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STABLE LIMIT THEOREM FOR U-STATISTIC PROCESSES INDEXED BY A RANDOM WALK

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Abstract. Let $(S_n)_{n\in\mathbb{N}}$ be a random walk in the domain of attraction of an α-stable Lévy process and $(\xi(n))_{n\in\mathbb{N}}$ a sequence of iid random variables (called scenery). We want to investigate U-statistics indexed by the random walk S_n, that is $U_n := \sum_{1\le i<j\le n} h(\xi(S_i),\xi(S_j))$ for some symmetric bivariate function h. We will prove the weak convergence without the assumption of finite variance. Additionally, under the assumption of finite moments of order greater than two, we will establish a law of the iterated logarithm for the U-statistic U_n.

1. Introduction

Random walks in random scenery were introduced by Kesten and Spitzer [9] and lead to a class of self-similar processes with scaling not equal to $\sqrt{n}$, which is the scaling for sums of independent or short range dependent random variables. We want to study not partial sums, but U-statistics indexed by a stable random walk, which is defined as follows:

Let $(X_n)_{n\in\mathbb{N}}$ be an iid (independent identically distributed) sequence of $\mathbb{Z}$-valued random variables with $EX_1 = 0$ and the property that the law $\mathcal{L}(X_1)$ is in the domain of attraction of an α-stable law F_α with $1 < \alpha \le 2$, i.e.: for $S_n := \sum_{m=1}^n X_m$ one has

$$P(n^{-\frac{1}{\alpha}} S_n \le x) \to F_\alpha(x).$$

It is then well known that the sequence of stochastic processes

$$S_t^{(n)} := n^{-\frac{1}{\alpha}} S_{[nt]};\quad t \ge 0, n \in \mathbb{N}$$

converges in distribution towards an α-stable Lévy process $S_t^\star$ (see Skorokhod [14], Theorem 2.7). Moreover, let $(\xi(i))_{i\in\mathbb{Z}}$ be a sequence of iid random variables. Kesten and Spitzer [9] studied the partial sum process $\sum_{i=1}^n \xi(S_i)$ and showed that it converges weakly to a self similar process. Instead of partial sums, we want to investigate the asymptotic behaviour of U-statistics, which can be regarded als generalized partial sums. Let h be a bivariate, measurable and symmetric function, such that

$$E[|h(\xi(1),\xi(2))|] < \infty \quad\text{and}\quad E[|h(\xi(1),\xi(1))|] < \infty.$$

The U-statistic indexed by the stable random walk $(S_n)_{n\in\mathbb{N}}$ is the given by

$$U_n := \sum_{1\le i<j\le n} h(\xi(S_i),\xi(S_j)).$$

Key words and phrases. random walk; random scenery; U-statistics; stabel limits; law of the iterated logarithm.

A classic approach for U-statistics in the case of finite second moments is the Hoeffding decomposition [8]. Without loss of generality, we can assume that $Eh(\xi(1),\xi(2)) = 0$ and we can write

$$U_n = (n-1)\sum_{i=1}^{n} h_1(\xi(S_i)) + \sum_{1\le i<j\le n} h_2(\xi(S_i),\xi(S_j))$$

with

$$\begin{aligned} h_1(x) &:= E(h(x,\xi(1))) \\ h_2(x,y) &:= h(x,y) - h_1(x) - h_1(y). \end{aligned}$$

We call $L_n := \sum_{i=1}^{n} h_1(\xi(S_i))$ the linear part of the U-statistic U_n and $R_n = \sum_{1\le i<j\le n} h_2(\xi(S_i),\xi(S_j))$ the remainder term. Note that under second moments, the summands of the remainder R_n are uncorrelated.

One might expect the linear part to dominate the asymptotic behaviour (and we will indeed show this). But this is not obvious, as the random walk in random scenery shows some long range dependent behaviour. For other models of long range dependence (e.g. Gaussian sequences with slowly decaying covariances), both the linear part and the remainder term might contribute to the limit distribution. Because of this, there are other methods like representing the U-statistics as a functional of the empirical distribution function, see Dehling and Taqqu [5] or Beutner and Zähle [1]. Furthermore, the Hoeffding decomposition uses the fact the summands of the remainder term are uncorrelated and thus it requires second moments in its original form.

