J-class operators on certain Banach spaces

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To my family Jaffar, Parvaneh and Parinaz

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Contents

Introduction

Linear dynamics is a fashionable and new branch of mathematics. An operator T defined on a separable Banach space X into itself is called hypercyclic if there exists some vector x in X such that the set $\mathrm{orb}(T, x) := \{T^n x \mid n \in \mathbb{N}\}$ is dense in X. As an application of Baire's theorem it follows that hypercyclicity is equivalent to topological transitivity, i.e. that for every two, non-empty open subsets U, V of X there exists a natural number n such that $T^n(V) \cap U$ is non-empty. The weighted unilateral backward shift or also known as the Rolewicz operator was one of the first hypercyclic operators found in the nineteen sixties. It is obvious that the definition of hypercyclicity or topological transitivity can be extended to general topological spaces. In the case where the underlying space is a separable Fréchet-space (i.e. a locally convex space whose topology is induced by a complete translation invariant metric), the equivalence of topological transitivity and hypercyclicity remains valid due to Baire's theorem. Examples like the differential operator on the space of entire functions or the translation operator on the same space are well known hypercyclic operators which have been the headstone for further research. Working on non-separable Banach spaces it is clear that the notion of hypercyclicity has to be replaced by the topological transitivity. Since every separable Banach space admits hypercyclic and hence topological transitive operators it is natural to ask whether every non-separable Banach space admits a topological transitive operator. Nowadays there exist several examples of Banach spaces of pathological character which admit no topologically transitive operator. So one might expect that at least the classical non-separable Banach spaces admit topological transitive operators. For Hilbert spaces the latter question is true as shown in [11] by T. Bermúdez and N. J. Kalton. However in the same paper it is also shown that on l^∞ (the space of bounded sequences with the usual sup-norm) there does not exist any topological transitive operator. This matter of fact leads to modifications which are for example replacing the norm-topology by the weak star topology on l^∞ and considering then hypercyclic operators on l^∞, see [13]. If one wants not to renounce the norm-topology one can replace topological transitivity by its localized version:

T is called locally topologically transitive if there exists some non-zero vector $x \in X$, such that for every open neighborhood U of x and any non-empty open $V \subseteq X$ there exist a positive integer n such that $T^n(U) \cap V \neq \emptyset$.

An equivalent formulation is the following which is the basis for the remainder of this thesis: There exists a non-zero vector $x \in X$ such that for every $y \in X$ there is a sequence $(x_k)_k$ in X and a strictly increasing sequence $(n_k)_k$ of positive integers such that $x_k \to x$ and $T^{n_k} x_k \to y$ holds.

Operators on Banach spaces satisfying the latter definition will be called J-class operators and the respective vector x is said to be a J-vector. It is immediately clear that topologically transitive operators are J-class. This localized dynamic behaviour is in particular interesting in the case of l^∞ because of the aforementioned reasons and has therefore led A. Manoussos and G. Costakis finding J-class operators on l^∞.
In the first chapter we will state briefly the main definitions and important properties about hypercyclic and topological transitive operators. Subsequently we introduce the notion of J-class operators and list up some basic properties which we extracted from [14] and [15] and extended these to more general assertions. The Spectral lemma for example is one of the key tools we will use frequently in the upcoming chapters. At the end of the first chapter we discuss the J-class property of a certain class of operators on l^∞.
As we mentioned before the existence of topologically transitive operators is only guaranteed on separable Banach spaces as the case l^∞ shows. On the other hand there exist J-class operators on l^∞ like unilateral backward shifts. So one might ask the natural question if every Banach space admits J-class operators. This problem was first raised by A. Manoussos and G. Costakis in [14]. We will tackle this question in the second chapter and give a negative answer by showing that on a non-separable hereditarily indecomposable Banach space constructed by A. Arvanitakis, S. Argyros and A. Tolias there does not exist any J-class operator. The main parts of this chapter are also published in [31].

Properties of the spectrum of operators on Banach spaces are necessary and sometimes sufficient conditions to ensure hypercyclicity. In this connection the Spectral lemma which appears its first time in chapter one will be the springboard for chapter 3. With this tool we will show that J-class operators on l^∞ can not have small spectrum in the sense that it has to be at least uncountable. To be more precise we will see that the spectrum of the adjoint of an operator on l^∞ which has totally disconnected spectrum consists entirely of eigenvalues. From this observation and our Spectral lemma it follows that the spectrum is at least uncountable as we will see. The second part of this chapter is devoted to adjoint operators on l^∞ and their J-class behaviour. That means that operators on l^∞ which are adjoints gained by some operators on l^1 will be completely characterized whenever they are J-class or not. We will get strong use of the geometry of l^1 like the Schur-property or the fact that l^1 contains no infinite dimensional reflexive subspace. A. Manoussos and G. Costakis give in [15] a necessary and sufficient condition for the weight sequence of B_w to be J-class. Here B_w is the unilateral positive weighted backward shift on l^∞. As an application of the characterization of the adjoints we will give necessary and sufficient conditions for the weight sequence of operators of the type $f(B_w)$ to be J-class. Here f is any non-constant holomorphic function defined on a neighborhood of the spectrum of B_w. If $f(B_w)$ is J-class the reader will ask the natural question how many vectors $x \in l^\infty$ are J-vectors. The answer for B_w is that any $x = (x_n)_n$ with $\lim_{n\to\infty} x_n = 0$ is a J-vector which is also shown in [15]. For $f(B_w)$ we will show the same but the reader will notice that this general case needs another approach as in [15] which is mainly based on spectral properties of B_w.

In the fourth and last chapter we will introduce the notion of locally topologically transitive C_0-semigroups. Hypercyclic C_0-semigroups were introduced by W. Desch, W. Schappacher, and G. F. Webb in [17] and further researches in this direction are done by several authors for instance in [22], [21]. And also here we ask in analogy to the discrete case if every Banach space admits a locally topologically transitive C_0-semigroup and show that the answer here is also negative. Finally in an additional section we will state some remarks and open questions which occurred during our researches and will act as a stimulus for further investigations.

Chapter 1

Basic facts about J-class and hypercyclic operators

In this chapter we list up the most important facts about hypercyclic and J-class operators which are the most work of G. Costakis and A. Manoussos in [14]. If one wants to work on non-separable Banach-spaces or more generally on non-separable Fréchet spaces it is clear that one has to replace the notion of hypercyclicity by a suitable definition. The notion of topological transitivity is frequently used in dynamical systems but unfortunately classical spaces like l^∞ or $L(l^2)$ do not admit any topologically transitive operators, see [11]. So one has to alleviate the definition of topological transitivity which we will state more precisely in the following chapter. Parts of this chapter are also published by the author in [31].

1.1 Basic definitions and properties of topologically transitive operators

Definition 1.1.1. Let X be a separable Banach space and $T \in L(X)$.

(i) The operator T is called hypercyclic if there exists some $x \in X$ such that the set

$$\operatorname{orb}(T, x) := \{T^n x : n \geq 0\}$$

is dense in X. The set $\operatorname{orb}(T, x)$ is called the orbit of T under x.

(ii) T is called chaotic if T is hyperyclic and the set of its periodic points $P(T)$ is dense in X. Recall that a vector $x \in X$ is periodic under T if there exists a positive integer $n \geq 1$, such that $T^n x = x$.

Definition 1.1.2. Let X be a Banach space and let $T \in L(X)$.

(i) The operator T is called topologically transitive if for every pair of non-empty open sets $U, V \subseteq X$ there exists a positive integer n such that

$$T^n(U) \cap V \neq \emptyset.$$

(ii) The operator T is called topologically mixing if for every pair of non-empty open sets $U, V \subseteq X$ there exists a positive integer N such that

$$T^n(U) \cap V \neq \emptyset \text{ for every } n \geq N.$$

The next theorem due to Birkhoff states in short that Definition 1.1.1 and Definition 1.1.2 are equivalent.

Theorem 1.1.3. ***(Birkhoff transitivity theorem, [20], p. 10)*** *Let X be a separable Banach space. Then the following assertions are equivalent.*

(i) *T is topologically transitive.*

(ii) *There exists some $x \in X$ such that $orb(T, x)$ is dense in X.*
If one of these conditions is satisfied then the set of points in X with dense orbit is a dense G_δ-set.

Recall that a set $A \subseteq X$ is called a G_δ-set if A can be written as a countable intersection of open sets.
The next criterions are fundamental in linear dynamics to determine several important classes of hyperyclic operators.

Theorem 1.1.4. ***(Hypercyclicity Criterion, [24])*** *Let X be a Banach space and $T \in L(X)$. Assume that there are dense sets $X_0, Y_0 \subseteq X$, a strictly increasing sequence $(n_k)_k$ of positive integers and maps $S_{n_k} : Y_0 \to X$, $k \geq 1$ such that the following conditions are satisfied for any $x \in X_0$ and $y \in Y_0$:*

(i) $T^{n_k} x \to 0$.

(ii) $S_{n_k} y \to 0$.

(iii) $T^{n_k} S_{n_k} y \to y$.

Then T is hypercyclic.

Theorem 1.1.5. ***(Godefroy-Shapiro Criterion, [20], p. 69)*** *Let X be a Banach-space and $T \in L(X)$. Suppose that the subspaces*

$$X_0 := span\ \{x \in X \mid Tx = \lambda x \text{ for some } \lambda \in \mathbb{C} \text{ with } |\lambda| < 1\}$$
$$Y_0 := span\ \{x \in X \mid Tx = \lambda x \text{ for some } \lambda \in \mathbb{C} \text{ with } |\lambda| > 1\}$$

are dense in X. Then T is mixing, and in particular hypercyclic. If moreover the subspace

$$Z_0 := span\ \{x \in X \mid Tx = e^{\alpha \pi i} x \text{ for some } \alpha \in \mathbb{Q}\}$$

is dense in X, then T is chaotic.

As we mentioned already the above theorems are powerful tools to determine whether certain classes, like the weighted backward shifts, are hypercyclic.

Example 1.1.6. ([35], [20] p. 97) Let $T := B_w : l^p \to l^p$, $1 \leq p < \infty$, be the operator defined by $B_w(x_1, x_2, \ldots) = (w_1 x_2, w_2 x_3, \ldots)$, where $w = (w_n)_n$ is a positive and bounded weight-sequence.

Then T is hyperyclic if and only if

$$\sup_{n \geq 1} \prod_{\nu=1}^{n} w_\nu = \infty.$$

T is mixing if and only if

$$\lim_{n \to \infty} \prod_{\nu=1}^{n} w_\nu = \infty.$$

1.2 Properties of J-class operators

In this section we present the most important properties and facts about J-class operators which are the most work by G. Costakis and A. Manoussos developed in [14].

Definition 1.2.1. Let X be a Banach space and $T \in L(X)$.

(i) For every $x \in X$ the set

$$\begin{aligned} J_T(x) := \{ y \in X : \ & \text{there exists a strictly increasing sequence of positive} \\ & \text{integers } (k_n)_n \text{ and a sequence } (x_n)_n \text{ in } X \text{ such that } x_n \to x \\ & \text{and } T^{k_n} x_n \to y \} \end{aligned}$$

denotes the J-set of T under x.

(ii) For every $x \in X$ the set

$$\begin{aligned} J_T^{mix}(x) := \{ y \in X : & \text{ there exists a sequence } (x_n)_n \text{ in } X \text{ such that } x_n \to x \\ & \text{and } T^n x_n \to y \} \end{aligned}$$

denotes the J^{mix}-set of T under x.

Definition 1.2.2. Let X be a Banach space and $T \in L(X)$.

(i) T is called J-class or J-class operator if there exists some $x \in X \backslash \{0\}$ such that

$$J_T(x) = X.$$

(ii) T is called J^{mix}-class or J^{mix}-class operator if there exists some $x \in X \backslash \{0\}$ such that

$$J_T^{mix}(x) = X.$$

The set of all x such that $J_T(x) = X$ ($J_T^{mix}(x) = X$) is denoted by A_T (A_T^{mix}) and its elements will be called J-vectors (J^{mix}-vectors).

Remark 1.2.3. (i) An equivalent definition of the J-set is the following.

$$J_T(x) = \{y \in X : \text{for every neighborhood } U \text{ of } x \text{ and every non-empty open } V \subseteq X \text{ there exists a positive integer } n \text{ such that } T^n(U) \cap V \neq \emptyset\}$$

(ii) The reason why we require in Definition 1.2.2 that $x \neq 0$ holds, is to omit certain trivialities. For example it is easy to see that $J_{\lambda I}(0) = X$ where I is the identity operator on X and $|\lambda| > 1$.

Proposition 1.2.4. *([14]) Let X be a Banach space and $T \in L(X)$.*

(i) *For every $x \in X$ the sets $J_T(x)$, $J_T^{mix}(x)$ are closed and T-invariant. Moreover the set $J_T^{mix}(x)$ is convex and in particular $J_T^{mix}(0)$ is a closed linear subspace of X.*

(ii) *The set A_T is a closed and T-invariant subset of X and A_T^{mix} is a closed T-invariant subspace of X.*

The next two theorems show the relation between topologically transitive (topologically mixing) and J-class (J_{mix}-class) operators.

Theorem 1.2.5. *[14] Let X be a Banach space and $T \in L(X)$. The following assertions are equivalent.*

(i) *T is topologically transitive.*

(ii) *$J_T(x) = X$ holds for every $x \in X$.*

(iii) *$A_T = X$.*

(iv) *A_T is dense in X.*

(v) *A_T has non-empty interior.*

Theorem 1.2.6. *([14]) Let X be a Banach space and $T \in L(X)$. The following assertions are equivalent.*

(i) *T is topologically mixing.*

(ii) *$J_T^{mix}(x) = X$ holds for every $x \in X$.*

(iii) *$A_T^{mix} = X$.*

(iv) *A_T^{mix} is dense in X.*

(v) *A_T^{mix} has non-empty interior.*

The next example can be found in [14]. To get more familiar with J-class operators we will state the proof.

Example 1.2.7. ([14]) Let $X = l^\infty$ and consider the operator $\lambda B : l^\infty \to l^\infty$ defined by $\lambda B(x_1, x_2, \ldots) = (\lambda x_2, \lambda x_3, \ldots)$ and $|\lambda| > 1$. Then λB is J-class and

$$A_{\lambda B}^{mix} = A_{\lambda B} = \{x \in l^\infty : J_{\lambda B}(x) = l^\infty\} = c_0,$$

where

$$c_0 = \{x = (x_n)_{n\in\mathbb{N}} \in l^\infty : \lim_{n\to\infty} x_n = 0\}.$$

Proof. We will first show that there exists a vector $x \in l^\infty$ with finite support such that $J_{\lambda B}(x) = l^\infty$. In fact, $x = e_1 = (1, 0, 0, \ldots)$ has this property: Consider any $y = (y_1, y_2, \ldots) \in l^\infty$ and define

$$x_n = (1, 0, \ldots, 0, \frac{y_1}{\lambda^n}, \frac{y_2}{\lambda^n}, \ldots)$$

where 0's are taken up to the n-th coordinate. It is $x_n \in l^\infty$, $x_n \to e_1$ and

$$(\lambda B)^n x_n = y \text{ for all } n.$$

Therefore

$$J_{\lambda B}^{mix}(e_1) = l^\infty.$$

The closure of the set of all vectors with finite support is c_0. An application of Proposition 1.2.4 yields that

$$c_0 \subseteq A_{\lambda B}^{mix} \subseteq A_{\lambda B}.$$

For the converse, assume that $x \in l^\infty$ is a non-zero J-vector for λB. Then there exists a sequence $y_n = (y_{n1}, y_{n2}, \ldots)$ and a strictly increasing sequence of positive integers k_n such that $y_n \to x$ and $(\lambda B)^{k_n} y_n \to 0$. Let $\varepsilon > 0$. Then there exists a positive integer n_0 such that

$$\|y_n - x\| < \varepsilon$$

and

$$\left\|(\lambda B)^{k_n} y_n\right\| = |\lambda|^{k_n} \sup_{m\geq k_n+1} |y_{nm}| < \varepsilon$$

for every $n \geq n_0$. Therefore, for every $m \geq k_{n_0} + 1$ and since $|\lambda| > 1$ it follows

$$|x_m| \leq \|y_{n_0} - x\| + |y_{n_0 m}| < 2\varepsilon.$$

This implies $x \in c_0$. Hence it follows that $A_{\lambda B}^{mix} = A_{\lambda B} = c_0$ holds. □

Proposition 1.2.8. *([14]) Let $T : X \to X$ be an operator on a Banach space X.*

(i) *For every positive integer m we have*

$$J_T(0) = J_{T^m}(0).$$

(ii) *Suppose that z is a non-zero periodic point for T. Then the following assertions are equivalent.*
(a) T is a J-class operator.
(b) $J_T(0) = X$.
(c) $J_T(z) = X$.

(iii) *Suppose that there exist a non-zero vector $z \in X$, a vector $w \in X$ and a sequence $(z_n)_n \subseteq X$ such that $z_n \to z$ and $T^n z_n \to w$, i.e. $J_T^{mix}(z) \neq \emptyset$. Then the following assertions are equivalent.*
(1) T is a J-class operator.
(2) $J_T(0) = X$.
(3) $J_T(z) = X$.
In particular, this statement holds for operators with non-trivial kernel or for operators having at least an eigenvalue with absolute value less than one.

(iv) *Suppose that T is J-class. Then $J_T(0) = X$.*

Moreover all statements are also valid for the J_T^{mix}-set instead of the J_T-set.

The authors of [14] showed that any operator T on a Banach space X which is power bounded (that means $\|T^n\| \leq M$ for all positive integers n and some $M \geq 0$) cannot be J-class. The next proposition is an improvement of the aforementioned statement for the case where the underlying Banach space is non-separable.

Proposition 1.2.9. *Let X be a non-separable Banach space and $U \in L(X)$ be power-bounded. Then for every $S \in L(X)$ with separable range the operator $T := U + S$ is never J-class.*

Proof. First of all observe that for each positive integer n the operator T^n has the representation $T^n = (U+S)^n = U^n + S_n$, where S_n has separable range. To see this, consider

$$\begin{aligned}(U+S)^{n+1} &= (U+S)(U+S)^n \\ &= (U+S)(U^n + S_n) \\ &= U^{n+1} + US_n + SU^n + SS_n.\end{aligned}$$

Then the operator $S_{n+1} := US_n + SU^n + SS_n$ has separable range. Assume now that T is J-class with J-vector $x \neq 0$. Then for every $y \in X$, there exists a strictly increasing sequence $(n_k)_k$ of natural numbers and a sequence $(x_k)_k$ in X with $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} T^{n_k} x_k = y$. In particular we get

$$\begin{aligned}\|U^{n_k} x_k - U^{n_k} x\| &\leq \|U^{n_k}\| \, \|x_k - x\| \\ &\leq M \, \|x_k - x\| \to 0.\end{aligned}$$

Thus for $\varepsilon > 0$ and k large enough it follows

$$\begin{aligned}T^{n_k} x_k &= U^{n_k} x_k - U^{n_k} x + U^{n_k} x + S_{n_k} x_k \\ &\in B_\varepsilon(0) + \overline{\operatorname{orb}(U,x) + \bigcup_{n\in\mathbb{N}} S_n(X)}.\end{aligned}$$

Since ε was arbitrary, we get

$$X = J_T(x) \subseteq \overline{\operatorname{orb}(U,x) + \bigcup_{n\in\mathbb{N}} S_n(X)} =: Y.$$

In particular, Y has non-empty interior, but this is not possible since Y is separable. This gives the desired contradiction. $\square$

The next easy observation will be very useful in many situations.

Proposition 1.2.10. *Let X, Y be Banach spaces and assume there exists a linear and continuous isomorphism $J : Y \to X$. Then the operator $S := J^{-1}TJ$ is J-class (J_{mix}-class) if T is J-class (J_{mix}-class) .*

Proof. We will show the statement for the case where T is J-class (the proof for J_{mix}-class is similar). Let T be J-class with J-vector $x \neq 0$ and consider any $y \in Y$. Then $z := J^{-1}x \neq 0$ is a J-vector for S. Indeed for $u := Jy$ there exists a sequence $(x_k)_k$ with $x_k \to x$ and a strictly increasing sequence of positive integers $(n_k)_k$ such that $T^{n_k}x_k \to u$. Therefore we get with $z_k := J^{-1}x_k$ that $z_k \to z$ and

$$S^{n_k}z_k = J^{-1}T^{n_k}Jz_k = J^{-1}T^{n_k}x_k \to J^{-1}u = y.$$

This shows $J_S(z) = Y$ and the proof is finished. □

The next well known theorem will be used frequently in the following chapters and can also be found in your favorite functional analysis book, for instance [1].

Theorem 1.2.11. ***(Riesz decomposition theorem, [1], p. 262)*** *Let X be a Banach space and $T \in L(X)$. Suppose $\sigma(T)$ can be written as a finite union of disjoint and closed sets $\sigma_1, \ldots, \sigma_n$. Then there exist T-invariant subspaces $M_1, \ldots, M_n$ such that $X = M_1 \oplus \ldots \oplus M_n$ and $\sigma(T|_{M_i}) = \sigma_i$ for $i = 1, \ldots, n$.*

The next lemma is one of the key tools for the following chapters. The first statement can be found in [14] and the second statement is improved by the author where we assume that $J_T(x)$ has non-empty interior and can be also found in [31].

Lemma 1.2.12. ***(Spectral lemma)*** *Let X be a Banach space and consider $T \in L(X)$.*

(i) *Assume there exists a vector $x \in X$ such that $J_T(x)$ has non-empty interior. Then for every $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$ the operator $T - \lambda I$ has dense range. This is equivalent to say that*

$$\sigma_p(T^*) \cap \overline{\mathbb{D}} = \emptyset. \tag{1}$$

(ii) *Assume there exists a non-zero vector $x \in X$ such that $J_T(x)$ has non-empty interior. Then*

$$\sigma(T) \cap \partial\mathbb{D} \neq \emptyset. \tag{2}$$

Proof. **(i):** Assume there exists some $|\lambda| \leq 1$ such that $T - \lambda I$ has non dense range. By a well known application of the Hahn-Banach-Theorem there exist some $x^* \in X^*$ with $x^* \neq 0$ such that

$$x^*(Tx - \lambda x) = 0$$

for every $x \in X$. From this we get

$$x^*(T^n x) = \lambda^n x^*(x) \tag{$*$}$$

for every $x \in X$ and $n \in \mathbb{N}$. Take any $y \in J_T(x)$ with the corresponding sequences $(x_k)_k$ and $(n_k)_k$ such that $T^{n_k}x_k \to y$. Then by continuity of x^* we get with $(*)$

$$x^*(y) = \lim_{k\to\infty} x^*(T^{n_k}x_k) = \lim_{k\to\infty} \lambda^{n_k}x^*(x_k).$$

For the case that $|\lambda| < 1$ we get $x^*(y) = 0$ for $y \in J_T(x)$ and therefore $x^* = 0$ since $J_T(x)$ has non-empty interior. For the case $|\lambda| = 1$ it is easy to see that $V := \{x^*(y) \mid y \in J_T(x)\}$ is a subset of a suitable circle in the complex plane, which is also not possible since V has non-empty interior.
(ii): Suppose $\sigma(T) \cap \partial\mathbb{D} = \emptyset$. We decompose $\sigma(T)$ in

$$\sigma_1 := \{\lambda \in \sigma : |\lambda| > 1\} \text{ and } \sigma_2 := \{\lambda \in \sigma : |\lambda| < 1\}.$$

Then σ_1 and σ_2 are disjoint and closed. By the Riesz decomposition theorem we can decompose $X = M_1 \oplus M_2$, where M_1 and M_2 are closed and T-invariant subspaces and $\sigma_1 = \sigma(T_{|M_1})$, $\sigma_2 = \sigma(T_{|M_2})$. Assume now that there exists a non-zero vector $x \in X$, such that $J_T(x)$ has non-empty interior. We can write $x = x_1 + x_2$, with $x_1 \in M_1$ and $x_2 \in M_2$. Since $x \neq 0$, either x_1 or x_2 is not equal to zero.
Consider the projection $P_1 : X \to M_1$ along M_2 onto M_1. Since

$$J_T(x) \subseteq J_{T|_{M_1}}(x_1) + J_{T|_{M_2}}(x_2)$$

it follows that $P_1(J_T(x)) \subseteq J_{T|_{M_1}}(x_1)$. By the open mapping theorem we get that $P_1(J_T(x))$ has non-empty interior and hence $(J_{T|_{M_1}}(x_1))^\circ \neq \emptyset$. From the spectral radius formula we obtain that

$$\|T_1^n\widetilde{x}\| \leq a^n \,\|\widetilde{x}\|$$

for some $a \in \mathbb{R}$ with $0 \leq a < 1$ and for all $\widetilde{x} \in M_1$. This implies that $J_{T|_{M_1}}(x_1) = \{0\}$ and therefore $M_1 = \{0\}$. So we get $x_1 = 0$. Again from the spectral radius formula we know that

$$\|T_1^n(\hat{x})\| \geq A^n \,\|\hat{x}\|$$

for an $A > 1$ and all $\hat{x} \in M_2$. This implies $x_2 = 0$, which is a contradiction to our assumption that $x \neq 0$. □

Corollary 1.2.13. *Let X be a finite dimensional Banach space. Then there is no $T \in L(X)$ for which there exists a non-zero vector $x \in X$ such that $J_T(x)$ has non-empty interior for some $x \in X\backslash\{0\}$. In particular there are no J-class operators on finite-dimensional Banach spaces and hence no topologically transitive operators.*

Proof. Assume that there exists some operator T and a vector $x \neq 0$ such that $J_T(x)$ has non-empty interior. Then by the Spectral lemma and the fact that the spectrum consists entirely of eigenvalues we get

$$\emptyset = \overline{\mathbb{D}} \cap \sigma_p(T^*) = \overline{\mathbb{D}} \cap \sigma(T^*) = \overline{\mathbb{D}} \cap \sigma(T)$$

which is a contradiction to (ii) of the previous lemma. □

If X is a real Banach space and $T \in L(X)$ the first statement of the Spectral lemma holds for the complexified operator $T_{\mathbb{C}}$.

Proposition 1.2.14. *Let X be a real Banach space and $T \in L(X)$. If $J_T(x)$ has non-empty interior for some x, then $\sigma_p(T_{\mathbb{C}}^*) \cap \overline{\mathbb{D}} = \emptyset$.*

Proof. Suppose $\sigma_p(T_{\mathbb{C}}^*) \cap \overline{\mathbb{D}} \neq \emptyset$ and choose $|\lambda| \leq 1$ from this set. Then there exist a non-zero functional ϕ on $X_{\mathbb{C}}$, such that $\phi(T_{\mathbb{C}}(x+iy)) = \lambda\phi(x+iy)$ for all $x+iy \in X_{\mathbb{C}}$. Now for any open subset $V \subseteq X$

$$\phi|_{V+i\{0\}} \neq 0 \text{ or } \phi|_{\{0\}+iV} \neq 0.$$

Otherwise $\phi|_{V+iV} \equiv 0$ would hold on an open subset of $X_{\mathbb{C}}$, which is not possible, since $\phi \neq 0$. So we can assume that

$$\phi|_{(J_T(x))^\circ+i\{0\}} \neq 0$$

Choose any $y \in (J_T(x))^\circ$ with $\phi(y+i0) \neq 0$. Then there exist sequences $(x_k)_k$ and $(n_k)_k$ such that $x_k \to x$ and $T^{n_k}x_k \to y$.

