A doubly non-linear system in small-strain visco-plasticity

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Abstract: We study a system of small strain visco-plasticity. We use an additive decomposition of the strain into elastic and plastic part, and allow for non-linear relations in the Hooke's law and in the flow rule. We show the existence of solutions, using a time-discrete approximation scheme. The limit procedure is based on a strong convergence result for the time-discrete solution sequence.
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1 Introduction

Elasto-plasticity is a commonly used constitutive model in the macroscopic handling of a wide variety of materials ranging from metals to granular materials.

In a small strain setting, that is when both stretches and rotations are assumed to be small in comparison to the size of the sample under consideration, elasto-plasticity is by now a mathematically established theory. In recent years it has been revisited within the rapidly expanding framework of variational evolutions; see in particular [2]. In that context, it is usually tackled through an time-incremental variational process as in [2]. However, the original existence proof for small strain elasto-plasticity [7] was based on a visco-plastic approximation which is not of a variational nature.

The setting of small strain elasto-plasticity is mathematically very rigid. In a nutshell, the linearized strain $\nabla^s u(t)$ of the displacement field $u(t)$, i.e., the symmetric part of $\nabla u(t)$ is decomposed additively into an elastic and a plastic part

$$\nabla^s u(t) = e(t) + p(t), \tag{1.1}$$

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the elastic part $e(t)$ is related linearly to the Cauchy stress $\sigma(t)$ through Hooke's law,

$$\sigma(t) = Ae(t), \quad \text{with } A \text{ a symmetric mapping on symmetric tensors,} \tag{1.2}$$

while the plastic part of the strain p is given through a differential inclusion

$$\partial_t p(t) \in N_{\mathcal{K}}(\sigma(t)), \tag{1.3}$$

where $\mathcal{K}$ is a convex, compact set of admissible stresses and $N_{\mathcal{K}}$ stands for the normal cone to $\mathcal{K}$ at $\sigma(t)$. Implicit in that relation is the constraint that $\sigma(t)$ must belong to $\mathcal{K}$.

If looking at a quasi-static evolution, then $\sigma(t)$ must be equilibrated by the external body loads $f(t)$, that is

$$\operatorname{div} \sigma(t) + f(t) = 0. \tag{1.4}$$

The visco-plastic approximation consists in replacing $N_{\mathcal{K}}(\sigma(t))$ in (1.3) by $\varepsilon^{-1}(\sigma - P_{\mathcal{K}}(\sigma))$, where $P_{\mathcal{K}}$ is the orthogonal projection onto $\mathcal{K}$ and $\varepsilon > 0$ is a small parameter. As is well-known to readers accustomed to convex duality, this is equivalent to replacing

$$\sigma(t) \in \partial H(\partial_t p(t)),$$

where $H(q) := \sup\{q \cdot \tau : \tau \in \mathcal{K}\}$ is the support function of $\mathcal{K}$ by

$$\sigma(t) - \varepsilon \partial_t p(t) \in \partial H(\partial_t p(t)).$$

In essence the positively one-homogeneous support function $H(q)$ – called the dissipation potential – is quadratically regularized, becoming $H(q) + \varepsilon |q|^2/2$.

In any case, at this time there is no existence theory for quasi-static elasto-plastic evolutions when one chooses to modify the linear stress-strain relation (1.2). One could easily argue against the propriety of investigating a non-linear stress-strain relation because the latter is viewed as ill-suited to a small strain setting.

However, we believe that such an investigation would be useful as a first step towards finite plasticity. Indeed, abandonment of the small strain assumption results in a slew of hotly debated models which will not be discussed here. Those have as common feature the competition between two non-linear terms: the stress-strain relation which controls the equilibrium and the dissipation term which controls the evolution of the plastic strain. Of course the difficulties are compounded by the apparent necessity of switching to a multiplicative decomposition of the deformation gradient, in lieu of the additive decomposition introduced earlier (see e.g. [6]).

In this light, the present contribution should be viewed as a necessary preliminary step if one wishes to mimic the visco-plastic approximation of elasto-plasticity in a non-linear setting. We thus propose to investigate a visco-plastic model where the stress-strain relation is no longer linear as in (1.2), but only monotone. The existence of a quasi-static evolution in such an environment is established in Theorem 2.4 and constitutes the main result of this contribution.

From the standpoint of visco-plasticity, there is no compelling mathematical rationale for limiting one's scope to a variational setting. This is why we prefer to set up a non-variational framework; we then proceed through a time-incremental procedure detailed in Section 3.

In Remark 3.1, we quickly illustrate the simplifications that a variational framework would bring to the analysis (at least at the time-incremental level). We recall that system (2.1)–(2.4) below is not of variational type since $\tilde{\sigma}$ and ψ are monotone functions, but not necessarily gradients. Furthermore, we emphasize that the system is not rate-independent.

The presented model is amenable to many mathematical generalizations: spatial dependence of the non-linear laws, replacement of monotone operators by suitable monotone graphs, etc .... We have decided to keep the model as simple as possible, because our admittedly distant goal is to carry the analysis over to the pure elasto-plastic setting.

In terms of notation, we denote the space of symmetric $n \times n$-matrices by $\mathbb{R}^{n\times n}_s$, the associated (Frobenius) scalar product $A : B = \mathrm{tr}\,(A^T \cdot B) = \sum_{i,j} a_{ij} b_{ij}$ and the associated norm by $\|A\| = \sqrt{A : A}$. As already seen $\nabla^s u$ denotes the symmetrized gradient, $\nabla^s u = (\nabla u + (\nabla u)^T)/2$ of a field $u : \Omega \to \mathbb{R}^n$.

2 The visco-plastic model

As already announced in the introduction, we follow the modeling of additive small strain elasto-plasticity and refer to e.g. [5] for a detailed exposition of that model (see also [1, 3]).

In particular, the symmetric gradient of the deformation $u : \Omega \to \mathbb{R}^n$ is decomposed additively into an elastic deformation tensor e and a plastic deformation tensor p; see (2.2) below. The Cauchy stress tensor, a symmetric tensor denoted by σ, is assumed to be non-linearly related to the elastic deformation e by a general monotone law; see (2.3) below. Because we are only considering quasi-static evolutions, it should also satisfy the equilibrium equations under the applied volumic loads f; see (2.1) below. The other non-linear relation is the flow-rule, which describes the evolution of the plastic deformation in terms of the stress tensor; see (2.4) below.

The quasi-static evolution system reads as

$$-\nabla \cdot \sigma = f \tag{2.1}$$

$$\nabla^s u = e + p \tag{2.2}$$

$$\sigma = \tilde{\sigma}(e) \tag{2.3}$$

$$\partial_t p = \psi(\sigma) \tag{2.4}$$

In this contribution, we derive an existence result for the above doubly non-linear set of equations.

