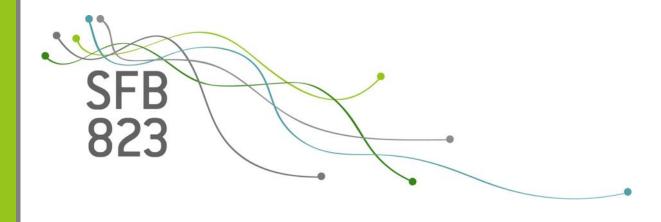# Nonparametric tests for constant tail dependence with an application to energy and finance

Axel Bücher, Stefan Jäschke, Dominik Wied

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Nonparametric tests for constant tail dependence with an application to energy and finance

Axel Bücher [a]

Université catholique de Louvain & Ruhr-Universität Bochum

Stefan Jäschke [*,b]   Dominik Wied [c]

RWE Supply & Trading GmbH   Technische Universität Dortmund

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Abstract

The present paper proposes new tests for detecting structural breaks in the tail dependence of multivariate time series using the concept of tail copulas. To obtain asymptotic properties, we derive a new limit result for the sequential empirical tail copula process. Moreover, consistency of both the tests and a change-point estimator are proven. We analyze the finite sample behavior of the tests by Monte Carlo simulations. Finally, and crucial from a risk management perspective, we apply the new findings to datasets from energy and financial markets.

Keywords: Change-point detection, Multiplier bootstrap, Tail dependence, Weak convergence

JEL classification: C12, C14, C32, C58, G32

1. INTRODUCTION

Modeling and estimating stochastic dependencies has attracted increasing attention over the last decades in various fields of applications, including mathematical finance, actuarial science or hydrology, among others. Of particular interest, especially in risk management, is a sensible quantitative description of the dependence between extreme events, commonly referred to as tail dependence; see for example Embrechts et al. (2003). A formal definition of this concept is given in Section 2 below.

In applications, tail dependence is often assessed by fitting a parametric copula family to the data and by subsequently extracting the tail behavior of that particular copula. Examples can be found in Breymann et al. (2003) and Malevergne and Sornette (2003), among others. More robust methods are based on the assumption that the underlying copula is an extreme-value copula. The class of these copulas can be regarded as a nonparametric copula family indexed by a function on the unit simplex (Gudendorf and Segers, 2010). Since the copula is a rather general measure for stochastic dependence, the estimation techniques for both of the latter approaches are usually based on the entire available dataset (see, for instance, Genest et al. (1995); Chen and Fan (2006) for parametric families or Genest and Segers (2009) for extreme-value copulas). However, due to the fact that the center of a distribution does not contain any information about the tail behavior, these techniques might in general yield biased estimates for the tail dependence. We refer to Frahm et al. (2005) for a more elaborated discussion of this issue. In order to circumvent the problem and to obtain estimators that are robust with respect to deviations in the center of the distribution, there are basically two important approaches: either one could extract the tail dependence from subsamples of block maximal data, for which extreme-value copulas provide a natural model (McNeil et al., 2005, Section 7.5.4), or one could rely on extreme-value techniques some of which are presented in Section 2 below. Applications of these procedures can be found in Breymann et al. (2003); Caillault and Guéégan (2005); Jäschke et al. (2012); Jäschke (2012), among others.

Most of the aforementioned applications to time series data are based on the implicit assumption that the tail dependence remains constant over time. Whereas nonparametric testing for constancy of the whole dependence structure, as for instance measured by the copula, has recently drawn some attention in the literature (Remillard, 2010; Busetti and Harvey, 2011; Krämer and van Kampen, 2011; van Kampen and Wied, 2012; Kojadinovic and Rohmer, 2012; Bücher and Ruppert, 2013), there does not seem to exist a unified approach to testing for constancy of the tail dependence. It is the main purpose of the present paper to fill this gap. Our proposed testing procedures are genuine extreme-value methods depending only on the tails of the data and are hence robust with respect to potential (non-)constancy of the dependence between the centers of the distributions. In particular, the presented tests do not rely on the assumption of a constant copula throughout the sample period.

Our procedures are based on new limit results for the sequential empirical tail copula process, formally defined in Section 3.1. We derive its asymptotic distribution under the null hypothesis and propose several variants to approximate the required critical values. When restricting to the case of testing for constancy of the simple tail dependence coefficient, the limiting process can be easily transformed

into a Brownian bridge. In this case, the asymptotic critical values of the tests can be obtained by direct calculations or simulations. In the more complicated case of testing for constancy of the whole extremal dependence structure as measured by the tail copula, we propose a multiplier bootstrap procedure to obtain approximate asymptotic quantiles. The finite-sample performance of all proposals is assessed in a simulation study, which reveals accurate approximations of the nominal level and reasonable power properties.

We apply our methods to two real datasets. The first application revisits a recent investigation in Jäschke (2012) on the tail dependence between WTI and Brent crude oil spot log-returns, which is based on the implicit assumption that the tail dependence remains constant over time. Our testing procedures show that this assumption cannot be rejected. The second application concerns the tail dependence between Dow Jones Industrial Average and the Nasdaq Composite time series around Black Monday on 19th of October 1987, it reveals a significant break in the tail dependence. However, our results do not show clear evidence for the hypothesis that this break takes place at the particular date of Black Monday.

The structure of the paper is as follows: in Section 2, we briefly summarize the concept of tail dependence and corresponding nonparametric estimation techniques. The new testing procedures for constancy of the tail dependence are introduced in Section 3. In particular, we derive the asymptotic distribution of the sequential empirical tail copula process, propose a multiplier bootstrap approximation of the latter and show consistency of various asymptotic tests. Additionally, we deal with the estimation of change-points in case the null hypothesis is rejected and propose a data-adaptive way for the necessary parameter choice, common to inference methods in extreme-value theory. A comprehensive simulation study is presented in Section 4, followed by the two elaborate empirical applications in Section 5. All proofs are deferred to an Appendix.

2. The concept of tail dependence and its nonparametric estimation

Let (X, Y) be a bivariate random vector with continuous marginal cumulative distribution functions (c.d.f.s) F and G. *Lower or upper tail dependence* concerns the tendency that extremely small or extremely large outcomes of X and Y occur simultaneously. Simple, widely used and intuitive scalar measures for these tendencies are provided by the well-established coefficients of tail dependence (TDC), defined as

$$\lambda_L = \lim_{t \searrow 0} \mathbb{P}\{F(X) \leq t \mid G(Y) \leq t\}, \qquad \lambda_U = \lim_{t \nearrow 1} \mathbb{P}\{F(X) \geq t \mid G(Y) \geq t\} \quad (1)$$

see for instance Joe (1997); Frahm et al. (2005), among others.

It is well-known that the joint c.d.f. H of (X, Y) can be written in a unique way as

$$H(x, y) = C\{F(x), G(y)\}, \qquad x, y \in \mathbb{R}, \quad (2)$$

where the copula C is a c.d.f. on $[0, 1]^2$ with uniform marginals. Elementary calcu-

lations show that the conditional probabilities in (1) can be written as

$$\lambda_L = \lim_{t \searrow 0} \frac{C(t,t)}{t}, \qquad \lambda_U = \lim_{t \searrow 0} \frac{\overline{C}(t,t)}{t},$$

where $\overline{C}$ denotes the survival copula of (X, Y). Therefore, the coefficients of tail dependence can be regarded as directional derivatives of C or $\overline{C}$ at the origin with direction $(1, 1)$. Considering different directions, we arrive at the so-called lower tail copulas, defined for any $(x, y) \in \mathbb{E} = [0, \infty]^2 \setminus \{(\infty, \infty)\}$ by

$$\Lambda_L(x, y) = \lim_{t \searrow 0} \frac{C(xt, yt)}{t}, \qquad \Lambda_U(x, y) = \lim_{t \searrow 0} \frac{\overline{C}(xt, yt)}{t}, \qquad (3)$$

see Schmidt and Stadtmüller (2006). Note that the upper tail copula of (X, Y) is the lower tail copula of $(-X, -Y)$, whence there is no conceptual difference between upper and lower tail dependence.