Using the Hoeffding decomposition and truncation arguments, Heinrich and Wolf studied U-statistics without finite second moments. An alternative approach using point processes was used by Dabrowski et al. [4]. The U-statistic indexed by a random walk was examined by Cabus and Guillotin-Plantard [2] and Guillotin-Plantard and Ladret [6], but only in the case of finite fourth moments. Our first aim is to show the convergence of the U-statistic process indexed by S_n towards stable, long-range dependent, selfsimilar processes in the case that this moment condition does not hold.

Furthermore, we want to establish a law of the iterated logarithm for the U-statistic process indexed by S_n, extending results from Lewis [12] and Zhang [15] for the partial sum indexed by a random walk.

2. Main Results

Our first theorem will establish the weak convergence of the U-statistic process not assuming that the summands of the linear part have second (or even higher) moments:

Theorem 1. *Assume that the law $\mathcal{L}(X_1)$ is in the domain of attraction of an α-stable law F_α with $1<\alpha\le 2$ and that the law $\mathcal{L}(h_1(\xi(1)))$ is in the domain of attraction of an β-stable law F_β with $1<\beta\le 2$. Furthermore, let $E|h(\xi(1),\xi(2))|^\eta<\infty$ with $\eta = \frac{2\beta'}{1+\beta'}$ for a $\beta'>\beta$. Then we have the weak convergence*

$$\left(n^{-2+\frac{1}{\alpha}-\frac{1}{\alpha\beta}}U_{[nt]}\right)_{t\in[0,1]} \to (\Delta_t)_{t\in[0,1]},$$

with Δ_t as defined in Kesten and Spitzer [9].

It is possible that $\eta < \beta$, that means that the summands h of the U-statistic might have less moments than h_1. Without loss of generality, we can assume that $\eta < \beta$, so $E|h_1(\xi(1))|^\eta < \infty$.

To give the definition of the process $(\Delta_t)_{t\in[0,1]}$, we have to introduce some notation. Let $(T_t(x))_{t\geq 0}$ be the local time of the limit process $(S_t^\star)_{t\geq 0}$ of the rescaled partial sum $(n^{-\frac{1}{\alpha}}\sum_{i=1}^{[nt]} X_i)_{t\geq 0}$, that means

$$\int_0^t \mathbb{1}_{[a,b)}(S_s^\star)ds = \int_a^b T_t(x)dx$$

almost surely. Let $(Z_+(t))_{t\geq 0}$ and $(Z_-(t))_{t\geq 0}$ be to independent copies of the limit process of the rescaled partial sum process $\left(n^{-\frac{1}{\beta}}\sum_{i=1}^{[nt]} h_1(\xi(i))\right)_{t\geq 0}$. Then the limit process of the random walk in random scenery is defined as

$$\Delta_t = \int_0^\infty T_t(x)dZ_+(x) + \int_0^\infty T_t(-x)dZ_-(x).$$

For random walks in random scenery, Lewis [12] and Khoshnevisan and Lewis [10] proved the law of the iterated logarithm. This was improved by Csáki et al. [3] and Zhang [15] using strong approximation methods. In our second theorem, we will extend these results to U-statistics:

Theorem 2. *Let the assumption of Theorem 1 hold with $\alpha = \beta = 2$ and additional $E|h_1(\xi(i))|^p < \infty$ and $E|X_i|^p < \infty$ for some $p > 2$. Then*

$$\limsup_{n\to\infty} \frac{U_n}{n^{\frac{7}{4}}(\log\log n)^{\frac{3}{4}}} = \frac{2^{\frac{1}{4}}\operatorname{Var}(\xi)^{\frac{1}{2}}}{3\operatorname{Var}(X)^{\frac{1}{4}}}$$
$$\liminf_{n\to\infty} \frac{U_n}{n^{\frac{7}{4}}(\log\log n)^{\frac{3}{4}}} = -\frac{2^{\frac{1}{4}}\operatorname{Var}(\xi)^{\frac{1}{2}}}{3\operatorname{Var}(X)^{\frac{1}{4}}}$$

almost surely.