Now assume $|\lambda| < 1$. Then

$$\begin{aligned}\phi(y+i0) &= \lim_{k\to\infty} \phi(T^{n_k}x_k + i0)\\ &= \lim_{k\to\infty} \lambda^{n_k}\phi(x_k+i0)\\ &= 0,\end{aligned}$$

which is a contradiction.
Assume now $|\lambda| = 1$. We choose $y \in J_T(x)$ such that $|\phi(y+i0)| \neq |\phi(x+i0)|$. It follows

$$\begin{aligned}\phi(y+i0) &= \lim_{k\to\infty} \phi(T^{n_k}x_k + i0)\\ &= \lim_{k\to\infty} \lambda^{n_k}\phi(x_k+i0).\end{aligned}$$

But $|\phi(y+i0)| = \left|\lim_{k\to\infty} \lambda^{n_k}\phi(x_k+i0)\right| = |\phi(x+i0)|$ and this again is a contradiction. □

Remark 1.2.15. (i) In [14] the question was raised whether any operator $T \in L(X)$ for which $(J_T(x))^\circ \neq \emptyset$ with some $x \neq 0$ is already J-class. M. R. Azimi and V. Müller constructed a counterexample to this question on l^1 which can be found in [7]. The latter implies that, if the operator T satisfies the statement of Lemma 1.2.12, then this does not imply that T is J-class.

(ii) Lemma 1.2.12 is clearly valid for J-class operators and therefore it is natural to ask the converse: Let X be a Banach-space and let $T \in L(X)$ satisfy (1) and (2) as in the above lemma. Under which additional conditions does it follow that T is a J-class operator? Conditions on the space X and on T will give a positive answer as we will see in Chapter 3.

1.3 Examples of J-class operators

In the previous section we have seen that the weighted backward shift λB on l^∞ is J-class for $|\lambda| > 1$. Considering λB on the spaces $c_0, l^p, 1 \leq p < \infty$ one has even that λB is hypercyclic for every $|\lambda| > 1$, (see [20], p. 40 or Example 1.1.6). In this section we consider the operator $T = I + \lambda B$. Our aim is to determine those λ for which T is J-class. The reader might ask why we are taking this operator into consideration and not something else. There are two reasons: First, T plays an important role for the existence of hypercyclic operators on an arbitrary Banach space (see [4] and [12]). Secondly, T has a simple form, which nevertheless is complicated enough to determine the set A_T. We start with an elementary lemma, where we calculate the preimage of a vector $x = (x_n)_n \in l^\infty$ under $I + \lambda B : l^\infty \to l^\infty$ for $\lambda \in \mathbb{C}$.

Lemma 1.3.1. *Let $B : l^\infty \to l^\infty$ be the unilateral backward shift where l^∞ is the Banach space of bounded sequences over $\mathbb{C}$ and let $y = (y_n)_n$ be a vector in l^∞. If $x = (x_n)_n$ is a vector, not necessarily in l^∞, such that $(I + \lambda B)x = y$, then*

$$x_n = \frac{-1}{(-\lambda)^n} \sum_{k=1}^{n-1} (-\lambda)^k y_k + \frac{x_1}{(-\lambda)^{n-1}} \tag{3.1}$$

holds for every $n \in \mathbb{N}$.

Proof. Notice that $x_n + \lambda x_{n+1} = y_n$, for every $n \in \mathbb{N}$. It is easy to show that the kernel of $I + \lambda B$ consists of all vectors $w = (w_n)_n$ such that $w_n = \frac{w_1}{(-\lambda)^{n-1}}$ holds for every $n \in \mathbb{N}$. Let $z = (z_n)_n$ be such that $z_n = \frac{-1}{(-\lambda)^n} \sum_{k=1}^{n-1} (-\lambda)^k y_k$. With straightforward calculations, it is easy to see that $(I + \lambda B)z = y$. Hence, if $x = (x_n)_n$ is such that $(I + \lambda B)x = y$, then $x_n = \frac{-1}{(-\lambda)^n} \sum_{k=1}^{n-1} (-\lambda)^k y_k + \frac{x_1}{(-\lambda)^{n-1}}$, for every $n \in \mathbb{N}$. $\square$

The next proposition was proposed to us by A. Manoussos.

Proposition 1.3.2. *Let $|\lambda| = 1$. Then the operator $I + \lambda B : l^\infty \to l^\infty$ does not have dense range.*

Proof. Consider the vector $y = (y_n)_n \in l^\infty$ with $y_n = (-\lambda)^{-n}$ for $n \in \mathbb{N}$. We will show that the open ball $B(y, 1/2)$ centered at y with radius $\frac{1}{2}$ does not intersect the range of $I + \lambda B$. Assume the contrary, i.e. there is a vector $w = (w_n)_n \in B(y, 1/2)$ and a vector $x \in l^\infty$ such that $(I + \lambda B)x = w$. Hence, by Lemma 1.3.1 we get that

$$x_n = \frac{-1}{(-\lambda)^n} \sum_{k=1}^{n-1} (-\lambda)^k w_k + \frac{x_1}{(-\lambda)^{n-1}} \tag{3.2}$$

holds for every $n \in \mathbb{N}$. Since $w \in U$ we have $w_k = y_k + \varepsilon_k$ with $|\varepsilon_k| < \frac{1}{2}$ for every $k \in \mathbb{N}$. By (3.2) we get

$$\left| \sum_{k=1}^{n-1} (-\lambda)^k (y_k + \varepsilon_k) - \lambda x_1 \right| \leq \|x\|_\infty$$

for every $n \in \mathbb{N}$. Since $y_k = (-\lambda)^{-k}$ it follows

$$\left| n - 1 + \sum_{k=1}^{n-1} (-\lambda)^k \varepsilon_k \right| \leq \|x\|_\infty + |x_1| \leq 2\,\|x\|_\infty.$$

Note that $|\sum_{k=1}^{n-1} (-\lambda)^k \varepsilon_k| \leq \frac{n-1}{2}$. Hence, from the triangle inequality we get

$$n - 1 - \frac{(n-1)}{2} \leq \left| n - 1 + \sum_{k=1}^{n-1} (-\lambda)^k \varepsilon_k \right| \leq 2\,\|x\|_\infty.$$

Therefore, $\frac{n-1}{2} \leq 2\,\|x\|_\infty$ holds for every $n \in \mathbb{N}$ which is a contradiction. □

Theorem 1.3.3. *Let $B : l^\infty \to l^\infty$ be the unilateral backward shift. Then*

(i) For $|\lambda| \leq 2$ the operator $T := I + \lambda B$ is not J-class. In particular, the set $J_T(x)$ has empty interior for every $x \in l^\infty$.

(ii) For $|\lambda| > 2$ the operator $T = I + \lambda B$ is a J^{mix}-class operator.

Proof. If $\lambda = 0$ then, clearly, the J-sets of the identity operator are singletons. So, from now on, we may assume that $\lambda \neq 0$.

(i) Let $0 < |\lambda| \leq 2$. We can rewrite the previous inequality as $||\lambda| - 1| \leq 1$. Assume that $J_T(x)$ has non-empty interior for some $x \in X$. Then, by Lemma 1.2.12, the operator $(I + \lambda B) - (1 - |\lambda|)I$ has dense range. But,

$$(I + \lambda B) - (1 - |\lambda|)I = |\lambda|(I + \frac{\lambda}{|\lambda|}B)$$

then implies that $I + \frac{\lambda}{|\lambda|}B$ has dense range which is a contradiction by Proposition 1.3.2.

(ii) Let $\lambda \in \mathbb{C}$ with $|\lambda| > 2$. Fix a vector $y = (y_n)_n \in l^\infty$ and let $x = (x_n)_n \in l^\infty$ be such that $(I + \lambda B)x = y$. Then, by Lemma 1.3.1,

$$x_n = \frac{-1}{(-\lambda)^n} \sum_{k=1}^{n-1} (-\lambda)^k y_k + \frac{x_1}{(-\lambda)^{n-1}}$$

holds for every $n \in \mathbb{N}$. If we take $x_1 = 0$ then it is easy to see that $\|x\|_\infty \leq \frac{\|y\|_\infty}{|\lambda|-1}$ holds. This actually shows also that $x \in l^\infty$. Put $w_1 := x$. If we repeat the same argument for $y := w_1$ we can find a vector $w_2 \in l^\infty(\mathbb{N})$ such that

$$(I + \lambda B)w_2 = w_1 \text{ and } \|w_2\|_\infty \leq \frac{\|w_1\|_\infty}{|\lambda| - 1} \leq \frac{\|y\|_\infty}{(|\lambda| - 1)^2}.$$

Proceeding in a similar way and given a positive integer n we can find a vector $w_n \in l^\infty$ such that $(I + \lambda B)^n w_n = y$ and $\|w_n\|_\infty \leq \frac{\|y\|_\infty}{(|\lambda|-1)^n}$. Since $|\lambda| > 2$ it follows $w_n \to 0$, hence $J^{mix}(0) = l^\infty$. Note that the kernel of $I + \lambda B$ in l^∞ is non-empty. Take any vector x in this kernel, then we get that $J^{mix}(x) = l^\infty$ holds by Proposition 1.2.8. □

Our next purpose is to determine A_T. As we have seen in Example 1.2.7 $A_{\lambda B} = c_0$ and therefore one expects that $A_T = c_0$. This is indeed true, but requires another technique as in Example 1.2.7. If one wants to imitate the procedure of Example 1.2.7, then one has to assume that $x = (x_k)_k \in l^\infty$ is a J-class vector for $T = I + \lambda B$. For $y = 0$ there exists some sequence $(x^{(n)})_n$ in l^∞ such that

$$\left\|x^{(n)} - x\right\|_\infty = \sup_{k\in\mathbb{N}} \left|x_k^{(n)} - x_k\right| \to 0 \quad \text{for} \quad n \to \infty$$

and

$$(*) \qquad (I+\lambda B)^n x^{(n)} = \left(\sum_{k=0}^{n} \binom{n}{k} x_{k+1}^{(n)},\ \sum_{k=0}^{n} \binom{n}{k} x_{k+2}^{(n)},\ \sum_{k=0}^{n} \binom{n}{k} x_{k+3}^{(n)}, \dots \right) \to 0$$

for $n \to \infty$. The problem occurs at that point when the values $x_k^{(n)}$ get negative and then it is difficult to argue that if the entries of $(*)$ are tending to zero, then this yields that $x \in c_0$.

Definition 1.3.4. Let X be a Banach space and $T \in L(X)$. Let M be a closed T-invariant subspace. Then we define the operator

$$\widehat{T} := \widehat{T}_M : X/M \to X/M \quad \text{by} \quad \widehat{T}\,[x]_M := [Tx]_M\,.$$

The operator $\widehat{T}$ is well-defined, linear and continuous in the induced quotient-topology. Moreover, we have $\left\|\widehat{T}\right\| \le \|T\|$.

In the sequel we will just write $[x]$ instead of $[x]_M$ if there is no risk of confusion.

Remark 1.3.5. If M is a closed subspace of X, we consider all $T \in L(X)$, for which M is T-invariant. Then the operators $\widehat{T}$ form a closed subspace of $L(X/M)$.

The map $T \to \widehat{T}$ is linear and continuous with respect to the operatornorms. In particular, if we consider any holomorphic function f in a neighborhood of $\sigma(T)$, then we have

$$\widehat{f(T)} = f(\widehat{T}).$$

Lemma 1.3.6. (i) *Let X be a Banach space and $T \in L(X)$. Assume that T is J-class and that there exists a T-invariant closed subspace $M \subseteq A_T$ such that $A_T \backslash M \neq \emptyset$. Then $\widehat{T}$ is J-class.*

(ii) *Furthermore if T is topologically transitive, then for every proper T-invariant subspace $M \subseteq X$ the induced operator $\widehat{T} : X/M \to X/M$ is also topologically transitive.*

Proof. (i) We choose any vector $x \in A_T \backslash M$, then $[x] \neq 0$. Now consider any $[y] \in X/M$. Then there exists a strictly increasing sequence $(n_k)_k$ of positive integers and a sequence $(x_k)_k$ with $x_k \to x$, such that $T^{n_k} x_k \to y$. Hence $[x_k] \to [x]$ and $\widehat{T}^{n_k}\,[x_k] = [T^{n_k} x_k] \to [y]$ as $k \to \infty$ and this shows that $\widehat{T}$ is J-class.
(ii) Assume now that T is topologically transitive. Then it follows that $X = A_T$ by

Theorem 1.2.5. Consider now any proper T-invariant and closed subspace $M \subseteq X$. Then $A_T \backslash M = X \backslash M \neq \emptyset$. From the above, we know that the set $\pi(X \backslash M)$ consists entirely of J-vectors for T_M, where $\pi : X \to X/M$ is the canonical map. Therefore $\pi(X \backslash M) \subseteq A_{T_M}$ holds. Since π is surjective, by the open mapping theorem we conclude that $\pi(X \backslash M)$ has an interior point, since $X \backslash M$ is an open set. Therefore A_{T_M} has an interior point and hence T_M is topologically transitive again by Theorem 1.2.5. This completes the proof. □

Corollary 1.3.7. *Let X be a Banach space and $T \in L(X)$ a topologically transitive operator. Then T does not admit any proper T-invariant closed subspace $M \subseteq X$ with $\dim X/M < \infty$.*

Proof. Suppose T admits a T-invariant subspace M with $\dim X/M < \infty$. Then $T_M : X/M \to X/M$ is topologically transitive. But this is not possible, since operators on finite dimensional spaces can never be topologically transitive by Corollary 1.2.13. □

We will determine at this point the spectrum of the operator $\widehat{B} : l^\infty/c_0 \to l^\infty/c_0$, where B is the backward shift on l^∞.

Proposition 1.3.8. *Let B be the backward shift on l^∞ and consider $\widehat{B} : l^\infty/c_0 \to l^\infty/c_0$. Then $\sigma(\widehat{B}) = \partial\mathbb{D}$.*

Proof. We will show first that $\partial\mathbb{D} \subseteq \sigma_p(\widehat{B}) \subseteq \sigma(\widehat{B})$. Choose any $\mu \in \partial\mathbb{D}$. Then $\mu = e^{i\theta}$ for some $\theta \in \mathbb{R}$. It is easy to see that the vector $[(x_n)_n]$ with $x_1 = 1$ and $x_{n+1} = e^{i\theta n}$ is the corresponding eigenvector. So we get $\partial\mathbb{D} \subseteq \sigma_p(\widehat{B})$. Now we take any $\mu \in \mathbb{C}$ with $|\mu| < 1$. Then $\widehat{B} - \mu I$ is injective. To see this, assume that there exists some vector $[x] \neq 0$ with

$$(*) \qquad (\widehat{B} - \mu I)\,[x] = 0.$$

Let $x = (x_1, x_2, x_3, ...)$. Then from $(*)$ it follows that

$$(**) \qquad (z_n)_n := (x_{n+1} - \mu x_n)_n \in c_0$$

holds. Let $(n_k)_k$ be a strictly increasing sequence of positive integers, such that

$$\delta := \lim_{k\to\infty} |x_{n_k+1}| = \limsup_{n\to\infty} |x_n|.$$

Then $\delta' := \limsup_{k\to\infty} |x_{n_k}| \leq \delta$. From $(**)$ it follows that there exists a null-sequence ε_n with $x_{n+1} - \mu x_n = \varepsilon_n$ for each $n \in \mathbb{N}$. Passing to the subsequence $(n_k)_k$, we obtain $|x_{n_k+1}| \leq |\mu||x_{n_k}| + |\varepsilon_{n_k}|$ and therefore

$$\delta = \limsup_{n\to\infty} |x_n| = \lim_{k\to\infty} |x_{n_k+1}| \leq |\mu| \limsup_{k\to\infty} |x_{n_k}| = |\mu|\delta' \leq \delta'.$$

Hence $\delta = 0$, and $(x_n)_n$ is a null sequence which implies $[x] = 0$. Now for every $|\mu| < 1$, $\widehat{B} - \mu I$ is surjective. Indeed take for any $[y]$ with $y = (y_1, y_2, ...)$ the vector $[x]$, where $x = (x_n)_n$ with $x_1 = 0$ and $x_{n+1} = \sum_{k=0}^{n-1} y_{n-k}\mu^k$. Then we get $(\widehat{B} - \mu I)\,[x] = [y]$. Finally we obtain $\mathbb{D} \cap \sigma(\widehat{B}) = \emptyset$ and since $\left\|\widehat{B}\right\| \leq \|B\| = 1$, we have $\sigma(\widehat{B}) = \partial\mathbb{D}$. □

Now we are prepared to prove the next theorem.

Theorem 1.3.9. *Consider the operator $T := I + \lambda B$ with $|\lambda| > 2$. Then*

$$A_T^{mix} = A_T = c_0.$$

Proof. We will first show that $c_0 \subseteq A_T^{mix}$. Since the restricted operator $T|_{c_0} : c_0 \to c_0$ is mixing ([20], p. 123), we get by Theorem 1.2.6 that every $x \in c_0$ is a J_{mix}-vector of $T|_{c_0}$. Hence by Proposition 1.2.8 every $x \in c_0$ is a J_{mix}-vector of T, since $J_T^{mix}(0) = l^\infty$. Now assume $A_T \backslash c_0 \neq \emptyset$. Then by Lemma 1.3.6, we get that $\widehat{T}$ is J-class. But $\sigma(\widehat{T}) = 1 + \sigma(\lambda \widehat{B}) = 1 + \{\lambda e^{i\theta} | \theta \in \mathbb{R}\}$. This set does not intersect with the unit circle, since $|\lambda| > 2$ and so $\widehat{T}$ is not J-class by Lemma 1.2.12, which is a contradiction. It follows that $A_T \subseteq c_0$ and since $A_T^{mix} \subseteq A_T$ we obtain the desired statement. □

It is natural to generalize the problem whether $T = I + \lambda B$ is J-class or not. For example, if we consider any polynomial p we might ask the following question: Is it possible to give a characterization for $p(B)$ being J-class? In Chapter 3 we will give a complete characterization of this problem using techniques from Spectral theory, Tauberian operator theory and some specific properties of the geometry of l^∞ and l^1.

Remark 1.3.10. If we consider the operator $I + \lambda B$ with $|\lambda| < 1$ then $(I + \lambda B)^{-1}$ exists and we have $\|(I + \lambda B)^{-1}\| \leq \frac{1}{1-|\lambda|}$. Then the solution $x \in l^\infty$ for the equation $(I + \lambda B)x = y$ satisfies the estimate

$$\|x\|_\infty = \left\|(I + \lambda B)^{-1} y\right\|_\infty \leq \frac{1}{1 - |\lambda|} \|y\|_\infty .$$

If $|\lambda| > 1$ one can find a solution x with

$$\|x\|_\infty \leq \frac{1}{|\lambda| - 1} \|y\|_\infty$$

by repeating the arguments of the proof of Theorem 1.3.3. Finally we find for $|\lambda| \neq 1$ and $y \in l^\infty$ a solution $x \in l^\infty$ of $(I + \lambda B)x = y$ with

$$\|x\|_\infty \leq \frac{1}{||\lambda| - 1|} \|y\|_\infty .$$

Theorem 1.3.11. *Consider the backward shift B on l^∞ and let*

$$p(z) = a(z - z_1) \cdot \ldots \cdot (z - z_n)$$

be a polynomial whose zeros $z_1, z_2 \ldots, z_n$ are not equal to zero and satisfy

$$|a| \prod_{k=1}^{n} ||z_k| - 1| > 1.$$

Then there exists a positive number $q < 1$, such that for all $y \in l^\infty$ there exist a solution $x \in l^\infty$ of the equation $p(B)x = y$ with $\|x\| \leq q \|y\|$.

Proof. We write

$$p(z) = (-1)^n a \cdot z_1 \cdot \ldots \cdot z_n \cdot \left(1 - \frac{z}{z_1}\right) \cdot \ldots \cdot \left(1 - \frac{z}{z_n}\right)$$

and therefore

$$p(B) = (-1)^n a \cdot z_1 \cdot \ldots \cdot z_n \left(I - \frac{1}{z_1} B\right) \circ \ldots \circ \left(I - \frac{1}{z_n} B\right).$$

Consider the equation $p(B)x = y$, which is equivalent to

$$\left(I - \frac{1}{z_1} B\right) \circ \left(I - \frac{1}{z_2} B\right) \circ \ldots \circ \left(I - \frac{1}{z_n} B\right) x = (-1)^n \frac{1}{a \prod_{k=1}^n z_k} y =: \tilde{y}.$$

Define $\lambda_k := \frac{1}{|z_k|}$. So we get

$$(I - \lambda_1 B) \circ (I - \lambda_2 I) \circ \ldots \circ (I - \lambda_n B)\, x = \tilde{y}.$$

By Remark 1.3.10 there exists a solution $x_1 \in l^\infty$ of $(I - \lambda_1 B)x_1 = \tilde{y}$ with $\|x_1\| \leq \frac{1}{||\lambda_1| - 1|} \|\tilde{y}\| = \frac{|z_1|}{|1 - |z_1||} \|\tilde{y}\|$. With the same arguments there exists a solution x_2 of $(I - \lambda_2 B)x_2 = x_1$ with

$$\|x_2\| \leq \frac{1}{||\lambda_2| - 1|} \|x_1\| \leq \frac{|z_2|}{|1 - |z_2||} \frac{|z_1|}{|1 - |z_1||} \|\tilde{y}\|.$$

Repeating this n-times we get a vector $x_n \in l^\infty$ with $(I + \lambda_n B)x_n = x_{n-1}$ and

$$\|x_n\| \leq \frac{|z_n|}{|1 - |z_n||} \cdot \ldots \cdot \frac{|z_2|}{|1 - |z_2||} \frac{|z_1|}{|1 - |z_1||} \|\tilde{y}\|.$$

By the definition of $\tilde{y}$ we get in particular

$$\|x_n\| \leq \frac{1}{|a|} \frac{1}{|1 - |z_n||} \cdot \ldots \cdot \frac{1}{|1 - |z_2||} \frac{1}{|1 - |z_1||} \|y\|$$

and therefore $\|x_n\| \leq q \|y\|$, where

$$q := \frac{1}{|a|} \frac{1}{|1 - |z_n||} \cdot \ldots \cdot \frac{1}{|1 - |z_2||} \frac{1}{|1 - |z_1||} < 1$$

by our assumption.
Finally we have $p(B)x_n = y$ with $\|x_n\| \leq q \|y\|$.

□

Proposition 1.3.12. *Let $p(z) = a \cdot (z - z_1) \cdot (z - z_2) \ldots \cdot (z - z_n)$ be a polynomial with zeros z_k for $k = 1, \ldots, n$. Assume that*

$$|a| \prod_{k=1}^n ||z_k| - 1| > 1$$

holds and $|z_k| < 1$ for at least one k. Then $T := p(B)$ is J-class.

Proof. Take any $y \in l^\infty$ and consider the case where 0 is a zero of p. Without loss of generality assume that $z_1 = ... = z_l = 0$. This means 0 is a zero of p with multiplicity $l \geq 1$. So we can write $p(z) = a(z - z_{l+1}) \cdot \ldots \cdot (z - z_n) \cdot z^l$. Then $q(z) := a(z - z_{l+1}) \cdot \ldots \cdot (z - z_n)$ satisfies the conditions of Theorem 1.3.11 and hence there exists a positive number $q < 1$, $x_1 \in l^\infty$ with $q(B)x_1 = y$ and $\|x_1\| \leq q\,\|y\|$. Define $\tilde{x}_1 := S^l x_1$, where $S : l^\infty \to l^\infty$ is the forward shift, i.e. $S(z_1, z_2, \ldots) = (0, z_1, z_2, \ldots)$. Then

$$p(B)\tilde{x}_1 = q(B)B^l\tilde{x}_1 = q(B)B^lS^lx_1 = q(B)x_1 = y$$

and

$$\|\tilde{x_1}\| = \left\|S^l x_1\right\| = \|x_1\| \leq q\,\|y\|$$

holds. Thus for $\tilde{x}_1$ we find in the same manner $\tilde{x}_2$ such that

$$p(B)\tilde{x}_2 = \tilde{x}_1$$

and

$$\|\tilde{x}_2\| \leq q\,\|\tilde{x}_1\| \leq q^2\,\|y\|$$

holds. By induction we can therefore find a sequence $(\tilde{x}_n)_n$ such that $\|\tilde{x}_n\| \leq q^n\,\|y\|$ and $P(B)^n\tilde{x}_n = y$. But this means $\lim\limits_{n\to\infty} \tilde{x}_n = 0$ and $\lim\limits_{n\to\infty} T^n\tilde{x}_n = y$. Since y was arbitrary we have $J_T(0) = l^\infty$. Now by assumption there exits some zero z_k of $p(z)$ with absolute value less than one. From $\mathbb{D} \subseteq \sigma_p(B)$ it follows

$$0 \in p(\mathbb{D}) \subseteq p(\sigma_p(B)) = \sigma_p(p(B)),$$

where we used the Spectral mapping theorem for the point spectrum (see A.1.2). This means that $p(B)$ has non-trivial kernel which implies with Proposition 1.2.8 that $T = p(B)$ is J-class. The case where 0 is not a zero of $p(B)$ is similar. Here we just have to observe that $p(z) = q(z)$. □

We have seen the set A_T of J-vectors is a separable set for all examples of operators on l^∞ which are J-class.Therefore, it is natural to ask if this is always the case. That means is A_T always separable where $X = l^\infty$? The next example shows that this not the case.

Example 1.3.13. There exists a J_{mix}-class operator T on l^∞, such that A_T^{mix} is non-separable.

Proof. Consider an isomorphism $J : l^\infty \to l^\infty \oplus l^\infty$ and the operator $U : l^\infty \oplus l^\infty \to l^\infty$ defined by $U(a \oplus b) := a$. Now choose $|\lambda| > \|J^{-1}\|$ and define the operator $T : l^\infty \to l^\infty$ by $T := \lambda B \circ U \circ J$, where B is the backward shift. Then T is J-class with $M := J^{-1}(\{0\} \oplus l^\infty) \subseteq A_T$. To see this take any $y \in l^\infty$ arbitrary. Define $x_1 := (0, \lambda^{-1}y_1, \lambda^{-1}y_2, \ldots)$. Then $\lambda Bx_1 = y$ and $\|x_1\| = \frac{1}{|\lambda|}\,\|y\|$. Now $U(x_1 \oplus 0) = x_1$ with $\|x_1 \oplus 0\|_{l^\infty \oplus l^\infty} = \|x_1\|_\infty$. Choose now $z_1 \in l^\infty$ with $Jz_1 = (x_1, 0)$. Then

$$\|z_1\| \leq \left\|J^{-1}\right\|\,\|(x_1, 0)\| \leq \left\|J^{-1}\right\| \cdot \|x_1\| \leq \frac{\|J^{-1}\|}{|\lambda|}\,\|y\| = q\,\|y\|$$

holds with $q = \|J^{-1}\|\,|\lambda|^{-1} < 1$. Repeating the above, we find z_2 with $Tz_2 = z_1$ and $\|z_2\| \leq q\,\|z_1\| \leq q^2\,\|y\|$.