Several variants of tail copulas have been proposed in the literature on multivariate extreme-value theory. For instance, $L(x, y) = x + y - \Lambda_U(x, y)$ denotes the stable tail dependence function, see, e.g., de Haan and Ferreira (2006). The function $A(t) = 1 - \Lambda_U(1 - t, t)$, which is simply the restriction of L to the unit sphere with respect to the $\| \cdot \|_1$-norm, is called Pickands dependence function, see Pickands (1981). All these variants are one-to-one and are known to characterize the extremal dependence of X and Y, see de Haan and Ferreira (2006). In the present paper we restrict ourselves to the case of tail copulas.

Nonparametric estimation of L and Λ has been addressed in Huang (1992); Drees and Huang (1998); Einmahl et al. (2006); de Haan and Ferreira (2006); Bücher and Dette (2011); Einmahl et al. (2012) for i.i.d. samples $(X_i, Y_i)_{i \in \{1,\dots,n\}}$. For instance, in the case of lower tail copulas, the considered estimators are slight variants, differing only up to a term of uniform order $O(1/k)$, of the function

$$(x, y) \mapsto \frac{1}{k} \sum_{i=1}^{n} \mathbb{1}\Big(R_i \le kx, S_i \le ky \Big) \qquad (4)$$

where R_i (resp. S_i) denotes the rank of X_i (resp. Y_i) among $X_1, \dots, X_n$ (resp. $Y_1, \dots, Y_n$), and where $k = k_n \to \infty$ denotes an intermediate sequence to be chosen by the statistician. Under suitable assumptions on k_n and on the speed of convergence in (3) the estimators are known to be $\sqrt{k_n}$-consistent. Additionally, under certain smoothness conditions on Λ, the corresponding process $\sqrt{k_n}(\hat{\Lambda} - \Lambda)$ converges to a Gaussian limit process.

3. Testing for constant tail dependence

3.1. Setting and test statistics Let $(X_i, Y_i)_{i \in \{1,\dots,n\}}$ be an independent sequence of bivariate random vectors with joint c.d.f. H_i and identical continuous marginal c.d.f.s F and G, respectively. According to Sklar's Theorem, see (2), we can decompose

$$H_i(x, y) = C_i\{F(x), G(y)\}, \qquad x, y \in \mathbb{R},$$

where $C_i(u, v) = \mathbb{P}(U_i \leq u, V_i \leq v)$ with $U_i = F(X_i)$ and $V_i = G(Y_i)$. We assume that the corresponding lower tail copulas

$$\Lambda_i(x, y) = \lim_{t \to \infty} tC_i(x/t, y/t) \tag{5}$$

exist for all $(x, y) \in \mathbb{E} = [0, \infty]^2 \setminus \{(\infty, \infty)\}$ and all $i = 1, \ldots, n$.

The assumption of i.i.d. marginal time series may appear somewhat restrictive. Nonetheless, in the literature on testing for constant copulas, it can be considered as a common practice, see for instance Busetti and Harvey (2011); Remillard (2010); van Kampen and Wied (2012); Kojadinovic and Rohmer (2012). In Section 5, the role of (X_i, Y_i) will be played by the unobservable, serially independent innovations of popular time series models such as AR or GARCH processes. In these cases, we will apply the proposed tests to the observable, standardized residuals (obtained by univariate filtering) and consider these residuals as *almost i.i.d.* Our extensive simulation study in Section 4 indicates that the additional estimation step does not influence the asymptotic behavior of our test statistics, i.e., the asymptotic distribution of the estimator based on residuals is the same as the one based on the unobservable, serially independent innovations. Note that this observation is supported by the results in Chen and Fan (2006); Remillard (2010); Chan et al. (2009), where it is shown that the asymptotic distributions of both semi- and non-parametric estimators in copula models are not influenced by marginal filtering.

It is our aim to develop tests for detecting changes in the tail dependence, i.e., to test for

$$H_0^\Lambda : \text{ there exists } \Lambda > 0 \text{ such that } \Lambda_i \equiv \Lambda \text{ for all } i = 1, \ldots, n$$

against alternatives involving the non-constancy of Λ_i. A special case of this null hypothesis is given by considering the conventional lower tail dependence coefficient $\lambda_i = \Lambda_i(1, 1)$. The corresponding null hypothesis reads as

$$H_0^\lambda : \text{ there exists } \lambda > 0 \text{ such that } \lambda_i = \lambda \text{ for all } i = 1, \ldots, n.$$

In order to motivate our test statistics, let us first recapitulate the *empirical tail copula* from Schmidt and Stadtmüller (2006) as the basic nonparametric estimator for Λ under H_0^Λ, see also (4) and the corresponding citations. Replacing the unknown copula in (5) by the empirical copula C_n, it is defined as

$$\hat{\Lambda}_n(x, y) = \frac{n}{k} C_n \left(\frac{kx}{n}, \frac{ky}{n} \right) = \frac{1}{k} \sum_{i=1}^{n} \mathbb{1} \left(\hat{U}_i \leq kx/n, \hat{V}_i \leq ky/n \right), \tag{6}$$

where $(\hat{U}_i, \hat{V}_i)$ denote *pseudo-observations* from the copula C, defined by

$$\hat{U}_i = \frac{n}{n+1} F_n(X_i), \qquad \hat{V}_i = \frac{n}{n+1} G_n(Y_i),$$

with F_n and G_n denoting the marginal empirical c.d.f.s. Additionally, $k = k_n \to \infty$, $k = o(n)$, represents a sequence of parameters discussed in detail below. The ratio k/n can be interpreted as the fraction of data that one considers as *being in the tail* and thus taken into account to estimate the tail dependence in Equation (6). Under suitable regularity conditions some of which are given in the subsequent Section 3.2, it is known that $\hat{\Lambda}_n$ is $\sqrt{k}$-consistent for Λ and that the corresponding *empirical*

tail copula process $(x, y) \mapsto \sqrt{k}\{\hat{\Lambda}_n(x, y) - \Lambda(x, y)\}$ converges weakly to a Gaussian limit process.

Now, in order to test for H_0^Λ, it is natural to consider a suitable sequential version of $\hat{\Lambda}_n$. We define

$$\hat{\Lambda}_n^\circ(s, x, y) = \frac{1}{k} \sum_{i=1}^{\lfloor ns \rfloor} \mathbb{1}\left(\hat{U}_i \leq kx/n, \hat{V}_i \leq ky/n\right)$$

as the sequential empirical tail copula. Under H_0^Λ, $\hat{\Lambda}_n^\circ$ should be regarded as an estimator for $\Lambda^\circ(s, x, y) = s\Lambda(x, y)$. Note that $\hat{\Lambda}_n^\circ(1, x, y) = \hat{\Lambda}_n(x, y)$. The crucial quantity for all test procedures in this paper is now given by the corresponding *sequential empirical tail copula process* $\{\mathbb{G}_n(s, x, y), s \in [0, 1], (x, y) \in \mathbb{E}\}$ with

$$\mathbb{G}_n(s, x, y) = \sqrt{k}\left\{\hat{\Lambda}_n^\circ(s, x, y) - s\hat{\Lambda}_n^\circ(1, x, y)\right\}.$$

Some simple calculations show that, for $s \in (0, 1)$, $\mathbb{G}_n$ can be written as

$$\mathbb{G}_n(s, x, y) = \sqrt{k}\{s(1-s)\} \left\{ \frac{1}{ks} \sum_{i=1}^{\lfloor ns \rfloor} \mathbb{1}\left(\hat{U}_i \leq kx/n, \hat{V}_i \leq ky/n\right) \right.$$
$$\left. - \frac{1}{k(1-s)} \sum_{i=\lfloor ns \rfloor + 1}^{n} \mathbb{1}\left(\hat{U}_i \leq kx/n, \hat{V}_i \leq ky/n\right) \right\}.$$

