

**Olshanski spherical functions for  
infinite dimensional motion groups  
of fixed rank**

Margit Rösler and Michael Voit

Preprint 2012-17

Oktober 2012



# OLSHANSKI SPHERICAL FUNCTIONS FOR INFINITE DIMENSIONAL MOTION GROUPS OF FIXED RANK

MARGIT RÖSLER AND MICHAEL VOIT

ABSTRACT. Consider the Gelfand pairs  $(G_p, K_p) := (M_{p,q} \rtimes U_p, U_p)$  associated with motion groups over the fields  $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$  with  $p \geq q$  and fixed  $q$  as well as the inductive limit  $p \rightarrow \infty$ , the Olshanski spherical pair  $(G_\infty, K_\infty)$ . We classify all Olshanski spherical functions of  $(G_\infty, K_\infty)$  as functions on the cone  $\Pi_q$  of positive semidefinite  $q \times q$ -matrices and show that they appear as (locally) uniform limits of spherical functions of  $(G_p, K_p)$  as  $p \rightarrow \infty$ . The latter are given by Bessel functions on  $\Pi_q$ . Moreover, we determine all positive definite Olshanski spherical functions and discuss related positive integral representations for matrix Bessel functions. We also extend the results to the pairs  $(M_{p,q} \rtimes (U_p \times U_q), (U_p \times U_q))$  which are related to the Cartan motion groups of non-compact Grassmannians. Here Dunkl-Bessel functions of type B (for finite  $p$ ) and of type A (for  $p \rightarrow \infty$ ) appear as spherical functions.

## 1. INTRODUCTION

Let  $M_{p,q} := M_{p,q}(\mathbb{F})$  be the vector space of  $p \times q$ -matrices over one of the fields  $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$  and  $U_p = U_p(\mathbb{F})$  the group of unitary  $p \times p$ -matrices over  $\mathbb{F}$ . Consider the Gelfand pairs  $(G_p, K_p) := (M_{p,q} \rtimes U_p, U_p)$  associated with motion groups over  $\mathbb{F}$  where  $q$  is fixed and  $p \geq q$  is increasing. The inductive limit  $(G_\infty, K_\infty) := (\lim_{p \rightarrow \infty} G_p, \lim_{p \rightarrow \infty} K_p) = (M_{\infty,q} \rtimes U_\infty, U_\infty)$  is an Olshanski spherical pair; see [F], [O1], [O2] for the notion. The main purpose of this paper is to classify all Olshanski spherical functions of this pair, i.e., the continuous  $K_\infty$ -biinvariant functions  $\phi : G_\infty \rightarrow \mathbb{C}$  which satisfy the product formula

$$\phi(g)\phi(h) = \lim_{p \rightarrow \infty} \int_{K_p} \phi(gkh) dk \quad \text{for } g, h \in G_\infty \quad (1.1)$$

with respect to the normalized Haar measures  $dk$  on  $K_p$ . We shall obtain a classification in terms of certain exponential functions which relies on the fact that for integers  $p \geq q$  as well for  $p = \infty$ , the double coset space  $G_p//K_p$  may be identified with the cone  $\Pi_q$  of positive semidefinite  $q \times q$ -matrices over  $\mathbb{F}$ , and that for finite  $p$ , formula (1.1) can be made explicit by the results of [R1]. It is known (see e.g. [H], [FK], [R1] and references cited there) that for finite  $p$ , Bessel functions  $J_\mu$  of index  $\mu = pd/2$  on the cone  $\Pi_q$  provide spherical functions of  $(G_p, K_p)$ ; here  $d := \dim_{\mathbb{R}} \mathbb{F} = 1, 2, 4$ . Moreover, it is also known that under suitable rescaling, the Bessel functions  $J_\mu$  converge (locally) uniformly to exponential functions, see [O1] and [RV2], and references therein. From this we obtain that all Olshanski spherical

---

2010 *Mathematics Subject Classification*. Primary 43A90; Secondary 33C80, 43A85, 22E66.

*Key words and phrases*. Spherical functions, Olshanski spherical pairs, Bessel functions on matrix cones, Dunkl theory, positive definite functions, multivariate beta distributions.

functions appear as (locally) uniform limits of spherical functions on  $(G_p, K_p)$  for  $p \rightarrow \infty$ .

The classification of those Olshanski spherical functions which are positive definite is also easily achieved. The connection between positive definite spherical functions of  $(G_p, K_p)$  with different values of  $p$  leads to the existence of positive integral representations for the involved matrix Bessel functions  $J_{pd/2}$  as already explained by Schoenberg [S] in a general setting. Following mainly [H] and [FK] we shall derive explicit formulas for these integral representations of  $J_\nu$  in terms of  $J_\mu$  for  $\nu \geq \mu$  in two different ways. A comparison of both approaches will imply an identity for projections of beta distributions on matrix cones.

Besides the Gelfand pairs  $(M_{p,q} \times U_p, U_p)$  we also consider the Cartan motion pairs  $(M_{p,q} \times (U_p \times U_q), U_p \times U_q)$  and the associated Olshanski spherical functions as  $p \rightarrow \infty$ . For finite  $p$ , the spherical functions are now Dunkl-Bessel functions of type B which converge for  $p \rightarrow \infty$  to certain Dunkl-Bessel functions of type A, and these in turn are just the Olshanski spherical functions of  $(M_{\infty,q} \times (U_\infty \times U_q), U_\infty \times U_q)$ ; see [BF], [Du], [O], [R1], [RV1] for the background.

The approach to Olshanski spherical functions taken in this paper is similar to the one for noncompact infinite dimensional Grassmannian manifolds in [RKV] where the finite dimensional spherical functions are Heckman-Opdam hypergeometric functions of type B which converge to hypergeometric functions of type A. In this case, the classification of the Olshanski spherical functions is based on a product formula in [R2]. The present paper as well as [RKV] were partly motivated by related results in [DOW].

Our approach via special functions does not use explicit representation theory for Olshanski spherical pairs in contrast to [O11], [O12], [P] or further related references. This has the slight advantage that we need not restrict to positive definite spherical functions first, but can deal with general spherical functions from the beginning.

The paper is organized as follows: In Sections 2 and 3 we collect some facts about matrix Bessel functions and Dunkl-Bessel functions of type B and type A. The Gelfand pairs  $(M_{p,q} \times U_p, U_p)$  and the corresponding Olshanski spherical functions will then be studied in Section 4, while Section 5 is devoted to the pairs  $(M_{p,q} \times (U_p \times U_q), U_p \times U_q)$ . In Section 6 we return to matrix Bessel functions and study their positive integral representations.

## 2. BESSEL FUNCTIONS ON MATRIX CONES

In this section we collect some facts about Bessel functions on matrix cones. The material is mainly taken from [FK], [H], and [R1], and is in part slightly generalized.

Let  $\mathbb{F}$  be one of the fields  $\mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$  with real dimension  $d = 1, 2$  or  $4$  respectively. Denote the usual conjugation in  $\mathbb{F}$  by  $t \mapsto \bar{t}$ , the real part of  $t \in \mathbb{F}$  by  $\Re t = \frac{1}{2}(t + \bar{t})$ .

For  $p, q \in \mathbb{N}$  we denote by  $M_{p,q} := M_{p,q}(\mathbb{F})$  the vector space of all  $p \times q$ -matrices over  $\mathbb{F}$  and put  $M_q := M_q(\mathbb{F}) := M_{q,q}(\mathbb{F})$  for abbreviation. Let further

$$H_q = H_q(\mathbb{F}) = \{x \in M_q(\mathbb{F}) : x = x^*\}$$

be the space of Hermitian  $q \times q$ -matrices over  $\mathbb{F}$ , where  $x^* = \bar{x}^t$  denotes the usual involution on  $M_q$ . The space  $H_q$  is a real Euclidean vector space with scalar product  $\langle x, y \rangle := \text{tr}(xy)$  and norm  $\|x\| = \langle x, x \rangle^{1/2}$ , where  $\text{tr}$  is the trace. Notice that  $\text{tr}(xy)$  is real for  $x, y \in H_q$ .  $H_q$  is a (Euclidean) Jordan algebra with the usual Jordan

product  $x \circ y = \frac{1}{2}(xy + yx)$ . The real dimension of  $H_q$  is

$$n := n(q, \mathbb{F}) := \dim_{\mathbb{R}} H_q = q + \frac{d}{2}q(q-1).$$

We also need the complexification  $H_q^{\mathbb{C}}$  of  $H_q$  which is a Jordan algebra over  $\mathbb{C}$  in the natural way. The extension of the scalar product  $\langle \cdot, \cdot \rangle$  to a hermitian scalar product on  $H_q^{\mathbb{C}}$  will again be denoted by  $\langle x, y \rangle$ . For  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ , the space  $H_q^{\mathbb{C}}$  may be written as

$$H_q^{\mathbb{C}} = \{a + ib : a, b \in H_q(\mathbb{F})\} \subset M_q(\mathbb{C}).$$

More precisely, for  $\mathbb{F} = \mathbb{R}$  we have  $H_q^{\mathbb{C}}(\mathbb{R}) = \{a \in M_q(\mathbb{C}) : a^t = a\}$ , while for  $\mathbb{F} = \mathbb{C}$ ,  $H_q^{\mathbb{C}}(\mathbb{C}) = M_q(\mathbb{C})$ ; c.f. Section VIII.5 of [FK]. The complexification of the Jordan algebra  $H_q(\mathbb{H})$  can be described as follows, c.f. Sections V.2 and VIII.5 of [FK]: One realizes matrices  $z \in H_q(\mathbb{H})$  as complex Hermitian matrices

$$z = \begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix} \in H_{2q}(\mathbb{C})$$

where  $x \in H_q(\mathbb{C})$  is Hermitian and  $y \in Skew_q(\mathbb{C})$  is skew-symmetric. Then with  $J := \begin{pmatrix} 0 & I_q \\ -I_q & 0 \end{pmatrix} \in M_{2q}$ , the mapping

$$z \mapsto -Jz = \begin{pmatrix} \bar{y} & -\bar{x} \\ x & y \end{pmatrix} \in Skew_{2q}(\mathbb{C})$$

defines a real Jordan algebra isomorphism from  $H_q(\mathbb{H})$  onto the Jordan algebra  $V_q := \{u \in Skew_{2q}(\mathbb{C}) : u^* = JuJ\}$  with Jordan product  $u \circ v := \frac{1}{2}(uJv + vJu)$ . The complexification of this Jordan algebra is just  $Skew_{2q}(\mathbb{C})$  (with the same Jordan-product). We thus identify  $H_q^{\mathbb{C}}(\mathbb{H})$  and  $Skew_{2q}(\mathbb{C})$  as complex Jordan algebras.