Inductively we get a sequence $T^n z^{(n)} = y$ and $z^{(n)} \to 0$. Since y was arbitrary we conclude $J_T^{mix}(0) = X$. It is easy to see that $M := J^{-1}(\{0\} \oplus l^\infty)$ is non separable and $M = N(U) \subseteq N(T)$, where $N(T)$ is the kernel of T. We thus obtain $M \subseteq N(T) \subseteq A_T$ by Proposition 1.2.8 (iii), since $J_T^{mix}(0) = X$. $\square$

Chapter 2

The existence of J-class operators on Banach spaces

2.1 Preliminaries and the main result

In this chapter we answer in a negative the question raised by G. Costakis and A. Manoussos in [14], whether there exists a J-class operator on every non-separable Banach space. In particular we show that there exists a non-separable Banach space constructed by A. Arvanitakis, S. Argyros and A. Tolias in [6] such that the J-set of every operator on this space has empty interior for each non-zero vector. On the other hand, on non-separable spaces which are reflexive there always exists a J-class operator. The main parts of this chapter are also published in [31]. Our main result is the following:

Theorem 2.1.1. *There exists a non-separable complex Banach space X on which the J-set of every operator has empty interior for every non-zero vector. Consequently there exists no J-class operator on X.*

As is clear from our proof, the conclusion of Theorem 2.1.1 is satisfied for every complex non-separable Banach space, for which every $T \in L(X)$ is of the form $T = \lambda I + S$ with $\lambda \in \mathbb{C}$ and S a strictly singular operator with separable range. A real non-separable HI (Hereditarily Indecomposable) Banach space containing no reflexive subspace for which every $T \in L(X)$ takes the form $T = \lambda I + S$ with $\lambda \in \mathbb{R}$ and S a weakly compact operator with separable range has been constructed by S. Argyros, A. Arvanitakis and A.Tolias in [6]. The complexification of this space is easily shown that satisfies our requirements, and thus the conclusion of Theorem 2.1.1. In contrast we show in Theorem 2.1.19 that every non-separable reflexive Banach space admits a J-class operator.

Definition 2.1.2. Let X, Y be infinite dimensional Banach spaces. A linear and bounded operator $S : X \to Y$ is called strictly singular, if for every infinite dimensional subspace $M \subseteq X$ the restriction $S|_M : M \to S(M)$ is not an isomorphism (i.e. a linear homeomorphism).

Theorem 2.1.3. *([1], p. 281) Assume that $S \in L(X)$ is strictly singular and that an operator $T \in L(X)$ has an at most countable spectrum. Then the spectrum of $S+T$*

is at most countable. Zero and the points of $\sigma(T)$ are the only possible accumulation points of $\sigma(S+T)$.

Remark 2.1.4. In the case that $T=0$ in Theorem 2.1.3, we conclude that $0 \in \sigma(S)$, with 0 as the only possible accumulation point. Therefore the spectrum of the operator $\lambda I + S$ is equal to $\lambda + \sigma(S)$ and is at most countable with λ as the only possible accumulation point.

Theorem 2.1.5. *([1], p. 175) Let X be a Banach space. The collection of all strictly singular operators is a closed subspace of $L(X)$, which is also a two-sided ideal.*

Proposition 2.1.6. *([1], p. 177) Let X be a real Banach space and $X_{\mathbb{C}}$ its complexification. A bounded operator $T : X \to X$ is strictly singular if and only if $T_{\mathbb{C}} : X_{\mathbb{C}} \to X_{\mathbb{C}}$ is strictly singular.*

Theorem 2.1.7. *Let X be a complex infinite dimensional Banach space. Then for every operator of the form $T = \lambda I + S$, where S is strictly singular and $|\lambda| > 1$, and every $x \neq 0$, the set $J_T(x)$ has empty interior.*

Proof. By Remark 2.1.4 it follows that $\lambda \in \sigma(T)$ and it is the only possible accumulation point. We decompose the spectrum in

$$\sigma_1 := \{\mu \in \sigma(T) : |\mu| \leq 1\} \text{ and } \sigma_2 := \{\mu \in \sigma(T) : |\mu| > 1\}.$$

Clearly $\lambda \in \sigma_2$ holds. Moreover, σ_1 is closed and since $\lambda \in \sigma_2$, and λ is the only possible accumulation point, we conclude that σ_2 is also closed. Furthermore σ_1 and σ_2 are disjoint. By the Riesz decomposition theorem we can decompose $X = M_1 \oplus M_2$, where M_1 and M_2 are closed and T-invariant subspaces and $\sigma_1 = \sigma(T_{|M_1})$, $\sigma_2 = \sigma(T_{|M_2})$. Now we assume that there exist a non-zero vector x such that $J_T(x)$ has non-empty interior. Let $x = x_1 + x_2$, where $x_1 \in M_1$ and $x_2 \in M_2$. Since x is non-zero, either x_1 or x_2 is not equal to zero. We claim that M_1 is finite dimensional. For a contradiction, assume that M_1 is infinite dimensional, then $S_{|M_1}$ is strictly singular and it follows from Remark 2.1.4, that $0 \in \sigma(S_{|M_1})$ and therefore $\lambda \in \sigma(T_{|M_1}) = \sigma_1$, which is not possible. Now consider the projection $P_1 : X \to M_1$ along M_2 onto M_1. Since

$$J_T(x) \subseteq J_{T_{|M_1}}(x_1) + J_{T_{|M_2}}(x_2),$$

it follows that

$$P_1(J_T(x)) \subseteq J_{T_{|M_1}}(x_1)$$

holds. By the open mapping theorem it follows that $P_1(J_T(x))$ has non empty interior and thus $J_{T_{|M_1}}(x_1)$ has non-empty interior. This is possible if $x_1 = 0$, since M_1 is finite dimensional, see [14].
As above we conclude that $J_{T_{|M_2}}(x_2)$ has non-empty interior. Therefore since $\sigma(T_{|M_2}) = \sigma_2 \subseteq \mathbb{C} \backslash \overline{\mathbb{D}}$, it follows by Lemma 1.2.12 (ii) that $x_2 = 0$. Thus $x = x_1 + x_2 = 0$, which is a contradiction.

□

S. Argyros, A. Arvanitakis and A. Tolias constructed a non-separable real Banach space, on which every operator T has the form $T = \lambda I + S$, where S is strictly singular and has separable range, see [6].

Theorem 2.1.8. ***(Argyros, Arvanitakis, Tolias, [6])*** *There exists a real non-separable Banach space X_A, containing no reflexive subspace, on which every operator T is of the form $T = \lambda I + S$ with $\lambda \in \mathbb{R}$ and S a weakly compact operator (and hence of separable range).*

The fact that X_A contains no reflexive subspace, implies that every weakly compact operator on X_A is strictly singular, thus every operator $T : X_A \to X_A$ is of the form $T = \lambda I + S$ with $\lambda \in \mathbb{R}$ and S a strictly singular operator with separable range.

Corollary 2.1.9. *Consider $X := (X_A)_{\mathbb{C}}$. Then every operator $T \in L(X)$ is of the form $T = wI + S$ $(w \in \mathbb{C})$, where S is strictly singular and has separable range.*

Proof. Every operator $T \in L(X)$ can be written as $T = T_1 + iT_2$, where $T_1, T_2 \in L(X_A)$. By the previous theorem $T_1 = \lambda I + S_1$ and $T_2 = \mu I + S_2$ $(\lambda, \mu \in \mathbb{R})$, where $S_1, S_2 \in L(X_A)$ are strictly singular and have separable range. Therefore we get

$$\begin{aligned} T &= T_1 + iT_2 \\ &= \lambda I + S_1 + i(\mu I + S_2) \\ &= (\lambda + i\mu) I_{\mathbb{C}} + (S_1)_{\mathbb{C}} + i(S_2)_{\mathbb{C}} \end{aligned}$$

By Proposition 2.1.6 the operator $(S_i)_{\mathbb{C}}$ is strictly singular for $i \in \{1, 2\}$ and by Theorem 2.1.5 the operator $S := (S_1)_{\mathbb{C}} + i(S_2)_{\mathbb{C}}$ is strictly singular and has separable range. With $w := \lambda + i\mu$ we get $T = wI + S$. □

Theorem 2.1.10. *There exists a non-separable complex Banach space X on which the J-set of every operator has empty interior for every non-zero vector. In particular there does not exist a J-class operator on X.*

Proof. We consider the space $X = (X_A)_{\mathbb{C}}$. Then every operator T is of the form $T = \lambda I + S$ by Corollary 2.1.9, where S is strictly singular and has separable range. If $|\lambda| > 1$, then it follows from Theorem 2.1.7 that the interior of $J_T(x)$ is empty for each non-zero vector x. Now consider $|\lambda| \leq 1$. Then by Lemma 1.2.12 the operator $T - \lambda I = S$ has dense range. This is not possible since S has separable range and X is non-separable. □

Our next aim is to show that on the space $Y := X \oplus X$, where $X = (X_A)_{\mathbb{C}}$, the J-set of every $T \in L(Y)$ has also empty interior for each non-zero vector in Y. The next Lemma gives us some information about the form of the operators in $L(Y)$.

Lemma 2.1.11. *Consider $Y := X \oplus X$, where $X = (X_A)_{\mathbb{C}}$. Then for every operator $T \in L(Y)$ there exists an isomorphism $J \in L(Y)$, such that $J^{-1}TJ$ has one of the two matrix representations*

$$J^{-1}TJ = \begin{pmatrix} \lambda I & I \\ 0 & \lambda I \end{pmatrix} + \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$$

or

$$J^{-1}TJ = \begin{pmatrix} \lambda_1 I & 0 \\ 0 & \lambda_2 I \end{pmatrix} + \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix},$$

where $S_i \in L(X)$ is strictly singular and has separable range for $i \in \{1, 2, 3, 4\}$.

Proof. Every operator $T \in L(Y)$ has the matrix representation

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

where $T_i \in L(X)$ for $i \in \{1, 2, 3, 4\}$. From Corollary 2.1.9 it follows that every $T_i = a_i I + \widetilde{S}_i$ with $\widetilde{S}_i$ strictly singular with separable range and $a_i \in \mathbb{C}$. Thus we get

$$T = \overbrace{\begin{pmatrix} a_1 I & a_2 I \\ a_3 I & a_4 I \end{pmatrix}}^{A} + \overbrace{\begin{pmatrix} \widetilde{S}_1 & \widetilde{S}_2 \\ \widetilde{S}_3 & \widetilde{S}_4 \end{pmatrix}}^{\widetilde{S}}$$

Applying the Jordan decomposition of matrices, there exists now an isomorphism such that

$$J^{-1}AJ = \begin{pmatrix} \lambda I & I \\ 0 & \lambda I \end{pmatrix} \text{ or } J^{-1}AJ = \begin{pmatrix} \lambda_1 I & 0 \\ 0 & \lambda_2 I \end{pmatrix}$$

By Theorem 2.1.5 the operator $S := J^{-1}\widetilde{S}J$ is also strictly singular and therefore there exist some $S_i \in L(X)$, $i \in \{1, 2, 3, 4\}$ strictly singular with separable range, such that

$$S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}, \text{see } [1], p.\ 177.$$

It now follows the desired statement. □

Definition 2.1.12. Let X, Y be Banach-spaces and consider $T \in L(X, Y)$.

(i) If $\dim N(T) < \infty$ and $\operatorname{codim} R(T) < \infty$, then T is said to be a Fredholm operator and we will write $T \in \Phi(X, Y)$. If $X = Y$ then $\Phi := \Phi(X) := \Phi(X, X)$.

(ii) The essential spectrum is defined as

$$\sigma_{ess}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\}.$$

Theorem 2.1.13. *([1], p. 175) Let X be a Banach space and $S, T \in L(X)$. If T is Fredholm and S is strictly singular, then the operator $T + S$ is also Fredholm.*

Theorem 2.1.14. *([1], p. 301) Let X be a Banach space and consider $T \in L(X)$. If $\lambda_0 \in \sigma(T) \backslash \sigma_{ess}(T)$ is an isolated point, then the corresponding T-invariant subspace for λ_0 resulting from the Riesz decomposition theorem is finite dimensional.*

Theorem 2.1.15. *Consider $Y = X \oplus X$ with $X = (X_A)_{\mathbb{C}}$ and $T \in L(Y)$. Then $(J_T((x, y)))^\circ = \emptyset$ for all $(x, y) \in Y \backslash \{(0, 0)\}$. Hence on Y there does not exist any J-class operator.*

Proof. We argue by contradiction. Assume that there exists some $T \in L(Y)$, such that $\overset{\circ}{J_T((x, y))} \neq \emptyset$ for some $(x, y) \in Y \backslash \{(0, 0)\}$. By the above lemma we will find an isomorphism $D \in L(Y)$ such that

$$(*) \qquad D^{-1}TD = \begin{pmatrix} \lambda I & I \\ 0 & \lambda I \end{pmatrix} + \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$$

or

$$(**) \qquad D^{-1}TD = \begin{pmatrix} \lambda_1 I & 0 \\ 0 & \lambda_2 I \end{pmatrix} + \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}.$$

Then for $\widetilde{T} := D^{-1}TD$ the J-set $J_{\widetilde{T}}(D^{-1}(x,y))$ has also non-empty interior by similar arguments as in the proof of Proposition 1.2.10.

Case 1 : $\widetilde{T} = (*)$
In this case $\lambda = \lambda_1 = \lambda_2$. If $|\lambda| \neq 1$ then we decompose $\sigma(\widetilde{T}) = \sigma_1 \cup \sigma_2$ with

$$\sigma_1 = \{\mu \in \sigma(\widetilde{T}) : |\mu| = 1\} \text{ and } \sigma_2 = \{\mu \in \sigma(\widetilde{T}) : |\mu| \neq 1\}.$$

σ_1 is closed and by Theorem 2.1.3, σ_2 is also closed, since λ and 0 are the only possible accumulation points of $\sigma(\widetilde{T})$ and hence of σ_2. Furthermore the corresponding $\widetilde{T}$-invariant closed subspace M_1 for σ_1, resulting from the Riesz decomposition theorem is finite dimensional. This follows from the fact that for $\lambda_0 \in \sigma_1$ the operator

$$\widetilde{T} - \begin{pmatrix} \lambda_0 I & 0 \\ 0 & \lambda_0 I \end{pmatrix} = \begin{pmatrix} (\lambda - \lambda_0)I & I \\ 0 & (\lambda - \lambda_0)I \end{pmatrix} + \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$$

is Fredholm by Theorem 2.1.13 and since λ_0 is an isolated point, it follows from Theorem 2.1.14 that the corresponding T-invariant closed subspace for λ_0 is finite dimensional. Now σ_1 is a finite set and therefore with the above arguments the corresponding T-invariant closed subspace for σ_1 is also finite-dimensional. The rest of the proof for $|\lambda| \neq 1$ is analogous to Theorem 2.1.7.
Now consider $|\lambda| = 1$. Then $\widetilde{T} - \begin{pmatrix} \lambda I & 0 \\ 0 & \lambda I \end{pmatrix}$ does not have a dense range, which is a contradiction to Lemma 1.2.12.

Case 2: $\widetilde{T} = (**)$.
If $\lambda_1 = \lambda_2$ the argumentation is almost identical to Case 1. Assume $\lambda_1 \neq \lambda_2$ and $|\lambda_1| = 1$ or $|\lambda_2| = 1$. Without loss of generality we have $|\lambda_1| = 1$. Then $\widetilde{T} - \begin{pmatrix} \lambda_1 I & 0 \\ 0 & \lambda_1 I \end{pmatrix}$ has not a dense range, which is a contradiction as in Case 1.
For $|\lambda_1| \neq 1$ and $|\lambda_2| \neq 1$ the argumentation is identical like Case 1. □

Remark 2.1.16. With some more technicalities it is also possible to prove the same result in Theorem 2.1.15 for $Y = \overbrace{X \oplus \ldots \oplus X}^{n-times}$, where $X = (X_A)_{\mathbb{C}}$.

We will now show that there is a large class of non-separable Banach spaces on which there always exists a J-class operator, namely the reflexive non-separable Banach spaces. The next theorem due to Costakis and Manoussos can be found in [14].

Theorem 2.1.17. *[14] Let X be a Banach space and Y be a separable Banach space. Consider $S \in L(X)$ with $\sigma(S) \subseteq \{\lambda : |\lambda| > 1\}$. Let $T \in L(Y)$ be hypercyclic. Then*

(1) *$S \times T : X \times Y \to X \times Y$ is a J-class operator, but not hypercyclic.*

(2) $A_{S\times T} = \{0\} \times Y$.

The next theorem by Lindenstrauss ([26]) gives us some information about the decomposition of reflexive non-separable Banach spaces.

Theorem 2.1.18. ***(Lindenstrauss, [26])*** *Let X be a non-separable reflexive Banach space and $Y \subseteq X$ be a separable and closed subspace. Then there exists a separable closed subspace W of X, which contains Y and a linear bounded projection $P_W : X \to W$ with $\|P_W\| = 1$.*

Theorem 2.1.19. *Let X be a non-separable reflexive Banach space. Then for every infinite dimensional separable and closed subspace Y and for every $\lambda \in (1, \infty)$ there exists a J-class operator T with $Y \subseteq A_T$ and $\|T\| = \lambda$.*

Proof. By Theorem 2.1.18 there exists a separable infinite dimensional subspace W, which contains Y and a linear bounded projection $P_W : X \to W$ with $\|P_W\| = 1$. There exists now a closed subspace U of X such that $X = U \oplus W$. For given $\varepsilon > 0$ we can find a hypercyclic operator $T_1 : W \to W$, $T_1 := I_W + K$, where I_W is the identity operator on W and K compact with $\|K\| < \varepsilon$, see ([4], [12]). Then by Theorem 2.1.17 the operator

$$T_\lambda := \lambda I_U \oplus T_1 = \lambda I + (1 - \lambda)P_W + K \circ P_W$$

is J-class for $\lambda > 1$. Furthermore $Y \subseteq W = A_T$. Now we define the function $g : (1, \infty) \to \mathbb{R}$ by $g(\delta) := \|T_\delta\|$. It is easy to see, that g is continuous. For given λ we choose $\delta > 1$ and $\varepsilon > 0$, such that $2\delta + \varepsilon < 1 + \lambda$. Therefore we get

$$\begin{aligned} g(\delta) &= \|T_\delta\| \\ &= \|\delta I + (1 - \delta)P_W + K \circ P_W\| \\ &\leq \delta + |1 - \delta| \, \|P_W\| + \|P_W\| \| K\| \\ &\leq 2\delta - 1 + \varepsilon \\ &< \lambda. \end{aligned}$$

On the other hand we can find $\mu > 1$ large enough, such that $g(\mu) > \lambda$. By the intermediate value theorem there exists a $\xi \in [\delta, \mu]$ with $g(\xi) = \|T_\xi\| = \lambda$. □

Chapter 3

J-class operators on l^∞

3.1 Spectral properties of J-class operators on l^∞

It follows immediately from the definition of a hypercyclic operator on a separable space that it is also J-class. The spectrum of a hypercyclic operator can vary very drastically. There are hypercyclic operators whose spectrum has an interior point (for example the weighted backward shift λB on l^p for $1 \leq p < \infty$) but also those where the spectrum just consists of one single point (see [4]). The next section will show that on l^∞, the spectrum can not be too small for J-class operators. We will prove that the spectrum has at least to be uncountable. Further assumptions such that the operator T is gained from l^1, for example if $T = S^*$, where S is any operator on l^1 guarantee that the point spectrum of T has even an interior point.

Definition 3.1.1. Consider $T \in L(X)$, where X is a Banach space. The residual space X^{co} is the quotient space $X^{**}/J(X)$ with the natural quotient topology. The operator

$$T^{co} : X^{**}/J(X) \to X^{**}/J(X),$$

defined by

$$T^{co}(x^{**} + J(X)) := T^{**}x^{**} + J(X)$$

is called the residuum operator, where J is the canonical embedding from X into X^{**}. If there is no necessity, we will write for short $X = J(X)$ and therefore $X^{co} = X^{**}/X$.

Remark 3.1.2. (1) Given any operator $T \in L(X)$, we can identify $(T^*)^{co}$ with $(T^{co})^*$ and therefore $(T^{co})^{**} \cong ((T^*)^{co})^* \cong (T^{**})^{co}$, see [19]. Here $T \cong S$ means that there exists a continuous isomorphism $J : X \to Y$ such that $T = J^{-1}SJ$ holds, where $T \in L(X)$, $S \in L(Y)$ and X, Y are Banach spaces.

(2) The spectrum of T^{co} is contained in the spectrum of T, that means

$$\sigma(T^{co}) \subseteq \sigma(T)$$

holds. This follows from the fact that for a T-invariant closed subspace Y of X we have

$$(*) \qquad \sigma(\widehat{T}_Y) \subseteq \sigma(T) \cup \sigma(T|_Y),$$

see [36], p. 11), where $\widehat{T}_Y : X/Y \to X/Y$ is the canonical operator induced by T. That means, $\widehat{T}_Y [x]_Y := [Tx]_Y$. Therefore T^{co} is the same operator as $\widehat{T^{**}_X}$. Hence from $(*)$ we get

$$\sigma(T^{co}) = \sigma(\widehat{T^{**}}_X) \subseteq \sigma(T^{**}|_X) \cup \sigma(T^{**}) = \sigma(T) \cup \sigma(T^{**}) = \sigma(T).$$

The last equation follows from the fact that $T^{**}|_X = T$.

Proposition 3.1.3. *Let M_1, M_2 be Banach spaces and consider $T_1 \times T_2 \in L(M_1 \times M_2)$. Then*

$$(T_1 \times T_2)^{co} \cong T_1^{co} \times T_2^{co}$$

holds.

Proof. Consider first the following operator:

i) $\psi_1 : (M_1 \times M_2)^* \to M_1^* \times M_2^*$ defined by

$$\psi_1(x^*) = (x_1^*, x_2^*),$$

where $x_1^* = x^* \circ J_1$, $x_2^* = x^* \circ J_2$ and $J_1 : M_1 \to M_1 \times M_2$, $J_2 : M_2 \to M_1 \times M_2$ are the canonical embeddings, that means $J_1(u) = (u, 0)$ and $J_2(v) = (0, v)$ for $u \in M_1$ and $v \in M_2$.

ii) Similar define $\psi_2 : (M_1^* \times M_2^*)^* \to M_1^{**} \times M_2^{**}$ by

$$\psi_2(x^{**}) = (x_1^{**}, x_2^{**}),$$

where $x_1^{**} = x^{**} \circ J_{1,*}$ and $x_2^{**} = x^{**} \circ J_{2,*}$ and $J_{1,*} : M_1^* \to M_1^* \times M_2^*$, $J_{2,*} : M_2^* \to M_1^* \times M_2^*$ are the canonical embeddings similarly defined as in (i).

iii) Define now $\phi = \psi_1^* \circ \psi_2^{-1}$. A short calculation shows that $\phi(i(M_1) \times i(M_2)) \subseteq i(M_1 \times M_2)$ holds, where i is the respective canonical map. Therefore the map

$$\widehat{\phi} : \frac{M_1^{**} \times M_2^{**}}{i(M_1) \times i(M_2)} \to \frac{(M_1 \times M_2)^{**}}{i(M_1 \times M_2)},$$

defined by

$$\widehat{\phi}([(m_1^{**}, m_2^{**})]) := \left[\psi_1^* \circ \psi_2^{-1}(m_1^{**}, m_2^{**})\right]$$

is well defined, continuous and invertible.

iv) One can also easily see that

$$(\widehat{\phi})^{-1}(T_1 \times T_2)^{co}\widehat{\phi}\,[(m_1^{**}, m_2^{**})] = [(T_1^{**}m_1^{**}, T_2^{**}m_2^{**})]$$

is true. Define now

$$\varphi : \frac{M_1^{**}}{i(M_1)} \times \frac{M_2^{**}}{i(M_2)} \to \frac{M_1^{**} \times M_2^{**}}{i(M_1) \times i(M_2)}$$

by $\varphi([m_1^{**}, m_2^{**}]) := ([m_1^{**}], [m_2^{**}])$.
Now ϕ is a continuous isomorphism and again a short calculation shows that

$$(\widehat{\phi}\varphi)^{-1}(T_1 \times T_2)^{co}\widehat{\phi}\varphi = T_1^{co} \times T_2^{co}$$

holds. This completes the proof. □

Definition 3.1.4. Let X, Y be Banach spaces and consider $T \in L(X, Y)$.

(1) T is called bounded below, if there exists a constant $c > 0$, such that

$$\|Tx\| \geq c\,\|x\|$$

for all $x \in X$.

(2) The approximate spectrum of $T \in L(X)$ is defined as the set

$$\begin{aligned}\sigma_a(T) &:= \{\lambda \in \mathbb{C} \mid \; T - \lambda I \text{ is not bounded below}\} \\ &= \{\lambda \in \mathbb{C} \mid \exists (x_n)_n \in X,\; \|x_n\| = 1 \text{ with } \lim_{n\to\infty} \|Tx_n - \lambda x_n\| = 0\}.\end{aligned}$$

Remark 3.1.5. It is immediately clear from the above definition that $\sigma_p(T) \subseteq \sigma_a(T)$. Moreover, if we consider any closed subspace $M \subseteq X$ which is invariant under T then it is also easy to see that $\sigma_a(T|_M) \subseteq \sigma_a(T)$.

Proposition 3.1.6. *([1], p. 249) For $T \in L(X)$ the set $\sigma_a(T)$ is a closed subset of $\sigma(T)$. Moreover, the boundary of the spectrum is contained in the approximate spectrum, that means*

$$\partial\sigma(T) \subseteq \sigma_a(T)$$

holds.

The next theorem can be found in [11]; in particular it states that the point spectrum of the adjoint of an operator defined on l^∞ can not be empty since we know by the previous proposition that the boundary of the spectrum is contained in the approximate spectrum.

Theorem 3.1.7. *(**Kalton, Bermudez**, [11]) Consider $T \in L(l^\infty)$. Then we have*

$$\sigma_a((T^*)^{co}) \subseteq \sigma_p(T^*).$$

Definition 3.1.8. An infinite-dimensional Banach space X is said to be prime if every infinite-dimensional complemented subspace of X is isomorphic to X.

Any separable Hilbert-space or more generally any l^p-space for $1 \leq p < \infty$ is prime, see [27], p. 57 . The next theorem due to Lindenstrauss states that this is also true for l^∞.

Theorem 3.1.9. *(**Lindenstrauss**, [27], p. 57)*
The space l^∞ is prime.

The next elementary lemma will be important for describing the spectral structure of operators with totally disconnected spectrum.