Since $ks \approx \lfloor ks \rfloor$, $ns \approx \lfloor ns \rfloor$ and $k/n \approx \lfloor ks \rfloor/\lfloor ns \rfloor$ for any $s \in (0, 1)$, the two summands in the brackets on the right-hand side can be interpreted as (slightly adapted) empirical tail copulas of the subsamples $(X_1, Y_1), \ldots, (X_{\lfloor ns \rfloor}, Y_{\lfloor ns \rfloor})$ and $(X_{\lfloor ns \rfloor + 1}, Y_{\lfloor ns \rfloor + 1}), \ldots, (X_n, Y_n)$, respectively, with corresponding sequence of parameters $k' = \lfloor ks \rfloor$ and $k'' = \lfloor k(1-s) \rfloor$. Under H_0^Λ, one would expect that the difference between these two estimators converges to 0. Therefore, any statistic that can be interpreted as a distance between $\mathbb{G}_n$ and the function being constantly equal to 0 is a reasonable candidate for a test statistic for the null hypothesis. A simulation study similar to one presented in Section 4 showed that a Cramér-von Mises functional yields to the best finite-sample performance, which is why we restrict ourselves to this case in the subsequent presentation. Consequently, in case of the simple null hypothesis H_0^λ, we propose the test statistic

$$\mathcal{S}_n := \{\hat{\Lambda}_n^\circ(1, 1, 1)\}^{-1} \int_0^1 \{\mathbb{G}_n(s, 1, 1)\}^2 \, ds \tag{7}$$

and to reject the null hypothesis whenever $\mathcal{S}_n$ is larger than an appropriate critical value to be determined later on.

For the construction of a test for the null hypothesis H_0^Λ, we make use of the fact that, by homogeneity, the lower tail copula is uniquely determined by its values on the sphere $S(c) = \{\mathbf{x} \in [0, \infty)^2 : \|\mathbf{x}\| = c\}$, where $\|\cdot\|$ denotes an arbitrary fixed norm on $\mathbb{R}^2$ and where $c > 0$ is an arbitrary fixed constant. The most popular choice in bivariate extreme value theory is $c = 1$ together with the $\|\cdot\|_1$-norm resulting in the function $B : [0, 1] \to [0, 1/2] : t \mapsto B(t) = \Lambda(1 - t, t)$. Note that $B(t) = 1 - A(t)$ with the Pickands dependence function A, see, e.g., Segers (2012).

In order to test for overall constancy of Λ_i it is sufficient to test for constancy of Λ_i on some sphere $S(c)$. In Section 3.5, we will propose a data-adaptive procedure for the choice of the parameter k, which will suggest to use a sphere that contains the point $(1, 1)$. For that reason, we introduce the following test statistic

$$\mathcal{T}_n := \int_0^1 \int_0^1 \{\mathbb{G}_n(s, 2 - 2t, 2t)\}^2 \, dt ds,$$

whose support corresponds to the $\| \cdot \|_1$-norm and $c = 2$, and let H_0^Λ again be rejected when $\mathcal{T}_n$ is larger than an appropriate critical value.

In order to determine the critical values, we will derive the asymptotic null distributions of the tests in the next subsection. For both statistics, they will rely on a limit result for the sequential empirical tail copula process.

3.2. Asymptotic null distributions   Let $\mathcal{B}_\infty([0, 1] \times \mathbb{E})$ denote the space of all functions $f : [0, 1] \times \mathbb{E} \to \mathbb{R}$ which are uniformly bounded on every compact subset of $[0, 1] \times \mathbb{E}$ (here and throughout, we understand $\mathbb{E} = [0, \infty]^2 \setminus \{(\infty, \infty)\}$ as the one-point uncompactification of the compact set $[0, \infty]^2$), equipped with the metric

$$d(f, g) = \sum_{m=1}^\infty 2^{-m}(\|f - g\|_{S_m} \wedge 1),$$

where $a \wedge b = \min(a, b)$, where the sets S_m are defined as $S_m = [0, 1] \times T_m$ with

$$T_m = [0, m]^2 \cup (\{\infty\} \times [0, m]) \cup ([0, m] \times \{\infty\})$$

and where $\| \cdot \|_S$ denotes the sup-norm on a set S. Note that convergence with respect to d is equivalent to uniform convergence on each S_m.

In the following we are going to show weak convergence of $\mathbb{G}_n$ as an element of the metric space $(\mathcal{B}_\infty([0, 1] \times \mathbb{E}), d)$. Similar as in related references on the estimation of tail copulas (see Section 2), we have to impose several regularity conditions. First, we need a second order condition quantifying the speed of convergence in (5) uniformly in i and (x, y).

Assumption 3.1. *We have $\Lambda_i \not\equiv 0$ and*

$$\Lambda_i(x, y) - tC_i(x/t, y/t) = O(B(t)), \quad t \to \infty, \tag{8}$$

uniformly on $\{(x, y) \in [0, 1]^2 : x + y = 1\}$ (and hence uniformly on each T_m) and uniformly in $i \in \mathbb{N}$, where $B : [0, \infty) \to [0, \infty)$ denotes a function satisfying $\lim_{t \to \infty} B(t) = 0$.

Second, the following conditions have to be imposed on the sequence $k = k_n$.

Assumption 3.2. *For some $\alpha > 0$, the non-decreasing sequence $k = k_n \to \infty$ satisfies the conditions*

(a) $k_n/n \downarrow 0$,   *(b)* $\sqrt{k_n}B(n/k_n) = o(1)$,   *(c)* $\limsup_{n \to \infty} k_{\lfloor n\delta \rfloor}/k_n \leq \delta^\alpha$

as n tends to infinity, where (c) has to hold for any $\delta \in (0, 1)$.

Condition (a) is needed anyway to define a meaningful estimator. Condition (b) allows to control appearing bias terms in the non-sequential empirical tail copula

process, see also Schmidt and Stadtmüller (2006) and Bücher and Dette (2011). Finally, Condition (c), which can be regarded as very light, will allow to transfer the results from the non-sequential to the sequential setting.

With these assumptions we can now state the main result of our paper.

Proposition 3.3. *Suppose that Assumptions 3.1 and 3.2 hold. Then, under H_0^Λ,*

$$\mathbb{G}_n \rightsquigarrow \mathbb{G}_\Lambda \quad in \ (\mathcal{B}_\infty([0,1] \times \mathbb{E}), d),$$

where $\mathbb{G}_\Lambda(s, x, y) = \mathbb{B}_\Lambda(s, x, y) - s\mathbb{B}_\Lambda(1, x, y)$. Here, $\mathbb{B}_\Lambda$ is a tight centered Gaussian process with continuous sample paths and with covariance structure

$$\mathbb{E}[\mathbb{B}_\Lambda(s_1, x_1, y_2)\mathbb{B}_\Lambda(s_2, x_2, y_2)] = (s_1 \wedge s_2)\Lambda(x_1 \wedge x_2, y_1 \wedge y_2).$$

As stated above, Assumption 3.2 (b) is needed to control bias terms occurring when estimating Λ by $\hat{\Lambda}_n$. As the process $\mathbb{G}_n$ does not involve the true tail copula Λ, the assertion of Proposition 3.3 actually holds if (b) is replaced by a quite technical, but less restrictive assumption, see Remark A.2 in the appendix. However, as an application of the proposed test procedures in this paper will usually be followed by the application of estimation techniques relying on (b), we do not feel that imposing this condition is too restrictive.

Proposition 3.3 immediately yields the asymptotic null distributions of $\mathcal{S}_n$ and $\mathcal{T}_n$.

Proposition 3.4. *Suppose that Assumptions 3.1 and 3.2 hold. Then, under H_0^Λ,*

$$\mathcal{S}_n \rightsquigarrow \mathcal{S} = \int_0^1 \{B(s)\}^2 ds,$$

where B is a one-dimensional standard Brownian bridge, and

$$\mathcal{T}_n \rightsquigarrow \mathcal{T} = \int_0^1 \int_0^1 \{\mathbb{G}_\Lambda(s, 2 - 2t, 2t)\}^2 dt ds,$$

where $\mathbb{G}_\Lambda$ is defined in Proposition 3.3.