We adopt the following

Assumption 2.1 (Properties of the constitutive relations)**.** *Assume that* $\tilde\sigma(0) = 0$ *and, for some* $\gamma, \delta > 0$*, the following strong monotonicity properties: For all matrices* $E_1, E_2 \in \mathbb{R}^{n\times n}_s$ *and* $\Sigma_1, \Sigma_2 \in \mathbb{R}^{n\times n}_s$

$$\begin{aligned} (\tilde\sigma(E_1) - \tilde\sigma(E_2)) \quad &: (E_1 - E_2) \quad \geq \gamma \quad \|E_1 - E_2\|^2 \\ ((\tilde\sigma^{-1}(\Sigma_1) - (\tilde\sigma^{-1}(\Sigma_2)) \quad &: (\Sigma_1 - \Sigma_2) \quad \geq \delta \quad \|\Sigma_1 - \Sigma_2\|^2 \end{aligned} \tag{2.5}$$

Also assume that

$\psi : \mathbb{R}^{n\times n}_s \to \mathbb{R}^{n\times n}_s$ *is monotone and Lipschitz continuous with constant* L*.*

Remark 2.2. Note that, if we were to attempt to pass to a purely elasto-plastic model, then the Lipschitz constant L would be of order ε^{-1}. In that case, the bounds in Lemma 4.1 would depend on ε and the strong convergence statement of Lemma 4.3 below would no longer hold in the $\varepsilon \searrow 0$-limit. ¶

Initial and boundary values. Let $T > 0$ be a time horizon and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. The boundary $\partial\Omega$ with exterior normal ν is decomposed into a Dirichlet boundary Γ_D and a Neumann boundary Γ_N, where Γ_D is open in the relative topology of $\partial\Omega$ and $\Gamma_N = \partial\Omega \setminus \bar\Gamma_D$. We introduce

$$H^1_D(\Omega;\mathbb{R}^n) := \left\{ v \in H^1(\Omega;\mathbb{R}^n) : v = 0 \text{ on } \Gamma_D \right\}.$$

For $p_0 \in L^2(\Omega;\mathbb{R}^{n\times n}_s)$, $w \in L^2(0,T;H^1(\Omega;\mathbb{R}^n))$, $g \in L^2(0,T;H^{-\frac12}(\partial\Omega;\mathbb{R}^n))$ we impose the initial and boundary conditions

$$p(0) = p_0 \quad \text{in } \Omega, \tag{2.6}$$

$$u(t) = w(t) \quad \text{on } \Gamma_D, \tag{2.7}$$

$$\sigma(t)\cdot\nu = g(t) \quad \text{on } \Gamma_N. \tag{2.8}$$

Definition 2.3 (Weak solution)**.** *We call* (u, e, p) *a weak solution of* (2.1)–(2.8) *iff the following holds:*

- *Regularity:*

$$u \in L^2(0,T;H^1(\Omega;\mathbb{R}^n)),\ e \in L^2(0,T;L^2(\Omega;\mathbb{R}^{n\times n}_s))$$
$$\sigma := \tilde\sigma(e) \in L^2(0,T;L^2(\Omega;\mathbb{R}^{n\times n}_s)),\ p \in W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^{n\times n}_s));$$

- *Equations* (2.2) *and* (2.4) *are satisfied, as well as the initial and boundary conditions* (2.6)–(2.7) *while* (2.1), (2.8) *are satisfied in the following sense: for a.e.* $t \in (0,T)$ *and any* $\varphi \in H^1_D(\Omega;\mathbb{R}^n)$,

$$\int_\Omega \sigma(t) : \nabla^s\varphi\, dx = \int_\Omega f(t)\cdot\varphi\, dx + \langle g(t), \varphi\rangle, \tag{2.9}$$

where $\langle . , . \rangle$ *denotes from now onward the duality product between* $H^{\frac{1}{2}}(\partial\Omega; \mathbb{R}^n)$ *and* $H^{-\frac{1}{2}}(\partial\Omega; \mathbb{R}^n)$.

We will prove existence of weak solutions to the doubly non-linear visco-plasticity system (2.1)–(2.8). We emphasize that our result is restricted to quasi-static evolutions and to a strongly monotone non-linearity $\tilde{\sigma}$.

Theorem 2.4 (Existence result)**.** *Under assumption 2.1, let the initial and boundary conditions be given by*

$$p_0 \in L^2(\Omega; \mathbb{R}^{n\times n}_s),\ w \in L^2(0, T; H^1(\Omega; \mathbb{R}^n)),\ g \in L^2(0, T; H^{-\frac{1}{2}}(\partial\Omega; \mathbb{R}^n))$$

while $f \in L^2(0, T; L^2(\Omega; \mathbb{R}^n))$. *Then, there exists a weak solution to system* (2.1)–(2.8) *in the sense of Definition 2.3.*

Remark 2.5. The monotonicity of ψ is only used for the existence of solutions to the time-discrete approximation in Subsection 3.1. ¶

3 The time-stepping scheme

We consider the following time-discrete approximation of system (2.1)–(2.4). With a number $N \in \mathbb{N}$ of time steps we discretize the interval $[0, T]$ with

$$0 = t_0 < t_1 < \ldots < t_N = T.$$

For example, we may use the time increment $\Delta t = T/N$ and choose equidistant points $t_k := k\,\Delta t,\ k = 0, \ldots, N$. The functions $u_k \in H^1(\Omega; \mathbb{R}^n)$ and $e_k, p_k \in L^2(\Omega; \mathbb{R}^{n\times n}_s)$ shall be approximations of the solution-values $u(t_k)$, $e(t_k)$, and $p(t_k)$ for $k \geq 1$. For $k = 0$, we use the initial data p_0 as the value in $t_0 = 0$. Initial data for u or e are not used. The loads and boundary values are discretized with time averages as

$$f_k := \frac{1}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} f(\tau)\,d\tau, \quad g_k := \frac{1}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} g(\tau)\,d\tau \tag{3.1}$$

$$w_k := \frac{1}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} w(\tau)\,d\tau.$$

We use the backward Euler discretization of (2.2)–(2.4), (2.6)–(2.7), (2.9), which reads, for all $k = 1, \ldots, N$, as

$$\int_\Omega \sigma_k : \nabla^s \varphi\,dx = \int_\Omega f_k \cdot \varphi\,dx + \langle g_k, \varphi \rangle,\ \forall \varphi \in H^1_D(\Omega; \mathbb{R}^n) \tag{3.2}$$

$$\nabla^s u_k = e_k + p_k \tag{3.3}$$

$$\sigma_k = \tilde{\sigma}(e_k) \tag{3.4}$$

$$\frac{p_k - p_{k-1}}{t_k - t_{k-1}} = \psi(\sigma_k) \tag{3.5}$$

$$u_k = w_k \text{ on } \Gamma_D \tag{3.6}$$

Remark 3.1 (Existence for the discrete scheme in the variational case)**.** Assume additionally that ψ is invertible and that $\tilde\sigma$ and ψ^{-1} are gradients of a potential. In other words, let $Q, R : \mathbb{R}^{n\times n}_s \to \mathbb{R}$ be differentiable and convex potentials such that

$$DQ = \tilde\sigma, \qquad DR = \psi^{-1}. \tag{3.7}$$

Note that Q is automatically strictly convex.

Then the discrete scheme (3.2)–(3.5) possesses a solution.