Lemma 3.1.10. *([9], p. 12.) Let K be a compact subset of $\mathbb{C}$ and let C be a connected component of K. Assume that C is contained in some open set $O \subseteq \mathbb{C}$. Then one can find a clopen (closed and open) subset σ of K, such that*

$$C \subseteq \sigma \subseteq O.$$

Lemma 3.1.11. (a) *Consider $A := \{a_i \mid i \in I\}$, where I is an uncountable index set and a_i are positive real numbers for each $i \in I$. Then there exists $\delta > 0$ and an uncountable index set $J \subseteq I$ such that*

$$a_j > \delta \text{ for every } j \in J.$$

(b) *Consider A as in a). Then*

$$\sup\{ \sum_{j\in J} a_j \mid J \subseteq I \text{ finite } \} = \infty.$$

(c) *Assume that $(I_n)_{n\in\mathbb{N}}$ are intervals in $\mathbb{R}^d$ and that there exists a positive integer k with the property that for $|m-n| \geq k$ the intervals I_m and I_n are disjoint. Then*

$$\sum_{n=1}^{\infty} |I_n| \leq k \cdot \lambda(I),$$

where $I = \bigcup_{n=1}^{\infty} I_n$ and λ is the Lebesgue measure on $\mathbb{R}^d$.

Proof. (a) Without loss of generality we may assume that $A \subseteq (0,1)$. Suppose that the statement is not valid. Then for every positive integer m there exist at most countable many $a_i \in A$, such that $a_i > \frac{1}{m}$. But then

$$A = A \cap (0,1) = \bigcup_{m=1}^{\infty} \left[\frac{1}{m+1}, \frac{1}{m}\right) \cap A$$

is at most countable. This gives a contradiction.
(b) follows immediately from (a) and (c) can be found in [8], Lemma 3.4. □

Lemma 3.1.12. *Let $K \subseteq \mathbb{C}$ be a compact subset of the complex plane. Then the set $K \backslash K_c$ is at most countable, where K_c stands for the set of all accumulation points of K.*

Proof. Regard K as a subset of $\mathbb{R}^2$ with the max-norm. Without loss of generality we may assume $K \subseteq (0,1) \times (0,1)$. We put $M := K\backslash K_c$. Assume that M is uncountable. For each $x := (x^{(1)}, x^{(2)}) \in M$ choose $\varepsilon_x > 0$, such that $I_{\varepsilon_x} \cap K = \{x\}$ (since x is isolated) and $I_{\varepsilon_x} \subseteq (0,1) \times (0,1)$, where

$$I_{\varepsilon_x} = (-\varepsilon_x + x^{(1)}, \varepsilon_x + x^{(1)}) \times (-\varepsilon_x + x^{(2)}, \varepsilon_x + x^{(2)}).$$

By Lemma 3.1.11 (a) there exists $0 < \delta < 1$ and uncountable many elements $x \in M$, such that $\varepsilon_x > \delta$. We denote this set by M_δ. Furthermore there exist positive integers m, k such that $\frac{1}{m} \leq \frac{\delta}{2} < \frac{\varepsilon_x}{2}$ and

$$S := M_\delta \cap \left[\frac{k-1}{m}, \frac{k}{m}\right]$$

is uncountable. By Lemma 3.1.11 (b) we can therefore choose finitely many elements $x_1, x_2, \dots, x_n$ from S, such that

$$\sum_{i=1}^{n} \varepsilon_{x_i} > 1.$$

Furthermore for $|i-j| > 2$ with $1 \leq i, j \leq n$ we have $I_i \cap I_j = \emptyset$, where

$$I_i := \left[\frac{k-1}{m}, \frac{k}{m}\right] \times (-\varepsilon_{x_i} + x_i^{(2)}, \varepsilon_{x_i} + x_i^{(2)})$$

Hence by Lemma 3.1.11 (c) we get

$$\frac{2}{m} < \sum_{i=1}^{n} \frac{2}{m}\varepsilon_{x_i} = \sum_{i=1}^{n} |I_i| \leq 2\lambda(\cup_{i=1}^{n} I_i) \leq \frac{2}{m}$$

which yields a contradiction. □

At this point we are ready for the main theorem of this section.

Theorem 3.1.13. *Let $T \in L(l^\infty)$ and suppose that the spectrum of T is totally disconnected. Then the following statements hold.*

(a) *The set*

$$\sigma(T)\backslash\sigma(T^{co})$$

is at most countable.

(b) *The spectrum of the adjoint T^* consists of eigenvalues. That means*

$$\sigma(T^*) = \sigma_p(T^*)$$

holds.

Proof. (a) Let α be a accumulation point of $\sigma(T)$. Assume that there exists some $r > 0$ such that

$$B_r(\alpha) \cap \sigma(T^{co}) = \emptyset,$$

where $B_r(\alpha)$ is the open disk centered at α with radius r. We will apply Lemma 3.1.10 to $K := \sigma(T)$, $O := B_r(\alpha)$ and $C = \{\alpha\}$; here we use the fact that every point is a connected component. Therefore there exists a clopen subset σ_1 of $\sigma(T)$, such that

$$\{\alpha\} \subseteq \sigma_1 \subseteq B_r(\alpha).$$

By the Riesz decomposition theorem applied to σ_1 and $\sigma_2 := \sigma(T)\backslash\sigma_1$ we get that the corresponding spectral subspace $M_1 := M_{\sigma_1}$ is infinite dimensional. This follows from the fact that σ_1 is an infinite set, since α is a accumulation point and σ_1 is open in $\sigma(T)$. Since l^∞ is a prime space, it follows that $M_1 \cong l^\infty$. Moreover, we get from

$$T = T|_{M_1} \oplus T|_{M_2}$$

the isomorphy

$$(*) \qquad T^{co} \cong (T|_{M_1})^{co} \oplus (T|_{M_2})^{co}$$

by Proposition 3.1.3. Since M_1 is non-reflexive (l^∞ is non-reflexive) it follows that $(M_1)^{co}$ is not the null-space and therefore $\sigma((T|_{M_1})^{co})$ is non-empty. In particular, we get

$$\sigma((T|_{M_1})^{co}) \subseteq \sigma(T^{co})$$

from $(*)$ but on the other hand it follows from Remark 3.1.2 (2)

$$\sigma((T|_{M_1})^{co}) \subseteq \sigma((T|_{M_1})) = \sigma_1$$

and therefore

$$\emptyset \neq \sigma((T|_{M_1})^{co}) = \sigma_1 \cap \sigma((T|_{M_1})^{co}) \subseteq B_r(\alpha) \cap \sigma(T^{co}) = \emptyset.$$

This is a contradiction and therefore for each $n \in \mathbb{N}$ there exists

$$a_n \in \sigma(T^{co}) \text{ with } |a_n - \alpha| < \frac{1}{n}.$$

Hence

$$\lim_{n\to\infty} a_n = \alpha \in \sigma(T^{co})$$

holds, since $\sigma(T^{co})$ is closed. Therefore, the set of all accumulation points of $\sigma(T)$ is contained in $\sigma(T^{co})$ and finally the set

$$\sigma(T)\backslash\sigma(T^{co}) \subseteq \sigma(T)\backslash(\sigma(T))_c$$

is at most countable by Lemma 3.1.12.

(b) If α is an accumulation point of $\sigma(T) = \sigma(T^*)$ then the proof of (a) shows that

$$\begin{aligned} \alpha \in \sigma(T^{co}) &= \sigma((T^{co})^*) \\ &= \sigma((T^*)^{co}) \overbrace{=}^{(*)} \partial\sigma((T^*)^{co}) \\ &= \sigma_a((T^*)^{co}) \subseteq \sigma_p(T^*), \end{aligned}$$

where the last inclusion follows from Theorem 3.1.7 and $(*)$ is valid since $\sigma(T)$ is totally disconnected and therefore $\sigma(T^{co})$.
Let α be an isolated point in $\sigma(T)$. Consider first the case, where $\dim M_{\sigma_\alpha} < \infty$, with $\sigma_\alpha = \{\alpha\}$. Then α is an eigenvalue of T and since $T^* \cong T_1^* \oplus T_2^*$ holds, we conclude that α is an eigenvalue of T^*. If $\dim M_{\sigma_\alpha} = \infty$, the argumentation is identical as in the proof of (a). This shows (b). □

Remark 3.1.14. (i) The above theorem is clearly valid for spaces which are isomorphic to l^∞, for example $L^\infty([0,1])$ (see [3], p. 85) or the weighted H^∞ spaces (see [29]).

(ii) If we consider any operator $T \in L((l^\infty)^*)$ with countable spectrum, the above theorem does not remain true. Indeed, for a Banach space X, we can find a closed subspace $M \subseteq X^{***}$ such that we have the following direct composition $X^{***} = M \oplus X^*$. To see this, consider the canonical embedding $i : X \to X^{**}$ and its adjoint $i^* : X^{***} \to X^*$. Denote by $J : X^* \to J(X^*)$ the canonical embedding of X^* onto its image. Then $(J \circ i^*)|_{J(X^*)}$ is the identity on $J(X^*)$ and therefore $J \circ i^*$ is a projection. Now for $X = c_0$ we obtain $(l^\infty)^* = M \oplus l^1$. Choose the identity operator on M and any compact operator K on l^1 with $\sigma(K) = \{0\}$ so that 0 is not an eigenvalue of K. Then the spectrum of $T := I \oplus K$ is equal to $\{0, 1\}$. But 0 is not an eigenvalue of T.

Using Lemma 1.2.12, we finally see that the spectrum of a J-class operator can not be too small.

Corollary 3.1.15. *Let $T \in L(l^\infty)$ be a J-class operator. Then*

(a) The spectrum of T is uncountable.

(b) For every $\lambda \in \mathbb{C}$ there exists a subspace $U_\lambda \subseteq l^\infty$ isomorphic to l^∞, such that

$$(T - \lambda I)|_{U_\lambda}$$

is an isomorphism.

Proof. a) Suppose there is a J-class operator with countable spectrum (hence totally disconnected). Then by Theorem 3.1.13 (b) $\sigma(T) = \sigma(T^*) = \sigma_p(T^*)$. By Lemma 1.2.12 (i) we have $\sigma_p(T^*) \cap \overline{\mathbb{D}} = \emptyset$. But then $\sigma(T) \cap \overline{\mathbb{D}} = \emptyset$, which is a contradiction to Lemma 1.2.12 (ii).

b) Suppose there exists some $\lambda \in \mathbb{C}$, such that $(T - \lambda I)|_U$ is not an isomorphism for all $U \subseteq l^\infty$ isomorphic to l^∞. Then by a result of Rosenthal $W := T - \lambda I$ is weakly compact, see [27], p. 106. Since W^2 is compact (see [27], p. 57) it follows that $T = \lambda I + W$ has countable spectrum which is a contradiction to (a). □

3.2 Spectral properties of J-class operators on l^∞ which are adjoints

We have seen in the previous section that the spectrum of a J-class operator has to be uncountable. The aim of this section is to describe the whole spectrum of those J-class operators on l^∞ which are "more or less" the adjoint of some operator on l^1. We will get strongly use of the geometry of l^1 and with this it will be possible to give a complete qualitative and quantitative characterization whenever these operators are J-class or not. Under additional but natural conditions we determine also the set A_T of J-vectors, which is the main part of Section 3.3. We begin with an elementary topological lemma, which we just state to get more familiar with the weak-star topology.

Lemma 3.2.1. *Let (X, τ) be a Hausdorff-space and consider any sequence $\{y_n \mid n \in \mathbb{N}\} \subseteq X$. Then*

$$\overline{\{y_n \mid n \in \mathbb{N}\}} = \overline{\{y_n \mid n \geq k\}} \cup \{y_1, y_2, \ldots, y_k\}$$

holds for all $k \in \mathbb{N}$.

Proof. Fix $k \in \mathbb{N}$ and consider any $x \in \overline{\{y_n \mid n \in \mathbb{N}\}}$. If $x \in \{y_1, y_2, \dots, y_k\}$, there is nothing to prove, otherwise assume that $x \notin \{y_1, y_2, \dots, y_k\}$. We have to show that $x \in \overline{\{y_n \mid n \geq k\}}$. Take therefore any neighborhood U of x and choose for $1 \leq j \leq k$ neighborhoods $U_{x,j}$ of x and U_{y_j} of y_j, such that $U_{x,j} \cap U_{y_j} = \emptyset$. Define now $U_x := U_{x,1} \cap U_{x,2} \dots \cap U_{x,k}$. Since $(U \cap U_x) \cap \{y_n \mid n \geq 1\} \neq \emptyset$, it follows that there exists y_m with $m \geq k$ and $y_m \in (U \cap U_x) \cap \{y_n \mid n \geq 1\}$. This means $y_m \in U \cap \{y_n \mid n \geq k\}$. □

Corollary 3.2.2. *Let X be a Banach space and consider $\{y_n^* \mid n \in \mathbb{N}\} \subseteq X^*$. If $x^* \in \overline{\{y_n^* \mid n \in \mathbb{N}\}}^{w^*} \setminus \{y_n^* \mid n \in \mathbb{N}\}$, it follows that $x^* \in \overline{\{y_n^* \mid n \geq k\}}$ for all $k \in \mathbb{N}$.*

Proof. This follows immediately from the above lemma. □

At this point we will introduce the surjectivity spectrum of an operator on a Banach space which will play an essential role for the rest of this chapter.

Definition 3.2.3. Let X be a Banach space and consider $T \in L(X)$. The surjectivity spectrum of T is the set

$$\sigma_s(T) := \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not surjective}\}.$$

The next proposition lists some properties of the surjectivity spectrum and can be found in [25], p. 34.

Proposition 3.2.4. *Consider $T \in L(X)$ where X is a Banach space. Then we have the following spectral properties.*

(1) $\sigma_s(T) = \sigma_a(T^*)$.

(2) $\sigma_a(T) = \sigma_s(T^*)$

Remark 3.2.5. From the above proposition and from Proposition 3.1.6 it follows that the boundary of the spectrum is contained in $\sigma_s(T)$. Moreover it is a closed set.

The next theorem due to Eberlein and Smulian states, that sequential compactness and compactness with respect to the weak topology on a Banach space are equivalent. This is not clear, since the weak topology is not metrizable.

Theorem 3.2.6. *(**Eberlein-Smulian**, [38], p. 49)*
(a) Let X be a Banach space. Then the set $A \subseteq X$ is relatively compact if and only if every sequence in A has a convergent subsequence.

(b) A subset $B \subseteq X$ is relatively weakly compact if and only if it is bounded and the $\sigma(X^{**}, X^*)$-closure of $i(B)$ in X^{**} is contained in $i(X)$, where i is the canonical embedding.

The next theorem and its proof was suggested by M. O. Gonzalez. It states that in the case $X = l^1$ any λ which is an approximate point of an operator T is an eigenvalue of its second adjoint. This is of course not true in general. For example consider $X = l^2$ and any compact operator K for which 0 is not an eigenvalue. Then 0 is an approximate point of K ($\sigma_a(K) = \partial\sigma(K) = \sigma(K)$, since K has countable spectrum) but 0 is likewise not an eigenvalue of $K^{**} = K$.

Theorem 3.2.7. *Let X be a Banach space with the Schur-property (that means every weakly convergent sequence $(x_n)_n \subseteq X$ is norm convergent) and consider $T \in L(X)$. Then the approximate spectrum of T is equal to the point spectrum of T^{**}. In other words*

$$\sigma_a(T) = \sigma_p(T^{**})$$

holds.

Proof. Take any $\lambda \in \sigma_a(T)$. Then there exists a sequence $x_n \in X$ with $\|x_n\| = 1$, such that $\|Tx_n - \lambda x_n\| \to 0$ holds as $n \to \infty$. Consider the case where x_n has a convergent subsequence, which we will denote again by $(x_n)_n$. It follows immediately that $Tz = \lambda z$, where $z := \lim\limits_{n\to\infty} x_n$ and since T^{**} is an extension of T, we get $T^{**}z = \lambda z$. Suppose at this point that x_n has no norm convergent subsequence and put $S := T - \lambda I$. Since X has the Schur-property, $(x_n)_n$ has no weak convergent subsequence. Hence the set $\{x_n \mid n \in \mathbb{N}\}$ is not relatively compact by the Eberlein-Smulian theorem and therefore by Theorem 3.2.6 b) the $\sigma(X^{**}, X^*)$ closure of $\{x_n \mid n \in \mathbb{N}\}$ is not contained in X. We choose

$$z^{**} \in \overline{\{x_n \mid n \in \mathbb{N}\}}^{w^*} \backslash X \subseteq \overline{\{x_n \mid n \in \mathbb{N}\}}^{w^*} \backslash \{x_n \mid n \in \mathbb{N}\},$$

i.e. in particular $z^{**} \neq 0$. For each $\varepsilon > 0$ and for each $y^* \in X^*$ there exist by Corollary 3.2.2 $n_k > k$, such that $|z^{**}(y^*) - x^{**}_{n_k}(y^*)| < \varepsilon$, where $x^{**}_{n_k} = i(x_{n_k})$ and i is the canonical embedding from X to X^{**}. By choosing $y^* = x^* \circ S$, we get $|z^{**}(x^* \circ S) - x^{**}_{n_k}(x^* \circ S)| < \varepsilon$. Hence we get

$$\begin{aligned}
|(S^{**}z^{**})x^*| &\leq |(S^{**}z^{**} - S^{**}x^{**}_{n_k})(x^*)| + |S^{**}x^{**}_{n_k}(x^*)| \\
&= |(z^{**} - x^{**}_{n_k})(x^* \circ S)| + |S^{**}x^{**}_{n_k}(x^*)| \\
&\leq \varepsilon + \left\|S^{**}x^{**}_{n_k}\right\| \cdot \|x^*\| \\
&= \varepsilon + \|Sx_{n_k}\| \cdot \|x^*\| \to \varepsilon. \ (k \to \infty)
\end{aligned}$$

Since ε and x^* were arbitrary, it follows that $S^{**}z^{**} = (T^{**} - \lambda I)z^{**} = 0$. This shows $\sigma_a(T) \subseteq \sigma_p(T^{**})$.
To establish the converse consider

$$\sigma_p(T^{**}) \subseteq \sigma_a(T^{**}) = \sigma_s(T^*) = \sigma_a(T)$$

which follows by Proposition 3.2.4. □

The above theorem can also be reformulated in the following sense: Consider $\lambda \in \mathbb{C}$ and let $T : l^\infty \to l^\infty$ be an adjoint operator, where $T - \lambda I$ has dense range. Then $T - \lambda I$ is surjective.
One may ask if this is true in general. Therefore we state the following problem:

Problem: Consider any $T \in L(l^\infty)$ with dense range. Is it true that T is surjective?

Unfortunately we can not give an answer to this question but we can improve Theorem 3.2.7 in the following sense: If $T = S^*U + W$ has dense range, where $S \in L(l^1)$, U is lower semi-Fredholm and $W \in L(l^\infty)$ is weakly compact, then T is actually surjective. To prove this we introduce some notions from Banach space geometry and Tauberian operator theory.

Definition 3.2.8. (i) Let X be a Banach space. We call X a Grothendieck space, if every weak-star convergent sequence $(x_n^*)_n$ in X^* is weakly convergent.

(ii) Let X, Y be Banach spaces and consider $T \in L(X, Y)$. We call T a Dunford-Pettis operator or completely continuous if $T(W)$ is a norm-compact subset of Y whenever W is a weakly compact subset of X.

(iii) A Banach space X has the Dunford-Pettis-Property (DPP), if every weakly compact operator T from X into a Banach space Y is Dunford-Pettis.

Remark 3.2.9. l^∞ is a Grothendieck space by a theorem of Grothendieck (see [18], p. 156). Since l^∞ can be identified with $C(K)$, where K is compact, the dual $(l^\infty)^*$ can be regarded as the space of all regular Borel measures $M(K)$ and the latter space has in fact the Dunford-Pettis-Property, see [3], p. 117. In terms of sequences the Dunford-Pettis-Property is equivalent to the fact that for every sequence $(x_n)_n$ in X converging weakly to 0 and every $(x_n^*)_n$ in X^* converging weakly to 0, the sequence of scalars $(x_n^*(x_n))_n$ converges to 0, see [3], p. 115.

Definition 3.2.10. Let $T \in L(X, Y)$, where X, Y are Banach spaces.

(i) T is called tauberian if

$$(T^{**})^{-1}(Y^{**} \backslash Y) \subseteq X^{**} \backslash X,$$

or equivalently

$$(T^{**})^{-1}(Y) \subseteq X.$$

(ii) T is called upper-semi Fredholm if its kernel $N(T)$ is finite-dimensional and its range $R(T)$ is closed. The class of all upper-semi Fredholm operators from X to Y is denoted by $\Phi_+(X, Y)$. For $X = Y$ we write $\Phi_+ := \Phi_+(X) := \Phi_+(X, X)$.

(iii) T is called lower-semi Fredholm if codim $R(T) < \infty$. The class of all lower-semi Fredholm operators from X to Y is denoted by $\Phi_-(X, Y)$. For $X = Y$ we write $\Phi_- := \Phi_-(X) := \Phi_-(X, X)$.

The next proposition can be found in [19], p. 12, p. 21 and p. 29.

Proposition 3.2.11. *Let X be a Banach space and $T \in L(X)$. Then the following assertions are equivalent.*

(i) *T is tauberian.*

(ii) *$N(T^{**}) = N(T)$ and $T(B_X)$ is closed.*

(iii) *$N(T)$ is reflexive and $\overline{T(B_E)} \subseteq T(E)$ holds for every closed subspace E of X.*

If in addition X has no infinite-dimensional reflexive subspace then T is tauberian if and only if T is upper-semi Fredholm.

Remark 3.2.12. (i) If $T \in L(X, Y)$ and $N(T^*) = N(T^{***})$ then by Proposition 3.2.11 (ii) the operator T^* is tauberian since $T^*(B_{X^*})$ is always closed.

(ii) If T is tauberian and M is a closed subspace of X then the restriction $S := T|_M : M \to Y$ is also tauberian. Indeed, since $N(T)$ is reflexive then $N(S)$ is reflexive. Moreover, $\overline{S(B_{\widehat{E}})} = \overline{T(B_{\widehat{E}})} \subseteq T(\widehat{E}) = S(\widehat{E})$ holds for every closed subspace $\widehat{E}$ of E. This shows that S is tauberian by Proposition 3.2.11.

Theorem 3.2.13. *Let $T : l^\infty \to l^\infty$ be an operator of the form $T = S^*U + W$ where $S : l^1 \to l^1$, $U \in \Phi_-$ and W is weakly compact. If T has dense range, then it is surjective.*

Proof. The fact that T has dense range implies that $N(T^*) = \{0\}$ and since l^∞ is a Grothendieck-space it follows that $N(T^{***}) = N(T^*) = \{0\}$, see [11], p. 1450. From Remark 3.2.12 (i) we get that T^* is tauberian. With $l^1 \cong J_{l^1}(l^1) \subseteq (l^\infty)^*$ (here J is the canonical embedding) we get that the restriction

$$T^*|_{l^1} = U^*S^{**}|_{l^1} + W^*|_{l^1}$$

is also tauberian by Remark 3.2.12 (ii). Now we notice that l^1 has no reflexive infinite-dimensional subspace ([27], 2.a.2) and therefore $T^*|_{l^1}$ is upper semi-Fredholm by the above proposition. Furthermore, W^* is also weakly compact by Gantmacher's theorem ([1], p. 89). Assume that there exists an infinite dimensional closed subspace Y of $(l^\infty)^*$ such that $Z := W^*|_Y$ is an isomorphism onto its image. Hence $Z^{-1}(Z(B_Y)) = B_Y$ is weakly compact in $(l^\infty)^*$ and therefore $W^*(B_Y)$ is norm-compact by the Dunford-Pettis-Property of $(l^\infty)^*$ (see Remark 3.2.9). It follows that Y is finite dimensional. From this we conclude that W^* is strictly singular (see Definition 2.1.2) and therefore also Z. Since $S^{**}|_{l^1} = S$ we get

$$U^*S = U^*S^{**}|_{l^1} = T^*|_{l^1} - W^*|_{l^1} \in \Phi_+,$$

since a strictly singular perturbation of an upper-semi Fredholm operator remains upper-semi Fredholm, see [23], Theorem 5.2. Hence from $U^*S \in \Phi_+$ it follows that $S \in \Phi_+$, [30] p. 151. Now the operator S^{**} belongs to Φ_+ (see [30], p. 150) and so $U^*S^{**} \in \Phi_+$, since $U^* \in \Phi_+$. Since W^* is strictly singular, with the same argument as above T^* itself is upper-semi Fredholm. In particular, $R(T^*)$ is closed and therefore $R(T)$. It follows that T is surjective. □

Corollary 3.2.14. *Suppose there exists some positive integer $m \in \mathbb{N}$ such that $V \in L(l^\infty)$ has the property that $V^m = S^*$ holds for some $S \in L(l^1)$. Assume further that $U \in \Phi$ commutes with V. If the operator $T = VU + W$ with W weakly compact has dense range, it is surjective.*

Proof. Since T has dense range it follows that T^m has dense range. Furthermore we have

$$T^m = V^mU^m + W_1 = S^*\overline{U} + W_1$$

where W_1 is weakly compact and $\overline{U} := U^m \in \Phi$ (see [1], p. 158). An application of the above theorem yields that T^m is surjective and so is T. □

Definition 3.2.15. If T is a weakly compact perturbation of an adjoint operator we will write for short

$$T \in L_A^W(l^\infty) := \{T \in L(l^\infty) \mid T = S^* + W, \text{ where } S \in L(l^1) \text{ and } W \in L(l^\infty) \text{ is weakly compact}\}.$$

Corollary 3.2.16. *Consider $T \in L(l^\infty)$ such that T can be written as $T = S^* + W$, where $W \in L(l^\infty)$ is weakly compact and $S : l^1 \to l^1$. Then $\sigma_a(T^*) = \sigma_p(T^*)$.*

Proof. By the above theorem it follows immediately that the surjectivity spectrum of T is contained in $\sigma_p(T^*)$. That means $\sigma_s(T) \subseteq \sigma_p(T^*)$. Since $\sigma_s(T) = \sigma_a(T^*)$ holds, the desired statement follows. □

Corollary 3.2.17. *Let $T \in L(l^\infty)$. Assume that there exists a surjective operator $U \in L(l^\infty)$ and an operator $S \in L_A^W(l^\infty)$, such that*

$$US = TU,$$

Then

$$\partial\sigma(T) \subseteq \sigma_s(T) \subseteq \sigma_p(S^*)$$

holds.

Proof. Take any $\lambda \in \sigma_s(T)$. Then we get

$$\begin{aligned} X \neq (T - \lambda I)(X) &= (T - \lambda I)(U(X)) \\ &= (TU - \lambda U)(X) = (US - \lambda U)(X) \\ &= U(S - \lambda I)(X) \end{aligned}$$

Since U is surjective we conclude that $S - \lambda I$ is not surjective. Hence $\sigma_s(T) \subseteq \sigma_s(S)$. In particular we get

$$\begin{aligned} \partial\sigma(T) \subseteq \sigma_s(T) &\subseteq \sigma_s(S) \\ &= \sigma_a(S^*) = \sigma_p(S^*) \end{aligned}$$

by Corollary 3.2.16, Proposition 3.2.4 and Remark 3.2.5. □

Definition 3.2.18. Let $T \in L(l^\infty)$. We say that T is quasi-similar to an operator $S \in L_A^W(l^\infty)$, if there exist surjective operators $U, V \in L(l^\infty)$, such that

$$US = TU$$

and

$$SV = VT.$$

Remark 3.2.19. Of course any operator $T = S^*$ for some $S \in L(l^1)$ trivially satisfies the condition in the above definition and these type of operators will be the main source of attention.