Let

$$\Pi_q := \{x^2 : x \in H_q\} = \{x^*x : x \in H_q\}$$

denote the set of all positive semidefinite matrices in  $H_q$ , and  $\Omega_q$  its interior consisting of all strictly positive definite matrices.  $\Omega_q$  is a symmetric cone; see [FK] for details. On  $H_q$  we use the standard partial ordering

$$x \leq y : \iff y - x \in \Pi_q.$$

To define Bessel functions, we need the spherical polynomials

$$\Phi_\lambda(x) = \int_{U_q} \Delta_\lambda(uxu^{-1}) du, \quad x \in H_q$$

which are indexed by partitions  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_q) \in \mathbb{N}_0^q$  (for short,  $\lambda \geq 0$ ). Here  $du$  denotes the normalized Haar measure of  $U_q = U_q(\mathbb{F})$ , and  $\Delta_\lambda$  is the power function

$$\Delta_\lambda(x) := \Delta_1(x)^{\lambda_1 - \lambda_2} \Delta_2(x)^{\lambda_2 - \lambda_3} \dots \Delta_q(x)^{\lambda_q} \quad (x \in H_q)$$

with the principal minors  $\Delta_i(x)$  of  $x$ ; see [FK]. There is a renormalization  $Z_\lambda = c_\lambda \Phi_\lambda$  with suitable constants  $c_\lambda > 0$  depending on  $\Pi_q$  such that

$$(tr x)^k = \sum_{|\lambda|=k} Z_\lambda(x) \quad \text{for } k \in \mathbb{N}_0; \quad (2.1)$$

see Section XI.5. of [FK] where the  $Z_\lambda$  are called zonal polynomials. By construction, the  $Z_\lambda$  are invariant under conjugation by  $U_q$  and thus depend only on

the eigenvalues of their argument. More precisely, for  $x \in H_q$  with eigenvalues  $\xi = (\xi_1, \dots, \xi_q) \in \mathbb{R}^q$ ,

$$Z_\lambda(x) = C_\lambda^\alpha(\xi) \quad \text{with} \quad \alpha = \frac{2}{d} \quad (2.2)$$

where the  $C_\lambda^\alpha$  are the Jack polynomials of index  $\alpha$  (c.f. [FK], [M], [R1]). These are homogeneous of degree  $|\lambda|$  and symmetric in their arguments. Notice that the zonal polynomials  $Z_\lambda$  naturally extend to  $H_q^{\mathbb{C}}$ .

The Bessel functions associated with the cone  $\Omega_q$  are defined as  ${}_0F_1$ -hypergeometric series

$$J_\mu(z) = \sum_{\lambda \geq 0} \frac{(-1)^{|\lambda|}}{(\mu)_\lambda |\lambda|!} Z_\lambda(z), \quad z \in H_q^{\mathbb{C}} \quad (2.3)$$

where the generalized Pochhammer symbol  $(\mu)_\lambda$  is given by

$$(\mu)_\lambda = (\mu)_\lambda^{2/d} \quad \text{with} \quad (\mu)_\lambda^\alpha := \prod_{j=1}^q \left(\mu - \frac{1}{\alpha}(j-1)\right)_{\lambda_j} \quad (\alpha > 0).$$

The index  $\mu \in \mathbb{C}$  is required to satisfy  $(\mu)_\lambda^\alpha \neq 0$  for all  $\lambda \geq 0$ . In this case,  $J_\mu$  is holomorphic on  $H_q^{\mathbb{C}}$ .

If  $q = 1$ , then  $\Pi_q = \mathbb{R}_+$ , and the Bessel function  $J_\mu$  is independent of  $d$  with

$$J_\mu\left(\frac{z^2}{4}\right) = j_{\mu-1}(z)$$

where  $j_\kappa(z) = {}_0F_1(\kappa + 1; -z^2/4)$  is the modified Bessel function in one variable.

We need the following product formula for the  $J_\mu$  from [R1]:

**Proposition 2.1.** *For  $\mu \in \mathbb{C}$  with  $\Re\mu > d(q - 1/2)$  and all  $r, s \in H_q^{\mathbb{C}}$ ,*

$$J_\mu(r^2)J_\mu(s^2) = \frac{1}{\kappa_\mu} \int_{B_q} J_\mu(r^2 + s^2 + rws + sw^*r) \Delta(I_q - w^*w)^{\mu-\gamma} dw$$

where  $\Delta$  denotes the determinant on  $H_q$ ,  $dw$  the Lebesgue measure on the ball

$$B_q := \{w \in M_q : w^*w < I_q\},$$

$$\gamma := d(q - 1/2) + 1, \quad \text{and} \quad \kappa_\mu := \left( \int_{B_q} \Delta(I_q - w^*w)^{\mu-\gamma} dw \right)^{-1}.$$

*Proof.* For  $r, s \in H_q$  we refer to Eq. (3.8) of [R1]. As both sides of the equation are holomorphic in  $r, s \in H_q^{\mathbb{C}}$ , the result is true in general.  $\square$

For the limit case  $\mu = d(q - 1/2)$  there exists a degenerated product formula; see Section 3.5 of [R1].

For real  $\mu \geq d(q - 1/2)$ , product formula 2.1 leads to a positive (hypergroup) convolution structure on the Banach space  $M_b(\Pi_q)$  of all bounded signed Borel measures on  $\Pi_q$ , as follows: define first the convolution of point measures

$$(\delta_r *_{\mu} \delta_s)(f) := \frac{1}{\kappa_\mu} \int_{B_q} f(\sqrt{r^2 + s^2 + swr + rw^*s}) \Delta(I - ww^*)^{\mu-\rho} dw \quad (r, s \in \Pi_q) \quad (2.4)$$

for  $f \in C_b(\Pi_q)$ , and then extend this convolution in a weakly continuous, bilinear way to  $M_b(\Pi_q)$ . This generates a probability preserving commutative Banach algebra  $(M_b(\Pi_q), *_{\mu})$ ; see [R1] for details. For  $\mu = pd/2$  with integers  $p \geq 2q$ , this convolution is just the double coset convolution associated with the Gelfand pair  $(M_{p,q} \rtimes U_p, U_p)$  (see [R1]) which we shall consider in more detail in Section 4.

For  $s \in H_q^{\mathbb{C}}$  we define the continuous function

$$f_s^\mu(r) := J_\mu\left(\frac{1}{4}rsr\right) \quad (r \in H_q^{\mathbb{C}}). \quad (2.5)$$

as well as the function

$$\phi_s^\mu(r) := f_{s^2}^\mu(r) = J_\mu\left(\frac{1}{4}rs^2r\right), \quad (2.6)$$

where the latter definition is in correspondence with the notion in [R1]. The following facts slightly generalize results from [R1].

**Lemma 2.2.** (1) For all  $r, s \in H_q^{\mathbb{C}}$ ,  $\phi_s^\mu(r) = \phi_r^\mu(s)$ .

(2) For all  $r, t \in \Pi_q$  and  $s \in H_q^{\mathbb{C}}$ ,

$$f_s^\mu(r)f_s^\mu(t) = \int_{\Pi_q} f_s^\mu(z) d(\delta_r *_{\mu} \delta_t)(z), \quad (2.7)$$

*i.e.*, the functions  $f_s^\mu$  with  $s \in H_q^{\mathbb{C}}$  are multiplicative with respect to  $*_{\mu}$ .

*Proof.* Both statements are known for  $r, s, t \in \Pi_q$ ; see the results for  $\phi_s$  in Section 3 of [R1] and notice for the second statement that each  $s \in \Pi_q$  may be written as  $s = \tilde{s}^2$  with some unique  $\tilde{s} \in \Pi_q$ . The general statements now follow easily by analytic continuation from  $\Pi_q$  to  $H_q^{\mathbb{C}}$ . For this, notice that  $H_q^{\mathbb{C}} \setminus \Pi_q$  is connected, and that the identity theorem can be applied because  $\Pi_q$  has non-empty interior in  $H_q$ .  $\square$

### 3. DUNKL-BESSEL FUNCTIONS ON WEYL CHAMBERS OF TYPE B AND TYPE A

There is a close connection between the Bessel functions on the cone  $\Pi_q$  and the theory of Dunkl operators associated with the root system  $B_q$ , see [R1]. We briefly review this connection. We do not go into details of Dunkl theory, but refer to [BF], [Du], [O] and [R1]. For a reduced root system  $R \subset \mathbb{R}^q$  and a multiplicity function  $k : R \rightarrow [0, \infty)$  (i.e.  $k$  is invariant under the action of the corresponding reflection group), we denote by  $J_k^R$  the Dunkl-Bessel function associated with  $R$  and  $k$ . It is obtained from the Dunkl kernel by symmetrization with respect to the underlying reflection group. Dunkl-Bessel functions generalize the spherical functions of Euclidean type symmetric spaces, which occur for crystallographic root systems and specific discrete values of  $k$ , see [O]. For the root system  $A_{q-1} = \{\pm(e_i - e_j) : i < j\} \subset \mathbb{R}^q$ , the multiplicity  $k$  is a single real parameter. If  $k > 0$ , then according to formulas (3.22) and (3.37) of [BF] the associated Dunkl-Bessel function is the generalized  ${}_0F_0$ -hypergeometric function

$$J_k^A(\xi, \eta) = {}_0F_0^\alpha(\xi, \eta) := \sum_{\lambda \geq 0} \frac{1}{|\lambda|!} \cdot \frac{C_\lambda^\alpha(\xi)C_\lambda^\alpha(\eta)}{C_\lambda^\alpha(\mathbf{1})} \quad (\xi, \eta \in \mathbb{C}^q),$$

with  $\mathbf{1} = (1, \dots, 1)$ ,  $\alpha = 1/k$ . For  $k = \frac{d}{2}$  with  $d = \dim_{\mathbb{R}} \mathbb{F}$ , the Dunkl-Bessel functions  $\xi \mapsto J_{d/2}^A(\xi, \eta)$  are known to be the spherical functions of the flat symmetric space  $H_q(\mathbb{F}) \times U_q(\mathbb{F})/U_q(\mathbb{F})$  and therefore have the Harish-Chandra type integral representation

$$J_{d/2}^A(\xi, \eta) = \int_{U_q(\mathbb{F})} e^{\text{tr}(\eta u \xi u^{-1})} du. \quad (3.1)$$

See also formula (7) of [RV1], where an alternative proof of (3.1) is given.

