T^*N)$.

**Proposition 2.7.** *Let $H, H' \in C_c^\infty([0,1] \times T^*N)$ be such that $\phi_H = \phi_{H'}$. Then the spectral invariants of $H, H'$ coincide, i.e. $\ell(A, H) = \ell(A, H')$ for any $A \in H_*(N)$.*

**Notation 2.8.** For any $\phi \in \mathrm{Ham}(T^*N)$ we denote by $\ell(A, \phi)$ the value $\ell(A, H)$ for any $H$ generating $\phi$. In particular, we still denote

$$\ell_+(\phi) = \ell([N], \phi) \text{ and } \ell_-(\phi) = \ell(\mathrm{pt}, \phi)$$

for generators $[N] \in H_n(N)$ and $\mathrm{pt} \in H_0(N)$.

For the proof of the proposition we need the following statement:

**Lemma 2.9.** *Let $H \in C_c^\infty([0,1] \times T^*N)$ be a Hamiltonian which generates a loop, i.e. $\phi_H = \mathrm{id}$. Then its spectral invariants all vanish.*

**Proof.** With the above smoothing procedure we can replace $H$ by a smoothed Hamiltonian (still denoted by $H$) such that $\phi_H$ is still the identity map without altering the spectral invariants. Any spectral invariant $\ell(A, H)$ is, by spectrality, the action of a Hamiltonian arc $\gamma \in \Omega(N)$. Since $H$ generates a loop and is smoothed, this arc is a smooth closed orbit. Now, a classical computation shows (see, for instance, [Sch]) that the actions $\mathcal{A}_H(\gamma_x)$ of $\gamma_x(t) = \phi_H^t(x)$ are all the same, that is, independent of the choice of $x$. Since we can take $x$ to be outside the support of $H$, the actions are all zero. Thus, we have $\mathcal{A}_H(\gamma) = \mathcal{A}_H(\gamma_{\gamma(0)}) = 0$ as claimed. $\square$

**Proof** (of Proposition 2.7)**.** Let $H, H'$ be such that $\phi_H = \phi_{H'}$. Again, we can assume that $H$ and $H'$ are smoothed so that both equal 0 near $t = 0$ and near $t = 1$. Suppose for a moment that we can show

$$\ell(A, H) = \ell(A, H \sharp \overline{H'} \sharp H'),$$

where $\overline{H'}(t, x) = -H'(1-t, x)$. Then we have

$$\ell(A, H) = \ell(A \cap [N], H \sharp \overline{H'} \sharp H') \leq \ell(A, H') + \ell([N], H \sharp \overline{H'}).$$

Since $H\sharp\overline{H'}$ generates a loop, its spectral invariants all vanish, and we obtain

$$\ell(A,H) \leq \ell(A,H')\,.$$

The reverse inequality follows by exchanging $H$ and $H'$.

Therefore, it remains to prove $\ell(A,H) = \ell(A, H\sharp\overline{H'}\sharp H')$ in order to prove the proposition. Recall that the flow of $\overline{H'}$ is generated by the flow of $H'$ by retracing it backwards. Thus, the Hamiltonian $\overline{H'}\sharp H'$ generates a contractible loop. Let $K^\tau$ be the Hamiltonian which generates the contraction, that is, the Hamiltonian $K^\tau$ generates a loop based at the identity for every $\tau \in [0,1]$ and $K^0 = 0$ and $K^1 = \overline{H'}\sharp H'$. We have $\phi_{H\sharp K^\tau} = \phi_H$ for every $\tau$. Since the action spectrum $\mathrm{Spec}(H\sharp K^\tau : N)$ only depends on the time-1 map of $H\sharp K^\tau$, it is independent of $\tau$. The spectrality and the continuity of the spectral invariants imply that $\ell(A, H\sharp K^\tau)$ is a continuous function of $\tau$ which takes values in the fixed compact and nowhere dense subset $\mathrm{Spec}(H : N)$. Thus, it is constant. □

## 2.3 Comparison of Lagrangian and Hamiltonian spectral invariants

Besides the Lagrangian spectral invariants there are well-defined spectral invariants in cotangent bundles coming from a version of Hamiltonian Floer homology which were introduced and studied by Frauenfelder and Schlenk [FS]. In this subsection we compare the Hamiltonian and Lagrangian spectral invariants; in fact, we prove an inequality between them which allows to prove that the functions $\mu_a$ which we introduce in Section 3 are partial quasi-morphisms and that they vanish on Hamiltonian diffeomorphisms with displaceable support.

In order to be able to state and prove the inequality between the spectral invariants in Subsection 2.3.2 we give a short review of the construction of Hamiltonian Floer homology and Hamiltonian spectral invariants in cotangent bundles in Subsection 2.3.1.

### 2.3.1 Hamiltonian spectral invariants

Hamiltonian spectral invariants arise from Hamiltonian Floer homology. In general, Hamiltonian Floer homology for a Hamiltonian $H$ can be viewed as Morse theory of the action functional associated to $H$ in an infinite-dimensional setting. On several closed symplectic manifolds (on monotone, or aspherically, or weakly exact closed symplectic manifolds, for instance), Hamiltonian Floer homology is well-defined and isomorphic to the singular homology of the manifold. The Floer chain complex is thereby generated by the 1-periodic orbits of the Hamiltonian flow which are in one-to-one correspondence with the critical points of the action functional, and the Floer differential counts perturbed pseudo-holomorphic cylinders connecting two 1-periodic orbits of index difference one. Originally, Floer homology is due to Floer, see [Fl1], [Fl2], [Fl3] for instance. On closed symplectic manifolds as above, where Floer homology is isomorphic to the singular homology of the manifold, one can associate Hamiltonian spectral invariants to homology classes, see [Sch] for instance. On more general closed symplectic manifolds, Hamiltonian Floer homology involves quantum effects and uses a Novikov ring; it is isomorphic to the quantum homology of the manifold; see [HS], [McS2], [Oh3], [Oh4] for the construction of Hamiltonian Floer homology on more general closed symplectic manifolds and the definition of spectral invariants. We also refer to Subsection 6.5 for a very short overview.

In case of non-closed or open symplectic manifolds, in particular, in case of cotangent bundles, one can set up a version of Hamiltonian Floer homology for compactly supported Hamiltonians. Standard Floer homology cannot be correctly defined for compactly supported Hamiltonians since they are degenerate, but following the work of Frauenfelder and Schlenk [FS], one can circumvent this difficulty on cotangent bundles by considering Hamiltonians which have support in some fixed cotangent ball bundle and a certain prescribed behavior near the boundary. In fact, in [FS] the authors define Hamiltonian Floer homology on weakly exact convex symplectic manifolds (note that any $T^*_rN$ and $T^*N = \bigcup_{r\in\mathbb{N}} T^*_rN$ are exact convex symplectic manifolds) by considering such types of Hamiltonians, provide an isomorphism between Floer homology and the homology of the manifold, and define Hamiltonian spectral invariants. Lanzat generalized this construction to strongly semi-positive compact convex symplectic manifolds [La].

In this subsection we present a sketch of the construction for cotangent bundles, referring to the aforementioned paper for details.

The class of Hamiltonians which is used to define Hamiltonian Floer homology on $T^*N$ is given by the following: Fix $R > 0$ and let $\varepsilon > 0$. Let $h\colon (-\varepsilon, \infty) \to \mathbb{R}$ be a smooth function such that $h(t) = 0$ for $t \geq 0$ and $h'(t) \geq 0$ for $t \leq 0$. Moreover, $h'(t)$ should be small enough so that the flow of $h(\|p\| - R)$ does not have non-constant periodic orbits of period $\leq 1$ for $\|p\| \in (-\varepsilon, 0)$. Let $H_t \in C^\infty_c(T^*N)$ be such that

$$H_t(q,p) = h(\|p\| - R)$$

for $\|p\| \geq R - \varepsilon$. The Floer complex $CF(H)$ is the vector space spanned over $\mathbb{Z}_2$ by the 1-periodic orbits of $H$ inside $T^*_{<R}N$; all of them are non-degenerate and $CF(H)$ is well-defined. It is graded by the Conley-Zehnder index[4)] $m_H$. The boundary operator lowers the degree by 1 and counts Floer cylinders connecting two of such orbits. Thereby, the behavior of $H$ near $\|p\| = R$ guarantees that all Floer cylinders are contained in $T^*_{\leq R-\varepsilon}N$. In summary, the Floer homology of $CF_*(H)$, which we denote by $HF_*(H; h, R)$, is well-defined. It is identified with the homology of $T^*N$, i.e.

$$HF_*(H; h, R) = H_*(T^*N)\,.$$

Since $HF_*(H; h, R)$ is filtered by action via the action functional $\mathcal{A}_H$, one can define spectral invariants in the standard fashion. Denote by $HF^{<a}_*(H; h, R)$ the homology of the subcomplex $CF^{<a}_*(H)$ in $CF_*(H)$ spanned by orbits of action $< a$, and consider the inclusion morphism

$$i^a\colon HF^{<a}_*(H; h, R) \to HF_*(H; h, R)\,.$$

For $A \in H_*(T^*N)$ one can define

$$c(A, H; h, R) = \inf\{a \mid A \in \operatorname{im} i^a\}\,.$$

[4)]It is normalized to equal the Morse index of critical points of a $C^2$-small Hamiltonian, considered as 1-periodic orbits.

These spectral invariants satisfy all the standard properties, including Lipschitz continuity, the triangle inequality, and spectrality. Moreover, they can be defined for arbitrary compactly supported Hamiltonians. Indeed, let $H \in C_c^\infty([0,1] \times T^*N)$ be such a Hamiltonian. It can be $C^0$-approximated by non-degenerate Hamiltonians $H_k$, $k \in \mathbb{N}$, whose behavior for $\|p\| \in [R - \varepsilon_k, \infty)$ is prescribed by the function $h$ as above, and $\varepsilon_k \to 0$. Then the sequence $c(A, H_k; h, R)$ is a Cauchy sequence and one can declare its limit to be the spectral invariant $c(A, H; h, R)$.

It can be shown that the spectral invariants are independent of the choices, that is, independent of $h$ and $R$, and it is proved in [FS] that the Hamiltonian spectral invariants only depend on the time-1 map $\phi_H$ of $H$. Therefore, for any $\phi \in \mathrm{Ham}(T^*N)$ and $A \in H_*(T^*N)$ one can extract Hamiltonian spectral invariants which we denote by $c(A, \phi)$. In particular, we denote

$$c_-(\phi) = c(\mathrm{pt}, \phi) \text{ and } c_+(\phi) = -c_-(\phi^{-1}),$$

where $\mathrm{pt} \in H_0(T^*N)$.

**Remark 2.10.** In [FS] the authors prove that $\Gamma$: $\mathrm{Ham}(T^*N) \to \mathbb{R}$ given by

$$\Gamma(\phi) = c_+(\phi) - c_-(\phi)$$

is a norm on $\mathrm{Ham}(T^*N)$ which is invariant under conjugation (in fact, $\Gamma$ is invariant under conjugation with compactly supported symplectomorphisms); it is called spectral norm. It gives rise to a bi-invariant metric on $\mathrm{Ham}(T^*N)$, referred to as spectral metric, via $\Gamma(\phi, \psi) = \Gamma(\phi\psi^{-1})$. Similarly to the Hofer norm there is an asymptotic version of the spectral norm given by

$$\Gamma_\infty(\phi) = \lim_{k\to\infty} \frac{\Gamma(\phi^k)}{k}.$$

It is shown in [FS] that if $\phi \in \mathrm{Ham}(U)$, where $U \subset T^*N$ is an open and displaceable subset and $\psi \in \mathrm{Ham}(T^*N)$ is such that $\psi(U) \cap \overline{U} = \varnothing$, then it is true that

$$-\Gamma(\psi) \le c_-(\phi) \le c_+(\phi) \le \Gamma(\psi).$$

**Remark 2.11.** For future use we introduce the spectral displacement energy of an open and displaceable subset $S \subset T^*N$; it is given by

$$e(S) = \inf\{\Gamma(\psi) \,|\, \psi(S) \cap \overline{S} = \varnothing\}.$$

The spectral displacement energy of a family $\mathcal{S} = \{S_i\}_i$ of subsets is given by $e(S) = \sup_i e(S_i)$. Note that the spectral displacement energy of a subset $S$ is invariant under $\mathrm{Ham}(T^*N)$ in the sense that $e(S) = e(\varphi(S))$ for any $\varphi \in \mathrm{Ham}(T^*N)$.

### 2.3.2 Comparison of Lagrangian and Hamiltonian spectral invariants

**Proposition 2.12.** *For any $\phi \in \mathrm{Ham}(T^*N)$ we have*

$$\ell_-(\phi) \ge c_-(\phi).$$

This implies the following chain of inequalities

$$c_-(\phi) \le \ell_-(\phi) \le \ell_+(\phi) \le c_+(\phi)\,,$$

where the rightmost inequality follows by duality.

**Proof.** The proof is essentially contained in [Al]. There the author provides a comparison homomorphism between Lagrangian and Hamiltonian Floer homology for certain degrees on closed symplectic manifolds and Lagrangian submanifolds under certain assumptions which guarantee that both, Hamiltonian and Lagrangian Floer theory, are well-defined.

The point of difference is that in [Al] the theory is restricted to closed symplectic manifolds. Albers' proofs rely on certain compactness arguments of moduli spaces of perturbed pseudo-holomorphic curves which are valid in the closed case. In our case there are no additional compactness issues beyond the closed case since the almost complex structure is assumed to coincide with the one coming from the Riemannian metric outside of a large compact set. Moreover, since the symplectic form is exact in our case, there is no bubbling off of spheres or disks, and the proofs are actually simpler. In particular, the homomorphisms which are constructed by Albers for certain degrees are defined for all degrees and are isomorphisms in our case. Therefore, we only present a sketch of the argument pointing out the essential steps for the comparison of the spectral invariants.

Let $H \in C_c^\infty([0,1] \times T^*N)$ be a compactly supported time-dependent Hamiltonian. Albers defines a map

$$\iota\colon CF_*(H : N) \to CF_*(H)$$

as follows: First, one can assume that the Hamiltonian $H$ is time-independent near $t = 0, 1$, and that the Floer homology for it is defined as above. Denote by $\Upsilon'$ the Riemann surface conformal to a closed disk with one boundary and one interior puncture which is obtained from the above strip with a slit $\Upsilon$ through identifying the top and the bottom boundary components. Let $\gamma$ be a Hamiltonian arc and $x$ a periodic orbit of $H$. Consider the moduli space $\mathcal{M}(\gamma, x)$ consisting of solutions of the Floer equation defined on $\Upsilon'$, that is, solutions to

$$\frac{\partial u}{\partial s} + J_t(u)\left(\frac{\partial u}{\partial t} - X_H\right) = 0\,,$$

where the boundary puncture is asymptotic to $\gamma$ and the interior puncture is asymptotic to $x$, while the boundary is mapped to the zero section. The above equation is well-defined because of the existence of global conformal coordinates $(s, t)$ on $\Upsilon'$. Albers shows that the moduli space $\mathcal{M}(\gamma, x)$ is a smooth manifold of dimension $m_{H:N}(\gamma) - m_H(x)$ which is compact in dimension 0. He also shows that $\iota$, which is the linear extension of

$$\iota(\gamma) = \sum_{m_H(x) = m_{H:N}(\gamma)} \#\mathcal{M}(\gamma, x)\, x\,,$$

is a chain map. The canonical identifications $HF_*(H : N) = H_*(N)$ and $HF_*(H) = H_*(T^*N)$ intertwine it with the isomorphism $H_*(N) \to H_*(T^*N)$ induced by the inclusion of the zero section into $T^*N$.

Now, for an element $u \in \mathcal{M}(\gamma, x)$ there is a sharp action-energy identity

$$\mathcal{A}_H(\gamma) - \mathcal{A}_H(x) = E(u) \geq 0\,.$$

Hence, $\iota$ maps $CF_*^{<a}(H:N) \to CF_*^{<a}(H)$ for $a \notin \mathrm{Spec}(H) \cup \mathrm{Spec}(H:N)$, and it follows that $c_-(H) \leq \ell_-(H)$. By the continuity of the spectral invariants we conclude that this inequality holds for arbitrary compactly supported smooth Hamiltonians. □

Proposition 2.12 and the inequality obtained by Frauenfelder and Schlenk given in Subsection 2.3.1 immediately imply:

**Corollary 2.13.** *Let $U$ be an open subset which is displaceable by $\psi \in \mathrm{Ham}(T^*N)$. Then we have*

$$-\Gamma(\psi) \leq \ell_-(\phi) \leq \ell_+(\phi) \leq \Gamma(\psi)$$

*for any $\phi \in \mathrm{Ham}(U)$.*

## 2.4 Lagrangian spectral invariants - summary

In summary, the Lagrangian spectral invariants have the following properties:

**Theorem 2.14.** *Let $N$ be a closed connected manifold. To each $A \in H_*(N) \setminus \{0\}$ we associate a function $\ell(A, \cdot)\colon \mathrm{Ham}(T^*N) \to \mathbb{R}$ such that:*

*(i)* $\ell(A, \phi) \in \mathrm{Spec}(\phi : N)$;

*(ii)* *if $H$, $K$ generate $\phi$, $\psi$, then*

$$\int_0^1 \min(H_t - K_t)\, dt \leq \ell(A, \phi) - \ell(A, \psi) \leq \int_0^1 \max(H_t - K_t)\, dt\,;$$

*(iii)* $\ell(A \cap B, \phi\psi) \leq \ell(A, \phi) + \ell(B, \psi)$ *for any $A, B \in H_*(N)$ such that $A \cap B \neq 0$; in particular,* $\ell_+(\phi\psi) \leq \ell_+(\phi) + \ell_+(\psi)$;

*(iv)* $\ell_-(\phi) \leq \ell(A, \phi) \leq \ell_+(\phi)$;

*(v)* $\ell_\pm(\phi) = -\ell_\mp(\phi^{-1})$, *and thus* $\ell_-(\phi\psi) \geq \ell_-(\phi) + \ell_-(\psi)$;

*(vi)* *if $H$ generates $\phi$ and $H|_N = c$ (respectively $H|_N \geq c$, $H|_N \leq c$) for some $c \in \mathbb{R}$, then $\ell(A, \phi) = c$ (respectively $\ell(A, \phi) \geq c$, $\ell(A, \phi) \leq c$);*

*(vii)* *if $U \subset T^*N$ is an open and displaceable subset and $\psi \in \mathrm{Ham}(T^*N)$ is such that $\psi(U) \cap \overline{U} = \varnothing$, then*

$$-\Gamma(\psi) \leq \ell_-(\phi) \leq \ell_+(\phi) \leq \Gamma(\psi)$$

*for any $\phi \in \mathrm{Ham}(U)$;*

*(viii)* $|\ell(A, \phi) - \ell(A, \psi\phi\psi^{-1})| \leq \ell_+(\psi) - \ell_-(\psi)$ *for any $\psi \in \mathrm{Ham}(T^*N)$;*

*(ix)* $\ell_+(\phi) + \ell_-(\psi) \leq \ell_+(\phi\psi)$.

**Proof.** With the exception of (iv), (viii), and (ix) the statements are proved in the previous subsections.
(iv) The triangle inequality and the fact that $\ell(A, \mathrm{id}) = 0$ imply

$$\ell(A, \phi) = \ell(A \cap [N], \mathrm{id} \circ \phi) \leq \ell(A, \mathrm{id}) + \ell([N], \phi) = \ell_+(\phi)\,.$$

(viii) With the triangle inequality we conclude

$$\begin{aligned}\ell(A, \psi\phi\psi^{-1}) &= \ell(A \cap [N], \psi\phi\psi^{-1}) \\ &\leq \ell(A, \psi\phi) + \ell_+(\psi^{-1}) \\ &\leq \ell(A, \phi) + \ell_+(\psi) + \ell_+(\psi^{-1}) \\ &= \ell(A, \phi) + \ell_+(\psi) - \ell_-(\psi)\,,\end{aligned}$$

which gives

$$\left|\ell(A, \psi\phi\psi^{-1}) - \ell(A, \phi)\right| \leq \ell_+(\psi) - \ell_-(\psi)\,.$$

(ix) The triangle inequality implies $\ell_+(\phi) = \ell_+(\phi\psi\psi^{-1}) \leq \ell_+(\phi\psi) + \ell_+(\psi^{-1})$ and with $-\ell_+(\psi^{-1}) = \ell_-(\psi)$ we have $\ell_+(\phi) + \ell_-(\psi) \leq \ell_+(\phi\psi)$. □

## 2.5 Lagrangian spectral invariants for Hamiltonians with complete flow

We can define the various Lagrangian spectral invariants $\ell(A, H : M)$ for Hamiltonians $H$ with complete flow, that is, for Hamiltonians whose flow exists for all times, and any closed connected submanifold $M \subset N$.

**Lemma 2.15.** *Let $H, H' \in C_c^\infty([0,1] \times T^*N)$ and assume that there are two open subsets $U \subset V \subset T^*N$ such that $N \subset U$, $\phi_H^t(U), \phi_{H'}^t(U) \subset V$ for all $t \in [0,1]$, and $H|_{[0,1]\times V} = H'|_{[0,1]\times V}$. Then the spectral invariants of $\phi_H$ and $\phi_{H'}$ coincide.*

**Proof.** The claim follows from the fact that $H$ can be continuously deformed into $H'$ such that the action spectrum stays intact during the deformation. More precisely, let $H^\tau = \tau H' + (1-\tau)H$. Then $H^\tau$ is a smooth Hamiltonian whose flow sends $U$ into $V$ for all times, and which coincides with $H$ and $H'$ when restricted to $V$. It follows that $H^\tau$ has the same set of Hamiltonian orbits in $\Omega(M)$ regardless of $\tau$, and those have actions independent of $\tau$. □

If $H$ has complete flow, there is $R > 0$ such that $\phi_H^t(N) \subset T^*_{<R}N$ for all $t \in [0,1]$. Any two compactly supported cutoffs $H', H''$ of $H$ outside $T^*_{<R}N$ satisfy the assumptions of the lemma and so have identical spectral invariants. The spectral invariant $\ell(A, H : M)$ is defined to be the common value $\ell(A, H' : M) = \ell(A, H'' : M)$. Thus, for any Hamiltonian $H$ with complete flow and any $A \in H_*(M) \setminus \{0\}$ the number $\ell(A, H : M)$ is well-defined as the corresponding spectral invariant of any suitable cutoff of $H$. Moreover, the spectral invariants for Hamiltonians with complete flow share the properties of the usual ones, that is, spectrality, continuity, and the triangle inequality. In particular, we have

- $\ell(A, H : M) \in \mathrm{Spec}(H : M)$;

· if $H, H'$ have complete flow and if $V \subset T^*N$ is an open subset such that $\phi^t_H(N), \phi^t_{H'}(N) \subset V$ for all $t \in [0,1]$, then

$$\int_0^1 \inf_V (H_t - H'_t)\,dt \leq \ell(A, H : M) - \ell(A, H' : M) \leq \int_0^1 \sup_V (H_t - H'_t)\,dt\,;$$

· if $H, H'$ are Hamiltonians with complete flow such that the concatenation $H \sharp H'$ is smooth, then

$$\ell(A \cap B, H \sharp H') \leq \ell(A, H) + \ell(B, H')$$

for $A, B \in H_*(N)$ with $A \cap B \neq 0$.

**Remark 2.16.** For Hamiltonians with complete flow it is no longer true that the corresponding spectral invariants only depend on the time-1 map. For instance, the constant Hamiltonians $H = 1$ and $H' = 0$ generate the same time-1 map, while we have $\ell(A, H) = 1$ and $\ell(A, H') = 0$ for all $A \in H_*(N) \setminus \{0\}$.

**Lemma 2.17.** *Let $H$ be a time-dependent Hamiltonian with complete flow, and assume that the flow keeps the zero section inside an open subset $U$ for all times. Let $G$ be any cutoff of $H$ outside $U$. Then we have $\ell(A, \phi^t_H : M) = \ell(A, \phi^t_G : M)$ for any $t$.* □

**Remark 2.18.** In general it is not true that one can consistently define the Floer complex for a Hamiltonian with complete flow; moduli spaces of Floer trajectories may fail to be compact without some additional assumptions on the behavior of the Hamiltonian at infinity. Nevertheless, the Floer complex of any cutoff of the Hamiltonian is well-defined. But it is not true that the Floer complexes of two different cutoffs are isomorphic. They are related by canonical chain maps which descend to level preserving isomorphisms on homology.

## 2.6 The product formula

In this subsection we prove a product formula for the Lagrangian spectral invariants. It yields a product formula for the partial quasi-morphisms $\mu_a$ and the partial symplectic quasi-integrals $\zeta_a$ which we define in Section 3; the latter is important for applications to symplectic rigidity which we state in Subsection 5.4.

Let $H, H'$ be time-dependent Hamiltonians on symplectic manifolds $Z, Z'$. The direct sum $H \oplus H'$ on $Z \times Z'$ is the time-dependent Hamiltonian given by

$$(H \oplus H')(t, z, z') = H(t, z) + H'(t, z')\,,$$

for $z \in Z, z' \in Z'$. Note that whenever $H$ and $H'$ have complete flows, so does $H \oplus H'$.

**Theorem 2.19.** *Let $H, H'$ be time-dependent Hamiltonians with complete flows on $T^*N$, $T^*N'$, respectively. For any $A \in H_*(N) \setminus \{0\}$ and $A' \in H_*(N') \setminus \{0\}$ we have*

$$\ell(A \otimes A', H \oplus H') = \ell(A, H) + \ell(A', H')\,,$$

*where $A \otimes A' \in H_*(N) \otimes H_*(N') = H_*(N \times N')$.*

For the proof of the theorem we introduce and make use of the following algebraic observation: In general, one can define spectral invariants for a filtered graded chain complex $\mathcal{V}$. A filtered graded chain complex $\mathcal{V}$ is a quadruple $\mathcal{V} = (V, \vec{v}, \mathcal{A}, \partial)$, where $\vec{v} = \{v_1, \ldots, v_k\}$ is a graded finite set, $V = \mathbb{Z}_2 \otimes \vec{v}$ is the $\mathbb{Z}_2$-vector space spanned by $\vec{v}$, $\mathcal{A}$: $\vec{v} \to \mathbb{R}$ is a 1-1 function (called the action) and $\partial$: $V \to V$ is a differential which lowers the grading by one. The corresponding homology is denoted by $H(V, \partial)$. The filtered chain complex is given by $V^{<a} = \mathbb{Z}_2 \otimes (\vec{v} \cap \{\mathcal{A} < a\})$ and the differential is supposed to respect the filtration, that is, it is supposed to preserve $V^{<a}$ for every $a \in \mathbb{R}$. Spectral invariants $\ell(A, \mathcal{V})$ of $\mathcal{V}$ can be defined for every homology class $A \in H(V, \partial) \setminus \{0\}$ in the standard fashion.

Given two such filtered graded chain complexes $\mathcal{V} = (V, \vec{v}, \mathcal{A}, \partial)$ and $\mathcal{V}' = (V', \vec{v}', \mathcal{A}', \partial')$ one can define the product filtered graded chain complex $\mathcal{V}'' := \mathcal{V} \otimes \mathcal{V}' = (V'', \vec{v}'', \mathcal{A}'', \partial'')$, where $V'' = V \otimes V'$, $\vec{v}'' = \vec{v} \otimes \vec{v}'$, $\partial'' = \partial \otimes \mathrm{id}_{V'} + \mathrm{id}_V \otimes \partial'$ and $\mathcal{A}'' = \mathcal{A} \oplus \mathcal{A}'$ with $(\mathcal{A} \oplus \mathcal{A}')(v_i, v_j') = \mathcal{A}(v_i) + \mathcal{A}'(v_j')$ if $\mathcal{A}''$ is still 1-1. It gives rise to the product homology $H(V'', \partial'') = H(V, \partial) \otimes H(V', \partial')$. For the corresponding spectral invariants we have the following statement which is essentially contained in Theorem 5.2 in [EP3]:

**Lemma 2.20.** *Let $\mathcal{V}, \mathcal{V}', \mathcal{V}''$ be graded filtered chain complexes as above. For $A \in H(V, \partial) \setminus \{0\}$, $A' \in H(V', \partial') \setminus \{0\}$ and $A \otimes A' \in H(V'', \partial'')$ we have*

$$\ell(A \otimes A', \mathcal{V}'') = \ell(A, \mathcal{V}) + \ell(A', \mathcal{V}') .$$

□

**Proof** (of Theorem 2.19)**.** Let $H$ be an arbitrary Hamiltonian with complete flow on $T^*N$. We can perturb $H$ such that the Floer complex of any cutoff of $H$ is a filtered graded chain complex; we refer to such a Hamiltonian as generic. Given two arbitrary Hamiltonians with complete flow, we can perturb both of them such that both perturbations and their direct sum are generic. Since spectral invariants are continuous in the $C^0$-norm, the above perturbations do not alter the corresponding spectral invariants. Therefore, we can assume that the Hamiltonians $H, H'$ on $T^*N, T^*N'$ and their direct sum $H \oplus H'$ are generic. We would like to make use of the above lemma. To wit, let $G$ be a cutoff of $H$ and $G'$ be a cutoff of $H'$. Note that we have $\ell(A, H) = \ell(A, G)$ for any $A \in H_*(N)$ and similar for $H'$ and $G'$. The direct sum $G \oplus G'$ has complete flow. Let $J$ be a regular almost complex structure on $T^*N$ which coincides with the almost complex structure induced by the Riemannian metric outside of a large compact set. Let $J'$ be such a regular almost complex structure on $T^*N'$. Let $r$ be sufficiently large so that $T_r^*N$ contains the images of all the critical points of $\mathcal{A}_G$ as well as all the Floer trajectories connecting critical points of $\mathcal{A}_G$ of index difference 1, and similarly for $r'$, $T^*N'$ and $G'$. Let $R > 0$ be large enough such that $T_R^*(N \times N')$ contains the product $T_r^*N \times T_{r'}^*N'$, and let $G''$ be a cutoff of $G \oplus G'$ outside $T_R^*(N \times N')$. Then $G''$ is also a cutoff of $H \oplus H'$, in particular, they have the same spectral invariants

$$\ell(A'', G'') = \ell(A'', H \oplus H') .$$

Moreover, $G''$ is generic by construction and $J'' = J \oplus J'$ is a regular almost complex structure. Thus, the Floer complex of $G''$ relative to $J''$ is a filtered graded chain complex which is the product of the Floer complexes of $G$ and $G'$ by construction. Together with Lemma 2.20 we conclude

$$\ell(A \otimes A', H \oplus H') = \ell(A \otimes A', G'') = \ell(A, G) + \ell(A', G') = \ell(A, H) + \ell(A', H')\,.$$

□

# 3 Partial quasi-morphisms and partial symplectic quasi-integrals

In this section we define the family of functions $\mu_a\colon \operatorname{Ham}(T^*N) \to \mathbb{R}$, parameterized by $a \in H^1(N;\mathbb{R})$, where each function has properties including those of a partial quasi-morphism. It is obtained by homogenizing a certain Lagrangian spectral invariant. Using the family of partial quasi-morphisms $\mu_a$ we deduce a family of functionals $\zeta_a\colon C_c^\infty(T^*N) \to \mathbb{R}$, where each functional has properties analogous to those of a partial symplectic quasi-integral. The properties of $\mu_a$ are summarized in Theorem 3.5, the ones of $\zeta_a$ in Theorem 3.10.