The next elementary lemma will be important for the next theorem.

Lemma 3.2.20. (a) *Let $K \subseteq \mathbb{C}$ be compact and suppose $K \cap \mathbb{R} \neq \emptyset$. Then*

$$\partial K \cap \mathbb{R} \neq \emptyset.$$

(b) *Let $K \subseteq \mathbb{C}$ be compact. Suppose $K \cap \overline{\mathbb{D}} \neq \emptyset$, but $\partial K \cap \overline{\mathbb{D}} = \emptyset$. Then*

$$\overline{\mathbb{D}} \subseteq K^\circ \subseteq K.$$

Proof. (a) If $K = \partial K$ there is nothing to prove. Otherwise write $K = \partial K \cup K^\circ$ and assume that $\partial K \cap \mathbb{R} = \emptyset$. Choose $x \in K^\circ \cap \mathbb{R} \neq \emptyset$ and $\varepsilon > 0$ such that $[x, x+\varepsilon] \subseteq K \cap \mathbb{R}$. Consider the set $A := \{\varepsilon > 0 \mid [x, x+\varepsilon] \subseteq K \cap \mathbb{R}\}$. Since $K \cap \mathbb{R}$ is compact $s := \max A$ exists and furthermore $x + s$ is a boundary point of K, which contradicts $\partial K \cap \mathbb{R} \neq \emptyset$.

(b) Take any $z \in \overline{\mathbb{D}} \backslash K^\circ$. Since $K^\circ \cap \overline{\mathbb{D}} \neq \emptyset$ we can choose $\tilde{z} \in K^\circ \cap \overline{\mathbb{D}}$. It follows that $[z, \tilde{z}] \subseteq \overline{\mathbb{D}}$ and as in (a), we see that $\partial K \cap [z, \tilde{z}] \neq \emptyset$ holds, which is a contradiction to our assumptions. □

Theorem 3.2.21. *Let $T \in L(l^\infty)$ be quasi-similar to an operator $S \in L_A^W(l^\infty)$. If T is J-class we have*

$$\overline{\mathbb{D}} \subseteq \sigma_p(T).$$

Proof. Let $U, V \in L(l^\infty)$ be surjective operators as in Definition 3.2.18. Then the equation

$$S^n V = V T^n$$

is satisfied for all positive integers n. Proposition 1.2.8 implies that $J_T(0) = l^\infty$ and since V is surjective we obtain $J_S(0) = l^\infty$. But then

$$\overline{\mathbb{D}} \cap \sigma_p(S^*) = \emptyset$$

holds by Lemma 1.2.12 (i). Moreover by Lemma 1.2.12 (ii) we have $\sigma(T) \cap \overline{\mathbb{D}} \neq \emptyset$. Recall now that $US = TU$ holds. Therefore by Corollary 3.2.17 we get

$$(*) \qquad \partial\sigma(T) \subseteq \sigma_s(T) \subseteq \sigma_p(S^*).$$

Since $\sigma_p(S^*) \cap \overline{\mathbb{D}} = \emptyset$, it follows that $\partial\sigma(T) \cap \overline{\mathbb{D}} = \emptyset$ and with regard to Lemma 3.2.20 we conclude that $\overline{\mathbb{D}} \subseteq \sigma(T)$. Now take any $\lambda \in \overline{\mathbb{D}}$. Since $\sigma(T) = \sigma_p(T) \cup \sigma_s(T)$ we have

$$\lambda \in \sigma(T) \backslash \sigma_p(S^*) \subseteq \sigma(T) \backslash \sigma_s(T) \subseteq \sigma_p(T).$$

It follows the desired statement. □

Remark 3.2.22. (i) Let us consider the case where $T \in L_A^W(l^\infty)$. Then we can choose $S = T$ and $V, U = I$ in Theorem 3.2.21. From the fact that $\sigma_a(T^*) = \sigma_s(T)$ holds, it is easy to see from the above proof that T being J-class can be replaced by $\sigma(T) \cap \overline{\mathbb{D}} \neq \emptyset$ and $\sigma_p(T^*) \cap \overline{\mathbb{D}} = \emptyset$ in Theorem 3.2.21.

(ii) Because of $\sigma_a(T^*) = \sigma_s(T)$ it is clear from the proof of the above theorem that $\sigma_a(T^*) \cap \overline{\mathbb{D}} = \emptyset$ holds, if T satisfies the condition in Theorem 3.2.21. If moreover $T \in L_A^W(l^\infty)$, then by (i) it follows that if T satisfies $\sigma(T) \cap \overline{\mathbb{D}} \neq \emptyset$ and $\sigma_p(T^*) \cap \overline{\mathbb{D}} = \emptyset$ then

$$\sigma_a(T^*) \cap \overline{\mathbb{D}} = \emptyset \text{ and } \overline{\mathbb{D}} \subseteq \sigma_p(T)$$

holds.

Corollary 3.2.23. *Let $T \in L(l^\infty)$ be J-class and quasi-similar to some $S \in L_A^W(l^\infty)$. Then the following assertions hold.*

(a) *T is surjective but not invertible.*

(b) *A_T contains an infinite dimensional closed subspace.*

(c) *T^2 is J-class.*

Proof. (a) follows immediately from Theorem 3.2.21 and Remark 3.2.22 (ii).
(b) Consider $\lambda \in \mathbb{D}$. Then by Theorem 3.2.21 those λ with absolute value smaller than one are eigenvalues and the corresponding eigenvectors x_λ are linearly independent for different λ. Hence we obtain

$$\overline{\operatorname{span}\{x_\lambda |\ \lambda \in \mathbb{D}\}} \subseteq A_T$$

where the inclusion follows from Proposition 1.2.8 and Proposition 1.2.4.
(c) Again by Proposition 1.2.8 it follows $J_{T^2}(0) = X$ if T is J-class and the existence of eigenvalues with absolute value smaller that one guarantees that T^2 is J-class. □

Remark 3.2.24. From the above corollary it follows that e^T is never J-class for $T \in L_A^W(l^\infty)$ since e^T is always invertible. It follows that e^{B_w} is not J-class since $S_w^* = B_w$, where $B_w : l^\infty \to l^\infty$ is the unilateral weighted backward shift, $S_w : l^1 \to l^1$ the unilateral forward shift on l^1 and $w := (w_n)_n$ a positive and bounded weight sequence. This example is very important since it is hypercylcic on c_0 and l^p for $1 \leq p < \infty$ and plays an essential role for the existence of hypercyclic C_0-semigroups on separable Banach spaces, see [17] and [10].

At this point we introduce some specific values which are in general important in Spectral theory and for the quantitative characterization of J-class operators on l^∞ which are quasi-similar to operators of $L_A^W(l^\infty)$.

Definition 3.2.25. Let X be a Banach space and $T \in L(X)$. The injectivity modulus (sometimes called the minimum modulus) is defined as

$$\kappa(T) := \inf\{\|Tx\| : x \in X, \|x\| = 1\}.$$

We define further

$$i(T) := \lim_{n\to\infty} \kappa(T^n)^{\frac{1}{n}}.$$

The surjectivity modulus of T is defined as

$$s(T) := \sup\{r \geq 0 : T(B_X) \supset r \cdot B_X\},$$

where B_X denotes the closed unit ball. Moreover we define

$$\delta(T) := \lim_{n\to\infty} s(T^n)^{\frac{1}{n}}.$$

Remark 3.2.26. For $x \in X$ we get from

$$\|STx\| \geq \kappa(S) \cdot \|Tx\| \geq \kappa(S)\kappa(T)\,\|x\|$$

the inequality

$$\kappa(ST) \geq \kappa(S)\kappa(T).$$

The same is true for the surjectivity modulus, i.e. $s(ST) \geq s(S)s(T)$, see also [30], p. 82.

We are ready to formulate the main theorem in this section.

Theorem 3.2.27. *Consider the operator $T \in L_A^W(l^\infty)$. Then the following statements are equivalent.*

(i) *T is J^{mix}-class.*

(ii) *T is J-class.*

(iii) *$\sigma(T) \cap \partial\mathbb{D} \neq \emptyset$ and $\sigma_p(T^*) \cap \overline{\mathbb{D}} = \emptyset$.*

(iv) *$i(T^*) > 1$ and $\overline{\mathbb{D}} \subseteq \sigma_p(T)$.*

(v) *$i(T^*) > 1$ and $0 \in \sigma_p(T)$.*

Proof. The implication $(i) \Rightarrow (ii)$ is clear and $(ii) \Rightarrow (iii)$ is the statement of the spectral lemma. The direction $(iv) \Rightarrow (v)$ is trivial. Let us prove that (iii) implies (iv). First observe that by Remark 3.2.22 (ii), the conditions in (iii) imply that $\sigma_a(T^*) \cap \overline{\mathbb{D}} = \emptyset$ and $\overline{\mathbb{D}} \subseteq \sigma_p(T)$. So we have just to show that $i(T^*) > 1$. Now for any bounded operator R on a complex Banach space X a deep result by C. Read states that

$$\mathrm{dist}(\{0\}, \sigma_a(R)) = i(R), \text{ [30], p. 91.}$$

Hence since $\sigma_a(T^*)$ is closed this implies

$$1 < \mathrm{dist}(0, \sigma_a(T^*)) = i(T^*).$$

This shows that (iii) implies (iv).

Assume that (v) holds. For arbitrarily operators T on Banach spaces we have the following relation:

$$\begin{aligned} i(T^*) &= \lim_{n\to\infty} \kappa((T^*)^n)^{\frac{1}{n}} = \lim_{n\to\infty} s((T)^n)^{\frac{1}{n}} \\ &= \delta(T) \end{aligned}$$

where we used the fact that $\kappa(T^*) = s(T)$, see [30], p. 83.
With our assumption that $i(T^*) > 1$ we can therefore find n_0 large enough and $\varepsilon > 0$, such that

$$s(T^{n_0}) > (1+\varepsilon)^{n_0}.$$

Choose at this point any $y \in l^\infty \setminus \{0\}$ and consider $\tilde{y} := \frac{y}{\|y\|}$. Looking at the definition of the surjectivity modulus, we can therefore find $x \in l^\infty$ with $\|x\| \leq 1$, such that

$$T^{n_0} x = (1+\varepsilon)^{n_0} \tilde{y},$$

or equivalently

$$T^{n_0} \tilde{x} = y,$$

where

$$\tilde{x} = \frac{\|y\|}{(1+\varepsilon)^{n_0}} \cdot x \quad \text{and so} \quad \|\tilde{x}\| \leq \frac{\|y\|}{(1+\varepsilon)^{n_0}}.$$

Define $x_1 := \tilde{x}$, and like above we can find x_2 with

$$T^{n_0} x_2 = x_1 \quad \text{and} \quad \|x_2\| \leq \frac{1}{(1+\varepsilon)^{n_0}} \|x_1\|.$$

Hence

$$T^{2n_0} x_2 = y \quad \text{and} \quad \|x_2\| \leq \frac{1}{(1+\varepsilon)^{2n_0}} \|y\|.$$

Inductively we are able to find a sequence $(x_m)_m$ in l^∞ with the property that

$$T^{mn_0} x_m = y \quad \text{and} \quad \|x_m\| \leq \frac{1}{(1+\varepsilon)^{mn_0}}.$$

With $z_m := T^{n_0} x_m$ we get in particular

$$\lim_{m\to\infty} z_m = 0 \quad \text{and} \quad \lim_{m\to\infty} T^m z_m = y.$$

Since y was arbitrarily it follows that $J_T^{mix}(0) = l^\infty$. Last but not least, take into consideration that $N(T) \neq \{0\}$, which implies together with $J_T^{mix}(0) = l^\infty$ that there exists some $x \neq 0$, such that $J_T^{mix}(x) = l^\infty$, see Proposition 1.2.8. This shows (i). $\square$

Remark 3.2.28. *a)* Remarkable are the equivalences of the statements $(i), (ii), (iii)$ in Theorem 3.2.27. It states in short that any operator which satisfies conditions (1) and (2) in our spectral-lemma (Lemma 1.2.12) is J-class and vice versa.

b) It follows from the proof of Theorem 3.2.27 that the direction $(v) \Rightarrow (i)$ is also valid for arbitrary Banach spaces.

At this point we will consider for the sake of simplicity those $T \in L(l^\infty)$ which are adjoints of operators on l^1, i.e. $T = S^*$, where $S \in L(l^1)$. Of course these T are, as mentioned before, trivially quasi-similar to some operator of $L_A^W(l^\infty)$. Moreover take into consideration that $i(S) = \delta(S^*) = \delta(T) = i(T^*)$ which can be easily seen as in the above theorem, since $\kappa(T^*) = s(T)$ and $\kappa(T) = s(T^*)$ (see [30], p. 83). Hence as a direct consequence of Theorem 3.2.27 we get the following important theorem for the rest of this chapter:

Theorem 3.2.29. *Consider the operator $T \in L(l^\infty)$ which is an adjoint operator, i.e. there exist some $S \in L(l^1)$ with $T = S^*$. Then the following statements are equivalent:*

(i) *T is J^{mix}-class.*

(ii) T *is* J*-class.*

(iii) $\sigma(T) \cap \partial\mathbb{D} \neq \emptyset$ *and* $\sigma_p(T^*) \cap \overline{\mathbb{D}} = \emptyset$.

(iv) $i(S) > 1$ *and* $\overline{\mathbb{D}} \subseteq \sigma_p(T)$.

(v) $i(S) > 1$ *and* $0 \in \sigma_p(T)$.

Corollary 3.2.30. *Let* $T_1, T_2 \in L(l^\infty)$ *be commuting adjoint operators, which are also* J*-class. Then* $T := T_1T_2$ *is also* J*-class.*

Proof. Let S_1 and S_2 be the corresponding operators on l^1, such that $(S_i)^* = T_i$ for $i \in \{1, 2\}$. We have $T = S_1^* S_2^* = (S_2S_1)^*$ and by the above theorem, we get that $i(S_i) > 1$ holds for $i = 1, 2$. Furthermore we get from Remark 3.2.26

$$\kappa((S_2S_1)^n) = \kappa(S_2^n S_1^n) \geq \kappa(S_2^n)\kappa(S_1^n)$$

for all $n \in \mathbb{N}$. Hence $i(S_2S_1) \geq i(S_2) \cdot i(S_1) > 1$ and $N(T_1T_2) \supset N(T_2) \neq \{0\}$. This shows that condition (v) of Theorem 3.2.29 is satisfied for T and is therefore J-class. $\square$

The next theorem can be found in [1], p. 73. It states that like the invertible operators on a Banach space, the set of all surjective operators which have non-trivial kernel is also open with respect to the norm topology.

Theorem 3.2.31. *([1], p. 73) Let* X *be a Banach space. Then the set of all surjective but not invertible operators is open in* $L(X)$ *with respect to the norm-topology.*

Proposition 3.2.32. *Let* X *be a Banach space. Then the set*

$$M := \{T \in L(X) : \delta(T) > 1 \text{ and } N(T) \neq \{0\}\}$$

is open with respect to the norm-topology.

Proof. Consider any $T \in M$. Then there exist some $m \in \mathbb{N}$ and a positive constant such that $s(T^m) > c > 1$. In particular T is surjective and $N(T) \neq \{0\}$. Define

$$g : L(X) \to L(X^*) \text{ by } g(S) := (S^*)^m.$$

Then g is continuous since the mappings $S \to S^*$ and $S \to S^m$ are continuous with respect to the operator norm-topologies. Now the set of all surjective operators with non-trivial kernel is open with respect to the norm-topology by the above theorem. For $\varepsilon := \frac{c-1}{2}$ we can therefore choose $\nu > 0$ small enough such that for all $S \in L(X)$ satisfying $\|S - T\| < \nu$ we have that S is surjective, $N(S) \neq \{0\}$ and

$$\|(T^*)^m - (S^*)^m\| < \varepsilon.$$

Hence it follows

$$\|(T^*)^m x^*\| < \|(S^*)^m x^*\| + \varepsilon \text{ for all } x^* \in X^* \text{ with } \|x^*\| = 1$$

and therefore

$$c < s(T^m) = \kappa((T^*)^m) \leq \kappa((S^*)^m) + \varepsilon = s(S^m) + \varepsilon.$$

This implies in particular $1 < \frac{c+1}{2} = c - \varepsilon < s(S^m)$ and therefore by Remark 3.2.26

$$\left(\frac{c+1}{2}\right)^n < (s(S^m))^n \leq s(S^{mn}).$$

Hence we get

$$1 < \left(\frac{c+1}{2}\right)^{\frac{1}{m}} < (s(S^{mn}))^{\frac{1}{mn}} \to \delta(S) \text{ for } n \to \infty.$$

It follows that the open ball with radius ν and center T is included in M. $\square$

Corollary 3.2.33. *The set*

$$\begin{aligned} JA(l^1) &:= \{S \in L(l^1) : S^* \textit{ is J-class}\} \\ &= \{S \in L(l^1) : \sigma(S) \cap \partial\mathbb{D} \neq \emptyset, \overline{\mathbb{D}} \cap \sigma_p(S^{**}) = \emptyset\} \end{aligned}$$

is open with respect to the norm-topology in $L(l^1)$.

Proof. Consider the linear isometry $\phi : L(l^1) \to L(l^\infty)$ defined by $\phi(S) = S^*$. Then we have

$$\begin{aligned} \phi^{-1}(M) &= \{S \in L(l^1) : \delta(S^*) > 1 \text{ and } N(S^*) \neq \emptyset\} \\ &= \{S \in L(l^1) : i(S) > 1 \text{ and } N(S^*) \neq \emptyset\} \\ &= \{S \in L(l^1) : S^* \text{ is J-class}\} \\ &= \{S \in L(l^1) : \sigma(S^*) \cap \partial\mathbb{D} \neq \emptyset, \overline{\mathbb{D}} \cap \sigma_p(S^{**}) = \emptyset)\} \\ &= \{S \in L(l^1) : \sigma(S) \cap \partial\mathbb{D} \neq \emptyset, \overline{\mathbb{D}} \cap \sigma_p(S^{**}) = \emptyset\}, \end{aligned}$$

where M is the set as in Proposition 3.2.32 applied to l^∞ and since ϕ is continuous the desired statement follows by Proposition 3.2.32 and the equivalences of Theorem 3.2.29. $\square$

Remark 3.2.34. The latter corollary shows the strong relation between adjoint J-class operators on l^∞ and their spectral behaviour. It is also interesting in its own regarding operators on l^1 and their spectrum.

As promised before we will now characterize a large class of operators on l^∞ using our new tools established above.

Theorem 3.2.35. (i) *Consider $T \in L_A^W(l^\infty)$ with the corresponding operators S, W as in Definition 3.2.15 and let f be holomorphic on a neighborhood of $K_R(0)$, where $R > \max\{r(S^* + W), r(S^*)\}$ and $K_R(0)$ is the open disk at zero with radius R. Then $f(T)$ is J-class if and only if*

$$f(\sigma_a(T^*)) \cap \overline{\mathbb{D}} = \emptyset \textit{ and } \overline{\mathbb{D}} \subseteq f(\sigma(T^*) \backslash \sigma_a(T^*)).$$

(ii) *Let B_w be the unilateral backward shift with a positive and bounded weight sequence $w := (w_n)_n$ on l^∞ and consider $f(B_w)$, where f is a holomorphic map defined on a neighborhood of $\sigma(B_w)$. Then $f(B_w)$ is J-class if and only if*

$$\overline{\mathbb{D}} \cap f(\overline{K_{r_1}} \backslash K_{r_2}) = \emptyset \textit{ and } \overline{\mathbb{D}} \subseteq f(K_{r_2}),$$

where

$$r_1 = \lim_{n\to\infty} \sup_{k\in\mathbb{N}} (w_k \cdot \ldots \cdot w_{k+n-1})^{\frac{1}{n}},$$
$$r_2 = \lim_{n\to\infty} \inf_{k\in\mathbb{N}} (w_k \cdot \ldots \cdot w_{k+n-1})^{\frac{1}{n}}$$

and K_{r_i} *denotes the open disc centered at* 0 *with radius* r_i *for* $i = 1, 2,$.

Proof. (i) Let $\sum_{n=0}^{\infty} a_n z^n$ be the power series expansion of $f(z)$. For each positive integer we can write $(S^* + W)^n = (S^*)^n + W_n$, where W_n is weakly compact, since the weakly compact operators form a closed two-sided ideal (see [1], p. 89). Since $R > r(S^*)$ the corresponding operator $f(S^*) = (f(S))^*$ exists and so we get

$$\begin{aligned} f(T) - f(S^*) &= \sum_{n=0}^{\infty} a_n (S^* + W)^n - \sum_{n=0}^{\infty} a_n (S^*)^n \\ &= \sum_{n=0}^{\infty} a_n ((S^*)^n + W_n) - \sum_{n=0}^{\infty} a_n (S^*)^n \\ &= \sum_{n=0}^{\infty} a_n W_n =: V \end{aligned}$$

This shows in particular that the series V exists and is weakly compact. Therefore we get that $f(T) = (f(S))^* + V \in L_A^W(l^\infty)$ holds. Assume that $f(T)$ is J-class. Then by Theorem 3.2.27, the Spectral theorem and the Spectral theorem for the approximate spectrum (see A.1.1 and A.1.3) we get that

$$\emptyset = \overline{\mathbb{D}} \cap \sigma_a(f(T^*)) = \overline{\mathbb{D}} \cap f(\sigma_a(T^*)) \tag{1}$$

and

$$\begin{aligned} \overline{\mathbb{D}} &\subseteq \sigma_p(f(T)) \subseteq \sigma(f(T)) \\ &= \sigma((f(T))^*) = \sigma(f(T^*)) = f(\sigma(T^*)). \end{aligned}$$

Hence with (1) we get that $\overline{\mathbb{D}} \subseteq f(\sigma(T^*)\backslash\sigma_a(T^*))$ holds. This shows the first direction. For the other direction read the steps in (1) backward which in particular shows that $i(f(T)^*) > 1$. Moreover we have that

$$\overline{\mathbb{D}} \subseteq f(\sigma(T^*)\backslash\sigma_a(T^*)) = f(\sigma(T)\backslash\sigma_s(T)) \subseteq f(\sigma_p(T)) = \sigma_p(f(T)),$$

where we used the Spectral theorem for the point spectrum (see A.1.2). Again by Theorem 3.2.27 we get that $f(T)$ is J-class.
(ii) Let $T := B_w$. Consider the forward shift S_w on l^1 with a positive and bounded weight sequence $w = (w_1, w_2, w_3, \ldots,)$, i.e.

$$S_w(x_1, x_2, x_3, \ldots) = (0, w_1x_1, w_2x_2, w_3x_3, \ldots).$$

Then $T = S_w^*$ and the approximate spectrum of T^* is exactly

$$\sigma_a(T^*) = \sigma_a(S_w^{**}) = \sigma_a(S_w) = \{\lambda \in \mathbb{C}|\ i(S_w) \le |\lambda| \le r(S_w)\} = \overline{K_{r_1}}\backslash K_{r_2}, \text{ see } A.2.2,$$

where

$$i(S_w) = r_2 \text{ and } r(S_w) = r_1.$$

Further the whole spectrum of S_w is the closed disk with center 0 and radius r_1, see A.2.2. In other words $\sigma(T^*) = \sigma(S_w) = \overline{K_{r_1}}$. Hence we get by (i) that $f(B_w)$ is J-class if and only if

$$f(\overline{K_{r_1}} \backslash K_{r_2}) \cap \overline{\mathbb{D}} = f(\sigma_a(T^*)) \cap \overline{\mathbb{D}} = \emptyset \text{ and } \overline{\mathbb{D}} \subseteq f(\sigma(T) \backslash \sigma_a(T^*)) = f(K_{r_2}).$$

□

The following theorem gives a characterization of operators of the form $T = f(B)$, whenever they are hypercyclic.

Theorem 3.2.36. *([16], [20] p. 122) Let X be one of the spaces l^p, $1 \leq p < \infty$ or c_0. Further let f be a holomorphic function on a neighborhood of $\overline{\mathbb{D}}$. Then the following assertions are equivalent:*

(i) $f(B)$ is chaotic.

(ii) $f(\mathbb{D}) \cap \partial\mathbb{D} \neq \emptyset$.

(iii) $f(B)$ has a non-trivial periodic point.

Referring to the above theorem, we are interested in the case $p = \infty$ and the J-class behaviour of $f(B)$. Geometrically, condition (ii) is not enough to get the J-class property on l^∞. More conditions are needed as the next corollary will show.

Corollary 3.2.37. *Consider the backward shift B on l^∞ and let f be a holomorphic function in a neighborhood of the closed unit disc. Then the following statements are equivalent.*

(i) $f(B)$ is J-class.

(ii) $f(\partial\mathbb{D}) \cap \overline{\mathbb{D}} = \emptyset$ and $\overline{\mathbb{D}} \subseteq f(\mathbb{D})$.

(iii) $\min_{\theta \in [0,2\pi)} |f(e^{i\theta})| > 1$ and f has a zero with absolute value less than one.

Proof. Note that for the operator B the values r_3, r_2, r_1 are equal to 1. Then the statement follows directly from Theorem 3.2.35. □

At this point, we will give more practical, necessary and sufficient conditions for types of operators to be J-class. We have seen in Chapter 1 that it was quite technical to determine those complex numbers λ for which the operator $T = I + \lambda B$ is J-class or not. With our tools we are able to determine quantitative whenever $I + B_w$ is J-class and other types of operators gained by B_w.

Example 3.2.38. (Costakis, Manoussos, [15]) Consider $T := B_w$ on l^∞. Then T is J-class if and only if

$$r_2 = \lim_{n\to\infty} \inf_{k\in\mathbb{N}} (w_k \cdot w_{k+1} \cdots w_{n+k-1})^{\frac{1}{n}} > 1.$$

Proof. Consider $p(z) = z$. Then by Theorem 3.2.35 we have that T is J-class if and only if

$$p(\overline{K_{r_1}}\backslash K_{r_2}) \cap \overline{\mathbb{D}} = \overline{K_{r_1}}\backslash K_{r_2} \cap \overline{\mathbb{D}} \overbrace{=}^{(*)} \emptyset \text{ and } \overline{\mathbb{D}} \subset K_{r_2}.$$

The last inclusion follows already by $(*)$ and therefore T is J-class if and only if $r_2 > 1$. This completes the proof. $\square$

Example 3.2.39. Consider for $m \in \mathbb{N}$ the operator $T := I + B_w^m$. Then T is J-class if and only if

$$r_2 = \lim_{n\to\infty} \inf_{k\in\mathbb{N}} (w_k \cdot w_{k+1} \cdots w_{n+k-1})^{\frac{1}{n}} > \sqrt[m]{2}.$$

Proof. We can write $T = p(B_w)$, where $p(z) = 1 + z^m$. If T is J-class then

$$\begin{aligned} 1 < i(I + S_w^m) &= \operatorname{dist}(\{0\}, \sigma_a(I + S_w^m)) \\ &= \min\{|1 + z^m| : r_2 \leq |z| \leq r_1\} \\ &= \min_{\theta\in[0,2\pi]} |1 + r_2^m e^{i\theta m}| \end{aligned}$$

follows from Theorem 3.2.29. Hence this is equivalent to

$$r_2 > 2^{\frac{1}{m}}.$$

In view of the the definition of r_2, the desired direction follows.

