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EXISTENCE OF SOLUTIONS FOR TWO TYPES OF GENERALIZED VERSIONS OF THE CAHN-HILLIARD EQUATION

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EXISTENCE OF SOLUTIONS FOR TWO TYPES OF GENERALIZED VERSIONS OF THE CAHN-HILLIARD EQUATION

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ABSTRACT. We show existence of solutions to two types of generalized Cahn-Hilliard problems: In the first case, we assume the mobility to be dependent on the concentration and its gradient, where the system is supplied with dynamic boundary conditions. In the second case, we treat with classical no-flux boundary conditions where the mobility depends on concentration u, gradient of concentration ∇u and the curvature $\Delta u - s'(u)$. Existence will be shown using a newly developed generalization of gradient flows by the author [16] and the theory of Young measures.

1. INTRODUCTION

This work deals with existence of solutions to a variety of Cahn-Hilliard models generalizing applications in [16]. In what follows, we will introduce the three types of equations that will be discussed in this paper, where we use some notation and Hilbert spaces as they are introduced below in section 2

1.1. **Introductory example: Cahn-Hilliard equations on a closed manifold.** The first problem in most parts was treated in [16] and we will not spend to effort discussing it; we rather consider it as an introductory exercise for the other two problems, as it will help to improve understanding of the method. In the aforementioned paper, the author developed and applied a generalized concept of gradient flows to the following problem:

Given $\Omega \subset \mathbb{R}^n$, $n \le 3$, a bounded and open domain with smooth boundary Γ and outer normal $\boldsymbol{n}_\Gamma$, show existence of solutions to the following problem in some suitable Hilbert space:

$$\begin{aligned} \partial_t u + \operatorname{div}\left[A(u,\nabla u)\,\nabla\left(\Delta u - s'(u)\right)\right] &= 0 && \text{on } (0,T]\times\Omega\,,\\ \left[A(u,\nabla u)\,\nabla\left(\Delta u - s'(u)\right)\right]\cdot \boldsymbol{n}_\Gamma = \nabla u\cdot\boldsymbol{n}_\Gamma &= 0 && \text{on } (0,T]\times\Gamma\,,\\ u(0) &= u_0 && \text{for } t=0\,. \end{aligned}$$

where we assume for some bounded interval $(a,b)\subset\mathbb{R}$, $0\in(a,b)$, that $u_0(x)\in(a,b)$ for all $x\in\Omega$, $s(u)=s_0(u)+s_1(u)$ with $s_0\in C^2((a,b))$ convex and $\lim_{x\to a}s_0'(x)=-\infty$, $\lim_{x\to b}s_0'(x)=+\infty$ as well as $s_1\in C^2(\mathbb{R})$.

Furthermore, we will assume that $A:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^{n\times n}$ is Lipschitz continuous, bounded and uniformly elliptic, which means there is a constant $C>0$ s.t. $C^{-1}\left|\xi\right|^2 \le (A(c,d)\xi)\cdot\xi \le C\left|\xi\right|^2$ for all $(c,d)\in\mathbb{R}\times\mathbb{R}^n$ and all $\xi\in\mathbb{R}^n$. We will use this problem in order to introduce the basic concepts of the theory. The weak formulation of the above problem reads

$$\begin{gathered} \int_0^T\int_\Omega \partial_t u\psi - \int_0^T\int_\Omega \left(A(u,\nabla u)\,\nabla\left(\Delta u - s'(u)\right)\right)\cdot\nabla\psi = 0 \qquad \forall\psi\in L^2(0,T;H^1_{(0)}(\Omega))\\ \nabla u\cdot\boldsymbol{n}_\Gamma = 0 \text{ on } (0,T]\times\Gamma\,,\qquad u(0)=u_0 \text{ for } t=0\,. \end{gathered} \tag{1.1}$$

and the existence result can be formulated as follows

Theorem 1.1. *For $0<T<+\infty$ and any $u_0\in H^1_{(0)}(\Omega)$ there exists $u\in H^1(0,T;H^{-1}_{(0)}(\Omega))\cap L^2(0,T;H^2(\Omega))$ satisfying* (1.1) *with $u(t,x)\in(a,b)$ for a.e. $(t,x)\in(0,T)\times\Omega$, and there is a positive constant $C\in\mathbb{R}$ such that the estimate*

$$\left\|\partial_t u\right\|^2_{L^2(0,t;H^{-1}_{(0)})} + \left\|\Delta u - s_0'(u)\right\|^2_{L^2(0,t;H^1_{(0)})} + \left\|u\right\|^2_{L^2(0,t;H^2)} \le C\left(\mathcal{S}(u_0)-\mathcal{S}(u(t))\right) \tag{1.2}$$

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holds for all $t \in (0,T)$, where

$$\mathcal{S}(u) := \int_\Omega \frac{1}{2} |\nabla u|^2 + \int_\Omega s(u)\,. \tag{1.3}$$

However, for Ω being a bounded domain with smooth boundary Γ, we can also ask for existence of a solution to the following problem

$$\begin{aligned} \partial_t u + \operatorname{div}_\Gamma \left(A(u, \nabla_\Gamma u) \nabla_\Gamma \left(\Delta_\Gamma u - s'(u) \right) \right) &= 0 && \text{on } (0,T] \times \Gamma\,, \\ u(0) &= u_0 && \text{for } t = 0\,, \end{aligned}$$

where $\operatorname{div}_\Gamma$, ∇_Γ and Δ_Γ are the tangential divergence, tangential gradient and Laplace-Beltrami operator on Γ. To this aim, let $T_x\Gamma$ be the tangential space to Γ in $x \in \Gamma$ and $T\Gamma := \bigcup_{x\in\Gamma} \{x\} \times T_x\Gamma$ the tangential bundle. We suppose that s has the properties as above and $A : T\Gamma \to \mathbb{R}^{n\times n}$ is Lipschitz continuous, bounded and uniformly elliptic, which means there is a constant $C > 0$ s.t. $C^{-1} |\xi|^2 \leq (A(c,d)\xi) \cdot \xi \leq C |\xi|^2$ for all $(c,d) \in T\Gamma$ and all $\xi \in T_c\Gamma$. The weak formulation reads

$$\begin{aligned} \int_0^T \int_\Gamma \partial_t u \psi + \int_0^T \int_\Gamma \left(A(u, \nabla_\Gamma u) \nabla_\Gamma \left(\Delta_\Gamma u - s'(u) \right) \right) \cdot \nabla_\Gamma \psi = 0 \qquad &\forall \psi \in L^2(0,T; H^1_{(0)}(\Gamma)) \\ u(0) = u_0 \text{ for } t = 0\,. & \end{aligned} \tag{1.4}$$

This problem is of particular interest for numerical simultions in vesicles formation in biological membranes, see Lowengrub, Rätz, Voigt [?], as well as Mercker and coworkers [?, ?, ?].

Theorem 1.2. *For $0 < T < +\infty$ and any $u_0 \in H^1_{(0)}(\Gamma)$ there exists $u \in H^1(0,T; H^{-1}_{(0)}(\Gamma)) \cap L^2(0,T; H^2(\Gamma))$ satisfying* (1.4) *and there is a positive constant $C \in \mathbb{R}$ such that the estimate*

$$\|\partial_t u\|^2_{L^2(0,t;H^{-1}_{(0)}(\Gamma))} + \|\Delta u - s_0'(u)\|^2_{L^2(0,t;H^1_{(0)}(\Gamma))} + \|u\|^2_{L^2(0,t;H^2(\Gamma))} \leq C \left(\mathcal{S}(u_0) - \mathcal{S}(u(t)) \right)$$

holds for all $t \in (0,T)$, where

$$\mathcal{S}(u) := \int_\Gamma \frac{1}{2} |\nabla_\Gamma u|^2 + \int_\Gamma s(u)\,.$$

The earliest proof of existence for the Cahn-Hilliard equation the author is aware of, is for $A(\cdot,\cdot) \equiv 1$, smooth convex function $s_0 : \mathbb{R} \to \mathbb{R}$ and small concave pertubation s_1 and was given in [9]. The first attempt to the Cahn-Hilliard equation using an energy functional $\mathcal{S}$ with s_0 like above and s_1 a small concave perturbation was in [1]. This form of s seems to be more physical (for a choice $(a,b) = (-1,1)$) as it forces the concentration of each constituent to remain between the fixed boundaries -1 and 1.

Though there is a hughe literature on Cahn-Hilliard equation (refer to [1, 3] and references therein), there seems to be only few results on concentration dependent mobility, among the most cited being Cahn, Elliot and Novick-Cohen [5]. Other works are by Liu [6], the one dimensional treatments by Dal Passo, Giacomelli and Novick-Cohen [7] and Liu [17] and the work by Novick-Cohen [21, 22] which both treat very special cases, but which are both not covered by our approach. Rossi [25] and Grasselli, Miranville, Rossi and Schimperna [12] deal with a Cahn-Hilliard equation of the form

$$\partial_t u - \Delta\alpha(w) = 0\,, \qquad w = s_0'(u) - \Delta u\,.$$

However, a dependence on w will also be included in the third part of the present framework, but with different form of α.

1.2. **Cahn-Hilliard equation with dynamic boundary conditions and nonlinear mobility.** The theory of Cahn-Hilliard equation with dynamic boundary condition is rather young. Mathematical studies and references can be found in Miranville and Zelik [18], Gilardi, Miranville and Schimperna [11], Gal [10] and the initial work by Racke and Zheng [24]. From the modeling point of view, note that the equations derived below fall within the modeling framework developed in Heida [14, 13] or by Qian, Wang and Sheng [23].

Here, we prove existence of a solution to the problem

$$\begin{aligned} \partial_t u &= \operatorname{div} \left(A(u, \nabla u) \nabla \left(s'(u) - \Delta u \right) \right) && \text{on } \Omega\,, \\ 0 &= A(u, \nabla u) \nabla \left(s'(u) - \Delta u \right) \cdot \boldsymbol{n}_\Gamma && \text{on } \Gamma\,, \\ \partial_t u &= A_\Gamma(u) \left(\Delta_\Gamma u - s_\Gamma'(u) - \nabla u \cdot \boldsymbol{n}_\Gamma \right) && \text{on } \Gamma\,, \end{aligned}$$

with $u(0,\cdot) = u_0(\cdot)$ for $t = 0$ on Ω and Γ and we assume A and s to be given like in section 1.1. A_Γ is assumed to be bounded and Lipschitz continuous with some $0 < C \leq A_\Gamma(\cdot)$ for some positive constant C and $s_\Gamma = s_0 + s_2$ with $s_2 \in C^2(\mathbb{R})$. Existence to above problem in case $A = Id$, $A_\Gamma = 1$ was treated in the above references for different forms of s and s_Γ. Note that the first and third equation of the problem are not coupled directly through boundary integrals but only through $\nabla u \cdot \boldsymbol{n}_\Gamma$. Thus, the weak formulation of the above problem splits up into two parts:

(1.5)
$$\begin{aligned} \int_0^T \int_\Omega \partial_t u \, \psi - \int_0^T \int_\Omega (A(u, \nabla u) \nabla (s'(u) - \Delta u)) \cdot \nabla \psi = 0 \\ \int_0^T \int_\Gamma \partial_t E(u) \, \varphi - \int_0^T \int_\Gamma A_\Gamma(E(u)) \left(\Delta_\Gamma E(u) - s_\Gamma'(E(u)) - \nabla u \cdot \boldsymbol{n}_\Gamma \right) \varphi = 0 \end{aligned} \qquad \forall \psi, \varphi \in C^1(0,T; C^\infty(\overline{\Omega}))$$

together with the inital condition, where we use $E(u)$ to denote the trace of u on Γ and P_0 the projection operator defined below in (2.2). Our existence result then reads as follows:

Theorem 1.3. *For $0 < T < +\infty$ and any $u_0 \in H^1_{(0)}(\Omega) \cap H^2(\Omega)$ there exists $u \in H^1(0,T; H^{-1}_{(0)}(\Omega)) \cap L^2(0,T; H^1(\Omega))$ with $E(u) \in H^1(0,T; L^2(\Gamma)) \cap L^2(0,T; H^1(\Gamma))$, as well as $P_0(s'(u) - \Delta u) \in L^2(0,T; H^1_{(0)}(\Omega)$ and $(\Delta_\Gamma u - \nabla u \cdot \boldsymbol{n}_\Gamma) \in L^2(0,T; L^2(\Gamma))$ satisfying (1.5) and there is a positive constant $C \in \mathbb{R}$ such that the estimate*

$$\begin{aligned} \|u\|^2_{H^1(0,T;H^{-1}_{(0)}(\Omega)) \cap L^2(0,T;H^1(\Omega))} + \|P_0 \left(\Delta u - s_0'(u) \right)\|^2_{L^2(0,t;H^1_{(0)})} + \|\Delta_\Gamma E(u) - \nabla u \cdot \boldsymbol{n}_\Gamma\|_{L^2(0,T;L^2(\Gamma))} \\ + \|Eu\|^2_{H^1(0,T;L^2(\Gamma)) \cap L^2(0,T;H^1(\Gamma))} \leq C \left(\mathcal{S}(u_0) - \mathcal{S}(u(t)) \right) \end{aligned}$$

holds for all $t \in (0,T)$, where

$$\mathcal{S}(u) := \int_\Omega \frac{1}{2} |\nabla u|^2 + \int_\Omega s(u) + \int_\Gamma \frac{1}{2} |\nabla_\Gamma E(u)|^2 + \int_\Gamma s_\Gamma(Eu) \, .$$

Note that the usual way for treating such equations is different and we will shortly skech it formally: Starting from the classical Cahn-Hilliard problem with dynamic boundary conditions

$$\begin{aligned} \partial_t u &= \operatorname{div} \left(\nabla \left(s'(u) - \Delta u \right) \right) && \text{on } \Omega \, , \\ 0 &= \nabla \left(s'(u) - \Delta u \right) \cdot \boldsymbol{n}_\Gamma && \text{on } \Gamma \, , \\ \partial_t u &= \left(\Delta_\Gamma u - s_\Gamma'(u) - \nabla u \cdot \boldsymbol{n}_\Gamma \right) && \text{on } \Gamma \, , \end{aligned}$$

it is convenient to reformulate this problem (for the moment informally) as

$$\begin{aligned} -\Delta_N^{-1} \partial_t u &= - \left(s'(u) - \Delta u \right) + \langle \mu \rangle \qquad \text{on } \Omega \, , \\ \langle \mu \rangle &= \langle s'(u) \rangle - \langle \Delta u \rangle \, , \end{aligned}$$

where $\langle u \rangle := \fint_\Omega u$. This formulation allows to perform partial integration in the term Δu and thus to treat the problem in one single weak formulation of the form

$$\int_0^T \int_\Omega -\Delta_N^{-1} \partial_t u \, \psi + \int_0^T \int_\Omega \left(s'(u)\psi + \nabla u \cdot \nabla \psi \right) + \int_0^T \int_\Gamma \partial_t u \, \psi + \int_0^T \int_\Gamma \left(\nabla_\Gamma u \cdot \nabla_\Gamma \psi + s_\Gamma'(u) \psi \right) = 0$$

However, for the nonlinear dependence of the mobility on $u, \nabla u$, the operator Δ_N^{-1} would have to be replaced by a time-dependent operator, imposing lots of technical difficulties.

1.3. **Cahn-Hilliard equation with curvature-dependent mobility.** The third type of Cahn-Hilliard equation is a generalization of the first type with an additional dependence on the "curvature" term $w := -\Delta u + s'(u)$ (see below). Thus, we write down the problem as

$$\begin{aligned} \partial_t u - \operatorname{div} \left(A(u, \nabla u, w) \, \nabla w \right) &= 0 && \text{on } (0,T] \times \Omega \, , \\ w + \Delta u - s'(u) &= 0 && \text{on } (0,T] \times \Omega \, , \\ \left(A(u, \nabla u, w) \, \nabla w \right) \cdot \boldsymbol{n}_\Gamma = \nabla u \cdot \boldsymbol{n}_\Gamma &= 0 && \text{on } (0,T] \times \Gamma \, , \\ u(0) &= u_0 && \text{for } t = 0 \, . \end{aligned}$$

where $s(u) = s_0(u) + s_1(u)$ with $s_0(u) = |u|^p$ for some $p \geq 2$ and $s_1 \in C_b^{3,1}(\mathbb{R})$ is a three times continuously differentiable mapping with bounded derivatives up to order 2.