Moreover, we prove a product formula for the partial quasi-morphisms $\mu_a$ and the partial symplectic quasi-integrals $\zeta_a$ which we use in the sequel to extract rigidity results. The applications following from the existence of both functions are discussed in Sections 4 and 5.

## 3.1 Partial quasi-morphisms

We define a function $\mu_0\colon \operatorname{Ham}(T^*N) \to \mathbb{R}$ by

$$\mu_0(\phi) = \lim_{k\to\infty} \frac{\ell_+(\phi^k)}{k}\,,$$

where $\ell_+ = \ell([N],\cdot)$ denotes the Lagrangian spectral invariant introduced in Notation 2.8 in Section 2.

**Proposition 3.1.** *The function* $\mu_0\colon \operatorname{Ham}(T^*N) \to \mathbb{R}$ *is well-defined and invariant under conjugation in* $\operatorname{Ham}(T^*N)$*.*

**Proof.** The above limit exists since the sequence $\{\ell_+(\phi^k)\}_k$ is subadditive according to point (iii) in Theorem 2.14, it is finite because of property (ii) in Theorem 2.14, and hence $\mu_0$ is well-defined. To prove the conjugation-invariance of $\mu_0$ we note that according to point (viii) in Theorem 2.14 we have $|\ell_+(\psi\phi\psi^{-1}) - \ell_+(\phi)| \le \ell_+(\psi) - \ell_-(\psi)$ for any $\phi, \psi \in \operatorname{Ham}(T^*N)$ which implies

$$\left|\ell_+((\psi\phi\psi^{-1})^k) - \ell_+(\phi^k)\right| = \left|\ell_+(\psi\phi^k\psi^{-1}) - \ell_+(\phi^k)\right| \le \ell_+(\psi) - \ell_-(\psi)\,.$$

Dividing by $k$ and taking the limit $k \to \infty$ yields

$$\left|\mu_0(\psi\phi\psi^{-1}) - \mu_0(\phi)\right| = 0\,.$$

□

The conjugation-invariance of the function $\mu_0$ implies that we can construct a well-defined function $\mu_a\colon \operatorname{Ham}(T^*N) \to \mathbb{R}$ for any $a \in H^1(N;\mathbb{R})$. For $a \in H^1(N;\mathbb{R})$ and a closed 1-form $\alpha \in a$ let $T_\alpha\colon T^*N \to T^*N$ be the symplectomorphism given by

$$T_\alpha(q,p) = (q, p + \alpha(q))\,.$$

Note that for $\phi \in \operatorname{Ham}(T^*N)$ it is true that $T_{-\alpha}\phi T_\alpha \in \operatorname{Ham}(T^*N)$. We define $\mu_a\colon \operatorname{Ham}(T^*N) \to \mathbb{R}$ by

$$\mu_a(\phi) = \mu_0(T_{-\alpha}\phi T_\alpha)$$

for any $\alpha \in a$.

**Proposition 3.2.** *The function* $\mu_a\colon \mathrm{Ham}(T^*N) \to \mathbb{R}$ *is well-defined, that is, independent of the choice of* $\alpha \in a$.

**Proof.** For $a \in H^1(N;\mathbb{R})$ and any $\alpha, \alpha' \in a$ we claim that $\mu_0(T_{-\alpha}\phi T_\alpha)$ equals $\mu_0(T_{-\alpha'}\phi T_{\alpha'})$. Assume that $\alpha - \alpha' = df$ for some $f \in C^\infty(N)$. Let $B$ be a cotangent ball large enough to contain the support of a Hamiltonian generating $T_{-\alpha'}\phi T_{\alpha'}$. Let $F$ be a compactly supported Hamiltonian obtained from $\pi^* f$ by cutting it off outside $B$. We then have $T_{-df}T_{-\alpha'}\phi T_{\alpha'}T_{df} = \phi_F T_{-\alpha'}\phi T_{\alpha'}\phi_{-F}$. Consequently

$$T_{-\alpha}\phi T_\alpha = T_{-df}T_{-\alpha'}\phi T_{\alpha'}T_{df} = \phi_F T_{-\alpha'}\phi T_{\alpha'}\phi_{-F}\,,$$

and so

$$\mu_0(T_{-\alpha}\phi T_\alpha) = \mu_0(\phi_F(T_{-\alpha'}\phi T_{\alpha'})\phi_{-F}) = \mu_0(T_{-\alpha'}\phi T_{\alpha'})\,.$$

□

**Remark 3.3.** The functions $\mu_a$ are defined via $\mu_0$ which in turn is the homogenization of the spectral invariant $\ell_+$. An equivalent construction of the $\mu_a$ can be achieved by "changing the zero section" in the construction of the spectral invariants. More precisely, for $a \in H^1(N;\mathbb{R})$ let $\alpha \in a$ and let $\Gamma_\alpha$ be the graph of $\alpha$. Define the 1-form $\lambda_\alpha = \lambda - \pi^*\alpha$. Then the Lagrangian submanifold $\Gamma_\alpha$ is exact in $(T^*N, \lambda_\alpha)$ and $\lambda_\alpha$ vanishes on $\Gamma_\alpha$. By replacing the zero section by $\Gamma_\alpha$ we can perform the construction of the Lagrangian spectral invariants in the same fashion. This leads to spectral invariants $\ell_{+,\alpha}$, and homogenization gives rise to the same functions $\mu_a$.

**Remark 3.4.** For the next theorem and future use we introduce Banyaga's fragmentation norm for Hamiltonian diffeomorphisms with respect to a family of open subsets. Let $\mathcal{V}$ be an open covering of $T^*N$. Banyaga's fragmentation lemma [Ba] states that any $\phi \in \mathrm{Ham}(T^*N)$ can be represented as a finite product $\phi = \prod_i \phi_i$, where $\phi_i \in \mathrm{Ham}(V_i)$ for some $V_i \in \mathcal{V}$. The fragmentation norm $\|\phi\|_{\mathcal{V}}$ relative to $\mathcal{V}$ is the minimal number of factors needed to represent $\phi$. If $\mathcal{U}$ is an arbitrary family of open subsets, one can consider the open covering $\mathcal{V}$ consisting of all open subsets $V$ for which there is $\psi \in \mathrm{Ham}(T^*N)$ such that $\psi(V) \in \mathcal{U}$. By $\|\phi\|_{\mathcal{U}}$ we denote the fragmentation norm relative to the covering $\mathcal{V}$ given by the family $\mathcal{U}$.

The next theorem lists the properties of the functions $\mu_a$. Recall the definition of the spectral displacement energy $e(\mathcal{U})$ of a family of subsets $\mathcal{U}$ from Remark 2.11.

**Theorem 3.5.** *Let* $N$ *be a closed connected manifold. For every class* $a \in H^1(N;\mathbb{R})$ *there is a function* $\mu_a\colon \mathrm{Ham}(T^*N) \to \mathbb{R}$ *with the following properties:*

*(i)* $\mu_a$ *is semi-homogeneous, i.e.* $\mu_a(\phi^l) = l\mu_a(\phi)$ *for all* $l \in \mathbb{Z}_{\geq 0}$;

*(ii)* $\mu_a$ *is invariant under conjugation in* $\mathrm{Ham}(T^*N)$;

*(iii) if* $\phi, \psi \in \mathrm{Ham}(T^*N)$ *are generated by the Hamiltonians* $H, K$, *then*

$$\int_0^1 \min(H_t - K_t)\,dt \leq \mu_a(\phi) - \mu_a(\psi) \leq \int_0^1 \max(H_t - K_t)\,dt\,;$$

*in particular,* $\mu_a$ *is Lipschitz continuous with respect to Hofer's metric;*

*(iv) if $U \subset T^*N$ is an open and displaceable subset, then $\mu_a|_{\mathrm{Ham}(U)} = 0$;*

*(v) for any collection $\mathcal{U}$ of open subsets with $e(\mathcal{U}) < \infty$ we have*

$$|\mu_0(\phi\psi) - \mu_0(\psi)| \leq e(\mathcal{U})\, \|\phi\|_{\mathcal{U}} \;;$$

*(vi) if $\phi \in \mathrm{Ham}(T^*N)$ is generated by a Hamiltonian $H$ such that $H = c$ (respectively $H \geq c$, $H \leq c$) when restricted to the graph of a closed 1-form in the class $a$, where $c \in \mathbb{R}$, then $\mu_a(\phi) = c$ (respectively $\mu_a(\phi) \geq c$, $\mu_a(\phi) \leq c$);*

*(vii) for commuting $\phi, \psi$ we have $\mu_a(\phi\psi) \leq \mu_a(\phi) + \mu_a(\psi)$;*

*(viii) for fixed $\phi \in \mathrm{Ham}(T^*N)$ the function $H^1(N;\mathbb{R}) \to \mathbb{R}$, $a \mapsto \mu_a(\phi)$, is Lipschitz continuous, the Lipschitz constant being given by a semi-norm.*

**Remark 3.6.** The functions $\mu_a$ are partial quasi-morphisms in the sense of Definition 1.2: Let $U \subset T^*N$ be an open and displaceable subset and $\phi, \psi \in \mathrm{Ham}(T^*N)$, where $\psi$ is generated by a Hamiltonian whose support is dominated by $U$ (recall from Subsection 1.2.1 that a subset $S \subset T^*N$ is dominated by an open subset $U$ if there is $\varphi \in \mathrm{Ham}(T^*N)$ such that $S \subset \varphi(U)$). For brevity we say that $\psi$ is dominated by $U$. Fix $a \in H^1(N;\mathbb{R})$ and let $\mathcal{U}$ be the family of open subsets which consists of $U$ and all the shifts $T_{-\alpha}(U)$, where $\alpha \in a$. Note that any $U \in \mathcal{U}$ is displaceable. Since the support of $\psi$ is dominated by $U$, we have $\mu_a(\psi) = 0$ according to property (iv), and with property (ii) we conclude $|\mu_a(\phi\psi) - \mu_a(\phi) - \mu_a(\psi)| = |\mu_a(\phi\psi) - \mu_a(\phi)| = |\mu_a(\psi\phi) - \mu_a(\phi)|$. Moreover, the definition of $\mu_a$ and property (v) yield

$$\begin{aligned}|\mu_a(\psi\phi) - \mu_a(\phi)| &= |\mu_0(T_{-\alpha}\psi\phi T_\alpha) - \mu_0(T_{-\alpha}\phi T_\alpha)| \\ &= |\mu_0(T_{-\alpha}\psi T_\alpha T_{-\alpha}\phi T_\alpha) - \mu_0(T_{-\alpha}\phi T_\alpha)| \\ &\leq e(\mathcal{U})\, \|T_{-\alpha}\psi T_\alpha\|_{\mathcal{U}} \;.\end{aligned}$$

Since $\psi$ is dominated by $U$, the diffeomorphism $T_{-\alpha}\psi T_\alpha$ is dominated by $T_{-\alpha}(U)$, and since we assumed $T_{-\alpha}(U) \in \mathcal{U}$, we have $\|T_{-\alpha}\psi T_\alpha\|_{\mathcal{U}} = 1$. Moreover, since the spectral displacement energy is invariant under the symplectomorphisms $T_\alpha$, we have $e(\mathcal{U}) = e(U)$, and thus we conclude

$$|\mu_a(\phi\psi) - \mu_a(\phi) - \mu_a(\psi)| \leq e(U)\,.$$

**Proof.** (i) The claim for $\mu_0$ follows by definition. With $\mu_a(\phi^l) = \mu_0(T_{-\alpha}\phi^l T_\alpha) = \mu_0((T_{-\alpha}\phi T_\alpha)^l)$ we conclude for any $\mu_a$ with the semi-homogeneity of $\mu_0$.
(ii) For the conjugation-invariance of $\mu_a$ we consider $\mu_a(\psi\phi\psi^{-1}) = \mu_0(T_{-\alpha}\psi\phi\psi^{-1}T_\alpha)$ and note that since $\mu_0$ is invariant under conjugation, the latter equals

$$\mu_0(T_{-\alpha}\psi^{-1}T_\alpha T_{-\alpha}\psi\phi\psi^{-1}T_\alpha T_{-\alpha}\psi T_\alpha) = \mu_a(\phi)\,.$$

(iii) We prove the upper bound for $\mu_0$; the rest follows similarly. According to (ii) in Theorem 2.14 we have

$$\ell_+(\phi) - \ell_+(\psi) = \ell_+(H) - \ell_+(K) \leq \int_0^1 \max(H_t - K_t)\, dt\,.$$

In order to pass to $\mu_0$ we need to homogenize and to this end to concatenate Hamiltonians. Let $f$ be a smoothing function as in Remark 2.6 and $H^f, K^f$ the corresponding smoothed Hamiltonians. Then

$$\ell_+(H) - \ell_+(K) = \ell_+(H^f) - \ell_+(K^f) \le \int_0^1 \max(H_t^f - K_t^f)\, dt\,.$$

For any $\varepsilon > 0$ there is such a smoothing function for which

$$\int_0^1 \max(H_t^f - K_t^f)\, dt \le \int_0^1 \max(H_t - K_t)\, dt + \varepsilon\,.$$

It follows that

$$\frac{\ell_+(\phi^k) - \ell_+(\psi^k)}{k} = \frac{\ell_+\big((H^f)^{\sharp k}\big) - \ell_+\big((K^f)^{\sharp k}\big)}{k} \le \int_0^1 \max(H_t - K_t)\, dt + \varepsilon\,,$$

and passing to the limit $k \to \infty$, and then letting $\varepsilon \to 0$, we obtain

$$\mu_0(\phi) - \mu_0(\psi) \le \int_0^1 \max(H_t - K_t)\, dt\,.$$

(iv) Let $U \subset T^*N$ be open and displaceable by $\psi \in \mathrm{Ham}(T^*N)$. By point (vii) in Theorem 2.14 we have

$$|\ell_+(\phi)| \le \Gamma(\psi)$$

for any $\phi \in \mathrm{Ham}(U)$, and thus

$$|\mu_0(\phi)| = \lim_{k\to\infty} \frac{|\ell_+(\phi^k)|}{k} \le \lim_{k\to\infty} \frac{\Gamma(\psi)}{k} = 0\,.$$

To prove the claim for $\mu_a$ it suffices to note that for $a \in H^1(N;\mathbb{R})$ and $\alpha \in a$ we have $T_{-\alpha}\phi T_\alpha \in \mathrm{Ham}(T_{-\alpha}(U))$, where $T_{-\alpha}(U)$ is displaceable by $T_{-\alpha}\psi T_\alpha$.
(v) Let $\mathcal{U}$ be a collection of open subsets with $e(\mathcal{U}) < \infty$ and $\phi, \psi \in \mathrm{Ham}(T^*N)$. Assume $\|\phi\|_{\mathcal{U}} = 1$. Then $\phi$ is generated by a Hamiltonian which has compact support in some $V \in \mathcal{V}$. Thus, by definition of $\mathcal{V}$, the support of the Hamiltonian is dominated by an open subset $U \in \mathcal{U}$. For brevity we say that $\phi$ is dominated by an open subset $U \in \mathcal{U}$. If we define $\phi_j = \psi^j \phi \psi^{-j}$, we have

$$(\phi\psi)^k = \phi_0\phi_1 \ldots \phi_{k-1}\psi^k$$

and point (ix) in Theorem 2.14 implies (using induction)

$$\sum_{j=0}^{k-1} \ell_-(\phi_j) + \ell_+(\psi^k) \le \ell_+((\phi\psi)^k) \le \sum_{j=0}^{k-1} \ell_+(\phi_j) + \ell_+(\psi^k)\,.$$

Every $\phi_j$ is dominated by one of the elements of $\mathcal{U}$ which is displaceable and has spectral displacement energy $\le e(\mathcal{U})$. Thus, with point (vii) of Theorem 2.14 and the invariance of the spectral displacement energy under Hamiltonian diffeomorphisms we conclude that for any $\phi_j$ we have

$$-e(\mathcal{U}) \le \ell_-(\phi_j) \le \ell_+(\phi_j) \le e(\mathcal{U})\,.$$

In summary, we get

$$\left|\ell_+((\phi\psi)^k) - \ell_+(\psi^k)\right| \le ke(\mathcal{U})\,.$$

Dividing by $k$ and taking the limit $k \to \infty$ gives

$$|\mu_0(\phi\psi) - \mu_0(\psi)| \le e(\mathcal{U})\,.$$

The claim follows by induction over $\|\phi\|_{\mathcal{U}}$.

(vi) It suffices to prove the claim for the zero section and $a = 0$. Let $H$ be such that, for example, $H|_N \ge c$. Following the proof of Lemma 2.3 let $K$ be another time-dependent compactly supported Hamiltonian with $H \ge K$ everywhere and $K|_N = c$. In order to conclude for $\mu_0$ we need to concatenate a smoothed version of $H$ with itself. Let $f$ be a smoothing function as in Remark 2.6 and $H^f$ and $K^f$ the corresponding smoothed Hamiltonians. We have $H^f \ge K^f$ since $f' \ge 0$. Moreover, recall that the spectral invariants of $H^f$ and of $K^f$ coincide with those of $H$ and of $K$, respectively. With the above considerations, the continuity of the spectral invariants, and property (vi) in Theorem 2.14 we conclude

$$\ell_+(\phi_H^k) = \ell_+((H^f)^{\sharp k}) \ge \ell_+((K^f)^{\sharp k}) = kc\,.$$

Dividing by $k$ and taking the limit $k \to \infty$ we obtain $\mu_0(H) \ge c$.

(vii) For commuting $\phi$ and $\psi$ we have $(\phi\psi)^k = \phi^k\psi^k$ and the triangle inequality for $\ell_+$ yields

$$\ell_+((\phi\psi)^k) = \ell_+(\phi^k\psi^k) \le \ell_+(\phi^k) + \ell_+(\psi^k)\,.$$

The claim follows by dividing by $k$ and taking the limit $k \to \infty$.

(viii) We have

$$|\mu_a(\phi_H) - \mu_b(\phi_H)| = |\mu_0(T_{-\alpha}\phi_H T_\alpha) - \mu_0(T_{-\beta}\phi_H T_\beta)|\,,$$

where $\alpha \in a, \beta \in b$. The right-hand side is bounded from above by

$$\int_0^1 \|H_t \circ T_\alpha - H_t \circ T_\beta\|_{C^0}\, dt\,.$$

For any 1-form $\chi$ on $N$ we have

$$\max_{T^*N} |H_t - H_t \circ T_\chi| \le |dH_t|(\chi)\,,$$

where we use the notation

$$|dH_t|(\chi) = \max_{(q,p)\in T^*N} \left|\left\langle d_{(q,p)}H_t|_{T^{\mathrm{vert}}_{(q,p)}T^*N}, \chi(q)\right\rangle\right|\,,$$

where we identified $T^{\mathrm{vert}}_{(q,p)}T^*N = T^*_qN$ and $\langle\cdot,\cdot\rangle$ is the pairing between $T^*_qN$ and $T_qN$. It follows that

$$|\mu_a(\phi_H) - \mu_b(\phi_H)| \le |dH|(a-b)\,,$$

where $|dH|\colon H^1(N;\mathbb{R}) \to \mathbb{R}$ is the semi-norm defined by

$$|dH|(a) = \inf_{\alpha\in a} \int_0^1 |dH_t|(\alpha)\, dt\,.$$

This means that $a \mapsto \mu_a(\phi)$ is Lipschitz continuous, the Lipschitz constant being the semi-norm $|dH|$. □

If $N$ is a torus, the partial quasi-morphisms $\mu_a$ are invariant under coverings for any $a \in H^1(\mathbb{T}^n;\mathbb{R}) = \mathbb{R}^n$. More precisely, let $\rho_k\colon T^*\mathbb{T}^n \to T^*\mathbb{T}^n$ be the covering given by $\rho_k(q,p) = (kq,p)$. It allows to pull back Hamiltonian vector fields via $\rho_k^*(X)(z) = (d_z\rho_k)^{-1}(X(\rho_k(z)))$, and thus defines a homomorphism

$$\rho_k^*\colon \operatorname{Ham}(T^*\mathbb{T}^n) \to \operatorname{Ham}(T^*\mathbb{T}^n)$$

which allows to pull back Hamiltonian flows in the following sense: Let $H$ be a time-dependent Hamiltonian generating $\phi \in \operatorname{Ham}(T^*\mathbb{T}^n)$, define $H_k(t,q,p) = H(kt,kq,p)$ and let $\phi_k$ be its time-1 map. The map $\phi_k$ can be viewed as the pull back of $\phi^k$ by the covering $\rho_k$, i.e. $\phi_k = \rho_k^*\phi^k$.

**Proposition 3.7.** *Let $N = \mathbb{T}^n$. Then the partial quasi-morphisms $\mu_a$ are invariant under coverings, that is, for any $k$ and $\phi \in \operatorname{Ham}(T^*\mathbb{T}^n)$ we have $\mu_a(\phi_k) = \mu_a(\phi)$, where $\phi_k \in \operatorname{Ham}(T^*\mathbb{T}^n)$ is defined as above.*

The proof of the proposition is given in Subsection 4.3; it can be extracted from the proof of the equivalence between Viterbo's symplectic homogenization and the partial quasi-morphisms on tori, Theorem 4.13.

**Remark 3.8.** We would like to note that if a certain conjecture due to Viterbo would be true, the partial quasi-morphisms $\mu_a$ would be homogeneous quasi-morphisms when restricted to $\operatorname{Ham}(T_r^*N)$.

In [Vi3] Viterbo claims that there is a certain bound, we refer to it as the *Viterbo bound*, on the Lagrangian spectral invariants for cotangent bundles of tori. In more detail, he claims that the following statement is true: *There is a constant $\kappa > 0$, depending only on the auxiliary Riemannian metric on $\mathbb{T}^n$, such that if $\phi \in \operatorname{Ham}(T_r^*\mathbb{T}^n)$, then*

$$\ell_+(\phi) - \ell_-(\phi) \leq \kappa r\,.$$

Unfortunately, there is a critical error in Viterbo's argument in [Vi3], and there is no other proof for the claim known so far. Thus, for now it is not clear whether the Viterbo bound holds for tori or if there are manifolds at all for which the Viterbo bound holds. But note that if the bound would be true on tori, the effect of reduction on Lagrangian spectral invariants, as in [Vi1], would imply that it would hold on manifolds which admit a finite connected covering which is a torus.

Although we do not know whether there are manifolds $N$ for which the Viterbo bound holds, we would like to point out that on such $N$ the functions $\ell_\pm|_{\operatorname{Ham}(T_r^*N)}$ would be genuine quasi-morphisms of defect $\leq \kappa r$ since the Viterbo bound would imply $\ell_+(\phi) + \ell_+(\psi) - \kappa r \leq \ell_+(\phi\psi)$ and $\ell_-(\phi) + \ell_-(\psi) + \kappa r \geq \ell_-(\phi\psi)$ for any $\phi \in \operatorname{Ham}(T^*N)$ and $\psi \in \operatorname{Ham}(T_r^*N)$. Therefore, the restricted function $\mu_0|_{\operatorname{Ham}(T_r^*N)}$ would be a homogeneous quasi-morphism with defect $\leq 2\kappa r$ (every quasi-morphism gives rise to a homogeneous one where the defect is increased by a factor at most 2, see [Ca]).

Moreover, for any $\varepsilon > 0$ there is $\alpha \in a$ whose graph is contained in $T^*_{<\|a\|+\varepsilon}N$, where $\|a\|$ denotes the dual of the Gromov-Federer stable norm on $H^1(N;\mathbb{R})$ which is expressible as

$$\|a\| = \inf_{\alpha\in a}\max_{q\in N}\|\alpha(q)\|$$

for $a \in H^1(N;\mathbb{R})$, see [Gro], [PPS]. Thus, we have $T_{-\alpha}\phi T_\alpha \in \mathrm{Ham}(T^*_{r+\|a\|+\varepsilon}N)$ for any $\phi \in \mathrm{Ham}(T^*_r N)$, and we can use the bound for $\mu_0$ to conclude that any $\mu_a|_{\mathrm{Ham}(T^*_r N)}$ would be a homogeneous quasi-morphism of defect $\leq 2\kappa(r + \|a\|)$.

Recall from Subsection 2.5 that we defined the Lagrangian spectral invariants $\ell(\alpha, H : M)$ for Hamiltonians with complete flow, and that the extended invariants share the properties of the usual ones (but actually depend on the Hamiltonian $H$ and not only on its time-1 map). Therefore, we can extend the definition of the functions $\mu_a$ to Hamiltonians with complete flow and these extended functions share the properties of the usual ones listed in Theorem 3.5. Since the values of the extended functions depend on the Hamiltonian $H$ and not just on its time-1 map, we use the notation $\mu_a(H)$. Recall the definition of the direct sum $H \oplus H'$ of two time-dependent Hamiltonians $H$ and $H'$ given in Subsection 2.6 and note that whenever $H, H'$ have complete flows, so does $H \oplus H'$. The product formula for the Lagrangian spectral invariants stated in Theorem 2.19 gives, together with homogenization and shifting by $T_\alpha$ for appropriate 1-forms $\alpha$, the following product formula for the partial quasi-morphisms

**Proposition 3.9.** *Let $N'' = N \times N'$. For $a \in H^1(N;\mathbb{R})$ and $a' \in H^1(N';\mathbb{R})$ let $\mu_a, \mu'_{a'}$ be the partial quasi-morphisms on $T^*N, T^*N'$, respectively, extended to the set of Hamiltonians with complete flow. Let $H, H'$ be time-dependent Hamiltonians with complete flow on $T^*N$, $T^*N'$, respectively. Then we have*

$$\mu_{a''}(H \oplus H') = \mu_a(H) + \mu'_{a'}(H')\,,$$

*where $a'' = (a, a') \in H^1(N;\mathbb{R}) \times H^1(N';\mathbb{R}) \subset H^1(N'';\mathbb{R})$.* □

## 3.2 Partial symplectic quasi-integrals

In this subsection we show that the partial quasi-morphisms $\mu_a$: $\mathrm{Ham}(T^*N) \to \mathbb{R}$ yield a family of functionals $\zeta_a$: $C^\infty_c(T^*N) \to \mathbb{R}$ via $\zeta_a(F) = \mu_a(\phi_F)$, where each functional has properties analogous to those of a partial symplectic quasi-integral. Recall from Subsection 1.2.1 that a subset $S \subset T^*N$ is dominated by an open subset $U$ if there is $\varphi \in \mathrm{Ham}(T^*N)$ such that $S \subset \varphi(U)$.

**Theorem 3.10.** *For any $a \in H^1(N;\mathbb{R})$ the functional $\zeta_a$: $C^\infty_c(T^*N) \to \mathbb{R}$ defined as $\zeta_a(F) = \mu_a(\phi_F)$ has the following properties:*

*(i)* *$\zeta_a$ is semi-homogeneous, i.e. $\zeta_a(\lambda F) = \lambda\zeta_a(F)$ for $\lambda \in \mathbb{R}_{\geq 0}$;*

*(ii)* *$\zeta_a$ is invariant under the natural action of* $\mathrm{Ham}(T^*N)$ *on $C^\infty_c(T^*N)$, i.e. $\zeta_a(F \circ \phi) = \zeta_a(F)$ for any $\phi \in \mathrm{Ham}(T^*N)$;*

*(iii)* *we have*

$$\min(F - G) \leq \zeta_a(F) - \zeta_a(G) \leq \max(F - G)\,,$$

*in particular, $\zeta_a$ is Lipschitz continuous with respect to the $C^0$-norm, i.e. $|\zeta_a(F) - \zeta_a(G)| \leq \|F - G\|_{C^0}$;*

*(iv)* *$\zeta_a$ is monotone, i.e. $\zeta_a(F) \leq \zeta_a(G)$ for $F \leq G$;*

*(v)* $\zeta_a(F) = 0$ *for any* $F$ *with displaceable support;*

*(vi)* *for any open and displaceable subset* $U \subset T^*N$*, any* $F$*, and* $G$ *with support dominated by* $U$ *we have*

$$|\zeta_a(F+G) - \zeta_a(F) - \zeta_a(G)| = |\zeta_a(F+G) - \zeta_a(F)| \le \sqrt{2e(U)\,\|\{F,G\}\|_{C^0}}\,,$$

*and in particular*

*(vii)* *if* $F, G$ *Poisson commute and the support of* $G$ *is displaceable, then*

$$\zeta_a(F+G) = \zeta_a(F) + \zeta_a(G) = \zeta_a(F)\,;$$

*(viii)* *if* $F = c$ *(respectively* $F \ge c$*,* $F \le c$*) when restricted to the graph of a closed 1-form in the class* $a$*, then* $\zeta_a(F) = c$ *(respectively* $\zeta_a(F) \ge c$*,* $\zeta_a(F) \le c$*);*

*(ix)* *if* $F, G$ *Poisson commute, then* $\zeta_a(F+G) \le \zeta_a(F) + \zeta_a(G)$*.*

**Proof.** (i) By semi-homogeneity of $\mu_a$, the identity $\zeta_a(\lambda F) = \lambda\zeta_a(F)$ is obtained for natural $\lambda$, then for all positive rational $\lambda$, and finally, using Lipschitz continuity, for all $\lambda \ge 0$. Points (ii),(iii),(iv),(v),(viii),(ix) are immediate consequences of the properties of the partial quasi-morphisms $\mu_a$. Point (vii) is a consequence of (vi). The proof of (vi) repeats verbatim the proof of Theorem 1.8 in [EPZ]. In fact, the proof in [EPZ] gives $|\zeta_a(F+G) - \zeta_a(F) - \zeta_a(G)| \le \sqrt{2C\,\|\{F,G\}\|_{C^0}}$, where $C$ is such that $|\mu_a(\phi_F\phi_G) - \mu_a(\phi_F) - \mu_a(\phi_G)| \le C$. According to the above discussion we have $C = e(U)$ if the support of $G$ is dominated by an open and displaceable subset $U$. □

**Remark 3.11.** Since $\zeta_a$ is Lipschitz continuous in the $C^0$-norm, it admits a unique extension to the space $C_c(T^*N)$ of continuous functions with compact support which is a partial symplectic quasi-integral in the sense of Definition 1.4. However, we refer to the functionals $\zeta_a\colon C_c^\infty(T^*N) \to \mathbb{R}$ as partial symplectic quasi-integrals as well.