Indeed, at every time step we minimize, for a given p_{k-1}, the functional

$$\int_\Omega \left\{ Q(e) + (t_k - t_{k-1})\, R\left(\frac{p - p_{k-1}}{t_k - t_{k-1}}\right) - f_k \cdot u \right\} \, dx - \langle g_k, u\rangle \tag{3.8}$$

in the variables (u, e, p) under the constraint

$$e + p = \nabla^s u \quad \text{ and } u \in H^1(\Omega; \mathbb{R}^n) \text{ satisfies } u = w_k \text{ on } \Gamma_D\,. \tag{3.9}$$

The existence of a minimizer can be shown with the direct method.

A minimizing triplet $(u_k, e_k, p_k) \in H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{R}^{n\times n}_s) \times L^2(\Omega; \mathbb{R}^{n\times n}_s)$ satisfies, by construction, equation (3.3) and the boundary condition. Furthermore, it satisfies the Euler-Lagrange equation

$$\int_\Omega \left\{ \tilde\sigma(e_k) : \xi + \psi^{-1}\left(\frac{p_k - p_{k-1}}{t_k - t_{k-1}}\right) : \zeta - f_k\, v \right\} \, dx - \langle g_k, v\rangle = 0 \quad \forall\, \xi + \zeta = \nabla^s v\,,$$

with $v \in H^1_D(\Omega; \mathbb{R}^n)$. We insert, for an arbitrary v, the variations $\xi = \nabla^s v$ and $\zeta = 0$ and obtain (3.2), (3.4). Finally, we insert $v = 0$ and arbitrary $\xi = -\zeta$ to obtain

$$\tilde\sigma(e_k) = \psi^{-1}\left(\frac{p_k - p_{k-1}}{t_k - t_{k-1}}\right)\,.$$

This provides relation (3.5). ¶

3.1 Galerkin scheme for the single time-step

Our aim is to show the solvability of the scheme (3.2)–(3.6) for general monotone non-linear relations. Concentrating on a single time instance k, we regard $p_{k-1} \in L^2(\Omega; \mathbb{R}^{n\times n}_s)$ as a given function. We introduce the (k-dependent) non-linear function

$$\tilde\psi(\Sigma) := p_{k-1} + (t_k - t_{k-1})\, \psi(\Sigma) \tag{3.10}$$

which is again monotone, since ψ is monotone. Omitting the subscript k, the system now reads as

$$\int_\Omega \sigma : \nabla^s \varphi \, dx = \int_\Omega f \cdot \varphi \, dx + \langle g, \varphi \rangle, \ \forall \varphi \in H^1_D(\Omega; \mathbb{R}^n) \tag{3.11}$$

$$\nabla^s u = e + p \ \text{ in } \Omega \tag{3.12}$$

$$\sigma = \tilde{\sigma}(e) \ \text{ in } \Omega \tag{3.13}$$

$$p = \tilde{\psi}(\sigma) \ \text{ in } \Omega \tag{3.14}$$

$$u = w \ \text{ on } \Gamma_D. \tag{3.15}$$

In order to solve system (3.11)–(3.15), we use finite dimensional approximations of the function spaces and weak formulations of the equations. We choose a sequence $\{w_h\}$ of $H^1(\Omega; \mathbb{R}^n)$ with

$$w_h \to w \text{ in } H^1(\Omega; \mathbb{R}^n) \tag{3.16}$$

and a sequence of finite dimensional spaces $V_h^0 \subset H^1_D(\Omega, \mathbb{R}^n), X_h \subset L^2(\Omega, \mathbb{R}^{n\times n}_s)$ such that

$$\nabla^s V_h^0 \subset X_h \tag{3.17}$$

$$\bigcup_h V_h^0 \text{ is dense in } H^1_D(\Omega; \mathbb{R}^n) \tag{3.18}$$

$$\|v_h^0\|_{H^1(\Omega)} \le C \|\nabla^s v_h^0\|_{L^2(\Omega)}, \quad \forall v_h^0 \in V_h^0. \tag{3.19}$$

Above C is a constant in Poincaré-Korn's inequality which is independent of h.

Remark 3.2. Such a sequence of finite-dimensional spaces can always be constructed upon considering an orthonormal basis of eigenvectors $\{u_j, j \in \mathbb{N}\}$ of $-\triangle$ with the boundary conditions $u = 0$ on Γ_D and $\partial u/\partial \nu = 0$ on Γ_N. With index $h \in \mathbb{N}$, the space V_h^0 is that generated by the h-tuple $(u_1, ..., u_h)$ and X_h is the space of symmetric gradients of functions in V_h^0. ¶

Remark 3.3. In the case when $\partial\lfloor_{\partial\Omega}\Gamma_D$ is a smooth $N-2$ -dimensional manifold, one possible choice of function spaces V_h^0 and X_h are finite element spaces: Let $\mathcal{T}_h$ be a triangulation of a polygonal superdomain $\Omega_h \supset \bar{\Omega}$ with simplices, $h > 0$ indicating the diameter of the largest simplex; the union should be such that no simplex has an empty intersection with Ω. We assume that Ω_h approximates the Lipschitz domain Ω from outside for $h \to 0$. On boundary elements of $\partial\Omega_h$ that approximate Γ_D (in the sense that the union of the closure of those elements contains $\bar{\Gamma}_D$), we impose a nul boundary condition. The space $V_h^0 \subset H^1_D(\Omega, \mathbb{R}^n)$ is then defined as that of the piecewise affine and continuous functions on the triangulation while elements of $X_h \subset L^2(\Omega, \mathbb{R}^{n\times n}_s)$ are piecewise constant functions on the triangulation. ¶

We construct a Galerkin scheme for the system (3.11)–(3.14) as follows. Find $(u_h, e_h, p_h, \sigma_h) \in (w_h + V_h^0) \times X_h \times X_h \times X_h$ such that, for all $\varphi_h^0 \in V_h^0$, $\eta_h \in X_h$

$$\int_\Omega \sigma_h : \nabla^s \varphi_h^0 \, dx = \int_\Omega f \cdot \varphi_h^0 \, dx + \langle g, \varphi_h^0 \rangle \tag{3.20}$$

$$\int_\Omega (\nabla^s u_h - e_h - p_h) : \eta_h \, dx = 0 \tag{3.21}$$

$$\int_\Omega (\sigma_h - \tilde{\sigma}(e_h)) : \eta_h \, dx = 0 \tag{3.22}$$

$$\int_\Omega (p_h - \tilde{\psi}(\sigma_h)) : \eta_h \, dx = 0. \tag{3.23}$$

Lemma 3.4 (Solvability of the space-discrete equations (3.20)–(3.23))**.** *Under Assumption 2.1, the system* (3.20)–(3.23) *admits a solution*

$$(u_h, e_h, p_h, \sigma_h) \in \big(w_h + V_h^0\big) \times X_h \times X_h \times X_h.$$

Proof. Step 1. There exists a Lipschitz continuous and strongly monotone map $\Sigma_h^{-1} : X_h \to X_h$ with the following solution property: for every $\sigma^h \in X_h$, the function $e_h := \Sigma_h^{-1}(\sigma_h)$ solves (3.22).