For the root system  $B_q = \{\pm e_i, \pm e_i \pm e_j : i < j\}$ , we have  $k = (k_1, k_2)$  where  $k_1$  and  $k_2$  correspond to the roots  $\pm e_i$  and  $\pm e_i \pm e_j$  respectively. The associated Dunkl-Bessel function is

$$J_k^B(\xi, \eta) = {}_0F_1^\alpha\left(\mu; \frac{\xi^2}{2}, \frac{\eta^2}{2}\right) \quad \text{with } \alpha = 1/k_2, \mu = k_1 + (q-1)k_2 + 1/2$$

where  $\xi^2 = (\xi_1^2, \dots, \xi_q^2)$  and

$${}_0F_1^\alpha(\mu; \xi, \eta) := \sum_{\lambda \geq 0} \frac{1}{(\mu)_\lambda^\alpha |\lambda|!} \cdot \frac{C_\lambda^\alpha(\xi) C_\lambda^\alpha(\eta)}{C_\lambda^\alpha(\mathbf{1})}.$$

The Dunkl-Bessel function  $J_k^B$  is invariant in both arguments under the action of the hyperoctahedral group which is generated by sign changes and permutations of the coordinates. For suitable multiplicity  $k$ , it is related to the matrix Bessel function  $J_\mu$  and to the functions  $\phi_s^\mu(r) = J_\mu(\frac{1}{4}sr^2s)$  of Section 2, as follows: Consider the  $B_q$ -Weyl chamber

$$C_q^B := \{\xi = (\xi_1, \dots, \xi_q) \in \mathbb{R}^q : \xi_1 \geq \dots \geq \xi_q \geq 0\} \quad (3.2)$$

For  $\xi \in C_q^B$  we denote by  $\underline{\xi} \in \Pi_q$  the diagonal matrix with entries  $\xi_1, \dots, \xi_q$ . Let

$$k(\mu, d) := (\mu - (d(q-1) + 1)/2, d/2).$$

Then for  $\xi, \eta \in C_q^B$ ,

$$J_{k(\mu, d)}^B(\xi, i\eta) = \int_{U_q} J_\mu\left(\frac{1}{4}\eta u \underline{\xi}^2 u^{-1} \eta\right) du; \quad (3.3)$$

see Section 4 of [R1]. Dunkl-Bessel functions of type  $B$  will play an important role in Section 5 of this paper, in connection with the study of the Olshanski spherical pairs  $(M_{\infty, q} \rtimes (U_\infty \times U_q), U_\infty \times U_q)$ .

#### 4. OLSHANSKI SPHERICAL FUNCTIONS RELATED TO $M_{\infty, q} \rtimes U_\infty$

In this section, we consider Olshanski spherical pairs associated with matrix cones. For a general background on Olshanski spherical pairs and their spherical functions we refer to Faraut [F] and Olshanski [O11], [O12].

We fix the field  $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$  and the rank  $q \in \mathbb{N}$  as in Section 2. Consider the Gelfand pairs  $(G_p, K_p)$  with  $G_p = M_{p, q} \rtimes U_p$  and  $K_p := U_p$  where  $U_p$  acts on  $M_{p, q}$  by left multiplication. In all cases,  $G_p$  will be regarded as a closed subgroup of  $G_{p+1}$  with  $K_p = G_p \cap K_{p+1}$ . Consider the inductive limits  $G_\infty := \lim_{p \rightarrow \infty} G_p$  and  $K_\infty := \lim_{p \rightarrow \infty} K_p$ . Then  $(G_\infty, K_\infty)$  is an Olshanski spherical pair. A continuous function  $\phi : G_\infty \rightarrow \mathbb{C}$  is called an Olshanski spherical function of  $(G_\infty, K_\infty)$  if  $\phi$  is  $K_\infty$ -biinvariant and satisfies the product formula

$$\phi(g)\phi(h) = \lim_{p \rightarrow \infty} \int_{K_p} \phi(gkh) dk \quad \text{for all } g, h \in G_\infty \quad (4.1)$$

with respect to the normalized Haar measure  $dk$ . Further, an Olshanski spherical function  $\phi$  of  $(G_\infty, K_\infty)$  is called positive definite if  $\phi$  is positive definite on  $G_\infty$  in the usual sense, i.e. if for all  $n \in \mathbb{N}$ ,  $g_1, \dots, g_n \in G_\infty$  and  $c_1, \dots, c_n \in \mathbb{C}$ ,

$$\sum_{k, l=1}^n c_k \bar{c}_l \phi(g_k g_l^{-1}) \geq 0.$$



Positive definite continuous functions on  $G_\infty$  are always bounded. Furthermore, in our situation

$$K_\infty g^{-1} K_\infty = K_\infty g K_\infty \quad \text{for } g \in G_\infty.$$

This implies that positive definite spherical functions on  $G_\infty$  are automatically  $\mathbb{R}$ -valued.

We shall now classify the Olshanski spherical functions of  $(G_\infty, K_\infty)$  as well as those which are positive definite in addition.

For this we recapitulate that for each  $p$ , the space of double cosets  $G_p//K_p$  can be topologically identified with the cone  $\Pi_q$  via the homeomorphism

$$K_p(x, k)K_p \mapsto \sqrt{x^*x} \quad \text{for } x \in M_{p,q}, k \in K_p.$$

In other words, if for  $a \in \Pi_q$  we consider the matrix  $\begin{pmatrix} a \\ 0 \end{pmatrix} \in M_{p,q}$ , then the inverse of the above homeomorphism can be written as

$$F_p : \Pi_q \rightarrow G_p//K_p, \quad r \mapsto K_p g_a K_p \quad \text{with } g_a := \left( \begin{pmatrix} a \\ 0 \end{pmatrix}, I_p \right).$$

By definition of the inductive limit topology, the mapping

$$F : \Pi_q \rightarrow G_\infty//K_\infty, \quad a \mapsto K_\infty g_a K_\infty \quad \text{with } g_a := \left( \begin{pmatrix} a \\ 0 \end{pmatrix}, I_\infty \right)$$

also provides a homeomorphism for  $p = \infty$ . We will therefore use the agreement that for all integers  $p \geq q$  as well as for  $p = \infty$ , continuous and  $K_p$ -biinvariant functions on  $G_p$  are identified with continuous functions on  $\Pi_q$ . When doing so, our notations immediately imply the following

**Lemma 4.1.** *Let  $p_1 \geq p_2 \geq q$ . If a continuous function on  $\Pi_q$  corresponds to a  $K_{p_1}$ -biinvariant, positive definite function on  $G_{p_1}$ , then it also corresponds to a  $K_{p_2}$ -biinvariant, positive definite function on  $G_{p_2}$ . Moreover, a continuous function on  $\Pi_q$  corresponds to a  $K_\infty$ -biinvariant, positive definite function on  $G_\infty$  if and only if there exists an integer  $p_0 \geq 1$  such that for all integers  $p \geq p_0$ , it corresponds to a  $K_p$ -biinvariant, positive definite function on  $G_p$ .*

*Proof.* The first statement follows immediately from the definition of positive definite functions and the fact that for  $p_1 \geq p_2 \geq q$  (where possibly  $p_1 = \infty$ ), the canonical projections

$$P_p : G_p \rightarrow G_p//K_p \cong \Pi_q, \quad P_p((x, k)) := \sqrt{x^*x}$$

satisfy  $P_{p_1}|_{G_{p_2}} = P_{p_2}$ . The remaining part of the second statement follows also easily by these arguments.  $\square$

We next turn to the (not necessarily positive definite) spherical functions of  $(G_p, K_p)$  for finite  $p \geq q$ . We know from Lemma 2.2 that the functions  $f_s^{pd/2}$  of Section 2 with  $s \in H_q^{\mathbb{C}}$  are spherical. On the other hand, using results of Wolf [W1] in combination with an integral representation of the matrix Bessel functions  $f_s^{pd/2}$  (see Eq. (3.4) of [R1] and Propos. XVI.2.3 of [FK]), we obtain the following classification:

**Theorem 4.2.** *Let  $p \geq q$  be finite. Then  $\{f_s^{pd/2} : s \in H_q^{\mathbb{C}}\}$  is the set of all spherical functions of  $(G_p, K_p)$ . Moreover, the set of positive definite spherical functions of  $(G_p, K_p)$  is given by  $\{f_s^{pd/2} : s \in \Pi_q\}$ .*

*Proof.* According to [W1] (see also [W2]), the complete set of spherical functions of  $(G_p, K_p)$  can be described as follows:

Consider  $M_{p,q}$  as a real vector space of dimension  $dpq$  with Euclidean scalar product

$$(x|y) := \Re tr(x^*y)$$

and extend this form in a bilinear way to the complexification  $M_{p,q}^{\mathbb{C}}$  of  $M_{p,q}$ . Then it is easily checked that for each  $y \in M_{p,q}^{\mathbb{C}}$ , the function

$$\tilde{\phi}_y(x, v) := \int_{U_p} e^{-i(ux|y)} du \quad (x \in M_{p,q}, v \in U_p) \quad (4.2)$$

defines a spherical function of  $(G_p, K_p)$ . Moreover, by Theorem 4.4 of [W1], all spherical functions are given in this way, and by Theorem 5.4 of [W1], the set of positive-definite spherical functions is made up by those  $\tilde{\phi}_y$  with  $y \in M_{p,q}$ .