Furthermore, we will assume that $A : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times n}$ is Lipschitz continuous, bounded and uniformly elliptic, which means there is a constant $C > 0$ s.t. $C^{-1} |\xi|^2 \leq (A(a, b, c)\xi) \cdot \xi \leq C |\xi|^2$ for all $(a, b, c) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ and all $\xi \in \mathbb{R}^n$. The weak formulation of the above problem reads

$$\begin{aligned} &\int_0^T \int_\Omega \partial_t u\psi + \int_0^T \int_\Omega (A(u, \nabla u, w) \nabla w) \cdot \nabla\psi = 0 \qquad \forall \psi \in L^2(0, T; H^1_{(0)}(\Omega)) \\ &w = -\Delta u + s'(u), \qquad \nabla u \cdot \boldsymbol{n}_\Gamma = 0 \text{ on } (0, T] \times \Gamma\,, \qquad u(0) = u_0 \text{ for } t = 0\,. \end{aligned} \tag{1.6}$$

for which the following existence theorem holds:

Theorem 1.4. *For $0 < T < +\infty$ and any $u_0 \in H^1_{(0)}(\Omega)$ there exists $u \in H^1(0, T; H^{-1}_{(0)}(\Omega)) \cap L^2(0, T; H^2(\Omega))$, $w \in L^2(0, T; H^1_{(0)}(\Omega))$ satisfying (1.6) and there is a positive constant $C \in \mathbb{R}$ such that the estimate*

$$\|\partial_t u\|^2_{L^2(0,t;H^{-1}_{(0)})} + \|\Delta u - P_0(s_0'(u))\|^2_{L^2(0,t;H^1_{(0)})} + \|u\|^2_{L^2(0,t;H^2)} \leq C\,(\mathcal{S}(u_0) - \mathcal{S}(u(t)))$$

holds for all $t \in (0, T)$, where

$$\mathcal{S}(u) := \int_\Omega \frac{1}{2} |\nabla u|^2 + \int_\Omega s(u)\,.$$

The last result is of particular interest for the sharp interface limit. This limit is obtained by replacing $\mathcal{S}$ with

$$\mathcal{S}^\varepsilon(u) := \int_\Omega \frac{1}{2} |\nabla u|^2 + \frac{1}{\varepsilon^2} \int_\Omega s(u)$$

and solving a sequence of problems

$$\begin{aligned} \partial_t u^\varepsilon - \operatorname{div}\,(A(u^\varepsilon, \nabla u^\varepsilon, w^\varepsilon) \nabla w^\varepsilon) &= 0 && \text{on } (0, T] \times \Omega\,, \\ w^\varepsilon + \Delta u^\varepsilon - \frac{1}{\varepsilon^2} s'(u^\varepsilon) &= 0 && \text{on } (0, T] \times \Omega\,, \\ (A(u^\varepsilon, \nabla u^\varepsilon, w^\varepsilon) \nabla w^\varepsilon) \cdot \boldsymbol{n}_\Gamma = \nabla u^\varepsilon \cdot \boldsymbol{n}_\Gamma &= 0 && \text{on } (0, T] \times \Gamma\,, \\ u^\varepsilon(0) &= u_0^\varepsilon && \text{for } t = 0\,. \end{aligned}$$

For the corresponding sequence of solutions u^ε, we expect

$$u^\varepsilon \to u$$

where $u \in BV(\Omega)$ with $u(\cdot) \in \{-1, 1\}$ almost surely, ∇u being equal to a varifold γ with curvature κ, satisfying $\partial_t \gamma = \kappa$ in a weak sense. We refer to the work by Röger and Schätzle [**?**], Mugnai and Röger [19, 20] or the survey by Serfaty [27]. Note that with regard to the limit equations, the dependence of A on u or ∇u makes limited sense as $u(x, t) \in \{-1, 1\}$ almost surely and ∇u is a lower dimensional Hausdorff measure indicating the interface. However, the quantity w^ε should converge to the curvature κ of ∇u and thus, the dependence of A on w may affect the limit equations. A rigorous study of these reflektions is, unfortunatly, beyond the scope of this article.

1.4. **Outline of the paper.** In section 2 we will introduce some standard Hilbert spaces which will be frequently used in this paper and collect some basic facts on them. We will furthermore introduce basic notations for the work with boundary derivatives. In section 3 we will introduce some functional analytical tools, in particular the theory of Young measures whereas in section 4, we will introduce the theory of gradient flows in the way it is presented in [16].

Since we introduced the three types of problems by complexity of their analysis, we will then go on first treating the problems from subsection 1.1, making the reader familiar with the method and notation in section 5. The second step will be generalization to the dynamic boundary conditions in section 6, making it necessary to look for a suitable Hilbert space in order to apply gradient flow theory. Finally, we will include the dependence of mobility on curvature and proof Theorem 1.4 in section 7.

2. Notations and Preliminaries

For any Hilbert space $\mathcal{H}$, we denote $L^p(0,T;\mathcal{H})$ the Bochner space of L^p-functions over $(0,T]$ having values in $\mathcal{H}$ and by $H^1(0,T;\mathcal{H})$ the space of functions $u \in L^2(0,T;\mathcal{H})$ having $\partial_t u \in L^2(0,T;\mathcal{H})$. Furthermore, by $C([0,T],\mathcal{H})$ we denote the continuous functions from $[0,T]$ to $\mathcal{H}$, by $C^k([0,T],\mathcal{H})$ the k-times continuously differentiable functions and by $AC([0,T];\mathcal{H})$ the set of absolutely continuous functions over $[0,T]$.

2.1. **Sobolev spaces on Ω.** In order to study the examples below, we will frequently make use of the following Banach and Hilbert spaces: We consider an open, bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\Gamma = \partial\Omega$ and outer normal vector $\boldsymbol{n}_\Gamma$. $W^k_p(\Omega)$ denotes the usual L^p-Sobolev space and $W^k_{p,0}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^k_p(\Omega)$. We will also make use of the notation

$$H^k(\Omega) := W^k_2(\Omega) \quad \text{and} \quad H^k_0(\Omega) := W^k_{2,0}(\Omega) \tag{2.1}$$

Following Adams [2], we introduce the fractional Sobolev spaces by interpolation: Let W_ν be the space of measurable functions $[0,\infty) \to L^2(\Omega)$ with $u \in W_\nu$ iff $t^\nu u(t) \in L^2(0,\infty;W^1_2(\Omega))$ and $t^\nu \partial_t u(t) \in L^2(0,\infty;L^2(\Omega))$. Then, for $\nu = \theta - \frac{1}{2}$ set

$$\left\|u;T^\theta(\Omega)\right\|^2 := \inf_{f\in W_\nu,\, f(0)=u} \max\left\{\int_0^\infty t^{2\nu}\left\|f(t)\right\|^2_{W^1_2},\, \int_0^\infty t^{2\nu}\left\|f'(t)\right\|^2_{L^2}\right\}$$

and for $s = m+\sigma \geq 0$ with $m \in \mathbb{N}$, $\sigma \in (0,1)$ define

$$\|u\|^2_{W^s_2(\Omega)} := \|u\|^2_{W^m_2} + \sum_{|\alpha|=m}\left\|\partial^\alpha u\,;\,T^{1-\sigma}(\Omega)\right\|^2$$

and $W^s_2(\Omega) = \overline{W^{m+1}_2(\Omega)}^{\|\cdot\|_{W^s_2(\Omega)}}$. For $s<0$, set $W^s_2(\Omega) = \left(W^{-s}_{2,0}(\Omega)\right)^{-1}$.

$H^{-1}(\Omega)$ denotes the dual of $H^1_0(\Omega)$. Furthermore, we introduce

$$H^1_{(0)}(\Omega) := \left\{\phi \in H^1(\Omega)\,:\,\int_\Omega \phi = 0\right\}$$

with the scalar product

$$\langle\phi,\psi\rangle_{H^1_{(0)}} := \int_\Omega \nabla\phi\cdot\nabla\psi \qquad \forall \phi,\psi \in H^1_{(0)}(\Omega)$$

and its dual space $H^{-1}_{(0)}(\Omega)$ with scalar product

$$\langle\phi,\psi\rangle_{H^{-1}_{(0)}} := \left\langle\nabla\Delta_N^{-1}\phi, \nabla\Delta_N^{-1}\psi\right\rangle_{L^2} \qquad \forall\phi,\psi\in H^{-1}_{(0)}(\Omega)\,,$$

where Δ_N is the Laplace operator with Neumann boundary conditions. More generally, define

$$L^2_{(m)}(\Omega) := \left\{f \in L^2(\Omega)\,:\,\int_\Omega f = m\right\}, \qquad C^k_{(0)}(\overline{\Omega}) := L^2_{(0)}(\Omega)\cap C^k(\overline{\Omega}) \quad \forall k \in \mathbb{N}\cup\{\infty\}$$

and

$$P_0 :\ L^2(\Omega) \to L^2_{(0)}(\Omega), \qquad f \mapsto f - \int_\Omega f \tag{2.2}$$

the orthogonal projection onto $L^2_{(0)}(\Omega)$. For simplicity, we may sometimes omit the (Ω) if the context is clear (e.g. H^1 instead of $H^1(\Omega)$). Then, $-\Delta_N : H^1_{(0)}(\Omega) \to H^{-1}_{(0)}(\Omega)$ is the Riesz isomorphism.

Lemma 2.1. [16] *Let $A \in L^\infty(\Omega;\mathbb{R}^{n\times n})$ having the property that there is $0 < C \leq 1$ such that $C|\xi|^2 \leq \xi A(x)\xi \leq C^{-1}|\xi|^2$for a.e. $x\in\Omega$ and for all $\xi\in\mathbb{R}^n$. For $\phi \in H^{-1}_{(0),n}(\Omega)$ let $p_\phi \in H^1_{(0)}(\Omega)$solve*

$$-div(A\nabla p_\phi) = \phi \text{ on } \Omega, \qquad (A\nabla p_\phi)\cdot\boldsymbol{n}_\Gamma = 0 \text{ on } \Gamma\,,$$

Then, there is $0 < G \leq 1$ only depending on C such that for all $\phi \in H^{-1}_{(0)}(\Omega)$holds

$$G\|\phi\|^2_{H^{-1}_{(0)}} \leq \int_\Omega \nabla p_\phi\cdot(A\nabla p_\phi) \leq G^{-1}\|\phi\|^2_{H^{-1}_{(0)}}\,,$$

2.2. **Sobolev spaces on Γ.** Since Γ is C^∞, we may introduce the tangential gradient ∇_Γ in the following way: On Γ, let $\boldsymbol{n}_\Gamma$ be the normal vector field and for each arbitrary C^∞-vector field $\boldsymbol{a} : \overline{\Omega} \to \mathbb{R}^3$, we define the normal part a_n and the tangential part $\boldsymbol{a}_\tau$ on Γ via

$$a_n := \boldsymbol{a} \cdot \boldsymbol{n}_\Gamma , \qquad \boldsymbol{a}_\tau := \boldsymbol{a} - a_n \boldsymbol{n}_\Gamma \,.$$

We define the normal derivative

$$\partial_n a := \nabla a \cdot \boldsymbol{n}_\Gamma$$

and the tangential gradient ∇_Γ for any scalar a through

$$\nabla_\Gamma a := (\nabla a)_\tau = \nabla a - \boldsymbol{n}_\Gamma \partial_n a \,.$$

For a smooth mannifold, this is equivalent with the Levi-Civita connection on Γ. Thus, we may understand any vector field $\boldsymbol{f}_\tau$ tangential to Γ as an element of the $T\Gamma$, and we define the divergence

$$\mathrm{div}_\Gamma \boldsymbol{f}_\tau := \mathrm{trace} \nabla_\Gamma \boldsymbol{f}_\tau \,,$$

where we find for any sufficiently regular $\boldsymbol{f}$:

$$\mathrm{div}\, \boldsymbol{f} = \mathrm{div}_\Gamma \boldsymbol{f}_\tau + \partial_n (\boldsymbol{f}_n) \,.$$

The mean curvature of Γ is defined as

$$\kappa_\Gamma := \mathrm{trace}\, (\nabla_\Gamma \boldsymbol{n}_\Gamma)$$

and we find the following important result:

Lemma 2.2. [4]*For any $f \in C^1(\Gamma)$ holds*

$$\int_\Gamma \nabla_\Gamma f = \int_\Gamma f \kappa_\Gamma \boldsymbol{n}_\Gamma + \int_{\partial\Gamma} f \boldsymbol{\nu}$$

where $\boldsymbol{\nu}$ is the unit vector tangent to Γ and normal to $\partial\Gamma$. Furthermore, for any tangentially differentiable field $\boldsymbol{q}$ holds

$$\int_\Gamma div_\Gamma \boldsymbol{q} = \int_\Gamma \kappa_\Gamma \boldsymbol{q} \cdot \boldsymbol{n}_\Gamma + \int_{\partial\Gamma} \boldsymbol{q} \cdot \boldsymbol{\nu}$$

The Laplace-Beltrami operator Δ_Γ on Γ is defined as $\Delta_\Gamma f := \mathrm{div}_\Gamma \nabla_\Gamma f$. For a nice introduction to surface gradients and the Laplace-Beltrami operator not based on the Levi-Civita connection, we refer to Buscaglia and Ausas [4].

Remark. Lemma 2.2 implies for the closed surface Γ that

$$-\int_\Gamma g \Delta_\Gamma f = \int_\Gamma \nabla_\Gamma g \cdot \nabla_\Gamma f \qquad \forall f, g \in C^2(\overline{\Omega}) \,.$$

Via localization, projection and interpolation, we can introduce $W_2^s(\Gamma)$ for $s \in \mathbb{R}$ [2]. Note that

$$\|u\|^2_{W_2^1(\Gamma)} = \int_\Gamma |\nabla_\Gamma u|^2 + \int_\Gamma u^2 \,.$$

For $u \in C^2(\overline{\Omega})$, we set $E_\Gamma(u) := u|_\Gamma$ the trace of u on Γ, and $\partial_n u := \nabla u \cdot \boldsymbol{n}_\Gamma$, with $E_\Gamma(u)$, $\partial_n u$ both being functions on Γ. Like in Ω, consider the space

$$H^1_{(0)}(\Gamma) := \left\{ u \in W_2^1(\Gamma) \,:\, \int_\Gamma u = 0 \right\} , \tag{2.3}$$

$$\|u\|^2_{H^1_{(0)}(\Gamma)} := \int_\Gamma |\nabla_\Gamma u|^2 \,.$$

and introduce $H^{-1}_{(0)}(\Gamma)$ in an obvious way. We summarise the main imbedding results of interest from [2] in a short lemma:

Lemma 2.3. *The operators $E_\Gamma : W_2^k(\Omega) \to W_2^{k-\frac{1}{2}}(\Gamma)$, $k \geq 1$, and $\partial_n : W_2^k(\Omega) \to W_2^{k-\frac{3}{2}}(\Gamma)$, $k \geq 2$, are continuous. Furthermore, $W_2^{k_1}(\Omega) \hookrightarrow W_2^{k_2}(\Omega)$, $W_2^{k_1}(\Gamma) \hookrightarrow W_2^{k_2}(\Gamma)$ are continuous and compact for all $k_1 > k_2$ and $k_1, k_2 \in \mathbb{R}$.*

Remark 2.4. Note that there is $0 < C < 1$ such that

$$C \|u\|_{W_2^1(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)} + \|E_\Gamma(u)\|_{L^2(\Gamma)} \leq C^{-1} \|u\|_{W_2^1(\Omega)}$$

i.e. the last chain of inequalities shows an equivalence of norms on $W_2^1(\Omega)$.

Furthermore, for simplicity of notation, we simply write

$$u \equiv E_\Gamma(u) \in L^2(\Gamma) \qquad \forall u \in W_2^1(\Omega) \tag{2.4}$$

and thus, we do not distinguish between $W_2^1(\Omega)$-functions and their traces, whenever this will not cause confusion. Finally, we have the following result, which can be found for example in the book by Temam [29]:

Lemma 2.5. *Let*

$$E(\Omega) := \left\{u \in L^2(\Omega)^n \,:\, div u \in L^2(\Omega)\right\} .$$

then, the operator

$$\partial_n \,:\, E(\Omega) \to L^2(\Omega) , \quad u \mapsto u \cdot \boldsymbol{n}_\Gamma$$

is continuous.