To prove this fact, we use the $L^2(\Omega)$-orthogonal projection $P_h : L^2(\Omega) \to L^2(\Omega)$ onto X_h, and study the map $\Sigma_h : X_h \to X_h$ given by $\Sigma_h(e_h) = P_h(\tilde{\sigma}(e_h))$. The map Σ_h is Lipschitz continuous and strongly monotone on X_h (equipped with the $L^2(\Omega)$-norm). This implies that Σ_h has an inverse, which provides the desired map Σ_h^{-1}. Lipschitz continuity and monotonicity of Σ_h^{-1} follow from the corresponding properties of Σ_h.

Step 2. There exists a Lipschitz continuous and strongly monotone map $\Phi_h^{-1} : X_h \to X_h$ with the following property: for an arbitrary $u_h \in (w_h + V_h^0)$, the functions $\sigma^h := \Phi_h^{-1}(\nabla^s u^h)$ together with $e_h := \Sigma_h^{-1}(\sigma_h)$ and $p_h := P_h(\tilde{\psi}(\sigma_h))$ solve (3.21)–(3.23).

We construct the map $\Phi_h : X_h \to X_h$ as $\Phi_h := \Sigma_h^{-1} + P_h \circ \tilde{\psi}$. Strong monotonicity of Σ_h^{-1} and monotonicity of $P_h \circ \tilde{\psi}$ imply the strong monotonicity of Φ_h. Since Φ_h is additionally Lipschitz continuous, Φ_h has an inverse Φ_h^{-1}.

It remains to verify the solution properties. The choice $e_h := \Sigma_h^{-1}(\sigma_h)$ implies that (3.22) is satisfied. The choice $p_h := P_h(\tilde{\psi}(\sigma_h))$ implies that (3.23) is satisfied. Because of $\nabla^s u^h = \Phi_h(\sigma_h) = e_h + p_h$, (3.21) is also satisfied.

Step 3. With the function $\Phi_h^{-1} : X_h \to X_h$, system (3.20)–(3.23) can be reduced to a single relation. Looking for the solution in the form $u_h = w_h + v_h$, the system (3.20)–(3.23) is equivalent to

$$\int_\Omega \Phi_h^{-1}(\nabla^s(w_h + v_h)) : \nabla^s \varphi_h^0 \, dx = \int_\Omega f \cdot \varphi_h^0 \, dx + \langle g, \varphi_h^0 \rangle \quad \forall \varphi_h^0 \in V_h^0, \tag{3.24}$$

where the unknown is $v_h \in V_h^0$. The left hand-side of relation (3.24) defines a map $F_h : V_h^0 \to (V_h^0)'$ from the finite dimensional space V_h^0 into its dual $(V_h^0)'$. The operator F_h is continuous and strongly monotone in view of the corresponding properties of Φ_h^{-1}; note that we have used Poincaré-Korn's inequality (3.19). We conclude that the operator F_h is invertible. Since the right hand-side of relation (3.24) defines a bounded linear functional on V_h^0, we have thus solved (3.20). □

Lemma 3.5 (Existence for the time-discrete scheme)**.** *Under Assumption 2.1, the single time-step system* (3.11)–(3.15) *admits a solution.*

Remark 3.6. Since only finitely many time-steps are performed, Lemma 3.5 implies the existence of a solution to the time-discrete scheme (3.2)–(3.6). ¶

Proof. Step 1: The sequence of space-discrete solutions. Lemma 3.4 provides a sequence of solutions, indexed by h, of the spatially discretized system (3.20)–(3.23). The solution satisfies the uniform bound

$$\|e_h\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} + \|p_h\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} + \|\sigma_h\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} + \|u_h\|_{H^1(\Omega;\mathbb{R}^n)} \leq C\,. \tag{3.25}$$

The bound above can be inferred from the construction in Lemma 3.4. We will provide a different derivation with a testing procedure. Once (3.25) is established, we can select a subsequence of $h \to 0$ so that those subsequences converge weakly to e, p, σ, and u respectively in the appropriate topologies.

Consider the sequence of (3.16). Using $\varphi_h^0 = u_h - w_h \in V_h^0$ as a test-function in (3.20) yields, in view of (3.21)–(3.23),

$$\int_\Omega f\cdot(u_h - w_h)\,dx + \langle g, u_h - w_h\rangle + \int_\Omega \sigma^h : \nabla^s w_h\,dx = \int_\Omega \sigma_h : \nabla^s u_h$$
$$= \int_\Omega \sigma_h : e_h\,dx + \int_\Omega \sigma_h : p_h\,dx = \int_\Omega \tilde\sigma(e_h) : e_h\,dx + \int_\Omega \sigma_h : \tilde\psi(\sigma_h)\,dx.$$

The strong monotonicity of $\tilde\sigma$ (see (2.5)) and the fact that $\tilde\sigma(0) = 0$ implies that the first integral on the right hand-side controls $\gamma\|e_h\|^2_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}$. Since, by monotonicity of $\tilde\psi$,

$$\Sigma : \tilde\psi(\Sigma) = (\Sigma - 0) : (\tilde\psi(\Sigma) - \tilde\psi(0)) + \Sigma : \tilde\psi(0) \geq \Sigma : \tilde\psi(0),$$

we get

$$\gamma\|e_h\|^2_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} \leq \|f\|_{L^2(\Omega;\mathbb{R}^n)}\|u_h - w_h\|_{L^2(\Omega;\mathbb{R}^n)} + \|g\|_{H^{-1/2}(\partial\Omega;\mathbb{R}^n)}\|u_h - w_h\|_{H^1(\Omega;\mathbb{R}^n)}$$
$$+ \|\sigma_h\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}\|\nabla^s w_h\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} + C_\psi\|\sigma_h\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}\,.$$

The strong monotonicity of $\tilde\sigma^{-1}$ immediately implies that, in the previous inequality, the linear terms in $\|\sigma_h\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}$ are bounded above by $\delta^{-1}\|e_h\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}$. For linear terms in $\|u_h\|_{H^1(\Omega)}$, we use the Poincaré-Korn inequality (3.19) and (3.21) (with $\eta_h = \nabla^s u_h$), which yields

$$\begin{aligned}\|u_h - w_h\|_{H^1(\Omega;\mathbb{R}^n)} &\leq C\|\nabla^s u_h - \nabla^s w_h\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} \\ &\leq C\left(\|\nabla^s w_h\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} + \|e_h\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} + \|p_h\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}\right) \\ &\leq C\left(\|\nabla^s w_h\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} + C'\|e_h\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}\right). \end{aligned} \tag{3.26}$$

We finally obtain, for a new constant $C > 0$,

$$\gamma\|e_h\|^2_{L^2(\Omega)} \leq C\left(1 + \|e_h\|_{L^2(\Omega)}\right), \tag{3.27}$$

hence the $L^2(\Omega;\mathbb{R}^{n\times n}_s)$-estimate for e_h. In turn, this provides the $L^2(\Omega;\mathbb{R}^{n\times n}_s)$-estimates for σ_h and p_h, and, in view of (3.26), an $H^1(\Omega;\mathbb{R}^n)$-estimate for u_h.