Note that, in fact, the weak convergence of $\mathcal{S}_n$ can be derived under a relaxation of H_0^Λ, as it suffices that $\Lambda_i(x, y) \not\equiv 0$ exists and is constant in time in an open neighborhood of $(1, 1)$. This is, however, *a bit more* than assumed in H_0^λ.

Since the limiting distribution for $\mathcal{S}_n$ in Proposition 3.4 is pivotal, we directly obtain an asymptotic level α test for H_0^λ.

TDC-Test 1. Reject H_0^λ for $\mathcal{S}_n \geq q_{1-\alpha}^C$, where $q_{1-\alpha}^C$ denotes the $(1 - \alpha)$-quantile of the Cramér-von Mises distribution, the latter being defined as the distribution of the random variable $\int_0^1 \{B(s)\}^2 \, ds$.

In order to derive critical values for the test based on $\mathcal{T}_n$, some more effort is needed. Its limiting distribution in Proposition 3.4 is not pivotal and cannot be easily transformed to a distribution which is independent of Λ. Therefore, we propose an appropriate bootstrap approximation for $\mathbb{G}_\Lambda$ which will also allow for the definition of an alternative test for H_0^λ.

Let $B \in \mathbb{N}$ be a large integer and let $\xi_1^{(1)}, \ldots, \xi_n^{(1)}, \ldots, \xi_1^{(B)}, \ldots, \xi_n^{(B)}$ be an independent sequence of $n \times B$ i.i.d. non-negative random variables with mean and variance 1 which are independent of the data $(X_1, Y_1), \ldots, (X_n, Y_n)$ and possess finite moments of any order. We will refer to $\xi_i^{(b)}$ as a multiplier. For $b \in \{1, \ldots, B\}$,

8

let $\bar{\xi}_n^{(b)} = n^{-1} \sum_{i=1}^n \xi_i^{(b)}$ denote the arithmetic mean of $\xi_1^{(b)}, \ldots, \xi_n^{(b)}$. Similar in spirit as in Bücher and Dette (2011) we define, for any $(s, x, y) \in [0,1] \times \mathbb{E}$ and $b \in \{1, \ldots, B\}$,

$$\mathbb{G}_{n,\xi^{(b)}}(s, x, y) = \mathbb{B}_{n,\xi^{(b)}}(s, x, y) - s\mathbb{B}_{n,\xi^{(b)}}(1, x, y),$$

where

$$\mathbb{B}_{n,\xi^{(b)}}(s, x, y) = \sqrt{k}\{\hat{\Lambda}_{n,\xi^{(b)}}^\circ(s, x, y) - \hat{\Lambda}_n^\circ(s, x, y)\}$$

and

$$\hat{\Lambda}_{n,\xi^{(b)}}^\circ(s, x, y) = \frac{1}{k} \sum_{i=1}^{\lfloor ns \rfloor} \frac{\xi_i^{(b)}}{\bar{\xi}_n^{(b)}} \mathbb{1}(\hat{U}_i \leq kx/n, \hat{V}_i \leq ky/n).$$

The following proposition states that, for large n, $\mathbb{G}_{n,\xi^{(1)}}, \ldots, \mathbb{G}_{n,\xi^{(B)}}$ can be regarded as *almost* independent copies of $\mathbb{G}_n$. To prove the result, one additional technical assumption on the sequence k_n is required, which, again, can be regarded as very light.

Assumption 3.5. *There exists some $p \in \mathbb{N}$ such that $n/k_n^p = o(1)$.*

Proposition 3.6. *Suppose that Assumptions 3.1, 3.2 and 3.5 hold. Then, under H_0^Λ,*

$$(\mathbb{G}_n, \mathbb{G}_{n,\xi^{(1)}}, \ldots, \mathbb{G}_{n,\xi^{(B)}}) \rightsquigarrow (\mathbb{G}_\Lambda, \mathbb{G}_\Lambda^{(1)}, \ldots, \mathbb{G}_\Lambda^{(B)})$$

in $(\mathcal{B}_\infty([0,1] \times \mathbb{E}), d)^{B+1}$, where $\mathbb{G}_\Lambda^{(1)}, \ldots, \mathbb{G}_\Lambda^{(B)}$ are independent copies of $\mathbb{G}_\Lambda$.

For $b = 1, \ldots, B$, define $\mathcal{S}_{n,\xi^{(b)}}$ and $\mathcal{T}_{n,\xi^{(b)}}$ by

$$\mathcal{S}_{n,\xi^{(b)}} = \hat{\lambda}_n^{-1} \int_0^1 \{\mathbb{G}_{n,\xi^{(b)}}(s, 1, 1)\}^2 ds, \quad \mathcal{T}_{n,\xi^{(b)}} = \int_0^1 \int_0^1 \{\mathbb{G}_{n,\xi^{(b)}}(s, 2 - 2t, 2t)\}^2 dt ds,$$

where $\hat{\lambda}_n = \hat{\Lambda}_n(1, 1, 1)$. We obtain the following tests for H_0^λ and H_0^Λ, respectively.

TDC-Test 2. Reject H_0^λ for $\mathcal{S}_n \geq \hat{q}_{\mathcal{S}_n, 1-\alpha}$, where $\hat{q}_{\mathcal{S}_n, 1-\alpha}$ denotes the $(1 - \alpha)$-sample quantile of $\mathcal{S}_{n,\xi^{(1)}}, \ldots, \mathcal{S}_{n,\xi^{(B)}}$.

TC-Test. Reject H_0^Λ for $\mathcal{T}_n \geq \hat{q}_{\mathcal{T}_n, 1-\alpha}$, where $\hat{q}_{\mathcal{T}_n, 1-\alpha}$ denotes the $(1 - \alpha)$-sample quantile of $\mathcal{T}_{n,\xi^{(1)}}, \ldots, \mathcal{T}_{n,\xi^{(B)}}$.

The final result of this subsection shows that all proposed tests in this paper asymptotically hold their level.

Corollary 3.7. *Suppose that Assumptions 3.1 and 3.2 hold and that H_0^Λ is valid. Then TDC-Test 1 is an asymptotic level α test for H_0^λ. If, additionally, Assumption 3.5 holds, then TDC-Test 2 and TC-Test are asymptotic level α test for H_0^λ and H_0^Λ, respectively, in the sense that, for any $\alpha \in (0, 1)$,*

$$\lim_{B \to \infty} \lim_{n \to \infty} \mathbb{P}(\mathcal{S}_n \geq \hat{q}_{\mathcal{S}_n, 1-\alpha}) = \alpha, \qquad \lim_{B \to \infty} \lim_{n \to \infty} \mathbb{P}(\mathcal{T}_n \geq \hat{q}_{\mathcal{T}_n, 1-\alpha}) = \alpha.$$

3.3. Asymptotics under a fixed alternative In the present subsection we are going to show consistency of the proposed test statistics under fixed alternatives. We observe a triangular array of row-wise independent random vectors $(X_{i,n}, Y_{i,n})$, $i = 1, \ldots, n$, such that $X_{i,n} \sim F$ and $Y_{i,n} \sim G$ for all i and n and such that the copula $C_{i,n}$ of $(X_{i,n}, Y_{i,n})$ may vary over time. Slightly abusing notation, we omit

the index n wherever it does not cause any ambiguity. For the sake of a clear exposition, we first consider the following two simple alternatives for H_0^λ and H_0^Λ.