3. Functional Analytical Tools and Young measures

3.1. **Tools from functional analysis.** We state two fundamental results from functional analysis which are known in various versions, among which we will use the following:

Theorem 3.1 (Egorov's theorem for $L^2(0,T;\mathcal{H})$)**.** *Let $\mathcal{H}$ be a Hilbert space and $(v_n)_{n\in\mathbb{N}} \subset L^2(0,T;\mathcal{H})$ be a sequence such that $v_n \to v \in L^2(0,T;\mathcal{H})$ strongly and pointwise for a.e. $t \in (0,T)$. Then, for any $\varepsilon > 0$ there is $K_\varepsilon \subset (0,T)$ compact with $\mathcal{L}((0,T)\backslash K_\varepsilon) < \varepsilon$ such that $v_n \to v$ uniformly on K_ε.*

Theorem 3.2 (Lusin)**.** *For a Banach space $\mathcal{B}$, let $f \in L^p(0,T;\mathcal{B})$ for some $1 \leq p < \infty$. Then, for each $\varepsilon > 0$ there is a compact set $K^\varepsilon \subset (0,T)$ such that $\mathcal{L}((0,T)\backslash K^\varepsilon) < \varepsilon$ and $f \in C(K^\varepsilon;\mathcal{B})$.*

3.2. **Young measures.** For a separable metric space E, we denote by $\mathcal{B}(E)$ the Borel-σ-algebra, where $\mathcal{L}(0,T)$ is the Lebesgue-σ-algebra on $(0,T)$ and $\mathcal{L}(0,T)\otimes\mathcal{B}(E)$ is the product σ-algebra. $\mathcal{M}(0,T;E)$ denotes the set of measurable functions over $(0,T)$ with values in E. A $\mathcal{L}(0,T)\otimes\mathcal{B}(E)$-measurable function $h : (0,T)\times E \to (-\infty,+\infty]$ is a normal integrand if $v \mapsto h(t,v)$ is lower semicontinuous for all $t \in (0,T)$.

For a Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ denote the Borel-sigma-algebra with respect to $\|\cdot\|_{\mathcal{H}}$. We say that a $\mathcal{L}\otimes\mathcal{B}(\mathcal{H})$-measurable functional $h :\, (0,T)\times\mathcal{H} \to (-\infty,+\infty]$ is a weakly normal integrand if

$$v \mapsto h_t(v) := h(t,v) \quad \text{is sequentially weakly l.s.c. for a.e. } t \in (0,T) .$$

Definition 3.3. (Time dependent parametrized measures) A parametrized measure in E is a family $\boldsymbol{\nu} := \{\nu_t\}_{t\in(0,T)}$ of Borel probability measures on E such that

$$t \in (0,T) \mapsto \nu_t(B) \quad \text{is } \mathcal{L} - \text{measurable for all } B \in \mathcal{B}(E) .$$

We denote by $\mathcal{Y}(0,T;E)$ the set of all parametrized measures.

For computations below, the most important result on parametrized measures is a generalization of Fubini's theorem [8]: For every parametrized measure $\boldsymbol{\nu} = \{\nu_t\}_{t\in(0,T)}$, there exists a unique measure ν on $\mathcal{L}(0,T)\otimes\mathcal{B}(E)$ defined by

$$\nu(I\times A) = \int_I \nu_t(A)dt \qquad \forall I \in \mathcal{L}(0,T),\ A \in \mathcal{B}(E) .$$

Moreover, for every $\mathcal{L}(0,T)\otimes\mathcal{B}(E)$-measurable function $h : (0,T)\times E \to [0,+\infty]$, the function

$$t \mapsto \int_E h(t,\xi)d\nu_t(\xi)$$

is $\mathcal{L}(0,T)$-measurable and the Fubini integral representation holds:

$$\int_{(0,T)\times E} h(t,\xi)d\nu(t,\xi) = \int_0^T \left(\int_E h(t,\xi)d\nu_t(\xi)\right) dt . \tag{3.1}$$

If ν is concentrated on the graph of a measurable function $u : (0,T) \to E$, then $\nu_t = \delta_{u(t)}$ for a.e. $t \in (0,T)$, where $\delta_{u(t)}$ denotes the dirac's measure carried by $\{u(t)\}$. In this case, by (3.1):

$$\int_{(0,T)\times E} h(t,\xi)d\nu(t,\xi) = \int_0^T h(t,u(t))dt\,.$$

For calculations below, we will study the following situation: given two Hilbert spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$, we will consider a mapping $g_\bullet(\cdot,\cdot) : \tilde{\mathcal{H}} \times \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ being continuous in $\tilde{\mathcal{H}}$ and bilinear continuous in $\mathcal{H}$ with

$$C^{-1}\|\xi\|^2_{\mathcal{H}} \leq g_u(\xi,\xi) \leq C\|\xi\|^2_{\mathcal{H}} \qquad \forall u \in \tilde{\mathcal{H}},\, \xi \in \mathcal{H}$$

for some constant C and

$$g_{u_m}(v_m,\varphi) \to g_u(v,\varphi) \qquad \forall \varphi \in \mathcal{H}, \tag{3.2}$$

whenever $u_m \to u$ strongly in $\tilde{\mathcal{H}}$ and $v_m \rightharpoonup v$ weakly in $\mathcal{H}$. Starting from section 4 below, we will assume $\tilde{\mathcal{H}} \hookrightarrow \mathcal{H}$ continuously, which is actually not needed for the results in this section.

Corollary 3.4. [16] *As a consequence of* (3.2)*, we find for $u_n \to u$ strongly in $\tilde{\mathcal{H}}$ and $\varphi_n \rightharpoonup \varphi$ weakly in $\mathcal{H}$:*

$$g_u(\varphi,\varphi) \leq \liminf_{n\to\infty} g_{u_n}(\varphi_n,\varphi_n)\,.$$

Theorem 3.5. [16] *Let $\{v_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $L^p(0,T;\mathcal{H})$, for some $p > 1$, and let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $L^p(0,T;\tilde{\mathcal{H}})$ with $u_n \to u \in L^p(0,T;\tilde{\mathcal{H}})$ pointwise a.e. in $(0,T)$. Then there exists a subsequence $k \mapsto v_{n_k}$ and a parameterized measure $\boldsymbol{\nu} = \{\nu_t\}_{t\in(0,T)} \in \mathcal{Y}(0,T;\mathcal{H})$ such that for a.e. $t \in (0,T)$*

$$\limsup_{k\to\infty} \|v_{n_k}(t)\|_{\mathcal{H}} < +\infty, \quad \nu_t \text{ is concentrated on } L(t) := \bigcap_{q=1}^{\infty} \overline{\{v_{n_k}(t) \,:\, k \geq q\}}^{w}$$

of weak limit points of $\{v_n\}_{n\in\mathbb{N}}$, and

$$\liminf_{k\to\infty} \int_0^T h(t,v_{n_k}(t))dt \geq \int_0^T \left(\int_{\mathcal{H}} h(t,\xi)d\nu_t(\xi)\right)dt$$

for every weakly normal integrand h such that $h^-(\cdot,v_{n_k}(\cdot))$ is uniformly integrable and there holds

$$\liminf_{k\to\infty} \int_0^T g_{u_m}(v_m(t),v_m(t))dt \geq \int_0^T \left(\int_{\mathcal{H}} g_u(\xi,\xi)d\nu_t(\xi)\right)dt\,. \tag{3.3}$$

In particular,

$$\int_0^T \left(\int_{\mathcal{H}} \|\xi\|^p_{\mathcal{H}}\, d\nu_t(\xi)\right) \leq \liminf_{k\to\infty} \int_0^T \|v_{n_k}\|^p_{\mathcal{H}}\, dt\,,$$

and, setting

$$v(t) := \int_{\mathcal{H}} \xi d\nu_t(\xi), \quad \text{we have } v_{n_k} \rightharpoonup v \text{ in } L^p(0,T;\mathcal{H})\,.$$

Finally, if $\nu_t = \delta_{v(t)}$ for a.e. $t \in (0,T)$, then

$$\langle v_{n_k}, w\rangle_{\mathcal{H}} \to \langle v, w\rangle_{\mathcal{H}} \quad \text{in } L^1(0,T) \quad \forall w \in L^q(0,T;\mathcal{H}), \quad \frac{1}{p}+\frac{1}{q} = 1\,,$$

and up to extraction of a further subsequence independent of t (still denoted by v_{n_k})

$$v_{n_k}(t) \rightharpoonup v(t) \quad \text{for a.e. } t \in (0,T)\,.$$

Remark 3.6. In the original theorem in [16], there was the assumption that $\tilde{\mathcal{H}} \hookrightarrow \mathcal{H}$ is continuously embedded for conceptual reasons of the paper (see also section 4 below). However, looking at the original proof, it is obvious that the assumption $\tilde{\mathcal{H}} \hookrightarrow \mathcal{H}$ is not needed. We also refer to Stefanelli [28] Theorem 4.3 for a more general result.

4. Gradient Flow Theory

The theory developed in [16] deals with equations of the form

$$\partial_t u \in -\nabla_{l,u}\mathcal{S}(u) + f(t) \tag{4.1}$$

with $\mathcal{S}$ being a (possible nonconvex) lower semicontinuous entropy functional on a Hilbert space $\mathcal{H}$, $\nabla_{l,u}\mathcal{S}$ being the *limiting subgradient* with respect to a *densly defined metric structure* $g_\bullet$ and $f \in L^2(0,T;\mathcal{H})$.

More precisely, consider Hilbert spaces $\mathcal{H}_0 \hookrightarrow \tilde{\mathcal{H}} \hookrightarrow \mathcal{H}$ with the set $B(\mathcal{H})$ of positive definite continuous bilinear forms. We then use the following terms and notations:

Definition 4.1. We call any tuple $(\mathcal{H}_0, \tilde{\mathcal{H}}, \mathcal{H}, g)$ of Hilbert spaces $\mathcal{H}_0$, $\tilde{\mathcal{H}}$, $\mathcal{H}$ and a mapping $g_\bullet : \tilde{\mathcal{H}} \to B(\mathcal{H})$ satisfying 1 and 2 an entropy space:

(1) $\mathcal{H}_0 \hookrightarrow \tilde{\mathcal{H}} \hookrightarrow \mathcal{H}$, where the embeddings are dense, and the embedding $\mathcal{H}_0 \hookrightarrow \tilde{\mathcal{H}}$ is compact. We denote $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{\tilde{\mathcal{H}}}$, $\|\cdot\|_{\mathcal{H}_0}$ the respective norms and by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ the scalar produkt on $\mathcal{H}$.
(2) g is a densly defined metric in the following sense: There are positive constants $1 \le G^* < +\infty$ such that

$$\sqrt{G^*}^{-1} \left|\langle x,y\rangle_{\mathcal{H}}\right| \le |g_u(x,y)| \le \sqrt{G^*}\left|\langle x,y\rangle_{\mathcal{H}}\right| \quad \forall u \in \tilde{\mathcal{H}}, \quad \forall x,y \in \mathcal{H}\,, \tag{4.2}$$

for all $u \in \tilde{\mathcal{H}}$ and $g_\bullet$ is strong-weak-continuous in the following sense: if $u_n \to u$ strongly in $\tilde{\mathcal{H}}$ and $\varphi_n \rightharpoonup \varphi$ weakly in $\mathcal{H}$ as $n \to \infty$, then

$$g_{u_n}(\varphi_n,\psi) \to g_u(\varphi,\psi) \quad \text{as } n\to\infty \qquad \forall \psi \in \mathcal{H}\,. \tag{4.3}$$

This means that to every point $u \in \tilde{\mathcal{H}}$ we associate a local scalar produkt and local norm

$$\langle x,y\rangle_{g(u)} := g_u(x,y)\,, \qquad \|x\|_{g(u)} := \sqrt{g_u(x,x)} \qquad \forall x,y \in \mathcal{H}\,.$$

We denote by $\tilde{g}_u$ the unique automorphism on $\mathcal{H}$ such that

$$g_u(v,\varphi) = \langle \tilde{g}_u(v),\varphi\rangle_{\mathcal{H}} \qquad \forall \varphi \in \mathcal{H}\,. \tag{4.4}$$

We will assume that $\mathcal{S}$ is a proper functional $\mathcal{S} : \mathcal{H} \to (-\infty,+\infty]$. Then, following Rossi and Savaré [26], we define the set valued subdifferential $d\mathcal{S}(u)$ at $u \in D(\mathcal{S}) \cap \tilde{\mathcal{H}}$ through

$$\delta \in d\mathcal{S}(u) \quad \Leftrightarrow \quad \langle \delta, v\rangle_{\mathcal{H}} \le \liminf_{h\searrow 0} \frac{\mathcal{S}(u+hv)-\mathcal{S}(u)}{h} \quad \forall v \in \mathcal{H} \tag{4.5}$$

and the subgradient $\nabla_u\mathcal{S}(u)$ of $\mathcal{S}$ in $u \in \tilde{\mathcal{H}} \cap D(d\mathcal{S})$ through

$$\delta \in \nabla_u\mathcal{S}(u) \quad \Leftrightarrow \quad \exists\tilde{\delta} \in d\mathcal{S}(u) \,:\, g_u(\delta,v) := \left\langle \tilde{\delta}, v\right\rangle_{\mathcal{H}} \qquad \forall v \in \mathcal{H}\,, \tag{4.6}$$

where the index u refers to the local metric. If no confusion occurs, we write $\nabla\mathcal{S}(u) = \nabla_u\mathcal{S}(u)$. Note that this concept of subdifferential coincides with the classical Fréchet subdifferential in case $\mathcal{S}$ is convex (see [15]) and will thus also coincide in our case of a continuous perturbation with signle valued L^2-subdifferential.

In what follows, we denote the *local slope* by

$$|\partial\mathcal{S}|(u) := \limsup_{w\to u,\, w\in D(\mathcal{S})} \frac{|\mathcal{S}(u)-\mathcal{S}(w)|}{\|u-w\|_{g(u)}}\,, \tag{4.7}$$

implying

$$\sup_{\delta\in\nabla_u\mathcal{S}(u)} \|\delta\|_{g(u)} \le |\partial\mathcal{S}|(u) \qquad \forall u \in D(d\mathcal{S}) \tag{4.8}$$

and in case $d\mathcal{S}$ is single valued, $|\partial\mathcal{S}|(u) = \|\nabla\mathcal{S}(u)\|_{g(u)}$.