We next show that all functions of the form (4.2) are in fact Bessel functions  $f_s^{pd/2}$  for suitable  $s \in H_q^{\mathbb{C}}$ . For this we again regard  $M_{p,q}$  and  $H_q$  as vector spaces over  $\mathbb{R}$ , and denote the complex-bilinear extension of the  $\mathbb{R}$ -bilinear mapping

$$M_{p,q} \times M_{p,q} \rightarrow M_q, \quad (x, y) \mapsto x^*y$$

(to a mapping  $M_{p,q}^{\mathbb{C}} \times M_{p,q}^{\mathbb{C}} \rightarrow M_q^{\mathbb{C}}$ ) again by  $x^*y$ . For  $y \in M_{p,q}^{\mathbb{C}}$  we then obtain easily that  $y^*y \in H_q^{\mathbb{C}}$ .

Now fix a matrix  $y \in M_{p,q}^{\mathbb{C}}$ . Then  $y^*y \in H_q^{\mathbb{C}}$ , and for all  $r \in \Pi_q$ ,

$$f_{y^*y}^{pd/2}(r) = J_{pd/2}\left(\frac{1}{4}ry^*yr\right) = J_{pd/2}\left(\frac{1}{4}(yr)^*yr\right).$$

We conclude from Eqs. (3.3) and (3.4) of [R1] (see also Propos. XVI.2.3 of [FK]) that for all  $x \in M_{p,q}$ ,

$$J_{pd/2}\left(\frac{1}{4}x^*x\right) = \int_{U_p} e^{-i(u\sigma_0|x)} du \quad \text{with } \sigma_0 = \begin{pmatrix} I_q \\ 0 \end{pmatrix} \in M_{p,q}. \quad (4.3)$$

Using the complex-bilinear and thus analytic extension above, we conclude that (4.3) remains correct for all  $x \in M_{p,q}^{\mathbb{C}}$ . Now let  $r \in \Pi_q$ ,  $y \in M_{p,q}^{\mathbb{C}}$ . Then

$$J_{pd/2}\left(\frac{1}{4}(yr)^*yr\right) = \int_{U_p} e^{-i(u\sigma_0|yr)} du.$$

As

$$(u\sigma_0|yr) = \Re tr((u\sigma_0)^*yr) = \Re tr((u\sigma_0r)^*y) = (u\sigma_0r|y),$$

we obtain

$$f_{y^*y}^{pd/2}(r) = \int_{U_p} e^{-i(u\sigma_0r|y)} du = \tilde{\phi}_y(\sigma_0r, v) \quad (4.4)$$

with arbitrary  $v \in U_p$ . As the functions on both sides are biinvariant, and the  $r \in \Pi_q$  form a set of representatives of all double cosets as described in the beginning of this section, the proof of the first statement of the theorem is complete. Moreover, equation (4.4) in combination with Theorem 5.4 of [W1] leads to the stated classification of the positive definite spherical functions.  $\square$

We mention that the statement about the positive definite spherical functions above can be also obtained by hypergroup methods from Theorem 3.12 of [R1] in combination with results of [J].

We now turn to the case  $p = \infty$ . The Olshanski spherical functions of  $(G_\infty, K_\infty)$  can be characterized as follows:

**Lemma 4.3.** *A continuous  $K_\infty$ -biinvariant function  $\phi : G_\infty \rightarrow \mathbb{C}$  is Olshanski spherical if and only if the continuous function  $\tilde{\phi}(b) := \phi(g_b)$  on  $\Pi_q$  satisfies the product formula*

$$\tilde{\phi}(a) \cdot \tilde{\phi}(b) = \tilde{\phi}(\sqrt{a^2 + b^2}), \quad a, b \in \Pi_q. \quad (4.5)$$

*Proof.* Let  $\phi$  be a continuous  $K_\infty$ -biinvariant function on  $G_\infty$  and  $\tilde{\phi} \in C(\Pi_q)$  as described in the lemma. Then, by (4.1) and the product formula (2.4),  $\phi$  is Olshanski spherical iff  $\tilde{\phi}$  satisfies

$$\tilde{\phi}(a) \cdot \tilde{\phi}(b) = \lim_{p \rightarrow \infty} \frac{1}{\kappa_{pd/2}} \int_{B_q} \tilde{\phi}(\sqrt{a^2 + b^2 + awb + bw^*a}) \cdot \Delta(I - w^*w)^{pd/2 - \gamma} dw \quad (4.6)$$

for  $a, b \in \Pi_q$  with  $\gamma = d(q - 1/2) + 1$ . The probability measures

$$\kappa_{pd/2}^{-1} \cdot \Delta(I - w^*w)^{pd/2 - \gamma} dw$$

are compactly supported in  $B_q$  and tend weakly to the point measure  $\delta_0$  for  $p \rightarrow \infty$ . Therefore (4.6) is equivalent to

$$\tilde{\phi}(a) \cdot \tilde{\phi}(b) = \tilde{\phi}(\sqrt{a^2 + b^2})$$

as claimed.  $\square$

We remark that precise estimates for the order of convergence of the probability measures  $\kappa_{pd/2}^{-1} \cdot \Delta(I - w^*w)^{pd/2 - \gamma} dw$  are given in [V2].

We now solve the functional equation (4.5):

**Lemma 4.4.** *A continuous function  $\tilde{\phi} \in C(\Pi_q)$  with  $\tilde{\phi}(0) = 1$  satisfies (4.5) if and only if there exists some  $b \in H_q^{\mathbb{C}}$  such that*

$$\tilde{\phi}(a) = \exp(-\langle a^2, b \rangle) =: \psi_b(a), \quad a \in \Pi_q.$$

*Proof.* Clearly, all  $\psi_b$  satisfy (4.5). Conversely, if a function  $\tilde{\phi} \in C(\Pi_q)$  satisfies (4.5), then  $\psi(a) := \tilde{\phi}(\sqrt{a})$  satisfies  $\psi(a)\psi(b) = \psi(a+b)$  for  $a, b \in \Pi_q$  and thus, because of  $\psi(0) = 1$  and continuity,  $\psi(a) \neq 0$  for all  $a \in \Pi_q$ . Each  $a \in H_q$  can be written as  $a = d - cI_q$  for some  $d \in \Pi_q$  and some  $c \in [0, \infty[$ , and it is easily checked that  $\psi(a) := \psi(d)/\psi(cI_q)$  defines a well-defined function  $\psi \in C(H_q)$  with  $\psi(a)\psi(b) = \psi(a+b)$  for  $a, b \in H_q^{\mathbb{C}}$ . The assertion now follows by the well-known characterization of the exponential function on a Euclidean space by its functional equation.  $\square$

The preceding lemmata immediately lead to the following characterization of the Olshanski spherical functions of  $(G_\infty, K_\infty)$ .

**Theorem 4.5.** *A continuous  $K_\infty$ -biinvariant function  $\phi : G_\infty \rightarrow \mathbb{C}$  is Olshanski spherical if and only if*

$$\phi(g_a) = \psi_b(a) = \exp(-\langle a^2, b \rangle) \quad (a \in \Pi_q)$$

for some  $b \in H_q^{\mathbb{C}}$ .

We next investigate which Olshanski spherical functions of  $(G_\infty, K_\infty)$  appear as limits of spherical functions on  $(G_p, K_p)$  for  $p \rightarrow \infty$ . For this we employ the known convergence of  $J_\mu(\mu y)$  to  $e^{-tr(y)}$  for  $\mu \rightarrow \infty$ :

**Lemma 4.6.** For  $y \in H_q^{\mathbb{C}}$ ,

$$\lim_{\mu \rightarrow \infty} J_{\mu}(\mu y) = e^{-\text{tr}(y)} \quad \text{locally uniformly.}$$

On the cone  $\Pi_q$ , this convergence is even uniform. More precisely, there exists a constant  $C = C(q, \mathbb{F})$  such that

$$|J_{\mu}(\mu y) - e^{-\text{tr}(y)}| \leq C/\mu \quad \text{for all } y \in \Pi_q, \mu \geq 2q.$$

*Proof.* For the second statement we refer to Proposition 3.2 of [RV2]; see also Lemma 4.8 of [O11] or references cited there for the uniform convergence.

The first statement can be obtained either from the integral representation (3.12) of [R1] for  $J_{\mu}$ , or by power series expansion as in the proof of Proposition 3.5 in [RV2]. We outline the second approach: Using the expansions (2.3) and (2.1), we have

$$J_{\mu}(\mu y) - e^{-\text{tr}(y)} = \sum_{\lambda \geq 0} \frac{1}{|\lambda|!} \left( \frac{\mu^{|\lambda|}}{(\mu)_{\lambda}} - 1 \right) \cdot Z_{\lambda}(-y)$$

where by Lemma 3.4 of [RV2]

$$\left| 1 - \frac{\mu^{|\lambda|}}{(\mu)_{\lambda}} \right| \leq dq \cdot 2^{dq(q-1)/2} \cdot \frac{|\lambda|^2}{\mu}.$$

It is easily checked that the series

$$\sum_{\lambda \geq 0} \frac{|\lambda|^2}{|\lambda|!} \cdot Z_{\lambda}(-y)$$

converges absolutely and locally uniformly for  $y \in H_q^{\mathbb{C}}$  (c.f. Section XV.1 of [FK] and our normalization of the  $Z_{\lambda}$  instead of the  $\Phi_{\lambda}$  there). This immediately leads to the locally uniform convergence of order  $1/\mu$ .  $\square$

We conclude from Lemma 4.6 that for all  $y \in \Pi_q$ ,  $b \in H_q^{\mathbb{C}}$  and for  $\mu = pd/2 \rightarrow \infty$ , the functions  $f_s^{\mu}$  of Section 2 satisfy

$$f_{\mu b}^{\mu}(y) = J_{\mu}\left(\frac{\mu}{4}yby\right) \rightarrow \exp(-\text{tr}(yby)/4) = \psi_{b/4}(y)$$

uniformly or locally uniformly depending on  $b$ . According to Theorem 4.2, the functions  $f_{\mu b}^{\mu}$  with  $b \in H_q^{\mathbb{C}}$  form the spherical functions of  $(G_p, K_p)$ . Considering biinvariant functions on  $G_p$  and  $G_{\infty}$  as functions on  $\Pi_q$  as above, we obtain

**Corollary 4.7.** All Olshanski spherical functions  $\psi_b$  of  $(G_{\infty}, K_{\infty})$  with  $b \in H_q^{\mathbb{C}}$  appear as locally uniform limits of spherical functions of  $(G_p, K_p)$ .