For the other direction (see also Figure 3.1 below for the case $m = 1$) take into consideration that

$$r_3 \geq r_2 > 2^{\frac{1}{m}}$$

holds, where $r_3 = \liminf_{n\to\infty} (w_1 \cdot \ldots \cdot w_n)^{\frac{1}{n}}$ and it is easy to see that $K_{r_3} \subseteq \sigma_p(B_w)$ (K_{r_3} is the open disk centered at 0 with radius r_3).
This in particular shows that

$$\min_{\theta\in[0,2\pi]} |1 + r_3^m e^{i\theta m}| \geq \min_{\theta\in[0,2\pi]} |1 + r_2^m e^{i\theta m}| > 1$$

holds. Therefore we get that $0 \in \sigma_p(I + B_w^m)$. By Theorem 3.2.29 it follows that T is J-class.

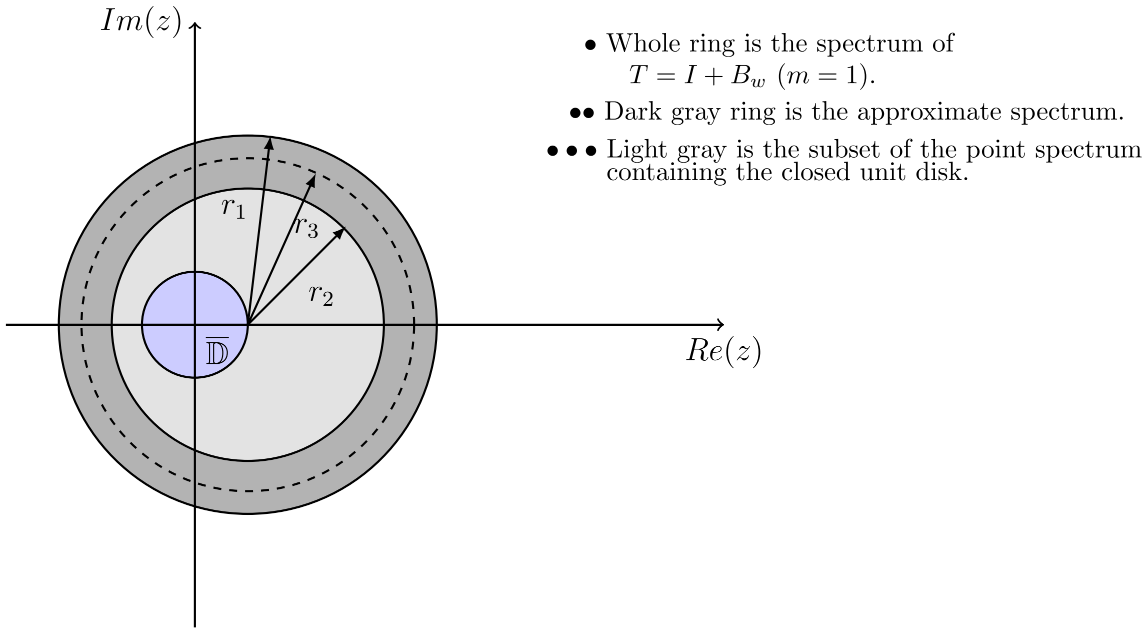

Figure 3.1: Spectrum of $T = I + B_w$ for the case $m = 1$ including the unit disk.

□

Example 3.2.40. Consider the operator $T := I - B_w + B_w^2$. Then T is J-class if and only if

$$r_2 = \lim_{n\to\infty} \inf_{k\in\mathbb{N}} (w_k \cdot w_{k+1} \cdots w_{n+k-1})^{\frac{1}{n}} > \sqrt{\frac{2+\sqrt{3}}{\sqrt{3}}}.$$

Proof. First consider the function $g : (1,\infty)\times[0,2\pi] \to \mathbb{R}$ defined by $g(r,t) := |p(re^{it})|^2$, where

$$p(z) = 1 - z + z^2 = (z - z_1)(z - \overline{z_1}) \text{ and } z_1 = \frac{1+i\sqrt{3}}{2}.$$

We get

$$\begin{aligned} g(r,t) &= |re^{it} - z_1|^2 |re^{it} - \overline{z_1}|^2 \\ &= |re^{it} - z_1|^2 |re^{-it} - z_1|^2 \\ &= |r^2 + z_1^2 - z_1 r(e^{it} + e^{-it})|^2 \\ &= |r^2 + \frac{1}{2}(-1 + i\sqrt{3}) - 2r\frac{1}{2}(1 + i\sqrt{3})\cos t|^2 \\ &= |r^2 - \frac{1}{2} - r\cos t + i\sqrt{3}(\frac{1}{2} - r\cos t)|^2 \\ &= (r^2 - \frac{1}{2} - r\cos t)^2 + 3(\frac{1}{2} - r\cos t)^2 \end{aligned}$$

A short calculation provides that

$$\frac{dg(r,t)}{dt} = 2r\sin(t)(1 + r^2 - 4r\cos t).$$

Hence,

$$2r\sin(t)(1 + r^2 - 4r\cos t) = 0 \;\Leftrightarrow\; t = 0, \pi \text{ or } \cos t = \frac{1+r^2}{4r}.$$

Our aim is now to estimate the value

$$\nu := \min\{|p(z)| : \ r_2 \le |z| \le r_1\}$$

with regard to g.

case 1:

$$\frac{1+r^2}{4r^2} \le 1 \ \Leftrightarrow \ 1 < r \le 2+\sqrt{3}.$$

For fixed r the function g takes its absolute minimum for those t for which we have $\cos t = \frac{1+r^2}{4r}$ ($t = 0, \pi$ are maxima!). Hence

$$g_{1,\min}(r) = \frac{3}{4}(r^2-1)^2.$$

case 2:

$$\frac{1+r^2}{4r^2} > 1 \ \Leftrightarrow \ r > 2+\sqrt{3}.$$

In this case, g takes its minimum at $t = 0$ and its maximum at $t = \pi$. Hence

$$g_{2,\min}(r) = (r^2 - \frac{1}{2} - r)^2 + 3(\frac{1}{2} - r)^2.$$

Suppose at this point that T is J-class and assume first that r_1, r_2 both satisfy the condition in case 1. Then we have $g_{1,\min}(r_2) \le g_{1,\min}(r_1)$ and therefore $\nu = g_{1,\min}(r_2)$. By assumption

$$\nu > 1 \iff \frac{3}{4}(r_2^2-1)^2 > 1 \iff r_2 > \sqrt{\frac{2+\sqrt{3}}{\sqrt{3}}}.$$

Consider now the case where r_2 satisfies the condition in case 2. Then

$$r_2 > 2+\sqrt{3} > \sqrt{\frac{2+\sqrt{3}}{\sqrt{3}}}$$

and there is nothing to prove. Assume now that r_1 satisfies the condition in case 2 and r_2 satisfies the condition in case 1. Then since $r_3 > r_2 > 1$ and $g_{2,\min}(r) \ge g_{1,\min}(r)$ for all $r \ge 0$ we get

$$g_{2,\min}(r_1) \ge g_{1,\min}(r_1) \ge g_{1,\min}(r_2) \text{ and therefore } \nu = g_{1,\min}(r_2).$$

Finally we get that

$$(*) \qquad r_2 > \sqrt{\frac{2+\sqrt{3}}{\sqrt{3}}}$$

holds if T is J-class. Suppose now that $(*)$ is satisfied. Reading the steps backward it is easy to see that

$$i(S) = \nu > 1 \text{ where } S = I - S_w + S_w^2$$

holds. Since $r_3 \ge r_2$ this implies $z_1 = \frac{1}{2}(1+i\sqrt{3}) \in K_{r_3}(0) \subseteq \sigma_p(B_w)$ and therefore $0 = p(z_1) \in \sigma_p(T)$. By an application of Theorem 3.2.29 we see that T is J-class. This completes the proof. $\square$

Let us leave for short the sequence spaces at this point and consider the space $L^1([0,\infty))$. The operator $S_\lambda : L^1([0,\infty)) \to L^1([0,\infty))$ defined by

$$(S_\lambda f)(x) := \lambda f(x-s) \text{ for } x \geq s \text{ and } (Sf)(x) = 0 \text{ for } 0 \leq x < s.$$

is the continuous counterpart of the forward shift on l^1, where $|\lambda| > 1$ and s are fixed. It is easy to see that $S^* = B$, where

$$B : L^\infty([0,\infty)) \to L^\infty([0,\infty)) \text{ and } Bf(s) := \lambda f(x+s).$$

Since $L^1([0,\infty))$ does not have the Schur property, it is not clear if $\sigma_a(S_\lambda) = \sigma_p(S_\lambda^{**})$. But the next proposition shows that this in fact is also the case by using the preceding properties of operators on l^1.

Proposition 3.2.41. *Consider the operator S_λ on $L^1([0,\infty))$. Then*

$$\sigma_a(S_\lambda) = \sigma_p(S_\lambda^{**}).$$

Proof. Consider the map $\Phi : l^1 \to X$, where

$$X := \overline{\text{span}}\left\{\chi_{[s(n-1),sn]} : \; n \in \mathbb{N}\right\}$$

and

$$\Phi(e_n) := \chi_{[s(n-1),sn]},$$

s is as above and χ is the characteristic function. Hence, X is isomorphic to l^1 and therefore admits the Schur-property. We conclude, that $S_\lambda(X) \subseteq X$ and $\sigma_a(S_\lambda|_X) = \lambda \cdot \partial\mathbb{D}$. Since $i(S_\lambda) = r(S_\lambda) = 1$ it follows that

$$\sigma_a(S_\lambda) = \sigma_a(S_\lambda|_X) = \sigma_p((S_\lambda|_X)^{**}) \subseteq \sigma_p(S_\lambda^{**})$$

holds. The other inclusion $\sigma_p(S_\lambda^{**}) \subseteq \sigma_a(S_\lambda)$ follows as in the proof of Theorem 3.2.7. □

With the previous observation and identical calculations as before we obtain that Corollary 3.2.37 is valid for the continuous backward shift B on $L^\infty([0,\infty))$, which is left for the reader.

Let us return to the sequence spaces, but consider now sequences indexed by $\mathbb{Z}$. We are interested in J-class operators on $l^\infty(\mathbb{Z})$ which are adjoints of operators on $l^1(\mathbb{Z})$. Obviously, $l^1(\mathbb{Z})$ is isomorphic to $l^1(\mathbb{N}) \oplus l^1(\mathbb{N}) \cong l^1(\mathbb{N})$. Hence Theorem 3.2.29 is valid for the space $l^\infty(\mathbb{Z})$. In particular we are interested in the J-class behaviour of the bilateral weighted shift on $l^\infty(\mathbb{Z})$. More precisely, consider any bounded weight sequence $w := (w_n)_{n\in\mathbb{Z}}$ and define $B_w : l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z})$ by

$$B_w((x_n)_{n\in\mathbb{Z}}) := (w_n \cdot x_{n+1})_{n\in\mathbb{Z}}.$$

In [15] G. Costakis and A. Manoussos showed that bilateral weighted shifts are never J-class. This is a little bit surprising, since on c_0 and l^p ($1 \leq p < \infty$) there exist bilateral weighted shifts, see [20], p. 102. However the next question would be: Is

$f(B_w)$ J-class, where f is a holomorphic function defined on a neighborhood of the spectrum of B_w? To avoid technical details we give an answer to this question for the operator $T = \omega I + B_w$ with $\omega > 0$. Let us examine the spectral properties of bilateral weighted shifts, which are similar to the unilateral weighted shifts but a little bit more complicated. At this point we introduce some abbreviations.

- $r_1^+ := \lim_{n\to\infty} \inf_{k\geq 0} (w_k \cdots w_{k+n-1})^{\frac{1}{n}},\ r^+ := \lim_{n\to\infty} \sup_{k\geq 0} (w_k \cdots w_{k+n-1})^{\frac{1}{n}}$
- $r_1^- := \lim_{n\to\infty} \inf_{k<0} (w_k \cdots w_{k-n})^{\frac{1}{n}},\ r^- := \lim_{n\to\infty} \sup_{k<0} (w_k \cdots w_{k-n})^{\frac{1}{n}}$
- $r_2^+ := \liminf_{n\to\infty}(w_0 \cdots w_{n-1})^{\frac{1}{n}},\ r_3^+ := \limsup_{n\to\infty}(w_0 \cdots w_{n-1})^{\frac{1}{n}}$
- $r_2^- := \liminf_{n\to\infty}(w_{-1} \cdots w_{-n})^{\frac{1}{n}},\ r_3^+ := \limsup_{n\to\infty}(w_{-1} \cdots w_{-n})^{\frac{1}{n}}$

We have that

$$(*)\qquad r_1^- \leq r_2^- \leq r_3^- \leq r^- \text{ and } r_1^+ \leq r_2^+ \leq r_3^+ \leq r^+.$$

Theorem 3.2.42. *([36], p. 70/ p. 71) Let S_w be the bilateral weighted forward shift on $l^p(\mathbb{Z})$, $1 \leq p < \infty$. That means $S_w((x_n)_n) = (w_{n-1}x_{n-1})_{n\in\mathbb{Z}}$ with a positive and bounded weight sequence $w = (w_n)_{n\in\mathbb{Z}}$. Then S_w has the following spectral properties.*

1. *If S_w is an invertible bilateral weighted shift, then the spectrum of S_w is the annulus*
$$\{z \in \mathbb{C} : (r(T^{-1}))^{-1} \leq |z| \leq r(T)\}.$$
2. *If S_w is not invertible then the spectrum is the disc centered at zero with radius $r(S_w)$.*
3. *If $r^- < r_1^+$ then*
$$\sigma_a(S_w) = \{z \in \mathbb{C} : r_1^- \leq |z| \leq r^-\} \cup \{z \in \mathbb{C} : r_1^+ \leq |z| \leq r^+\}$$
holds. Otherwise we have
$$\sigma_a(S_w) = \{z \in \mathbb{C} : \min\{r_1^-, r_1^+\} \leq |z| \leq \max\{r^-, r^+\}\}.$$
4. - $\{z \in \mathbb{C} : r_3^+ < |z| < r_2^-\} \subseteq \sigma_p(S_w) \subseteq \{z \in \mathbb{C} : r_3^+ \leq |z| \leq r_2^-\}$
   - $\{z \in \mathbb{C} : r_3^- < |z| < r_2^+\} \subseteq \sigma_p(B_w) \subseteq \{z \in \mathbb{C} : r_3^- \leq |z| \leq r_2^+\}$
   - *At least one of $\sigma_p(S_w)$, $\sigma_p(B_w)$ is empty.*

Remark 3.2.43. From the previous theorem we get the following observation: If $r^- \geq r_1^+$ then $\sigma_a(S_w) = \{z \in \mathbb{C} : \min\{r_1^-, r_1^+\} \leq |z| \leq \max\{r^-, r^+\}\} = \sigma(S_w)$. To see this consider first the case when S_w is invertible. Then $(r(T^{-1}))^{-1} = \min\{r_1^-, r_1^+\}$ and therefore the annuli in 1) and 3) of the previous theorem must be the same. If S_w is not invertible, then we have to show that $\min\{r_1^-, r_1^+\} = 0$. If this is false then S_w has closed range since it is bounded below. Furthermore since S_w is injective but not invertible we conclude that S_w is not surjective. Thus S_w has not dense range which implies that $0 \in \sigma_p(B_w)$. Hence $r_3^- = 0$ by 4) of the previous theorem and because of $(*)$ we get $0 = r_1^- = \min\{r_1^-, r_1^+\}$.

Theorem 3.2.44. *Let B_w be the bilateral backward shift on $l^\infty(\mathbb{Z})$ and let $\omega > 0$. Then the following statements are equivalent.*

(1) $T := \omega I + B_w$ is J-class.

(2) The numbers r_1^+, r^- satisfy

$$r^- + 1 < \omega < r_1^+ - 1$$

Proof. Assume first that T is J-class and put $S := \omega I + S_w$. Then $r^- < r_1^+$. If this is not true, then by Remark 3.2.43 we get

$$\sigma(S_w) = \sigma_a(S_w) = \{z \in \mathbb{C} : \min\{r_1^-, r_1^+\} \leq |z| \leq \max\{r^-, r^+\}\}.$$

But then

$$1 < i(S) = \operatorname{dist}(\{0\}, \sigma_a(S)) = \operatorname{dist}(\{0\}, \sigma(S)) = \operatorname{dist}(\{0\}, \sigma(T)),$$

which in turn means that $\sigma(T) \cap \overline{\mathbb{D}} = \emptyset$ and this contradicts Lemma 1.2.12. By Theorem 3.2.42 3) we have

$$\sigma_a(S_w) = \{z \in \mathbb{C} : r_1^- \leq |z| \leq r^-\} \cup \{z \in \mathbb{C} : r_1^+ \leq |z| \leq r^+\}.$$

Hence,

$$\begin{aligned}
1 &< i(\omega I + S_w) \\
&= \operatorname{dist}(\{0\}, \sigma_a(\omega I + S_w)) \\
&= \operatorname{dist}(\{0\}, \omega + \{\{z \in \mathbb{C} : r_1^- \leq |z| \leq r^-\} \cup \{z \in \mathbb{C} : r_1^+ \leq |z| \leq r^+\}\}) \\
&= \min\{\operatorname{dist}(\{0\},\ \{\omega + e^{i\theta} r^- : \theta \in \mathbb{R}\}),\ \operatorname{dist}\ (\{0\},\ \{\omega + e^{i\theta} r_1^+ : \theta \in \mathbb{R}\})\}
\end{aligned}$$

where the last equality follows from the fact that

$$\overline{\mathbb{D}} \subseteq \omega + \{z \in \mathbb{C} : r^- < |z| < r_1^+\} = \sigma(T) \backslash \sigma_a(S) = \sigma(T) \backslash \sigma_s(T) \subseteq \sigma_p(T),$$

which can be seen as in the proof of Theorem 3.2.21. Hence Figure 3.2 below shows that the outer dark gray ring surrounds the closed unit disk and it is then obvious that $\omega > r^- + 1$ and $\omega < r_1^+ - 1$.
For the other direction just go through the steps backward and use Theorem 3.2.29. $\square$

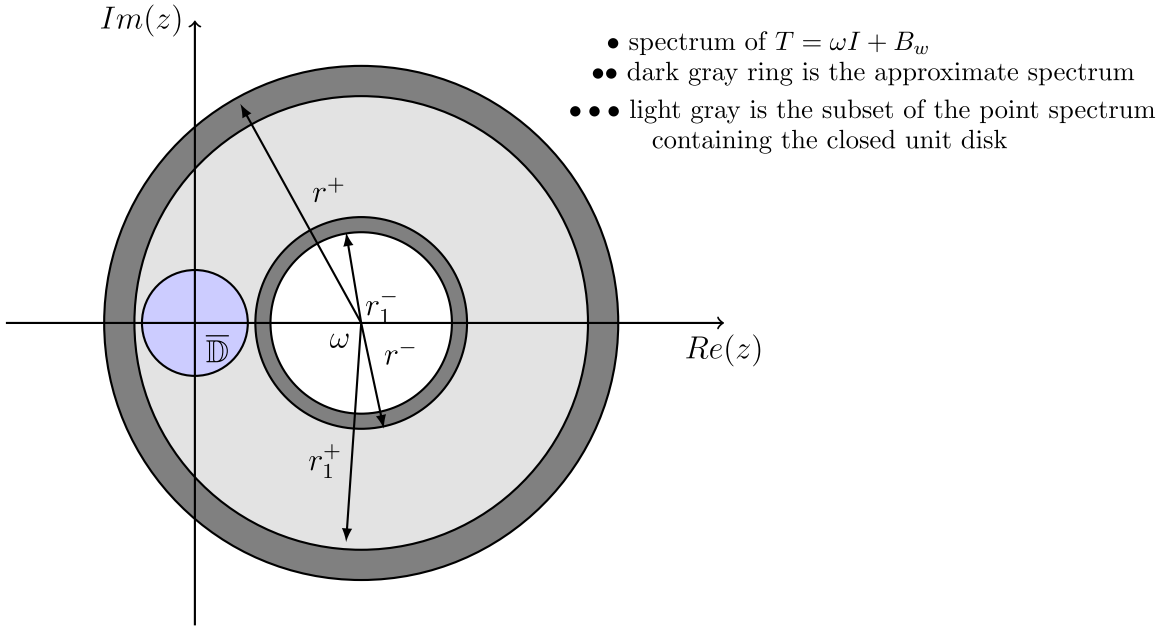

Figure 3.2: Spectrum of $T = \omega I + B_w$ including the closed unit disk.

Remark 3.2.45. From the above theorem we get immediately that B_w is never J-class on $l^\infty(\mathbb{Z})$ if we choose $\omega = 0$. This result was also obtained by A. Manoussos and G. Costakis in [15] as already mentioned.

3.3 The set of J-vectors for operators on l^∞ which are double adjoints

We have seen in Chapter 1 that the set A_T of J-vectors is equal to c_0, where $T = I + \lambda B$. Therefore we are interested in the more general case $f(B_w)$, where again B_w is the unilateral backward shift and f is any non-constant holomorphic function defined on a neighborhood of the spectrum of B_w. And indeed in this section we will show that $A_{f(B_w)} = c_0$ holds. We will line out two different ways to obtain this result. The first one is more elementary in itself and is based on the idea by calculating directly the spectrum of the induced operator $\widehat{B_w}$, where $\widehat{B_w} : l^\infty/c_0 \to l^\infty/c_0$ and $\widehat{B_w}\,[x]_{c_0} = [B_w x]_{c_0}$. The second proof is based on Kaltons and Bermúdez theorem (see Theorem 3.1.7), but has the advantage that we can say more about $A_{T^{**}}$, where $T : c_0 \to c_0$ is any bounded linear operator. As a main result we will see that $f(B_w)$ being J-class on l^∞ implies that $f(B_w)$ is mixing on c_0.

Proposition 3.3.1. *Consider the positive weighted unilateral backward shift $B_w : l^\infty \to l^\infty$. Then we have*

$$\sigma(\widehat{B_w}) \subseteq \sigma_a(S_w) = \{\lambda \in \mathbb{C} |\ i(S_w) \le |\lambda| \le r(S_w)\},$$

where S_w is the forward shift on l^1.

Proof. Consider any $\mu \in \mathbb{C}$ with $c := |\mu| < i(S_w)$. We choose $n_0 \in \mathbb{N}$ large enough such that

$$\inf_{k\in\mathbb{N}} w_k \cdot \ldots \cdot w_{k+n_0-1} > c^{n_0}.$$

Therefore we can find $\delta < 1$ such that

$$1 > \delta > \frac{c^{n_0}}{w_k \cdot \ldots \cdot w_{k+n_0-1}}$$

holds for all $k \in \mathbb{N}$. We will show that $(\widehat{B_w})^{n_0} - \mu^{n_0} I$ is bijective. Take into consideration that

$$\begin{aligned} B_w^{n_0}(x_1, x_2, \ldots) &= ((w_1 w_2 \ldots w_{n_0} \cdot x_{n_0+1}), (w_2 w_3 \ldots w_{n_0+1} \cdot x_{n_0+2}), \ldots) \\ &= (w_k w_{k+1} \ldots w_{n_0+k-1} \cdot x_{k+n_0})_{k\in\mathbb{N}}. \end{aligned}$$

Consider the equation

$$((\widehat{B_w})^{n_0} - \mu^{n_0} I)\,[x] = 0 \text{ with } [x] = [(x_1, x_2, \ldots)]\,,$$

which is equivalent to say that

$$B_w^{n_0}(x_1, x_2, \ldots) - \mu^{n_0}(x_1, x_2, \ldots) \in c_0,$$

or more precisely

$$w_k \cdot \ldots \cdot w_{k+n_0-1} x_{k+n_0} - \mu^{n_0} x_k = \varepsilon_k \text{ for all } k \ge 1,$$

where $\lim_{k\to\infty} \varepsilon_k = 0$. Hence we get the estimate

$$\begin{aligned}|x_{k+n_0}| &\leq \frac{c^{n_0}}{w_k \cdot \ldots \cdot w_{k+n_0-1}}|x_k| + \frac{\varepsilon_k}{w_k \cdot \ldots \cdot w_{k+n_0-1}} \\ &\leq \delta|x_k| + \frac{\varepsilon_k}{c^{n_0}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\limsup_{k\to\infty} |x_{k+n_0}| &\leq \delta \limsup_{k\to\infty} |x_k| + \frac{1}{c^{n_0}} \lim_{k\to\infty} \varepsilon_k \\ &= \delta \limsup_{k\to\infty} |x_k|.\end{aligned}$$

Since $\limsup_{k\to\infty} |x_{k+n_0}| = \limsup_{k\to\infty} |x_k|$ and δ is smaller than 1, it follows that $\limsup_{k\to\infty} |x_k| = 0$ and therefore $\lim_{k\to\infty} x_k = 0$. In other words, $[x] = 0$ holds and hence $(\widehat{B_w})^{n_0} - \mu^{n_0} I$ is injective.

Next we show that $(\widehat{B_w})^{n_0} - \mu^{n_0} I$ is surjective. For that purpose consider the forward shift S_w on l^1. Since

$$\sigma_s(B_w) = \sigma_a(S_w) = \{\lambda \in \mathbb{C} \mid i(S_w) \leq |\lambda| \leq r(S_w)\}$$

it follows that $B_w - e^{i\theta_k} cI$ is surjective for $k = \{1, 2, ..., n_0\}$, where $e^{i\theta_k} c$ are the roots of the equation $z^{n_0} = \mu^{n_0}$. Now notice that

$$B_w^{n_0} - \mu^{n_0} I = (B_w - e^{i\theta_1} cI) \cdot \ldots \cdot (B_w - e^{i\theta_n} cI)$$

and therefore $B_w^{n_0} - \mu^{n_0} I$ is surjective. Since the surjectivity of every operator T implies the surjectivity of the induced operator $\widehat{T}$, it follows that $(\widehat{B_w})^{n_0} - \mu^{n_0} I$ is surjective, hence bijective. This means $\mu^{n_0} \notin \sigma((\widehat{B_w})^{n_0})$ and therefore $\mu \notin \sigma(\widehat{B_w})$. From

$$\left\|\widehat{B_w}^n\right\| \leq \|B_w^n\| = \|S_w^n\|$$

we get $r(\widehat{B_w}) \leq r(S_w)$. Finally we obtain with the preceding calculations that

$$\sigma(\widehat{B_w}) \subseteq \{\lambda \in \mathbb{C} \mid i(S_w) \leq |\lambda| \leq r(\widehat{B_w})\} \subseteq \{\lambda \in \mathbb{C} \mid i(S_w) \leq |\lambda| \leq r(S_w)\} = \sigma_a(S_w)$$

holds. □

Lemma 3.3.2. *Let $(w_n)_n$ be a positive and bounded sequence. Suppose there exists $\lambda_0 \in \mathbb{C}$ and $\varepsilon > 0$ such that the open disk $K_\varepsilon(\lambda_0)$ is contained in $K_m(0)$, where*

$$m := \liminf_{n\to\infty} \left(\prod_{k=1}^{n} w_k\right)^{\frac{1}{n}} > 0.$$

Then the span of the set consisting of the vectors

$$e_\lambda := \left(\frac{\lambda}{w_1}, \frac{\lambda^2}{w_1 w_2}, \ldots\right),$$

with $\lambda \in K_\varepsilon(\lambda_0)$ is dense in c_0.