Similarly to the partial quasi-morphisms $\mu_a$ we can define the partial symplectic quasi-integrals $\zeta_a$ on autonomous Hamiltonians with complete flow. For these we have the following product formula which follows directly from the one for the partial quasi-morphisms $\mu_a$.

**Proposition 3.12.** *Let* $N'' = N \times N'$ *and for* $a \in H^1(N;\mathbb{R})$ *and* $a' \in H^1(N';\mathbb{R})$ *let* $\zeta_a, \zeta'_{a'}$ *be the partial symplectic quasi-integrals on* $T^*N, T^*N'$*, respectively, extended to the set of autonomous Hamiltonians with complete flow. Let* $F, F'$ *be Hamiltonians with complete flow on* $T^*N, T^*N'$*, respectively. Then we have*

$$\zeta_{a''}(F \oplus F') = \zeta_a(F) + \zeta'_{a'}(F')\,,$$

*where* $a'' = (a, a') \in H^1(N;\mathbb{R}) \times H^1(N';\mathbb{R}) \subset H^1(N'';\mathbb{R})$*.* □

**Remark 3.13.** Recall from Subsection 2.3 the definition of the Hamiltonian spectral invariant $c_+\colon \mathrm{Ham}(T^*N) \to \mathbb{R}$ . Since $c_+$ satisfies the triangle inequality, it can be homogenized to yield a function $\nu\colon \mathrm{Ham}(T^*N) \to \mathbb{R}$, and the latter can be used to define a function $\eta\colon C_c^\infty(T^*N) \to \mathbb{R}$. These two functions $\nu$ and $\eta$ coincide with Lanzat's functions on $T^*N$ which are defined in [La] via quantum and Floer homology, and they enjoy properties analogous to those of the partial quasi-morphisms $\mu_a$ and the partial symplectic quasi-integrals $\zeta_a$. Moreover, using the comparison described in Proposition 2.12 we can conclude $\mu_a \leq \nu$ and $\zeta_a \leq \eta$ for all $a \in H^1(N;\mathbb{R})$. Lanzat actually uses this relation between $\nu$ and $\mu_a$ in order to prove that $\nu$ is a genuine partial quasi-morphism (and not a homogeneous quasi-morphism) on $\mathrm{Ham}(T^*N)$.

# 4 Symplectic homogenization

Symplectic homogenization on cotangent bundles of tori is originally due to Viterbo [Vi4]. The construction of a symplectic notion of homogenization was thereby motivated by the classical one of homogenization in the sense that Viterbo associates a "rescaled" Hamiltonian $H_k(t,q,p) = H(kt,kq,p)$ to a Hamiltonian $H \in C_c^\infty([0,1] \times T^*\mathbb{T}^n)$ and asks, whether $H_k$ converges to some Hamiltonian $\widehat{H}$ which only depends on the fiber variable $p$. Thereby, the convergence is understood in the way that the limit of the time-1 maps of $H_k$ with respect to Viterbo's metric is generated by $\widehat{H}$ in a certain precise sense.

In fact, in [Vi4] Viterbo defines symplectic homogenization as an operator

$$\mathcal{H}\colon C_c^\infty([0,1] \times T^*\mathbb{T}^n) \to C_c(\mathbb{R}^n)$$

which indeed sends a Hamiltonian $H$ to a continuous function $\widehat{H}$ which only depends on the fiber variable and which is related to the limit of the rescaled Hamiltonian $H_k$. The operator is constructed in terms of Lagrangian spectral invariants from generating functions. More precisely, in [Vi4] Viterbo claims that the symplectic homogenization operator $\mathcal{H}$ exists and that it has various nice properties which include, for instance, a notion of convergence of the time-1 maps of $H_k$ to the time-1 map of $\widehat{H}$.

In this section we give an alternative description of Viterbo's symplectic homogenization operator. We prove that it is equivalent to the partial quasi-morphisms $\mu_a$ introduced above. Using this equivalence on $T^*\mathbb{T}^n$ we can give a definition of symplectic homogenization on cotangent bundles $T^*N$, where $N$ is any closed connected manifold, and conclude the properties of symplectic homogenization from the ones of the partial quasi-morphisms $\mu_a$.

Moreover, we will point out the resulting effects of the fact that Viterbo uses his unproven bound (see Remark 3.8) in the construction of symplectic homogenization and in the proofs of some of its properties. For now let us just mention that the existence of the symplectic homogenization operator can be proved without the Viterbo bound. The latter is just needed in the proof of some properties of symplectic homogenization, for instance, in the proof of the convergence of the time-1 maps. This observation is consistent with the equivalence between symplectic homogenization and the partial quasi-morphisms $\mu_a$. The equivalence can be seen as a way to define symplectic homogenization, and the properties of symplectic homogenization, which in this case are just the adapted ones of the partial quasi-morphisms $\mu_a$, are exactly the ones in whose proofs Viterbo does not use his bound.

To be able to state and prove the equivalence between Viterbo's symplectic homogenization operator and the partial quasi-morphisms, we give a short overview of the definition of Lagrangian spectral invariants from generating functions in Subsection 4.1, and state a result of Milinković and Oh describing an equality between spectral invariants coming from Floer homology and generating function homology in Subsection 4.1.1. The effect of the fact that we use a different sign convention than Milinković and Oh is discussed in Subsection 4.1.2. In Subsection 4.2 we give an overview about Viterbo's approach to symplectic homogenization, in Subsection 4.3 we formulate the equivalent description of symplectic homogenization using the partial quasi-morphisms coming from Lagrangian spectral invariants and define symplectic homogenization for more general cotangent bundles. The equivalence is proved in Subsection 4.3.1.

## 4.1 Spectral invariants from generating functions

According to Viterbo one can define spectral invariants associated to a Lagrangian submanifold Hamiltonian isotopic to the zero section of a cotangent bundle via a finite-dimensional approach [Vi1]. These Lagrangian spectral invariants are constructed using the homology theory of generating functions quadratic at infinity. In this subsection we briefly recall Viterbo's construction of these spectral invariants; the general reference for this is [Vi1]. Moreover, we review the relation between the Lagrangian spectral invariants coming from generating functions and from Floer homology given by Milinković and Oh in [MO1], [MO2], and discuss the effect of the different sign conventions.

Consider a closed connected $n$-dimensional manifold $N$ and the symplectic manifold $(T^*N, \omega = d\lambda)$, where $\lambda = p\,dq$ is the Liouville form. A generating function quadratic at infinity, or gfqi for short, is a function

$$S\colon N(q) \times E(\xi) \to \mathbb{R}\,,$$

where $E$ is a finite-dimensional vector space, such that $\|\partial_\xi S - \partial_\xi Q\|_{C^0}$ is bounded, where $Q\colon E \to \mathbb{R}$ is a non-degenerate quadratic form.

If $L \subset T^*N$ is a Lagrangian submanifold Hamiltonian isotopic to the zero section, a gfqi $S\colon N \times E \to \mathbb{R}$ is said to generate $L$ if the map $(q;\xi) \mapsto \partial_\xi S(q;\xi)$ has 0 as a regular value, and the map $i_S\colon \Sigma_S \to T^*N$ given by $(q;\xi) \mapsto (q, \partial_q S(q;\xi))$ has image $i_S(\Sigma_S) = L$, where $\Sigma_S = \{(q;\xi) \in N \times E \,|\, \partial_\xi S(q;\xi) = 0\}$ is a compact submanifold in $N \times E$. If $S$ is a gfqi for $L$, the critical points of $S$ are in one-to-one correspondence to the intersection points of $L$ and $\mathcal{O}_N$. The definition of generating functions for Lagrangian submanifolds was thereby motivated by the simple example that the graph of the differential of a smooth function $F \in C_c^\infty(N)$, that is, the set $\Gamma_{dF} = \{(x, dF(x)) \,|\, x \in N\}$, is a Lagrangian submanifold in $T^*N$ which is generated by $F$. In general, every Lagrangian submanifold Hamiltonian isotopic to the zero section admits a gfqi [Sik], and such a gfqi is unique up to the following elementary operations: addition of a non-degenerate quadratic form (*stabilization*), application of a fiber-preserving diffeomorphism (*gauge transformation*) and addition of a constant (*translation*) [Vi1], [Th].

Let $S\colon N \times E \to \mathbb{R}$ be a gfqi, $S^a = \{(q;\xi) \,|\, S(q;\xi) \le a\}$, and let $E = E^+ \oplus E^-$ be the splitting into the positive and negative eigenspaces of the quadratic form $Q$. The relative homology $H_*(S^a, S^b)$ is independent of $a$ and $b$ for sufficiently large $a$ and sufficiently small $b$, and we denote this group by $H_*(S : N)$. It is canonically isomorphic to $H_*(N) \otimes H_*(E^-, E^- \setminus \{0\}) \simeq H_{*+d}(N)$, where $d = \dim E^-$ and the last isomorphism ("Thom isomorphism") is given by tensoring with the generator of $H_d(E^-, E^- \setminus \{0\}) \simeq \mathbb{Z}_2$. There is a natural inclusion

$$i^b\colon H_*(S^b) \to H_*(S : N)$$

and to each $A \in H_*(N)$ one can associate the spectral invariant

$$\ell(A, S) = \inf\{b \,|\, A \in \operatorname{im} i^b\}\,.$$

**Notation 4.1.** We denote

$$\ell_+(S) = \ell([N], S) \text{ and } \ell_-(S) = \ell(\mathrm{pt}, S)$$

for generators $[N] \in H_n(N)$ and $\mathrm{pt} \in H_0(N)$.

If $M \subset N$ is a closed submanifold, one can consider the restriction $S|_{M\times E}$ as a gfqi with base $M$ and define spectral invariants $\ell(A, S|_{M\times E})$ associated to homology classes in $H_*(M)$.

In case $S\colon N \times E \to \mathbb{R}$ is a gfqi of a Lagrangian submanifold $L \subset T^*N$ Hamiltonian isotopic to the zero section, it is unique up to gauge transformation, stabilization and translation. With the exception of translation, the elementary operations do not alter the spectral invariants. So one can say that the spectral invariants are attached to the Lagrangian submanifold $L$ and that they are all defined up to simultaneous addition of a constant.

### 4.1.1 Relation between spectral invariants from Lagrangian Floer homology and from generating functions

In [MO1], [MO2] Milinković and Oh prove that the spectral invariants coming from Lagrangian Floer homology and those coming from generating functions coincide if the generating function is suitably normalized; this is needed to relate symplectic homogenization and the partial quasi-morphisms in the sequel. In this subsection we review the comparison of the spectral invariants due to Milinković and Oh. Thereby, we follow their sign convention which differs from ours, and denote objects defined with their convention (in this subsection as well as in the rest of this work) with the overline. In particular, we denote by $\overline{\ell}(A, H : M)$ the Lagrangian spectral invariants in the sign convention of Milinković and Oh. The effect of the different conventions regarding spectral invariants is discussed in Subsection 4.1.2.

The above definition of a generating function is a particular case of the general definition of a generating function [Vi1] as a function $S\colon E \to \mathbb{R}$ defined on the total space of a submersion $\pi\colon E \to N$. If $S\colon E \to \mathbb{R}$ is a generating function for a Lagrangian embedding $L \subset T^*N$, the set $\Sigma_S := \{e \in E \,|\, \partial_\xi S(e) = 0\}$ is a smooth manifold, and there exists a Lagrangian embedding $i_S\colon \Sigma_S \to T^*N$ given by $e \mapsto (\pi(e), dS(e))$ such that $i_S(\Sigma_S) = L$. In this case it is true that $i_S^*\lambda = d(S|_{\Sigma_S})$. Thus, $S$ induces a function, denoted by $S|L$, on the image of the embedding via the formula $S|L := S \circ (i_S)^{-1}\colon L \to \mathbb{R}$. The differential $d(S|L)$ coincides with $\lambda|_L$ and two generating functions of $L$ induce functions on $L$ whose difference is constant.

**Remark 4.2.** The action functional $\overline{\mathcal{A}}_H(\gamma) = -\int_0^1 H_t(\gamma(t))\,dt + \int \gamma^*\lambda$ defined on the path space $\Omega(N)$ is a particular example of a generating function (see [Oh1]); the submersion is given by $\Omega(N) \to N$, $\gamma \mapsto \gamma(1)$. The action functional $\overline{\mathcal{A}}_H$ generates $\phi_H(N)$. This observation, which is originally due to Weinstein, lead Oh to define his Lagrangian spectral invariants using Lagrangian Floer homology relative to the action functional $\overline{\mathcal{A}}_H(\gamma)$.

The relation between the Lagrangian spectral invariants is given by the following lemma which is essentially contained in [MO1], [MO2]:

**Lemma 4.3.** *Let $L \subset T^*N$ be a Lagrangian submanifold Hamiltonian isotopic to the zero section and $H \in C_c^\infty([0,1] \times T^*N)$ a time-dependent compactly supported Hamiltonian such that $\phi_H(N) = L$. Let $S\colon N \times E \to \mathbb{R}$ be a gfqi of $L$. If the induced functions $\overline{\mathcal{A}}_H|L$ and $S|L$ coincide, then*

$$\bar{\ell}(A, H : M) = \ell(A, S|_{M\times E})$$

*for any closed connected submanifold $M \subset N$ and any $A \in H_*(M)$.*

**Remark 4.4.** The proof is essentially contained in [MO1], [MO2] but since Milinković and Oh use a normalization which is in terms of wavefronts, we describe their proof and their normalization in order to see that our normalization equals theirs.

**Remark 4.5.** A gfqi $S$ of $L$ determines a wavefront of $L$ given by

$$W_S := \{(\pi(p), (S|L)(p)) \in N \times \mathbb{R} \,|\, p \in L\}\,,$$

where $\pi\colon T^*N \to N$ denotes the canonical projection. Similarly, when a Hamiltonian $H$ generates $L$ in the sense that $\phi_H(N) = L$, it determines a wavefront of $L$ by

$$W_H := \{(\pi(p), (\mathcal{A}_H|L)(p)) \in N \times \mathbb{R} \,|\, p \in L\}\,.$$

If $S$ and $H$ generate the same Lagrangian $L$, the two wavefronts are the same up to a vertical translation.

**Proof.** Milinković and Oh define an "action" functional $\overline{\mathcal{A}}_{H,S}$ on a space of paths with Lagrangian boundary conditions (relative to a closed connected submanifold $M \subset N$) which generates the Lagrangian submanifold $\phi_H(L_S)$, where $S$ is a gfqi of a Lagrangian submanifold $L_S$ and $H$ a Hamiltonian. In particular, if $H = 0$, $\overline{\mathcal{A}}_{0,S}$ generates $L_S$, and if $S = Q$, where $Q$ is a non-degenerate quadratic form, $\overline{\mathcal{A}}_{H,Q}$ generates $\phi_H(N)$. According to Milinković and Oh, the Floer homology of the action functional $\overline{\mathcal{A}}_{H,S}$ is well-defined, isomorphic to $H_*(M)$, and one can extract spectral invariants $\sigma(A, H, S : M)$ for any $A \in H_*(M)$. In case $S = Q$, these spectral invariants coincide with those of $\overline{\mathcal{A}}_H$, i.e. $\sigma(A, H, Q : M) = \bar{\ell}(A, H : M)$, and in case $H = 0$, they coincide with those of $S$, i.e. $\sigma(A, 0, S : M) = \ell(A, S|_{M\times E})$.

If $S$ is a gfqi for $L = \phi_H(N)$, one can consider a continuous family (the existence of such a family is proved *ibid.*) of gfqi $S(t)$ of $\phi_H^t(L)$, where $S_0 = S$ and $S_1 = Q$, and Hamiltonians $H(t)$ such that $\phi^1_{H(t)} = \phi^t_H$. If $L_{S(t)}$ denotes the Lagrangian generated by $L(t)$, one can calculate $\phi_{H(t)}(L_{S(t)}) = L$, and thus the action functional $\overline{\mathcal{A}}_{H(t),S(t)}$ generates the fixed Lagrangian $L$. In this situation Milinković and Oh show that if the isotopy $(H(t), S(t))$ is normalized such that the wavefront of $\overline{\mathcal{A}}_{H(t),S(t)}$ remains fixed as $t$ varies, the spectral invariants $\sigma(A, H(t), S(t) : M)$ are independent of $t$, and thus

$$\ell(A, S|_{M\times E}) = \sigma(A, 0, S : M) = \sigma(A, H, Q : M) = \bar{\ell}(A, H : M)\,.$$

By definition of $S(t)$ and $H(t)$ we see that the wavefront of $\overline{\mathcal{A}}_{0,S}$, which is the wavefront of $S$, and the one of $\overline{\mathcal{A}}_{H,Q}$, which is the wavefront of $H$, coincide. By our normalization, the wavefronts of $H$ and $S$ coincide by assumption which means that there is a unique continuous choice of normalization for the isotopy $(H(t), S(t))$, defined by these wavefronts. It follows that our normalization equals the normalization of Milinković and Oh, and the lemma follows from [MO1], [MO2]. □

### 4.1.2 Sign conventions

In this subsection we finally discuss the effect of the different sign conventions regarding spectral invariants. Recall that we denote objects defined with the sign conventions of Milinković and Oh with the overline, with the exception of $\overline{H}$, which we reserve for the "reversed" Hamiltonian.

Our sign convention follows the philosophy that the Floer theory of the action functional is a perturbation of the Morse theory of a function on a closed manifold, in particular, the Hamiltonian enters the action functional with a positive sign.

Let $H$ be a compactly supported time-dependent Hamiltonian on $T^*N$. The action functional $\overline{\mathcal{A}}_H \colon \Omega(M) \to \mathbb{R}$ is defined as

$$\overline{\mathcal{A}}_H(\gamma) = -\int_0^1 H_t(\gamma(t))\,dt + \int \gamma^*\lambda = -\mathcal{A}_H(\gamma)\,.$$

The symplectic form $\overline{\omega} = -d\lambda = -dp\wedge dq = -\omega$. The Hamiltonian vector field $\overline{X}_H$ is defined by the equation $\overline{\omega}(\overline{X}_H,\cdot) = dH$ and so $\overline{X}_H = X_H$. In particular, the flows in the two sign conventions coincide.

If $H$ is regular, that is $\phi_H(N)$ intersects $\nu^*M$ transversely, we have $\operatorname{Crit}(\overline{\mathcal{A}}_H : M) = \operatorname{Crit}(H : M)$, while the action spectrum is flipped: $\operatorname{Spec}(\overline{\mathcal{A}}_H : M) = -\operatorname{Spec}(H : M)$. Milinković and Oh use the negative gradient flow of $\overline{\mathcal{A}}_H$ to produce the Floer equation. In their sign conventions, an almost complex structure $\overline{J}$ is compatible with $\overline{\omega}$ if $\overline{\omega}(\cdot, \overline{J}\cdot)$ is a Riemannian metric. This is the case if and only if the almost complex structure $J = -\overline{J} = (\overline{J})^{-1}$ is compatible with $\omega$ in our sense. Therefore, their negative gradient flow corresponds to our positive gradient flow. It follows that there is a canonical identification

$$\overline{\mathcal{M}}(\gamma_+, \gamma_-) = \mathcal{M}(\gamma_-, \gamma_+)$$

for $\gamma_\pm \in \operatorname{Crit}(H : M)$. Consequently, their Floer boundary operator is the dual of ours. Their convention for the Conley-Zehnder index is $\overline{m}_{H:M}(\gamma) = \dim M - m_{H:M}(\gamma)$ for $\gamma \in \operatorname{Crit}(H : M)$. Thus, their Floer complex

$$(\overline{CF}_*(H : M), \overline{\partial}_{H:M})$$

is canonically isomorphic to

$$(CF^{\dim M - *}(H : M), (\partial_{H:M})^*)\,,$$

and so the homology they obtain is in fact the singular cohomology $H^{\dim M - *}(M)$ which by Poincaré duality is isomorphic to $H_*(M)$. Using this latter identification and the fact that $\overline{\partial}$ decreases the action $\overline{\mathcal{A}}_H$, they define spectral invariants by the usual recipe; we denote them by $\overline{\ell}(A, H : M)$ for $A \in H_*(M)$.

We will only need the relation between $\ell$ and $\overline{\ell}$ in the case $M = N$. By Poincaré duality described in Subsection 2.2.2 we have

$$(\overline{CF}_*(H : N), \overline{\partial}_{H:N}) = (CF^{n-*}(H : N), (\partial_{H:N})^*) = (CF_*(\overline{H} : N), \partial_{\overline{H}:N})\,.$$

Since $\overline{\mathcal{A}}_H(\gamma) = \mathcal{A}_{\overline{H}}(\overline{\gamma})$, the action filtration on $(\overline{CF}_*(H : M), \overline{\partial}_{H:M})$ induced by $\overline{\mathcal{A}}$ coincides with the filtration on $(CF_*(\overline{H} : M), \partial_{\overline{H}:M})$ induced by $\mathcal{A}_{\overline{H}}$. Thus, the relation between the spectral invariants is given by

$$\overline{\ell}(\mathrm{pt}, H) = \ell(\mathrm{pt}, \overline{H}) = -\ell([N], H) \quad \text{and} \quad \overline{\ell}([N], H) = \ell([N], \overline{H}) = -\ell(\mathrm{pt}, H)\,.$$

Using the above notation we can write these relations as

$$\overline{\ell}_{\pm}(H) = -\ell_{\mp}(H)\,.$$

## 4.2 Viterbo's approach to symplectic homogenization

Viterbo's symplectic homogenization is an operator which sends a compactly supported time-dependent Hamiltonian $H \in C_c^\infty([0,1] \times T^*\mathbb{T}^n)$ to a continuous function $\widehat{H}$ which is related to the rescaled Hamiltonian $H_k(t,q,p) = H(kt,kq,p)$ and only depends on the fiber variable $p$. To define the symplectic homogenization $\widehat{H}$ of $H$, Viterbo constructs a function $h_k'\colon \mathbb{R}^n \to \mathbb{R}$ via spectral invariants from a generating function of a Lagrangian submanifold which is obtained from $H_k$ in an appropriate cotangent bundle. Viterbo then proves that the sequence $(h_k'(p))$ converges to a continuous function for fixed $p$, and he defines $\widehat{H}(p) = \lim_{k\to\infty} h_k'(p)$. The general reference for Viterbo's construction, which we describe in more detail below, is Viterbo's unpublished manuscript [Vi4].

Let $H \in C_c^\infty([0,1] \times T^*\mathbb{T}^n)$ be a compactly supported time-dependent Hamiltonian and denote by $\phi^t = \phi_H^t$ the Hamiltonian isotopy generated by $H$ and by $\phi = \phi^1$ its time-1 map. Define $H_k \in C_c^\infty([0,1] \times T^*\mathbb{T}^n)$ as [5)]

$$H_k(t,q,p) := H(kt,kq,p)\,,$$

and denote by $\phi_k^t = \phi_{H_k}^t$ its flow and by $\phi_k = \phi_k^1$ its time-1 map.

The Lagrangian submanifold associated to $H_k$ is constructed in terms of the time-1 map $\phi_k$ of $H_k$ which is related to the time-1 map $\phi$ of $H$. Namely, consider the covering map

$$\rho_k\colon T^*\mathbb{T}^n \to T^*\mathbb{T}^n$$

given by $\rho_k(q,p) = (kq,p)$. The covering $\rho_k$ allows to pull back Hamiltonian vector fields via $\rho_k^*(X)(z) = (d_z\rho_k)^{-1}(X(\rho_k(z)))$ and thus it defines a homomorphism $\rho_k^*\colon \operatorname{Ham}(T^*\mathbb{T}^n) \to \operatorname{Ham}(T^*\mathbb{T}^n)$. The time-1 map of $H_k$ is given by

$$\phi_k = \rho_k^* \phi^k\,.$$

To associate a Lagrangian submanifold to the time-1 map $\phi_k$ of $H_k$ which is Hamiltonian isotopic to the zero section of an appropriate cotangent bundle, one can follow a particular construction which we describe in the following remark.

**Remark 4.6.** For any $H \in C_c^\infty([0,1] \times T^*\mathbb{T}^n)$ the graph $\Gamma_{\phi^t}$ of $\phi^t$ is a Lagrangian submanifold in $(T^*\mathbb{T}^n \times \overline{T^*\mathbb{T}^n}, \omega \oplus -\omega)$, where the overline $\overline{T^*\mathbb{T}^n}$ indicates that the symplectic form is the negative of the usual one. In particular, $\Gamma_{\phi^t}$ is given by the image of the diagonal $\Delta = \Delta_{T^*\mathbb{T}^n} = T^*\mathbb{T}^n$ in $T^*\mathbb{T}^n \times \overline{T^*\mathbb{T}^n}$ under the Hamiltonian isotopy

$$\Phi^t = \operatorname{id} \times \phi^t\colon T^*\mathbb{T}^n \times \overline{T^*\mathbb{T}^n} \to T^*\mathbb{T}^n \times \overline{T^*\mathbb{T}^n}\,,$$

i.e. $\Phi^t(\Delta) = \Gamma_{\phi^t}$. Note that the Hamiltonian isotopy $\Phi^t$ is generated by the Hamiltonian $K_t = 0 \oplus (-H_t)$ which means $K_t(z,z') = -H_t(z')$ for $z, z' \in T^*\mathbb{T}^n$.

[5)] Formally, one should assume that $H$ is time-periodic for this to make sense but we suppress such considerations below.

Consider the symplectic covering

$$\tau \colon T^*\Delta \to T^*\mathbb{T}^n \times \overline{T^*\mathbb{T}^n}$$

given by $\tau(u, v; U, V) = (u - V, v; u, v - U)$ which sends the zero section $\mathcal{O}_\Delta$ diffeomorphically onto $\Delta$. Here $(u, v)$ are the coordinates on the diagonal $\Delta = T^*\mathbb{T}^n = \mathbb{T}^n(u) \times \mathbb{R}^n(v)$ and $(U, V)$ are the dual coordinates on cotangent fibers $T_{(u,v)}\Delta = T_{(u,v)}T^*\mathbb{T}^n = T^*_{(u,v)}(\mathbb{T}^n \times \mathbb{R}^n) = T^*_u\mathbb{T}^n(U) \times T^*_v\mathbb{R}^n(V)$. All coordinates correspond to the splitting $T^*\Delta = T^*(\mathbb{T}^n \times \mathbb{R}^n) = \mathbb{T}^n(u) \times \mathbb{R}^n(v) \times \mathbb{R}^n(U) \times \mathbb{R}^n(V)$. On $T^*\mathbb{T}^n \times \overline{T^*\mathbb{T}^n} = \mathbb{T}^n(q) \times \mathbb{R}^n(p) \times \overline{\mathbb{T}^n(Q) \times \mathbb{R}^n(P)}$ the coordinates are given by $(q, p; Q, P)$. The symplectic form on $T^*\Delta$ is $\omega^{T^*\Delta} = dU \wedge du + dV \wedge dv$ and the symplectic form on $T^*\mathbb{T}^n \times \overline{T^*\mathbb{T}^n}$ is $\omega^{T^*\mathbb{T}^n} \oplus -\omega^{T^*\mathbb{T}^n} = dp \wedge dq - dP \wedge dQ$. A calculation shows

$$\tau^*(\omega^{T^*\mathbb{T}^n} \oplus -\omega^{T^*\mathbb{T}^n}) = \omega^{T^*\Delta}\,,$$

and therefore $\tau$ is indeed symplectic. Define the Hamiltonian $\widetilde{H}_t = K_t \circ \tau$ on $T^*\Delta$. It generates a Hamiltonian isotopy (which has no longer compact support)

$$\widetilde{\Phi}^t \colon T^*\Delta \to T^*\Delta$$

such that $\Phi^t \circ \tau = \tau \circ \widetilde{\Phi}^t$. The image of the zero section $\mathcal{O}_\Delta$ under $\widetilde{\Phi}^t$ is a Lagrangian submanifold in $T^*\Delta$,

$$\widetilde{\Phi}^t(\mathcal{O}_\Delta) =: L(t) \subset T^*\Delta\,.$$

In particular, this Lagrangian submanifold maps diffeomorphically onto the graph of $\phi^t$ under $\tau$, i.e. $\tau(L(t)) = \Gamma_{\phi^t}$.

Following the above construction for $\phi_k$ one can consider

$$\Phi_k := \mathrm{id} \times \phi_k \colon T^*\mathbb{T}^n \times \overline{T^*\mathbb{T}^n} \to T^*\mathbb{T}^n \times \overline{T^*\mathbb{T}^n}$$

and its lift

$$\widetilde{\Phi}_k \colon T^*\Delta \to T^*\Delta\,.$$

The image of the zero section $\mathcal{O}_\Delta$ under $\widetilde{\Phi}_k$ is a Lagrangian submanifold

$$\widetilde{\Phi}_k(\mathcal{O}_\Delta) =: L'(k) \subset T^*\Delta$$

which maps diffeomorphically to the graph of $\phi_k$ under $\tau$, i.e. $\tau(L'(k)) = \Gamma_{\phi_k}$. Since the Hamiltonian $H_k$ has compact support, the Lagrangian $L'(k)$ differs from the zero section $\mathcal{O}_\Delta$ only inside a compact subset of $T^*\Delta$. Thus, $L'(k) \subset T^*\Delta$ admits a generating function quadratic at infinity

$$S_k \colon \Delta \times E \to \mathbb{R}\,,$$

where $E$ is a finite-dimensional vector space, which is, up to gauge transformation and stabilization, uniquely determined by the requirement that it coincides with a quadratic form on $E$ on a complement of $K \times E$, where $K \subset \Delta$ is a certain compact subset. This implies that its spectral invariants $\ell(\cdot, S_k)$ are uniquely determined by $L'(k)$, and thus by $\phi_k$.

**Remark 4.7.** Actually, in [Vi4] Viterbo gives another way of constructing a generating function $S_k$ associated to the Lagrangian $\widetilde{\Phi}_k(\mathcal{O}_\Delta) = L'(k)$ from which he obtains a particular formula for $S_k$. But since we do not need the particular formula for $S_k$, it suffices to obtain the existence of $S_k$ by the above considerations.