Moreover, those Olshanski spherical functions  $\psi_b$  with  $b \in \Pi_q$  even appear as uniform limits of positive definite spherical functions of  $(G_p, K_p)$  as  $p \rightarrow \infty$ .

We finally determine the Olshanski spherical functions  $\psi_b$  which are positive definite.

**Theorem 4.8.** The positive definite Olshanski spherical functions of  $(G_{\infty}, K_{\infty})$  are precisely given by the functions  $\psi_b$  with  $b \in \Pi_q$ .

*Proof.* Assume that  $\psi_b$  is a positive definite Olshanski spherical function. Then  $\psi_b$  must be  $\mathbb{R}$ -valued. From  $\psi_b(a) = \exp(-\langle a^2, b \rangle)$  for  $a \in \Pi_q$  we infer that  $b \in H_q$ . Moreover, as  $\psi_b$  is in addition bounded on  $\Pi_q$ , it follows easily that  $b \in \Pi_q$ . In fact, if  $b$  would have a negative eigenvalue with eigenvector  $u$ , we could choose a matrix

$a \in \Pi_q$  with the same eigenvector  $u$  associated to the eigenvalue 1, and with all other eigenvalues equal to 0. It is then clear that  $\psi_b(ca) = \exp(-c^2\langle a^2, b \rangle)$  tends to  $\infty$  for  $c \rightarrow \infty$ .

For the converse statement consider  $b \in \Pi_q$ . Let  $p \geq q$  be a fixed integer. Then for each integer  $\tilde{p} \geq p$ , the spherical function  $f_b^{\tilde{p}d/2} \in C(\Pi_q)$  corresponds to a  $K_{\tilde{p}}$ -biinvariant positive definite function on  $G_{\tilde{p}}$ , and thus by Lemma 4.1, to a  $K_p$ -biinvariant positive definite function on  $G_p$ . As positive definiteness is preserved under limits, it follows from Corollary 4.7 that  $\psi_b \in C(\Pi_q)$  is a  $K_p$ -biinvariant positive definite function on  $G_p$ . This holds for all  $p$ , and therefore Lemma 4.1 ensures that  $\psi_b$  is a positive definite function on  $G_\infty$  as claimed.  $\square$

In summary, the Olshanski spherical functions of  $(G_\infty, K_\infty)$  admit a classification which is in complete accordance with that for finite  $p$  in Theorem 4.2.

### 5. OLSHANSKI SPHERICAL FUNCTIONS RELATED TO $M_{\infty,q} \rtimes (U_\infty \times U_q)$

In this section we consider the Gelfand pairs  $(G_p, K_p)$  with  $G_p = M_{p,q} \rtimes (U_p \times U_q)$  and  $K_p := U_p \times U_q$  for fixed  $q \geq 1$ , where the groups  $U_p$  and  $U_q$  act on  $M_{p,q}$  by multiplication from the left and right, respectively. Consider the Olshanski spherical pairs  $(G_\infty, K_\infty)$  with  $G_\infty := \lim_{\rightarrow} G_p$  and  $K_\infty := \lim_{\rightarrow} K_p$ .

Again, we investigate the Olshanski spherical functions of  $(G_\infty, K_\infty)$ . We recapitulate first that for each  $p$ , the double coset space  $G_p//K_p$  can be topologically identified with the Weyl chamber

$$C_q^B := \{\xi = (\xi_1, \dots, \xi_q) \in \mathbb{R}^q : \xi_1 \geq \dots \geq \xi_q \geq 0\}$$

of type  $B$  via

$$K_p(x, k)K_p \mapsto \sigma(\sqrt{x^*x}) \quad \text{for } x \in M_{p,q}, k \in K_p$$

independently of  $p$  where  $\sigma$  stands for the ordered spectrum of a positive semi-definite matrix. In other words, if for  $\xi \in C_q^B$  we consider the diagonal matrix  $\underline{\xi} := \text{diag}(\xi_1, \dots, \xi_q) \in M_q$  as well as  $\begin{pmatrix} \underline{\xi} \\ 0 \end{pmatrix} \in M_{p,q}$ , then the inverse of this homeomorphism can be written as

$$F_p : C_q^B \rightarrow G_p//K_p, \quad b \mapsto K_p g_\xi K_p \quad \text{with} \quad g_\xi := \left( \begin{pmatrix} \underline{\xi} \\ 0 \end{pmatrix}, I_p \times I_q \right).$$

By definition of the inductive limit topology, the  $F_p$  induce a homeomorphism  $F : C_q^B \rightarrow G_\infty//K_\infty$ . Again we use the agreement that for all integers  $p$  and for  $p = \infty$ ,  $K_p$ -biinvariant continuous functions on  $G_p$  will be identified with continuous functions on  $\Pi_q$ . When doing so, the statement of Lemma 4.1 transfers to the present setting without changes.

We now turn to the classification of spherical functions:

**Proposition 5.1.** *The spherical functions of  $(G_p, K_p)$ , considered as functions on the chamber  $C_q^B$ , are precisely given by the Dunkl-Bessel functions  $\phi_\eta^{pd/2}(\xi) := J_{k(\mu,d)}^B(\xi, i\eta)$  with  $\eta \in \mathbb{C}^q$ , where  $\mu = pd/2$ . Moreover, the positive definite spherical functions of  $(G_p, K_p)$  are given by the  $\phi_\eta^{pd/2}$  with  $\eta \in \mathbb{R}^q$ .*

*Proof.* The first statement is known from Dunkl theory, see [O]. The second then again follows from Theorem 5.4 of [W1].  $\square$

For the case  $p = \infty$ , we start with the following

**Lemma 5.2.** *A continuous  $K_\infty$ -biinvariant function  $\phi : G_\infty \rightarrow \mathbb{C}$  is Olshanski spherical if and only if the continuous function  $\tilde{\phi}(\xi) := \phi(g_\xi)$  on  $C_q^B$  satisfies*

$$\tilde{\phi}(\xi) \cdot \tilde{\phi}(\eta) = \int_{U_q} \tilde{\phi}(\sigma(\sqrt{\xi^2 + u\eta^2 u^{-1}})) du, \quad \xi, \eta \in C_q^B. \quad (5.1)$$

*Proof.* Let  $\phi$  be a continuous  $K_\infty$ -biinvariant function on  $G_\infty$  and  $\tilde{\phi}$  defined as above. Then, by (4.1) and the product formula for the spherical functions of  $(G_p, K_p)$  (see e.g., p. 771 of [R1]),  $\phi$  is Olshanski spherical iff  $\tilde{\phi}$  satisfies

$$\begin{aligned} \tilde{\phi}(\xi) \cdot \tilde{\phi}(\eta) = \lim_{p \rightarrow \infty} \frac{1}{\kappa_{pd/2}} \int_{U_q} \int_{B_q} \tilde{\phi}(\sigma(\sqrt{\xi^2 + u\eta^2 u^* + \xi w u \eta u^* + u \eta u^* w^* \xi})) \\ \cdot \Delta(I - w^* w)^{pd/2 - \gamma} dw du, \quad \xi, \eta \in C_q^B. \end{aligned} \quad (5.2)$$

As the probability measure

$$\kappa_{pd/2}^{-1} \cdot \Delta(I - w^* w)^{pd/2 - \gamma} dw$$

on  $B_q$  tends weakly to the point measure  $\delta_0$  for  $p \rightarrow \infty$ , (5.2) is equivalent to the condition of the lemma.  $\square$

We now solve the functional equation (5.1) by using the Dunkl-Bessel functions  $J_k^A$  of type A on the Weyl chamber

$$C_q^A := \{\xi \in \mathbb{R}^q : \xi_1 \geq \dots \geq \xi_q\} \supset C_q^B.$$

For this we identify the space of double cosets of the Gelfand pairs  $(H_q \rtimes U_q, U_q)$  (where  $U_q$  acts on  $H_q$  by conjugation) with  $C_q^A$  and recall from Section 3 that the spherical functions of  $(H_q \rtimes U_q, U_q)$  are given by the functions  $x \mapsto J_{d/2}^A(\xi, \eta)$  with  $\eta \in \mathbb{C}^q$  by [O].

**Lemma 5.3.** *A continuous function  $\tilde{\phi}$  on  $C_q^B$  with  $\tilde{\phi}(0) = 1$  satisfies (5.1) if and only if there exists some  $b \in \mathbb{C}^q$  such that  $\tilde{\phi}(\xi) = J_{d/2}^A(\xi^2, b)$  for all  $\xi \in C_q^B$ . Here  $\xi^2 \in C_q^B$  means the vector which is obtained from  $\xi$  by taking squares in each component.*

*Proof.* Let  $\tilde{\phi} \in C(C_q^B)$  with  $\tilde{\phi}(0) = 1$ . Then  $\tilde{\phi}$  satisfies (5.1) if and only if the function  $\psi(\xi) := \tilde{\phi}(\sqrt{\xi})$  on  $C_q^B$  satisfies

$$\psi(\xi) \cdot \psi(\eta) = \int_{U_q} \psi(\sigma(\xi + u\eta u^{-1})) du, \quad \xi, \eta \in C_q^B. \quad (5.3)$$

In particular, for  $a, b \in [0, \infty[$  we have

$$\psi((a, \dots, a))\psi((b, \dots, b)) = \psi((a + b, \dots, a + b)).$$

As  $\psi$  is continuous with  $\psi(0) = 1$ , this implies that there exists some  $c \in \mathbb{C}$  such that

$$\psi((a, \dots, a)) = e^{ca} \quad \text{for all } a \in [0, \infty[.$$

Precisely as in the proof of Lemma 4.4, it is now seen that  $\psi$  can be uniquely extended from  $C_q^B$  to a continuous function on  $C_q^A$  which satisfies (5.3) for all  $\xi, \eta \in C_q^A$ , namely by putting

$$\psi(\xi_1, \dots, \xi_q) := \psi(\xi_1 - \xi_q, \dots, \xi_{q-1} - \xi_q, 0) \cdot e^{c\xi_q} \quad \text{for } (\xi_1, \dots, \xi_q) \in C_q^A.$$

Thus the extension  $\psi \in C(C_q^A)$  is a spherical function of the Gelfand pair  $(H_q \rtimes U_q, U_q)$ . On the other hand, we know that the spherical functions of this Gelfand pair are precisely the Dunkl-Bessel functions  $\xi \mapsto J_{d/2}^A(\xi, b)$  with  $b \in \mathbb{C}^q$ . This proves the claim.  $\square$

The preceding lemmata yield the following characterization of the Olshanski spherical functions.