Proof. Consider any $y^* \in {c_0}^* \cong l^1$. Assume $y^*(e_\lambda) = 0$. We can find a sequence $(y_n)_n \in l^1$ which can be uniquely identified with y^*. That means

$$y^*(e_\lambda) = \sum_{n=0}^{\infty} y_n \cdot \frac{\lambda^n}{(\prod_{k=1}^n w_k)}$$

holds. The above equation defines a power series $\sum_{n=0}^\infty a_n \lambda^n$ on $K_m(0)$ with $a_n := \frac{y_n}{(\prod_{k=1}^n w_k)}$, which is identically zero for all $\lambda \in K_\varepsilon(\lambda_0)$. Therefore $a_n = 0$ for all $n \in \mathbb{N}$, which in turn implies that $y_n = 0$ for all $n \in \mathbb{N}$. This shows that $y^* = 0$. By a well-known application of the Hahn-Banach theorem it follows that the span of the e_λ is dense in c_0. □

Theorem 3.3.3. *Consider the weighted backward shift on l^∞. Assume that $T := f(B_w)$ is J-class, where f is a holomorphic function defined on a open neighborhood of $\sigma(B_w)$. Then*

$$A_T = c_0.$$

Proof. To show that $A_T \subseteq c_0$, we consider the induced operator $\widehat{T} : l^\infty/c_0 \to l^\infty/c_0$ and assume that $A_T \backslash c_0 \neq \emptyset$. By Lemma 1.3.6, the operator $\widehat{T}$ is J-class. Then we get from the Spectral mapping theorems (see A.1.1 and A.1.3) and Proposition 3.3.1

$$(*) \qquad \sigma(\widehat{T}) = \sigma(\widehat{f(B_w)})) = f(\sigma(\widehat{B_w})) \subseteq f(\sigma_a(S_w)) = \sigma_a(f(S_w)).$$

Since $f(B_w)$ is J-class it follows from Remark 3.2.22, that

$$\sigma_a(f(S_w)) \cap \overline{\mathbb{D}} = \emptyset$$

holds. Hence by $(*)$ we get that

$$\sigma(\widehat{T}) \cap \overline{\mathbb{D}} = \emptyset$$

holds, which is a contradiction to Lemma 1.2.12 (Spectral lemma), since $\widehat{T}$ is J-class.

To show the other inclusion, take into consideration that $r_2 \leq m$ where r_2 is the value as in Theorem 3.2.35. Hence by the same theorem we get that $\mathbb{D} \subseteq \overline{\mathbb{D}} \subseteq f(K_{r_2}(0)) \subseteq f(K_m(0))$ holds, where $K_{r_2}(0)$ is the open disk with center zero and radius r_2. The set $f^{-1}(\mathbb{D}) \cap K_m(0)$ is open and non-empty. Choose at this point any λ_0 and $\varepsilon > 0$ such that $K_\varepsilon(\lambda_0) \subseteq f^{-1}(\mathbb{D}) \cap K_m(0)$. Then, $f(K_\varepsilon(\lambda_0)) \subseteq f(f^{-1}(\mathbb{D}) \cap K_m(0)) \subseteq \mathbb{D}$. Now each e_λ, defined as in Lemma 3.3.2 is an eigenvector of B_w to the corresponding eigenvalue $\lambda \in K_\varepsilon(\lambda_0)$. Hence, e_λ is an eigenvalue of $f(B_w)$ to the corresponding eigenvalue $f(\lambda) \in f(K_\varepsilon(\lambda_0))$. Since $|f(\lambda)| < 1$, it follows that e_λ is a J-vector by Proposition 1.2.8. By Lemma 3.3.2 the set $\{e_\lambda : \lambda \in K_\varepsilon(\lambda_0)\}$ is dense in c_0. This shows $c_0 \subseteq A_T$, since A_T is closed by Proposition 1.2.4. □

Corollary 3.3.4. *Let $f(B_w)$ be J-class on l^∞. Then $f(B_w)$ is hypercyclic on c_0.*

Proof. Let $f(B_w)$ be J-class on l^∞. It follows that $i(f(S_w)) > 1$ and therefore

$$i((f(B_w))^*) = i(f(S_w)) > 1,$$

where this time $f(B_w)$ is considered on c_0. Hence, $f(B_w)$ is J-class on c_0 by Remark 3.2.28 b). Moreover the set $\{e_\lambda : \lambda \in K_\varepsilon(\lambda_0)\}$, as in the previous theorem, is dense in c_0 and $A_{f(B_w),c_0} = c_0$ (the index c_0 stresses that we are considering the operator on c_0). This latter fact can be seen by arguments to those in the previous theorem. We conclude with Theorem 1.2.5 that $f(B_w)$ is hypercyclic on c_0. □

The second proof of Theorem 3.3.3 is mainly based on a deep result of Kalton and Bermúdez stated in Theorem 3.1.7.

Proposition 3.3.5. *Consider a surjective operator $S : c_0 \to c_0$. Assume that S^{co} is invertible. Then*

$$\sigma(S^{co}) \subseteq \{\lambda \in \mathbb{C}|\ i(S^*) \leq |\lambda| \leq r(S^*)\}$$

holds.

Proof. Define $T := S^{**} : l^\infty \to l^\infty$. Hence, we obtain with Theorem 3.1.7 and Remark 3.1.2 that

$$\begin{aligned}\partial\sigma(T^{co}) &= \partial\sigma((T^{co})^*) = \partial\sigma((T^*)^{co}) \subseteq \sigma_a((T^*)^{co}) \\ &\subseteq \sigma_p(T^*) \subseteq \sigma_a(T^*) = \sigma_a(S^{***}) = \sigma_s(S^{**}) \\ &= \sigma_a(S^*)\end{aligned}$$

holds. Note at this point that we have for any operator R on any Banach space the relation:

$$\sigma_a(R) \subseteq \{\lambda \in \mathbb{C}|\ i(R) \leq |\lambda| \leq r(R)\}, (\text{ see } A.2.1).$$

From this we get $\partial\sigma(T^{co}) \subseteq \{\lambda \in \mathbb{C}|\ i(S^*) \leq |\lambda| \leq r(S^*)\}$. Again from Remark 3.1.2 we have $\sigma(S^{co}) = \sigma((S^{co})^{**}) = \sigma((S^{**})^{co}) = \sigma(T^{co})$. Therefore by the preceding statements the boundary of the spectrum of S^{co} is contained in the annulus. The fact that S^{co} is invertible implies that $0 \notin \sigma(S^{co})$ and therefore by a similar argument as in Lemma 3.2.20 we conclude

$$\sigma(S^{co}) \subseteq \{\lambda \in \mathbb{C} \mid i(S^*) \leq |\lambda| \leq r(S^*)\}.$$

□

Proposition 3.3.6. *Consider $S \in L(c_0)$. Assume that S^{co} is an isomorphism. If $T : l^\infty \to l^\infty$ defined as $T = S^{**}$ is J-class, then*

$$A_T \subseteq c_0.$$

If in addition S is a mixing operator, then

$$c_0 = A_T.$$

Proof. Let T be J-class. Then by Theorem 3.2.29 (v) we have $i(S^*) > 1$. Assume at this point that $A_T \backslash c_0$ is not empty. Then by Lemma 1.3.6, the induced operator $\widehat{T} = S^{co}$ is J-class. But then by our Spectral lemma we get that

$$\sigma(\widehat{T}) \cap \partial\mathbb{D} \neq \emptyset$$

holds. By Proposition 3.3.5 we also have

$$\sigma(\widehat{T}) \subseteq \{\lambda \in \mathbb{C} \mid i(S^*) \leq |\lambda| \leq r(S^*)\}.$$

This is not possible since $i(S^*) > 1$. This shows the first statement.

If S is a mixing operator then $A_S^{mix} = c_0$ holds by Proposition 1.2.5. Therefore, for each $x \in c_0$ there exist a sequence $(x_n)_n$ in c_0 such that $(x_n)_n$ converges to x and $S^n x_n$ converges to zero. Note that $S^{**} = T$ is the extension of S and thereby $T^n x_n$ converges to zero. With Proposition 1.2.8 (iii) we conclude that $J_T(x) = l^\infty$. Hence $A_T = c_0$. □

Remark 3.3.7. Consider any Banach space X. If S is Fredholm, then S^{co} is invertible. This can be seen by the following easy observation: If S is Fredholm, we can find closed subspaces W, V of finite dimension and respectively finite co-dimension, such that $X = N(S) \oplus V = W \oplus R(S)$. Furthermore we find operators S_1 and S_2 such that $S = S_1 \oplus S_2$, where $S_1 : N(S) \to W$ is the null-operator and $S_2 : V \to R(S)$ is invertible. Hence we get $X^{co} \cong N(S)^{co} \oplus V^{co} \cong V^{co}$ since $N(S)$ is finite dimensional and therefore $N(S)^{co} = \{0\}$. Therefore $S^{co} \cong S_1^{co} \oplus S_2^{co} = 0 \oplus S_2^{co} \cong S_2^{co}$ (actually this can be seen as in Proposition 3.1.3) is invertible, since S_2 is invertible.

We are ready to formulate Theorem 3.3.3 with a different proof.

Corollary 3.3.8. *Let $f(B_w)$ be as in Theorem* 3.3.3*. Then $c_0 = A_{f(B_w)}$.*

Proof. We show the inclusion $A_{f(B_w)} \subseteq c_0$; the other inclusion can be proved as in Theorem 3.3.3. Since $f(B_w)$ is J-class, we get that $\sigma_a(f(S_w)) \cap \overline{\mathbb{D}} = \emptyset$ by Remark 3.2.22. The non-trivial equation $\sigma_{ess}(S_w) = \sigma_a(S_w)$ (see [25], p. 289), implies with the Spectral theorems for the essential and approximate spectrum (see A.1.4 and A.1.3) that

$$\sigma_{ess}(f(S_w)) \cap \overline{\mathbb{D}} = f(\sigma_{ess}(S_w)) \cap \overline{\mathbb{D}} = f(\sigma_a(S_w)) \cap \overline{\mathbb{D}} = \sigma_a(f(S_w)) \cap \overline{\mathbb{D}} = \emptyset$$

holds. In particular, $0 \notin \sigma_{ess}(f(S_w))$. Hence, $f(S_w) = (f(B_w))^*$ (here B_w is defined on c_0) is Fredholm and therefore $f(B_w)$. It follows that $\widehat{f(B_w)}$ is invertible by the above remark and so the conditions in the above proposition are satisfied. It follows the desired statement. □

Chapter 4

Locally topologically transitive C_0-semigroups

4.1 Local topologically transitive C_0-semigroups

In this section, we introduce the notion of locally topologically transitive C_0-semigroups. Hypercyclic, or equivalently, topologically transitive C_0-semigroups on separable Banach space have been studied extensively in the last 15 years. We refer to [17] and [20] for more details. For the existence of hypercyclic semigroups on separable Banach spaces, the weighted backward shift plays a key role as was shown in [10]. At the end of this chapter we will see that on the space $(X_A)_{\mathbb{C}}$ introduced in Chapter 2, there does not exist any locally topologically transitive semigroup.
We begin with some basic facts and definitions.

Definition 4.1.1. Let X be a Banach space. A system $\mathfrak{T} = \{T_t \mid t \geq 0\}$ of operators $T_t : X \to X$ is called C_0-semigroup, if the following conditions are satisfied

(1) $T_0 = I$.

(2) $T_{s+t} = T_s \circ T_t$ for all $s, t \geq 0$.

(3) $\lim\limits_{t \to 0} T_t x = x$.

Moreover the generator of $\mathfrak{T}$ is defined as the operator

$$Ax := \lim_{t \to 0} \frac{T_t x - x}{t}$$

defined on the domain

$$\mathrm{dom}(A) = \{x \in X \mid \lim_{t \to 0} \frac{T_t x - x}{t} \text{ exists}\}$$

Remark 4.1.2. It is clear that the operators in the above definition must commute with each other.

The next lemma can be found in your favorite functional analysis book, see for instance [37], p. 357.

Lemma 4.1.3. *Let X be a Banach space and $\mathcal{T} = \{T_t \mid t \geq 0\}$ a C_0-semigroup. Then there exist constants $M \geq 1$, $\omega \in \mathbb{R}$ such that*

$$\|T_t\| \leq Me^{\omega t}$$

for all $t \geq 0$.

Definition 4.1.4. Let X be a Banach space, and $\mathcal{T} = \{T_t \mid t \geq 0\}$ a C_0-semigroup. $\mathcal{T}$ is called topologically transitive, if for all non-empty and open subsets $U, V \subseteq X$ there exists a positive real number t, such that

$$T_t(U) \cap V \neq \emptyset$$

holds.

Remark 4.1.5. If X is separable, it follows as in the discrete case from the Baire category theorem, that the above definition is equivalent to the fact that there exists a vector $x \in X$ such that $\{T_t x \mid t \geq 0\}$ is dense in X, see also [20], p. 186. In this case x is called a hypercyclic vector for $\mathcal{T}$.

We now define the locally topologically transitivity of operators analogous to that of J-class operators. For $x \in X$ the symbol $\mathcal{U}(x)$ denotes the family of all open neighborhoods of x.

Definition 4.1.6. Let X be a Banach space, and $\mathcal{T} = \{T_t \mid t \geq 0\}$ a C_0-semigroup. $\mathcal{T}$ is called locally topologically transitive, if there exists a vector $x \neq 0$, such that for all non-empty and open subsets $U \in \mathcal{U}(x)$, $V \subseteq X$, there exists a positive real number t with

$$T_t(U) \cap V \neq \emptyset.$$

Definition 4.1.7. Let X be a Banach space and $\mathcal{T} = \{T_t \mid t \geq 0\}$ a C_0-semigroup. The J-set of $\mathcal{T}$ under $x \in X$ is defined as

$$\begin{aligned} J_{\mathcal{T}}(x) := \{y \in X :\ & \text{there exists a strictly increasing sequence} \\ & \text{of positive real numbers } (t_n)_n \text{ with } t_n \to \infty \\ & \text{and a sequence } (x_n)_n \text{ in } X, \\ & \text{such that } x_n \to x \text{ and } T_{t_n} x_n \to y\}. \end{aligned}$$

We put $A_{\mathcal{T}} := \{x \in X \mid J_{\mathcal{T}}(x) = X\}$. The elements of $A_{\mathcal{T}}$ will be called J-vectors.

The next proposition shows us, that for a C_0 semigroup being locally topologically transitive is equivalent to existence of a non-zero vector x such that $J_{\mathcal{T}}(x) = X$.

Proposition 4.1.8. *Let X be a Banach space and $\mathcal{T} = \{T_t \mid t \geq 0\}$ a C_0-semigroup on X. Then $\mathcal{T}$ is locally topologically transitive if and only if there exists a vector $x \neq 0$, such that $J_{\mathcal{T}}(x) = X$.*

Proof. Assume that $\mathcal{T}$ is locally topologically transitive with the corresponding vector $x \in X$. Consider any $y \in X$ and assume first that $y \notin \{T_t x \mid t \geq 0\}$. Now for each $n \in \mathbb{N}$, there exists $t_n \geq 0$ with

$$T_{t_n}(B_{\frac{1}{n}}(x)) \cap B_{\frac{1}{n}}(y) \neq \emptyset,$$

which in particular means that there exists $x_n \in X$ with $\|x_n - x\| < \frac{1}{n}$ and $\|T_{t_n}x_n - y\| < \frac{1}{n}$. This implies that $\lim_{n\to\infty} T_{t_n}x_n = y$. We have to show that $(t_n)_n$ has a strictly increasing subsequence tending to infinity. Suppose the converse, i.e. that $(t_n)_n$ is bounded by some $\nu > 0$. Then there is a subsequence of t_n converging to some t. For the sake of simplicity, we assume that $t_n \to t$. It follows from the continuity of the map $g : [0,\infty) \times X \to X$, $g(t,x) := T_t x$ (see for example [37], p. 357) that $\lim_{n\to\infty} T_{t_n}x_n = T_t x = y$, which is not possible since $y \notin \{T_t x \mid t \geq 0\}$.

Now let $y \in \{T_t x \mid t \geq 0\}$. Since g is continuous, the image of g under t for fixed $x \in X$ is a countable union of compact sets, in particular $g_x([0,\infty)) = \bigcup_{n=1}^{\infty} g_x([0,n])$ and therefore of first category. Hence

$$B_{\frac{1}{n}}(y)\backslash\{T_t x \mid t \geq 0\} \neq \emptyset$$

holds for each $n \in \mathbb{N}$. Choose $y_n \in B_{\frac{1}{n}}(y)\backslash\{T_t x| \ t \geq 0\}$. Then by the first part of the proof there exists $t_n > n$, such that $\|T_{t_n}x_n - y_n\| < \frac{1}{n}$. Now we choose $y_{n+1} \in B_{\frac{1}{n+1}}(y)\backslash\{T_t x| \ t \geq 0\}$. Then again we choose $t_{n+1} > \max\{t_n, n+1\}$ and x_{n+1} with $\|x_{n+1} - x\| < \frac{1}{n+1}$, such that $\left\|T_{t_{n+1}}x_{n+1} - y_{n+1}\right\| < \frac{1}{n+1}$. It follows that $(t_n)_n$ is a strictly increasing sequence of positive real numbers tending to infinity and

$$\begin{aligned}
\|T_{t_n}x_n - y\| &\leq \|T_{t_n}x_n - y_n\| + \|y_n - y\| \\
&\leq \frac{1}{n} + \frac{1}{n} \\
&= \frac{1}{2n}
\end{aligned}$$

This implies $\lim_{n\to\infty} T_{t_n}x_n = y$ and this shows the first direction. The other direction is obvious. □

Proposition 4.1.9. *Let X be a Banach space and $\mathfrak{T}$ a C_0-semigroup. Consider sequences $(x_n)_n$, $(y_n)_n$ with the property that $y_n \in J_{\mathfrak{T}}(x_n)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n$, $\lim_{n\to\infty} y_n$ exist. Then $y \in J_{\mathfrak{T}}(x)$. Hence, $J_{\mathfrak{T}}(x)$ and $A_{\mathfrak{T}}$ are closed subsets of X.*

Proof. Take y_{n_1} with $n_1 \geq 1$ large enough, such that $\|y_{n_1} - y\| < 1$. Since $y_{n_1} \in J_{\mathfrak{T}}(x_{n_1})$, choose z_1 with $\|z_1 - x_{n_1}\| < 1$ and $t_1 \geq 1$ such that $\|T_{t_1}z_1 - y_{n_1}\| < 1$. Next choose y_{n_2} with $n_2 > n_1$ and $\|y_{n_2} - y\| < \frac{1}{2}$ and as above, since $y_{n_2} \in J_{\mathfrak{T}}(x_{n_2})$, choose z_2 and $t_2 \geq \max\{2, t_1\}$ with $\|z_2 - x_{n_2}\| < \frac{1}{2}$ and $\|T_{t_2}z_2 - y_{n_2}\| < \frac{1}{2}$. Inductively we get sequences $(n_m)_m, (t_m)_m, (z_m)_m$ with the property that $t_m \to \infty$ and

$$\begin{aligned}
\|z_m - x_{n_m}\| &< \frac{1}{m}, \\
\|y - y_{n_m}\| &< \frac{1}{m} \\
\|T_{t_m}z_m - y_{n_m}\| &< \frac{1}{m}
\end{aligned}$$

hold for each $m \in \mathbb{N}$. Now,

$$\|T_{t_m}z_m - y\| \leq \|T_{t_m}z_m - y_{n_m}\| + \|y_{n_m} - y\| \leq \frac{1}{m} + \frac{1}{m} = \frac{1}{2m}$$

and therefore $\lim_{m\to\infty} z_m = x$ and $\lim_{m\to\infty} T_{t_m}z_m = y$. So we proved $y \in J_{\mathfrak{T}}(x)$. □

Remark 4.1.10. If X is separable and $\mathcal{T}$ a hypercyclic C_0-semigroup on X, then $A_{\mathcal{T}} = X$. This follows from the fact that $HC(T) := \{x \in X \mid \overline{\operatorname{orb}(\mathcal{T}, x)} = X\}$ is a dense G_δ-set in X (see [20], p. 193). Obviously it holds that $HC(T) \subseteq A_{\mathcal{T}}$ and therefore $A_{\mathcal{T}} = X$ since $A_{\mathcal{T}}$ is closed by the above proposition.

Proposition 4.1.11. *Let $\mathcal{T} := \{T_t \mid t \geq 0\}$ be a C_0-semigroup on a Banach space X, such that $J_{\mathcal{T}}(x) = X$ for some $x \neq 0$. If $x^* \in X^*$, $x^* \neq 0$, then the orbit $\{T_t^* x^* \mid t \geq 0\}$ is unbounded.*

Proof. Assume that for some $x^* \in X^*$, $x^* \neq 0$, the orbit is bounded, i.e $\|T_t^* x^*\| \leq M$ for some $M > 0$ and all $t \geq 0$. Let $y \in X$ be such $|x^*(y)| > 3M\|x\|$ and choose $t > 0$, $x' \in X$ such that $\|x - x'\| < \frac{\|x\|}{3}$ and $\|T_t x' - y\| \leq M\frac{\|x\|}{\|x^*\|}$. We get the estimate

$$\begin{aligned}
3M\|x\| &\leq |x^*(y)| \\
&\leq |x^*(T_t x')| + |x^*(T_t x' - y)| \\
&= |T_t^* x^*(x')| + |x^*(T_t x' - y)| \\
&\leq M\|x'\| + \|x^*\| \frac{M\|x\|}{\|x^*\|} \\
&= M\|x'\| + M\|x\| \\
&\leq \frac{4}{3}M\|x\| + M\|x\| \\
&= M\frac{7}{3}\|x\|.
\end{aligned}$$

This gives a contradiction. □

In analogy to the Spectral lemma (Lemma 1.2.12 (i)) we have the following:

Proposition 4.1.12. *Let X be a Banach space and $\mathcal{T} = \{T_t \mid t \geq 0\}$ a locally topologically transitive C_0-semigroup. Then for all $t > 0$ and for all $|\lambda| \leq 1$ the operator*

$$T_t - \lambda I$$

has dense range.

Proof. Assume that there exists some $t > 0$ and $|\lambda| \leq 1$, such that $T_t - \lambda I$ has not a dense range. By an application of the Hahn-Banach theorem there exists a non-zero functional such that $\phi((T_t - \lambda I)x) = 0$ for all $x \in X$ or equivalently

$$\phi(T_t x) = \lambda \phi(x)$$

for all $x \in X$.
Since $\mathcal{T}$ is locally topological transitive, there exists for each $y \in X$ a strictly increasing sequence of positive real numbers t_n and a sequence x_n converging to x, such that $T_{t_n} x_n \to y$. We write $t_n = k_n \cdot t + s_n$, where $s_n \in [0, t)$ and $k_n \in \mathbb{N}_0$. Hence we get

$$(*) \qquad \phi(T_{t_n} x_n) = \phi(T_{k_n t}(T_{s_n} x_n)) = \lambda^{k_n} \phi(T_{s_n} x_n).$$

So by Lemma 4.1.3 there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$, such that

$$\begin{aligned} |\phi(T_{s_n} x_n)| &\leq \|\phi\| \, \|T_{s_n}\| \, \|x_n\| \\ &\leq \|\phi\| \, \|x_n\| \, M e^{\omega s_n} \\ &\leq \|\phi\| \, C \cdot M e^{|\omega| t} \end{aligned}$$

holds, where $\|x_n\| \leq C$ for all $n \in \mathbb{N}$ and some positive constant C, since x_n converges. This inequality in particular means that the right hand side of $(*)$ is bounded, since $|\lambda| \leq 1$. But this is a contradiction, since the left hand side of $(*)$ is dense in $\mathbb{C}$. □

It follows immediately from the definition of a topologically transitive C_0-semigroup, that it is also locally topologically transitive. Thus it is natural to ask if there exists a locally topologically C_0-semigroup, which is not topologically transitive. Our next theorem gives a description how to construct such C_0-semigroups.

Theorem 4.1.13. *Let X, Y be complex Banach spaces, where X is separable. Consider $S \in L(Y)$ with $\sigma(S) \subseteq \mathbb{H}$, where $\mathbb{H} = \{z \in \mathbb{C} : \; Re\ (z) > 0\}$ is the right half plane. Let $\mathcal{T} = \{T_t \mid t \geq 0\}$ be a hypercyclic (topologically transitive) C_0-semigroup on X. Then the system $\mathcal{B} := \{B_t \mid t \geq 0\}$, where $B_t := e^{tS} \oplus T_t$ is a locally topologically transitive C_0-semigroup on $Y \oplus X$. Furthermore $A_{\mathcal{B}} = \{0\} \oplus X$ holds and if in addition Y is separable, then $\mathcal{B}$ is not hypercyclic (topologically transitive).*

Proof. We will first show that every vector of the form $0 \oplus x$ is a J-vector for $\mathcal{B}$, where x is hypercyclic for $\mathcal{T}$. Take any $v \oplus u$, where $v \in Y$ and $u \in X$. Then there exists a strictly increasing sequence of positive real numbers $(t_n)_n$ such that $\lim\limits_{n \to \infty} T_{t_n} x = u$. Since $\sigma(S) \subseteq \mathbb{H}$ it follows from the Spectral mapping theorem (A.1.2) that

$$(*) \qquad \sigma(e^S) = e^{\sigma(S)} \subseteq \mathbb{C} \backslash \overline{\mathbb{D}}.$$

Hence e^S is invertible and $\sigma((e^S)^{-1}) \subseteq \mathbb{D}$. Define now

$$y_n := (e^{t_n S})^{-1} v = e^{-t_n S} v.$$

Then we may write $t_n = k_n + s_n$ for each $n \in \mathbb{N}$, where k_n is a positive integer and $s_n \in [0, 1)$. With constants M, ω as in Lemma 4.1.3 we get that

$$\begin{aligned} \|y_n\| &= \left\| e^{-t_n S} v \right\| \\ &= \left\| e^{-k_n S - s_n S} v \right\| \\ &\leq \left\| e^{s_n(-S)} \right\| \cdot \left\| e^{-k_n S} v \right\| \\ &\leq M e^{\omega s_n} \cdot \left\| e^{-k_n S} v \right\| \\ &\leq M e^{|\omega|} \cdot \left\| e^{-k_n S} v \right\| \end{aligned}$$

holds. From this inequality it follows that y_n converges to zero, since $\left\| e^{-k_n S} v \right\| \to 0$, where the latter limit follows from $(*)$. In particular we get

$$B_{t_n}(y_n \oplus x) = e^{t_n S} y_n \oplus T_{t_n} x = v \oplus T_{t_n} x \to v \oplus u,$$

where $y_n \oplus x \to 0 \oplus x$. This shows $\{0\} \oplus HC(\mathcal{T}) \subseteq A_{\mathcal{B}}$ and since $HC(\mathcal{T})$ is dense in X and $A_{\mathcal{B}}$ is closed by Remark 4.1.5 we get $\{0\} \oplus X \subseteq A_{\mathcal{B}}$.