3. Auxiliary Results

We define the occupation times $N_n(x) := \sum_{i=1}^n \mathbb{1}_{\{S_i = x\}}$.

Lemma 3.1. *For $k, p \geq 1$*

$$E\left(\sum_{x\in\mathbb{Z}} N_n^p(x)\right)^k = O(n^{kp(1-\frac{1}{\alpha})+k\frac{1}{\alpha}})$$

This is Lemma 2.1 of Guillotin-Plantard, Ladret [6]

Proposition 3.2. *Under the conditions of Theorem 1, we hvae that*

$$\max_{k\leq n} R_k = o(n^{1-\frac{1}{\alpha}+\frac{1}{\alpha\beta}})$$

almost surely.

Proof. We define $a_l = 2^{l\frac{1+\beta'}{\alpha\beta'}}$ and the truncated kernel

$$h_{0,l}(x,y) := \begin{cases} h(x,y) & \text{if } |h(x,y)| \leq a_l \\ 0 & \text{else} \end{cases}.$$

We also need the Hoeffding decomposition for the truncated kernel:

$$h_{1,l}(x) := E(h_{0,l}(x, \xi(1)))$$
$$h_{2,l}(x,y) := h_{0,l}(x,y) - h_{1,l}(x) - h_{1,l}(y).$$

We introduce the following notation:

$$\tilde{L}_{l,n} := \sum_{i=1}^{n} h_{1,l}(\xi(S_i))$$
$$\tilde{U}_{l,n} := \sum_{1 \le i < j \le n} h_{0,l}(\xi(S_i), \xi(S_j))$$
$$\tilde{R}_{l,n} := \sum_{1 \le i < j \le n} h_{2,l}(\xi(S_i), \xi(S_j))$$

Recall the Hoeffding decomposition

$$U_n = (n-1)L_n + R_n.$$

Similar, we have that

$$\tilde{U}_{l,n} = (n-1)\tilde{L}_{l,n} + \tilde{R}_{l,n}.$$

We now obtain the following representation for the remainder term:

$$R_n = U_n - (n-1)L_n = (U_n - \tilde{U}_{l,n}) - (n-1)L_n + \tilde{U}_{l,n}$$
$$= (U_n - \tilde{U}_{l,n}) - (n-1)L_n + (n-1)\tilde{L}_{l,n} + \tilde{R}_{l,n} = (U_n - \tilde{U}_{l,n}) - (n-1)(L_n - \tilde{L}_{l,n}) + \tilde{R}_{l,n}$$

We will treat the three summands separately, so we have to show that

(1)
$$\max_{n \le 2^l} |U_n - \tilde{U}_{l,n}| = o_{a.s.}(2^{l(2 - \frac{1}{\alpha} + \frac{1}{\alpha\beta})}),$$

(2)
$$\max_{n \le 2^l} 2^l |L_n - \tilde{L}_{l,n}| = o_{a.s.}(2^{l(1 - \frac{1}{\alpha} + \frac{1}{\alpha\beta})}),$$

(3)
$$\max_{n \le 2^l} |\tilde{R}_{l,n}| = o_{a.s.}(2^{l(2 - \frac{1}{\alpha} + \frac{1}{\alpha\beta})}).$$

With $a.s.$ we indicate that the convergence holds almost surely. In the proof of (1), we have to deal with the problem that we might have $S(i) = S(j)$ for $i \ne j$, so we will treat these two cases separately:

$$|U_n - \tilde{U}_{l,n}| \le \left| \sum_{\substack{1 \le i < j \le n \\ S(i) \ne S(j)}} (h(\xi(S_i), \xi(S_j)) - h_{0,l}(\xi(S_i), \xi(S_j))) \right|$$
$$+ \left| \sum_{\substack{1 \le i < j \le n \\ S(i) = S(j)}} (h(\xi(S_i), \xi(S_j)) - h_{0,l}(\xi(S_i), \xi(S_j))) \right| = |A_{l,n}| + |B_{l,n}|$$