Finally, for every subset $A \subset \mathcal{H}$ we define the affine hull aff A and its minimal section A° through

$$\text{aff } A := \left\{ \sum_i t_i a_i \,:\, a_i \in A,\, t_i \in \mathbb{R},\, \sum_i t_i = 1 \right\}\,,$$
$$|A^\circ| := \inf_{\xi\in A} \|\xi\|_{\mathcal{H}}\,, \qquad A^\circ := \left\{\xi \in A \,:\, \|\xi\|_{\mathcal{H}} = |A^\circ|\right\}\,.$$

Definition 4.2. [26, 16] We say that for any $u \in \mathcal{H}$, $\xi \in \mathcal{H}$ is an element of the limiting subdifferential $d_l\mathcal{S}(u)$ of $\mathcal{S}$ in u if there are $u_n \in \mathcal{H}$ with $u_n \to u$ strongly and $\xi_n \in d\mathcal{S}(u_n)$ such that $\xi_n \rightharpoonup \xi$ weakly in $\mathcal{H}$. The limiting subgradient and the weakly lower semicontinuous envelope of $|\partial\mathcal{S}|$ are defined through

$$\nabla_{l,u}\mathcal{S}(u) = \tilde{g}_u^{-1}(d_l\mathcal{S}(u)) \,,$$

$$|\nabla_l\mathcal{S}(u)^\circ| := \inf_{\xi\in\nabla_l\mathcal{S}(u)} \|\xi\|_{g(u)} \qquad \nabla_l\mathcal{S}(u)^\circ := \left\{\xi \in \nabla_l\mathcal{S}(u) \,:\, \|\xi\|_{g(u)} = |\nabla_l\mathcal{S}(u)^\circ|\right\} .$$

Thus, equation (4.1) has to be understood in the sense of

$$g_u(\partial_t u,\varphi) \in \langle d_l S(u),\varphi\rangle_{\mathcal{H}} + g_u(f,\varphi) \qquad \forall\varphi \in L^2(0,T;\mathcal{H}) \tag{4.9}$$

Note that in case the graph of $(\mathcal{S}, d\mathcal{S})$ is strongly-weakly closed in $\mathcal{H}\times\mathcal{H}\times\mathbb{R}$, i.e.

$$\left.\begin{array}{l}\xi_n \in d\mathcal{S}(v_n), \quad r_n = \mathcal{S}(v_n)\\ v_n \to v, \quad \xi_n \rightharpoonup \xi, \quad r_n \to r\end{array}\right\} \quad\Rightarrow\quad \xi \in d\mathcal{S}(v), \quad r = \mathcal{S}(v), \tag{4.10}$$

we find $d_l\mathcal{S} = d\mathcal{S}$. As explained by Rossi and Savaré [26], this condition yields closedness and convexity of $d\mathcal{S}$, the continuity condition

$$v_n \to v,\ \sup_n \left(|\partial\mathcal{S}(v_n)|\,,\mathcal{S}(v_n)\right) < +\infty \ \Rightarrow\ \mathcal{S}(v_n) \to \mathcal{S}(v) \quad \text{as } n \nearrow \infty \tag{4.11}$$

and the the following chain rule: If $v \in H^1(0,T;\mathcal{H})$, $\xi \in L^2(0,T;\mathcal{H})$ with $\xi(t) \in d_l\mathcal{S}(v(t))$ for a.e. $t \in (0,T)$, and $\mathcal{S}\circ v$ is a.e. equal to a function s of bounded variation, then

$$\frac{d}{dt}s(t) = \langle \xi, v'(t)\rangle_{\mathcal{H}} \,. \tag{4.12}$$

Lemma 4.3 ((See [26]))**.** *If $\mathcal{S}$ is convex, condition* (4.10) *is fulfilled.*

For the rest of the paper, we assume that $\mathcal{S}$ is an entropy functional in the following sense:

Definition 4.4. Let $(\mathcal{H}_0, \tilde{\mathcal{H}}, \mathcal{H}, g)$ be an entropy space with $G^* > 1$. We say that $\mathcal{S} : \mathcal{H} \to (-\infty, +\infty]$ is an entropy functional on $(\mathcal{H}_0, \tilde{\mathcal{H}}, \mathcal{H}, g)$ if it satisfies :

1. $D(\mathcal{S}) \subset \tilde{\mathcal{H}}$ and $\mathcal{S} : \mathcal{H} \to \mathbb{R}$ being proper, lower semicontinuous, i.e. the domain $D(\mathcal{S})$ of $\mathcal{S}$ is non-empty.
2. $\mathcal{S} + \|\cdot\|_{\mathcal{H}}$ has compact sublevels, i.e. there exists $\tau_* > 0$ such that sets
$$\left\{v \in \mathcal{H} \,:\, \mathcal{S}(v) + \frac{1}{2\tau}\min\left\{1, \sqrt{G^*}^{-1}\right\}\|v\|_{\mathcal{H}}^2 < C\right\}$$
are compact for any $\tau < \tau_*$ and any $C > 0$ and there is a constant $S_0 > 0$ such that
$$\mathcal{S}(v) + \frac{1}{2\tau_*}\min\left\{1, \sqrt{G^*}^{-1}\right\}\|v\|_{\mathcal{H}}^2 \geq -S_0 \tag{4.13}$$
3. $\mathcal{S}$ satisfies the estimate
$$\|u\|_{\mathcal{H}_0} \leq C\left(\mathcal{S}(u) + |\partial S|^2(u) + 1\right)$$

We close this section stating the first of three existence theorems from [16] which we will use below:

Theorem 4.5. *Let $\mathcal{H}_0$, $\tilde{\mathcal{H}}$, $\mathcal{H}$, g and $\mathcal{S}$ satisfy definitions 4.1 and 4.4 with $d_l\mathcal{S}(u)$ being convex and closed for all $u \in \mathcal{H}$.*

$$\mathcal{S}(u) = \mathcal{S}_{\mathcal{H}}(u) + \mathcal{S}_{\tilde{\mathcal{H}}}(u) \tag{4.14}$$

with functionals $\mathcal{S}_{\mathcal{H}} : \mathcal{H} \to \mathbb{R}$ being proper, lower semicontinuous, and $\mathcal{S}_{\tilde{\mathcal{H}}} : D(\mathcal{S}) \subset \tilde{\mathcal{H}} \to \mathbb{R}$ being continuous w.r.t. $\tilde{\mathcal{H}}$. Furthermore, let $f \in L^2(0,T;\mathcal{H})$. Then, for each $u_0 \in \mathcal{H}_0$ and every $0 < T \in \mathbb{R}$, there exists a solution $u \in H^1(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{H}_0)$ to (4.9)*, satisfying the Lyapunov inequality*

$$\frac{1}{2}\int_0^t \|\partial_t u\|_{g(u)}^2 + \frac{1}{2}\int_0^t |(f - \nabla_l\mathcal{S}(u))^\circ|^2 + \mathcal{S}(u(t)) \leq \mathcal{S}(u(0)) + \int_0^t \langle f, u\rangle_{\mathcal{H}} \qquad \textit{for a.e. } t \in (0,T)\,. \tag{4.15}$$

If $\mathcal{S}$ additionally fulfills the continuity assumption (4.11) *then, there is a negligible set $\mathcal{N} \subset (0,T)$ such that*

$$\frac{1}{2}\int_s^t |\partial_t u|^2 + \frac{1}{2}\int_s^t |(f - \nabla_l\mathcal{S}(u))^\circ|^2 + \mathcal{S}(u(t)) \leq \mathcal{S}(u(s)) + \int_0^t \langle f, u\rangle_{\mathcal{H}} \qquad \forall t \in (s,T),\ \forall s \in (0,T)\backslash\mathcal{N}\,.$$

5. Proofs of Theorems 1.1 and 1.2

We introduce the following spaces

$$\mathcal{H} := H^{-1}_{(0)}\,, \qquad \tilde{\mathcal{H}} := H^1_{(0)}(\Omega)\,, \qquad \mathcal{H}_0 := H^2(\Omega)\,,$$

such that we find $\mathcal{H}_0 \hookrightarrow \tilde{\mathcal{H}} \hookrightarrow L^2(\Omega) \hookrightarrow \mathcal{H}$ with all embeddings being dens and compact.

Definition 5.1. Let $\mathcal{S} : \mathcal{H} \to (-\infty, +\infty]$ be given through (1.3) with $\mathcal{S}(u) := +\infty$ for all $u \notin \tilde{\mathcal{H}}$. Then, we consider the restriction of $\tilde{\mathcal{S}} := \mathcal{S}\big|_{L^2}$ of $\mathcal{S}$ to $L^2(\Omega)$ and define the set valued L^2-subdifferentials $\frac{\delta \mathcal{S}}{\delta u}(u) \subset L^2(\Omega)$ and $\frac{\delta^0 \mathcal{S}}{\delta u}(u) \subset L^2_{(0)}(\Omega)$ at $u \in D(\tilde{\mathcal{S}})$ through:

$$u \in D(\tilde{\mathcal{S}}) : \quad \delta \in \frac{\delta \mathcal{S}}{\delta u}(u) \quad \Leftrightarrow \quad \langle \delta, v\rangle_{L^2} \le \lim_{h\searrow 0} \frac{\tilde{\mathcal{S}}(u+hv) - \tilde{\mathcal{S}}(u)}{h} \quad \forall v \in L^2(\Omega)$$

$$u \in D(\tilde{\mathcal{S}}) \cap L^2_{(0)}(\Omega) : \quad \delta \in \frac{\delta^0 \mathcal{S}}{\delta u}(u) \quad \Leftrightarrow \quad \langle \delta, v\rangle_{L^2} \le \lim_{h\searrow 0} \frac{\tilde{\mathcal{S}}(u+hv) - \tilde{\mathcal{S}}(u)}{h} \quad \forall v \in L^2_{(0)}(\Omega)$$

We only proof theorem 1.1 and start with two lemmata by Abels and Wilke. Theorem 1.2 is proved likewise.

Lemma 5.2. [1, Lemma 4.1, Corollary 4.4] *Assume $s_1 \equiv 0$, then $\mathcal{S} : L^2_{(0)}(\Omega) \to \mathbb{R}$ and $\mathcal{S} : \mathcal{H} \to \mathbb{R}$ are proper, lower semicontinuous and convex.*

Abels and Wilke [1] identified the L^2- and $\mathcal{H}$- subdifferential of $\mathcal{S}$ in the Frechet-sense:

Lemma 5.3. [1] *Assume $s_1 \equiv 0$ and set $s_0' = +\infty$ for $x \notin (a,b)$. Then, for the L^2-subdifferential of $\mathcal{S}$ defined through* (1.3) *holds*

$$D\Big(\frac{\delta^0 \mathcal{S}}{\delta u}\Big) = \Big\{ c \in H^2(\Omega) \cap L^2_{(0)}(\Omega) \,:\, s'(c) \in L^2(\Omega),\, s''(c)\,|\nabla c|^2 \in L^1(\Omega),\, \partial_n c\big|_{\partial\Omega} = 0 \Big\} \tag{5.1}$$

and

$$\frac{\delta^0 \mathcal{S}}{\delta u}(\tilde{u}) = -\Delta \tilde{u} + P_0 s'(\tilde{u})\,. \tag{5.2}$$

Moreover,

$$\|\tilde{u}\|^2_{H^2} + \|s'(\tilde{u})\|^2_{L^2} + \int_\Omega s''(\tilde{u})\,|\nabla \tilde{u}|^2 \le C\left(\left\| \frac{\delta^0 \mathcal{S}}{\delta u}(\tilde{u}) \right\|^2_{L^2} + \|\tilde{u}\|^2_{L^2} + 1 \right) \tag{5.3}$$

for some constant C independent of $\tilde{u}$.

For the $\mathcal{H}$-Subdifferential holds

$$D(d\mathcal{S}) = \left\{ c \in D\Big(\frac{\delta^0 \mathcal{S}}{\delta u}\Big) \,:\, \frac{\delta^0 \mathcal{S}}{\delta u}(c) \in H^1_{(0)}(\Omega) \right\} \tag{5.4}$$

$$d\mathcal{S}(\tilde{u}) = \Delta_N \left(-\Delta \tilde{u} + P_0 s'(\tilde{u}) \right)\,, \tag{5.5}$$

and in particular,

$$\|\tilde{u}\|^2_{H^2(\Omega)} \le C \left(\|d\mathcal{S}(\tilde{u})\|^2_{\mathcal{H}} + \|\tilde{u}\|^2_{L^2(\Omega)} + 1 \right)\,. \tag{5.6}$$

Note that the term $+1$ in (5.3) and (5.6) was not present in the orginal statements. As $\mathcal{S}$ in the setting of lemma 5.3 is convex, the graph of $(d\mathcal{S}, \mathcal{S})$ is strongly-weakly closed in the sense of (4.10). In particular, this implies the chain-rule condition (4.12) and convexity of $d\mathcal{S}(u)$ for all $u \in D(d\mathcal{S})$.

In case $s_1 \not\equiv 0$, $\mathcal{S} : \mathcal{H} \to \mathbb{R}$ remains lower semicontinuous and equations (5.1)-(5.5) still hold with modified constants. We finally have the following lemma:

Lemma 5.4. *$d\mathcal{S}$ is single valued and strong-weak closed.*

Proof. It is easy to veryfy that $d\mathcal{S}(u)$ is single valued for all $u \in D(d\mathcal{S})$. For $u_n \to u$ strongly in $\mathcal{H}$ and $\xi_n = d\mathcal{S}(u_n)$ such that $\xi_n \rightharpoonup \xi$ weakly in $\mathcal{H}$, note that due to boundedness of the sequences u_n and ξ_n we find boundedness of $\|u_n\|_{\mathcal{H}_0}$ and thus $u_n \rightharpoonup u$ weakly in $H^2(\Omega)$, $u_n \to u$ strongly in $H^1_{(0)}$ and $u_n \to u$ a.s. in Ω up to a subsequence. Furthermore, for $w_n := -\Delta u_n + P_0 s'(u_n)$ we find $w_n \rightharpoonup \omega$ weakly in $H^1_{(0)}$ for some $\omega \in H^1_{(0)}$.

Now, let

$$\tilde{\mathcal{S}}(u) := \mathcal{S}(u) - \int_\Omega s_1(u)\,,$$

then $\tilde{\mathcal{S}}(\cdot)$ is convex and therefore, the graph of $d\tilde{\mathcal{S}}$ is strongly weakly closed by lemma 4.3. For a further subsequence and for $\zeta_n := d\tilde{\mathcal{S}}(u_n)$ we get weak convergence of $\zeta_n \rightharpoonup \zeta = d\tilde{\mathcal{S}}(u) = -\Delta u + P_0 s_0'(u)$ in $\mathcal{H}$ and $P_0(s_1'(u_n)) \to P_0(s_1'(u))$ strongly in L^2. Thus,

$$\xi_n = \zeta_n + s_1'(u_n) \rightharpoonup \zeta + P_0(s_1'(u)) = -\Delta u + P_0(s'(u))$$

weakly in $\mathcal{H}$. □

For $u \in \tilde{\mathcal{H}}$, we define for $r_1, r_2 \in \mathcal{H}$:

$$g_u(r_1,r_2) = \int_\Omega \nabla p_1^u A(u,\nabla u)\nabla p_2^u = \int_\Omega r_1 p_2^u = \langle r_1, p_2\rangle_{H^{-1}_{(0)},H^1_{(0)}} = \int_\Omega r_2 p_1^u = \langle r_2, p_1\rangle_{H^{-1}_{(0)},H^1_{(0)}}\,, \tag{5.7}$$

where p_i^u solves

$$-\mathrm{div}\,\left(A(u,\nabla u)\nabla p_i^u\right) = r_i \quad \text{for } i = 1,2\,. \tag{5.8}$$

It is immediate to check that g is a densly defined metric in the sense of definition 4.1.

Above considerations together with (4.8) yield that $\mathcal{S}$ fulfills all requierements of definition 4.4. As a consequence of theorem 4.5 we get existence of a solution $u \in H^1(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{H}_0)$ to (4.9) and it remains to reconstruct an expression of the form (4.1):

For any $u \in D(\mathcal{S})$, $r \in L^2(\Omega)$ with p from (5.8), $\gamma \in AC(0,T;L^2(\Omega))$ with $\gamma(0) = u$, $\gamma'(0) = r$ we formally write

$$g_{\tilde u}(\nabla_{\tilde u}\mathcal{S}, r) = \frac{d}{dt}\mathcal{S}(\gamma(t))\Big|_0 = \int_\Omega \frac{\delta^0\mathcal{S}}{\delta u}(\tilde u)\, r = \int -\mathrm{div}\,\left(A(u,\nabla u)\nabla\frac{\delta^0\mathcal{S}}{\delta u}(\tilde u)\right) p$$

to obtain the specific form of (4.9) and equation (4.1) in the present setting reads (note that $g_u(\partial_t u, r_2) = \langle \partial_t u, p_2\rangle_{H^{-1}_{(0)},H^1_{(0)}}$,

$$\partial_t u \in \mathrm{div}\,\left(A(u,\nabla u)\nabla\frac{\delta^0\mathcal{S}}{\delta u}(\tilde u)\right)$$

$$\text{or:}\qquad g_u(\partial_t u,\varphi) \in -\langle d_l\mathcal{S}(u),\varphi\rangle_{\mathcal{H}} \qquad \forall\varphi \in L^2(0,T;\mathcal{H})\,. \tag{5.9}$$

Estimate (4.15) together with the above calculations yields (1.2). Theorem 1.2 can be prooved similarly having in mind that the proof of lemma 5.3 presented by Abels and Wilke [1] is the same for a closed surface Γ with $H^1_{(0)}(\Gamma)$ defined through (2.3).