Step 2: Limit process in the linear relations. It remains to check that the weak limit functions e, p, σ, and u solve the original system (3.11)–(3.15). Regarding (3.11) we observe that, as a consequence of (3.20), the limit function σ satisfies

$$\int_\Omega \sigma : \nabla^s\varphi\, dx = \int_\Omega f\cdot\varphi\, dx + \langle g,\varphi\rangle \qquad \forall\varphi\in V^0_h,\ \forall h. \tag{3.28}$$

Since, by (3.18), any $\varphi\in H^1_D(\Omega;\mathbb{R}^n)$ belongs to $\overline{\cup_h V^0_h}$, relation (3.28) holds true for φ. The Dirichlet boundary condition on Γ_D is satisfied by construction (the Neumann boundary condition on Γ_N is encoded in (3.28)). Similarly, we can take the limit in (3.21) and obtain $\nabla^s u = e + p$, i.e. (3.12).

Step 3: Limit process in the non-linear relations with Minty's lemma. First, we claim that

$$\int_\Omega\{\tilde\sigma(e_h) : e_h + \tilde\psi(\sigma_h) : \sigma_h\}\, dx = \int_\Omega\{\sigma_h : e_h + p_h : \sigma_h\}\, dx \to \int_\Omega\{\sigma : e + p : \sigma\}\, dx. \tag{3.29}$$

Indeed, test (3.20) with $\varphi^0_h = u_h - w_h \in V^0_h$. In view of (3.28) with $\varphi = u - w$ we obtain

$$\begin{aligned}\int_\Omega \sigma_h : (e_h + p_h)\, dx &= \int_\Omega \sigma^h : \nabla^s u_h\, dx \\ &= \int_\Omega \sigma_h : \nabla^s w_h\, dx + \int_\Omega f\cdot(u_h - w_h)\, dx + \langle g, u_h - w_h\rangle \\ &\to \int_\Omega \sigma : \nabla^s w\, dx + \int_\Omega f\cdot(u - w)\, dx + \langle g, u - w\rangle \\ &= \int_\Omega \sigma : \nabla^s u\, dx = \int_\Omega \sigma : (e + p)\, dx,\end{aligned}$$

hence (3.29).

The monotonicity of $\tilde{\sigma}$ and $\tilde{\psi}$ and (3.29) imply that, for an arbitrary pair $(E, \Sigma) \in [L^2(\Omega; \mathbb{R}^{n\times n}_s)]^2$

$$0 \le \int_\Omega \left\{ (\tilde{\sigma}(e_h) - \tilde{\sigma}(E)) : (e_h - E) + (\tilde{\psi}(\sigma_h) - \tilde{\psi}(\Sigma)) : (\sigma_h - \Sigma) \right\} dx$$
$$\to \left\{ \int_\Omega (\sigma - \tilde{\sigma}(E)) : (e - E) + (p - \tilde{\psi}(\Sigma)) : (\sigma - \Sigma) \right\} dx.$$

But, since $\tilde{\sigma}$ and $\tilde{\psi}$ are Lipschitz, we can choose (E, Σ) to be of the form $(e + \varepsilon\bar{e}, \sigma + \varepsilon\bar{\sigma})$, $\varepsilon > 0$ with $(\bar{e}, \bar{\sigma})$ arbitrary which yields, upon dividing by ε and letting ε tend to 0, that

$$(\sigma, p) = (\tilde{\sigma}(e), \tilde{\psi}(\sigma)) . \tag{3.30}$$

This verifies the non-linear relations (3.13) and (3.14). □

3.2 Construction of time-continuous approximations

The next step in our construction is to introduce interpolations of the time-discrete values. We proceed as follows.

Set X to be a given Hilbert space and consider a finite number of elements $r_k \in X$, $k = 0, \ldots, N$ and the corresponding times $0 = t_0 < t_1 < \ldots < t_N = T$. We define the piecewise affine continuous interpolation $\hat{r}^N : [0, T] \to X$ and the piecewise constant left-continuous interpolation $\bar{r}^N : [0, T] \to X$ as

$$\bar{r}^N(t) := r_k \qquad \forall t \in (t_{k-1}, t_k],$$
$$\hat{r}^N(t) := \mu r_{k-1} + (1-\mu) r_k \qquad t = \mu t_{k-1} + (1-\mu)t_k,\ \mu \in [0, 1].$$

Let the sequence $(u_k, e_k, p_k, \sigma_k)_{k=1,\ldots,N}$ be a solution to the family (3.2)–(3.6) of time-discrete equations. Recalling the definition (3.10) of $\tilde{\psi}$, we have solved the following system for almost every $t \in (0, T)$:

$$\int_\Omega \bar{\sigma}^N : \nabla^s\varphi \, dx = \int_\Omega \bar{f}^N \cdot \varphi \, dx + \langle \bar{g}^N, \varphi\rangle,\ \forall \varphi \in H^1_D(\Omega; \mathbb{R}^n) \tag{3.31}$$
$$\nabla^s \bar{u}^N = \bar{e}^N + \bar{p}^N \text{ in } \Omega \tag{3.32}$$
$$\bar{\sigma}^N = \tilde{\sigma}(\bar{e}^N) \text{ in } \Omega \tag{3.33}$$
$$\partial_t \hat{p}^N = \psi(\bar{\sigma}^N) \text{ in } \Omega \tag{3.34}$$
$$\bar{u}^N = \bar{w}^N \text{ on } \Gamma_D. \tag{3.35}$$

In the next section we will derive uniform estimates for solutions of (3.31)–(3.35) and perform the limit $N \to \infty$. The difficulty in the limit procedure is in the handling of (3.33) and (3.34) because limit functions do not necessarily satisfy the same non-linear relations. The limit is achieved through a strong convergence result for $\bar{e}^N$ and $\bar{\sigma}^N$.

4 The time-continuous limit

We first establish some a priori bounds for $\bar{u}^N, \bar{e}^N, \bar{p}^N, \bar{\sigma}^N$.

Lemma 4.1 (Estimates for the time-discrete scheme)**.** *Under Assumption 2.1 there exists $C > 0$ independent of N such that*

$$\begin{aligned}\|\bar{e}^N\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{n\times n}_s))} + \|\bar{p}^N\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{n\times n}_s))} + \\ \|\bar{\sigma}^N\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{n\times n}_s))} + \|\bar{u}^N\|_{L^2(0,T;H^1(\Omega;\mathbb{R}^n))} \leq C\,.\end{aligned} \tag{4.1}$$

A similar estimate holds true for the piecewise affine interpolations.