In order to establish bounds for the maximum, we have to control the increments of $A_{l,n}$. Let $n_1, n_2 \in \mathbb{N}$ with $n_1 \le n_2 \le 2^l$, then

$$A_{l,n_2} - A_{l,n_1} = \sum_{\substack{1\le i<j\le n \\ n_1<j\le n_2 \\ S(i)\ne S(j)}} (h(\xi(S_i),\xi(S_j)) - h_{0,l}(\xi(S_i),\xi(S_j))),$$

so we have at most $2^l(n_2-n_1)$ summands of $A_{l,n_2} - A_{l,n_1}$ and for every summand

$$\begin{aligned} E\,|h(\xi(S_i),\xi(S_j)) - h_{0,l}(\xi(S_i),\xi(S_j))| &= E|h(\xi(1),\xi(2))\mathbb{1}_{\{|h(\xi(1),\xi(2))|>a_l\}}| \\ &\le a_l^{1-\eta} E|h(\xi(1),\xi(2))\mathbb{1}_{\{|h(\xi(1),\xi(2))|>a_l\}}|^\eta \le a_l^{1-\eta}E|h(\xi(1),\xi(2))|^\eta. \end{aligned}$$

Consequently, we have by the triangular inequality that

$$\begin{aligned} E|A_{l,n_2} - A_{l,n_1}| &\le 2^l(n_2-n_1)E|h(\xi(1),\xi(2))\mathbb{1}_{\{|h(\xi(1),\xi(2))|>a_l\}}| \\ &\le 2^l(n_2-n_1)a_l^{1-\eta}E|h(\xi(1),\xi(2))|^\eta. \end{aligned}$$

We can write $A_{l,n} = \sum_{i=1}^n (A_{i,n} - A_{i-1,n})$ (with $A_{l,0} := 0$) and in the same way $A_{l,n_2} - A_{l,n_1} = \sum_{i=n_1+1}^{n_2}(A_{i,n} - A_{i-1,n})$, so we can apply the maximal inequality in Theorem 3 of Móricz [13] and obtain

$$E\left|\max_{k\le 2^l} A_{l,n}\right| \le C2^{2l}a_l^{1-\eta}l$$

for some constant C. Recall that $a_l = 2^{l\frac{1+\beta'}{\alpha\beta'}}$ and $\eta = \frac{2\beta'}{1+\beta'}$, so $1-\eta = \frac{1-\beta'}{1+\beta'}$. It follows from the Markov inequality that

$$\begin{aligned} \sum_{l=1}^\infty P\left(\frac{1}{2^{l(2-\frac{1}{\alpha}+\frac{1}{\alpha\beta})}}\max_{k\le 2^l} A_{l,n} \ge \epsilon\right) &\le \frac{C}{\epsilon}\sum_{l=1}^\infty \frac{2^{2l}a_l^{1-\eta}l}{2^{l(2-\frac{1}{\alpha}+\frac{1}{\alpha\beta})}} \\ &= \frac{C}{\epsilon}\sum_{l=1}^\infty \frac{2^{l\frac{1+\beta'}{\alpha\beta'}\frac{1-\beta'}{1+\beta'}}l}{2^{l(-\frac{1}{\alpha}+\frac{1}{\alpha\beta})}} = \frac{C}{\epsilon}\sum_{l=1}^\infty \frac{2^{l(\frac{1}{\alpha\beta'}-\frac{1}{\alpha})}l}{2^{l(-\frac{1}{\alpha}+\frac{1}{\alpha\beta})}} = \frac{C}{\epsilon}\sum_{l=1}^\infty 2^{l(\frac{1}{\alpha\beta'}-\frac{1}{\alpha\beta})}l < \infty, \end{aligned}$$