H_1^λ : there exists $\bar{s} \in (0,1), \lambda^{(1)} \neq \lambda^{(2)}$ such that
$$\lambda_i = \lambda^{(1)} \text{ for } i = 1, \ldots, \lfloor n\bar{s} \rfloor \text{ and } \lambda_i = \lambda^{(2)} \text{ for } i = \lfloor n\bar{s} \rfloor + 1, \ldots, n.$$
H_1^Λ : there exists $\bar{s} \in (0,1), \Lambda^{(1)} \not\equiv \Lambda^{(2)}$ such that
$$\Lambda_i = \Lambda^{(1)} \text{ for } i = 1, \ldots, \lfloor n\bar{s} \rfloor \text{ and } \Lambda_i = \Lambda^{(2)} \text{ for } i = \lfloor n\bar{s} \rfloor + 1, \ldots, n.$$

Proposition 3.8. *Suppose that Assumptions 3.1 and 3.2 hold.*

(i) If H_1^λ and H_1^Λ are true, then
$$\sup_{s \in [0,1]} \left| \frac{1}{\sqrt{k_n}} \mathbb{G}_n(s,1,1) - G(s) \right| = o_P(1)$$

where $G(s) = s(1-\bar{s})(\lambda^{(1)} - \lambda^{(2)})$ for $s \leq \bar{s}$ and $G(s) = \bar{s}(1-s)(\lambda^{(1)} - \lambda^{(2)})$ for $s > \bar{s}$. Moreover, $\mathcal{S}_n$ converges to infinity in probability.

(ii) If H_1^Λ is true, then
$$\sup_{s \in [0,1],(x,y) \in T_m} \left| \frac{1}{\sqrt{k_n}} \mathbb{G}_n(s,x,y) - H(s,x,y) \right| = o_P(1)$$

for any $m \in \mathbb{N}$, where $H(s,x,y) = s(1-\bar{s})\{\Lambda^{(1)}(x,y) - \Lambda^{(2)}(x,y)\}$ for $s \leq \bar{s}$ and $H(s,x,y) = \bar{s}(1-s)\{\Lambda^{(1)}(x,y) - \Lambda^{(2)}(x,y)\}$ for $s > \bar{s}$. Moreover, $\mathcal{T}_n$ converges to infinity in probability.

As already mentioned after Proposition 3.4, it is not necessary to assume global constancy of the tail copulas in the respective subsamples in part (i) of Proposition 3.8, constancy in a neighborhood of $(1,1)$ is sufficient. Moreover, Proposition 3.8 implies consistency of the proposed tests.

Corollary 3.9. *Suppose that Assumptions 3.1 and 3.2 are satisfied. Then TDC-Test 1 is consistent for H_1^λ. If, additionally, Assumption 3.5 holds, then TDC-Test 2 and TC-Test are consistent for H_1^λ and H_1^Λ, respectively, in the sense that, for any $B \in \mathbb{N}$ and $\alpha \in (0,1)$,*
$$\lim_{n \to \infty} \mathbb{P}(\mathcal{S}_n \geq \hat{q}_{\mathcal{S}_n, 1-\alpha}) = 1, \qquad \lim_{n \to \infty} \mathbb{P}(\mathcal{T}_n \geq \hat{q}_{\mathcal{T}_n, 1-\alpha}) = 1.$$

Under H_1^λ and H_1^Λ, consistent estimator for the change-point fraction $\bar{s}$ are given by $\hat{s}^\lambda := \text{argmax}_{s \in [0,1]} |\mathbb{G}_n(s,1,1)|$ and $\hat{s}^\Lambda := \text{argmax}_{s_{[0,1]}} \sup_{t \in [0,1]} |\mathbb{G}_n(s, 2-2t, 2t)|$, respectively.

Proposition 3.10. *Suppose that Assumptions 3.1 and 3.2 hold.*

(i) If H_1^λ and H_1^Λ are true, $\hat{s}^\lambda \to_p \bar{s}$.

(ii) If H_1^Λ is true, $\hat{s}^\Lambda \to_p \bar{s}$.

Note that, if one of the alternatives H_1^λ or H_1^Λ holds, then the other one cannot hold with a different value for $\bar{s}$. Hence, the change-point $\bar{s}$ in Proposition 3.10 (i) is well-defined.

10

Up to now, we have assumed the existence of at most one single break-point. An analog consistency result for the test can be obtained in the case of an arbitrary finite number of break-points between which the tail copula is constant, respectively. For example, a corresponding alternative for H_0^λ would then read as: there exists a finite number of points $0 = s_0 < s_1 < \ldots < s_\ell < \ldots < s_L = 1$ such that, for any $\ell \in \{1, \ldots, L\}$, the TDC of the sample $(X_{\lfloor ns_{\ell-1}\rfloor+1}, Y_{\lfloor ns_{\ell-1}\rfloor+1}), \ldots, (X_{\lfloor ns_\ell \rfloor}, Y_{\lfloor ns_\ell \rfloor})$ is given by $\lambda^{(\ell)}$, with $\lambda^{(\ell)} \neq \lambda^{(\ell+1)}$.

Estimating the change-points $s_1, s_2, \ldots$ is slightly more complicated than it is in the case of just one break-point. In principal, it is also possible to work with the argmax-estimator $s^{\hat\lambda}$ here, but, by construction, this estimator only estimates a single change-point. The number and the location of the other change-points can be estimated by a binary segmentation algorithm going back to Vostrikova (1981). This procedure is for instance applied in Galeano and Wied (2013) to the problem of detecting changing correlations. The basic principle is as follows: at first, the test is applied to the whole sample. If the null hypothesis gets rejected, the argmax-estimator $s^{\hat\lambda}$ can be shown to be a consistent estimator for the *dominating* change-point (see Galeano and Wied, 2013). In the next step, the sample is divided into two parts with the split point given by $\lfloor ns^{\hat\lambda} \rfloor$. The test is applied to both parts separately to decide whether one gets additional change-points in the corresponding subsamples. In that case, the respective subsample is further divided at the corresponding estimated break-point. This procedure is repeated until no further change-points are detected.

3.4. Testing for a break at a specific time point

In certain applications, one might have a reasonable guess for a potential break-point in the tail dependence of a time series. Important econometric examples can be seen in Black Monday on 19th of October 1987, the introduction of the Euro on 1st of January 1999 or the bankruptcy of Lehman Brothers Inc. on 15th of September 2008. In that case, it might be beneficial to test for constancy against a break at that specific time point rather than testing against the existence of *some* unspecified break-point. The results in the previous sections easily allow to obtain simple tests in this setting.

Under the situation of Section 3.1, let $\bar{s} \in (0,1)$ be some fixed time point of interest. Suppose we know that the tail dependence is constant in the two subsamples before $\lfloor n\bar{s} \rfloor$ and after $\lfloor n\bar{s} \rfloor + 1$, which, in practice, can be verified by the tests in the preceding sections. Then, to test for H_0^λ against

$H_1^\lambda(\bar{s}):$ there exists $\lambda^{(1)} \neq \lambda^{(2)}$ such that
$$\lambda_i = \lambda^{(1)} \text{ for } i = 1, \ldots, \lfloor n\bar{s} \rfloor \text{ and } \lambda_i = \lambda^{(2)} \text{ for } i = \lfloor n\bar{s} \rfloor + 1, \ldots, n,$$

we propose to use the test statistic

$$\mathcal{S}_n(\bar{s}) = \{\bar{s}\Lambda_n^\circ(1,1,1)\}^{-1}\mathbb{G}_n(\bar{s},1,1)^2. \tag{9}$$

It easily follows from Proposition 3.3 that, under the null hypothesis, $\mathcal{S}_n(\bar{s})$ weakly converges to a chi-squared distribution with one degree of freedom. Under the alternative, it follows from Proposition 3.8 that $\mathcal{S}_n(\bar{s})$ converges to infinity, in probability. Hence, rejecting H_0 if $\mathcal{S}_n(\bar{s})$ exceeds a corresponding quantile of the chi-squared distribution, yields a consistent test for H_0^λ against $H_1^\lambda(\bar{s})$, which asymptotically holds its significance level. Similar results can be obtained for the bootstrap analog

and for the test for constancy of the entire tail copula, the details are omitted for the sake of brevity.