Associated to the Hamiltonian isotopy $\phi^t$ of $H$ Viterbo considers the unique lift $\widetilde{\mathrm{id}\times\phi^t}$: $T^*\mathbb{R}^n \times \overline{T^*\mathbb{R}^n} \to T^*\mathbb{R}^n \times \overline{T^*\mathbb{R}^n}$ of the symplectic isotopy $\mathrm{id}\times\phi^t$. It can be composed with a map $T^*\mathbb{R}^n \times \overline{T^*\mathbb{R}^n} \to T^*\Delta_{T^*\mathbb{R}^n}$ to yield a function which descends, evaluated on the diagonal $\Delta_{T^*\mathbb{R}^n}$, to a well-defined Lagrangian embedding $\Delta_{T^*\mathbb{T}^n} \to T^*\Delta_{T^*\mathbb{T}^n}$. The image of this embedding in $T^*\Delta_{T^*\mathbb{T}^n}$ is exactly the Lagrangian submanifold $L(t)$ given by $\widetilde{\Phi}^t(\mathcal{O}_\Delta)$, see Remark 4.6. In particular, $L(1) = \widetilde{\Phi}(\mathcal{O}_\Delta) = L'(1)$. Using this construction Viterbo obtains a formula for a gfqi $S$: $\Delta_{T^*\mathbb{T}^n} \times E \to \mathbb{R}$ for the Lagrangian submanifold $L'(1)$ from which he deduces a formula for a generating function $S_k$: $\Delta_{T^*\mathbb{T}^n} \times E \to \mathbb{R}$ for the Lagrangian $L'(k)$.

With the generating function $S_k$: $\Delta \times E \to \mathbb{R}$ for the Lagrangian $L'(k) \subset T^*\Delta$ at hand, Viterbo defines, for any $p \in \mathbb{R}^n$, a function

$$(S_k)_p\colon \mathbb{T}^n \times E \to \mathbb{R}$$

via

$$(S_k)_p = S_k|_{\mathbb{T}^n \times \{p\} \times E}\,,$$

i.e. $(S_k)_p(q;\xi) = S_k(q,p;\xi)$, where we view $\mathbb{T}^n \times \{p\}$ as a subset of $\Delta = T^*\mathbb{T}^n$. He then defines the function

$$h'_k\colon \mathbb{R}^n \to \mathbb{R}$$

to be

$$h'_k(p) = \ell_+(S_k|_{\mathbb{T}^n \times \{p\} \times E})\,,$$

and proves that the sequence $(h'_k(p))$ converges for fixed $p$ (in the proof of the equivalence between symplectic homogenization and the partial quasi-morphisms we will see that the convergence of the sequence follows a posteriori), and that the limit is a continuous function. Finally, he defines the symplectic homogenization $\widehat{H}$ of $H$ to be the $q$-independent function $\widehat{H}(p) = \lim_{k\to\infty} h'_k(p)$. Thus, in summary, Viterbo's construction gives an operator

$$\mathcal{H} : C_c^\infty([0,1] \times T^*\mathbb{T}^n) \to C_c(\mathbb{R}^n)$$

given by

$$\mathcal{H}(H) = \widehat{H}\,.$$

**Remark 4.8.** Up to this point the Viterbo bound introduced in Remark 3.8 was not needed in the construction. Nevertheless, in addition to the existence of the symplectic homogenization operator $\mathcal{H}$, Viterbo claims in [Vi4] that it has the following properties:

- the time-1 maps $\phi_k$ form a Cauchy sequence with respect to Viterbo's metric $\gamma$ (see the following remark, Remark 4.9, for the definition of $\gamma$) and its limit with respect to this metric is, in a certain precise sense, generated by the $q$-independent Hamiltonian $\widehat{H}$;

- $\mathcal{H}(H)$ only depends on the time-1 map $\phi$ of $H$;
- $\mathcal{H}$ is Lipschitz continuous with respect to Viterbo's metric $\gamma$.

Moreover, he claims that the above operator gives rise to a symplectic homogenization operator for time-independent Hamiltonians $\mathcal{H}\colon C_c^\infty(T^*\mathbb{T}^n) \to C_c(\mathbb{R}^n)$ which has the following properties:

- $\mathcal{H}(F) \leq \mathcal{H}(G)$ if $F \leq G$;
- $\mathcal{H}(F \circ \phi) = \mathcal{H}(F)$ for all $\phi \in \mathrm{Ham}(T^*\mathbb{T}^n)$;
- $\mathcal{H}(-F) = -\mathcal{H}(F)$;
- if $L$ is a Lagrangian submanifold Hamiltonian isotopic to $L_{p_0} = \{(q, p_0) \in T^*\mathbb{T}^n\}$ and if $F$ is such that $F|_L \geq c$ ($\leq c$), then $\mathcal{H}(F)(p_0) \geq c$ ($\leq c$);
- if $F, G$ are such that $\{F, G\} = 0$, then $\mathcal{H}(F + G) = \mathcal{H}(F) + \mathcal{H}(G)$.

However, a consideration of Viterbo's proofs shows that some of these properties rely on the Viterbo bound. In fact, Viterbo uses it in the proof of the convergence of the time-1 maps, in the proof of the Lipschitz continuity, in the proof of $\mathcal{H}(-F) = -\mathcal{H}(F)$, and in the proof of the strong quasi-additivity property.

**Remark 4.9.** Using his spectral invariants from generating functions Viterbo defines a metric on the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(T^*\mathbb{T}^n)$ [Vi1], [Vi2]. For a Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(T^*\mathbb{T}^n)$ let $\Gamma_\phi \subset T^*\mathbb{T}^n \times \overline{T^*\mathbb{T}^n}$ be its graph. It can be lifted to a Lagrangian submanifold $\widetilde{\Gamma}_\phi \subset T^*\Delta$ via a symplectic covering $\tau\colon T^*\Delta \to T^*\mathbb{T}^n \times \overline{T^*\mathbb{T}^n}$ as above. Since $\phi$ is compactly supported, one can compactify $\widetilde{\Gamma}_\phi$ to a Lagrangian $\overline{\Gamma}_\phi$ in $T^*(S^n \times \mathbb{T}^n)$. The latter Lagrangian admits a gfqi $S$, and the formula

$$\gamma(\phi) = \ell([S^n] \otimes [\mathbb{T}^n], S) - \ell(\mathrm{pt} \otimes \mathrm{pt}, S)$$

defines a norm on $\mathrm{Ham}(T^*\mathbb{T}^n)$ which gives a metric, referred to as Viterbo's metric, via $\gamma(\phi, \psi) = \gamma(\psi\phi^{-1})$. The associated asymptotic Viterbo norm $\gamma_\infty$ on $\mathrm{Ham}(T^*\mathbb{T}^n)$ is given by

$$\gamma_\infty(\phi) = \lim_{k\to\infty} \frac{\gamma(\phi^k)}{k}\,.$$

In general, it is true that the (asymptotic) Viterbo norm is bounded from above by the (asymptotic) Hofer norm, i.e. $\gamma_{(\infty)} \leq \rho_{(\infty)}$ [Vi1], [Vi2].

### Another definition of the function $h'_k$

Combining the above construction, the relation between the time-1 maps $\phi_k$ and $\phi$, and Viterbo's approach, it is intuitively clear that we can define the function $h'_k$ using spectral invariants coming from a generating function obtained in terms of $\phi^k$ instead of $\phi_k$.

Recall the definition of the Lagrangian submanifold $L(t) = \widetilde{\Phi}^t(\mathcal{O}_\Delta) \subset T^*\Delta$ associated to the Hamiltonian isotopy $\phi^t$ coming from $H$ from Remark 4.6. Again, the Lagrangian $L(t) \subset T^*\Delta$ is Hamiltonian isotopic to the zero section $\mathcal{O}_\Delta$ and coincides with the zero section outside of a compact subset of $T^*\Delta$. Therefore, it admits a gfqi

$$S(t)\colon \Delta \times E \to \mathbb{R}$$

which is up to gauge transformation and stabilization uniquely determined by the requirement that it coincides with a quadratic form on $E$ outside of a compact subset, and this implies that its spectral invariants are uniquely determined by $\phi^t$.

For $k \in \mathbb{N}$ and any $p \in \mathbb{R}^n$ we define

$$S(k)_p := S(k)|_{\mathbb{T}^n \times \{p\} \times E}\,,$$

where we view $\mathbb{T}^n \times \{p\}$ as a subset of $\Delta = T^*\mathbb{T}^n$. Thus, we have a function

$$S(k)_p\colon \mathbb{T}^n \times E \to \mathbb{R}\,,$$

where $S(k)_p(q;\xi) = S(k)(q,p;\xi)$, and we define

$$h_k\colon \mathbb{R}^n \to \mathbb{R}$$

to be

$$h_k(p) = \frac{1}{k}\ell_+(S(k)_p)\,.$$

By construction it is intuitively clear that the two functions $h_k$ and $h_k'$ coincide, that is, that we have the following

**Proposition 4.10.**

$$h_k'(p) = \ell_+(S_k|_{\mathbb{T}^n \times \{p\} \times E}) = \frac{1}{k}\ell_+(S(k)|_{\mathbb{T}^n \times \{p\} \times E}) = h_k(p)\,.$$

To prove the proposition we use the following lemma which is contained in [Vi4].

**Lemma 4.11.** *Let $\phi \in \mathrm{Ham}(T^*\mathbb{T}^n)$ and $\psi = \rho_k^*\phi$. Let $S\colon \Delta \times E \to \mathbb{R}$ be a gfqi for the lift of the graph of $\phi$ to $T^*\Delta$. Define $T\colon \Delta \times E \to \mathbb{R}$ by $T(q,p,\xi) = \frac{1}{k}S(kq,p,\xi)$. Then $T$ is a gfqi for the lift of the graph of $\psi$ to $T^*\Delta$.* □

The lemma yields that the spectral invariants of $T$ are $\frac{1}{k}$ times the spectral invariants of $S$.

**Proof** (of Proposition 4.10)**.** Our situation is exactly the situation of the above lemma. In particular, we have that $S(k)\colon \Delta \times E \to \mathbb{R}$ is a gfqi for $L(k)$, where $L(k) = \widetilde{\Phi}^k(\mathcal{O}_\Delta)$ and $\tau(L(k)) = \Gamma_{\phi^k}$, while $S_k\colon \Delta \times E \to \mathbb{R}$ is a gfqi for $L'(k)$, where $L'(k) = \widetilde{\Phi}_k(\mathcal{O}_\Delta)$ and $\tau(L'(k)) = \Gamma_{\phi_k}$. Moreover, $\phi_k = \rho_k^*\phi^k$. Thus, we conclude $\ell_+(S_k) = \frac{1}{k}\ell_+(S(k))$, and in particular,

$$\ell_+(S_k|_{\mathbb{T}^n \times \{p\} \times E}) = \frac{1}{k}\ell_+(S(k)|_{\mathbb{T}^n \times \{p\} \times E})\,.$$

□

## 4.3 Symplectic homogenization via partial quasi-morphisms

In this section we formulate the equivalence between Viterbo's symplectic homogenization operator $\mathcal{H}$ and the partial quasi-morphisms $\mu_a$ for $T^*\mathbb{T}^n$. With this equivalence we define a symplectic homogenization operator for cotangent bundles $T^*N$, where $N$ is any closed connected manifold, and extract its properties from the ones of the partial quasi-morphisms $\mu_a$.

**Theorem 4.12.** *For each $k$ we have*

$$h_k(0) = \frac{1}{k}\ell_+(\phi^k)\,,$$

*where $\ell_+$ is the Lagrangian spectral invariant coming from Lagrangian Floer homology, see Notation 2.8 in Section 2.*

Postponing the proof of the theorem we conclude that the existence of the limit of the sequence $(h_k(0))$ follows a posteriori, and that

$$\widehat{H}(0) = \lim_{k\to\infty} h_k(0) = \lim_{k\to\infty} \frac{\ell_+(\phi^k)}{k} = \mu_0(\phi)\,.$$

If we define $H_p(t,q,\cdot) = H(t,q,\cdot + p)$, we have $\widehat{H}_p(0) = \widehat{H}(p)$, and since the same property holds for $\mu_a$, i.e. $\mu_0(\phi_{H_p}) = \mu_p(\phi)$, we have $\widehat{H}(p) = \mu_p(\phi)$. In summary, Theorem 4.12 implies

**Theorem 4.13.** *Let $N = \mathbb{T}^n$ and identify $H^1(\mathbb{T}^n;\mathbb{R}) = \mathbb{R}^n$. Then $\widehat{H}(p)$ equals the value of $\mu_p$ on the time-1 map $\phi_H$ of $H$, for any $H \in C_c^\infty([0,1]\times T^*\mathbb{T}^n)$ and $p \in \mathbb{R}^n$, i.e.*

$$\widehat{H}(p) = \mu_p(\phi_H)\,.$$

□

With the above equivalence at hand we can, in particular, prove that the partial quasi-morphisms $\mu_a$ are invariant under coverings as stated in Proposition 3.7 in Subsection 3.1.

**Proof** (of Proposition 3.7)**.** From the construction of symplectic homogenization as a limit process over $k$ it is clear that the operator $\mathcal{H}$ is invariant under the covering $\rho_k : T^*\mathbb{T}^n \to T^*\mathbb{T}^n$ given by $(q,p) \mapsto (kq,p)$, that is, that the symplectic homogenization of the Hamiltonian $H$ is the same as the symplectic homogenization of the Hamiltonian $H_k$, where $H_k(t,q,p) = H(kt,kq,p)$. With the above equivalence we conclude that the same is true for the partial quasi-morphisms $\mu_a$. For any $k$ we have $\mu_a(\phi) = \mu_a(\phi_k)$ for $\phi \in \mathrm{Ham}(T^*N)$ and $\phi_k = \rho_k^*\phi^k$. □

Since the functions $\mu_a$ exist on $T^*N$, where $N$ is any closed connected manifold, it makes sense to make the following general definition which can be interpreted to generalize Viterbo's construction from $\mathbb{T}^n$ to arbitrary closed connected $N$: For any closed connected manifold $N$, the symplectic homogenization $\widehat{H}$ of $H \in C_c^\infty([0,1]\times T^*N)$ is given by

$$\widehat{H}(p) := \mu_p(\phi_H)\,,$$

where $p \in H^1(N;\mathbb{R})$. Note that $\widehat{H}$ is indeed continuous because of property (viii) in Theorem 3.5.

Thus, symplectic homogenization yields an operator

$$\mathcal{H}\colon C_c^\infty([0,1]\times T^*N) \to C_c(H^1(N;\mathbb{R}))$$
$$H \mapsto \widehat{H}$$

which is (according to the Lipschitz continuity of the partial quasi-morphisms $\mu_a$) Lipschitz continuous with respect to Hofer's metric, i.e. $\|\mathcal{H}(H)-\mathcal{H}(G)\|_{C^0} \leq \rho(\phi_H,\phi_G)$.

Moreover, using the fact that the partial quasi-morphisms $\mu_a$ yield partial symplectic quasi-integrals $\zeta_a\colon C_c^\infty(T^*N)\to\mathbb{R}$ with properties as listed in Theorem 3.10, we conclude that the operator $\mathcal{H}$ yields an operator $\mathcal{H}\colon C_c^\infty(T^*N)\to C_c(H^1(N;\mathbb{R}))$ via $\mathcal{H}(F)(a)=\zeta_a(F)=\mu_a(\phi_F)$ such that the following holds:

**Lemma 4.14.** *The symplectic homogenization operator $\mathcal{H}\colon C_c^\infty(T^*N)\to C_c(H^1(N;\mathbb{R}))$ has the following properties:*

*(i) $\mathcal{H}$ is monotone, i.e. $\mathcal{H}(F)\leq\mathcal{H}(G)$ for all $F\leq G$;*

*(ii) $\mathcal{H}$ is invariant under Hamiltonian diffeomorphisms of $T^*N$, i.e. $\mathcal{H}(F\circ\phi)=\mathcal{H}(F)$ for all $\phi\in\mathrm{Ham}(T^*N)$;*

*(iii) $\mathcal{H}$ is Lipschitz continuous with respect to the $C^0$-norm, i.e. $\|\mathcal{H}(F)-\mathcal{H}(F)\|_{C^0}\leq\|F-G\|_{C^0}$;*

*(iv) if the restriction of $F$ to the graph of a closed 1-form in the class $a$ is $\geq c$ ($\leq c, = c$), for some $c\in\mathbb{R}$, then $\mathcal{H}(F)(a)\geq c$ ($\leq c, = c$);*

*(v) if $F, G$ are such that $\{F,G\}=0$ and the support of $G$ is displaceable, then $\mathcal{H}(F+G)=\mathcal{H}(F)+\mathcal{H}(G)$.*

□

**Remark 4.15.** One should compare the above considerations with the observations concerning the Viterbo bound given in Remark 4.8. From the equivalence and the properties of the partial quasi-morphisms $\mu_a$ we implicitly get that the symplectic homogenization $\widehat{H}$ just depends on the time-1 map of $H$ and that the operator $\mathcal{H}$ is Lipschitz with respect to Hofer's metric. The properties which we cannot deduce from the equivalence are the notion of convergence of the time-1 maps $\phi_k$ to the time-1 map of $\widehat{H}$, and the Lipschitz continuity with respect to Viterbo's metric; these properties are exactly the ones in whose proof Viterbo does use his bound.

Moreover, properties (i)-(iv) of the symplectic homogenization operator listed in the above proposition are exactly the ones in whose proofs Viterbo does not need his bound. As for quasi-additivity, we can prove the partial quasi-additivity, property (v), instead of the strong quasi-additivity in whose proof Viterbo uses his unproven bound.

**Remark 4.16.** In Section 6 we prove the existence and uniqueness of an operator $\mathcal{H}\colon C_c(T^*S^1) \to C_c(\mathbb{R})$ which has the properties listed in Lemma 4.14, where partial quasi-additivity is replaced by the stronger assumption of strong quasi-additivity. The proof thereby relies on the existence and uniqueness of a particular symplectic quasi-integral on $T^*S^1$ which is constructed using a representation theorem for quasi-integrals and topological measures (which is also developed in Section 6). Thus, in two dimensions, the Viterbo bound is not needed in order to prove the strong quasi-additivity and Viterbo's symplectic homogenization operator can be proved to exist and to be unique by an axiomatic approach.

### 4.3.1 Proof of Theorem 4.12

In order to prove Theorem 4.12 we need to show

$$\ell_+(S(k)|_{\mathbb{T}^n\times\{0\}\times E}) = \ell_+(\phi^k)\,.$$

To prove the above equality we want to make use of the equality between spectral invariants from generating functions and those from Lagrangian Floer homology due to Milinković and Oh stated in Lemma 4.3. To do so, we need to prove the following two statements.

**Lemma 4.17.** *The function $S(k)_0 = S(k)|_{\mathbb{T}^n\times\{0\}\times E}$ generates the Lagrangian submanifold $\overline{\phi^k(\mathcal{O}_{\mathbb{T}^n})}$ in $T^*\mathbb{T}^n$ which is also generated by the action functional of $H^{\#k}$. In particular, $S(1)_0$ and $\mathcal{A}_H$ generate the Lagrangian submanifold $\overline{\phi(\mathcal{O}_{\mathbb{T}^n})}$.*

**Lemma 4.18.** *The induced functions $S(k)_0|\overline{\phi^k(\mathcal{O}_{\mathbb{T}^n})}$ and $\mathcal{A}_{H^{\#k}}|\overline{\phi^k(\mathcal{O}_{\mathbb{T}^n})}$ coincide. In particular,*

$$S(1)_0|\overline{\phi(\mathcal{O}_{\mathbb{T}^n})} = \mathcal{A}_H|\overline{\phi(\mathcal{O}_{\mathbb{T}^n})}\,.$$

Here and in the sequel $\overline{L}$ denotes the image of the Lagrangian $L$ under the involution $(q,p)\mapsto(q,-p)$; $\overline{L}$ is still a Lagrangian submanifold.

Postponing the proof of the two lemmata for the moment we conclude:

**Proof** (of Theorem 4.12)**.** We can assume $k=1$ since the whole construction can be performed with $\phi$ replaced by $\phi^k$ (see also the proof of the lemmata). According to the above lemmata, $S(1)_0$ and $\mathcal{A}_H$ both generate $\overline{\phi(\mathcal{O}_{\mathbb{T}^n})}$, and $S(1)_0|\overline{\phi(\mathcal{O}_{\mathbb{T}^n})} = \mathcal{A}_H|\overline{\phi(\mathcal{O}_{\mathbb{T}^n})}$. Thus, $\overline{\mathcal{A}}_H = -\mathcal{A}_H$ and $-S(1)_0$ generate the Lagrangian $\phi(\mathcal{O}_{\mathbb{T}^n})$, and $\overline{\mathcal{A}}_H|\phi(\mathcal{O}_{\mathbb{T}^n}) = -S(1)_0|\phi(\mathcal{O}_{\mathbb{T}^n})$. Using Lemma 4.3 we conclude

$$\overline{\ell}_\pm(H) = \ell_\pm(-S(1)_0)\,.$$

In summary, we have the following chain of equalities

$$\ell_+(\phi) = \ell_+(H) = -\overline{\ell}_-(H) = -\ell_-(-S(1)_0) = \ell_+(S(1)_0)\,,$$

where we have $\overline{\ell}_\pm(H) = -\ell_\mp(H)$ due to the sign conventions (see Subsection 4.1.2) and $\ell_\pm(-S) = -\ell_\mp(S)$ by standard duality considerations (see [Vi1]). □

**Proof** (of Lemma 4.17)**.** We can assume $k = 1$ since we can proceed for general $k$ in the same fashion. We have the following commutative diagram:

$$\begin{array}{ccc} T^*\mathbb{R}^n \times \overline{T^*\mathbb{R}^n} & \longrightarrow & T^*\Delta_{T^*\mathbb{R}^n} \\ \downarrow & & \downarrow \\ T^*\mathbb{T}^n \times \overline{T^*\mathbb{T}^n} & \xleftarrow{\ \tau\ } & T^*\Delta_{T^*\mathbb{T}^n} \end{array}.$$

Here we view explicitly $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. The left and the right arrows are induced from the quotient maps[6)] $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{T}^n$ and $T^*\mathbb{R}^n \to T^*\mathbb{T}^n$. The top map is given by $(q, p; Q, P) \mapsto (Q, p; p - P, Q - q)$. Using the commutative diagram we conclude that $S(1)_p$ generates the Lagrangian submanifold in $T^*\mathbb{T}^n$ given by

$$\{(Q(q,p), p - P(q,p)) \,|\, q \in \mathbb{T}^n, (Q(q,p), P(q,p)) = \phi(q,p)\}\,.$$

Thus, $S(1)_0$ generates

$$\{(Q(q,0), -P(q,0)) \,|\, q \in \mathbb{T}^n, (Q(q,0), P(q,0)) = \phi(q,0)\} = \overline{\phi(\mathcal{O}_{\mathbb{T}^n})}\,.$$

But $\overline{\phi(\mathcal{O}_{\mathbb{T}^n})}$ is generated by the action functional $\mathcal{A}_H$ as well (recall Remark 4.2), proving the claim. □

**Proof** (of Lemma 4.18)**.** Again, we assume $k = 1$. To prove the equality of the induced functions $S(1)_0|\overline{\phi(\mathcal{O}_{\mathbb{T}^n})}$ and $\mathcal{A}_H|\overline{\phi(\mathcal{O}_{\mathbb{T}^n})}$ it suffices to show that they are equal at one point since their difference is constant. To do so, we first of all prove that we can define an action functional corresponding to $\widetilde{H}$ which generates $L(1)$, and that the induced function on $L(1)$ coincides with $S(1)|L(1)$ on the covering space $T^*\Delta$. With this equality we can conclude the claim for the functions $S(1)_0$ and $\mathcal{A}_H$.

Recall the definition of the Hamiltonian $\widetilde{H}$ on $T^*\Delta$ which generates $\widetilde{\Phi}^t$ from Remark 4.6. It is not compactly supported but for finite $t$ we can cut it off outside a large enough ball and consider the action functional corresponding to that function (which we also denote by $\widetilde{H}_t$), $\mathcal{A}_{\widetilde{H}}$. This action functional has the same values on all Hamiltonian arcs starting at the zero section and following the flow $\widetilde{\Phi}^t$ as the original function before the cutoff. The action functional $\mathcal{A}_{\widetilde{H}}$ generates the Lagrangian submanifold $\overline{\widetilde{\Phi}(\mathcal{O}_\Delta)}$ which is $\overline{L(1)}$. Thus, $\overline{\mathcal{A}_{\widetilde{H}}}$ generates the Lagrangian $L(1) = \widetilde{\Phi}(\mathcal{O}_\Delta)$. But the Lagrangian $L(1)$ is also generated by $S(1)$ (according to the definition, see Subsection 4.2). Thus, $\overline{\mathcal{A}_{\widetilde{H}}}$ and $S(1)$ both generate the Lagrangian submanifold $L(1) \subset T^*\Delta$ and therefore they induce functions on $L(1)$ which differ by a constant.

Since $L(1)$ differs from the zero section only inside a compact subset of $T^*\Delta$, we can compactify all objects to $T^*(\mathbb{T}^n \times S^n) = T^*(\Delta \cup \mathbb{T}^n \times \{\infty\})$. For simplicity, we denote all compactified objects by the same letters. From the definition of $\widetilde{H}$ it is clear that the points of $\mathbb{T}^n \times \{\infty\}$, considered as constant curves, are Hamiltonian

[6)] Usually if there is a smooth map $f\colon X \to Y$, there is no natural way of associating a smooth map between the corresponding cotangent bundles, however if this map is a local diffeomorphism, then we get the induced map $f_*\colon T^*X \to T^*Y$ given by $f_*(\alpha) = \alpha \circ (d_x f)^{-1}$ for $\alpha \in T^*_x X$, and it is symplectic: $(f_*)^*\omega^{T^*Y} = \omega^{T^*X}$.

arcs with respect to $\widetilde{H}$ starting and ending at the zero section. Moreover, $\widetilde{H}_t$ equals zero on an open neighborhood of $\mathbb{T}^n \times \{\infty\}$ inside $T^*(\mathbb{T}^n \times S^n)$ which means, in particular, that the action of a point in $\mathbb{T}^n \times \{\infty\}$, considered as a Hamiltonian arc, is zero. But the generating function $S(1)$ also equals zero at a point of $\mathbb{T}^n \times \{\infty\}$, and thus the induced functions on $L(1)$ coincide, i.e.

$$\overline{\mathcal{A}}_{\widetilde{H}}|L(1) = S(1)|L(1)\,.$$

In particular, if $\gamma : [0,1] \to T^*(\mathbb{T}^n \times S^n)$ is a Hamiltonian arc relative to $\widetilde{H}$ such that $\gamma(0) \subset \mathcal{O}_\Delta$ and $\gamma(1) = z \in L(1)$, then

$$(S(1)|L(1))(z) = \overline{\mathcal{A}}_{\widetilde{H}}(\gamma) = -\mathcal{A}_{\widetilde{H}}(\gamma)\,.$$

With this equality of the "lifted" functions we conclude that the functions $\mathcal{A}_H|\overline{\phi(\mathcal{O}_{\mathbb{T}^n})}$ and $S(1)_0|\overline{\phi(\mathcal{O}_{\mathbb{T}^n})}$ are equal at one point of $\overline{\phi(\mathcal{O}_{\mathbb{T}^n})}$ and therefore coincide.

Choose a point $z \in \overline{\phi(\mathcal{O}_{\mathbb{T}^n})} \cap \mathcal{O}_{\mathbb{T}^n}$. It exists by Lagrangian intersection theory. Let $\gamma$ be the Hamiltonian arc ending at $z$ relative to the flow $\phi^t$, i.e.

$$\gamma(t) = \phi^t(\gamma(0)) \text{ and } \gamma(1) = z\,,$$

where $\gamma(0) \in T^*\mathbb{T}^n$. In coordinates we denote $(q,0) = \gamma(0) \in T^*\mathbb{T}^n$ and $\gamma(t) = (Q_t, P_t)$. Note that the curve $t \mapsto Q_t \in \mathbb{T}^n$ has lifts to $\mathbb{R}^n$, and for any such lift, say, $\delta(t)$, the difference $\delta(t) - \delta(0)$ is independent of the lift. We denote this difference by $Q_t - q \in \mathbb{R}^n$.