as $\frac{1}{\alpha\beta'} - \frac{1}{\alpha\beta} < 0$. With the Borel-Cantelli Lemma, we can now conclude that

$$P\left(\frac{1}{2^{l(2-\frac{1}{\alpha}+\frac{1}{\alpha\beta})}}\max_{k\le 2^l} A_{l,n} \ge \epsilon \text{ infinitely often}\right) = 0$$

and thus $\max_{k\le 2^l} A_{l,n} = o_{a.s.}(2^{l(2-\frac{1}{\alpha}+\frac{1}{\alpha\beta})})$. For $B_{l,n}$, we use the fact that the sequences $(S_n)_{n\in\mathbb{N}}$ and $(\xi(n))_{n\in\mathbb{N}}$ are independent observe that

$$\begin{aligned} E|\max_{n\le 2^l} B_{l,n}| &\le E\max_{n\le 2^l}\sum_{\substack{1\le i<j\le n \\ S(i)=S(j)}} |(h(\xi(S_i),\xi(S_j)) - h_{0,l}(\xi(S_i),\xi(S_j)))| \\ &\le E\sum_{\substack{1\le i<j\le 2^l \\ S(i)=S(j)}} |(h(\xi(S_i),\xi(S_j)) - h_{0,l}(\xi(S_i),\xi(S_j)))| \\ &\le E\#\left\{1\le i<j\le 2^l | S(i)=S(j)\right\} E|h(\xi(1),\xi(1))\mathbb{1}_{\{|h(\xi(1),\xi(1))|>a_l\}}| \\ &\le E[\sum_{x\in\mathbb{Z}} N_{2^l}^2(x)]E|h(\xi(1),\xi(1))| \le C2^{l(2-\frac{1}{\alpha})} \end{aligned}$$

for some constant C, where we used Lemma 2.1 of Guillotin-Plantard and Ladret [6] for the ocupation times $N_n(x) := \sum_{i=1}^n \mathbb{1}_{\{S_i=x\}}$. Again using the Markov inequality, we arrive at

$$\sum_{l=1}^\infty P\left(\frac{1}{2^{l(2-\frac{1}{\alpha}+\frac{1}{\alpha\beta})}} \max_{k\le 2^l} B_{l,n} \ge \epsilon\right) \le \frac{C}{\epsilon}\sum_{l=1}^\infty \frac{2^{l(2-\frac{1}{\alpha})}}{2^{l(2-\frac{1}{\alpha}+\frac{1}{\alpha\beta})}} \le \frac{C}{\epsilon}\sum_{l=1}^\infty 2^{-\frac{1}{\alpha\beta}l} < \infty$$

and the Borel-Cantelli Lemma leads to $\max_{k\le 2^l} B_{l,n} = o_{a.s.}(2^{l(2-\frac{1}{\alpha}+\frac{1}{\alpha\beta})})$ as above, which completes the proof of (1). To prove (2), note that

$$\begin{aligned}
E|h_1(\xi(1)) - h_{1,l}(\xi(1))| &= E|E[h(\xi(1),\xi(2))|\xi(1)] - E[h(\xi(1),\xi(2))\mathbb{1}_{\{|h(\xi(1),\xi(2))|\le a_l\}}|\xi(1)]| \\
&= E|E[h(\xi(1),\xi(2))\mathbb{1}_{\{|h(\xi(1),\xi(2))|>a_l\}}|\xi(1)]| \\
&\le E\left[E\left[|h(\xi(1),\xi(2))\mathbb{1}_{\{|h(\xi(1),\xi(2))|>a_l\}}|\big|\xi(1)\right]\right] \\
&= E\left|h(\xi(1),\xi(2))\mathbb{1}_{\{|h(\xi(1),\xi(2))|>a_l\}}\right| \le \frac{1}{a_l^{\eta-1}} E|h(\xi(1),\xi(2))|^\eta.
\end{aligned}$$