3.5. Choice of the parameter k As usual in extreme-value theory, the choice of k_n plays a crucial role for statistical applications. The asymptotic properties of the tests proposed in this paper hold as long as the assumptions on the sequence k_n from Assumption 3.2 (and of course other assumptions) hold. This, of course, allows for a large number of possible choices of k_n. However, the results of the testing procedures may depend crucially on the specific choice of k_n.

The common approach in extreme-value theory to cope with this problem is to consider the outcome of statistical procedures, for instance of an estimator, for several different values of k. The set of all these outcomes should give a clearer picture of the underlying data-generating process. This, for instance, is the basic motivation for the Hill plot used in univariate extreme-value theory for estimating the extreme-value index, see, e.g., Embrechts et al. (1997). Additionally, in certain univariate settings some refined data-adaptive choices to estimate an *optimal k* have been developed, see for instance Drees and Kaufmann (1998) or Danielsson et al. (2001).

In the specific context of estimating tail dependence, Frahm et al. (2005) use plots of the function $k \mapsto \mathrm{TDC}(k)$ to define an *plateau-finding* algorithm that provides a single data-adaptive choice of k. In most of the application in this paper, we closely follow their approach for which reason we briefly summarize this algorithm in the following.

The aim of the algorithm is to search for a value k^* such that the TDC, as a function of k, is *as constant as possible* in a suitable neighborhood of k^*. This is achieved by accomplishing the following steps: first, the function $k \mapsto \mathrm{TDC}(k)$ is smoothed by a box kernel depending on a bandwidth b; we denote the smoothed plot by $k \mapsto \tilde{\lambda}_b(k), k = 1, \ldots, n - 2b$. In our simulation study, we use $b = \lfloor 0.005n \rfloor$. In a second step, we consider a rolling window of vectors or plateaus (having length $\ell = \lfloor \sqrt{n - 2b} \rfloor$) with their entries consisting of successive values of the smoothed TDC-plot, formally defined as $P(k) = (\tilde{\lambda}_b(k), \tilde{\lambda}_b(k+1), \ldots, \tilde{\lambda}_b(k+\ell-1)) \in \mathbb{R}^\ell$, where $k = 1, \ldots, n - 2b - \ell + 1$. We calculate the sum of the absolute deviations between all entries and the first entry in each vector, i.e., $\mathrm{MAD}(k) = \sum_{j=1}^{\ell} |(P(k))_1 - (P(k))_j|$. The algorithm searches for the first vector such that $\mathrm{MAD}(k)$ is smaller than two times the sample standard deviation of all values of the smoothed TDC-plot $\tilde{\lambda}_b(1), \ldots, \tilde{\lambda}_b(n - 2b)$. Finally, k^* is defined as the index which corresponds to the middle entry (the floor function if the length is even) of this vector. For further details, we refer to Frahm et al. (2005).

3.6. Higher dimensions   Although we have focused on the case of two dimensions so far, it is basically straightforward (although notationally more involved) to deal with d-dimensional random vectors for a fixed number d. Consider a sequence of marginally i.i.d. random vectors $(X_{i1}, \ldots, X_{id})_{i \in \{1, \ldots, n\}}$ with continuous marginal c.d.f.s $F_1, \ldots, F_d$ and d-dimensional copulas C_i. We suppose that the corresponding lower tail copulas

$$\Lambda_i(x_1, \ldots, x_d) = \lim_{t \to \infty} t C_i(x_1/t, \ldots, x_d/t).$$

12

exist for all $x = (x_1, \ldots, x_d) \in \mathbb{E}_d = [0, \infty]^d \setminus \{(\infty, \ldots, \infty)\}$. Define pseudo-observations $(\hat{U}_{i1}, \ldots, \hat{U}_{id})$ from the copula C_i by $\hat{U}_{ij} = \frac{n}{n+1} F_{nj}(X_{ij})$, $j = 1, \ldots, d$, where F_{nj} denote the marginal empirical c.d.f.s. The d-dimensional sequential empirical tail copula process is defined, for any $(s, x_1, \ldots, x_d) \in [0, 1] \times \mathbb{E}_d$, by

$$\mathbb{G}_n(s, x_1, \ldots, x_d) = \sqrt{k} \left\{ \hat{\Lambda}_n^\circ(s, x_1, \ldots, x_d) - s \hat{\Lambda}_n^\circ(1, x_1, \ldots, x_d) \right\},$$

where $\hat{\Lambda}_n^\circ(s, x_1, \ldots, x_d) = \frac{1}{k} \sum_{i=1}^{\lfloor ns \rfloor} \mathbb{1}(\hat{U}_{i1} \le kx_1/n, \ldots, \hat{U}_{id} \le kx_d/n)$. Then, using the test statistics

$$\mathcal{S}_n := \{\hat{\Lambda}_n^\circ(1, 1, \ldots, 1)\}^{-1} \int_0^1 \{\mathbb{G}_n(s, 1, \ldots, 1)\}^2 ds$$

and

$$\mathcal{T}_n := \int_0^1 \int_{\Delta_{d-1}} \{\mathbb{G}_n(s, 1 - t_1 - \cdots - t_{d-1}, t_1, \ldots, t_{d-1})\}^2 d(t_1, \ldots, t_{d-1}) ds,$$

where $\Delta_{d-1} = \{(t_1, \ldots, t_{d-1}) \in [0, 1]^d \mid \sum_{j=1}^{d-1} t_j = 1\}$, we obtain basically the same tests as in the two-dimensional case. Note that for testing constancy of Λ_i, by similar arguments as in the two-dimensional case, it suffices to show constancy of B_i for $B_i(t_1, \ldots, t_{d-1}) = \Lambda_i(1 - \sum_{j=1}^{d-1} t_j, t_1, \ldots, t_{d-1})$. For the asymptotic results, one has to modify the metric defined in the beginning of Section 3.2 such that

$$T_m = \bigcup_{j=0}^{d-1} \bigcup_{\ell=1}^{\binom{d}{j}} U_{m,j,\ell},$$

where, for each $m \in \mathbb{N}$ and $j = 0, \ldots, d-1$, the $U_{m,j,\ell}$ are the $\binom{d}{j}$ different d-fold cartesian products that contain j times $\{\infty\}$ and $d - j$ times $[0, m]$.

4. Evidence in finite samples

This section investigates the finite sample properties of the proposed testing procedures by means of a simulation study. We observe that the tests are slightly conservative and that they have reasonable power properties. As a main conclusion, we obtain that the tests based on i.i.d. observations and on time series residuals show the same asymptotic behavior.

4.1. Setup As outlined in Jäschke (2012) (see also McNeil et al., 2005, Section 7.5), many commonly applied symmetric tail copulas exhibit a quite similar behavior. When comparing, for instance, the Gumbel model (Gumbel, 1960), the Galambos model (Galambos, 1975) or the Hüsler-Reiss model (Hüsler and Reiss, 1989), the plots of $t \mapsto \Lambda(1 - t, t)$, which uniquely determine the tail copula by homogeneity, are nearly indistinguishable. We therefore stick to two cases of one common symmetric and one common asymmetric tail copula model as follows.

($\Lambda 1$) The **negative logistic** or Galambos model (Galambos, 1975), given by

$$\Lambda(1 - t, t) = \left\{ (1 - t)^{-\theta} + t^{-\theta} \right\}^{-1/\theta}, \quad t \in [0, 1],$$

13

where we chose the parameter $\theta \in [1, \infty)$ such that $\lambda = \Lambda(1,1) = 2^{-1/\theta}$ varies in the set $\{0.25, 0.50, 0.75\}$.

(Λ2) The **asymmetric negative logistic** model (Joe, 1990), defined by

$$\Lambda(1 - t, t) = \left\{ (\psi_1(1 - t))^{-\theta} + (\psi_2 t)^{-\theta} \right\}^{-1/\theta}, \quad t \in [0, 1],$$

with two fixed parameters $\psi_1 = 2/3$, $\psi_2 = 1$ and parameter $\theta \in [1, \infty)$ such that $\lambda = \Lambda(1, 1) = 2 \left((\psi_1/2)^{-\theta} + (\psi_2/2)^{-\theta} \right)^{-1/\theta}$ varies in the set $\{0.2, 0.4, 0.6\}$.