**Theorem 5.4.** *A continuous  $K_\infty$ -biinvariant function  $\phi : G_\infty \rightarrow \mathbb{C}$  is Olshanski spherical if and only if for some  $b \in \mathbb{C}^q$ ,*

$$\phi(g_\xi) = J_{d/2}^A(\xi^2, -b) =: \psi_b(\xi) \quad \text{for all } \xi \in C_q^B.$$

We next study which Olshanski spherical functions of  $(G_\infty, K_\infty)$  appear as limits of spherical functions on  $(G_p, K_p)$  for  $p \rightarrow \infty$ . For this we employ the known convergence of the Dunkl-Bessel functions of type B to those of type A from [RV1]:

**Lemma 5.5.** *For each  $b \in \mathbb{C}^q$ ,*

$$\lim_{\mu \rightarrow \infty} J_{k(\mu, d)}^B(2\sqrt{\mu}\xi, ib) = J_{d/2}^A(\xi^2, -b^2)$$

*locally uniformly in  $\xi \in C_q^B$ . For  $(\xi, b) \in C_q^B \times C_q^B$ , the convergence is even uniform.*

We mention that this result also follows immediately from Lemma 4.6 by taking the means (3.3) and (3.1) for the Dunkl-Bessel functions of type B and A respectively. Lemma 5.5 in combination with Proposition 5.1 and Theorem 5.4 implies the following result (again, we consider biinvariant functions on  $G_p$  and  $G_\infty$  as functions on  $C_q^B$ ):

**Theorem 5.6.** *All Olshanski spherical functions  $\psi_b$ ,  $b \in \mathbb{C}^q$  of  $(G_\infty, K_\infty)$  appear as locally uniform limits of spherical functions of  $(G_p, K_p)$  for  $p \rightarrow \infty$ .*

*Moreover, those Olshanski spherical functions  $\psi_b$  with  $b \in C_q^B$  appear even as uniform limits of the spherical functions of  $(G_p, K_p)$  for  $p \rightarrow \infty$ .*

We finally turn to the question which of the Olshanski spherical functions  $\psi_b$  are positive definite. The same argument as in the second part of the proof of Theorem 4.8 immediately implies:

**Proposition 5.7.** *The functions  $\psi_b$  with  $b \in C_q^B$  are positive definite Olshanski spherical functions of  $(G_\infty, K_\infty)$ .*

We conjecture that the converse statement holds as well. For  $q = 1$  we are in fact in the situation of Theorem 4.8, and the conjecture is clear.

For  $q \geq 2$  however, there exist spherical functions  $\psi_b$  with  $b \in \mathbb{C}^q \setminus \mathbb{R}^q$  which are  $\mathbb{R}$ -valued and bounded, and at present, we are unable to decide whether these functions are positive definite on  $G_\infty$ .

We give an example for  $q = 2$  and  $\mathbb{F} = \mathbb{R}$ : Let  $b = (i, -i)$  and  $\xi = (\xi_1, \xi_2) \in C_2^B$ , i.e.  $\xi_1 \geq \xi_2 \geq 0$ . Then in view of integral representation (3.1),

$$\begin{aligned} \psi_b(\xi) &= \int_{O(2)} \exp\left(-i \cdot \text{tr}(\underline{\xi}^2 \cdot u \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u^{-1})\right) du \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-i(\xi_1^2 - \xi_2^2)(\cos^2 t - \sin^2 t)) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos((\xi_1^2 - \xi_2^2)(\cos^2 t - \sin^2 t)) dt. \end{aligned}$$

It is unclear to us whether this function, which is a Bessel function, is positive definite on  $G_\infty$ , i.e., whether according to the version of Lemma 4.1 in the context of this section, this function is positive definite on  $G_p$  for all  $p \geq q$ .

## 6. POSITIVE INTEGRAL REPRESENTATIONS OF MATRIX BESSEL FUNCTIONS

Consider integers  $p_2 \geq p_1 \geq q$  and the associated indices  $\mu_k := p_k d/2$  ( $k = 1, 2$ ) of the matrix Bessel functions. Then the functions  $\phi_s^{\mu_2}$  with  $s \in \Pi_q$  as introduced in (2.6) represent positive definite biinvariant functions on  $G_{p_1}$ , and thus by Lemma 4.1, positive definite biinvariant functions on  $G_{p_2}$ . Therefore, by Bochner's theorem for hypergroups (see [J]), which may be applied to the associated matrix Bessel hypergroup on  $\Pi_q$  of index  $\mu_1$ , the function  $\phi_s^{\mu_2}$  has a representation

$$\phi_s^{\mu_2}(x) = \int_{\Pi_q} \phi_t^{\mu_1}(x) d\nu_{p_1, p_2; s}(t) \quad (x \in \Pi_q) \quad (6.1)$$

with some unique probability measure  $\nu_{p_1, p_2; s} \in M^1(\Pi_q)$ .

In this section we shall determine these measures explicitly in two different ways. Comparison of these results will then lead to a projection result for multivariate beta distributions. Our first approach was already carried out in [H] for  $\mathbb{F} = \mathbb{R}$ ; it relies on the following Laplace transform identity for the Bessel functions  $J_\mu$  which holds for general  $\mathbb{F}$ ; see Proposition XV.2.1 and Corollary VII.1.3 of [FK]:

**Proposition 6.1.** *For all  $\mu \in \mathbb{C}$  with  $\Re\mu > d(q-1)/2$  and  $y \in \Omega_q$ ,*

$$\int_{\Pi_q} J_\mu(x) e^{-\langle x, y \rangle} \Delta(x)^{\mu-n/q} dx = \Gamma_\Omega^q(\mu) \cdot \Delta(y)^{-\mu} \cdot e^{-tr(y^{-1})}$$

with the Gamma function

$$\Gamma_\Omega^q(\mu) = \int_{\Pi_q} e^{-tr(x)} \Delta(x)^{\mu-n/q} dx = (2\pi)^{(n-q)/2} \Gamma(\mu) \Gamma(\mu-d/2) \cdots \Gamma(\mu-(q-1)d/2).$$

By the transformation formula for linear maps, we have for  $m \in \Pi_q$

$$\int_{\Pi_q} f(mxm) dx = \Delta(m)^{2n/q} \int_{\Pi_q} f(x) dx.$$

We thus obtain for  $\mu \in \mathbb{C}$  with  $\Re\mu > d(q-1)/2$ ,  $m \in \Pi_q$  and  $y \in \Omega_q$ , that

$$\int_{\Pi_q} J_\mu(xm) e^{-\langle x, y \rangle} \Delta(x)^{\mu-n/q} dx = \Gamma_\Omega^q(\mu) \cdot \Delta(y)^{-\mu} \cdot e^{-tr(my^{-1})}; \quad (6.2)$$

c.f. equation (2.5) of [H] for  $\mathbb{F} = \mathbb{R}$ , where a minus sign in the exponential is missing. We notice that (6.2) may be also interpreted as a well-known formula for the Hankel transforms of Wishart distributions; see for instance [FK] or Section 5 of [V1]. Using injectivity and the convolution theorem for the Laplace transform, we deduce from (6.2) the following addition theorem:

**Proposition 6.2.** *For all  $\mu, \nu \in \mathbb{C}$  with  $\Re\mu > d(q-1)/2$  and  $\Re\nu > d(q-1)/2$  and all  $m_1, m_2 \in \Pi_q$ ,*

$$\begin{aligned} & J_{\mu+\nu}(x(m_1 + m_2)) \cdot \Delta(x)^{\mu+\nu-n/q} = \\ &= \frac{\Gamma_\Omega^q(\mu + \nu)}{\Gamma_\Omega^q(\mu) \Gamma_\Omega^q(\nu)} \int_{\{y \in \Pi_q: y \leq x\}} J_\mu(y m_1) \Delta(y)^{\mu-n/q} J_\nu((x-y)m_2) \Delta(x-y)^{\nu-n/q} dy. \end{aligned} \quad (6.3)$$



Taking  $m_1 = m$ ,  $m_2 = 0$ ,  $x = I_q$  (the identity matrix), and defining

$$\Pi_q^I := \{y \in \Pi_q : y \leq I_q\},$$

we obtain the following Sonine-type integral representation of  $J_{\mu+\nu}$  in terms of  $J_\mu$ , which was for  $\mathbb{F} = \mathbb{R}$  already proven in [H]:

**Corollary 6.3.** *For all  $\mu, \nu \in \mathbb{C}$  with  $\Re\mu, \Re\nu > d(q-1)/2$  and  $m \in \Pi_q$ ,*

$$J_{\mu+\nu}(m) = \frac{\Gamma_\Omega^q(\mu+\nu)}{\Gamma_\Omega^q(\mu)\Gamma_\Omega^q(\nu)} \int_{\Pi_q^I} J_\mu(y m) \Delta(y)^{\mu-n/q} \Delta(I_q - y)^{\nu-n/q} dy.$$