With the triangular inequality and the assumption that $E|h(\xi(1),\xi(2))|^\eta < \infty$, it follows that for some constant C and any $n_1, n_2 \in \mathbb{N}$ with $n_1 \le n_2$

$$E|\sum_{i=n_1+1}^{n_2} (h_1(\xi(S_i)) - h_{1,l}(\xi(S_i)))| \le C(n_2-n_1)a_l^{1-\eta}.$$

Again, we apply the maximal inequality in Theorem 3 of Móricz [13] and obtain

$$E\max_{n\le 2^l} 2^l|L_n - \tilde{L}_{l,n}| \le C2^{2l}a_l^{1-\eta}l$$

for some constant C and we can proceed in the same way as we proved almost sure convergence as for $A_{l,n}$. So it remains to show the last part (3). We will proof that

$$\max_{n\le 2^l} |E\tilde{R}_{l,n}| = o(2^{l(2-\frac{1}{\alpha}+\frac{1}{\alpha\beta})})$$
$$\max_{n\le 2^l} |\tilde{R}_{l,n} - E\tilde{R}_{l,n}| = o_{a.s.}(2^{l(2-\frac{1}{\alpha}+\frac{1}{\alpha\beta})}).$$

We obtain with a short calculation that

$$\tilde{R}_{l,n} = \tilde{U}_{l,n} - (n-1)\tilde{L}_{l,n} = (\tilde{U}_{l,n} - U_n) + (n-1)(L_n - \tilde{L}_{l,n}) + R_n$$

and consequently

$$\begin{aligned}
\max_{n\le 2^l}|E\tilde{R}_{l,n}| &= \max_{n\le 2^l}\left|E(\tilde{U}_{l,n} - U_n) + (n-1)E(L_n - \tilde{L}_{l,n}) + ER_n\right| \\
&\le E\max_{n\le 2^l}|U_n - \tilde{U}_{l,n}| + (n-1)E\max_{n\le 2^l}|L_n - \tilde{L}_{l,n}| + \max_{n\le 2^l}|ER_n|.
\end{aligned}$$

We have already shown in (1) and (2) that the first two summands are of order $o(2^{l(2-\frac{1}{\alpha}+\frac{1}{\alpha\beta})})$. For the last summand, we use the fact that $Eh_2(\xi(1),\xi(2)) = Eh(\xi(1),\xi(2)) - Eh_1(\xi(1)) - Eh_1(\xi(2)) = 0$ and that $E|h_{2,l}(\xi(1),\xi(1))| \le E|h_{0,l}(\xi(1),\xi(1))| + 2E|h_{1,l}(\xi(1))| \le 3E|h(\xi(1),\xi(1))| < \infty$, so

$$ER_n = E\left[\sum_{\substack{1\le i<j\le n\\ S(i)=S(j)}} h_2(\xi(S_i),\xi(S_j))\right]$$

and

$$\max_{n\le 2^l}|ER_n| \le \max_{n\le 2^l} E\#\{1\le i<j\le n | S(i)=S(j)\}\, E|h_2(\xi(1),\xi(1))|$$
$$\le E\#\left\{1\le i<j\le 2^l | S(i)=S(j)\right\} E|h_2(\xi(1),\xi(1))| \le C2^{l(2-\frac{1}{\alpha})} = o(2^{l(2-\frac{1}{\alpha}+\frac{1}{\alpha\beta})})$$

with Lemma 2.1 of Guillotin-Plantard and Ladret. To show the convergence of the remaining part, we first decompose it as

$$\tilde{R}_{l,n} - E\tilde{R}_{l,n} = \sum_{\substack{1\le i<j\le n\\ S(i)\neq S(j)}} (h_{2,l}(\xi(S_i),\xi(S_j)) - Eh_{2,l}(\xi(1),\xi(2)))$$
$$+ \sum_{\substack{1\le i<j\le n\\ S(i)=S(j)}} (h_{2,l}(\xi(S_i),\xi(S_j)) - Eh_{2,l}(\xi(1),\xi(1))) =: C_{l,n} + D_{l,n}.$$