Tail copulas being directional derivatives of copulas in the origin, there are of course many copulas that result in the same tail copula. In our simulation study, we stick to simulating from one of following two copula families.

(C1) The **Clayton copula**, given by

$$C(u, v) = \left(u^{-\theta} + v^{-\theta} - 1 \right)^{-1/\theta}, \quad u, v \in [0, 1],$$

possesses the negative logistic tail copula as specified in (Λ1). The Clayton copula is widely used for modeling negative tail dependent data.

(C2) The survival copula of the extreme-value copula

$$C(u, v) = \exp \left\{ \log(uv) A \left(\frac{\log(v)}{\log(uv)} \right) \right\} \quad u, v \in [0, 1], \tag{10}$$

where $A(t) = 1 - \Lambda(1 - t, t)$ with Λ as in (Λ2), see Segers (2012), possesses the asymmetric negative logistic tail copula specified in (Λ2).

Our simulation results will show that the distribution of the test statistic based on estimated *almost i.i.d.* residuals is the same as the one of the test statistic based on the unobservable i.i.d. innovations. Regarding the marginal time series behavior, we consider three different cases. We begin with a consideration of i.i.d. marginals. Subsequently, the simulation results in this case will serve as a benchmark for the application of the tests to *almost i.i.d.* residuals of AR and GARCH time series models.

(T1) **i.i.d. setting.** Here, we simply generate an independent sample (U_i, V_i), $i = 1, \ldots, n$, of one of the aforementioned copulas (C1) or (C2). Note that, without loss of generality, the marginal distribution can be chosen as standard uniform in this case, since all estimators in this paper are rank-based and hence invariant with respect to monotone transformations.

(T2) **AR(1) residuals.** This setting considers the stationary solution (Q_i, R_i) of the first order autoregressive process

$$Q_i = \beta_1 Q_{i-1} + X_i, \qquad R_i = \beta_2 R_{i-1} + Y_i, \tag{11}$$

where (X_i, Y_i) are i.i.d. bivariate random vectors (innovations) whose marginals are either standard normally distributed or t-distributed with $\nu = 3$ degrees of freedom and whose copula is from model (C1) or (C2). The coefficients (β_1, β_2) of the lagged variables vary in the set $\{1/3, 1/2, 2/3\}$. We simulate a time series of length n of this model as follows:

(a) choose some reasonably large number M, e.g., $M = -100$;

(b) generate an i.i.d. sample (U_i, V_i), $i = M, \ldots, n$, of the copula C and apply the inverse of the marginal c.d.f.s F and G to the copula sample, vis. $(X_i, Y_i) = (F^{-1}(U_i), G^{-1}(V_i))$;

(c) calculate recursively the values (Q_i, R_i) according to (11) for all $i = M + 1, \ldots, n$, starting with $(Q_M, R_M) = (X_M, Y_M)$; the last n observations form the final sample.

Since we do not observe the innovations (X_i, Y_i), we estimate β_1 and β_2 by the Yule-Walker estimators and obtain an *almost i.i.d.* sample having copula C (see Section 3.1) by considering the time series $(\hat{X}_i, \hat{Y}_i)$ of corresponding estimated residuals defined as

$$\hat{X}_i = Q_i - \hat{\beta}_1 Q_{i-1}, \qquad \hat{Y}_i = R_i - \hat{\beta}_2 R_{i-1}.$$

(T3) **GARCH(1,1) residuals.** The final setting analyses a two-dimensional time series model which is based on the frequently applied univariate GARCH(1,1) model. More precisely, we consider the solution (Q_i, R_i) of

$$\begin{cases} Q_i = \sigma_{i,1} X_i, & \sigma_{i,1}^2 = \omega_1 + \alpha_1 X_{i-1}^2 + \beta_1 \sigma_{i-1,1}^2, \\ R_i = \sigma_{i,2} Y_i, & \sigma_{i,2}^2 = \omega_2 + \alpha_2 Y_{i-1}^2 + \beta_2 \sigma_{i-1,2}^2, \end{cases} \tag{12}$$

where (X_i, Y_i) are i.i.d. bivariate random vectors (innovations) as in the AR(1) scenario. Following the empirical application of modeling volatility of S&P 500 and DAX daily log-returns in Jondeau et al. (2007) we set the coefficients $\omega_1 = 0.012$, $\omega_2 = 0.037$, $\alpha_1 = 0.072$, $\alpha_2 = 0.115$, $\beta_1 = 0.919$ and $\beta_2 = 0.868$. The long run average variances in this model are given by $\sigma_{M,j} = \sqrt{\omega_j/(1 - \alpha_j - \beta_j)}$ which also serve as initial values for simulating a sample from (12). The simulation algorithm reads as follows:

(a) generate an independent sample (X_i, Y_i), $i = M, \ldots, n$ as described in steps (a) and (b) of the previous AR(1) setting;

(b) recursively calculate the values (Q_i, R_i) according to (12) for all $i = M + 1, \ldots, n$, starting with $(Q_M, R_M) = (X_M, Y_M)$; again, the last n observations form the final sample.

An *almost i.i.d.* sample $(\hat{X}_i, \hat{Y}_i)$ to which we apply the tests is obtained by estimating the standardized residuals

$$\hat{X}_i = \sigma_{i,1}^{-1}(\hat{\omega}_1, \hat{\alpha}_1, \hat{\beta}_1) Q_i, \qquad \hat{Y}_i = \sigma_{i,2}^{-1}(\hat{\omega}_2, \hat{\alpha}_2, \hat{\beta}_2) R_i,$$

where the estimates $\hat{\omega}_j, \hat{\alpha}_j$ and $\hat{\beta}_j$, $j = 1, 2$, are calculated by applying standard constraint non-linear optimization routines.

4.2. Results and discussion   The target values of our finite sample study are the simulated rejection probabilities (s.r.p.) of the Cramér-von Mises tests described in Subsection 3.2 under the null hypothesis and under various alternatives. Based on $N = 5{,}000$ repetitions, we estimate the s.r.p. for three different levels of significance, $\alpha \in \{1\%, 5\%, 10\%\}$, for two different sample sizes $n = 1{,}000$ and $n = 3{,}000$ and for all of the previously described models. Due to the close similarity of some of the results, we report them only partially.

In Table 1, we present the results for TDC-Test 1 under 7×3 different null hypotheses. The estimated s.r.p. for the different levels are stated in columns 3–5 ($n = 1,000$) and 8–10 ($n = 3,000$), respectively. The parameter k is determined by the plateau algorithm described in Section 3.5. The properties of this algorithm are summarized in Columns 6 and 7 ($n = 1,000$) and 11 and 12 ($n = 3,000$), where we state the mean and the sample standard deviation of the estimate k^*. We observe an accurate approximation of the nominal level in all cases, with a tendency of a slight underestimation of the significance level in most of the cases. As already mentioned in Subsection 3, the additional initial estimation step of applying univariate filtering to the time series does not significantly influence the finite sample properties. The slight conservative behavior of the test can be explained by the constancy of the copula in most of our settings: defining $C_n(s, u, v) = \frac{1}{n} \sum_{i=1}^{\lfloor ns \rfloor} \mathbb{1}(\hat{U}_i \leq u, \hat{V}_i \leq v)$ the test statistic $\mathcal{S}_n$ from Equation (7) can be rewritten as

$$\mathcal{S}_n = \{C_n(1, k/n, k/n)\}^{-1} \int_0^1 \left[\sqrt{n}\{C_n(s, k/n, k/n) - sC_n(1, k/n, k/n)\} \right]^2 ds.$$

If k was chosen such that $u = k/n > 0$ is constant in n and if, additionally to the tail copula, the copula remained constant over time, it would follow from Corollary 3.3 (a) in Bücher and Volgushev (2013) that $\mathcal{S}_n$ weakly converges to $\{1 - C(u, u)\} \int_0^1 B^2(s) \, ds$, where B denotes a standard Brownian bridge. Since the critical values of TDC-Test 1 are the quantiles of $\int_0^1 B^2(s) \, ds$, we can easily see that the test rejects too rarely, provided that $C(u, u) > 0$. Note that this argument remains valid if the copula is constant over time only in a neighborhood of (u, u).