Recall the definition of the symplectic covering $\tau\colon T^*\Delta \to T^*\mathbb{T}^n \times \overline{T^*\mathbb{T}^n}$ and the Hamiltonian isotopy $\widetilde{\Phi}^t\colon T^*\Delta \to T^*\Delta$ from Remark 4.6. Consider the arc $\widetilde{\gamma}\colon [0,1] \to T^*\Delta$ given by

$$\widetilde{\gamma}(t) = \widetilde{\Phi}^t(\gamma(0))\,,$$

where we view $\gamma(0) \in T^*\mathbb{T}^n = \Delta = \mathcal{O}_\Delta \subset T^*\Delta$. Since $\tau(\widetilde{\Phi}^t(\mathcal{O}_\Delta)) = \Gamma_{\phi^t}$, we have

$$(\tau \circ \widetilde{\gamma})(t) == (\gamma(0); \phi^t(\gamma(0))) = (q, 0; \gamma(t)) = (q, 0; Q_t, P_t) \in T^*\mathbb{T}^n \times \overline{T^*\mathbb{T}^n}\,.$$

Moreover, using the symplectic covering $\tau$ one can calculate

$$\widetilde{\gamma}(t) = (Q_t, 0; -P_t, Q_t - q) \in T^*\Delta\,.$$

We compute the action of the arc $\widetilde{\gamma}$ relative to the Hamiltonian $\widetilde{H}$, i.e.

$$\mathcal{A}_{\widetilde{H}}(\widetilde{\gamma}) = \int_0^1 \widetilde{H}_t(\widetilde{\gamma}(t))\,dt - \int \widetilde{\gamma}^* \lambda^\Delta\,,$$

where $\lambda^\Delta$ is the Liouville form on $T^*\Delta$. With the definition of $\widetilde{H}_t = K_t \circ \tau = (0 \oplus (-H_t)) \circ \tau$ (recall Remark 4.6) and the above computation we have in the first integral

$$\int_0^1 \widetilde{H}_t(\widetilde{\gamma}(t))\,dt = \int_0^1 (K_t \circ \tau \circ \widetilde{\gamma})(t) = \int_0^1 (0 \oplus -H_t)(q, 0; \gamma(t))\,dt = -\int_0^1 H_t(\gamma(t))\,dt\,.$$

The second integral equals

$$-\int_0^1 \lambda^\Delta(\dot{\widetilde{\gamma}}(t))\,dt = -\int_0^1 \langle(-P_t, Q_t - q), \tfrac{d}{dt}(Q_t - q, 0)\rangle\,dt = \int_0^1 \langle P_t, \dot{Q}_t\rangle\,dt = \int \gamma^*\lambda\,.$$

In total, we get

$$-\mathcal{A}_{\widetilde{H}}(\widetilde{\gamma}) = \mathcal{A}_H(\gamma)\,.$$

Denoting $\widetilde{z} = \widetilde{\gamma}(1)$, we have

$$(S(1)_0|\overline{\phi(\mathcal{O}_{\mathbb{T}^n})})(z) = (S(1)|L(1))(\widetilde{z}) = -\mathcal{A}_{\widetilde{H}}(\widetilde{\gamma}) = \mathcal{A}_H(\gamma)\,.$$

The first of these equalities follows from the fact that $\overline{\phi(\mathcal{O}_{\mathbb{T}^n})}$ is obtained from $L(1)$ by symplectic reduction (which is just a reformulation of the fact that $\overline{\phi(\mathcal{O}_{\mathbb{T}^n})}$ is generated by the gfqi $S(1)_0$ which itself is the restriction of $S(1)$ to the zero section $\mathcal{O}_{\mathbb{T}^n} \subset T^*\mathbb{T}^n = \Delta$), and that $\widetilde{z}$ is mapped to $z$ under this reduction. □

# 5 Applications

The existence and the properties of the functions $\mu_a$: $\mathrm{Ham}(T^*N) \to \mathbb{R}$ and $\zeta_a$: $C_c^\infty(T^*N) \to \mathbb{R}$ lead to various applications. In Subsection 5.1 we present a lower bound on Banyaga's fragmentation norm relative to displaceable subsets, in Subsection 5.2 we prove that the partial quasi-morphisms $\mu_a$ are equivalent to Mather's alpha function and deduce the invariance of the latter under Hamiltonian diffeomorphisms. In Subsection 5.3 we present the applications to Hofer and spectral geometry. Finally, in Subsection 5.4 we deduce rigidity results of subsets using the partial symplectic quasi-integral $\zeta_0$.

## 5.1 Fragmentation norm

Recall from Remark 3.4 that the fragmentation norm $\|\phi\|_{\mathcal{U}}$ of a Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(T^*N)$ relative to an arbitrary family $\mathcal{U}$ of open subsets is given by the following: Associated to $\mathcal{U}$ we consider an open covering $\mathcal{V}$ of $T^*N$ consisting of open subsets $V$ for which there is $\psi \in \mathrm{Ham}(T^*N)$ such that $\psi(V) \in \mathcal{U}$; the fragmentation norm $\|\phi\|_{\mathcal{U}}$ is the fragmentation norm relative to the covering $\mathcal{V}$.

**Proposition 5.1.** *The fragmentation norm of $\phi \in \mathrm{Ham}(T^*N)$ relative to a family of open and displaceable subsets $\mathcal{U}$ satisfies*

$$\|\phi\|_{\mathcal{U}} \geq \frac{|\mu_0(\phi)|}{e(\mathcal{U})}\,,$$

*where $e(\mathcal{U})$ is the spectral displacement energy of the family $\mathcal{U}$ introduced in Remark 2.11. In particular, if $\phi \in \mathrm{Ham}(T^*N)$ is generated by a Hamiltonian $H$ such that $H|_L \geq c$, for some $c \in \mathbb{R}$, where $L$ is a Lagrangian Hamiltonian isotopic to the zero section $N$, then*

$$\|\phi\|_{\mathcal{U}} \geq \frac{c}{e(\mathcal{U})}\,.$$

**Proof.** Point (v) of Theorem 3.5 states $|\mu_0(\phi\psi) - \mu_0(\psi)| \leq e(\mathcal{U})\,\|\phi\|_{\mathcal{U}}$ and with $\psi = \mathrm{id}$ we obtain

$$|\mu_0(\phi)| \leq e(\mathcal{U})\,\|\phi\|_{\mathcal{U}}\;,$$

and the claim follows. For the particular case it suffices to note that according to the conjugation invariance of $\mu_0$ under $\mathrm{Ham}(T^*N)$ we have $|\mu_0(\phi)| \geq c$ for any such $\phi$. □

**Remark 5.2.** Similar results are proved in [EP1] for closed manifolds and in [La] for certain types of open convex manifolds. In the first reference the Hamiltonian diffeomorphism $\phi$ is required to have displaceable support. This has to do with the fact that the Calabi quasi-morphism used there coincides with the Calabi homomorphism on displaceable subsets (see Subsection 6.5), while our partial quasi-morphisms (and Lanzat's functions) vanish on displaceable subsets.

## 5.2 Mather's alpha function

Before stating and proving the equality between Mather's alpha function and the partial quasi-morphisms $\mu_a$ we briefly recall the definition of Mather's alpha function; we refer to [Ma] for details.

Mather's alpha function is a function

$$\alpha_H \colon H^1(N;\mathbb{R}) \to \mathbb{R}$$

which can be associated to any time-periodic Tonelli Hamiltonian $H\colon S^1 \times T^*N \to \mathbb{R}$. Thereby, a Hamiltonian $H\colon S^1 \times T^*N \to \mathbb{R}$ is called Tonelli if it is fiberwise strictly convex, i.e. the fiberwise Hessian of $H$ is positive definite, superlinear, i.e. $\lim_{\|p\|\to\infty} H(t,q,p)/\|p\| = \infty$ for all $(t,q) \in S^1 \times N$, and has complete flow. In general, Mather's theory, in whose context the alpha function is constructed, is about action minimizing invariant measures for certain Hamiltonian systems on cotangent bundles $T^*N$. In fact, the alpha function is the conjugate of a function $\beta_H\colon H_1(N;\mathbb{R}) \to \mathbb{R}$ which can be interpreted to represent the minimal average Lagrangian action needed in order to carry out motions with a given rotation vector.

Let $H\colon S^1 \times T^*N \to \mathbb{R}$ be a time-periodic Tonelli Hamiltonian. Consider the associated Lagrangian $L\colon S^1 \times TN \to \mathbb{R}$ which is given by the Legendre duality by the formula

$$L(t,q,v) = \sup\{\langle p, v\rangle - H(t,q,p) \,|\, p \in T_q^*N\}\,.$$

The Lagrangian $L$ is Tonelli and defines a so-called Euler-Lagrange flow $\phi_L$ on $S^1 \times TN$ which is given by the solution of the equation

$$\frac{d}{dt}\partial_v L(t,q,v) = \partial_q L(t,q,v)\,.$$

Let $P = S^1 \times (TN \cup \{\infty\})$ denote the one-point compactification of $S^1 \times TN$. The Euler-Lagrange flow $\phi_L$ extends to a flow on $P$ which fixes $\infty$. Let $\mathcal{M}_L$ denote the set of probability measures on $P$ which are invariant under $\phi_L$. To each $\mu \in \mathcal{M}_L$ one can associate a unique class $p(\mu) \in H_1(N;\mathbb{R})$, called the rotation vector, as follows: Let $\lambda$ be a closed 1-form on $N$ and $[\lambda] \in H^1(N;\mathbb{R})$ its cohomology class. One can view $\lambda$ as a map $TN \to \mathbb{R}$ which is linear in the fibers and compose it with the projection $S^1 \times TN \to TN$; the resulting map is still denoted by $\lambda$. Define a functional $H^1(N;\mathbb{R}) \to \mathbb{R}$ by $[\lambda] \mapsto \int_P \lambda\, d\mu$. For every $\mu \in \mathcal{M}_L$ there exists $p(\mu) \in H_1(N;\mathbb{R})$, referred to as rotation vector of $\mu$, such that

$$\int_P \lambda\, d\mu = \langle [\lambda], p(\mu)\rangle\ ,$$

for all closed 1-forms $\lambda$, where $\langle\cdot,\cdot\rangle$ denotes the canonical pairing between cohomology and homology classes.

For $\mu \in \mathcal{M}_L$ define its average action by

$$\mathcal{A}_L(\mu) = \int_P L\, d\mu\,,$$

where we set $L(\infty) = \infty$. According to Mather there exists $\mu \in \mathcal{M}_L$ such that $\mathcal{A}_L(\mu) < \infty$; the set of all such $\mu$ is denoted by $\widehat{\mathcal{M}}_L$. Moreover, there exists a minimal value of the average action over the set of probability measures $\widehat{\mathcal{M}}_L$ with a given rotation vector. Thus, there is a well-defined map

$$\begin{aligned} \beta_H\colon H_1(N;\mathbb{R}) &\to \mathbb{R} \\ h &\mapsto \min\{A_L(\mu)\,|\,\mu\in\widehat{\mathcal{M}}_L, p(\mu)=h\}\,. \end{aligned}$$

The function $\beta_H$ is convex and superlinear, and therefore one can consider its conjugate function (given by the Legendre duality)

$$\begin{aligned} \alpha_H\colon H^1(N;\mathbb{R}) &\to \mathbb{R} \\ c &\mapsto \max_{h\in H_1(N;\mathbb{R})}(\langle c,h\rangle-\beta_H(h))\,, \end{aligned}$$

which is known as Mather's alpha function.

We have the following relation between the partial quasi-morphisms (extended to the set of Hamiltonians with complete flow) and Mather's alpha function:

**Proposition 5.3.** *Let $H$ be a time-periodic Tonelli Hamiltonian. For any $a\in H^1(N;\mathbb{R})$ we have*

$$\alpha_H(a)=\mu_a(\phi_H)\,.$$

**Remark 5.4.** Note that the above result gives a way to define the alpha function for arbitrary Hamiltonians on $T^*N$ with complete flow as $\alpha_H(a):=\mu_a(\phi_H)$.

**Remark 5.5.** The above result was first established in [Vi4] for $N=\mathbb{T}^n$ in the formulation of symplectic homogenization (there the author proves that the alpha function of $H$ equals the symplectic homogenization of $H$).

**Proof.** It suffices to show the equality for $a=0$. According to Mather (this is implicit in [Ma]; see for example the proof of the proposition on page 178) there is the following expression of the alpha function at 0:

$$\alpha_H(0)=-\lim_{k\to\infty}\frac{1}{k}\inf\{\mathcal{A}_L^k(\gamma)\,|\,\gamma\colon[0,k]\to N\}\,,$$

where $k\in\mathbb{N}$ and

$$\mathcal{A}_L^k(\gamma)=\int_0^k L(t,\gamma(t),\dot\gamma(t))\,dt\,,$$

where $L\colon S^1\times TN\to\mathbb{R}$ is the time-periodic Lagrangian function associated to $H$. We claim that the infimum on the right-hand side equals $-\ell_+(\phi_H^k)$. Assuming this claim for a moment we obtain

$$\alpha_H(0)=\lim_{k\to\infty}\frac{\ell_+(\phi_H^k)}{k}=\mu_0(\phi_H)\,.$$

To prove the claim we consider the space of smooth paths $\mathcal{P}_k=\{\gamma\colon[0,k]\to N\}$ and the functional $\mathcal{A}_L^k\colon\mathcal{P}_k\to\mathbb{R}$. Since the evaluation map $\pi_k\colon\mathcal{P}_k\to N$ given by $\gamma\mapsto\gamma(k)$ is a submersion, one can consider the functional $\mathcal{A}_L^k$ as a generating function which generates the Lagrangian submanifold $\phi_H^k(N)$. The above infimum is a minimum and therefore it is a critical value of $\mathcal{A}_L^k$. Thus, it is the action of a Hamiltonian arc in $\mathcal{P}_k$ running from the zero section back to itself.

Moreover, it is possible to find a finite-dimensional generating function $S_k$ for $\phi_H^k(N)$ which equals a positive-definite quadratic form outside of a compact subset (this can be found in the latest version of [Vi4], appendix D, or in [MVZ], appendix A). In this case we have

$$\min S_k = \ell_-(S_k)\,.$$

Thus, we have two generating functions of $\phi_H^k(N)$ and both of them induce functions on the Lagrangian $\phi_H^k(N)$; their difference is a constant. By normalization of $S_k$ we can assume that its critical values coincide with those of $\mathcal{A}_L^k$, and thus

$$\min \mathcal{A}_L^k = \min S_k = \ell_-(S_k)\,.$$

Now, our sign conventions imply that the Hamiltonian action functional is the negative of the Lagrangian one when evaluated at a critical point. Therefore, we have

$$S_k|\phi_H^k(N) = \mathcal{A}_L^k|\phi_H^k(N) = -\mathcal{A}_H^k|\phi_H^k(N)\,,$$

and it follows that the spectral invariants of $S_k$ coincide with those of $-\mathcal{A}_H^k$. In summary, we have

$$\min \mathcal{A}_L^k = \min S_k = \ell_-(S_k) = -\ell_+(\phi_H^k)\,,$$

as claimed. □

As a consequence of the above result we get the invariance of the alpha function under Hamiltonian diffeomorphisms.

**Corollary 5.6.** *For a time-periodic Tonelli Hamiltonian $H$ and a Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(T^*N)$ such that $H \circ \phi$ is still Tonelli* [7] *we have*

$$\alpha_{H\circ\phi} = \alpha_H\,.$$

**Proof.** The functions $\mu_a$ are invariant under conjugation in $\mathrm{Ham}(T^*N)$ (when extended to the set of Hamiltonians with complete flow). Since $H\circ\phi$ generates $\phi^{-1}\phi_H\phi$, the claim follows. □

**Remark 5.7.** The proof of the symplectic invariance of the alpha function for Tonelli Hamiltonians can be found in [Be] and the references therein. Before that it was also shown in [PPS] in case $H$ is autonomous. The advantage of our approach is that the proof follows from the properties of the partial quasi-morphisms and that it applies to any Hamiltonian with complete flow.

## 5.3 Hofer and spectral geometry

Recall from Subsection 1.5 that the Hofer norm of $\phi \in \mathrm{Ham}(T^*N)$ is given by

$$\rho(\phi) = \inf_H \int_0^1 \operatorname{osc} H_t\, dt\,,$$

[7] Since $\phi$ has compact support, $H\circ\phi$ has automatically complete flow and is superlinear.

where the infimum goes over all compactly supported Hamiltonians generating $\phi$. It gives rise to the asymptotic Hofer norm $\rho_\infty(\phi) = \lim_{k\to\infty} \rho(\phi^k)/k$ and defines Hofer's metric on $\mathrm{Ham}(T^*N)$ via $\rho(\phi,\psi) = \rho(\phi\psi^{-1})$.

Furthermore, recall from Remark 2.10 that the spectral norm of $\phi \in \mathrm{Ham}(T^*N)$ is given by

$$\Gamma(\psi) = c_+(\psi) - c_-(\psi)\,,$$

where $c_\pm\colon \mathrm{Ham}(T^*N) \to \mathbb{R}$ are Hamiltonian spectral invariants. The corresponding asymptotic spectral norm is given by $\Gamma_\infty(\phi) = \lim_{k\to\infty} \Gamma(\phi^k)/k$, while the spectral metric on $\mathrm{Ham}(T^*N)$ is given by $\Gamma(\phi,\psi) = \Gamma(\phi\psi^{-1})$.

In [FS] it is proved that

$$\Gamma(\phi,\psi) \le \rho(\phi,\psi)\,.$$

We get the following chain of inequalities for the (asymptotic) Hofer and the (asymptotic) spectral norm (related results can be found in [PS] and in [Sib2]).

**Proposition 5.8.** *For $\phi \in \mathrm{Ham}(T^*N)$ we have*

$$\mathrm{osc}_{a\in H^1(N;\mathbb{R})}\, \mu_a(\phi) \le \Gamma(\phi) \le \rho(\phi)$$

*and*

$$\mathrm{osc}_{a\in H^1(N;\mathbb{R})}\, \mu_a(\phi) \le \Gamma_\infty(\phi) \le \rho_\infty(\phi)\,.$$

**Proof.** To prove the claim about the spectral metric recall from Proposition 2.12 that we have the following comparison inequality

$$c_-(\phi) \le \ell_+(\phi) \le c_+(\phi)\,.$$

Since the triangle inequality holds for the spectral invariants $c_\pm$, the sequence $\{c_+(\phi^k)\}_{k\ge1}$ is subadditive and the sequence $\{c_-(\phi^k)\}_{k\ge1}$ is superadditive. We get

$$c_-(\phi) \le \frac{\ell_+(\phi^k)}{k} \le c_+(\phi)\,,$$

and thus

$$c_-(\phi) \le \mu_0(\phi) \le c_+(\phi)\,.$$

Since the spectral invariants $c_\pm$ are invariant under the symplectomorphisms $T_\alpha$ which were used to define $\mu_a$, we have

$$\mathrm{osc}_{a\in H^1(N;\mathbb{R})}\, \mu_a(\phi) \le c_+(\phi) - c_-(\phi) = \Gamma(\phi)\,.$$

Finally, note that the spectral norm satisfies $\Gamma(\phi) \le \rho(\phi)$ which proves the estimate for the Hofer metric. □

For the next theorem note that oscillation is a norm on the subspace $C_c^\infty((0,1))$, the corresponding metric space is denoted by $(C_c^\infty((0,1)), \mathrm{osc})$.

**Proposition 5.9.** *(i) If $N$ does not admit a nowhere vanishing closed 1-form, there is an isometric embedding of $\mathbb{R}$ into $\mathrm{Ham}(T^*N)$.*

*(ii) If $N$ admits a nowhere vanishing closed 1-form, then there is an isometric embedding from $(C_c^\infty((0,1)), \mathrm{osc})$ into $(\mathrm{Ham}(T^*N), \rho)$, that is, there is a map $\iota\colon C_c^\infty((0,1)) \to \mathrm{Ham}(T^*N)$ such that $\rho(\iota(F), \iota(G)) = \mathrm{osc}(F-G)$.*

**Proof.** (i) Let $H \in C_c^\infty(T^*N)$ be such that $H|_N = 1$ and $0 \le H \le 1$ everywhere. Define $\iota\colon \mathbb{R} \to \mathrm{Ham}(T^*N)$ by $t \mapsto \phi_{tH}$. On the one hand, we have

$$\rho(\iota(t), \iota(t')) = \rho(\phi_{tH}, \phi_{t'H}) \le \mathrm{osc}((t-t')H) = |t-t'|\,.$$

On the other hand, we have according to (iii) in Theorem 3.5

$$\rho(\iota(t), \iota(t')) \ge |\mu_0(\phi_{tH}) - \mu_0(\phi_{t'H})|$$

but since $tH|_N = t$ and $t'H|_N = t'$, we have with (vi) in Theorem 3.5 that $\mu_0(\phi_{tH}) = t$ and $\mu_0(\phi_{t'H}) = t'$, and thus

$$\rho(\iota(t), \iota(t')) \ge |t-t'|\,.$$

(ii) Let $\alpha$ be a nowhere vanishing closed 1-form and $a = [\alpha] \in H^1(N;\mathbb{R})$. Since $\alpha$ has no zeros, we can fix a smooth $H\colon T^*N \to \mathbb{R}$ such that $H|_{\Gamma_{t\alpha}} = t$ for $t \in [0,1]$, where $\Gamma_{t\alpha}$ denotes the graph of $t\alpha$. Define a map $C_c^\infty((0,1)) \to C_c^\infty(T^*N)$ by $f \mapsto H_f := f \circ H$, where we implicitly extend $f \in C_c^\infty((0,1))$ to $\mathbb{R}$ by zero; the map is linear. Note that we have $\max H_f = \max f$ and the same for min and osc. Moreover, $f \circ H$ equals $f(t)$ on $\Gamma_{t\alpha}$. Define a map

$$\iota : C_c^\infty((0,1)) \to \mathrm{Ham}(T^*N)$$

by

$$\iota(f) \equiv \phi_{H_f}\,;$$

the map is a group homomorphism. Now, we have

$$\rho(\iota(f), \iota(g)) = \rho(\phi_{H_f}, \phi_{H_g}) \le \mathrm{osc}(H_f - H_g) = \mathrm{osc}(f-g)\,.$$

To prove the claim we need to prove the reversed inequality $\mathrm{osc}(f-g) \le \rho(\iota(f), \iota(g))$. Let $H, G$ be time-dependent Hamiltonians generating $\phi_{H_f}$ and $\phi_{H_g}$ respectively. According to (iii) in Theorem 3.5 we have

$$\mu_{ta}(\phi_{H_f}) - \mu_{ta}(\phi_{H_g}) \le \int_0^1 \max(F_t - G_t)\,dt\,.$$

By (vi) in Theorem 3.5 and the fact that $H_f$ equals $f(t)$ on $\Gamma_{t\alpha}$, and similar for $g$, we have

$$\mu_{ta}(\phi_{H_f}) - \mu_{ta}(\phi_{H_g}) = f(t) - g(t) = (f-g)(t)\,,$$

and thus

$$\max(f-g) \le \int_0^1 \max(F_t - G_t)\,dt\,.$$

Similarly,

$$\min(f-g) \ge \int_0^1 \min(F_t - G_t)\,dt\,.$$

The above inequalities imply

$$\mathrm{osc}(f-g) \le \int_0^1 \mathrm{osc}(F_t - G_t)\,dt\,,$$

and since this is true for any $F, G$ generating $\phi_{H_f}, \phi_{H_g}$, we have

$$\mathrm{osc}(f-g) \le \rho(\phi_{H_f}, \phi_{H_g}) = \rho(\iota(f), \iota(g))\,.$$

□

### 5.3.1 Mather's alpha function and Hofer geometry

There is a connection between Aubry-Mather theory and Hofer geometry as studied by Siburg in [Sib1]; in particular, there is an inequality between the Hofer norm and the minimum of the alpha function. It can be interpreted as to provide a relation between the dynamical and the geometric point of view to the study of Hamiltonian systems.

Recall that Aubry-Mather theory (or the existence of the alpha function) is established for Tonelli Hamiltonians. Since these Hamiltonians do not have compact support, they do not generate an element in $\mathrm{Ham}(T^*N)$. To circumvent this difficulty and to be able to relate the Hofer norm and the alpha function, one considers nicely behaved Tonelli Hamiltonians such that both, the notion of Hofer norm and the alpha function, are well-defined.

Let $B^*N \subset T^*N$ denote the closed unit disk cotangent bundle. Let $\mathcal{H}$ be the space of time-periodic Hamiltonians $H : S^1 \times B^*N \to \mathbb{R}$ which vanish on the boundary of $B^*N$ and which admit smooth extensions to the whole $S^1 \times T^*N$ which only depend on $||p||$ and $t$ outside $B^*N$. Then there is an associated notion of the Hofer norm for any Hamiltonian diffeomorphism $\phi_H \colon B^*N \to B^*N$, where $H \in \mathcal{H}$, given by

$$\rho_{\mathcal{H}}(\phi) = \inf \int_0^1 \operatorname{osc} H_t \, dt \,,$$

where the infimum goes over all $H \in \mathcal{H}$ generating $\phi_H$.

**Proposition 5.10.** *Let $\widetilde{H}$ be a Tonelli Hamiltonian which vanishes for $\|p\| = 1$ and which only depends on $\|p\|$ for $\|p\| \geq 1$. Let $H = \widetilde{H}|_{B^*N} \in \mathcal{H}$. Then*

$$\rho_{\mathcal{H}}(\phi_H) \geq - \min_{H^1(N;\mathbb{R})} \alpha_{\widetilde{H}} \,.$$

**Proof.** Consider a smooth function $f \colon [0, \infty) \to [0, 1]$ such that $f(t) = t$ for $t \in [0, \frac{1}{2}]$, $f(t) = 1$ for $t \geq 2$ and $f'(t) \geq 0$ for all $t$. For $\varepsilon > 0$ define $f_\varepsilon(t) = \varepsilon f(\frac{t}{\varepsilon})$ and $K_\varepsilon = f_\varepsilon \circ \widetilde{H}$. According to [SV] we have

$$\rho_{\mathcal{H}}(\phi_H) = \lim_{\varepsilon \to 0} \rho(\phi_{K_\varepsilon}) \,.$$

Moreover, for any $a \in H^1(N;\mathbb{R})$ such that $\|a\| < 1$, where $\|\cdot\|$ denotes the Gromov-Federer stable norm (see Remark 3.8), we have

$$\alpha_{K_\varepsilon}(a) = \alpha_{\widetilde{H}}(a) \,.$$

Note that the minima $\min \alpha_{\widetilde{H}}$ and $\min \alpha_{K_\varepsilon}$ are both negative and attained on $\{\|a\| < 1\} \subset H^1(N;\mathbb{R})$. Thus, for any $\|a\| < 1$ we have

$$\rho(\phi_{K_\varepsilon}) \geq -\mu_a(\phi_{K_\varepsilon}) = -\alpha_{K_\varepsilon}(a) \,,$$

and therefore

$$\rho(\phi_{K_\varepsilon}) \geq - \min \alpha_{K_\varepsilon} = - \min \alpha_{\widetilde{H}} \,.$$

Taking $\varepsilon \to 0$ gives the desired inequality. □

**Remark 5.11.** Note that the minimum on the right-hand side only depends on $H$.

**Remark 5.12.** In [Sib1] the above is proved for $N = \mathbb{T}^n$. A proof for cotangent bundles over a general base can be found in [ISM].

**Remark 5.13.** In [Sib1] the author gives a further relation between Hofer geometry and Aubry-Mather theory in terms of the asymptotic Hofer norm $\rho_\infty$ and the beta function. In fact, Siburg proves that if $\widetilde{H}$ is a Tonelli Hamiltonian such that $H = \widetilde{H}|_{B^*\mathbb{T}^n} \in \mathcal{H}$, then it is true that $\rho_\infty(\phi_H) \geq \beta_{\widetilde{H}}(0)$.

In [SV] the authors claim to prove the strict inequality using the asymptotic Viterbo norm $\gamma_\infty$ on $\mathrm{Ham}(T^*\mathbb{T}^n)$ (recall its definition from Remark 4.9) and the known relation $\gamma_\infty \leq \rho_\infty$. In fact, they prove that for a Tonelli Hamiltonian $\widetilde{H}$ and $H \in \mathcal{H}$ as above it is true that $\gamma_\infty(\phi_H) = \beta_{\widetilde{H}}(0)$. Furthermore, in Theorem 5.5 of [SV] the authors claim to give a construction of a Tonelli Hamiltonian $\widetilde{H}$ such that $H = \widetilde{H}|_{B^*\mathbb{T}^n} \in \mathcal{H}$ for which $\gamma_\infty$ is strictly less than the asymptotic Hofer norm $\rho_\infty$. This would imply $\rho_\infty(\phi_H) > \gamma_\infty(\phi_H) = \beta_{\widetilde{H}}(0)$. But again, the construction in the proof of Theorem 5.5 uses properties of the symplectic homogenization operator which rely on the unproven Viterbo bound.

## 5.4 Symplectic rigidity

Following the methods of [EP2] and [EP3] we can extract rigidity results using the existence of the partial symplectic quasi-integral $\zeta = \zeta_0\colon C_c^\infty(T^*N) \to \mathbb{R}$ introduced in Theorem 3.10. Rigidity of subsets is thereby a phenomenon in symplectic topology which means that certain subsets of symplectic manifolds cannot be completely displaced from another one or from themselves by a Hamiltonian diffeomorphism while it is possible to replace them by a genuine diffeomorphism.

**Definition 5.14.** A compact subset $X \subset T^*N$ is called $\zeta$-superheavy (or superheavy) if $\zeta(F) = c$ for any $F \in C_c^\infty(T^*N)$ with $F|_X = c \in \mathbb{R}$.

Note that the zero section $N$ is superheavy according to property (viii) in Theorem 3.10 and that the collection of superheavy sets is invariant under the action of $\mathrm{Ham}(T^*N)$ according to property (ii) in Theorem 3.10.

**Remark 5.15.** The notion of superheavy sets was introduced in [EP3] for closed symplectic manifolds. There the authors also introduce the notion of heavy sets for closed symplectic manifolds but since we could not find an example of a heavy set which is not superheavy, we do not introduce this notion at this point.

Superheavy sets are rigid in the sense that any two must intersect.

**Proposition 5.16.** *For two superheavy sets $X, Y \subset T^*N$ we have $X \cap Y \neq \varnothing$.*

**Proof.** Assume on the contrary $X \cap Y = \varnothing$ and choose $F, G \in C_c^\infty(T^*N)$ with disjoint supports such that $F|_X = G|_Y = -1$ and $\|F\|_{C^0} = \|G\|_{C^0} = 1$. This means that $F, G$ Poisson commute. Thus, we have, using (ix) in Theorem 3.10 and superheaviness of $X$ and $Y$,

$$\zeta(F + G) \leq \zeta(F) + \zeta(G) \leq -2.$$

But this is a contradiction to

$$|\zeta(F-(-G))| \le \|F-(-G)\|_{C^0} = \|F+G\|_{C^0} = 1.$$

□

**Corollary 5.17.** *Superheavy sets are non-displaceable.*

**Proof.** The statement follows from the above proposition and the fact that superheavy sets are invariant under the action of $\mathrm{Ham}(T^*N)$. □

The following proposition allows to construct examples of superheavy sets:

**Proposition 5.18.** *Let $X \subset T^*N$ be a compact subset such that $T^*N \setminus X = U_\infty \cup \bigcup_i U_i$ is a finite disjoint union with $U_\infty$ being the unbounded connected component (the union of the two unbounded connected components in case* $\dim N = 1$*). Assume that $U_\infty$ is disjoint from the zero section and that each $U_i$ is displaceable. Then $X$ is superheavy.*

**Proof.** For any $F \in C_c^\infty(T^*N)$ such that $F|_X = c$ we need to show $\zeta(F) = c$. By the Lipschitz continuity of $\zeta$ it suffices to show this for all $F$ which equal $c$ on an open neighborhood of $X$. Let $F$ be such a function. Denote $\hat{X} = X \cup \bigcup_i U_i = T^*N \setminus U_\infty$. Define a function $\hat{F}$ as follows: Let $\hat{F} = F$ on $U_\infty$ and $\hat{F} = c$ on $\hat{X}$. Then $\hat{F}$ is smooth and since $N \subset \hat{X}$, we have $\hat{F}|_N = c$. Thus, $\zeta(\hat{F}) = c$. If we can prove $\zeta(F) = \zeta(\hat{F})$, the claim follows. Define functions $F_i$ by $F_i|_{U_i^c} = 0$ and $F_i|_{U_i} = c - F$ for all $i$. The functions $F_i$ are smooth and have compact support in $U_i$, where $U_i$ is displaceable; thus, $\zeta(F_i) = 0$. Moreover, we have

$$\hat{F} = F + \sum_i F_i.$$

Since all $F_i$ commute with each other and with $F$, we conclude with property (vii) in Theorem 3.10

$$\zeta(\hat{F}) = \zeta(F + \sum_i F_i) = \zeta(F)\,.$$

□

**Remark 5.19.** Note that we can replace the assumption that any $U_i$ is displaceable by the weaker one saying that any $U_i$ is $\zeta$-null. Here we call an open subset $U \subset T^*N$ $\zeta$-null if $\zeta|_{C_c^\infty(U)} \equiv 0$. Note that displaceable subsets are $\zeta$-null according to (iv) in Theorem 3.10. More generally, if every compact subset of an open subset $U \subset T^*N$ is displaceable, then $U$ is $\zeta$-null.