For $D_{l,n}$, we have by the independence of $(S_n)_{n\in\mathbb{N}}$ and $(\xi(n))_{n\in\mathbb{N}}$

$$E\max_{n\le 2^l}|D_n| \le E\#\left\{1\le i<j\le 2^l | S(i)=S(j)\right\} E|h_{2,l}(\xi(1),\xi(1))| \le C2^{l(2-\frac{1}{\alpha})}$$

as $E|h_{2,l}(\xi(1),\xi(1))| < \infty$ and in the same way as for $B_{l,n}$ we can conclude that $\max_{k\le 2^l} D_{l,n} = o_{a.s.}(2^{l(2-\frac{1}{\alpha}+\frac{1}{\alpha\beta})})$. Finally, we will deal with $C_{l,n}$. Recall that $h_{0,l}$ is bounded by a_l, so $h_{2,l}$ is bounded by $3a_l$. By the triangular inequality for the L_η-norm, we have that

$$E|h_{2,l}(\xi(1),\xi(2))|^\eta \le \left(\|h_{0,l}(\xi(1),\xi(2))\|_\eta + 2\,\|h_{1,l}(\xi(1))\|_\eta\right)^\eta$$
$$\le \left(3\,\|h(\xi(1),\xi(2))\|_\eta\right)^\eta,$$

and as a consequence for some constant $C>0$

$$E\left(h_{2,l}(\xi(1),\xi(2))\right)^2 \le (3a_l)^{2-\eta} E|h_{2,l}(\xi(1),\xi(2))|^\eta \le C2^{l\frac{1+\beta'}{\alpha\beta'}(2-\frac{2\beta'}{1+\beta'})} = C2^{l\frac{2}{\alpha\beta'}}.$$

Furthermore we have the property of the Hoeffding-decomposition that the random variables $h_{2,l}(\xi(1),\xi(2))$ and $h_{2,l}(\xi(1),\xi(3))$ are uncorrelated, see Lee [11], page 30. So we can find bounds for the conditional variance of the increments of $C_{l,n}$. To simplify the notation, we write

$$Y(i,j) := h_{2,l}(\xi(i),\xi(j))\mathbb{1}_{\{i\neq j\}} - (Eh_{2,l}(\xi(i),\xi(j)))\,\mathbb{1}_{\{i\neq j\}}$$

and obtain for $n_1\le n_2\le 2^l$

$$E\left[(C_{l,n_2}-C_{l,n_1})^2 \,|(X_k)_{k\in\mathbb{N}}\right] = E\left[\left(\sum_{\substack{1\le i<j\le n\\ n_1<j\le n_2}} Y(S(i),S(j))\right)^2 |(X_k)_{k\in\mathbb{N}}\right]$$
$$= \sum_{\substack{x,y\in\mathbb{Z}\\ x<y}} (\#\{1\le i<j\le n | n_1<j\le n_2,\ S(i)=x,\ S(j)=y\}$$
$$+\#\{1\le i<j\le n | n_1<j\le n_2,\ S(i)=y,\ S(j)=x\})^2\, E(Y(x,y))^2$$
$$\le C2^{l\frac{2}{\alpha\beta'}} 2\sum_{x,y\in\mathbb{Z}} (N_n(x)(N_{n_2}(y)-N_{n_1}(y)))^2.$$