6. Proof of Theorem 1.3

6.1. **The Entropy space.** We introduce the space $\tilde V$ through

$$\tilde V := H^1_{(0)}(\Omega) \times L^2(\Gamma)\,, \qquad \|u = (u_\omega, u_\gamma)\|^2_{\tilde V} := \|u_\omega\|^2_{H^1(\Omega)} + \|u_\gamma\|^2_{L^2(\Gamma)}\,,$$

where $L^2(\Gamma)$ is with respect to the Hausdorff measure on Γ. Note that

$$V := \left\{u = (u_\omega, u_\gamma) \in \tilde V \,:\, E_\Gamma(u_\omega) = u_\gamma\right\}$$

is a closed subspace of $\tilde V$, being isomorph with $H^1_{(0)}(\Omega)$ and with the equivalent norm (cf. remark 2.4)

$$\|u = (u_\omega, u_\gamma)\|^2_V := \|\nabla u_\omega\|^2_{L^2(\Omega)} + \|u_\gamma\|^2_{L^2(\Gamma)}\,.$$

We furthermore introduce

$$\|(u_\omega,u_\gamma)\|^2_{H^1_\Gamma(\Omega)} := \int_\Omega |\nabla u_\omega|^2 + \int_\Gamma |\nabla_\Gamma u_\gamma|^2\,,$$

$$H^1_\Gamma := \overline{\{(u_\omega,u_\gamma) \in V \,:\, u_\omega \in H^2(\Omega)\}}^{\|\cdot\|_{H^1_\Gamma(\Omega)}}$$

and the dual space $H^*_\Gamma := \left(H^1_\Gamma\right)^{-1}$. For any function $v \in H^{-1}_{(0)}(\Omega)$ having the property that there is $\tilde v \in L^2_{(0)}(\Omega)$ with

$$\int_\Omega v\psi = \int_\Omega \tilde v\psi \qquad \forall\psi \in H^1_{(0)}(\Omega)$$

we formally write $\tilde{v} = P_0(v)$. We finally introduce the space $H^1_\Delta(\Omega)$ through

$$\|(u_\omega, u_\gamma)\|^2_{H^1_\Delta} := \int_\Omega (P_0(\Delta u_\omega))^2 + \int_\Gamma (\partial_n u_\omega - \Delta_\Gamma u_\gamma)^2 + \|u\|^2_{H^1_\Gamma(\Omega)} ,$$
$$H^1_\Delta := \overline{\{(u_\omega, u_\gamma) \in H^1_\Gamma \,:\, u_\omega \in H^3(\Omega)\}}^{\|\cdot\|_{H^1_\Delta}} .$$

Remark 6.1. Since for $u \in V$, $u \in H^1_\Gamma$ or $u \in H^1_\Delta$ holds $u_\gamma = E_\Gamma(u_\omega)$ like in (2.4), we will sometimes abuse notation and not destinguish between u_γ and $E_\Gamma(u_\omega)$, i.e. we will often write $u \simeq u_\gamma \simeq u_\omega$ whenever the meaning is clear from the context.

In what follows, we will say that $u \in H^2(\Omega)$ weakly solves the system

$$\begin{aligned} -\Delta u_\omega &= f \quad \text{in } \Omega \\ -\Delta_\Gamma u_\gamma + \partial_n u &= g \quad \text{on } \Gamma \end{aligned}$$

iff it is a solution to the problem

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma \varphi = \int_\Omega f \varphi + \int_\Gamma g \varphi \qquad \forall \varphi \in H^2(\Omega) . \tag{6.1}$$

In particular, we infer in case $g = 0$ for $\varphi \equiv 1$ that $\int_\Omega f = 0$.

Lemma 6.2.

$$H^1_\Delta = \left\{ u \in H^1_\Gamma \,:\, \Delta u \in L^2(\Omega) \,,\, (\partial_n u_\omega - \Delta_\Gamma u_\gamma) \in L^2(\Gamma) \right\}$$

Proof. We show for $(u_\omega, u_\gamma) \in H^1_\Gamma$ with $u_\omega \in H^3(\Omega)$ that there is $C > 0$ independent on u such that

$$\|\Delta u\|_{L^2} \leq C \left(\|P_0(\Delta u)\|_{L^2} + \|\partial_n u_\omega - \Delta_\Gamma u_\gamma\|_{L^2(\Gamma)} + \|u\|^2_{H^1_\Gamma(\Omega)} \right) .$$

If not, there was a sequence of functions $(u_m)_{m\in\mathbb{N}} \subset H^1_\Gamma$, $u_m \in H^3(\Omega)$ for all m, such that $\|\Delta u_m\|_{L^2} = 1$, and for $\tilde{f}_m := -P_0(\Delta u_m)$, $f_m := -\Delta u_m$, $g_m := \partial_n u_m - \Delta_\Gamma u_m$ holds $\tilde{f}_m \to 0$ strongly in L^2, $g_m \to 0$ strongly in $L^2(\Gamma)$, $u_m \to 0$ strongly in H^1_Γ and $f_m \rightharpoonup f$ weakly in L^2. However, as $\tilde{f}_m = P_0(f_m)$, we can assume w.l.o.g. that $h_m := f_m - \tilde{f}_m \in \mathbb{R}$ for all m and thus for a subsequence $h_m \to h \in \mathbb{R}$ such that we find $f_m \to h \neq 0$ strongly in L^2. Note that due to regularity of u and definitions above, for any m there holds

$$\int_\Omega \nabla u_m \cdot \nabla \varphi + \int_\Gamma \nabla_\Gamma u_m \cdot \nabla_\Gamma \varphi = \int_\Omega f_m \varphi + \int_\Gamma g_m \varphi \qquad \forall \varphi \in H^2(\Omega) ,$$

and thus, in the limit, u is a solution to

$$\int_\Omega f \varphi = \int_\Omega \Delta u \varphi = 0 \qquad \forall \varphi \in H^2(\Omega),$$

implying $\Delta u = P_0(\Delta u)$, a contradiction. Now, considering $u \in H^1_\Delta$ and any sequence $(u_m)_{m\in\mathbb{N}} \subset H^3(\Omega)$ such that $u_m \to u$ in $\|\cdot\|_{H^1_\Delta}$, we find $\Delta u \in L^2$. Since $\partial_n u_{m,\omega} - \Delta_\Gamma u_{m,\gamma} \to \tilde{f}$ for some $\tilde{f} \in L^2(\Gamma)$, we find for all sufficiently regular $\psi \in C^3_{(0)}(\overline{\Omega})$:

$$\begin{aligned} \int_\Gamma f \psi - \int_\Omega \psi \Delta u &= \lim_{m\to\infty} \left(\int_\Gamma (\partial_n u_{m,\omega} - \Delta_\Gamma u_{m,\gamma})\, \psi - \int_\Omega \psi \Delta u_m \right) \\ &= \lim_{m\to\infty} \left(\int_\Gamma (\nabla_\Gamma u_{m,\gamma}) \cdot \nabla_\Gamma \psi + \int_\Omega \nabla \psi \cdot \nabla u_m \right) \\ &= \left(\int_\Gamma (\nabla_\Gamma u_\gamma) \cdot \nabla_\Gamma \psi + \int_\Omega \nabla \psi \cdot \nabla u \right) \\ &= \left(\int_\Gamma (\partial_n u_\omega - \Delta_\Gamma u_\gamma)\, \psi - \int_\Omega \psi \Delta u \right) . \end{aligned}$$

□

In order to construct an entropy space in sense of definition 4.1, we make the following choice of the tripple of function spaces:

$$\mathcal{H}_0 := H^1_\Delta , \qquad \tilde{\mathcal{H}} := H^1_\Gamma , \qquad \mathcal{H} := H^{-1}_{(0)}(\Omega) \times L^2(\Gamma) .$$

With the additional space

$$\|u\|_\Gamma := \int_\Omega u_\omega^2 + \int_\Gamma u_\gamma^2\,, \qquad L^2_\Gamma := L^2_{(0)}(\Omega)\times L^2(\Gamma)$$

the chain of dense embeddings $\mathcal{H}_0 \hookrightarrow \tilde{\mathcal{H}} \hookrightarrow L^2_\Gamma \hookrightarrow \mathcal{H}$ holds with with the first and second being compact, as lemma 2.3 and the proof of the following corollary show.

Corollary 6.3. *The triple* $(\mathcal{H}_0, \tilde{\mathcal{H}}, \mathcal{H})$ *satisfies 4.1 point* (1).

Proof. The embedding $\tilde{\mathcal{H}} \hookrightarrow L^2_\Gamma$ evidently is compact as lemma 2.3 shows. Consider a bounded sequence $(u_m)_{m\in\mathbb{N}} \subset \mathcal{H}_0$. In the following we will pass to subsequences keeping the original index parameter m. Recalling remark 6.1 we show $u_m \rightharpoonup u$ weakly in $\mathcal{H}_0$ implies $u_m \to u$ strongly in $\tilde{\mathcal{H}}$. Due to Lemma (2.3), we find $u_{m,\omega} \to u_\omega$ strongly in $L^2_{(0)}(\Omega)$ and $u_{m,\gamma} \to u_\gamma$ strongly in $W_2^{1/2}(\Omega)$ for a subsequence. Furthermore, comparing to the proof of lemma 6.2, it is not difficult to check for $m \to \infty$ that w.l.o.g. $\Delta u_{m,\omega} \rightharpoonup \Delta u_\omega$ weakly in $L^2(\Omega)$ and

$$\partial_n u_{m,\omega} - \Delta_\Gamma u_{m,\gamma} \rightharpoonup \partial_n u_\omega - \Delta_\Gamma u_\gamma \qquad \text{weakly in } L^2(\Gamma),$$

and the statement follows from strong convergence of u_m in L^2_Γ and

$$\begin{aligned}\lim_{m\to\infty}\left(\int_\Omega |\nabla u_{m,\omega}|^2 + \int_\Gamma |\nabla u_{m,\gamma}|^2\right) &= \lim_{m\to\infty}\left(-\int_\Omega u_{m,\omega}\Delta u_{m,\omega} + \int_\Gamma u_{m,\omega}\left(\partial_n u_{m,\omega} - \Delta_\Gamma u_{m,\gamma}\right)\right)\\ &= \left(-\int_\Omega \Delta u_\omega u_\omega + \int_\Gamma \left(\partial_n u_\omega - \Delta_\Gamma u_\gamma\right) u_\gamma\right) = \int_\Omega |\nabla u_\omega|^2 + \int_\Gamma |\nabla_\Gamma u_\gamma|^2\end{aligned}$$

□

Note that $\mathcal{H}^{-1} = H^1_{(0)}(\Omega)\times L^2(\Gamma)$ and on $\mathcal{H}$ we introduce the local scalar products

$$\begin{aligned}g_u(r_1,r_2) &:= \int_\Omega \nabla p^u_{1,\omega} A(u,\nabla u)\nabla p^u_{2,\omega} + \int_\Gamma p^u_{1,\gamma} A_\Gamma(u) p^u_{2,\gamma} = \langle r_1, p_2\rangle_{\mathcal{H},\mathcal{H}^{-1}}\\ &= \int_\Omega r_{1,\omega} p^u_{2,\omega} + \int_\Gamma r_{1,\gamma} p^u_{2,\gamma} = \int_\Omega r_{2,\omega} p^u_{1,\omega} + \int_\Gamma r_{2,\gamma} p^u_{1,\gamma} = \langle r_2, p_1\rangle_{\mathcal{H},\mathcal{H}^{-1}}\,,\end{aligned} \tag{6.2}$$

where $p^u_i = (p^u_{i,\omega}, p^u_{i,\gamma}) \in \mathcal{H}^{-1}$ satisfy the equations

$$\int_\Omega \left(A(u,\nabla u)\nabla p^u_{i,\omega}\right)\nabla\varphi_\omega + \int_\Gamma A_\Gamma(u) p^u_{i,\gamma}\varphi_\gamma = \langle r_i, \varphi\rangle_{\mathcal{H},\mathcal{H}^{-1}} \quad \text{for } i = 1,2 \text{ and } \forall\varphi\in\mathcal{H}^{-1}\,, \tag{6.3}$$

with the constraint

$$\left(A(u,\nabla u)\nabla p^u_{i,\omega}\right)\cdot \boldsymbol{n}_\Gamma = 0\,.$$

In other words, $p^u_i \in \mathcal{H}^{-1}$ solves

$$\begin{aligned}-\operatorname{div}\left(A(u,\nabla u)\nabla p^u_{i,\omega}\right) &= r_{i,\omega}\,, \quad \text{on } \Omega \quad \text{and} \quad A(u,\nabla u)\nabla p^u_{i,\omega}\cdot\boldsymbol{n}_\Gamma = 0 \text{ on } \Gamma,\\ A_\Gamma(u) p^u_{i,\gamma} &= r_{i,\gamma}\,, \quad \text{on } \Gamma\,,\end{aligned}$$

Note that in general $p^u_{i,\gamma} \neq E_\Gamma(p^u_{i,\omega})$.

Corollary 6.4. $g_\bullet : \tilde{\mathcal{H}} \to B(\mathcal{H})$ *satisfies 4.1-*(2).

Proof. For fixed r_2 consider $r_{1,m}$ and $p_{1,m} = (p_{1,m,\omega}, p_{1,m,\gamma})$ solutions of (6.3) for $r_{1,m}$, s.t. $r_{1,m} \to r_1$ in $\mathcal{H}$ and $(u_m)_{m\in\mathbb{N}} \subset \tilde{\mathcal{H}}$ with $u_m \to u$. We check that $p_{1,m} \rightharpoonup \tilde{p}_1$ and $\tilde{p}_1$ solves (6.3) for u and r_1. Thus, from the representation in (6.2), we conclude

$$g_{u_m}(r_{1,m}, r_2) \to g_u(r_1, r_2)\,.$$

□

6.2. **The entropy functional and existence of solutions.** In this part, we shall rigorously use notations announced in remark 6.1 for functions $u \in \tilde{\mathcal{H}} = V$. Note that this notation is not applicable to L^2_Γ, $\mathcal{H}$ or $\mathcal{H}^{-1}$, which is, why we still use full notations in that spaces.

Definition 6.5. Let $\mathcal{S}$ be proper functional $\mathcal{S} : \mathcal{H} \to (-\infty, +\infty]$. Then, we consider the restriction of $\tilde{\mathcal{S}} := \mathcal{S}\big|_{L^2_\Gamma}$ of $\mathcal{S}$ to L^2_Γ and define the set valued L^2-subdifferential $\frac{\delta_\Gamma \mathcal{S}}{\delta u}(u) \subset L^2_\Gamma$ at $u \in D(\tilde{\mathcal{S}})$ through:

$$\delta \in \frac{\delta_\Gamma \mathcal{S}}{\delta u}(u) \quad \Leftrightarrow \quad \langle \delta, v \rangle_{L^2_\Gamma} \le \liminf_{h \searrow 0} \frac{\tilde{\mathcal{S}}(u + hv) - \tilde{\mathcal{S}}(u)}{h} \quad \forall v \in L^2_\Gamma$$

Remark 6.6. Comparing to section 5, due to the Riesz isomorphism $-\Delta_N : H^1_{(0)}(\Omega) \to H^{-1}_{(0)}(\Omega)$, we find

$$d\mathcal{S}(u) = \left\{ (s_\omega, s_\gamma) \,:\, (-\Delta_N^{-1} s_\omega, s_\gamma) \in \frac{\delta_\Gamma \mathcal{S}}{\delta u}(u) \right\} . \tag{6.4}$$

We introduce the following functional on L^2_Γ, resp. $\mathcal{H}$:

$$\mathcal{S}(u) := \begin{cases} \int_\Omega \left(s(u_\omega) + \frac{1}{2} |\nabla u_\omega|^2 \right) + \int_{\partial\Omega} \left(s_\Gamma(u_\gamma) + \frac{1}{2} |\nabla_\Gamma u_\gamma|^2 \right) & \text{for } u \in H^1_\Gamma(\Omega) \\ +\infty & \text{otherwise} \end{cases} , \tag{6.5}$$

with s, s_Γ as introduced in subsection 1.2.