**Example 5.20.** Let $q_0 \in N$ be a fixed point and consider $D(q_0) := \{(q,p) \in T^*N \,|\, q \in N,\, \|p\| = 1\} \cup \{(q_0,p) \,|\, \|p\| \le 1\}$. The complement of $D(q_0)$ is given by the union of $U_\infty = \{(q,p) \,|\, q \in N,\, \|p\| > 1\}$ and the open unit disk cotangent bundle over $N \setminus \{q_0\}$. The latter set is null and therefore $D(q_0)$ is superheavy according to the above proposition.

Finally, we have the following result which allows to obtain yet more examples. Note that here the partial symplectic quasi-integral $\zeta$ is the one which is extended to the set of autonomous Hamiltonians with complete flow.

**Theorem 5.21.** *Let $X_i \subset T^*N_i$, $i = 1, 2$, be superheavy subsets. Then the product*

$$X_1 \times X_2 \subset T^*N_1 \times T^*N_2 = T^*(N_1 \times N_2)$$

*is superheavy.*

**Proof.** Let $X = X_1 \times X_2$ and $\zeta_1, \zeta_2, \zeta$ denote the partial symplectic quasi-integrals on $T^*N_1, T^*N_2, T^*(N_1 \times N_2)$, respectively. For any $F \in C_c^\infty(T^*(N_1 \times N_2))$ such that $F|_X = c$ we need to show $\zeta(F) = c$. On the one hand, we have

$$c = \min_X F = -\max_X(-F) \leq -\zeta(-F)$$

and with property (ix) in Theorem 3.10 we conclude $0 = \zeta(F - F) \leq \zeta(F) + \zeta(-F)$, and thus $-\zeta(-F) \leq \zeta(F)$ which yields

$$c \leq \zeta(F)\,.$$

Thus, it suffices to show $\zeta(F) \leq c$. Moreover, since $\zeta$ is Lipschitz continuous, it suffices to show the above claim for any function $F$ which equals $c$ on a neighborhood of $X$. Let $F$ be such a function and $U$ the corresponding neighborhood.

Roughly speaking, the idea is to bound $\zeta(F)$ from above by $\zeta(F_1 \oplus F_2)$, where $F_i|_{X_i} = c/2$ for $i = 1, 2$, and to use the superheaviness of $X_1$ and $X_2$ and the product formula for $\zeta$ to conclude. For $i = 1, 2$ let $U_i \subset N_i$ be neighborhoods of $X_i$ such that $\overline{U_1 \times U_2} \subset U$. Let $S_i$ be a closed cotangent disk bundle in $T^*N_i$ which contains the image of the support of $F$ under the projection $T^*(N_1 \times N_2) \to T^*N_i$. Consider functions $F_i \in C_c^\infty(T^*N_i)$ such that $F_i|_{X_i} = c/2$, $F_i|_{U_i \setminus X_i} \geq c/2$, $F_i|_{S_i \setminus U_i} = M$ and $F_i|_{S_i^c} \geq 0$, where $M > 0$ is a real number such that $\min(2M, M + c/2) \geq \max F$. Note that since $X_i$ is superheavy, we have $\zeta_i(F_i) = c/2$ for $i = 1, 2$. For the direct sum of the $F_i$'s we have $F_1 \oplus F_2 \geq F$ on $S = S_1 \times S_2$ and $F_1 \oplus F_2 > 0$ on the boundary of $S_1 \times S_2$. Therefore, there is a neighborhood $V$ of $S_1 \times S_2$ such that $F_1 \oplus F_2$ is positive on $V \setminus S_1 \times S_2$. Moreover, the flow of $F_1 \oplus F_2$ keeps the zero section inside $S$. Let $G$ denote a cutoff of $F_1 \oplus F_2$ outside $V$. The function $G$ is compactly supported and its flow keeps the zero section inside $S$. According to Lemma 2.17 this implies that $\zeta(F_1 \otimes F_2) = \zeta(G)$. Since $G \geq F$, we conclude

$$\zeta(F) \leq \zeta(G) = \zeta(F_1 \otimes F_2) = \zeta_1(F_1) + \zeta_2(F_2) = c/2 + c/2 = c\,.$$

□

The theorem implies the following

**Corollary 5.22.** *Let $X_i, X_i' \subset T^*N_i$, $i = 1, \ldots, k$, be subsets as in Proposition 5.18. Then*

$$\prod_i X_i \cap \phi(\prod_i X_i') \neq \varnothing$$

*for any Hamiltonian diffeomorphism $\phi$ on $T^* \prod_i N_i$. In particular, $\prod_i X_i$ is non-displaceable.* □

**Remark 5.23.** If the Viterbo bound would hold on $N$, the partial quasi-morphisms $\mu_a$ would be genuine homogeneous quasi-morphisms when restricted to $\mathrm{Ham}(T_r^* N)$ and the functionals $\zeta_a$ would be genuine symplectic quasi-integrals (recall Remark 3.8). In this case there would be yet more applications following from the existence of the functions. In fact, there would be applications to the second bounded cohomology, the (stable) commutator norm, and to asymptotics of Hamilton-Jacobi equations.

# 6 Comparison of symplectic quasi-integrals in two dimensions

In this part of the work we are interested in the comparison of two particular symplectic quasi-integrals in two dimensions. On the one hand, we prove that there exists a symplectic quasi-integral $\eta_0$ on $T^*S^1$ which is uniquely characterized by its additional properties. On the other hand, there exists, due to Entov and Polterovich, a Calabi quasi-state $\zeta_{EP}$ on $S^2$ which is uniquely characterized by its additional properties as well [EP1], [EP2]. Thus, there are two symplectic quasi-integrals, one on $T^*S^1$ and one on $S^2$, which are both universal in some sense, and therefore it is an interesting question whether they are equal on an open neighborhood of the zero section in $T^*S^1$. More precisely, we consider a symplectic embedding $S^1 \times (-r, r) \to S^2$, pull $\zeta_{EP}$ back via this embedding and ask, whether this pull back coincides with the restriction of $\eta_0$ to $S^1 \times (-r, r)$. We provide a necessary and sufficient condition for these symplectic quasi-integrals to be equal.

Moreover, it will turn out that the symplectic quasi-integral $\eta_0$ on $T^*S^1$ is closely related to Viterbo's symplectic homogenization operator in two dimensions. Using the symplectic quasi-integral $\eta_0$ we can prove the existence and uniqueness of an operator

$$\mathcal{H}\colon C_c(T^*S^1) \to C_c(\mathbb{R}^n)$$

which has the properties of symplectic homogenization by an axiomatic approach. In fact, we can prove the existence and uniqueness of an operator on $T^*S^1$ which has the properties of Viterbo's symplectic homogenization operator introduced in Section 4, where the partial quasi-additivity is replaced by the stronger property of strong quasi-additivity. The operator $\mathcal{H}$ allows to define symplectic quasi-integrals $\eta_\sigma$ on $T^*S^1$ by integration against Radon measures $\sigma$, that is, we prove that

$$\eta_\sigma(F) = \int_{\mathbb{R}} \mathcal{H}(F)\, d\sigma$$

is a symplectic quasi-integral for any Radon measure $\sigma$. If we take the Radon measure $\sigma$ to be the Dirac measure centered at zero, i.e. $\eta_0(F) = \mathcal{H}(F)(0)$, we obtain the unique symplectic quasi-integral $\eta_0$ on $T^*S^1$.

In order to prove the existence of the symplectic quasi-integral $\eta_0$ on $T^*S^1$ and in order to compare it with the Calabi quasi-state $\zeta_{EP}$, we define the notion of quasi-integrals and topological measures for locally compact Hausdorff spaces and develop a representation theory for quasi-integrals in terms of topological measures. It is a generalization of the representation theorem for quasi-states and quasi-measures on compact Hausdorff spaces due to Aarnes [Aa1]. In addition, we introduce a reduction argument for topological measures and prove a statement about the symplecticity of quasi-integrals on surfaces without boundary; both are needed to prove the existence of $\eta_0$ and to compare the symplectic quasi-integrals.

The notions of quasi-integrals and topological measures for locally compact Hausdorff spaces and the representation theorem are subject of Subsection 6.1. In Subsection 6.2 we give the statement about the symplecticity of quasi-integrals on surfaces without boundary, and in Subsection 6.3 we introduce the reduction argument for topological measures.

In Subsection 6.4 we prove the existence and uniqueness of the partial symplectic quasi-integral $\eta_0$ on $T^*S^1$ and relate it to Viterbo's symplectic homogenization on $T^*S^1$. The construction and properties of the Calabi quasi-state $\zeta_{EP}$ due to Entov and Polterovich are subject of Subsection 6.5. In Subsection 6.6 we finally compare the symplectic quasi-integrals $\eta_0$ and $\zeta_{EP}$.

## 6.1 Quasi-integrals and topological measures on locally compact Hausdorff spaces

In this section we define quasi-integrals and topological measures for locally compact Hausdorff spaces and prove a representation theorem for them which is needed in order to prove the existence of the symplectic quasi-integral $\eta_0$ on $T^*S^1$ and in order to compare the two quasi-integrals $\eta_0$ and $\zeta_{EP}$ on an open neighborhood of the zero section in $T^*S^1$. The representation theorem states that there is a bijection between the set of quasi-integrals and the set of topological measures for locally compact Hausdorff spaces; it is a generalization of Aarnes' representation theorem for quasi-states and quasi-measures on compact Hausdorff spaces.

### 6.1.1 Quasi-integrals and topological measures

Let $X$ be a locally compact Hausdorff space. Denote by $C(X)$ the space of real valued continuous functions on $X$ and by $C_c(X) \subset C(X)$ the subspace of all continuous functions with compact support on $X$. On $C_c(X)$ we use the $C^0$-norm given by $\|F\|_{C^0} = \sup_{x\in X} |F(x)|$ for $F \in C_c(X)$. We make the following definition.

**Definition 6.1.** Let $X$ be a locally compact Hausdorff space. A functional $\zeta\colon C_c(X) \to \mathbb{R}$ is called a quasi-integral if it satisfies:

(i) *Monotonicity*: $\zeta(F) \le \zeta(G)$ for all $F, G \in C_c(X)$ with $F \le G$;

(ii) *Quasi-linearity*: $\zeta$ is linear on every subspace of $C_c(X)$ of the form $\{\phi \circ F \mid \phi \in C(\mathbb{R}), \phi(0) = 0\}$, where $F \in C_c(X)$;

(iii) *Lipschitz continuity*: For every compact subset $K \subset X$ there is a number $N_K \ge 0$ such that $|\zeta(F) - \zeta(G)| \le N_K \|F - G\|_{C^0}$ for all $F, G \in C_c(X)$ with support contained in $K$.

In case $X$ is compact and $\zeta$ is normalized, i.e. $\zeta(1) = 1$, it is called a quasi-state. It is called simple if $\zeta(F^2) = (\zeta(F))^2$ for any $F \in C(X)$.

**Remark 6.2.** In case $X$ is compact, the definition of quasi-states was introduced and first studied by Aarnes [Aa1]. In his sense, a quasi-state is a normalized, quasi-linear functional such that $\zeta(F) \ge 0$ for all $F \ge 0$. It is also proved in [Aa1] that these properties yield monotonicity, which in turn implies the Lipschitz continuity of $\zeta$. Thus, a quasi-state of Aarnes is the same as a quasi-integral on a compact Hausdorff space in the sense of the above definition. Moreover, it is proved in [Aa1] that a quasi-state $\zeta$ on a compact Hausdorff space satisfies

$$\zeta(F) = \zeta(F^+) - \zeta(F^-)$$

for every $F \in C(X)$, where $F = F^+ - F^-$ denotes the decomposition of $F$ into its positive and negative part, i.e. $F^+(x) = \max(0, F(x))$ and $F^-(x) = -\min(0, F(x))$.

**Remark 6.3.** Recall the definition of a symplectic quasi-integral on $T^*N$, Definition 1.3, and extend it to arbitrary locally compact symplectic manifolds $(M, \omega)$. We would like to note that every symplectic quasi-integral is a quasi-integral. In fact, strong quasi-additivity and Lipschitz continuity imply quasi-linearity: Let $\zeta\colon C_c(M) \to \mathbb{R}$ be strong quasi-additive and Lipschitz continuous. Let $F \in C_c(M)$ and $\phi, \psi \in C(\mathbb{R})$ such that $\phi(0) = \psi(0) = 0$. Note that we can replace $\phi$ and $\psi$ by functions with compact support without altering $\phi \circ F$ and $\psi \circ F$. There are functions $F_k \in C_c^\infty(M)$ and $\phi_k, \psi_k \in C_c^\infty(\mathbb{R})$ with $\phi_k(0) = \psi_k(0) = 0$ for all $k$ such that $F_k \to F$, $\phi_k \to \phi$ and $\psi_k \to \psi$ for $k \to \infty$, where the limit is with respect to the $C^0$-norm. We have $\{\phi_k \circ F_k, \psi_k \circ F_k\} = 0$ for all $k$, and thus

$$\zeta(\phi_k \circ F_k + \psi_k \circ F_k) = \zeta(\phi_k \circ F_k) + \zeta(\psi_k \circ F_k)$$

due to the strong quasi-additivity. But since $\zeta$ is Lipschitz continuous in the $C^0$-norm, we have

$$\zeta(\phi_k \circ F_k + \psi_k \circ F_k) \to \zeta(\phi \circ F + \psi \circ F)\,,$$

and

$$\zeta(\phi_k \circ F_k) \to \zeta(\phi + F)\,, \zeta(\psi_k \circ F_k) \to \zeta(\psi \circ F)\,,$$

proving the quasi-linearity of $\zeta$.

Let $\mathcal{K}(X)$ be the family of compact subsets of $X$, $\mathcal{O}(X)$ the family of open subsets of $X$ with compact closure, and $\mathcal{A}(X) = \mathcal{K}(X) \cup \mathcal{O}(X)$.

**Definition 6.4.** A function $\tau\colon \mathcal{A}(X) \to [0, \infty)$ is called a topological measure if it satisfies:

(i) *Additivity*: If $A, A' \in \mathcal{A}(X)$ are disjoint and $A \cup A' \in \mathcal{A}(X)$, then $\tau(A \cup A') = \tau(A) + \tau(A')$;

(ii) *Monotonicity*: If $A, A' \in \mathcal{A}(X)$ such that $A \subset A'$, then $\tau(A) \leq \tau(A')$;

(iii) *Regularity*: For any $K \in \mathcal{K}(X)$ we have $\tau(K) = \inf\{\tau(O) \,|\, O \in \mathcal{O}(X), K \subset O\}$ (outer regularity). For any $O \in \mathcal{O}(X)$ we have $\tau(O) = \sup\{\tau(K) \,|\, K \in \mathcal{K}(X), O \supset K\}$ (inner regularity).

**Remark 6.5.** In case $X$ is compact, a topological measure is the same as a quasi-measure in the sense of Aarnes [Aa1]. According to Aarnes, a quasi-measure is a function $\tau\colon \mathcal{A}(X) \to [0, \infty)$ that satisfies monotonicity and additivity for pairs of compact subsets $K \in \mathcal{K}(X)$, inner regularity, and normalization $\tau(K) + \tau(X \setminus K) = \tau(X)$ for $K \in \mathcal{K}(X)$. It is proved in [Aa1] that these four properties imply monotonicity and additivity for general subsets of $\mathcal{A}(X)$, and outer regularity. Therefore, a quasi-measure in the sense of Aarnes is the same as a topological measure on a compact Hausdorff space in the sense of the above definition.

### 6.1.2 Representation theory

The above definitions generalize the ones of quasi-states and quasi-measures on compact Hausdorff spaces due to Aarnes. In case $X$ is compact, there is a representation theory for quasi-states in terms of quasi-measures due to Aarnes in the sense that every quasi-measure determines a quasi-state, and all quasi-states arise in this way [Aa1]. We generalize this result and obtain:

**Theorem 6.6.** *There is a natural bijection between the set of quasi-integrals and the set of topological measures on a locally compact Hausdorff space.*

In the original work [Aa1] Aarnes used delicate analysis in order to prove his representation theorem. In order to prove the above theorem for the locally compact case we rely on results valid in the compact case by using a reduction argument via one-point compactifications.

Therefore, we briefly recall the representation theory for compact Hausdorff spaces; the general reference is [Aa1]. Aarnes proved that to each quasi-measure $\tau$ there corresponds a unique quasi-state $\zeta$ as follows: Let $F \in C(X)$ and consider the compact subset $\{F \geq t\} = \{x \in X \,|\, F(x) \geq t\} \subset X$. Then the function $t \mapsto \tau(\{F \geq t\})$ is non-increasing and we have $\tau(\{F \geq t\}) = 1$ for $t \leq \min F$ and $\tau(\{F \geq t\}) = 0$ for $t > \max F$. Thus, the functional

$$\zeta(F) = \tau(X) \cdot \min F + \int_{\min F}^{\max F} \tau(\{F \geq t\})\, dt$$

is well-defined, and it is a quasi-state on $C(X)$ according to Aarnes. Vice versa, for any quasi-state $\zeta$ on $C(X)$ there is a unique quasi-measure $\tau$ on $X$ with $\tau(X) = 1$ such that $\zeta$ is the quasi-state corresponding to $\tau$. The quasi-measure $\tau$ associated to $\zeta$ is given by

$$\tau(K) = \inf\{\zeta(F) \,|\, F \in C(X), F \geq \mathbb{1}_K\}\,,$$

for any compact subset $K \in \mathcal{K}(X)$ and $\tau(U) = 1 - \tau(X \setminus U)$ for any open subset $U \in \mathcal{O}(X)$. Here and in the sequel, $\mathbb{1}$ stands for the characteristic function of a set.

#### One-point compactifications

Let $X$ be a locally compact Hausdorff space. Any open subset of $X$ is a locally compact space as well. Moreover, since $X$ is a locally compact Hausdorff space, it is completely regular, meaning that for any $x \in X$ and any closed subset $A \subset X$ such that $x \notin A$ there is a continuous function $f\colon X \to [0, 1]$ such that $f(x) = 0$ and $f(a) = 1$ for any $a \in A$.

Fix an open subset $O \in \mathcal{O}(X)$ and let $\widehat{O} = O \cup \{\infty\}$ be its one-point compactification. For a topological measure $\tau$ on $X$ we can define a topological measure $\widehat{\tau}_O$ on $\widehat{O}$. Similarly, given a quasi-integral $\zeta$ on $X$ we can define a quasi-integral $\widehat{\zeta}_O$ on $\widehat{O}$. These topological measures and quasi-integrals on the compactified space $\widehat{O}$ allow to prove the representation theorem for locally compact Hausdorff spaces by using the representation theory for the compact case.

A topological measure on the compactified space $\widehat{O}$ is a map on the union $\mathcal{A}(\widehat{O})$ of the open and compact subsets of $\widehat{O}$. Here we have

$$\mathcal{O}(\widehat{O}) = \{U \subset O \text{ open}\} \cup \{(O \setminus K) \cup \{\infty\} \mid K \in \mathcal{K}(O)\}$$

and

$$\mathcal{K}(\widehat{O}) = \mathcal{K}(O) \cup \{(O \setminus U) \cup \{\infty\} \mid U \subset O \text{ open}\}.$$

For a topological measure $\tau$ on $X$ we define

$$\widehat{\tau}_O \colon \mathcal{A}(\widehat{O}) \to [0, \infty)$$

by

$$\widehat{\tau}_O(U) = \tau(U)\,, \quad \widehat{\tau}_O(K) = \tau(K)$$

and

$$\widehat{\tau}_O((O \setminus K) \cup \{\infty\}) = \tau(O \setminus K)\,, \quad \widehat{\tau}_O((O \setminus U) \cup \{\infty\}) = \tau(O) - \tau(U)$$

for $U \subset O$ open and $K \in \mathcal{K}(O)$.

**Lemma 6.7.** *$\widehat{\tau}_O$ is a topological measure on $\widehat{O}$.*

**Proof.** First of all, we note that $\widehat{\tau} = \widehat{\tau}_O$ is well-defined. Because of Remark 6.5 it suffices to prove that $\widehat{\tau}$ is a topological measure in the sense of Aarnes, that is, it suffices to show:

(i) *Normalization*: $\widehat{\tau}(\widehat{O} \setminus K) + \widehat{\tau}(K) = \widehat{\tau}(\widehat{O})$ for $K \in \mathcal{K}(\widehat{O})$;

(ii) *Additivity*: $\widehat{\tau}(K \cup K') = \widehat{\tau}(K) + \widehat{\tau}(K')$ for disjoint $K, K' \in \mathcal{K}(\widehat{O})$;

(iii) *Monotonicity*: $\widehat{\tau}(K) \le \widehat{\tau}(K')$ for $K, K' \in \mathcal{K}(\widehat{O})$ with $K \subset K'$;

(iv) *Inner regularity*: $\widehat{\tau}(K) = \inf\{\widehat{\tau}(U) \mid U \in \mathcal{O}(\widehat{O}),\, U \supset K\}$ for $K \in \mathcal{K}(\widehat{O})$.

If $K, K' \in \mathcal{K}(\widehat{O})$ are compact subsets of $O$, all of the above properties follow immediately from the definition of $\widehat{\tau}$ and the corresponding properties of $\tau$. Thus, it remains to consider the following:
(i) Let $K \in \mathcal{K}(X)$ be such that $K = (O \setminus U) \cup \{\infty\}$, where $U \subset O$ is open. Then

$$\widehat{\tau}(\widehat{O} \setminus K) + \widehat{\tau}(K) = \tau(U) + (\tau(O) - \tau(U)) = \tau(O) = \widehat{\tau}(\widehat{O})\,.$$

(ii) For disjoint $K, K' \in \mathcal{K}(\widehat{O})$ we must have $K \in \mathcal{K}(O)$ and $K' = (O \setminus U) \cup \{\infty\} \in \mathcal{K}(\widehat{O})$, where $U \subset O$ is open. Then

$$\widehat{\tau}(K \cup K') = \widehat{\tau}(O \setminus (U \setminus K) \cup \{\infty\}) = \tau(O) - \tau(U \setminus K) = \underbrace{(\tau(O) - \tau(U))}_{=\widehat{\tau}(K')} + \underbrace{\tau(K)}_{=\widehat{\tau}(K)}\,.$$

(iii) Let $K, K' \in \mathcal{K}(\widehat{O})$ be such that $K \subset K'$. We have two cases to consider. On the one hand, let $K \in \mathcal{K}(O)$ and $K' = (O \setminus U) \cup \{\infty\} \in \mathcal{K}(\widehat{O})$, where $U \subset O$

is open. Then $K \subset K'$ implies $K \subset O \setminus U$ which implies $U \subset O \setminus K$ and hence $\tau(U) \leq \tau(O \setminus K) = \tau(O) - \tau(K)$. Using the above inequality we have

$$\widehat{\tau}(K') = \tau(O) - \tau(U) \geq \tau(K) = \widehat{\tau}(K)\,.$$

On the other hand, let $K = (O \setminus V) \cup \{\infty\} \in \mathcal{K}(\widehat{O})$ with $V \subset O$ open and $K'$ as above. Then $K \subset K'$ implies $V \supset U$, and so

$$\widehat{\tau}(K) = \tau(O) - \tau(V) \leq \tau(O) - \tau(U) = \widehat{\tau}(K')\,.$$

(iv) Let $K = (O \setminus U) \cup \{\infty\} \in \mathcal{K}(\widehat{O})$, where $U \subset O$ is open. If an open set $V \subset \widehat{O}$ contains $K$, it has to be of the form $V = (O \setminus L) \cup \{\infty\}$, where $L \subset O$ is compact, and then it follows that $L \subset U$. Thus,

$$\inf\{\widehat{\tau}(V) \,|\, V \text{ open}, V \supset K\} = \inf\{\tau(O) - \tau(L) \,|\, L \subset U \text{ compact}\}$$

which equals

$$\tau(O) - \sup\{\tau(L) \,|\, L \subset U \text{ compact}\} = \tau(O) - \tau(U) = \widehat{\tau}(K)\,,$$

where the first equality follows from the inner regularity of $\tau$. □

Let $\zeta\colon C_c(X) \to \mathbb{R}$ be a quasi-integral on $X$. Since $\zeta$ is Lipschitz continuous on $C_c(X)$, the restriction $\zeta|_{C_c(O)}$ of $\zeta$ to $C_c(O)$ is Lipschitz continuous as well. We extend $F \in C_c(O)$ to $\widehat{O}$ by setting $F(\infty) = 0$ and consider the space $C_c(O)$ as a dense subset of $C_0(\widehat{O}) = \{F \in C(\widehat{O}) \,|\, F(\infty) = 0\}$ in the $C^0$-norm. This yields, together with the Lipschitz continuity, that there is a unique extension $\widehat{\zeta}_O$ of $\zeta$ to $C_0(\widehat{O})$. Moreover, $\widehat{\zeta}_O$ is also Lipschitz continuous with the same Lipschitz constant as $\zeta|_{C_c(O)}$.

We extend $\widehat{\zeta}_O$ to $C(\widehat{O})$, where we use the same notation $\widehat{\zeta}_O$. For $F \in C(\widehat{O})$ we put

$$\widehat{\zeta}_O(F) = \widehat{\zeta}_O(F - F(\infty)) + \lambda_O F(\infty)\,,$$

where $\lambda_0 = \sup\{\zeta(G) \,|\, G \in C_c(O), G \leq \mathbb{1}_O\}$ and $F - F(\infty) \in C_0(\widehat{O})$. Note that $\lambda_O = \tau_\zeta(O)$.

**Lemma 6.8.** *$\widehat{\zeta}_O$ is a quasi-integral on $\widehat{O}$.*

**Proof.** Abbreviate $\widehat{\zeta} = \widehat{\zeta}_O$ and $\lambda = \lambda_O$. According to Remark 6.2 it suffices to show:

(i) $\widehat{\zeta}(F) \geq 0$ for $F \in C(\widehat{O})$ such that $F \geq 0$;

(ii) $\widehat{\zeta}$ is linear on every subspace of $C(\widehat{O})$ of the form $\{\phi \circ F \,|\, \phi \in C(\mathbb{R})\}$, where $F \in C(\widehat{O})$.

(i) Let $F \in C(\widehat{O})$ be such that $F \geq 0$. Put $\widetilde{F} = F - F(\infty) \in C_0(\widehat{O})$. By definition of $\widehat{\zeta}$ we have $\widehat{\zeta}(F) = \widehat{\zeta}(\widetilde{F}) + \lambda F(\infty)$. Let $\varepsilon > 0$. Since the subspace $C_c(O)$ is dense in $C_0(\widehat{O})$ and $F \geq 0$, there is $G \in C_c(O)$ such that $\|\widetilde{F} - G\| < \varepsilon$ and $0 \geq \min G =$

$\min \widetilde{F} \geq -F(\infty)$. Moreover, there exists $H \in C_c(O)$ such that $0 \geq H \geq \min G$ and $H = \min G$ on the support of $G$. We then have $G \geq H \geq \min G \cdot \mathbb{1}_O$ and so

$$\widehat{\zeta}(G) = \zeta(G) \geq \zeta(H) \geq \lambda \cdot \min G = \lambda \cdot \min \widetilde{F} \geq -\lambda \cdot F(\infty)$$

by the definition of $\lambda$ and the linearity of $\zeta$ on $\mathbb{R} \cdot H \subset C_c(O)$. Since $\widehat{\zeta}|_{C_0(\widehat{O})}$ is Lipschitz continuous with Lipschitz constant $C$, we have

$$\widehat{\zeta}(\widetilde{F}) \geq \widehat{\zeta}(G) - C\varepsilon \geq -\lambda \cdot F(\infty) - C\varepsilon \,.$$

Thus, we obtained, for any $\varepsilon > 0$,

$$\widehat{\zeta}(F) = \widehat{\zeta}(\widetilde{F}) + \lambda \cdot F(\infty) \geq -C\varepsilon$$

which proves the claim.
(ii) Let $F \in C(\widehat{O})$ and $\phi, \psi \in C(\mathbb{R})$. We need to prove $\widehat{\zeta}(\phi \circ F + \psi \circ F) = \widehat{\zeta}(\phi \circ F) + \widehat{\zeta}(\psi \circ F)$. Note that

$$\begin{aligned}\widehat{\zeta}(\phi \circ F) &= \widehat{\zeta}(\phi \circ F - \phi(F(\infty))) + \lambda\phi(F(\infty)) \\ &= \hat{\zeta}((\phi - \phi(F(\infty))) \circ F) + \lambda\phi(F(\infty))\end{aligned}$$

and similarly for $\psi \circ F$ and $\phi \circ F + \psi \circ F = (\phi + \psi) \circ F$. Thus, proving that $\widehat{\zeta}(\phi \circ F + \psi \circ F) = \widehat{\zeta}(\phi \circ F) + \widehat{\zeta}(\psi \circ F)$ is equivalent to proving that $\widehat{\zeta}((\phi - \phi(F(\infty))) \circ F + (\psi - \psi(F(\infty))) \circ F) = \widehat{\zeta}((\phi - \phi(F(\infty))) \circ F) + \widehat{\zeta}((\psi - \psi(F(\infty))) \circ F)$. Therefore, we may assume that $\phi(F(\infty)) = \psi(F(\infty)) = 0$. Again, the idea is to use the quasi-integral $\zeta$ restricted to $C_c(O)$. Therefore, let $\widetilde{F} = F - F(\infty) \in C_0(\widehat{O})$, and let $\widetilde{\phi}$, $\widetilde{\psi}$ be defined by $\widetilde{\phi}(t) = \phi(t + F(\infty))$ and similarly for $\widetilde{\psi}$. We then have $\widetilde{\phi}(0) = \widetilde{\psi}(0) = 0$ and $\widetilde{\phi} \circ \widetilde{F} = \phi \circ F$ and the same for $\psi$. Again, $C_c(O)$ is dense in $C_0(\widehat{O})$ and we can choose a sequence $F_k \in C_c(O)$ whose limit is $\widetilde{F}$. Since $\zeta$ is quasi-linear (on $C_c(O)$), we have

$$\zeta(\widetilde{\phi} \circ F_k + \widetilde{\psi} \circ F_k) = \zeta(\widetilde{\phi} \circ F_k) + \zeta(\widetilde{\psi} \circ F_k) \,.$$

When $k \to \infty$, the left-hand side tends to $\widehat{\zeta}(\widetilde{\phi} \circ \widetilde{F} + \widetilde{\psi} \circ \widetilde{F}) = \widehat{\zeta}(\phi \circ F + \psi \circ F)$, while the right-hand side tends to $\widehat{\zeta}(\widetilde{\phi} \circ \widetilde{F}) + \widehat{\zeta}(\widetilde{\psi} \circ \widetilde{F}) = \widehat{\zeta}(\phi \circ F) + \widehat{\zeta}(\psi \circ F)$. □

### 6.1.3 Proof of the representation theorem

In this subsection we prove the representation theorem, Theorem 6.6. We establish procedures of going from quasi-integrals to topological measures and vice versa and prove that these procedures are inverse to each other.