So the conditions of Theorem 3 of Móricz [13] hold for the (random) superadditive function
$$g(F_{n_1,n_2}) = \sum_{x,y\in\mathbb{Z}} N_n(x)^2(N_{n_2}(y) - N_{n_1}(y))^2.$$
It follows that
$$E\left[\max_{n\le 2^l}\left(\sum_{1\le i<j\le n} Y(S(i),S(j))\right)^2 \big|(X_k)_{k\in\mathbb{N}}\right] \le C2^{l\frac{2}{\alpha\beta'}}\sum_{x\in\mathbb{Z}} N_{2^l}^2(x)\sum_{y\in\mathbb{Z}} N_{2^l}^2(y)l^2$$
Taking the expectation with respect to $(X_k)_{k\in\mathbb{N}}$, we get the following bound using Lemma 2.1 of Guillotin-Plantard and Ladret [6] at
$$E\left(\max_{n\le 2^l} C_{l,n}\right)^2 = E\left[E\left[\max_{n\le 2^l}\left(\sum_{1\le i<j\le n} Y(S(i),S(j))\right)^2 \big|(X_k)_{k\in\mathbb{N}}\right]\right]$$
$$\le C2^{l\frac{2}{\alpha\beta'}} l^2 E\left(\sum_{x\in\mathbb{Z}} N_{2^l}^2(x)\right)^2 \le C2^{l\frac{2}{\alpha\beta'}} l^2 2^{l(4-\frac{2}{\alpha})} = C2^{l2(2-\frac{1}{\alpha}+\frac{1}{\alpha\beta'})}l^2.$$
We can now use the Chebyshev inequality and arrive at
$$\sum_{l=1}^{\infty} P\left(\frac{1}{2^{l(2-\frac{1}{\alpha}+\frac{1}{\alpha\beta})}}\max_{k\le 2^l} C_{l,n}\ge\epsilon\right) \le \frac{C}{\epsilon^2}\sum_{l=1}^{\infty}\frac{2^{l2(2-\frac{1}{\alpha}+\frac{1}{\alpha\beta'})}l^2}{2^{l2(2-\frac{1}{\alpha}+\frac{1}{\alpha\beta})}}$$
$$\le \frac{C}{\epsilon^2}\sum_{l=1}^{\infty} 2^{l2(\frac{1}{\alpha\beta'}-\frac{1}{\alpha\beta})}l^2 < \infty$$
and the Borel-Cantelli Lemma completes the proof. □

4. Proofs of Main Results

Proof of Theorem 1. Recall the Hoeffding decomposition
$$n^{-2+\frac{1}{\alpha}-\frac{1}{\alpha\beta}}U_{[nt]} = \frac{n-1}{n}n^{-1+\frac{1}{\alpha}-\frac{1}{\alpha\beta}}L_{[nt]} + n^{-2+\frac{1}{\alpha}-\frac{1}{\alpha\beta}}R_{[nt]}.$$
For the linear part, we apply Theorem 1.1 of Kesten and Spitzer [9] to the random variables $h_1(\xi(i))$ and conclude that $\left(n^{-1+\frac{1}{\alpha}-\frac{1}{\alpha\beta}}L_{[nt]}\right)_{t\in[0,1]}$ converges weakly to $(\Delta_t)_{t\in[0,1]}$. For the remainder term, we have proved in Proposition 3.2 that
$$\sup_{t\in[0,1]}|n^{-2+\frac{1}{\alpha}-\frac{1}{\alpha\beta}}R_{[nt]}| \to 0$$
in probability. The statement of the theorem follows by Slutzky's theorem.

□

Proof of Theorem 2. We use again the Hoeffding decomposition
$$\frac{U_n}{n^{\frac{7}{4}}(\log\log n)^{\frac{3}{4}}} = \frac{n-1}{n}\frac{L_n}{(n\log\log n)^{\frac{3}{4}}} + \frac{R_n}{n^{\frac{7}{4}}(\log\log n)^{\frac{3}{4}}}.$$
For the remainder term we use Proposition 3.2 with $\alpha=\beta=2$:
$$R_n = o_{a.s.}(n^{2-\frac{1}{2}+\frac{1}{4}}) = o_{a.s.}(n^{\frac{7}{4}}(\log\log n)^{\frac{3}{4}}).$$

As $L_n = \sum_{i=1}^n h_1(\xi(S_i))$, we can aplly Corollary 1 of Zhang [15]

$$\limsup_{n\to\infty} \pm \frac{L_n}{(n\log\log n)^{\frac{3}{4}}} = \frac{2^{\frac{1}{4}}\operatorname{Var}(\xi)^{\frac{1}{2}}}{3\operatorname{Var}(X)^{\frac{1}{4}}}$$

almost surely, which leads to the statement of the Theorem. □

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