Lemma 6.7. *The functional $\mathcal{S}$ is lower semicontinuous on $\mathcal{H}$ and L^2_Γ. If $s_1 \equiv s_2 \equiv 0$, $\mathcal{S}$ is convex on both spaces.*

Proof. If $s_1 \equiv s_2 \equiv 0$, convexity is trivial. Furthermore, for any sequence $u_n \in \mathcal{H}$ with a constant $C > 0$ s.t. $\mathcal{S}(u_n) < C$, we find u_n to be bounded in $\tilde{\mathcal{H}}$, i.e. due to the particular structure of $s(\cdot)$, a short calculation yields

$$\mathcal{S}(u) \le \liminf_{n \to \infty} \mathcal{S}(u_n) .$$

In case $s_1, s_2 \not\equiv 0$, note that up to a minimizing subsequence $u_n \to u$ strongly in L^2_Γ and the statement follows from the Lipschitz-continuity of s_1 and s_2. □

Lemma 6.8. *Let $\mathcal{S}$ be given through* (6.5)*, then*

$$D(\frac{\delta_\Gamma \mathcal{S}}{\delta u}) = \left\{ c \in H^1_\Gamma(\Omega) \,:\, s'(c) \in L^2_\Gamma,\, s''(c) |\nabla c|^2 \in L^1(\Omega),\, s''(c) |\nabla_\Gamma c|^2 \in L^1(\Gamma) \right\} \tag{6.6}$$

and

$$\|\tilde{u}\|_{H^1_\Delta(\Omega)} \le C \left(\left\| \frac{\delta_\Gamma \mathcal{S}}{\delta u}(\tilde{u}) \right\|^2_{L^2_\Gamma} + \|\tilde{u}\|^2_{L^2(\Omega)} + 1 \right) . \tag{6.7}$$

Furthermore, $\tilde{u} \in D(d\mathcal{S})$ implies $\frac{\delta \mathcal{S}}{\delta_\Gamma u}(\tilde{u}) \in H^1_{(0)} \times L^2(\Gamma)$,

$$\|\tilde{u}\|_{H^1_\Delta(\Omega)} \le C \left(\|d\mathcal{S}_\Gamma(\tilde{u})\|^2_{\mathcal{H}} + \|\tilde{u}\|^2_{L^2(\Omega)} + 1 \right) , \tag{6.8}$$

and for any $u \in D(\mathcal{S})$, the L^2_Γ-subdifferential is given through

$$\left\langle \frac{\delta_\Gamma \mathcal{S}}{\delta u}, \psi \right\rangle_{L^2_\Gamma} = \langle P_0(s'(u)), \psi_\omega \rangle_{L^2(\Omega)} - \langle P_0(\Delta u), \psi_\omega \rangle_{L^2(\Omega)} + \langle \nabla u \cdot \boldsymbol{n}_\Gamma + s'_\Gamma(u) - \Delta_\Gamma u, \psi_\gamma \rangle_{L^2(\Gamma)} \tag{6.9}$$

for all $\psi = (\psi_\omega, \psi_\gamma) \in L^2_\Gamma$.

Remark 6.9. Thus, as the last lemma yields $\|u\|_{\mathcal{H}_0} \le (\mathcal{S}(u) + 1 + \|d\mathcal{S}\|_{\mathcal{H}})$, we have shown that $\mathcal{S}$ satisfies all claims of definition 4.4.

The proof of the following lemma follows the proof of lemma 5.4 and is left to the reader.

Lemma 6.10. *$d\mathcal{S}$ is single valued and strong-weak closed.*

Now, for any $u \in D(d\mathcal{S}_\Gamma)$, $r \in L^2_\Gamma$ with p from (6.3), $\gamma \in AC(0,T;L^2_\Gamma)$ with $\gamma(0) = \tilde{u}$, $\gamma'(0) = r$ we formally write

$$\frac{d}{dt}\mathcal{S}(\gamma(t))\Big|_0 \geq \int_\Omega \left(\frac{\delta_\Gamma \mathcal{S}_\Gamma}{\delta u}(u)\right)_\omega r_\omega + \int_\Gamma \left(\frac{\delta_\Gamma \mathcal{S}_\Gamma}{\delta u}(u)\right)_\gamma r_\gamma \,.$$

In particular, the last inequality holds for $r \in \mathcal{H}$ and we thus find

$$\lim_{h\to 0} \frac{\mathcal{S}(u+hv) - \mathcal{S}(u)}{h} \geq \int_\Omega -\text{div}\,\left(A(u,\nabla u)\nabla\left(s_0'(u) - \Delta u\right)\right) p_\omega + \int_\Gamma \left(\frac{\delta_\Gamma \mathcal{S}_\Gamma}{\delta u}(u)\right)_\gamma A_\Gamma(u) p_\gamma = \langle \nabla \mathcal{S}, r\rangle_{g(u)}$$

where p is the solution for r in (6.3). Similar to section 5 we deduce that the gradient flow (4.9) is equivalent with

$$\langle \partial_t u, p\rangle_{\mathcal{H},\mathcal{H}^{-1}} = \int_\Omega \text{div}\,\left(A(u,\nabla u)\nabla\left(s'(u) - \Delta u\right)\right) p_\omega - \int_\Gamma \left(\frac{\delta_\Gamma \mathcal{S}_\Gamma}{\delta u}(u)\right)_\gamma A_\Gamma(u) p_\gamma \quad \forall p \in L^2_\Gamma, \tag{6.10}$$

or, as p_ω and p_γ are independent, the last equation is also equivalent with (1.5). Theorem 1.3 is then a consequence of theorem 4.5.

Remark. Even though $(\partial_t u)_\omega$ and $(\partial_t u)_\gamma$ are not directly related with each other, note that still the condition $u \in L^2(0,T;H^1(\Omega))$ relates the values on Γ with those in Ω.

6.3. **Proof of Lemma 6.8.** In the following, recall $0 \in (a,b)$ and assume w.l.o.g. $s_0'(0) = s_0(0) = 0$ (shift s_0, s_1 and s_2 by affine functions) and define $s_0^+(x) := \max\{0, s_0(x)\}$, $s_0^-(x) := \min\{0, s_0(x)\}$. Furthermore, assume for the moment $s_1 \equiv s_2 \equiv 0$. Due to the assumptions on s_0, for any $n \in \mathbb{N}$ large enough there exist $a_n \in (a, \frac{a}{2})$ with $s_0'(a_n) = -n$ and $b_n \in (\frac{b}{2}, b)$ with $s_0'(b_n) = n$ and we introduce the following functions:

$$f_n^+(u) := \begin{cases} s_0'(u) & \text{for } c \in (\frac{b}{2}, b_n) \\ n + s_0''(b_n)(u - b_n) & \text{for } c \geq b_n \\ 0 & \text{for } c \leq 0 \end{cases},$$

$$f_n^-(u) := \begin{cases} s_0'(u) & \text{for } c \in (a_n, \frac{a}{2}) \\ n + s_0''(a_n)(u - a_n) & \text{for } c \leq a_n \\ 0 & \text{for } c \leq 0 \end{cases},$$

and extend $f_n^+(\cdot)$, $f_n^-(\cdot)$ to $(0, \frac{b}{2})$, resp. $(\frac{a}{2}, 0)$, monotone and $C^2(\mathbb{R})$, such that they are approximating $\left(s_0^+\right)'$ and $\left(s_0^-\right)'$. Note that also $y \mapsto y + f_n^+(y)$ is strictly monotone and we introduce $M_n := \sup_{c\in[a,b]} |f_n^+(u)'|$.

Now, let $u \in D(\mathcal{S}_\Gamma)$, i.e. $u \in H^1_\Gamma$ and $0 < t \leq 2/M_n$. By continuity and strict monotonicity we get unique existence of

$$\tilde{u}_t(x) = u(x) - t f_n^+(\tilde{u}_t(x))$$

and the theorem of the inverse function yields $\tilde{u}_t(x) = F_t^n(u(x))$, where $F_t^n : [a,b] \to [a,b]$ is a continuously monotone differentiable mapping with

$$F_t^n(x) \to x\,, \quad \left(F_t^n\right)'(x) \to 1\,, \quad \text{as} \quad t \to 0 \quad \text{uniformly on } [a,b]\,.$$

Thus, we see that for $u \in H^1_\Gamma(\Omega)$, also $\tilde{u}_t \in H^1(\Omega) \times L^2(\Gamma)$. Furthermore, the properties of F_t^n yield $\tilde{u}_t \to u$ in $H^1(\Omega) \times L^2(\Gamma)$ as $t \to 0$. Finally, monotonicity of $f_n^+(\cdot)$ yields $0 < \tilde{u}_t < u$ if $u > b_n$.

For $\phi \in C^2(\mathbb{R})$ being monotone decreasing with $\phi(x) = 1$ for $x < 0$, $\phi(x) = 0$ for $\phi > \frac{b}{2}$ and $\phi' \geq -4/b$ define $\psi_u(x) := \phi(u(x))/m(\phi(u(x)))$ such that

$$\int_\Omega \nabla\psi_u \cdot \nabla u = \int_\Omega \frac{\phi'(u)}{m(\phi(u))}|\nabla u|^2 \leq 0\,, \qquad \int_\Gamma \nabla_\Gamma \psi_u \cdot \nabla_\Gamma u = \int_\Gamma \frac{\phi'(u)}{m(\phi(u))}|\nabla_\Gamma u|^2 \leq 0$$

and $u_t := \tilde{u}_t + t\, m(f_n^+(\tilde{u}_t))\psi_u \in H^1_\Gamma \cap D(\mathcal{S})$ for t small enough, i.e. $\int_\Omega u_t = 0$.

Thus, we can easily calculate using the notation $d_n := m(f_n^+(\tilde{u}_t))\psi_u$

$$\begin{aligned} \mathcal{S}(u) - \mathcal{S}(u_t) \geq \int_\Omega \left(s_0(u) - s_0(u_t)\right) + t\int_\Omega \nabla u \cdot \nabla f_n^+(u_t) - t^2\, m(f_n^+(\tilde{u}_t))^2 \frac{1}{2}\int_\Omega |\nabla\psi_u|^2 \\ + \int_\Gamma \left(s_0(u) - s_0(u_t)\right) + t\int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma f_n^+(u_t) - t^2\, m(f_n^+(\tilde{u}_t))^2 \frac{1}{2}\int_\Gamma |\nabla\psi_u|^2 \,. \end{aligned}$$

For the first part of the above expression we get

$$
\begin{aligned}
&\int_\Omega (s_0(u)-s_0(u_t)) + t\int_\Omega \nabla u\cdot\nabla f_n^+(u_t) - t^2\, m(f_n^+(\tilde u_t))^2\frac{1}{2}\int_\Omega |\nabla\psi_u|^2\\
&\qquad\qquad \geq \int_{\Omega\cap\{u>b/2\}} t\, s_0'(u_t) f_n^+(u_t) + \int_{\Omega\cap\{a/2<u<b/2\}} (s_0(u)-s_0(\tilde u_t+d_n))\\
&\qquad - t^2\, m(f_n^+(\tilde u_t))^2\frac{1}{2}\int_\Omega|\nabla\psi_u|^2 + \int_{\Omega\cap\{u<a/2\}}(s_0(u)-s_0(\tilde u_t+d_n)) + t\int_\Omega \nabla u\cdot\nabla f_n^+(u_t)\\
&\quad \geq \int_{\Omega\cap\{u>b/2\}} t\, s_0'(u_t) f_n^+(u_t) + \int_{\Omega\cap\{a/2<u<b/2\}} (s_0(u)-s_0(\tilde u_t+d_n)) + t\int_\Omega \nabla u\cdot\nabla f_n^+(u_t)\\
&\qquad\qquad\qquad\qquad - t^2\, m(f_n^+(\tilde u_t))^2\frac{1}{2}\int_\Omega|\nabla\psi_u|^2
\end{aligned}
$$

where we used $s_0(u(x)) - s_0(u_t(x)) \geq s_0'(u_t(x))\,(u(x)-u_t(x))$ and $u_t(x) < u(x)$ if $u(x) > b/2$, $s_0'(u(x)) \geq f_n^+(u_t(x))$ as well as $s_0(u(x)) - s_0(u(x)+td_n(x)) \geq 0$ if $u(x)\leq a/2$ and $t\leq a/(2M_n)$. We similarly conclude

$$
\begin{aligned}
&\int_\Gamma (s_0(u)-s_0(u_t)) + t\int_\Gamma \nabla_\Gamma u\cdot\nabla_\Gamma f_n^+(u_t) - t^2\, m(f_n^+(\tilde u_t))^2\frac{1}{2}\int_\Gamma |\nabla\psi_u|^2\\
&\qquad\qquad \geq \int_{\Gamma\cap\{u>b/2\}} t\, s_0'(u_t) f_n^+(u_t) + \int_{\Gamma\cap\{a/2<u<b/2\}} (s_0(u)-s_0(\tilde u_t+d_n))\\
&\qquad\qquad\qquad + t\int_\Gamma \nabla_\Gamma u\cdot\nabla_\Gamma f_n^+(u_t) - t^2\, m(f_n^+(\tilde u_t))^2\frac{1}{2}\int_\Gamma|\nabla\psi_u|^2\ .
\end{aligned}
$$

Now, let $w\in\frac{\delta_\Gamma \mathcal{S}}{\delta u}(u)$, we then get by definition (note that $\mathcal{S}$ is convex in case $s_1\equiv s_2\equiv 0$)

$$
\begin{aligned}
&\left\langle w, f_n^+(\tilde u_t)-d_n\right\rangle_{L^2_\Gamma} \geq \frac{1}{t}\left(\mathcal{S}(u)-\mathcal{S}(\tilde u_t)\right)\\
&\quad = \int_{\Omega\cap\{u>b/2\}} s_0'(u_t) f_n^+(u_t) + t^{-1}\int_{\Omega\cap\{a/2<u<b/2\}} (s_0(u)-s_0(\tilde u_t+d_n)) + \int_\Omega \nabla u\cdot\nabla f_n^+(u_t)\\
&\quad + \int_{\Gamma\cap\{u>b/2\}} s_0'(u_t) f_n^+(u_t) + t^{-1}\int_{\Gamma\cap\{a/2<u<b/2\}} (s_0(u)-s_0(\tilde u_t+d_n)) + \int_\Gamma \nabla_\Gamma u\cdot\nabla_\Gamma f_n^+(u_t)\\
&\qquad\qquad - t\, m(f_n^+(\tilde u_t))^2\frac{1}{2}\int_\Omega|\nabla\psi_u|^2 - t\, m(f_n^+(\tilde u_t))^2\frac{1}{2}\int_\Gamma|\nabla\psi_u|^2
\end{aligned}
$$

which yields for $t\to 0$:

$$
\begin{aligned}
\left\langle w, f_n^+(u)-d_n\right\rangle_{L^2_\Gamma} \geq &\int_{\Omega\cap\{u>b/2\}} s_0'(u) f_n^+(u) + \int_{\Omega\cap\{a/2<u<b/2\}} s_0'(u)(f_n^+(u)-d_n) + \int_\Omega \nabla u\cdot\nabla f_n^+(u)\\
&\int_{\Gamma\cap\{u>b/2\}} s_0'(u) f_n^+(u) + \int_{\Gamma\cap\{a/2<u<b/2\}} s_0'(u)(f_n^+(u)-d_n) + \int_\Gamma \nabla_\Gamma u\cdot\nabla_\Gamma f_n^+(u)
\end{aligned}
$$

respectively

$$
\begin{aligned}
\left\langle w, f_n^+(u)-d_n\right\rangle_{L^2_\Gamma} \geq &\int_{\Omega\cap\{u>b/2\}} f_n^+(u)^2 + \int_{\Omega\cap\{a/2<u<b/2\}} s_0'(u)(f_n^+(u)-d_n) + t\int_\Omega (f_n^+)'(u)\nabla u\cdot\nabla u_t\\
&\int_{\Gamma\cap\{u>b/2\}} f_n^+(u)^2 + \int_{\Gamma\cap\{a/2<u<b/2\}} s_0'(u)(f_n^+(u)-d_n) + t\int_\Gamma (f_n^+)'(u)\nabla_\Gamma u\cdot\nabla_\Gamma u_t
\end{aligned}
$$

We make use of the simple extimate $\|m(f_n^+(u))\|_{L^2_\Gamma}\leq C\,\|f_n^+(u)\|_{L^2_\Gamma}$, following directly from the definition of $m(f_n^+(u))$, yielding for $n\to\infty$

$$
\|w\|^2_{L^2_\Gamma} \gtrsim \int_\Omega \left(s_0^+\right)'(u)^2 + \int_\Omega \left(s_0^+\right)''(u)\,|\nabla u|^2 + \int_\Gamma \left(s_0^+\right)'(u)^2 + \int_\Gamma \left(s_0^+\right)''(u)\,|\nabla_\Gamma u|^2\ ,
$$

which is together with the similar calculation for f_n^- the estimate (6.6). In particular, $s_0'(u)\in L^2(\Omega)\times L^2(\Gamma)$ implies $u\in(a,b)$ almost surely with respect to L^2_Γ.