#### From quasi-integrals to topological measures

Let $\zeta$ be a quasi-integral on $X$. Define a function

$$\tau_\zeta \colon \mathcal{A}(X) \to [0, \infty)$$

by

$$\tau_\zeta(K) = \inf\{\zeta(F) \mid F \in C_c(X), F \geq \mathbb{1}_K\}$$

for $K \in \mathcal{K}(X)$ and

$$\tau_\zeta(O) = \sup\{\zeta(F) \,|\, F \in C_c(X), F \leq \mathbb{1}_O\}$$

for $O \in \mathcal{O}(X)$.

**Lemma 6.9.** *$\tau_\zeta$ is a topological measure on $X$.*

**Proof.** Abbreviate $\tau = \tau_\zeta$.
*Monotonicity*: For pairs of compact subsets and for pairs of open subsets, monotonicity follows from the definition. Let $K \in \mathcal{K}(X)$ and $O \in \mathcal{O}(X)$ such that $O \subset K$. Then for any function $F$ such that $F \geq \mathbb{1}_K$ and any function $G$ with $G \leq \mathbb{1}_O$ we have $F \geq G$ and so $\tau(K) = \inf \zeta(F) \geq \sup \zeta(G) = \tau(O)$, the infimum and supremum being taken over all such $F, G$. Assume now that $K \subset O$. Then there exists $F \in C_c(X)$ with values in $[0, 1]$ such that $F|_K = 1$ and $F|_{X\setminus O} = 0$. Thus, $F \leq \mathbb{1}_O$ and $\tau(K) \leq \zeta(F) \leq \tau(O)$.
*Regularity*: Let $K \in \mathcal{K}(X)$. For outer regularity we have to prove that $\tau(K) = \inf\{\tau(O) \,|\, O \in \mathcal{O}(X),\, O \supset K\}$. Denote the infimum by $I$. It follows from monotonicity that $\tau(K) \leq I$. Thus, we need to show that $\tau(K) \geq I$. Let $\varepsilon > 0$ and fix a compact set $L$ containing $K$ in its interior. By the definition of the infimum and the fact that $X$ is completely regular there is a function $F$ such that $F|_K = 1$, $F = 0$ outside the interior of $L$ and $\tau(K) \geq \zeta(F) - \varepsilon$. By continuity of $F$, compactness of $K$, and local compactness of $X$, there is $O \in \mathcal{O}(X)$ such that $K \subset O \subset L$ and $F|_O > 1 - \varepsilon$. This means that any function $G$ with $G \leq \mathbb{1}_O$ satisfies $\frac{F}{1-\varepsilon} > G$, and so $\frac{1}{1-\varepsilon}\zeta(F) \geq \tau(O)$. Putting this together, and using the monotonicity of $\tau$ established above, we obtain

$$\tau(K) \geq \zeta(F) - \varepsilon \geq (1 - \varepsilon)\tau(O) - \varepsilon \geq \tau(O) - \varepsilon(1 + \tau(L)) \geq I - \varepsilon(1 + \tau(L))\,.$$

Since $\varepsilon$ was arbitrary and $L$ is fixed, we get $\tau(K) \geq I$. A similar argument shows inner regularity.
*Additivity*: For pairs of disjoint compact subsets and for pairs of disjoint open subsets, additivity follows from the definitions and the properties of the infimum and supremum. It remains to establish additivity for a disjoint pair $K \in \mathcal{K}(X)$ and $O \in \mathcal{O}(X)$ such that $K \cup O$ is either open or compact. Let us assume that $U = K \cup O$ is open (and then necessarily with compact closure); the case when the union is compact is treated similarly. Note that regularity implies $\tau(U) \geq \tau(O) + \tau(K)$ since for any compact $K' \subset O$ the union $K' \cup K$ is disjoint, compact, and contained in $U$, so $\tau(U) \geq \tau(K) + \tau(K')$. Taking the supremum over all such $K'$ we obtain the statement. Thus, it remains to show $\tau(U) \leq \tau(O) + \tau(K)$. Also by the regularity of $\tau$ we have the following statement: for any $\varepsilon > 0$ there is an open neighborhood $P$ of $K$ with compact closure such that whenever $F$ satisfies $\mathbb{1}_K \leq F \leq \mathbb{1}_P$, it is true that $\zeta(F) \geq \tau(K) \geq \zeta(F) - \varepsilon$. We can choose this $P$ to lie inside any open neighborhood of $K$. Similarly, for any $\varepsilon > 0$ there is a compact set $L \subset O$ such that if $G$ satisfies $\mathbb{1}_L \leq G \leq \mathbb{1}_O$, we have $\zeta(G) \leq \tau(O) \leq \zeta(G) + \varepsilon$, and this $L$ can be chosen to contain any prescribed compact subset of $O$.

Let $\varepsilon > 0$. Let $L$ be a compact subset of $O$ as we just described. Similarly, let $P$ be an open neighborhood of $K$ with compact closure. We may assume, $X$

being Hausdorff, that $\overline{P}$ is contained in $U \setminus L$. And finally, let $M$ be a compact subset of $U$, containing $L \cup \overline{P}$, which has the same property with respect to $U$, i.e. $\mathbb{1}_M \leq H \leq \mathbb{1}_U$ implies $\zeta(H) \leq \tau(U) \leq \zeta(H) + \varepsilon$. Let $H$ be such a function. Let $H''\colon X \to [0, \varepsilon]$ be such that $H''|_K = \varepsilon$ and $H'' = 0$ outside $P$ and set $H' = H + H''$. Then $H'|_K = 1 + \varepsilon$, $1 \leq H' \leq 1 + \varepsilon$ on $\overline{P}$ and $H' = H$ outside $P$. Consider two continuous functions $\phi, \psi\colon [0, 1+\varepsilon] \to [0, 1]$ such that $\phi(t) = 0$ for $t \in [0, 1]$, $\phi(1 + \varepsilon) = 1$ and $\psi(t) = t$ for $t \in [0, 1]$ and $\phi(t) + \psi(t) = 1$ for $t \in [1, 1 + \varepsilon]$. Define $F = \phi \circ H'$, $G = \psi \circ H'$. These functions have the following properties: $\mathbb{1}_K \leq F \leq \mathbb{1}_P$, $\mathbb{1}_L \leq G \leq \mathbb{1}_O$, $\mathbb{1}_M \leq F + G \leq \mathbb{1}_U$. It follows that

$$\tau(K) + \tau(O) \geq \zeta(F) + \zeta(G) - \varepsilon = \zeta(F + G) - \varepsilon \geq \tau(U) - 2\varepsilon\,,$$

where the equality is due to the quasi-linearity of $\zeta$. Thus, we obtained the required inequality. □

**From topological measures to quasi-integrals**

Let $\tau$ be a topological measure on $X$. We can define the corresponding quasi-integral $\zeta_\tau\colon C_c(X) \to \mathbb{R}$ on $X$ using the one-point compactification procedure to $\widehat{O}$ since any $F \in C_c(X)$ has support in some $O \in \mathcal{O}(X)$. Namely, let $O \in \mathcal{O}(X)$ and $\widehat{\tau}_O$ be the corresponding topological measure on $\widehat{O}$ given by the one-point compactification procedure. Consider the corresponding quasi-integral on $C(\widehat{O})$ (which exists according to the representation theorem for compact Hausdorff spaces)

$$\zeta_O\colon C(\widehat{O}) \to \mathbb{R}$$

given by

$$\zeta_O(F) = \widehat{\tau}_O(\widehat{O}) \cdot \min F + \int_{\min F}^{\max F} \widehat{\tau}_O(\{F \geq t\})\, dt\,.$$

Since $\zeta_O$ is a quasi-integral on $\widehat{O}$, it is monotone, quasi-linear, and Lipschitz continuous with constant $N_O = \widehat{\tau}_O(\widehat{O}) = \tau(O)$. The same properties hold for the restriction of $\widehat{\zeta}$ to $C_c(O) \subset C(\widehat{O})$. Thus, for $F \in C_c(X)$ with support in some $O \in \mathcal{O}(X)$ we can define

$$\zeta_\tau(F) = \zeta_O(F)\,.$$

**Lemma 6.10.** *$\zeta_\tau$ is a quasi-integral on $X$.*

**Proof.** Since we already know that $\zeta_O|_{C_c(O)}$ is monotone, quasi-linear, and Lipschitz continuous, it remains to check that the definition of $\zeta_\tau$ is correct, that is, it remains to check that if the support of $F$ is contained in $O' \in \mathcal{O}(X)$, then $\zeta_O(F) = \zeta_{O'}(F)$. Since $O \cap O'$ is also in $\mathcal{O}(X)$ and still contains the support of $F$, we see that it suffices to consider the case $O \subset O'$.

Since $\zeta_O$ and $\zeta_{O'}$ are quasi-integrals on a compact space, they respect the decomposition $\zeta_O(F) = \zeta_O(F^+) - \zeta_O(F^-)$, and similarly for $\zeta_{O'}$, where $F^+(x) = \max(0, F(x))$ and $F^-(x) = -\min(0, F(x))$ (see Remark 6.2). Thus, we may assume $F \geq 0$. Since $F$ has compact support, we have $\min F = 0$ on $\widehat{O}$ and $\widehat{O'}$.

Moreover, $F$ has the same maximum on $\widehat{O}$ and $\widehat{O'}$. For $t > 0$ the set $\{F \geq t\}$ is compact and contained in the support of $F$ and by definition of $\widehat{\tau}_O$ and $\widehat{\tau}_{O'}$ we have

$$\widehat{\tau}_O(\{F \geq t\}) = \tau(\{F \geq t\}) = \widehat{\tau}_{O'}(\{F \geq t\}).$$

Thus, the two functions $t \mapsto \widehat{\tau}_O(\{F \geq t\})$ and $t \mapsto \widehat{\tau}_{O'}(\{F \geq t\})$ coincide on $(0, \max F]$, and hence so do their integrals on $(0, \max F]$. In summary, the functions $\zeta_O(F)$ and $\zeta_{O'}(F)$ are equal. □

**The bijection**

To prove the theorem it remains to show that the above procedures of going from quasi-integrals to topological measures and vice versa are inverse to each other.

**Lemma 6.11.** *(i) Let $\tau$ be a topological measure on $X$. Then $\tau_{\zeta_\tau} = \tau$.*

*(ii) Let $\zeta$ be a quasi-integral on $X$. Then $\zeta_{\tau_\zeta} = \zeta$.*

**Proof.** (i) Let $\tau$ be a topological measure on $X$. For $K \in \mathcal{K}(X)$ we need to prove $\tau(K) = \tau_{\zeta_\tau}(K)$. For a fixed $K \in \mathcal{K}(X)$ there is $O \in \mathcal{O}(X)$ such that $K \subset O$. But for any $O \in \mathcal{O}(X)$ the topological measure $\tau$ induces a topological measure $\widehat{\tau}_O$ on the compactified space $\widehat{O}$. We have $K \in \mathcal{K}(\widehat{O})$ and by definition $\widehat{\tau}_O(K) = \tau(K)$. According to the representation theory for the compact case there is a quasi-integral $\zeta_O$ on $\widehat{O}$ associated to $\widehat{\tau}_O$. By the representation theory for the compact case we conclude

$$\tau(K) = \widehat{\tau}_O(K) = \inf\{\zeta_O(F) \mid F \in C(\widehat{O}), F \geq \mathbb{1}_K\}.$$

Moreover, the value of the infimum remains unchanged if we only consider functions with compact support in $O$ and thus we have

$$\tau(K) = \inf\{\zeta_O(F) \mid F \in C_c(O), F \geq \mathbb{1}_K\}.$$

Now, recall the definition of the quasi-integral $\zeta_\tau$ on $X$ associated to $\tau$. It is constructed in terms of the quasi-integral $\zeta_O$ on $\widehat{O}$. In particular, $\zeta_\tau$ restricted to $C_c(O) \subset C(\widehat{O})$ coincides with the restriction of $\zeta_O$ to $C_c(O)$, and thus going from quasi-integrals to topological measures we get that

$$\tau_{\zeta_\tau}(K) = \inf\{\zeta_O(F) \mid F \in C_c(O),\ F \geq \mathbb{1}_K\} = \tau(K)$$

which gives the desired equality. By inner regularity, the same is true on $\mathcal{O}(X)$.
(ii) Since both $\zeta_{\tau_\zeta}$ and $\zeta$ are quasi-integrals, they respect the decomposition of functions into the positive and the negative part, namely $\zeta(F) = \zeta(F^+) - \zeta(F^-)$ and similarly for $\zeta_{\tau_\zeta}$ (see Remark 6.2). Therefore, it suffices to show that $\zeta$ and $\zeta_{\tau_\zeta}$ coincide on non-negative functions. Any function $F \in C_c(X)$ has support in some $O \in \mathcal{O}(X)$. Fix $O \in \mathcal{O}(X)$ and let $F \in C_c(O)$ be non-negative. The quasi-integral $\zeta$ on $X$ induces a quasi-integral $\widehat{\zeta}_O$ on $\widehat{O}$ by the one-point compactification procedure and the restriction of $\zeta$ to $C_c(O)$ coincides with the restriction of $\widehat{\zeta}_O$ to $C_c(O)$. Thus, we have $\zeta(F) = \widehat{\zeta}_O(F)$. Moreover, the quasi-integral $\widehat{\zeta}_O$ on $\widehat{O}$ is represented by a

topological measure $\tau'$ on $\widehat{O}$ according to the representation theory for the compact case and we get

$$\zeta(F) = \widehat{\zeta}_O(F) = \int_0^{\max F} \tau'(\{F \geq t\})\, dt\,.$$

The quasi-integral $\zeta_{\tau_\zeta}$ is by definition given by

$$\zeta_{\tau_\zeta}(F) = \int_0^{\max F} \tau_\zeta(\{F \geq t\})\, dt\,,$$

and it suffices to show $\tau_\zeta(\{F \geq t\}) = \tau'(\{F \geq t\})$ to prove the claim. But since $\zeta$ and $\widehat{\zeta}_O$ coincide on $C_c(O)$, so do the corresponding topological measures $\tau_\zeta$ and $\tau'$ on any $K \subset O$. □

This completes the proof of Theorem 6.6. □

**Remark 6.12.** Aarnes' representation theorem was generalized to various other settings; we refer to [Bo], [GL], [Wh]. Topological measures as defined here are a generalization of both, Aarnes quasi-measures on compact spaces and Radon measures on locally compact spaces. Furthermore, we would like to mention that the theory of quasi-integrals and topological measures on locally compact spaces, as developed here, has been established by Rustad in [Ru] using different methods.

**Remark 6.13.** A topological measure $\tau$ on a locally compact Hausdorff space $X$ extends to a unique topological measure $\widehat{\tau}$ on the one-point compactification $\widehat{X}$, such that $\widehat{\tau}(\infty) = 0$, if and only if $\tau$ is bounded. This is the case if and only if the corresponding quasi-integral is globally Lipschitz continuous, and the Lipschitz constant evidently equals $\widehat{\tau}(\widehat{X}) = \sup_{\mathcal{A}(X)} \tau$.

## 6.2 Symplecticity of quasi-integrals on surfaces without boundary

In this subsection we prove a statement concerning the symplecticity of quasi-integrals on surfaces without boundary. It is needed in order to prove the existence of the symplectic quasi-integral $\eta_0$ on $T^*S^1$.

Recall the definition of a symplectic quasi-integral, Definition 1.3, and extend it to arbitrary locally compact symplectic manifolds. Moreover, recall from Remark 6.3 that every symplectic quasi-integral is a quasi-integral. In case $\Sigma$ is a closed surface with an area form, the opposite direction is proved in [EP2], that is, every quasi-state on $\Sigma$ is symplectic, meaning that quasi-linearity implies strong quasi-additivity. In [Za2] this result is extended to the notion of Poisson commutativity for continuous functions. It is proved there that a quasi-state on a closed surface with an area form is linear on Poisson commuting subspaces of the space of continuous functions. Thereby, the definition of Poisson commutativity for continuous functions, which is due to Cardin and Viterbo [CV], is the following: Two continuous functions $F, G \in C(\Sigma)$ are said to Poisson commute if there are functions $F_k, G_k \in C^\infty(\Sigma)$ such that $F_k \to F$, $G_k \to G$, $\{F_k, G_k\} \to 0$, all in the $C^0$-norm. It is proved in [CV] that a pair of compactly supported smooth functions Poisson commutes according to this definition if and only if their Poisson bracket vanishes. Therefore, the above definition is a genuine generalization of the classical notion of Poisson commutativity from smooth to continuous functions.

We formulate a yet more general result for quasi-integrals on surfaces without boundary. Therefore, we use a particular version of Poisson commutativity for continuous functions in which all the smooth functions approximating a given continuous function have support in a fixed compact subset. Again, this definition is a generalization of the usual notion of Poisson commutativity.

**Definition 6.14.** Let $\Sigma$ be a surface without boundary and let $\omega$ be an area form on it. We say that $F, G \in C_c(\Sigma)$ Poisson commute if there is an open set $O \subset \Sigma$ with compact closure, and for $k \in \mathbb{N}$ functions $F_k, G_k \in C_c^\infty(O) \subset C_c^\infty(\Sigma)$ such that $F_k \to F$, $G_k \to G$, $\{F_k, G_k\} \to 0$, all in the $C^0$-norm.

**Proposition 6.15.** *Let $\Sigma$ be a surface without boundary with an area form $\omega$. Any quasi-integral on $\Sigma$ is linear on Poisson commutative subspaces of $C_c(\Sigma)$, and in particular, it is symplectic.*

**Proof.** Let $\eta$ be a quasi-integral on $\Sigma$. Since a quasi-integral is homogeneous by definition, it remains to prove that $\eta$ is additive on Poisson commuting functions, that is, it suffices to prove $\eta(F+G) - \eta(F) - \eta(G) = 0$ for Poisson commuting $F, G \in C_c(\Sigma)$. According to the definition of Poisson commutativity there is an open subset $O \subset \Sigma$ with compact closure and functions $F_k, G_k \in C_c(O) \subset C_c(\Sigma)$ for $k \in \mathbb{N}$ such that $F_k \to F$, $G_k \to G$ and $\{F_k, G_k\} \to 0$, where the limits are with respect to the $C^0$-norm. We assume $\int_O \omega = 1$ without loss of generality. Define $\varepsilon_k = \|\{F_k, G_k\}\|_{C^0}$. According to the main theorem in [Za2] there are functions $F'_k, G'_k \in C_c(O)$ such that $\|F_k - F'_k\|_{C^0}, \|G_k - G'_k\|_{C^0} \leq \sqrt{\varepsilon_k}$. Moreover, denote by $\Phi_k\colon O \to \mathbb{R}^2$ the evaluation map given by $\Phi_k = (F'_k, G'_k)$. The image of $\Phi_k$ is a compact subset of $\mathbb{R} \times \sqrt{\varepsilon_k}\mathbb{Z} \cup \sqrt{\varepsilon_k}\mathbb{Z} \times \mathbb{R}$, in particular, it has covering dimension $\leq 1$. Let $\widehat{O}$ denote the one-point compactification of $O$. The evaluation map $\Phi_k$ extends to a map $\Phi_k\colon \widehat{O} \to \mathbb{R}^2$ by sending $\infty$ to 0. The quasi-integral $\eta$ induces a quasi-integral $\widehat{\eta}_O$ on $\widehat{O}$ by the procedure introduced in Subsection 6.1.2. According to [Wh] the quasi-integral $(\Phi_k)_* \widehat{\eta}_0$ on $\operatorname{im} \Phi_k$ is linear since $\operatorname{im} \Phi_k$ has covering dimension $\leq 1$. Let $(x, y)$ be the coordinates in $\mathbb{R}^2$. We obtain

$$\begin{aligned} 0 &= (\Phi_k)_* \widehat{\eta}_O(x+y) - (\Phi_k)_* \widehat{\eta}_O(x) - (\Phi_k)_* \widehat{\eta}_O(y) \\ &= \widehat{\eta}_O(F'_k + G'_k) - \widehat{\eta}_O(F'_k) - \widehat{\eta}_O(G'_k)\,. \end{aligned}$$

Since $F'_k \to F$ and $G'_k \to G$ with respect to the $C^0$-norm, we obtain, together with the Lipschitz continuity of $\widehat{\eta}_O$, the claim

$$\eta(F+G) - \eta(F) - \eta(G) = 0\,.$$

□

## 6.3 A reduction argument for topological measures

In this subsection we introduce a reduction argument for topological measures on manifolds without boundary. It gives a recipe how one can try to determine the values of a topological measure; it is particularly useful in two dimensions and we will use it multiple times in the sequel. In particular, it is needed in order to prove the existence of the symplectic quasi-integral $\eta_0$ on $T^*S^1$.

Let $M$ be a manifold without boundary and let $\mathcal{B}(M)$ denote the collection of codimension zero compact connected submanifolds with boundary of $M$. Given a topological measure $\tau$ on $M$, its restriction $\tau|_{\mathcal{B}(M)}$ has the following properties:

(i) $\tau|_{\mathcal{B}(M)}$ is monotone;

(ii) $\tau|_{\mathcal{B}(M)}$ is additive under finite disjoint unions;

(iii) $\tau|_{\mathcal{B}(M)}$ is regular in the sense that for $W \in \mathcal{B}(M)$ it is true that

$$\tau(W) = \inf\{\tau(W') \mid W' \in \mathcal{B}(M) \text{ contains } W \text{ in its interior}\}.$$

Adapting the arguments in [Za1] one can show that if $\tau'\colon \mathcal{B}(M) \to [0, \infty)$ is a function satisfying (i)-(iii) above, then it is the restriction of a unique topological measure. This implies the following:

**Proposition 6.16.** *The map $\tau \mapsto \tau|_{\mathcal{B}(M)}$ is a bijection between the set of topological measures and the set of functions $\mathcal{B}(M) \to [0, \infty)$ satisfying (i)-(iii) above. In particular, a topological measure on a manifold without boundary is determined by its values on codimension zero compact connected submanifolds with boundary.* □

## 6.4 A unique symplectic quasi-integral on $T^*S^1$

In this subsection we prove the existence of a genuine symplectic quasi-integral $\eta_0\colon C_c(T^*S^1) \to \mathbb{R}$ which is uniquely characterized by its additional properties. The proof of the existence and uniqueness relies on the representation theory for quasi-integrals and topological measures on locally compact Hausdorff spaces, as well as on the statement about the symplecticity of quasi-integrals on surfaces without boundary, and the reduction argument for topological measures developed in the previous subsections. In fact, we prove the existence and uniqueness of a certain topological measure $\tau$ on $T^*S^1$ which yields, according to the representation theorem, the existence and uniqueness of a corresponding quasi-integral $\eta_0$; it is symplectic according to the symplecticity of quasi-integrals on $T^*S^1$.

Moreover, we show that the symplectic quasi-integral $\eta_0$ is closely related to Viterbo's symplectic homogenization. In fact, it can be seen as to arise from symplectic homogenization. To be more precise, we prove the existence and uniqueness of an operator $\mathcal{H}\colon C_c(T^*S^1) \to C_c(\mathbb{R})$ which can be interpreted as Viterbo's symplectic homogenization operator in two dimensions using the symplectic quasi-integral $\eta_0$. This operator $\mathcal{H}$ gives rise to symplectic quasi-integrals on $T^*S^1$ by integration against Radon measures. In particular, if one takes the Dirac measure centered at 0, the symplectic quasi-integral given by $\eta_0(F) = \mathcal{H}(F)(0)$ is the unique symplectic quasi-integral on $T^*S^1$.

The construction of the symplectic quasi-integral $\eta_0$ is based on the following

**Proposition 6.17.** *There exists a unique topological measure $\sigma$ on $T^*S^1$ which is invariant under Hamiltonian diffeomorphisms and satisfies*

$$\sigma(S^1 \times [a,b]) = \mathbb{1}_{[a,b]}(0) \,.$$

*It satisfies $\sigma(D) = 0$ for any closed smoothly embedded disk $D \subset T^*S^1$.*

Before giving the proof of the above proposition we note the following:

**Remark 6.18.** According to Proposition 6.16 we know that a topological measure on $T^*S^1$ is uniquely determined by its values on codimension zero compact connected submanifolds with boundary. In $T^*S^1$ there are two types of such submanifolds; there are disks with holes and non-contractible annuli with holes (where holes are deleted disks). The properties of a topological measure allow to fill in these holes, that is, regularity implies that the values of a topological measure on open disks are determined by the values on closed disks, and then additivity implies that it suffices to know the values of the topological measure on closed smoothly embedded disks and non-contractible annuli.

**Proof** (of Proposition 6.17). *Uniqueness*: Assume that such a topological measure $\sigma$ exists. Following the above remark it suffices to know the values of $\sigma$ on closed smoothly embedded disks and non-contractible annuli in order to determine it completely. We claim that these values are determined by the above properties of the topological measure.

Let $D \subset T^*S^1$ be a smoothly embedded closed disk. Then there is another disk $D'$ which is Hamiltonian isotopic to $D$ such that $D' \subset S^1 \times [a,b]$ where $b > a > 0$. By Hamiltonian invariance and monotonicity of $\sigma$ we conclude

$$\sigma(D) = \sigma(D') \leq \sigma(S^1 \times [a,b]) = \mathbb{1}_{[a,b]}(0) = 0 \,.$$

Let $A \subset T^*S^1$ be an annulus. Then $A$ is Hamiltonian isotopic to a unique annulus of the form $S^1 \times [a,b]$, and thus by Hamiltonian invariance of $\sigma$ the value

$$\sigma(A) = \sigma(S^1 \times [a,b]) = \mathbb{1}_{[a,b]}(0)$$

is uniquely determined. In summary, $\sigma$, if it exists, is unique.
*Existence*: To prove the existence we define the values of $\sigma$ on disks and annuli as above and extend it, by regularity and additivity, to a function on the collection of codimension zero compact connected submanifolds with boundary of $T^*S^1$. Using Proposition 6.16 and the properties of the extended function we conclude that we can extend $\sigma$ to $\mathcal{A}(T^*S^1)$. Thus, we just have constructed a topological measure with the above properties which proves that $\sigma$ exists. □

As a consequence we get:

**Proposition 6.19.** *There exists a unique symplectic quasi-integral $\eta_0$ on $T^*S^1$ which is invariant under Hamiltonian diffeomorphisms and has the Lagrangian property, i.e. $\eta_0(F) = c$ for any $F \in C_c(T^*S^1)$ such that $F|_{S^1 \times \{0\}} = c \in \mathbb{R}$. In particular, it is represented by the topological measure $\sigma$ from Proposition 6.17.*

**Proof.** *Uniqueness*: Let $\eta_0$ be a symplectic quasi-integral on $T^*S^1$ as above and denote by $\sigma$ the corresponding topological measure (which exists according to Theorem 6.6). It is invariant under Hamiltonian diffeomorphisms and we claim

$$\sigma(S^1 \times [a,b]) = \mathbb{1}_{[a,b]}(0)\,.$$

In order to prove the claim, recall that for a compact subset $K \subset T^*S^1$ we have

$$\sigma(K) = \inf\{\eta_0(K) \mid F \in C_c(T^*S^1), F \geq \mathbb{1}_K\}\,.$$

Now, if $0 \in [a,b]$, we have for any $F \in C_c(T^*S^1)$ with $F \geq \mathbb{1}_{S^1\times[a,b]}$ that $\eta_0(F) \geq 1$, due to the Lagrangian property, and thus $\sigma(S^1 \times [a,b]) = 1$. If $0 \notin [a,b]$, there is $F \in C_c(T^*S^1)$ with $F \geq \mathbb{1}_{S^1\times[a,b]}$ and $F|_{S^1\times\{0\}} = 0$ which gives $\eta_0(F) = 0$, and thus $\sigma(S^1 \times [a,b]) = 0$. In summary, $\eta_0$ is represented by the unique topological measure $\sigma$ from the previous proposition and in particular, $\eta_0$ is unique itself.
*Existence*: According to the above, the topological measure $\sigma$ and its properties are dictated by a quasi-integral $\eta_0$ which is invariant under Hamiltonian diffeomorphisms and has the Lagrangian property. Since we have just proved the existence of the topological measure $\sigma$ in Proposition 6.17, the quasi-integral $\eta_0$ exists. Moreover, according to Proposition 6.15 the quasi-integral $\eta_0$ is symplectic. □

### 6.4.1 Symplectic homogenization on $T^*S^1$

In this subsection we explain the relation between the symplectic quasi-integral $\eta_0$ on $T^*S^1$ and Viterbo's symplectic homogenization in two dimensions. According to Section 4 the latter yields an operator $\mathcal{H}\colon C_c^\infty(T^*\mathbb{T}^n) \to C_c(\mathbb{R}^n)$, where $\mathcal{H}(F)(p) = \mu_p(\phi_F)$, which is monotone, Lipschitz continuous, partial quasi-additive, invariant under Hamiltonian diffeomorphisms, and has the Lagrangian property. In two dimensions $T^*\mathbb{T}^1 = T^*S^1 = S^1 \times \mathbb{R}$ we can prove the existence and uniqueness of such an operator by an axiomatic approach using the symplectic quasi-integral $\eta_0$, where partial quasi-additivity is replaced by the stronger property of strong quasi-additivity.