Proof. The proof is almost identical to that of Step 1 in the proof of Lemma 3.5. We fix a time instance $t \in (0,T)$ for which (3.31)–(3.35) are satisfied. We insert $(\bar{u}^N - \bar{w}^N)$ into (3.31) and obtain, for the time instance t,

$$\begin{aligned}\int_\Omega \bar{\sigma}^N : \nabla^s \bar{w}^N \,dx + \int_\Omega \bar{f}^N \cdot (\bar{u}^N - \bar{w}^N)\,dx + \langle \bar{g}^N, \bar{u}^N - \bar{w}^N\rangle = \int_\Omega \bar{\sigma}^N : \nabla^s \bar{u}^N \,dx \\ = \int_\Omega \bar{\sigma}^N : \bar{e}^N \,dx + \int_\Omega \bar{\sigma}^N : \bar{p}^N \,dx = \int_\Omega \tilde{\sigma}(\bar{e}^N) : \bar{e}^N \,dx + \int_\Omega \bar{\sigma}^N : \bar{p}^N \,dx.\end{aligned}$$

The first integral in the right hand-side controls $\gamma\|\bar{e}^N\|^2_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}$. The strong monotonicity of $\tilde{\sigma}^{-1}$ implies that $\|\bar{\sigma}^N\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} \leq \delta^{-1}\|\bar{e}^N\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}$ and allows one to conclude that

$$\begin{aligned}\gamma\|\bar{e}^N\|^2_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} &\leq \delta^{-1}\|\nabla^s \bar{w}^N\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}\|\bar{e}^N\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} \\ &+ \|\bar{f}^N\|_{L^2(\Omega;\mathbb{R}^n)}\|\bar{u}^N - \bar{w}^N\|_{L^2(\Omega;\mathbb{R}^n)} + \|\bar{g}^N\|_{H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^n)}\|\bar{u}^N - \bar{w}^N\|_{H^1(\Omega;\mathbb{R}^n)} \\ &+ \delta^{-1}\|\bar{e}^N\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}\|\bar{p}^N\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}.\end{aligned} \tag{4.2}$$

The Poincaré-Korn inequality, together with (3.32), yields, for some $C > 0$ independent of N,

$$\|\bar{u}^N\|_{H^1(\Omega;\mathbb{R}^n)} \leq C\left(1 + \|\bar{e}^N\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} + \|\bar{p}^N\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}\right).$$

Inserting that inequality into (4.2) yields with Young's inequality, for some possibly different $C > 0$, still independent of N,

$$\|\bar{e}^N(t)\|^2_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} \leq C\left(1 + F_N(t) + \|\bar{p}^N(t)\|^2_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}\right), \tag{4.3}$$

where

$$F_N(t) := \|\bar{w}^N(t)\|^2_{H^1(\Omega;\mathbb{R}^n)} + \|\bar{f}^N(t)\|^2_{L^2(\Omega;\mathbb{R}^n)} + \|\bar{g}^N(t)\|^2_{H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^n)}.$$

The norm of $\bar{p}^N(t)$ is computed as follows, for some $C > 0$, independent of N:

$$\|\bar{p}^N(t)\|^2_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} = \left\| p_0 + \int_0^t \partial_t \hat{p}^N(\tau)\, d\tau \right\|^2_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}$$
$$\leq C\left(1 + \left\| \int_0^t \psi(\bar{\sigma}^N(\tau))\, d\tau \right\|^2_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}\right)$$
$$\leq C\left(1 + \delta^{-2}L^2 \left\| \int_0^t |\bar{e}^N(\tau)|\, d\tau \right\|^2_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}\right)$$
$$\leq C\left(1 + \delta^{-2}L^2 T \int_0^t \|\bar{e}^N(\tau)\|^2_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}\, d\tau\right).$$

Inserting this into (4.3), we find

$$\|\bar{e}^N(t)\|^2_{L^2(\Omega)} \leq C\left(1 + F_N(t) + \int_0^t \|\bar{e}^N(\tau)\|^2_{L^2(\Omega)}\, d\tau\right).$$

An application of a type of Gronwall's inequality (see Lemma 4.2 below) finally implies that

$$\|\bar{e}^N\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{n\times n}_s))} \leq C\|1 + F_N\|_{L^1(0,T;\mathbb{R})} \leq C\Big(1 + \|\bar{f}^N\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{n\times n}_s))}$$
$$+\|\bar{w}^N(t)\|^2_{L^2(0,T;H^1(\Omega;\mathbb{R}^n))} + \|\bar{g}^N(t)\|^2_{L^2(0,T;H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^n))}\Big).$$

This is the desired estimate for $\bar{e}^N$. As already used, bounds for $\|\bar{e}^N(t)\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}$ provide in turn bounds for $\|\bar{\sigma}^N(t)\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}$, hence for $\|\bar{p}^N(t)\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}$, and finally for $\|\bar{u}^N(t)\|_{H^1(\Omega;\mathbb{R}^n)}$.

Since the norms of the piecewise affine interpolations are controlled by the norms of the piecewise constant interpolations, the analogous estimate also holds true for the piecewise affine interpolations. □

Lemma 4.2 (Gronwall in $L^1(0,T;\mathbb{R})$)**.** *Let $Y : [0,T] \to \mathbb{R}$ be non-negative and satisfy the estimate*

$$Y(t) \leq C_0 \int_0^t Y(\tau)\, d\tau + F(t), \qquad F \in L^1(0,T;\mathbb{R}),\ t \in [0,T].$$

Then, for some constant C independent of F,

$$\|Y\|_{L^1(0,T;\mathbb{R})} \leq C\, \|F\|_{L^1(0,T;\mathbb{R})}.$$

Proof. Set $Z(t_0) := \int_0^{t_0} Y(t)\, dt$. Then,

$$Z(t_0) \leq C_0 \int_0^{t_0} Z(t)\, dt + \|F\|_{L^1(0,T;\mathbb{R})}.$$

The classical Gronwall's inequality implies that

$$Z(t) \leq \|F\|_{L^1(0,T;\mathbb{R})} \exp(C_0 t)\,.$$

Inserting $t = T$ yields the desired result. □

The uniform estimate of Lemma 4.1 enables us to choose a subsequence of $\{N\}$ (which will not be relabeled), as well as weak limits e, p, σ, u (resp. $\hat{e}$, $\hat{p}$, $\hat{\sigma}$, $\hat{u}$) such that, e.g.

$$\bar{e}^N \rightharpoonup e \text{ in } L^2(0,T;L^2(\Omega;\mathbb{R}^{n\times n}_s)) \text{ (resp. } \hat{e}^N \rightharpoonup \hat{e} \text{ in } L^2(0,T;L^2(\Omega;\mathbb{R}^{n\times n}_s))).$$

Since weak convergence does not ensure the stability of the non-linear relations (3.33) and (3.34) in the limit process, we propose to prove the following strong convergence property.