Notice that for  $m = 0$  this formula implies the known beta integral

$$\int_{\Pi_q^I} \Delta(y)^{\mu-n/q} \Delta(I_q - y)^{\nu-n/q} dy = \frac{\Gamma_\Omega^q(\mu)\Gamma_\Omega^q(\nu)}{\Gamma_\Omega^q(\mu+\nu)} =: B_\Omega^q(\mu, \nu). \quad (6.4)$$

Using these notions, we define the beta distributions  $d\beta_{q;\mu,\nu} \in M^1(\Pi_q^I)$  on  $\Pi_q^I$  for real parameters  $\mu, \nu > d(q-1)/2$  by

$$d\beta_{q;\mu,\nu}(y) := \frac{1}{B_\Omega^q(\mu, \nu)} \Delta(y)^{\mu-n/q} \Delta(I_q - y)^{\nu-n/q} dy$$

with  $n = n(q, \mathbb{F})$  as in Section 2. Using Lemma 2.2(1), we obtain the following explicit form of relation (6.1):

**Corollary 6.4.** *For all  $\mu, \nu \in \mathbb{C}$  with  $\Re\mu, \Re\nu > d(q-1)/2$  and  $s, x \in \Pi_q$ ,*

$$\phi_s^{\mu+\nu}(x) = \int_{\Pi_q^I} \phi_{\sqrt{sy}s}^\mu(x) d\beta_{q;\mu,\nu}(y).$$

*Remarks.* (1) We conjecture that for all real parameters  $\nu \geq 0$ ,  $\mu > d(q-1)/2$  and all  $s \in \Pi_q$  there is a probability measure  $m_{\mu,\nu,s} \in M^1(\Pi_q)$  with compact support such that

$$\phi_s^{\mu+\nu}(x) = \int_{\Pi_q} \phi_y^\mu(x) dm_{\mu,\nu,s}(y) \quad \text{for all } x \in \Pi_q$$

- (2) One can easily combine Corollary 6.4 with Lemma 4.6 for  $\nu \rightarrow \infty$ . This yields an explicit integral representation of the Olshanski spherical functions  $\psi_b$  of Section 4 for  $b \in \Pi_q$  in terms of the  $\phi_s^\mu$ . This formula describes  $\psi_b$  as the Hankel transform of order  $\mu$  of some Wishart distribution and is well-known, see Proposition XV.2.1 of [FK] and Lemma 5.4. of [V1], which is adapted to our notation.

Next, we derive Corollaries 6.3 and 6.4 in a different, more geometric way as follows: Consider integers  $1 \leq q \leq \tilde{p}$  and  $p \geq 2\tilde{p}$ . For  $p, \tilde{p}$  define the associated parameters  $\mu := pd/2$  and  $\tilde{\mu} := \tilde{p}d/2$  of the Bessel functions on  $\Pi_q$ . From the fact that the  $\phi_\lambda^\mu$  are spherical functions we conclude that for  $\lambda, x \in \Pi_q$

$$\phi_\lambda^\mu(x) = \int_{U_p} \exp\left(i \cdot \Re tr\left((x, 0) u \begin{pmatrix} \lambda \\ 0 \end{pmatrix}\right)\right) du$$

with  $(x, 0) \in M_{q,p}$  and  $\begin{pmatrix} \lambda \\ 0 \end{pmatrix} \in M_{p,q}$  (see also [R1]). Now let  $w \in B_{\tilde{p}}$  be the upper left  $\tilde{p} \times \tilde{p}$ -block of  $u$ , i.e.  $u = \begin{pmatrix} w & * \\ * & * \end{pmatrix}$ . In the following,  $C_1, C_2, C_3$  are constants

depending on  $q, \tilde{p}, p, d$ . We first conclude from Lemma 2.1 of [R2] that for  $p \geq 2\tilde{p}$ ,

$$\phi_\lambda^\mu(x) = C_1 \int_{B_{\tilde{p}}} \exp\left(i \cdot \Re \operatorname{tr}\left((x, 0)w \begin{pmatrix} \lambda \\ 0 \end{pmatrix}\right)\right) \cdot \Delta(I - w^*w)^{pd/2 - d(\tilde{p}-1/2) - 1} dw$$

where now  $(x, 0) \in M_{q, \tilde{p}}$  and  $\begin{pmatrix} \lambda \\ 0 \end{pmatrix} \in M_{\tilde{p}, q}$ . Further, integration with respect to polar coordinates on  $M_{\tilde{p}}$  according to [FT] (see also p.759 of [R1]) gives

$$\int_{M_{\tilde{p}}} f(w) dw = C_2 \int_{\Omega_{\tilde{p}}} \int_{U_{\tilde{p}}} f(u\sqrt{r}) \cdot \Delta(r)^{d/2-1} du dr.$$

Therefore

$$\begin{aligned} \phi_\lambda^\mu(x) &= C_3 \int_{\Pi_{\tilde{p}}^I} \left( \int_{U_{\tilde{p}}} \exp\left(i \cdot \Re \operatorname{tr}\left((x, 0)u\sqrt{r} \begin{pmatrix} \lambda \\ 0 \end{pmatrix}\right)\right) du \right) \\ &\quad \cdot \Delta(r)^{d/2-1} \Delta(I - r)^{pd/2 - d(\tilde{p}-1/2) - 1} dr. \end{aligned}$$

By the definition of  $\phi_\lambda^{\tilde{\mu}}$ , we obtain

$$\phi_\lambda^\mu(x) = C_3 \int_{\Pi_{\tilde{p}}^I} \phi_{\lambda(r)}^{\tilde{\mu}}(x) \Delta(r)^{d/2-1} \Delta(I - r)^{pd/2 - d(\tilde{p}-1/2) - 1} dr$$

with  $\lambda(r) := \sqrt{(\lambda, 0)r} \begin{pmatrix} \lambda \\ 0 \end{pmatrix} \in \Pi_q$ .

Putting  $x = 0$  and using (6.4) for  $\tilde{p}$  instead of  $q$ , we finally obtain from the definition of  $n$  that

$$\phi_\lambda^\mu(x) = \int_{\Pi_{\tilde{p}}^I} \phi_{\lambda(r)}^{\tilde{\mu}}(x) d\beta_{\tilde{p}; \tilde{p}d/2, d(p-\tilde{p})/2}(r). \quad (6.5)$$

We now compare equation (6.5) with Corollary 6.4 for  $\mu = \tilde{p}d/2$ ,  $\nu = (p - \tilde{p})d/2$  and arbitrary  $x \in \Pi_q$ . By the injectivity of the Hankel transform of index  $\tilde{p}$  on  $\Pi_q$  and by analytic continuation with respect to  $\mu$  and  $\nu$ , we shall obtain the following projection result for multivariate beta distributions:

**Proposition 6.5.** *For integers  $\tilde{p} \geq q \geq 1$ , consider the projection*

$$P_{\tilde{p}, q} : \Pi_{\tilde{p}}^I \rightarrow \Pi_q^I, \quad r = \begin{pmatrix} y & * \\ * & * \end{pmatrix} \mapsto y.$$

*Then for all real-valued parameters  $\mu, \nu > d(\tilde{p} - 1)/2$ , the measure  $\beta_{q; \mu, \nu}$  is the push-forward of the measure  $\beta_{\tilde{p}; \mu, \nu}$  under  $P_{\tilde{p}, q}$ ,*

$$P_{\tilde{p}, q}(\beta_{\tilde{p}; \mu, \nu}) = \beta_{q; \mu, \nu}.$$

*Proof.* Assume first that  $\mu = d\tilde{p}/2$  and  $\nu = d(p - \tilde{p})/2$  for some integer  $p \geq 2\tilde{p}$ . In this case, the statement follows from the preceding arguments.

Now consider integers  $r \geq \tilde{p} \geq q$  and  $p \geq 2r$ . We obtain

$$\begin{aligned} \beta_{q; dr/2, d(p-r)/2} &= P_{r, q}(\beta_{r; dr/2, d(p-r)/2}) = P_{\tilde{p}, q} \circ P_{r, \tilde{p}}(\beta_{r; dr/2, d(p-r)/2}) \\ &= P_{\tilde{p}, q}(\beta_{\tilde{p}; dr/2, d(p-r)/2}). \end{aligned}$$

This equality means that for the parameters  $\mu = dr/2$  and  $\nu = d(p - r)/2$  with integers  $p, r$  with  $r \geq \tilde{p}$  and  $p \geq 2r$ , we have for all bounded continuous functions

$f \in C_b(\Pi_q^I)$  the identity

$$\int_{\Pi_{\tilde{p}}^I} f(P(x)) d\beta_{\tilde{p};\mu,\nu}(x) = \int_{\Pi_q^I} f(y) d\beta_{q;\mu,\nu}(y). \quad (6.6)$$

By definition of the beta distributions, both sides of this identity are, for fixed  $f$ , analytic in the variables  $\mu, \nu \in \mathbb{C}$  with  $\Re\mu, \Re\nu > d(\tilde{p}-1)/2$ . Precisely as in [R1], it is easily checked that both sides also satisfy the exponential growth conditions of Carlson's Theorem (p. 186 of [T]). This yields that (6.6) holds for general  $\mu$  and  $\nu$ , which finishes the proof.  $\square$

We mention that Proposition 6.5 can also be derived, at least for  $\mathbb{F} = \mathbb{R}$  and  $\mu = p/2$  and  $\nu = r/2$  with suitable integers  $p, r$ , from the construction of the multivariate beta distributions in statistics; see Section 10.2 of [Fa] and references cited there. Let us sketch this approach: we consider  $M_{p,\tilde{p}}$ - and  $M_{r,\tilde{p}}$ -valued independent random variables  $X$  and  $Y$  respectively such that all entries of  $X$  and  $Y$  are i.i.d. standard-normal distributed. Then  $S := X^t X$ ,  $T := Y^t Y$ , and  $S+T$  are  $\Pi_{\tilde{p}}$ -valued and Wishart-distributed. Now form the unique lower triangular matrix  $C$  with nonnegative entries satisfying  $CC^t = S+T$ . It is well-known (Section 10.2 of [Fa] and references cited there) that  $L := C^{-1}S(C^t)^{-1}$  is a  $\Pi_{\tilde{p}}^I$ -valued variable with distribution  $\beta_{\tilde{p};p/2,r/2}$ .