Thus, we find for some $\delta > 0$ that $|\{x : u(x) \in (a+\delta, b-\delta)\}| > 0$ and for some non negative $\phi \in C_0^\infty((a+\delta, b-\delta))$, with $spt\phi = [a+\delta, b-\delta]$, define $\varphi_u := \phi(u(x))/m(\phi(u(x)))$, being in $H^1(\Omega)\times L^2(\Gamma)$.

Now, let $M \in \mathbb{N}$ and $\psi_M \in C^\infty(\mathbb{R})$ such that $\psi_M(x) = 0$ for $|x| > M+1$, $\psi_M(x) = 1$ for $|x| < M$ and $\psi_M'(x) \leq 2$ for all x. Note that by the properties of s_0, $u \in H^1_\Gamma$ implies $\chi_M := \psi_M(s_0'(u)) \in \left(H^1_\Gamma \oplus \mathbb{R}\right)$ and $\chi_M = 0$ if $|s_0'(u)| > M+1$. Thus, we find for φ_u as above and any $\psi \in C^\infty_{(0)}(\overline{\Omega})$, $u \in D(\mathcal{S})$, some $t_0 > 0$ such that also $\tilde{u} := u + t\chi_M\psi - t\varphi_u m(\chi_M\psi) \in D(\mathcal{S})$ for all $0 < t < t_0$.

Thus, we find for $w \in \frac{\delta_\Gamma \mathcal{S}}{\delta u}(u)$:

$$\begin{aligned}
\langle w, \chi_M\psi - \varphi_u m(\chi_M\psi)\rangle &\geq \lim_{t\to 0}\frac{1}{t}\left(\mathcal{S}(u) - \mathcal{S}(\tilde{u})\right)\\
&= \lim_{t\to 0}\left(\int_\Omega \frac{1}{t}\left(s_0(u) - s_0(\tilde{u})\right) + \int_\Omega \nabla u\cdot\nabla(\chi_M\psi - \varphi_u m(\chi_M\psi))\right)\\
&\quad + \lim_{t\to 0}\left(\int_\Gamma \frac{1}{t}\left(s_0(u) - s_0(\tilde{u})\right) + \int_\Gamma \nabla_\Gamma u\cdot\nabla_\Gamma(\chi_M\psi - \varphi_u m(\chi_M\psi))\right)\\
&\geq \int_\Omega (s_0'(u)(\chi_M\psi - \varphi_u m(\chi_M\psi))) + \int_\Omega \nabla u\cdot\nabla(\chi_M\psi - \varphi_u m(\chi_M\psi))\\
&\qquad + \int_\Gamma (s_0'(u)(\chi_M\psi - \varphi_u m(\chi_M\psi))) + \int_\Gamma \nabla_\Gamma u\cdot\nabla_\Gamma(\chi_M\psi - \varphi_u m(\chi_M\psi))
\end{aligned}$$

In order to investigate the behavior as $M\to\infty$, note that trivially $m(\chi_M\psi)) \to 0$ and $\chi_M \to 1$ pointwise and due to boundedness by 1 also in $L^2(\Omega)\times L^2(\Gamma)$. Furthermore, as ψ_M' is bounded by 2 and $\psi_M'(s_0'(u)) \to 0$ pointwise for $M\to\infty$, it is straight forward to see

$$\begin{aligned}
\int_\Omega \nabla u\cdot\nabla(\psi_M(s_0'(u))\psi) &= \int_\Omega \nabla u\cdot(\chi_M\nabla\psi) + \int_\Omega s_0''(u)\psi_M'(s_0'(u))\,|\nabla u|^2\\
&\to \int_\Omega \nabla u\cdot\nabla\psi \quad \text{as } M\to\infty
\end{aligned}$$

and similar for $\int_\Gamma \nabla_\Gamma u\cdot\nabla_\Gamma(\psi_M(s_0'(u))\psi)$. Thus, we find

$$\langle w,\psi\rangle_{L^2_\Gamma} \geq \int_\Omega (s_0'(u)\psi) + \int_\Omega \nabla u\cdot\nabla\psi + \int_\Gamma s_0'(u)\psi + \int_\Gamma \nabla_\Gamma u\cdot\nabla_\Gamma\psi\,.$$

Replacing ψ by $-\psi$, we find equality. Using partial integration, Definition (6.1) and Lemma 6.2, we get

$$\langle w,\psi\rangle_{L^2_\Gamma} = \int_\Omega (s_0'(u)\psi) - \int_\Omega \Delta u\psi + \int_\Gamma \left(\nabla u\cdot\boldsymbol{n}_\Gamma + s_0'(u) - \Delta_\Gamma u\right)\psi\,.$$

and hence $w_\omega = P_0\left(s_0'(u)\right) - P_0\left(\Delta u\right)$, $w_\gamma = \left(\nabla u\cdot\boldsymbol{n}_\Gamma + s_0'(u) - \Delta_\Gamma u\right)$ in the weak sense yielding (6.9) and $u\in\mathcal{H}_0$. (6.7) follows immediately from the calculation whereas (6.8) follows from (6.7) and (6.4).

It is elementary to verify that the statement still holds in case $s_1 \not\equiv s_2 \not\equiv 0$: To this aim, note that the domain $D(d\mathcal{S})$ remains the same and that u is essentially bounded by $a < u < b$. In particular, calculating the $\frac{\delta_\Gamma}{\delta u}$-derivative of

$$\hat{\mathcal{S}}(u) := \int_\Omega s_1(u) + \int_\Gamma s_2(Eu)$$

for $u\in D(d\mathcal{S})$, it is easy to see that estimate (6.9) remains valid. Thus, having in mind above estimates in case $s_1\equiv s_2\equiv 0$, it is easy to verify that (6.7) still holds.

7. Proof of Theorem 1.4

We will now proof Theorem 1.4 in four steps: First we will construct an approximate problem, that can be directly solved using Theorem 4.5. Then, we will show convergence of a subsequence of the approximate solutions as the approximation parameter tends to zero and demonstrate that the limit function solves the original problem. We then finally proof a technical lemma on the subdifferentials.

Before starting, note that the experiences from sections 5 and 6 tell us, that the probably correct choice for the three Hilbert spaces are

$$\mathcal{H} := H^{-1}_{(0)}(\Omega) \qquad \tilde{\mathcal{H}} := H^1_{(0)}(\Omega) \qquad \mathcal{H}_0 := H^2(\Omega)$$

Note that one is tempted to directly consider the problem as a generalized gradient flow

$$\partial_t u = -\nabla \mathcal{S}(u)$$

where the gradient is with respect to the metric structure $g_\bullet(\cdot,\cdot)$ defined through

$$\begin{array}{rccc} g_\bullet : & \mathcal{H}_0 & \to & B(\mathcal{H}) \\ & u & \mapsto & g_u(\cdot,\cdot) \end{array}$$

where with $w = -\Delta u + s'(u)$:

$$g_u(r_1, r_2) = \int_\Omega \nabla p_1^u A(u, \nabla u, w) \nabla p_2^u = \int_\Omega r_1 p_2^u = \int_\Omega r_2 p_1^u\,, \tag{7.1}$$

where p_i^u solves

$$-\mathrm{div}\,\left(A(u, \nabla u, w)\nabla p_i^u\right) = r_i \quad \text{for } i = 1, 2\,.$$

However, $g_\bullet$ then is defined on $\mathcal{H}_0$ instead of $\tilde{\mathcal{H}}$ and thus theorem 4.5 does not apply. Nevertheless, the additional information $w \in H^1_{(0)}(\Omega)$ will be sufficient to cope with that problem.

The basic formal idea behind the following proof is to identify a set $A \subset L^2(0,T;\mathcal{H}_0)$ that is not compact in $L^2(0,T;\tilde{\mathcal{H}})$ but still has sufficiently nice properties in order to guaranty (4.3) resp. (3.3).

7.1. **An approximate problem.** We start by considering the following problem: Like in section 5, we choose

$$\mathcal{H}_0 := H^2(\Omega) \cap H^1_{(0)}(\Omega), \quad \tilde{\mathcal{H}} := H^1_{(0)}\,, \quad \text{and} \quad \mathcal{H} = H^{-1}_{(0)}(\Omega)\,.$$

We extend w to $\mathbb{R}^n$ by 0 and for any $\eta > 0$ we consider $w * \varphi_\eta$, where φ_η is the standard mollifier.

For any $u \in H^1_{(0)}(\Omega) \cap H^2(\Omega)$, we then consider the following scalar product on $\mathcal{H}$: we define for $r_1, r_2 \in \mathcal{H}$:

$$g_u^\eta(r_1, r_2) = \int_\Omega \nabla p_1^u A(u, \nabla u, w * \varphi_\eta) \nabla p_2^u = \int_\Omega r_1 p_2^u = \int_\Omega r_2 p_1^u\,, \tag{7.2}$$

where p_i^u solves

$$-\mathrm{div}\,\left(A(u, \nabla u, w * \varphi_\eta)\nabla p_i^u\right) = r_i \quad \text{for } i = 1, 2\,. \tag{7.3}$$

It is immediate to check that g is a densly defined metric in the sense of definition 4.1. For convenience of notation, we write the gradient with respect to g^η as ∇_η, i.e.

$$g_u^\eta(\nabla_\eta \mathcal{S}(u), \psi) = \langle d\mathcal{S}(u), \psi\rangle_{\mathcal{H}} \qquad \forall \psi \in \mathcal{H}$$

and denote by $\nabla_{\eta,l}$ the corresponding limiting subgradient w.r.t. ∇_η according to definition 4.2.

Instead of lemma 5.3, we this time consider the following:

Lemma 7.1. *Let $\mathcal{S}$ and s be as introduced in subsection 1.3. Then, for the L^2-subdifferential holds*

$$D\Big(\frac{\delta^0 \mathcal{S}}{\delta u}\Big) = \left\{ c \in H^2(\Omega) \cap L^2_{(0)}(\Omega) \,:\, s'(c) \in L^2(\Omega),\ s''(c)\,|\nabla c|^2 \in L^1(\Omega),\ \partial_n c\Big|_{\partial\Omega} = 0 \right\} \tag{7.4}$$

and

$$\frac{\delta^0 \mathcal{S}}{\delta u}(\tilde{u}) = -\Delta \tilde{u} + P_0 s'(\tilde{u})\,. \tag{7.5}$$

Moreover,

$$\|\tilde{u}\|^2_{H^2(\Omega)} + \|s'(\tilde{u})\|^2_{L^2(\Omega)} + \int_\Omega s''(\tilde{u})\,|\nabla \tilde{u}|^2 \le C\left(\left\| \frac{\delta^0 \mathcal{S}}{\delta u}(\tilde{u}) \right\|^2_{L^2(\Omega)} + \|\tilde{u}\|^2_{L^2(\Omega)} + 1 \right) \tag{7.6}$$

for some constant C independent of $\tilde{u}$.

For the $\mathcal{H}$-Subdifferential holds

$$D(d\mathcal{S}) = \left\{ c \in D\Big(\frac{\delta^0 \mathcal{S}}{\delta u}\Big) \,:\, \frac{\delta^0 \mathcal{S}}{\delta u}(c) \in H^1_{(0)}(\Omega) \right\} \tag{7.7}$$

$$d\mathcal{S}(\tilde{u}) = \Delta\left(-\Delta \tilde{u} + P_0 s'(\tilde{u})\right)\,, \tag{7.8}$$

i.e. $d\mathcal{S}(\tilde{u})$ is single valued and

$$\|\tilde{u}\|^2_{H^2(\Omega)} \le C\left(\|d\mathcal{S}(\tilde{u})\|^2_{\mathcal{H}} + \|\tilde{u}\|^2_{L^2(\Omega)} + 1 \right)\,. \tag{7.9}$$

Furthermore, we find

Lemma 7.2. *$d\mathcal{S}$ is strongly-weakly closed.*

Similar to section 5, we observe that $g^\eta_\bullet$ and $\mathcal{S}$ satisfy all conditions of Theorem 4.5, so we get existence of a $u_\eta \in H^1(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{H}_0)$ solution to the equation

$$\int_0^T g^\eta_{u_\eta}(\partial_t u_\eta, \psi) = -\int_0^T \langle d\mathcal{S}(u_\eta), \psi\rangle_{\mathcal{H}} \qquad \forall \psi \in L^2(0,T;\mathcal{H})\,, \tag{7.10}$$

$$\text{or} \qquad \partial_t u_\eta = -\nabla_\eta \mathcal{S}(u_\eta)\,, \tag{7.11}$$

where $u(0) = u_0$ for $t = 0$. This is a weak formulation to the problem

$$\begin{aligned}
\partial_t u_\eta - \mathrm{div}\,\left(A(u_\eta, \nabla u_\eta, w_\eta * \varphi_\eta)\,\nabla w_\eta\right) &\ni 0 && \text{on } (0,T]\times U\,,\\
w_\eta + \Delta u_\eta - s'(u_\eta) &= 0 && \text{on } (0,T]\times U\,,\\
\left(A(u_\eta, \nabla u_\eta, w_\eta * \varphi_\eta)\,\nabla w_\eta\right)\cdot \boldsymbol{n}_\Gamma = \nabla u_\eta \cdot \boldsymbol{n}_\Gamma &= 0 && \text{on } (0,T]\times \partial U\,,\\
u_\eta(0) &= u_0 && \text{for } t = 0\,.
\end{aligned}$$

Note that the solution satisfies the apriori estimate

$$\frac{1}{2}\int_0^t \left\|u_\eta'\right\|^2_{g^\eta(u_\eta)} + \frac{1}{2}\int_0^t \|\nabla_{\eta,l}\mathcal{S}(u_\eta)\|^2_{g^\eta(u_\eta)} + \mathcal{S}(u_\eta(t)) \le \mathcal{S}(u(0)) \qquad \text{for a.e. } t \in (0,T)\,. \tag{7.12}$$

However, we whish to study the behavior of solutions as $\eta \to 0$. In this context, note that we cannot decide whether $w_\eta * \varphi_\eta \to w$ in $L^2(0,T;L^2(\Omega))$ as we do not know whether $w_\eta \to w$ in $L^2(0,T;L^2(\Omega))$. (As w_η depends nonlinearly on u_η and s' is not Lipschitz in $\mathbb{R}$.)

7.2. **Convergence of the approximate problem.** It is thus necessary to repeat some of the steps in [16]. First, as $n \le 3$, we find $\mathcal{H}_0 \hookrightarrow\hookrightarrow C(\overline{\Omega})$ compactly and thus $u_\eta \in L^2(0,T;C(\overline{\Omega}))$.

We find a subsequence $(u_{\eta_k})_{k\in\mathbb{N}}$ with $\eta_k \to 0$ as $k \to \infty$ such that there is $u \in H^1(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{H}_0)$ with

$$\begin{aligned}
u_{\eta_k} &\rightharpoonup u \quad \text{weakly in } H^1(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{H}_0)\,,\\
u_{\eta_k} &\to u \quad \text{strongly in } L^2(0,T;\tilde{\mathcal{H}}) \cap L^2(0,T;C(\overline{\Omega}))\,,\\
u_{\eta_k}(t) &\to u(t) \quad \text{in } C(\overline{\Omega}) \cap H^1(\Omega) \text{ for a.e. } t \in (0,T)\,.
\end{aligned}$$

Now, let $\varepsilon > 0$. By Egorov's theorem, there is a compact set $K_0 \subset (0,T)$ with $\mathcal{L}((0,T)\backslash K_0) < \frac{\varepsilon}{2}$ s.t. uniformly for all $t \in K_0$ we find $u_{\eta_k}(t) \to u(t)$ strongly in $C(\overline{\Omega}) \cap H^1(\Omega)$. For each $k \in \mathbb{N}\backslash\{0\}$, Lusin's theorem yields existence of a kompakt set $K_k \subset (0,T)$ with $\mathcal{L}\left((0,T)\backslash K_k\right) \le 2^{-k-1}\varepsilon$ and $u_{\eta_k} \in C(K_k; C(\overline{\Omega}))$. Defining $K_\varepsilon := \bigcap_{k=0}^\infty K_k$, we find $\mathcal{L}\left((0,T)\backslash K_\varepsilon\right) \le \varepsilon$, $u_{\eta_k} \in C(K_\varepsilon; C(\overline{\Omega}))$ for all k and by the pointwise convergence also $u_{\eta_k} \to u$ uniformly in $C(K_\varepsilon; C(\overline{\Omega}))$ and strongly in $L^2(0,T;H^1(\Omega))$. In particular, we find $|u(t,x)| \le C_\varepsilon$, $|u_{\eta_k}(t,x)| \le C_\varepsilon$ for all k for some constant $C_\varepsilon > 0$ for all $(t,x) \in K \times \overline{\Omega}$. Now, it is evident that $s_0'(u_{\eta_k}) \to s_0'(u)$ strongly in $L^2(K_\varepsilon; L^2(\Omega))$as well as $\Delta u_{\eta_k} \rightharpoonup \Delta u$ weakly in $L^2(0,T;L^2(\Omega))$, implying $w_{\eta_k} \rightharpoonup w = -\Delta u + s_0(u)$ weakly in $L^2(K_\varepsilon; H^1_{(0)}(\Omega))$.