Lemma 4.3 (Strong convergence)**.** *Under Assumption 2.1, let the data p_0, w, g and f be as in Theorem 2.4. Consider a dyadic discretization of* $[0,T]$ *and the associated piecewise constant interpolation* $(\bar{u}^N, \bar{e}^N, \bar{p}^N, \bar{\sigma}^N)$ *of the time-discrete solutions to* (3.2)-(3.6)*.*

Then,

$$\bar{\sigma}^N \to \sigma,\ \bar{e}^N \to e \quad \textit{strongly in } L^2(0,T;L^2(\Omega;\mathbb{R}^{n\times n}_s)).$$

Proof. We study two different indices N and K with $N > K$, so that the N^{th}-discretization is a refinement of the K^{th}-discretization. We consider a point in the fine grid, $t = t_n^N$ for some $n \leq N$. We use (3.31) with indices N and K , take the difference, and use $[(\bar{u}^N - \bar{w}^N) - (\bar{u}^K - \bar{w}^K)](t)$ as a test function. We obtain

$$\begin{aligned}
&\int_\Omega (\bar{\sigma}^N - \bar{\sigma}^K)(t) : \nabla^s(\bar{u}^N - \bar{u}^K)(t)\,dx = \int_\Omega (\bar{f}^N - \bar{f}^K)(t)\cdot(\bar{u}^N - \bar{u}^K)(t)\,dx \\
&\quad - \int_\Omega (\bar{f}^N - \bar{f}^K)(t)\cdot(\bar{w}^N - \bar{w}^K)(t)\,dx + \int_\Omega (\bar{\sigma}^N - \bar{\sigma}^K)(t) : \nabla^s(\bar{w}^N - \bar{w}^K)(t)\,dx \\
&\qquad + \langle\, (\bar{g}^N - \bar{g}^K)(t),\ [(\bar{u}^N - \bar{u}^K)(t) - (\bar{w}^N - \bar{w}^K)(t)]\,\rangle =: \varepsilon_1(t)\,.
\end{aligned}$$

Inserting the decomposition (3.32) in this relation yields

$$\int_\Omega (\bar{\sigma}^N - \bar{\sigma}^K)(t) : (\bar{e}^N - \bar{e}^K)(t)\,dx + \int_\Omega (\bar{\sigma}^N - \bar{\sigma}^K)(t) : (\bar{p}^N - \bar{p}^K)(t)\,dx = \varepsilon_1(t)\,.$$

Because of the strong monotonicity of $\tilde{\sigma}$ and of relation (3.33), the first integrand is positive and provides an upper bound for $\gamma\|(\bar{e}^N - \bar{e}^K)(t)\|^2_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}$. With Young's inequality, we obtain, for an arbitrary $1 > \zeta > 0$, the estimate

$$\begin{aligned}
&\gamma\|(\bar{e}^N - \bar{e}^K)(t)\|^2_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} \leq \delta^{-1}\|(\bar{e}^N - \bar{e}^K)(t)\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}\|(\bar{p}^N - \bar{p}^K)(t)\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} \\
&\quad + \delta^{-1}\|\nabla^s(\bar{w}^N - \bar{w}^K)(t)\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}\|(\bar{e}^N - \bar{e}^K)(t)\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} \\
&\quad + \|(\bar{w}^N - \bar{w}^K)(t)\|^2_{H^1(\Omega;\mathbb{R}^n)} + \zeta\|(\bar{u}^N - \bar{u}^K)(t)\|^2_{H^1(\Omega;\mathbb{R}^n)} \\
&\quad + C(\zeta)\left\{\|(\bar{f}^N - \bar{f}^K)(t)\|^2_{L^2(\Omega;\mathbb{R}^n)} + \|(\bar{g}^N - \bar{g}^K)(t)\|^2_{H^{-\frac{1}{2}}(\Omega;\mathbb{R}^n)}\right\}
\end{aligned}$$

for some constant $C(\zeta)$ depending on ζ. The term containing $(\bar{u}^N - \bar{u}^K)(t)$ is bounded from above using Poincaré-Korn's inequality. Collecting all terms, we end up with

$$\begin{aligned} &\|(\bar{e}^N - \bar{e}^K)(t)\|^2_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} \\ &\qquad \le C\Big(\|(\bar{w}^N - \bar{w}^K)(t)\|^2_{H^1(\Omega;\mathbb{R}^n)} + \|(\bar{f}^N - \bar{f}^K)(t)\|^2_{L^2(\Omega;\mathbb{R}^n)} + \\ &\qquad\qquad + \|(\bar{g}^N - \bar{g}^K)(t)\|^2_{H^{-\frac{1}{2}}(\Omega;\mathbb{R}^n)} + \|(\bar{p}^N - \bar{p}^K)(t)\|^2_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}\Big) \end{aligned} \tag{4.4}$$

We recall (3.34), the identity $\bar{p}^N(t) = \hat{p}^N(t)$ at a grid point t of the (fine) N-grid, and obtain, at that point t,

$$\begin{aligned} \|(\bar{p}^N - \bar{p}^K)(t)\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} &\le \|(\hat{p}^N - \hat{p}^K)(t)\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} + \|(\hat{p}^K - \bar{p}^K)(t)\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} \\ &= \left\|\int_0^t (\psi(\bar{\sigma}^N) - \psi(\bar{\sigma}^K))(\tau)\,d\tau\right\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} + \|(\hat{p}^K - \bar{p}^K)(t)\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} \\ &\le \delta^{-1}L\int_0^t \|(\bar{e}^N - \bar{e}^K)(\tau)\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}\,d\tau + \|(\hat{p}^K - \bar{p}^K)(t)\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}\,. \end{aligned}$$

The norm of $(\hat{p}^K - \bar{p}^K)(t)$ is estimated using time $t_k^K \ge t = t_n^N$ of the K-grid, $t_k^K - t \le \Delta t_K$, where $\Delta t_K := \max_{k\le K}(t_k^K - t_{k-1}^K)$.

$$\begin{aligned} \left\|(\hat{p}^K - \bar{p}^K)(t)\right\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} &= \left\|\hat{p}^K(t) - \hat{p}^K(t_k^K)\right\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} \\ &= \left\|\int_t^{t_k^K} \partial_t\hat{p}^K(\tau)\,d\tau\right\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} \le \Delta t_K\,\left\|\psi(\bar{\sigma}^K(t))\right\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} \\ &\le \Delta t_K L\left\|\bar{\sigma}^K(t)\right\|_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}\,. \end{aligned}$$

Inserting the two estimates above into (4.4) finally yields

$$\begin{aligned} \frac{1}{C}\|(\bar{e}^N - \bar{e}^K)(t)\|^2_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} &\le \|(\bar{w}^N - \bar{w}^K)(t)\|^2_{H^1(\Omega;\mathbb{R}^n)} + \|(\bar{f}^N - \bar{f}^K)(t)\|^2_{L^2(\Omega;\mathbb{R}^n)} \\ &+ \|(\bar{g}^N - \bar{g}^K)(t)\|^2_{H^{-\frac{1}{2}}(\Omega;\mathbb{R}^n)} + 2L^2\Delta t_K^2\left\|\bar{\sigma}^K(t)\right\|^2_{L^2(\Omega;\mathbb{R}^{n\times n}_s)} \\ &+ 2\delta^{-2}L^2\int_0^t \|(\bar{e}^N - \bar{e}^K)(\tau)\|^2_{L^2(\Omega;\mathbb{R}^{n\times n}_s)}\,d\tau\,. \end{aligned}$$

We can now apply the Gronwall inequality of Lemma 4.2. With a new constant C, we obtain

$$\begin{aligned} \frac{1}{C}\|\bar{e}^N - \bar{e}^K\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^{n\times n}_s))} &\le \|\bar{w}^N - \bar{w}^K\|^2_{L^2(0,T;H^1(\Omega;\mathbb{R}^n))} \\ &+ \|\bar{f}^N - \bar{f}^K\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^n))} + \|\bar{g}^N - \bar{g}^K\|^2_{L^2(0,T;H^{-\frac{1}{2}}(\Omega;\mathbb{R}^n))} \\ &+ \Delta t_K^2\left\|\bar{\sigma}^K\right\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^{n\times n}_s))}\,. \end{aligned} \tag{4.5}$$

Since, by Lemma 4.1, $\bar{\sigma}^K$ is bounded in $L^2(0,T;L^2(\Omega;\mathbb{R}^{n\times n}_s))$, letting $K\to\infty$ in (4.5) we conclude that $\|\bar{e}^N - \bar{e}^K\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{n\times n}_s))} \to 0$ for $N \geq K \to \infty$.