**Theorem 6.20.** *There is a unique operator $\mathcal{H}\colon C_c(T^*S^1) \to C_c(\mathbb{R})$ such that for all $F, G \in C_c(T^*S^1)$:*

*(i) $\mathcal{H}$ is monotone, i.e. $\mathcal{H}(F) \leq \mathcal{H}(G)$ for $F \leq G$;*

*(ii) $\mathcal{H}$ is Lipschitz continuous with respect to the $C^0$-norm, i.e. $\|\mathcal{H}(F) - \mathcal{H}(G)\|_{C^0} \leq \|F - G\|_{C^0}$;*

*(iii) the restriction of $\mathcal{H}$ to any Poisson commutative subspace of $C_c^\infty(T^*S^1)$ is linear;*

*(iv) if there is a constant $c \in \mathbb{R}$ and $p \in \mathbb{R}$ such that $F = c$ on $S^1 \times \{p\}$, then $\mathcal{H}(F)(p) = c$;*

*(v) $\mathcal{H}$ is invariant under the natural action of $\mathrm{Ham}(T^*N)$, i.e. $\mathcal{H}(F \circ \phi) = \mathcal{H}(F)$ for all $\phi \in \mathrm{Ham}(T^*S^1)$.*

**Remark 6.21.** Recall from Section 4 that Viterbo claims in [Vi4] that the symplectic homogenization operator $\mathcal{H}$ is strong quasi-additive for any $T^*\mathbb{T}^n$ and that we have pointed out that his proof of this property relies on the unproven Viterbo bound. Therefore, it is an interesting observation that, in two dimensions, one can include the strong quasi-additivity to the properties of symplectic homogenization and prove the uniqueness and existence of the latter by an axiomatic approach which does not use the Viterbo bound.

Assuming the theorem for the moment we conclude, using Viterbo's idea in [Vi4], that the above operator $\mathcal{H}$ gives rise to symplectic quasi-integrals on $T^*S^1$.

**Proposition 6.22.** *Let $\sigma$ be a Radon measure on $\mathbb{R}$ (that is, a locally finite regular Borel measure). The functional $\eta_\sigma\colon C_c(T^*S^1) \to \mathbb{R}$ given by*

$$\eta_\sigma(F) = \int_{\mathbb{R}} \mathcal{H}(F)\, d\sigma$$

*is a symplectic quasi-integral. The Lipschitz constant of the restriction of $\eta_\sigma$ to functions with support in $S^1 \times K$, where $K \subset \mathbb{R}$ is compact, is bounded from above by $\sigma(K)$.*

**Proof.** *Monotonicity*: This follows from the monotonicity of $\mathcal{H}$.
*Lipschitz continuity*: We need to establish Lipschitz continuity for functions which are compactly supported in a subset of $T^*S^1$ of the form $S^1 \times K$, where $K \subset \mathbb{R}$ is compact. If a function $F$ has support in $S^1 \times K$, property (iv) of $\mathcal{H}$ implies that $\mathcal{H}(F)(p) = 0$ for $p \notin K$. Let $G$ be another function with support in $S^1 \times K$. We have

$$\begin{aligned}|\eta_\sigma(F) - \eta_\sigma(G)| &\le \int_{\mathbb{R}} |\mathcal{H}(F)(p) - \mathcal{H}(G)(p)|\, d\sigma(p)\\ &\le \sigma(K)\|\mathcal{H}(F) - \mathcal{H}(G)\|_{C^0} \le \sigma(K)\|F - G\|_{C^0}\end{aligned}$$

which proves Lipschitz continuity and the bound on the Lipschitz constant.
*Strong quasi-additivity*: The fact that $\eta_\sigma$ is strong quasi-additive follows from the fact that $\mathcal{H}$ is linear on Poisson commutative subspaces of $C_c^\infty(T^*S^1)$. □

If we take $\sigma$ to be the Dirac measure centered at 0, we can extract a symplectic quasi-integral by

$$\eta_0(F) = \mathcal{H}(F)(0)$$

which is invariant under Hamiltonian diffeomorphisms and has the Lagrangian property. Therefore, it is the unique symplectic quasi-integral $\eta_0$ on $T^*S^1$ which we introduced above.

**Proof** (of Theorem 6.20)**.** *Uniqueness*: Assume that the operator $\mathcal{H}\colon C_c(T^*S^1) \to C_c(\mathbb{R})$ exists. Proposition 6.22 states that the properties of $\mathcal{H}$ listed in the theorem imply that $\eta_0 = \mathcal{H}(\cdot)(0)$ is a symplectic quasi-integral which is invariant under Hamiltonian diffeomorphisms and has the Lagrangian property. Moreover, according to Proposition 6.19, the symplectic quasi-integral $\eta_0$ is unique. Now, for $p \in \mathbb{R}$, we define $\eta_p \colon C_c(T^*S^1) \to \mathbb{R}$ by $\eta_p = \mathcal{H}(\cdot)(p)$. For any $F \in C_c(T^*S^1)$ and $F_p(q,\cdot) = F(q, \cdot + p)$ we have

$$\eta_p(F) = \eta_0(F_p)\,.$$

Thus, the operator $\mathcal{H}$, if it exists, is determined by $\eta_0$ and therefore it is the unique operator satisfying the properties listed in the theorem.
*Existence*: Let $\eta_0$ be the unique symplectic quasi-integral on $T^*S^1$ which is invariant under Hamiltonian diffeomorphisms and has the Lagrangian property. We define $\mathcal{H}$ by

$$\mathcal{H}(F)(p) = \eta_p(F) = \eta_0(F_p)\,,$$

where we define $F_p(q,\cdot) = F(q,\cdot + p)$. Now, in order to prove the existence of an operator which has the properties listed in the theorem, it suffices to show that $\mathcal{H}$ just defined has these properties. Points (i) and (v) follow from the monotonicity and invariance of $\eta_p$. Since the topological measure $\sigma$ only takes values 0 and 1, the corresponding quasi-integral $\eta_0$ is globally Lipschitz continuous with constant 1, see Remark 6.13. Therefore, $\mathcal{H}$ is Lipschitz continuous with constant 1 as well, proving (ii). Point (iii) follows from the strong quasi-additivity of $\eta_0$. Property (iv) is satisfied tautologically. We also have to show that $\mathcal{H}$ indeed takes values in $C_c(\mathbb{R})$. This follows from the fact that for any $F \in C_c(T^*S^1)$ we have

$$\lim_{p\to 0} \|F - F_p\|_{C^0} = 0\,.$$

Therefore, the proof of Theorem 6.20 is complete. □

**Remark 6.23.** Using techniques similar to the ones in [Za3] one can give an explicit formula for $\mathcal{H}$ in terms of its Reeb graph, it is, in terms of its level sets, see [MZ] for details. With this formula it is proved *ibid.* that the asymptotic Hofer norm $\rho_\infty$ (recall Subsection 1.5 for definitions) of $F \in C_c^\infty(T^*S^1)$ satisfies $\rho_\infty(F) = \max \mathcal{H}(F) - \min \mathcal{H}(F)$.

Moreover, in [Vi4] Viterbo claims that the quantity $\max \mathcal{H}(F) - \min \mathcal{H}(F)$ also equals the asymptotic Viterbo norm $\gamma_\infty(F)$ for $F \in C_c^\infty(T^*S^1)$ (recall Remark 4.9 for the definition). But again, as far as we understand, the proof of this equality relies on the Viterbo bound (in particular, it uses $\mathcal{H}(-F) = -\mathcal{H}(F)$). Nevertheless, if we assume that Viterbo's claim holds, the above discussion shows that in the autonomous case we would have $\gamma_\infty(F) = \rho_\infty(F)$ for $F \in C_c^\infty(T^*S^1)$.

**Remark 6.24.** The above remark should be contrasted with Remark 5.13. If we assume that the Viterbo bound holds, then both, the equality $\gamma_\infty = \rho_\infty$ for autonomous Hamiltonians on $T^*S^1$ introduced in the above remark and the strict inequality between the asymptotic norms given in [SV] introduced in Remark 5.13, are true. One should note that this would not be a contradiction as one could assume since the point of difference would be the following: In [SV] the authors use Hamiltonians on $B^*\mathbb{T}^n$ which vanish on the boundary and which admit a smooth extension to $T^*\mathbb{T}^n$ depending only on time and on $\|p\|$ outside $B^*\mathbb{T}^n$ (this particular flavor of asymptotic Hofer geometry was introduced in Subsection 5.3). The fact that $B^*\mathbb{T}^n$ has finite volume then allows to use the Calabi invariant of the Hamiltonian as a lower bound for its asymptotic Hofer norm which is impossible on $T^*S^1$.

## 6.5 The Calabi quasi-state on $S^2$

In this subsection we introduce the Calabi quasi-state $\zeta_{EP}\colon C(S^2) \to \mathbb{R}$ due to Entov and Polterovich in order to compare it to the symplectic quasi-integral $\eta_0\colon C_c(T^*S^1) \to \mathbb{R}$ on an open neighborhood of the zero section in the next subsection. The Calabi quasi-state $\zeta_{EP}$ stems from a Calabi quasi-morphism which is given in terms of spectral invariants coming from Hamiltonian Floer homology. Therefore, we give a very short overview of Hamiltonian Floer homology, Hamiltonian spectral invariants and the construction of the Calabi quasi-morphism and the Calabi quasi-state on $\mathbb{C}P^n$. The general references are [EP1] and [EP2].

### The Calabi quasi-morphism on $\mathbb{C}P^n$

Consider the symplectic manifold $(\mathbb{C}P^n, \omega)$ with its standard symplectic structure $\omega$ given by the Fubini-Study form normalized such that $\int_{\mathbb{C}P^n} \omega^n = 1$.

In [EP1] Entov und Polterovich construct a nontrivial homogeneous quasi-morphism $\widetilde{\mu}_{EP}$ on the universal cover of the group of Hamiltonian diffeomorphisms $\widetilde{\mathrm{Ham}}(\mathbb{C}P^n)$ which descends to a homogeneous quasi-morphism $\mu_{EP}$ on the group $\mathrm{Ham}(\mathbb{C}P^n)$. When restricted to $\widetilde{\mathrm{Ham}}(U)$, the quasi-morphism $\widetilde{\mu}_{EP}$ coincides with the Calabi homomorphism on $\widetilde{\mathrm{Ham}}(U)$ for every open and displaceable subset $U \subset \mathbb{C}P^n$. It is obtained by homogenizing a certain spectral invariant coming from Hamiltonian Floer homology which is, in this setting, isomorphic to the quantum homology. We refer to [HS], [McS2], [Oh3], [Oh4] for details concerning Hamiltonian Floer and quantum homology with coefficients in a Novikov ring of closed symplectic manifolds and spectral invariants coming from Hamiltonian Floer homology; briefly, the construction is as follows:

Denote by $\Lambda$ the space of smooth contractible loops $\gamma\colon S^1 \to \mathbb{C}P^n$ and by $\widetilde{\Lambda}$ its covering which consists of equivalence classes of pairs $(\gamma, u)$, where $\gamma \in \Lambda$ and $u$ is a disk spanning $x$. Thereby, $(\gamma_1, u_1)$ and $(\gamma_2, u_2)$ are equivalent if and only if $\gamma_1 = \gamma_2$ and the disks $u_1$ and $u_2$ are homotopic with fixed boundary.

Let $H\colon S^1 \times \mathbb{C}P^n \to \mathbb{R}$ be a time-periodic generic Hamiltonian. Assume that $H$ is *normalized*, i.e. $\int_{\mathbb{C}P^n} H_t \, \omega^n = 0$ for any $t \in S^1$. Consider the action functional $\mathcal{A}_H\colon \widetilde{\Lambda} \to \mathbb{R}$ associated to $H$ given by $\mathcal{A}_H([\gamma, u]) = \int_0^1 H(t, \gamma(t))dt - \int_D u^*\omega$, where $D$ denotes the standard unit disk in $\mathbb{R}^2$. The lift $\widetilde{P}_H \subset \widetilde{\Lambda}$ of the set $P_H$ of contractible 1-periodic orbits of the Hamiltonian flow generated by $H$ is in one-to-one correspondence to the set of critical points $\mathrm{Crit}(H)$ of $\mathcal{A}_H$. The Floer complex $CF(H)$ is the complex vector space which is generated by $\mathrm{Crit}(H)$, and the differential is defined by counting isolated gradient trajectories of the negative gradient flow of $\mathcal{A}_H$ connecting critical points of $\mathcal{A}_H$ of index difference one. The full Hamiltonian Floer homology $HF_*(H)$ is well-defined, and there is an isomorphism between $HF_*(H)$ and $QH_*(\mathbb{C}P^n)$ which is grading preserving. Here $QH_*(\mathbb{C}P^n)$ denotes the quantum homology of $\mathbb{C}P^n$ which is isomorphic to $H_*(\mathbb{C}P^n; \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[[s]$ as a vector space over $\mathbb{C}$, where $\mathbb{C}[[s]$ is the field whose elements are formal Laurent series $\sum_{j \in \mathbb{Z}} z_j s^j$ in the formal variable $s$, where $z_j \in \mathbb{C}$ vanishes for large enough positive $j$. On $QH_*(\mathbb{C}P^n)$ there exists a quantum product which makes the quantum homology into an associative algebra. In fact, on $\mathbb{C}P^n$, the quantum homology $QH_*(\mathbb{C}P^n)$ is a field, and the quantum product is commutative (on more general closed symplectic manifolds one needs to consider the even-dimensional quantum homology to obtain an algebra which is commutative).

The filtered version of Hamiltonian Floer homology $HF^a_*(H)$ is obtained by the filtered Floer complex which is generated by critical points of $\mathcal{A}_H$ with action $< a$ and the differential which is induced by the full differential; Floer trajectories are negative gradient flow lines and thus the differential decreases the action. There is an inclusion homomorphism

$$i_a\colon HF^a(H) \to HF(H)\,,$$

and for generic normalized $H$ and $A \in HF_*(H) = QH_*(\mathbb{C}P^n)$ one can define spectral invariants

$$c(A,H) := \inf\{a \mid A \in \operatorname{im} i_a\}$$

which can be extended to arbitrary normalized $H$. Actually, the spectral invariants descend to $\widetilde{\mathrm{Ham}}(\mathbb{C}P^n)$ and have all the standard properties including spectrality, continuity, Hamiltonian invariance, and the triangle inequality; we denote them by $c(A,\widetilde{\phi})$. Let $e \in QH(\mathbb{C}P^n)$ denote the unit element of the quantum homology $QH_*(\mathbb{C}P^n)$ (which is a field); it is given by the fundamental class $[\mathbb{C}P^n]$. According to Entov and Polterovich, the map

$$\widetilde{\mu}_{EP}\colon \widetilde{\mathrm{Ham}}(\mathbb{C}P^n) \to \mathbb{R}$$

given by

$$\widetilde{\mu}_{EP}(\widetilde{\phi}) = -\lim_{k\to\infty} \frac{c(e,\widetilde{\phi}^k)}{k}$$

is a homogeneous quasi-morphism [EP1]. Moreover, the restriction of $\widetilde{\mu}_{EP}$ to $\widetilde{\mathrm{Ham}}(U)$ coincides with the Calabi homomorphism for any open and displaceable subset $U \subset \mathbb{C}P^n$, where the Calabi homomorphism $\widetilde{Cal}_U\colon \widetilde{\mathrm{Ham}}(U) \to \mathbb{R}$ is given by $\widetilde{Cal}_U(\widetilde{\phi}) = \int_0^1 \int_{\mathbb{C}P^n} H_t\,\omega^n\,dt$, where $H$ is a Hamiltonian with compact support in $U$ which generates $\widetilde{\phi}$ (see [McS1] for instance). Therefore, $\widetilde{\mu}_{EP}$ is called Calabi quasi-morphism. In addition, it is proved in [EPZ] that

$$\int_0^1 \min_M (F_t - G_t)\,dt \le \widetilde{\mu}_{EP}(\widetilde{\phi}_G) - \widetilde{\mu}_{EP}(\widetilde{\phi}_F) \le \int_0^1 \max_M (F_t - G_t)\,dt$$

for all normalized Hamiltonians $F, G$ generating $\widetilde{\phi}_F, \widetilde{\phi}_G$, respectively.

Moreover, the quasi-morphism $\widetilde{\mu}_{EP}$ descends to a homogeneous quasi-morphism

$$\mu_{EP}\colon \mathrm{Ham}(\mathbb{C}P^n) \to \mathbb{R}$$

which coincides with the Calabi homomorphism $Cal_U\colon \mathrm{Ham}(U) \to \mathbb{R}$ on any open and displaceable subset $U \subset \mathbb{C}P^n$ such that $\omega$ is exact on $U$ (if $\omega$ is exact on $U$, the Calabi-homomorphism is well-defined, meaning that it does not depend on the specific choice of the Hamiltonian generating the element in $\mathrm{Ham}(U)$). The quasi-morphism $\mu_{EP}$ is invariant under Hamiltonian diffeomorphisms and Lipschitz continuous with respect to Hofer's metric, i.e. $|\mu_{EP}(\phi) - \mu_{EP}(\psi)| \le \rho(\phi,\psi)$ for $\phi, \psi \in \mathrm{Ham}(\mathbb{C}P^n)$.

**The Calabi quasi-state on $\mathbb{C}P^n$**

In [EP2] Entov and Polterovich prove that the Calabi quasi-morphism $\widetilde{\mu}_{EP}$ on $\mathbb{C}P^n$ yields a functional

$$\zeta_{EP}\colon C^\infty(\mathbb{C}P^n) \to \mathbb{R}$$

by setting

$$\zeta_{EP}(F) = \int_{\mathbb{C}P^n} F\,\omega^n - \widetilde{\mu}_{EP}(\widetilde{\phi}_F) = \int_{\mathbb{C}P^n} F\,\omega^n + \lim_{k\to\infty} \frac{c(e,\widetilde{\phi}_F^k)}{k}$$

which satisfies the axioms of a symplectic quasi-state. Indeed, $\zeta_{EP}$ is monotone since $\widetilde{\mu}_{EP}$ has the continuity property. The strong quasi-additivity of $\zeta_{EP}$ follows from the fact that the restriction of a homogeneous quasi-morphism to an abelian subgroup is a homomorphism (see [Ca]) and that for $F, G \in C^\infty(\mathbb{C}P^n)$ with $\{F,G\} = 0$ it is true that $\phi_F$ and $\phi_G$ commute. The normalization $\zeta_{EP}(1) = 1$ follows by construction.

Since monotonicity gives Lipschitz continuity of $\zeta_{EP}$ in the $C^0$-norm, one can extend $\zeta_{EP}$ to a functional $\zeta_{EP}\colon C(\mathbb{C}P^n) \to \mathbb{R}$ which is a symplectic quasi-state; it is called Calabi quasi-state. According to the additional properties of the Calabi quasi-morphism $\widetilde{\mu}_{EP}$, the Calabi quasi-state $\zeta_{EP}$ satisfies [EP2]:

(i) *Vanishing*: $\zeta_{EP}(F) = 0$ if the support of $F$ is displaceable;

(ii) $\zeta_{EP}$ is invariant under the natural action of $\mathrm{Ham}(\mathbb{C}P^n)$, i.e. $\zeta_{EP}(F \circ \phi) = \zeta_{EP}(F)$ for any $\phi \in \mathrm{Ham}(\mathbb{C}P^n)$.

**The Calabi quasi-state on $S^2$**

Consider $S^2 = \mathbb{C}P^1$ with an area form $\omega$ such that $\int_{S^2} \omega = 1$. The group $\mathrm{Ham}(S^2)$ admits a Calabi quasi-morphism which has additional properties as mentioned above. According to Entov and Polterovich [EP1] any two such Calabi quasi-morphisms on $\mathrm{Ham}(S^2)$ coincide on the set of elements generated by autonomous Hamiltonians; in fact, one can explicitly compute $\mu_{EP}(\phi_F)$ in terms of the data of the level sets of $F$.

The Calabi quasi-morphism on $\mathrm{Ham}(S^2)$ gives rise to a symplectic quasi-state on $C(S^2)$ which has the vanishing property and is invariant under Hamiltonian diffeomorphisms. Since the quasi-state is Lipschitz continuous in the $C^0$-norm, it suffices to know its values on a dense subset of $C(S^2)$; in particular, it suffices to know its values on the set of smooth Morse functions on $S^2$ with distinct critical values. But any two Calabi quasi-morphisms on $\mathrm{Ham}(S^2)$ coincide on the set of smooth Morse functions and thus the Calabi quasi-states do. Therefore, the Calabi quasi-state $\zeta_{EP}$ is the unique symplectic quasi-state on $C(S^2)$ which is invariant under Hamiltonian diffeomorphisms and has the vanishing property.

Moreover, Entov and Polterovich prove that for a smooth Morse function $F$ with distinct critical values, the value of $\zeta_{EP}(F)$ is the value of the unique connected component $m_F$ of a level set of $F$ such that every component of $S^2 \setminus m_F$ has area $\leq \frac{1}{2}$ [EP2]. More general, if $\mathcal{A}_F$ denotes the space consisting of all smooth functions $G \in C^\infty(S^2)$ such that $\{F,G\} = 0$ (note that every $G \in \mathcal{A}_F$ is constant on connected components of level sets of $F$), it is true that $\zeta_{EP}(G) = G(m_F)$ for $G \in \mathcal{A}_F$. In addition, it is proved in [EP2] that the restriction of $\zeta_{EP}$ to $\mathcal{A}_F$ is multiplicative, i.e. $\zeta_{EP}(GH) = \zeta_{EP}(G)\zeta_{EP}(H)$ for any $G, H \in \mathcal{A}_F$, and the Lipschitz continuity of $\zeta_{EP}$ implies that $\zeta_{EP}$ is multiplicative on $\mathcal{A}_F$ for any $F \in C(S^2)$. In particular, $\zeta_{EP}$ is simple.

In summary, the Calabi quasi-state $\zeta_{EP}\colon C(S^2) \to \mathbb{R}$ is the unique symplectic quasi-state on $S^2$ which is invariant under Hamiltonian diffeomorphisms and has the vanishing property (and is simple).

## 6.6 Comparison

In the previous subsections we introduced two symplectic quasi-integrals in two dimensions which are uniquely characterized by their additional properties. The first one is the symplectic quasi-integral $\eta_0\colon C_c(T^*S^1) \to \mathbb{R}$ which is related to Viterbo's symplectic homogenization on $T^*S^1$; it is invariant under Hamiltonian diffeomorphisms and has the Lagrangian property. The second one is the Calabi quasi-state $\zeta_{EP}\colon C(S^2) \to \mathbb{R}$ due to Entov and Polterovich; it is invariant under Hamiltonian diffeomorphism and has the vanishing property (and is simple).

In this section we compare the two symplectic quasi-integrals. In particular, we ask whether the quasi-integrals are equal on an open neighborhood of the zero section of $T^*S^1$.

In order to give an answer to the above question we make use of the representation theorem of quasi-integrals and topological measures, Theorem 6.6, developed in Section 6.1; in fact, we compare the corresponding topological measures.

To be able to compare $\zeta = \zeta_{EP}\colon C(S^2) \to \mathbb{R}$ and $\eta = \eta_0\colon C_c(T^*S^1) \to \mathbb{R}$ we need them to be defined on the same space. Let $r \in (0, \frac{1}{2}]$ and $U_r = S^1 \times (-r, r) \subset T^*S^1$. Consider a symplectic embedding

$$j_r\colon U_r \to S^2$$

such that $j_r(S^1 \times \{0\})$ is the equator in $S^2$. The symplectic forms $\omega'$ on $S^2$ and $\omega = d\lambda$ on $T^*S^1$ are normalized so that $\operatorname{area}(S^2) = \int_{S^2} \omega' = 1$ and $\operatorname{area}(U_r) = \int_{U_r} \omega = 2r$. The symplectic embedding $j_r$ induces a map

$$j_r^!\colon C_c(U_r) \to C(S^2)$$

and we can pull $\zeta$ back to $C_c(U_r)$ by

$$\zeta_r := j_r^* \zeta = \zeta \circ j_r^! \,.$$

The following observation motivates the question whether $\zeta_r$ and $\eta$ are equal.

**Remark 6.25.** The two functionals $\zeta_r$ and $\eta$ coincide on functions which only depend on the vertical coordinate: Let $(q, p)$ be the standard coordinates on $T^*S^1$ and for $F \in C_c((-r, r))$ define $f \in C_c(U_r)$ by $f(q, p) = F(p)$. We have

$$\eta(f) = \zeta_r(f) = F(0)\,.$$

For $\eta$ this follows immediately from the definition, the statement about $\zeta$ is contained in [EP2].

In general, we have the following result:

**Theorem 6.26.** *The restriction of $\eta$ to $C_c(U_r)$ coincides with $\zeta_r$ if and only if $r \in (0, \frac{1}{4}]$.*

**Remark 6.27.** Recall from Proposition 6.22 that the operator $\mathcal{H}\colon C_c(T^*S^1) \to C_c(\mathbb{R})$ yields more general symplectic quasi-integrals $\eta_\sigma$ on $T^*S^1$. For those it can be seen that $\eta_\sigma|_{C_c(U_r)} \neq \zeta_r$ if the restriction of $\sigma$ to $(-r, r)$ does not coincide with the Dirac measure centered at 0.

**Remark 6.28.** The above result has to do with the general question of uniqueness of symplectic quasi-states and quasi-integrals. As the above theorem shows, there is no uniqueness on a neighborhood of the zero section in $T^*S^1$, even not if we impose additional properties of the symplectic quasi-integrals like Hamiltonian invariance.

To prove Theorem 6.26 we make use of the representation theorem of quasi-integrals in terms of topological measures, Theorem 6.6.

Recall that the topological measure which is associated to $\eta$ is the unique topological measure $\sigma$ on $T^*S^1$ given by Proposition 6.17. It is invariant under Hamiltonian diffeomorphisms and satisfies $\sigma(S^1 \times [a,b]) = \mathbb{1}_{[a,b]}(0)$, in particular, $\sigma$ vanishes on smoothly embedded closed disks.

Let $\tau$ be the topological measure on $S^2$ corresponding to $\zeta$. It is invariant under Hamiltonian diffeomorphisms. Moreover, according to a theorem of Aarnes [Aa2], it is simple, that is, it only takes values 0 and 1 since the corresponding quasi-state $\zeta$ is multiplicative on $\mathcal{A}_F$ for any $F \in C(S^2)$. In addition, according to [AR], [Aa3], the topological measure $\tau$ is completely determined by its values on smoothly embedded closed disks $D \subset S^2$ as follows: Let $D \subset S^2$ be a closed disk, then

$$\tau(D) = \begin{cases} 1, & \text{area}(D) \geq \frac{1}{2} \\ 0, & \text{area}(D) < \frac{1}{2} \end{cases} .$$

Now, let $\tau_r$ be the pull-back of $\tau$ by $j_r$, i.e.

$$\tau_r(A) = \tau(j_r(A))$$

for $A \in \mathcal{A}(U_r)$.

According to the representation theorem, Theorem 6.6, Theorem 6.26 is equivalent to

**Theorem 6.29.** *We have $\sigma|_{\mathcal{A}(U_r)} = \tau_r$ if and only if $r \in (0, \frac{1}{4}]$.*

**Proof** (of Theorem 6.29)**.** We prove the theorem in the language of topological measures.
(i) We need to show that $r > \frac{1}{4}$ implies $\sigma|_{\mathcal{A}(U_r)} \neq \tau_r$. If $r > \frac{1}{4}$, we have $\text{area}(U_r) \geq \frac{1}{2}$. Thus, there is a smoothly embedded closed disk $D \subset U_r$ such that $\text{area}(D) \geq \frac{1}{2}$. Thus, $j_r(D) \subset S^2$ is a closed smoothly embedded disk of area $\geq \frac{1}{2}$. Now, on the one hand, we have

$$\tau_r(D) = \tau(j_r(D)) = 1\,.$$

On the other hand,

$$\sigma(D) = 0$$

which proves that $\sigma|_{\mathcal{A}(U_r)} \neq \tau_r$ if $r > \frac{1}{4}$.
(ii) It remains to prove that we have $\sigma|_{\mathcal{A}(U_r)} = \tau_r$ if we assume $r \leq \frac{1}{4}$. Let $r \leq \frac{1}{4}$ and recall from Remark 6.18 that a topological measure on $U_r \subset T^*S^1$ is completely determined by its values on smoothly embedded closed disks and non-contractible annuli. Let $D \subset U_r$ be a smoothly embedded closed disk. Its image $j_r(D) \subset S^2$ is a disk of area $< \frac{1}{2}$ and we have

$$\tau_r(D) = \tau(j_r(D)) = 0 = \sigma(D)\,.$$

Let $A \subset U_r$ be a non-contractible annulus. It can be isotoped to a unique standard annulus of the form $S^1 \times [a,b]$, where $-r < a < b < r$, by a Hamiltonian isotopy with compact support in $U_r$. Since both, $\tau_r$ and $\sigma|_{\mathcal{A}(U_r)}$, are invariant under Hamiltonian diffeomorphisms, it suffices to consider annuli in $U_r$ of the form $S^1 \times [a,b]$. Now, there are two cases to consider: $0 \in [a,b]$ and $0 \notin [a,b]$. Let $0 \notin [a,b]$, then $j_r(S^1 \times [a,b])$ is contained in a disk of area $< \frac{1}{2}$ and so

$$\tau_r(S^1 \times [a,b]) = \tau(j_r(S^1 \times [a,b])) = 0$$

and

$$\sigma(S^1 \times [a,b]) = \mathbb{1}_{[a,b]}(0) = 0\,.$$

If $0 \in S^1 \times [a,b]$, then

$$\sigma(S^1 \times [a,b]) = \mathbb{1}_{[a,b]}(0) = 1\,.$$

For the evaluation of $\tau_r$ note that the complement of $j_r(S^1 \times [a,b])$ in $S^2$ is the disjoint union of two open disks of area $< \frac{1}{2}$, i.e. $S^2 \setminus j_r(S^1 \times [a,b]) = D \cup D'$. By additivity we get

$$\begin{aligned}\tau_r(S^1 \times [a,b]) &= \tau(j_r(S^1 \times [a,b])) = 1 - \tau(S^2 \setminus j_r(S^1 \times [a,b])) \\ &= 1 - \tau(D) - \tau(D') = 1\,.\end{aligned}$$

□

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