To see the other inclusion, assume that $y \oplus x$ is a J-vector for $\mathcal{B}$. Then there exists a strictly increasing sequence of positive real numbers $(t_n)_n$ and sequences $(y_n)_n, (x_n)_n$ with $y_n \to y$ and $x_n \to x$ such that $B_{t_n}(y_n \oplus x_n) \to 0 \oplus 0$. This in particular means that

$$e^{t_n S} y_n \to 0$$

holds. We write again $t_n = k_n + s_n$, where k_n can be chosen as a strictly increasing sequence of positive integers and $s_n \in [0, 1)$. Then we get

$$\begin{aligned} \left\|e^{t_n S} y_n\right\| &= \left\|e^{k_n S} \cdot e^{s_n S} y_n\right\| \\ &\geq A^{k_n} \left\|e^{s_n S} y_n\right\|, \end{aligned} \tag{$**$}$$

where $A > 1$ follows from the fact that $\sigma(e^S) \subseteq \mathbb{C} \backslash \overline{\mathbb{D}}$.
Since s_n is bounded we can assume without loss of generality that s_n converges to some δ, otherwise we pass to a subsequence. Therefore we get

$$\begin{aligned} \left\|e^{s_n S} y_n - e^{\delta S} y\right\| &\leq \left\|e^{s_n S} y_n - e^{s_n S} y\right\| + \left\|e^{s_n S} y - e^{\delta S} y\right\| \\ &\leq \left\|e^{s_n S}\right\| \cdot \|y_n - y\| + \left\|(e^{s_n S} - e^{\delta S}) y\right\|, \end{aligned}$$

which shows that $\lim_{n\to\infty} e^{s_n S} y_n = e^{\delta S} y$. From $(**)$ it follows, that $y = 0$ since $e^{\delta S}$ is invertible and the left hand side of $(**)$ converges to zero.
This shows $\{0\} \oplus X = A_{\mathcal{B}}$. If now Y is separable, then by Remark 4.1.10 we get that $\mathcal{B}$ is not hypercyclic. Hence the proof is finished. □

Lemma 4.1.14. *Let X be a Banach space and let $\mathcal{T}$ be a locally topologically transitive C_0-semigroup, with a bounded generator A. Then*

$$\sigma(A) \cap i\mathbb{R} \neq \emptyset$$

holds.

Proof. Suppose $\sigma(A) \cap i\mathbb{R} = \emptyset$. Define $\sigma_1 := \{\lambda \in \sigma(A) \mid \text{ Re } (\lambda) > 0\}$ and $\sigma_2 := \{\lambda \in \sigma(A) \mid \text{ Re } (\lambda) < 0\}$. Then σ_1 and σ_2 are closed and disjoint subsets of the complex plane. By the Riesz decomposition theorem we get the decomposition $X = M_1 \oplus M_2$ and $A = A_1 \oplus A_2$, where $\sigma(A_1) = \sigma_1$ and $\sigma(A_2) = \sigma_2$. Now, it is easy to see that $e^A = e^{A_1} \oplus e^{A_2}$ and hence by A.1.1 we obtain $\sigma(e^{A_1}) = e^{\sigma_1(A)} \subseteq \mathbb{C} \backslash \overline{\mathbb{D}}$ and $\sigma(e^{A_2}) = e^{\sigma_2(A)} \subseteq \mathbb{D}$. Now $e^{tA} = e^{tA_1} \oplus e^{tA_2}$ holds for $t \geq 0$ and it is easy to see that at least one of $\{e^{tA_1} \mid t \geq 0\}$ or $\{e^{tA_2} \mid t \geq 0\}$ must be locally topologically transitive. Similar arguments as in the proof of Theorem 4.1.13 show that e^{tA_1} is not locally topologically transitive. If we suppose that e^{tA_2} is locally topologically transitive, then there exists a non-zero vector $x \in M_2$ such that for any $y \in M_2$ with $y \neq 0$, we find a sequence $(x_n)_n$ in M_2 and positive real numbers $t_n \to \infty$ with $x_n \to x$ and $e^{t_n A_2} \to y$. Since $\sigma(e^{A_2}) \subseteq \mathbb{D}$ holds there exist some $0 < a < 1$ and $n_0 \in \mathbb{N}$ such that $\left\|e^{nA} x\right\| \leq a^n \|x\|$ for all $n \geq n_0$. Writing $t_n = k_n + s_n$ with $k_n \in \mathbb{N}$ and $s_n \in [0, 1)$ we have

$$\left\|e^{t_n A_2} x_n\right\| \leq \left\|e^{s_n A_2}\right\| \left\|e^{k_n A_2} x_n\right\| \leq M e^{\omega} a^{k_n} \|x_n\| \to 0,$$

where we used Lemma 4.1.3 for the last estimate. Hence, $y = 0$ which is a contradiction and this completes the proof. $\square$

As in the chapters before it is natural to ask about the existence of locally topologically transitive C_0-semigroups on Banach spaces. In analogy to the existence of J-class operators on reflexive non-separable Banach spaces, the next theorem shows us, that the situation for locally topologically transitive C_0-semigroups on non-separable reflexive Banach spaces is also true. In particular, we will make use of the next theorem by Bermúdez, Bonilla and Martinon.

Theorem 4.1.15. *(Bermúdez, Bonilla, Martinon) [10] Every separable infinite dimensional Banach space X admits a hypercyclic uniformly continuous semigroup.*

Corollary 4.1.16. *Let X be a non-separable reflexive Banach space and $Y \subseteq X$ a closed and separable subspace. Then there exists a locally topologically C_0-semigroup $\mathcal{T}$, with $Y \subseteq A_{\mathcal{T}}$.*

Proof. By Theorem 2.1.18 there exists a separable infinite dimensional closed subspace W, which contains Y and a linear bounded projection $P_W : X \to W$. Hence there exists a closed subspace U such that $X = U \oplus W$. Theorem 4.1.15 ensures the existence of a hypercyclic C_0-semigroup $\mathcal{T} = \{T_t \mid t \geq 0\}$ on W. Take now any $S \in L(U)$, such that $\sigma(S) \subseteq \mathbb{H} = \{z \in \mathbb{C} : \ \text{Re}\,(z) > 0\}$. Then Theorem 4.1.13 implies that the C_0-semigroup $\mathcal{B} = \{B_t \mid t \geq 0\}$ defined by $B_t := e^{tS} \oplus T_t$ for $t \geq 0$ is locally topologically transitive with $Y \subseteq W = A_{\mathcal{B}}$, where the last equality follows from Remark 4.1.10, since $\mathcal{T}$ is hypercyclic. $\square$

4.2 A Banach space which admits no local topological transitive operator

Theorem 4.1.15 states that every separable Banach space admits a hypercyclic (topologically transitive) C_0-semigroup, and obviously then a locally topologically transitive C_0-semigroup. At the beginning of our survey, we have seen that there are no J-class operators on $(X_A)_{\mathbb{C}}$ (there are even no operators T on $(X_A)_{\mathbb{C}}$ with $(J_T(x))^\circ \neq \emptyset$). It is therefore natural to ask, if there are any locally topologically transitive C_0-semigroups on $(X_A)_{\mathbb{C}}$. Indeed in this section we will show that there are also no locally topologically transitive semigroups on $(X_A)_{\mathbb{C}}$. For this purpose we need some preparation.

Definition 4.2.1. Let X be a Banach space. A system $\mathcal{T} = \{T_t \mid t \geq 0\}$ of operators $T_t : X \to X$ is called a C_0-group, if the following conditions are satisfied

(1) $T_0 = I$

(2) $T_{s+t} = T_s \circ T_t$ for all $-\infty < s, t < \infty$

(3) $\lim\limits_{t \to 0} T_t x = x$.

Obviously every C_0-group is also a C_0-semigroup.
Räbinger has studied intensively the spectral properties of C_0-semigroups on HI-Banach spaces in [33]. He states the following theorem in [33], which he formulated for HI-Banach spaces. From an inspection of the proof it is clear that it is enough to assume that the operator algebra of the underlying Banach space consists of operators which are strictly singular perturbations of multiples of the identity.

Theorem 4.2.2. *[33] Let X be a Banach space and suppose $L(X) = \{\lambda I + S \mid \lambda \in \mathbb{C}, S$ is strictly singular $\}$, i.e. every operator $T \in L(X)$ is a strictly singular perturbation of a multiple of the identity. Consider any C_0-semigroup $\mathcal{T} := \{T_t \mid t \geq 0\}$. Then the generator A of $\mathcal{T}$ is a bounded linear operator.*

Remark 4.2.3. From this theorem it follows, that $\mathcal{T} = \{e^{tA} \mid t \in \mathbb{R}\}$, since A is bounded (see also [37], p. 364).

Theorem 4.2.4. *([32], p. 24, Theorem 6.5) Let X be a Banach space and $\mathcal{T} = \{T_t \mid t \geq 0\}$ be C_0-semigroup. If there exists some $t_0 > 0$, such that $0 \notin \sigma(T(t_0))$, then $0 \notin \sigma(T(t))$ for all $t > 0$ and $\mathcal{T}$ can be embedded in a C_0-group. That is, there exists a C_0-group $\tilde{\mathcal{T}} := \{\tilde{T}_t \mid t \in \mathbb{R}\}$, such that $\tilde{T}_t = T_t$ for $t \geq 0$.*

We turn back to the space $X := (X_A)_{\mathbb{C}}$. We have seen in Chapter 2 that the operator algebra of this space consists of operators which are strictly singular perturbations of multiples of the identity. Now we are ready to prove our main result in this chapter.

Theorem 4.2.5. *Consider the Banach space $X = (X_A)_{\mathbb{C}}$. Then there exists no locally topologically transitive C_0-semigroup on X.*

Proof. We assume, that there exists a locally topologically C_0-semigroup on X with the corresponding J-vector $x \neq 0$.

Step 1: For every $t \geq 0$ we have $T_t = \lambda_t I + S_t$, where $\lambda_t \neq 0$ and S_t is strictly singular. As we already know, every operator T on X has the form $T = \lambda I + S$ for some $\lambda \in \mathbb{C}$ and S strictly singular. So we must show that $\lambda_t \neq 0$ for every t. For $t = 0$ we have $T_0 = I$, so there is nothing to prove. Assume that $T_{t_0} = S$ for some $t_0 > 0$, where S is strictly singular. Then T_t is not locally topologically transitive, which is the desired contradiction. To see this take any $y \in X$ and assume that $\{T_t \mid t \geq 0\}$ is locally topologically transitive. Then there exists a strictly increasing sequence of positive real numbers $(t_n)_n$ and a sequence $(x_n)_n$ with $x_n \to x$ and $T_{t_n} x_n \to y$. Write $t_n = k_n t_0 + s_n$, where $k_n \in \mathbb{N}_0$ and $0 \leq s_n < t_0$. Then

$$\begin{aligned} T_{t_n} x_n &= T_{k_n t_0 + s_n} x_n \\ &= T_{k_n t_0} T_{s_n} x_n \\ &= T_{t_0}^{k_n} T_{s_n} x_n \\ &= S^{k_n}(T_{s_n} x_n) \\ &\in S(X) \end{aligned}$$

Therefore $T_{t_n} x_n \to y \in \overline{S(X)}$. Since y was arbitrary it follows that $X = \overline{S(X)}$, which in turn means that X is separable, since S has separable range. A contradiction.

Step 2: $0 \notin \sigma(T_t)$ for all $t \geq 0$. Every strictly singular operator S is a Riesz-operator (see [1], p. 314), an operator T is called Riesz if $\sigma_{ess}(T) = \{0\}$). This in particular means, that S^* is also a Riesz-operator and hence its spectrum is at most countable and 0 is the only accumulation point (see [1], p. 303). Furthermore every point of $\sigma(S^*)\backslash\{0\}$ is an eigenvalue, that means $\sigma(S^*) = \sigma_p(S^*) \cup \{0\}$ (see [1], p. 303). Therefore we get

$$\begin{aligned} \sigma(T_t) &= \sigma(T_t^*) \\ &= \{\lambda_t + \sigma_p(S_t^*)\} \cup \{\lambda_t\} \\ &= \sigma_p(T_t^*) \cup \{\lambda_t\}. \end{aligned}$$

We know that $\sigma_p(T_t^*) \cap \overline{\mathbb{D}} = \emptyset$ for all $t > 0$ by Proposition 4.1.12 and since $\lambda_t \neq 0$ for all $t \geq 0$ by Step 1, it follows that $0 \notin \sigma(T_t)$ for all $t \geq 0$.

Step 3: $\mathcal{T} = \{e^{tA} \mid t \geq 0\}$, where A is a bounded operator and hence is of the form $A = \lambda I + S$, with $\lambda \in \mathbb{C}$ and S is strictly singular.
Since $0 \notin \sigma(T_t)$ for all $t \geq 0$, it follows that $\mathcal{T}$ can be embedded in a C_0-group by Theorem 4.2.4. Hence A is bounded by Theorem 4.2.2 and therefore $A = \lambda I + S$, where $\lambda \in \mathbb{C}$ and S is strictly singular. It follows that $T_t = e^{tA}$ for all $t \geq 0$.

Step 4: $\mathcal{T}$ is not locally topological transitive. By Step 3 we know that the generator A of $\mathcal{T}$ is bounded and hence can be written as $A = \lambda I + S$.

Case 1: $\mathrm{Re}\lambda \leq 0$. Then $|e^\lambda| = e^{\mathrm{Re}\lambda} \leq 1$ and therefore

$$\begin{aligned} T_1 - e^\lambda I &= e^A - e^\lambda I \\ &= e^{\lambda I + S} - e^\lambda I \\ &= e^\lambda I + e^\lambda \sum_{n=1}^{\infty} \frac{S^n}{n!} - e^\lambda I \\ &= e^\lambda \sum_{n=1}^{\infty} \frac{S^n}{n!}, \end{aligned}$$

is strictly singular by Theorem 2.1.5 with separable range. Hence $T_1 - e^\lambda$ does not have dense range in X, which is a contradiction to Proposition 4.1.12.

Case 2: $\mathrm{Re}\lambda > 0$. We know that

$$\begin{aligned} \sigma_p((e^A)^*) \cap \overline{\mathbb{D}} &= \sigma_p(e^{A^*}) \cap \overline{\mathbb{D}} \\ &= e^{\sigma_p(A^*)} \cap \overline{\mathbb{D}} \\ &= \emptyset, \end{aligned}$$

where we used Theorem A.1.2. Therefore $\sigma_p(A^*) \cap \mathrm{i}\mathbb{R} = \emptyset$.

Now

$$\begin{aligned}\sigma(A) &= \sigma(A^*) \\ &= \{\lambda + \sigma_p(A^*)\} \cup \{\lambda\} \\ &= \sigma_p(A^*) \cup \{\lambda\}.\end{aligned}$$

Therefore since $\mathrm{Re}\lambda > 0$, it follows that $\sigma(A) \cap \mathrm{i}\mathbb{R} = \emptyset$, which is a contradiction to Lemma 4.1.14.
This completes the proof. □

4.3 Final remarks and open problems

A remarkable Theorem of Lotz (see [28]) states that every C_0-semigroup on l^∞ is uniformly continuous and hence takes the form $\{e^{tT} \mid t \geq 0\}$ for a bounded linear operator $T : l^\infty \to l^\infty$. From Chapter 3 we know that any operator e^T where $T \in L_A^W(l^\infty)$ is not J-class. These examples show that it will be very hard finding any locally topologically transitive C_0-semigroups on l^∞, since most of the important known operators on l^∞ are adjoints of operators on l^1. Therefore it is natural to ask the following question.

Problem: Does there exist any locally topologically transitive C_0-semigroup on l^∞?

We have the feeling that the answer to the above question is negative. To justify this, it would be enough to show that the boundary of the spectrum of any bounded operator on l^∞ is contained in the point spectrum of its adjoint. Therefore the following question is of great interest:

Problem: Consider $T \in L(l^\infty)$. Is it true that $\partial\sigma(T) \subseteq \sigma_p(T^*)$?

S. Ansari showed in [5] that every power of a hypercyclic operator is also hypercyclic with the same hypercyclic vector. So it is natural to ask whether every power of a J-class operator is also J-class.

Problem: Let X be a Banach space and $T \in L(X)$. If T is J-class, is it true that T^p is also J-class for $p > 1$?

In Chapter 3, Corollary 3.2.23 we have seen that A_T contains an infinite dimensional closed subspace under certain assumptions.

Problem: If X is a Banach space and $T \in L(X)$ is J-class, is it true that A_T contains an infinite dimensional subspace? The same question can also be formulated for locally topologically transitive C_0-semigroups.

Appendix A

Properties of certain subsets of the spectrum

A.1 spectral mapping theorems

Theorem A.1.1. ***(Spectral mapping theorem, [20], p. 365)***
Let X be a Banach space and consider $T \in L(X)$. Let f be a holomorphic function defined on an open neighborhood of $\sigma(T)$. Then

$$\sigma(f(T)) = f(\sigma(T)).$$

Theorem A.1.2. ***(Point spectral mapping theorem, [20], p. 365)***
Let X be a Banach space and consider $T \in L(X)$. Let f be a holomorphic function defined on an open neighborhood O of $\sigma(T)$ that is not constant on any component of O. Then

$$\sigma_p(f(T)) = f(\sigma_p(T)).$$

Recall that the approximate spectrum $\sigma_a(T)$ is defined as the subset of the complex plane such that for $\lambda \in \sigma_a(T)$ there exist some sequence $(x_n)_n$ of norm one and $\|(T - \lambda I)x_n\|$ tends to zero. The next theorem can be found in [2], p. 83.

Theorem A.1.3. ***(Approximate point spectral mapping theorem)*** *Let X be a Banach space an consider $T \in L(X)$. Let f be a holomorphic function defined on an open neighborhood O of $\sigma(T)$. Then*

$$\sigma_a(f(T)) = f(\sigma_a(T)).$$

Theorem A.1.4. ***(Essential point spectral mapping theorem, [2], p. 51)*** *Let X be a Banach space an consider $T \in L(X)$. Let f be a holomorphic function defined on an open neighborhood O of $\sigma(T)$. Then*

$$\sigma_{ess}(f(T)) = f(\sigma_{ess}(T)).$$

A.2 Approximate spectrum of the forward shift on l^p for $1 \leq p < \infty$

We begin with a general result for the approximate spectrum and state its proof for the reader to get more familar with the approximate spectrum.

Proposition A.2.1. *([25], p. 77) Let X be a Banach space and consider $T \in L(X)$. Then the approximate spectrum of T is contained in the annulus $\{\lambda \in \mathbb{C} \mid i(T) \leq |\lambda| \leq r(T)\}$.*

Proof. We have $\sigma_a(T) \subseteq \sigma(T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq r(T)\}$. Hence it suffices to show that a $\lambda < i(T)$ does not belong to $\sigma_a(T)$. Choose therefore a $c > 0$ which satisfies $|\lambda| < c < i(T)$ and an positive integer n for which $c^n \leq \kappa(T^n)$. By definition of $\kappa(T^n)$ we therefore obtain $c^n \|x\| \leq \|T^n x\|$ for all $x \in X$. Hence we get

$$\|(T^n - \lambda^n)x\| \geq \|T^n x\| - |\lambda^n| \|x\| \geq (c^n - |\lambda|^n) \|x\| .$$

This shows that $\lambda^n \notin \sigma_a(T^n)$. It follows with the identity

$$T^n - \lambda^n I = (\sum_{k=1}^{n} \lambda^{n-k} T^{k-1})(T - \lambda)$$

that indeed $\lambda \notin \sigma_a(T)$. □

Theorem A.2.2. ***(Approximate spectrum of the forward shift, [25], p. 86)*** *Let S_w be the forward shift on $l^p(1 \leq p < \infty)$ with a positive and bounded weight sequence $w = (w_n)_n$, i.e. $S_w(x_1, x_2, \ldots) = (0, w_1 x_1, w_2 x_2, \ldots)$. Then*

$$\sigma_a(S_w) = \{\lambda \in \mathbb{C} \mid i(T) \leq |\lambda| \leq r(T)\}$$

and

$$\sigma(S_w) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq r(T)\}.$$

Proof. Assume at this point that $i(T) < r(T)$. It is sufficient to show that any $\lambda \in \mathbb{C}$ which satisfies $i(T) < |\lambda| < r(T)$ is contained in $\sigma_a(T)$ since the latter set is always closed. Define $c := |\lambda|$ and choose $a, b \in \mathbb{R}$ such that $i(T) < a < c < b < r(T)$. Further choose some $\varepsilon > 0$ such that $i(T) < (1 - \varepsilon)a$ holds. Since $c < b < r(T)$ we can find positive integers n, k with the property that

$$\left(\frac{c}{b}\right)^n < \varepsilon \quad \text{and} \quad b^n < w_k \cdots w_{k+n-1}.$$

Likely, since $i(T) < (1 - \varepsilon)a$ and $a < c$, there exist positive integers r, m such that

$$\left(\frac{a}{c}\right)^r < \varepsilon, \ (1 - \varepsilon)^r < w_1 \cdots w_{k+n} \quad \text{and} \quad w_m \cdots w_{m+r-1} < (1 - \varepsilon)^r a^r.$$

Now define $q := \max\{m, k + n + 1\}$. Then we get

$$w_q \cdots w_{q+r-1} < a^r \quad \text{and} \quad q > k + n.$$

The latter inequalities are clear for the case where $m > k+n+1$, otherwise we have $q = k+n+1$ and hence

$$w_q \cdots w_{q+r-1} = \frac{w_1 \cdots w_{k+n+r}}{w_1 \cdots w_{k+n}} \leq \frac{w_m \cdots w_{m+r-1}}{w_1 \cdots w_{k+n}} < a^r,$$

since by assumption $w_j \leq 1$ for all $j \in \mathbb{N}$. We will now construct a finitely supported element $x = (x_j)_j \in l^p$ (i.e. x has only finitely many entries not equal to zero) with $\|(T-\lambda I)x\| < \varepsilon \|x\| \|T\|$.Take $x_j = 0$ for $j < k$ and $j > q+r$. Let $x_k = 1$ and $x_j = \lambda^{k-j} w_k \cdots w_{j-1}$ for $j = k+1, \cdots, q+r$. Since $w_j x_j = \lambda x_{j+1}$ for $j = k+1, \cdots, q+r$ it is easy to see that our constructed sequence x satisfies

$$(T-\lambda I)x = \sum_{j=k}^{q+r} w_j x_j e_{j+1} - \sum_{j=k}^{q+r} \lambda x_j e_j = w_{q+r} x_{q+r} e_{q+r+1} - \lambda e_k$$

and therefore

$$\|(T-\lambda I)x\|^p = w_{q+r}^p |x_{q+r}|^p + c^p \leq (|x_{q+r}|^p + 1) \|T\|^p .$$

Moreover since $q > k+n$ we achieve the estimates

$$\left| \frac{x_{q+r}}{x_q} \right| = \frac{c^k w_k \cdots w_{q+r-1}}{c^{k-q} w_k \cdots w_{q-1}} = \frac{w_q \cdots w_{q+r-1}}{c^r} < \frac{a^r}{c^r} < \varepsilon,$$

and also

$$|x_{k+n}| = c^{-n} w_k \cdots w_{k+n-1} > \frac{b^n}{c^n} > \frac{1}{\varepsilon}.$$

Altogether we conclude that

$$|x_{q+r}|^p + 1 \leq \varepsilon |x_q|^p + \varepsilon |x_{k+n}|^p \leq \varepsilon \|x\|^p$$

and hence $\|(T-\lambda)x\| \leq \varepsilon \|x\| \|T\|$. This shows that $\lambda \in \sigma_a(T)$. □

Symbol Directory

$\mathbb{N}$	set of natural numbers
$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$	
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$X, Y, Z, \dots$	Banach space
X^*	dual space: set of all continuous functionals on X
X/U	quotient space
$X^{co} := X^{**}/X$	residual space of X
c_0	space of all sequences which converge to zero
$l^p := l^p(\mathbb{N}), 1 \leq p < \infty$	space of all p-summable sequences indexed by the set of positive integers
$l^p(\mathbb{Z})$	space of all p-summable sequences indexed by the set of integers
$l^\infty := l^\infty(\mathbb{N})$	space of bounded sequences indexed by the set of positive integers
$l^\infty(\mathbb{Z})$	space of bounded sequences indexed by the set of integers
B_X	closed unit ball: $\{x \in X : \|x\| \leq 1\}$
$\sigma(X, X^*)$	weak topology on X
$\sigma(X^*, X)$	weak*-topology on X^*
$L(X, Y)$	space of all linear and bounded operators from X into Y
$L(X)$	space of all linear and bounded operators from X into X
$\Phi(X, Y)$	set of all Fredholm operators from X into Y
$\Phi := \Phi(X) := \Phi(X, X)$	set of all Fredholm operators from X into X (Def. 2.1.12)
$\Phi_+(X, Y)$	set of all upper-semi Fredholm operators from X into Y
$\Phi_+ := \Phi_+(X) := \Phi_+(X, X)$	set of all upper-semi Fredholm operators from X into X
$\Phi_-(X, Y)$	set of all lower-semi Fredholm operators from X into Y (Def. 3.2.10)
$\Phi_- := \Phi_-(X) := \Phi_-(X, X)$	set of all lower-semi Fredholm operators from X into X
T	linear and bounded operator from X into Y
T^*	adjoint of $T \in L(X, Y)$
$T \cong S$	there exists a continuous isomorphism $J : X \to Y$ such that $T = J^{-1}SJ$, where $T \in L(X)$ and $S \in L(Y)$
$\widehat{T}_M$	induced operator on the quotient space X/M (Def. 1.3.4)
T^{co}	residuum operator (Def. 3.1.1)
$N(T)$	kernel of T: $\{x \in X : Tx = 0\}$
$R(T)$	range of T: $\{Tx \in Y : x \in X\}$
$J_T(x)$	J-set of T under $x \in X$ (Def. 1.2.1)
$J_T^{mix}(x)$	J_{mix}-set of T under $x \in X$ (Def. 1.2.1)

A_T	set of all J-vectors of T: $\{x \in X : J_T(x) = X\}$ (Def. 1.2.2)
A_T^{mix}	set of all J_{mix}-vectors of T: $\{x \in X : J_T^{mix}(x) = X\}$ (Def. 1.2.2)
$\sigma(T)$	spectrum of T: $\{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$
$\sigma_p(T)$	point spectrum of T $\{\lambda \in \mathbb{C} : T - \lambda I \text{ is not injective}\}$
$\sigma_a(T)$	approximate spectrum of T (Def. 3.1.4)
$\sigma_s(T)$	surjectivity spectrum of T (Def. 3.2.3)
$\sigma_{ess}(T)$	essential spectrum of T (Def. 2.1.12)
$\kappa(T)$	injectivity modulus
$i(T) := \lim\limits_{n\to\infty} (\kappa(T^n))^{\frac{1}{n}}$	
$s(T)$	surjectivity modulus
$\delta(T) := \lim\limits_{n\to\infty} (s(T^n))^{\frac{1}{n}}$	

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