Relation (3.33) and the Lipschitz continuous character of ψ then imply the corresponding strong convergence $\bar{\sigma}^N \to \sigma$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^{n\times n}_s))$. This concludes the proof. □

With the above results, we have all necessary ingredients to conclude the proof of Theorem 2.4.

Lemma 3.5 provides the existence of a solution sequence $(\bar{u}^N, \bar{e}^N, \bar{p}^N, \bar{\sigma}^N)$ of the time-discrete system (3.31)–(3.35). Lemma 4.1 provides a subsequence $N\to\infty$ and limit functions (u,e,p,σ). Lemma 4.3 provides the strong convergence $\bar{e}^N \to e$ and $\bar{\sigma}^N \to \sigma$. We take a further subsequence $N\to\infty$, such that the convergence also holds pointwise almost everywhere.

It remains to show that the limit functions (u,e,p,σ) is a weak solution of (2.1)–(2.8). This is immediate, except for (2.4). To obtain (2.4), we must pass to the limit in (3.34). We obtain

$$\partial_t \hat{p} = \psi(\sigma).$$

In order to conclude that (2.4) holds true, we must have $p = \hat{p}$. The strong convergence of the right hand-side in the evolution equation (3.34) implies that

$$\partial_t \hat{p}^N \to \partial_t \hat{p}, \text{ strongly in } L^2(0,T;L^2(\Omega;\mathbb{R}^{n\times n}_s)).$$

Since $p(0) = \hat{p}(0) = p_0$,

$$\hat{p}^N \to \hat{p} \text{ strongly in } L^2(0,T;L^2(\Omega;\mathbb{R}^{n\times n}_s)).$$

Lemma 4.4 below permits to conclude $p = \hat{p}$.

The proof of Theorem 2.4 is complete.

The following lemma is taken from [4], where it is shown for equidistant discretizations. We repeat the simple proof for the reader's convenience.

Lemma 4.4 (Comparison of interpolations)**.** *Let X be a Hilbert space and $T>0$. Let with points $0 = t_0^N < t_1^N < \ldots < t_N^N = T$ be such that $\max_{0\leq k\leq N-1} |t_{k+1}^N - t_k^N| \leq \Delta t_N \to 0$ for $N\to\infty$. Let $f_k^N \in X$ be given values for $k = 0,1,...,N$. We consider the piecewise affine interpolation $\hat{f}^N$ and the piecewise constant interpolation $\bar{f}^N$ of the point values, $\bar{f}^N(t_k^N) = \hat{f}^N(t_k^N) = f_k^N$. If*

$$\hat{f}^N \to g \text{ in } L^2(0,T;X)$$

then,

$$\bar{f}^N \to g \text{ in } L^2(0,T;X). \tag{4.6}$$

Proof. First,

$$\|\bar{f}^N\|^2_{L^2(0,T;X)} \leq 6\,\|\hat{f}^N\|^2_{L^2(0,T;X)} \tag{4.7}$$

The computation is elementary: we omit the superscript N and use $h_k := t_{k+1} - t_k$,

$$\begin{aligned}
\|\hat{f}^N\|^2_{L^2(0,T;X)} &= \sum_k h_k \int_0^1 \|t f_k^N + (1-t) f_{k+1}^N\|_X^2 \\
&= \sum_k h_k \int_0^1 (t^2 \|f_k^N\|_X^2 + (1-t)^2 \|f_{k+1}^N\|_X^2) + 2h_k \int_0^1 t(1-t)\langle f_k^N, f_{k+1}^N\rangle \\
&\geq \frac{1}{6} \sum_k h_k (\|f_k^N\|_X^2 + \|f_{k+1}^N\|_X^2) \geq \frac{1}{6} \int_0^T \|\bar{f}^N\|_X^2.
\end{aligned}$$

Given $g \in L^2(0,T;X)$, we construct the averages

$$g_k^N := \frac{1}{h_k} \int_{t_{k-1}}^{t_k} g(t)\, dt$$

and the corresponding piecewise affine and constant interpolations $\hat{g}^N$ and $\bar{g}^N$. The interpolations $\hat{g}^N$ and $\bar{g}^N$ converge strongly to g in $L^2(0,T;X)$. By assumption,

$$\|\hat{f}^N - \hat{g}^N\|_{L^2(0,T;X)} \leq \|\hat{f}^N - g\|_{L^2(0,T;X)} + \|g - \hat{g}^N\|_{L^2(0,T;X)} \to 0.$$

Inequality (4.7) implies that

$$\|\bar{f}^N - \bar{g}^N\|^2_{L^2(0,T;X)} \leq 6 \|\hat{f}^N - \hat{g}^N\|^2_{L^2(0,T;X)} \to 0.$$

Another application of the triangle inequality yields

$$\|\bar{f}^N - g\|_{L^2(0,T;X)} \leq \|\bar{f}^N - \bar{g}^N\|_{L^2(0,T;X)} + \|\bar{g}^N - g\|_{L^2(0,T;X)} \to 0,$$

and hence the claim. □

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References

[1] H.-D. Alber. *Materials with memory*, volume 1682 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1998. Initial-boundary value problems for constitutive equations with internal variables.

[2] G. Dal Maso, A. DeSimone, and M. G. Mora. Quasistatic evolution problems for linearly elastic-perfectly plastic materials. *Arch. Ration. Mech. Anal.*, 180(2):237–291, 2006.

[3] W. Han and B. D. Reddy. *Plasticity*, volume 9 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, 1999. Mathematical theory and numerical analysis.

[4] M. Lenzinger and B. Schweizer. Two-phase flow equations with outflow boundary conditions in the hydrophobic-hydrophilic case. *Nonlinear Anal.*, 73(4):840–853, 2010.

[5] J. Lubliner. *Plasticity Theory*. Macmillan Publishing, New York, 1990.

[6] P. M. Naghdi. A critical review of the state of finite plasticity. *Z. Angew. Math. Phys.*, 41(3):315–394, 1990.

[7] P.-M. Suquet. Sur les équations de la plasticité: existence et régularité des solutions. *J. Mécanique*, 20(1):3–39, 1981.

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