Now consider some integer  $1 \leq q \leq \tilde{p}$ . For any matrix  $A \in M_{\tilde{p}}$  we denote by  $\tilde{A}$  its upper left  $q \times q$ -block. It is now easily checked from the triangular structure of  $C$  and  $C^{-1}$  that

$$\tilde{L} = \widetilde{C^{-1}S(C^t)^{-1}} = (\tilde{C})^{-1}\tilde{S}((\tilde{C})^t)^{-1}$$

with  $\tilde{C}\tilde{C}^t = \tilde{S} + \tilde{T}$ . This observation readily leads to a further proof of Proposition 6.5 for  $\mu = p/2$  and  $\nu = r/2$ . Again, application of Carlson's theorem implies the general result.

## REFERENCES

- [BF] T. Baker, P. Forrester, The Calogero-Sutherland model and generalized classical polynomials. *Comm. Math. Phys.* 188 (1997), 175216.
- [DOW] M. Dawson, G. Olafsson, J.A. Wolf, Direct systems of spherical functions and representations. Preprint 2011, arXiv:1110.0655v1.
- [Du] C.F. Dunkl, Differential-difference operators associated to reflection groups. *Trans. Amer. Math. Soc.* 311 (1989), 167–183.
- [F] J. Faraut, Infinite dimensional Spherical Analysis. COE Lecture Notes Vol. 10 (2008), Kyushu University.
- [FK] J. Faraut, A. Korányi, Analysis on symmetric cones. Oxford Science Publications, Clarendon press, Oxford 1994.
- [FT] J. Faraut, G. Travaglini, Bessel functions associated with representations of formally real Jordan algebras. *J. Funct. Anal.* 71 (1987), 123–141.
- [Fa] R.H. Farrell, Multivariate Calculus: Use of Continuous Groups. Springer Verlag, New York 1985.
- [H] C.S. Herz, Bessel functions of matrix argument. *Ann. Math.* 61 (1955), 474–523.
- [J] R.I. Jewett, Spaces with an abstract convolution of measures, *Adv. Math.* 18 (1975), 1–101.
- [M] I.G. Macdonald, Commuting differential operators and zonal spherical functions. In: Algebraic groups (Utrecht 1986), eds. a.M. Cohen et al, *Lecture Notes in Mathematics* 1271, Springer-Verlag, Berlin, 1987.
- [OI1] G.I. Olshanskii, Infinite-dimensional classical groups of finite  $r$ -rank: Description of representations and asymptotic theory. *Funct. Anal. Appl.* 18:1 (1984), 22–34.

- [Ol2] G.I. Olshanskii, Unitary representations of infinite dimensional pairs  $(G, K)$  and the formalism of R. Howe. In: A. Vershik and D. Zhelobenko (eds.), Representations of Lie Groups and Related topics. Adv. Stud. Contemp. Math. 7, Gordon and Breach, 1990.
- [O] E. Opdam, Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group. *Compos. Math.* 85 (1993), 333–373.
- [P] D. Pickrell, Mackey analysis of infinite classical motion groups. *Pacific. J. Math.* 150 (1991), 139–166.
- [R1] M. Rösler, Bessel convolutions on matrix cones. *Compos. Math.* 143 (2007), 749–779.
- [R2] M. Rösler, Positive convolution structure for a class of Heckman-Opdam hypergeometric functions of type  $BC$ . *J. Funct. Anal.* 258 (2010), 2779–2800.
- [RKV] M. Rösler, T. Koornwinder, M. Voit, Limit transition between hypergeometric functions of type  $BC$  and type  $A$ . Preprint 2012, arXiv:1207.0487.
- [RV1] M. Rösler, M. Voit, A limit relation for Dunkl-Bessel functions of type  $A$  and  $B$ . *Symmetry Integrability Geom. Methods Appl.* 4, Paper 083 (2008).
- [RV2] M. Rösler, M. Voit, Limit theorems for radial random walks on  $p \times q$ -matrices as  $p$  tends to infinity *Math. Nachr.* 284 (2011), 87–104.
- [S] I.J. Schoenberg, Metric spaces and completely monotone functions; *Ann. Math.* 39 (1938), 811–841.
- [T] E.C. Titchmarsh, The theory of functions. Oxford Univ. Press, London, 2nd ed., 1939.
- [V1] M. Voit, Bessel convolutions on matrix cones: Algebraic properties and random walks. *J. Theor. Probab.* 22 (2009), 741–771.
- [V2] M. Voit, Central limit theorems for radial random walks on  $p \times q$  matrices for  $p \rightarrow \infty$ . *Adv. Pure Appl. Math.* 3 (2012), 231–246.
- [W1] J.A. Wolf, Spherical functions on Euclidean space, *J. Funct. Anal.* 239 (2006), 127–136.
- [W2] J.A. Wolf, Harmonic analysis on commutative spaces. American Mathematical Society (AMS), Providence, 2007.

INSTITUT FÜR MATHEMATIK, UNIVERSITÄT PADERBORN, WARBURGER STR. 100, 33098 PADERBORN, GERMANY

*E-mail address:* roesler@math.upb.de

FAKULTÄT MATHEMATIK, TECHNISCHE UNIVERSITÄT DORTMUND, VOGELPOTHSWEG 87, D-44221 DORTMUND, GERMANY

*E-mail address:* michael.voit@math.tu-dortmund.de

## Preprints ab 2010/15

- 2012-17 **Margit Rösler and Michael Voit**  
Olshanski spherical functions for infinite dimensional motion groups of fixed rank
- 2012-16 **Selim Esedoğlu, Andreas Rätz, Matthias Röger**  
Colliding Interfaces in Old and New Diffuse-interface Approximations of Willmore-flow
- 2012-15 **Patrick Henning, Mario Ohlberger and Ben Schweizer**  
An adaptive multiscale finite element method
- 2012-14 **Andreas Knauf, Frank Schulz, Karl Friedrich Siburg**  
Positive topological entropy for multi-bump magnetic fields
- 2012-13 **Margit Rösler, Tom Koornwinder, and Michael Voit**  
Limit transition between hypergeometric functions of type BC and Type A
- 2012-12 **Alexander Schnurr**  
Generalization of the Blumenthal-Gettoor Index to the Class of Homogeneous Diffusions with Jumps and some Applications
- 2012-11 **Wilfried Hazod**  
Remarks on pseudo stable laws on contractible groups
- 2012-10 **Waldemar Grundmann**  
Limit theorems for radial random walks on Euclidean spaces of high dimensions
- 2012-09 **Martin Heida**  
A two-scale model of two-phase flow in porous media ranging from porespace to the macro scale
- 2012-08 **Martin Heida**  
On the derivation of thermodynamically consistent boundary conditions for the Cahn-Hilliard-Navier-Stokes system
- 2012-07 **Michael Voit**  
Uniform oscillatory behavior of spherical functions of  $GL_n/U_n$  at the identity and a central limit theorem
- 2012-06 **Agnes Lamacz and Ben Schweizer**  
Effective Maxwell equations in a geometry with flat rings of arbitrary shape
- 2012-05 **Frank Klinker and Günter Skoruppa**  
Ein optimiertes Glättungsverfahren motiviert durch eine technische Fragestellung
- 2012-04 **Patrick Henning, Mario Ohlberger, and Ben Schweizer**  
Homogenization of the degenerate two-phase flow equations
- 2012-03 **Andreas Rätz**  
A new diffuse-interface model for step flow in epitaxial growth
- 2012-02 **Andreas Rätz and Ben Schweizer**  
Hysteresis models and gravity fingering in porous media
- 2012-01 **Wilfried Hazod**  
Intrinsic topologies on H-contraction groups with applications to semistability

- 2011-14 **Guy Bouchitté and Ben Schweizer**  
Plasmonic waves allow perfect transmission through sub-wavelength metallic gratings
- 2011-13 **Waldemar Grundmann**  
Moment functions and Central Limit Theorem for Jacobi hypergroups on  $[0, \infty[$
- 2011-12 **J. Koch, A. Rätz, and B. Schweizer**  
Two-phase flow equations with a dynamic capillary pressure
- 2011-11 **Michael Voit**  
Central limit theorems for hyperbolic spaces and Jacobi processes on  $[0, \infty[$
- 2011-10 **Ben Schweizer**  
The Richards equation with hysteresis and degenerate capillary pressure
- 2011-09 **Andreas Rätz and Matthias Röger**  
Turing instabilities in a mathematical model for signaling networks
- 2011-08 **Matthias Röger and Reiner Schätzle**  
Control of the isoperimetric deficit by the Willmore deficit
- 2011-07 **Frank Klinker**  
Generalized duality for k-forms
- 2011-06 **Sebastian Aland, Andreas Rätz, Matthias Röger, and Axel Voigt**  
Buckling instability of viral capsides - a continuum approach
- 2011-05 **Wilfried Hazod**  
The concentration function problem for locally compact groups revisited: Non-dissipating space-time random walks,  $\tau$ -decomposable laws and their continuous time analogues
- 2011-04 **Wilfried Hazod, Katrin Kosfeld**  
Multiple decomposability of probabilities on contractible locally compact groups
- 2011-03 **Alexandra Monzner\* and Frol Zapolsky†**  
A comparison of symplectic homogenization and Calabi quasi-states
- 2011-02 **Stefan Jäschke, Karl Friedrich Siburg and Pavel A. Stoimenov**  
Modelling dependence of extreme events in energy markets using tail copulas
- 2011-01 **Ben Schweizer and Marco Veneroni**  
The needle problem approach to non-periodic homogenization
- 2010-16 **Sebastian Engelke and Jeannette H.C. Woerner**  
A unifying approach to fractional Lévy processes
- 2010-15 **Alexander Schnurr and Jeannette H.C. Woerner**  
Well-balanced Lévy Driven Ornstein-Uhlenbeck Processes