Thus, we may perform the following calculation:

$$\begin{aligned}
\lim_{k\to\infty}\int_{K_\varepsilon}\int_\Omega w_{\eta_k}^2 &= -\lim_{k\to\infty}\int_{K_\varepsilon}\int_\Omega \Delta u_{\eta_k} w_{\eta_k} + \lim_{k\to\infty}\int_{K_\varepsilon}\int_\Omega s'(u_{\eta_k}) w_{\eta_k}\\
&= \lim_{k\to\infty}\int_{K_\varepsilon}\int_\Omega \nabla u_{\eta_k}\nabla w_{\eta_k} + \lim_{k\to\infty}\int_{K_\varepsilon}\int_\Omega s'(u_{\eta_k}) w_{\eta_k}\\
&= \int_{K_\varepsilon}\int_\Omega \nabla u \nabla w + \int_{K_\varepsilon}\int_\Omega s'(u) w\\
&= \int_{K_\varepsilon}\int_\Omega w^2
\end{aligned}$$

where we used boundedness of u_{η_k} to get local Lipschitz continuity of $s'(\cdot)$. In particular, we find for fixed ε a further subsequence $w^\varepsilon_{\eta_k}$ s.t. $w^\varepsilon_{\eta_k}(t) \to w^\varepsilon(t)$ in $L^2(\Omega)$ for a.e. $t \in K^\varepsilon$. A standard diagonalization argument yields the existence of a subsequence such that $w_{\eta_k}(t) \to w(t)$ in $L^2(\Omega)$ for a.e. $t \in (0,T)$.

We consider the space $\hat{\mathcal{H}} := H^1_{(0)}(\Omega) \times L^2(\Omega)$ and

$$\begin{array}{rccc} \hat{g}_\bullet : & \hat{\mathcal{H}} & \to & B(\mathcal{H}) \\ & (u,w) & \mapsto & \hat{g}_{(u,w)}(\cdot,\cdot) \end{array}$$

where

$$\hat{g}_{u,w}(r_1,r_2) = \int_\Omega \nabla p_1^u A(u,\nabla u,w)\nabla p_2^u = \int_\Omega r_1 p_2^u = \int_\Omega r_2 p_1^u \,,$$

where p_i^u solves

$$-\mathrm{div}\,\left(A(u,\nabla u,w)\nabla p_i^u\right) = r_i \quad \text{for } i=1,2\,,$$

and we immediately check with (7.1) and (7.2) that

$$g_u^\eta(\cdot,\cdot) = \hat{g}_{(u,w*\varphi_\eta)}(\cdot,\cdot)\,, \qquad g_u(\cdot,\cdot) = \hat{g}_{(u,w)}(\cdot,\cdot)\,.$$

We find with the above estimates and theorem 3.5 two Young measures $\boldsymbol{\mu},\boldsymbol{\nu} \in \mathcal{Y}(0,T;\mathcal{H})$ associated with u'_{η_k} and $\nabla_{\eta_k}\mathcal{S}(u_{\eta_k})$ such that $u'_{\eta_k} \rightharpoonup \int_{\mathcal{H}} \xi d\mu_t(\xi)$ and $\nabla_{\eta_k}\mathcal{S}(u_{\eta_k}) \rightharpoonup \int_{\mathcal{H}} \xi d\nu_t(\xi)$ weakly in $L^2(0,T;\mathcal{H})$. Our final aim is now to identify the sets of concentration of $\boldsymbol{\mu},\boldsymbol{\nu}$:

We find with help of theorem 3.5 and corollary 3.4 that

$$\liminf_{k\to\infty}\int_0^T \hat{g}_{(u_{\eta_k},w_{\eta_k})}(\partial_t u_{\eta_k},\partial_t u_{\eta_k}) \geq \int_0^T\int_{\mathcal{H}} \hat{g}_{(u,w)}(\xi,\xi)\,d\mu_t(\xi) \quad \text{and}$$
$$\liminf_{k\to\infty}\int_0^T \left(\nabla_{\eta_k}\mathcal{S}(u_{\eta_k})\right)^2 \geq \int_0^T\int_{\mathcal{H}} \hat{g}_{(u,w)}(\xi,\xi)\,d\nu_t(\xi)\,.$$

Also, with help of (7.11) as well as the following corollary 7.3 below, arguing as in the proof of theorem 4.5 in [16], we find that μ_t, ν_t are concentrated on $\left(\tilde{\hat{g}}_{u,w}\right)^{-1}(d_l\mathcal{S}(u)) = \tilde{g}_u^{-1}(d_l\mathcal{S}(u))$ for $t \in K_\varepsilon$ for all $\varepsilon > 0$. As $d_l\mathcal{S}(u)$ is convex for all u and ε was arbitrary, the theorem is prooved.

Corollary 7.3. [16] *For a bounded sequence $\varphi_n \in \mathcal{H}$ and $u_n \to u$ strongly in $\tilde{\mathcal{H}}$, we find $\varphi_n \rightharpoonup \varphi$ weakly in $\mathcal{H}$ iff $\tilde{g}_{u_n}(\varphi_n) \rightharpoonup \tilde{g}_u(\varphi)$ weakly in $\mathcal{H}$, where $\tilde{g}_u$ is defined through* (4.4).

7.3. **Proof of Lemma 7.1.** The proof is similar to subsection 6.3: This time, $s_0'(0) = s_0(0) = 0$ and define $s_0^+(x) := \max\{0, s_0(x)\}$, $s_0^-(x) := \min\{0,s_0(x)\}$. For $a_0 \in (s_0')^{-1}(-1/2)$, $b_0 \in (s_0')^{-1}(1/2)$, there are for any $n \in \mathbb{N}$ $a_n \in (-\infty, a_0)$ with $s_0'(a_n) = -n$ and $b_n \in (b_0,+\infty)$ with $s_0'(b_n) = n$ and we introduce f_n^+ and f_n^- similar to subsection 6.3, such that $f_n^+(\cdot)$, $f_n^-(\cdot)$ are both monotone and $C^2(\mathbb{R})$ with $y \mapsto y + f_n^+(y)$ being strictly monotone and $C^2(\mathbb{R}^n)$, too.

Now, let $u \in D(\mathcal{S})$, i.e. $u \in H^1_\Gamma$ and define $\tilde{u}_t := u - f_n^+(\tilde{u}_t)$

$$u_t := \tilde{u}_t + t\, m(f_n^+(\tilde{u}_t))/\mathcal{L}^n(\Omega) \in H^1_\Gamma \cap D(\mathcal{S})$$

for t small enough.

Thus, we can easily calculate using the notation $d_n := m(f_n^+(\tilde{u}_t))/\mathcal{L}^n(\Omega)$ following the outline of section 6.3 or the proof of theorem 4.3 in [1] that for $w \in \frac{\delta_\Gamma\mathcal{S}}{\delta u}(u)$, we get by definition

$$\begin{aligned}\left\langle w, f_n^+(\tilde{u}_t) - d_n\right\rangle_{L^2_\Gamma} &\geq \frac{1}{t}\left(\mathcal{S}(u) - \mathcal{S}(u_t)\right) \\ &\geq \int_{\Omega\cap\{u>b/2\}} f_n^+(\tilde{u}_t)^2 + t^{-1}\int_{\Omega\cap\{a/2<u<b/2\}}\left(s_0(u) - s_0(\tilde{u}_t + d_n)\right) + \int_\Omega \nabla u\cdot\nabla f_n^+(u_t)\end{aligned}$$

which yields for $t \to 0$:

$$\left\langle w, f_n^+(u) - d_n\right\rangle_{L^2_\Gamma} \geq \int_{\Omega\cap\{u>b/2\}} f_n^+(u)^2 + \int_{\Omega\cap\{a/2<u<b/2\}} s_0'(u)f_n^+(u) + t\int_\Omega (f_n^+)'(u_t)\nabla u\cdot\nabla u_t$$

and for $n \to \infty$ by monotone convergence together with the similar calculation for f_n^-:

$$1 + \|w\|^2_{L^2_\Gamma} \gtrsim \int_\Omega s_0'(u)^2 + \int_\Omega s_0''(u)\left|\nabla u\right|^2\,,$$

which is (6.6).

We find for any $\psi \in C^\infty_{(0)}(\overline{\Omega})$ and $u \in D(d\mathcal{S})$ some $t_0 > 0$ such that $\tilde{u} := u + t\psi \in D(\mathcal{S})$ for all $0 < t < t_0$.

Thus, we find for $w \in \frac{\delta_\Gamma \mathcal{S}}{\delta u}(u)$:

$$\langle w, \psi \rangle \geq \lim_{t \to 0} \frac{1}{t} \left(\mathcal{S}(u) - \mathcal{S}(\tilde{u}) \right)$$

$$= \lim_{t \to 0} \left(\int_\Omega \frac{1}{t} \left(s_0(u) - s_0(\tilde{u}) \right) + \int_\Omega \nabla u \cdot \nabla \psi \right) \geq \int_\Omega s_0'(u)\psi + \int_\Omega \nabla u \cdot \nabla \psi$$

Replacing ψ by $-\psi$, we find equality. Using partial integration, we get

$$\langle w, \psi \rangle_{L^2_\Gamma} = \int_\Omega P_0 \left(s_0'(u) \right) \psi - \int_\Omega \Delta u \psi \qquad \forall \psi \in C^\infty(\overline{\Omega}) \,,$$

and hence, the standard theory of elliptic equations tells us that u solves $w_\omega - P_0\left(s_0'(u)\right) = -\Delta u$ with $\partial_\nu u = 0$, implying $u \in H^2(\Omega)$ and $\|u\|_{H^2(\Omega)} \leq C \|w\|_{L^2}$ (See also Abels and Wilke [1], Section 2).

Again, $\mathcal{S}$ in the setting of the last lemma is convex and the graph of $(d\mathcal{S}, \mathcal{S})$ is strongly-weakly closed in the sense of (4.10), implying convexity of $d\mathcal{S}(u)$ for all $u \in D(d\mathcal{S})$. This convexity remains even in case $s_1 \not\equiv 0$, whereas the subdifferentials remain in the form (7.5) and (7.8).

References

[1] Helmut Abels and Mathias Wilke. Convergence to equilibrium for the cahn-hilliard equation with a logarithmic free energy. *Nonlinear Analysis: Theory, Methods & Applications*, 67(11):3176–3193, 2007.

[2] R. Adams. *Sobolev spaces, Acad.* 1975.

[3] M. Brokate, N. Kenmochi, I. Müller, J.F. Rodriguez, and C. Verdi. *Phase transitions and hysteresis. Lectures given at the 3rd session of the Centro Internazionale Matematico Estivo (C.I.M.E.) held in Montecatini Terme, Italy, July 13-21, 1993.* Lecture Notes in Mathematics. 1584. Berlin: Springer-Verlag. vii, 1994.

[4] G. C. Buscaglia and R. F. Ausas. Variational formulations for surface tension, capillarity and wetting. *Computer Methods in Applied Mechanics and Engineering*, 2011.

[5] JW Cahn, CM Elliott, and A. Novick-Cohen. The cahn-hilliard equation with a concentration dependent mobility: motion by minus the laplacian of the mean curvature. *European Journal of Applied Mathematics*, 7(3):287–302, 1996.

[6] L. Changchun. Cahn-hilliard equation with terms of lower order and non-constant mobility. *Electronic Journal of Qualitative Theory of Differential Equations*, 10, 2003.

[7] R. Dal Passo, L. Giacomelli, and A. Novick-Cohen. *Existence for an Allen-Cahn/Cahn-Hilliard system with degenerate mobility.* Carnegie Mellon University, Department of Mathematical Sciences [Center for Nonlinear Analysis, 1999.

[8] C. Dellacherie and P.A. Meyer. *Probabilities and potential.* North-Holland Publishing Co., Amsterdam, 1978.

[9] C.M. Elliott and S. Zheng. On the Cahn-Hilliard equation. *Archive for Rational Mechanics and Analysis*, 96(4):339–357, 1986.

[10] C.G. Gal. Global well-posedness for the non-isothermal cahn-hilliard equation with dynamic boundary conditions. *Advances in Differential Equations*, 12(11):1241–1274, 2007.

[11] G. Gilardi, A. Miranville, and G. Schimperna. On the Cahn-Hilliard equation with irregular potentials and dynamic boundary conditions. *Communications on Pure and Applied Analysis (CPAA)*, 8(3):881–912, 2009.

[12] M. Grasselli, A. Miranville, R. Rossi, and G. Schimperna. Analysis of the Cahn–Hilliard equation with a chemical potential dependent mobility. *Communications in Partial Differential Equations*, 36(7):1193–1238, 2011.

[13] M. Heida. *Modeling multiphase flow in porous media with an application to permafrost soil.* PhD thesis, University of Heidelberg, 2011.

[14] M. Heida. On the derivation of thermodynamically consistent boundary conditions for the Cahn-Hilliard-Navier-Stokes system. *submitted to IJES*, 2011.

[15] M. Heida. On gradient flows of nonconvex functionals in hilbert spaces with riemannian metric and application to cahn-hilliard equations. *submitted*, 2012.

[16] M. Heida. A two-scale model of two-phase flow in porous media ranging from porespace to the macro scale. *submitted to SIAM MMS*, 2012.

[17] C. Liu. On the convective cahn–hilliard equation with degenerate mobility. *Journal of Mathematical Analysis and Applications*, 344(1):124–144, 2008.

[18] A. Miranville and S. Zelik. The Cahn-Hilliard equation with singular potentials and dynamic boundary conditions. *Discrete and continuous dynamical systems*, 28:275–310, 2010.

[19] L. Mugnai and M. Röger. The allen-cahn action functional in higher dimensions. *arXiv preprint arXiv:0704.1954*, 2007.

[20] L. Mugnai and M. Röger. Convergence of perturbed allen cahn equations to forced mean curvature flow. *preprint online at mis.mpg.de*, 2009.

[21] A. Novick-Cohen. The cahn–hilliard equation. *Handbook of differential equations: evolutionary equations*, 4:201–228, 2008.

[22] A. Novick-Cohen. *The Cahn-Hilliard Equation: From Backwards Diffusion to Surface Diffusion.* Cambridge University Press, to appear.

[23] T. Qian, X.P. Wang, and P. Sheng. A variational approach to moving contact line hydrodynamics. *Journal of Fluid Mechanics*, 564:333–360, 2006.

[24] R. Racke and S. Zheng. The cahn-hilliard equation with dynamic boundary conditions. In *Advances in Differential Equations.* Citeseer, 2001.

[25] R. Rossi. On two classes of generalized viscous Cahn-Hilliard equations. *Commun. Pure Appl. Anal*, 4(2):405–430, 2005.

[26] R. Rossi and G. Savaré. Gradient flows of non convex functionals in Hilbert spaces and applications. *ESAIM: Control, Optimisation and Calculus of Variations*, 12(03):564–614, 2006.

[27] Sylvia Serfaty. Gamma-convergence of gradient flows on hilbert and metric spaces and applications. *Disc. Cont. Dyn. Systems, A*, 31:1427–1451, 2009.

[28] U. Stefanelli. The brezis-ekeland principle for doubly nonlinear equations. *SIAM Journal on Control and Optimization*, 47(3):1615–1642, 2008.

[29] R. Temam. *Navier-Stokes equations: theory and numerical analysis*, volume 2. Amer Mathematical Society, 2001.

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