A more enlightening view on this issue can be gained from the results in the third block of Table 1. Here, we first simulate the first half of the dataset from model (C1) whereas the second half is simulated from model (C2). The parameters are chosen in such a way that both models exhibit the same tail dependence coefficient. Hence, we are still simulating under the null hypothesis but this time the hybrid model is not constant (over time) at any point on the diagonal of the interior of the unit square. Within the i.i.d. setting we observe that this is the only case where the s.r.p. (slightly) exceed some levels of significance.

In Table 2, we present simulation results for TDC-Test 1 under 8×3 different alternatives. We consider only the case of one break-point, which is either located at $\bar{s} = 0.25$ or at $\bar{s} = 0.5$, and of three different upward jumps. Note that, for symmetry reasons, the results are essentially the same for corresponding downward jumps at $1 - \bar{s}$. The second column of the table indicates the coefficient of tail dependence before and after the break-point. As one might have expected, higher jumps in the TDC are detected more frequently. Also, breaks at $\bar{s} = 0.5$ are more likely to be detected than breaks at $\bar{s} = 0.25$. Similar as for the null hypotheses presented in Table 1, the discrepancy between the corresponding results for the i.i.d. case and for the time series residuals appears to be negligible. Overall, one can conclude that TDC-Test 1 shows reasonable power properties.

Table 3 briefly presents simulation results for TDC-Test 2 and the TC-Test. For the sake of brevity, we only report the s.r.p. for the Clayton tail copula model and the i.i.d. case, since the results for the other cases do not convey any additional insights. The estimated s.r.p. are based on $N = 1,000$ simulation runs, while the sample size is again either $n = 1,000$ or $n = 3,000$ with $B = 500$ bootstrap replications ($B = 300$ for the TC-Test) and multipliers $\xi_i^{(b)}$ that are uniformly

distributed on the set $\{0, 2\}$. In comparison to its competitor TDC-Test 1, we observe that the TDC-Test 2 is slightly more conservative and has almost the same power properties. The results for the TC-Test are basically similar to TDC-Test 2 although the power is slightly lower.

The final results of this section, presented in Table 4, concern a setting, where the tail dependence coefficient stays constant over time whereas the tail copula may change at points $(x, y) \neq (1, 1)$ (cf. third block of Table 1). From theory, one would expect that the TC-Test should be able to detect those breaks, whereas the TDC-Tests should hold the nominal size. We only consider breaks at $\bar{s} = 0.5$ and model ($\Lambda 2$) (i.e., we simulate from (C2)) which will allow to construct tail copulas that are equal in $(1, 1)$, but sufficiently different in other points. More precisely, for a given $\lambda \in \{0.2, 0.4, 0.6\}$, we choose $\psi_1 = \lambda$, $\psi_2 = 1$ and $\theta = 100$ for $s \leq \bar{s}$ and we set $\psi_1 = 1$, $\psi_2 = \lambda$ and $\theta = 100$ for $s > \bar{s}$. For $\lambda = 0.4$, the corresponding graphs of $t \mapsto \Lambda(2 - 2t, 2t)$ are shown in Figure 1. Note that, for fixed ψ_1, ψ_2, we have

$$\Lambda_\infty(1 - t, t) := \lim_{\theta \to \infty} \Lambda(1 - t, t) = \{\psi_1(1 - t)\} \wedge (\psi_2 t).$$

The corresponding *limit* copula defined in (10) is the well-known Marshall–Olkin copula, whose TDC is given by $\min(\psi_1, \psi_2)$, see Segers (2012). With our choice of $\theta = 100$ in ($\Lambda 2$), the difference between the TDC and $\min(\psi_1, \psi_2) = \lambda$ is less than the machine accuracy 10^{-16}.

The results in Table 4 confirm the expectations: the TC-Test has considerable power while the TDC-Test 1 basically keeps the nominal size. As a conclusion, the developed testing procedures allow for empirically distinguishing between constant tail dependence coefficient and constant tail copula.
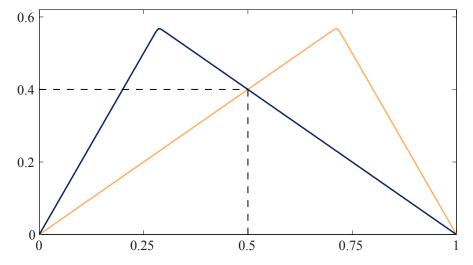

Figure 1: Negative asymmetric logistic model ($\Lambda 2$) for $\psi_1 = 0.4$, $\psi_2 = 1$, $\theta = 100$ (blue) and $\psi_1 = 1$, $\psi_2 = 0.4$, $\theta = 100$ (yellow) evaluated on the straight line $(2 - 2t, 2t)$, $t \in [0, 1]$. Both models exhibit the same tail dependence coefficient $\lambda = 0.4$.

tail copula	Λ(1,1)	n = 1,000					n = 3,000				
		α = 1%	α = 5%	α = 10%	avg(k*)	std(k*)	α = 1%	α = 5%	α = 10%	avg(k*)	std(k*)
i.i.d. setting											
(Λ1)	0.25	0.0076	0.0460	0.0922	52	23	0.0076	0.0442	0.0934	97	49
	0.50	0.0072	0.0444	0.0924	71	29	0.0088	0.0472	0.0906	134	59
	0.75	0.0068	0.0394	0.0846	127	46	0.0082	0.0376	0.0880	237	97
(Λ2)	0.20	0.0086	0.0474	0.0946	44	20	0.0090	0.0448	0.0998	80	40
	0.40	0.0100	0.0416	0.0914	61	26	0.0094	0.0466	0.0926	113	52
	0.60	0.0066	0.0460	0.0896	84	34	0.0082	0.0452	0.0976	153	67
(Λ1), (Λ2)	0.20	0.0098	0.0534	0.1060	47	21	0.0098	0.0558	0.0980	86	44
	0.40	0.0084	0.0444	0.0914	62	26	0.0100	0.0480	0.0922	114	52
	0.60	0.0094	0.0466	0.0920	87	33	0.0096	0.0476	0.0930	159	69
AR(1) residuals											
(Λ1)	0.25	0.0076	0.0462	0.0976	52	23	0.0106	0.0472	0.0992	97	49
	0.50	0.0086	0.0488	0.0906	72	29	0.0110	0.0494	0.0956	134	60
	0.75	0.0066	0.0358	0.0812	126	46	0.0084	0.0410	0.0904	235	95
(Λ2)	0.20	0.0068	0.0422	0.0926	44	20	0.0088	0.0490	0.0994	81	40
	0.40	0.0074	0.0462	0.0920	61	25	0.0090	0.0452	0.0978	112	42
	0.60	0.0088	0.0426	0.0906	83	34	0.0104	0.0454	0.0944	154	67
GARCH(1,1) residuals											
(Λ1)	0.25	0.0074	0.0444	0.0908	52	24	0.0084	0.0428	0.0932	99	50
	0.50	0.0068	0.0426	0.0890	72	29	0.0084	0.0468	0.0946	133	59
	0.75	0.0060	0.0374	0.0750	124	45	0.0094	0.0424	0.0806	232	93
(Λ2)	0.20	0.0080	0.0410	0.0922	44	21	0.0080	0.0418	0.0866	80	41
	0.40	0.0078	0.0426	0.0880	61	26	0.0100	0.0454	0.0908	112	52
	0.60	0.0076	0.0404	0.0858	84	33	0.0074	0.0442	0.0936	154	68

Table 1: Simulated rejection probabilities of the TDC-Test 1 under various null hypotheses. In each of the residual scenarios, the marginals F and G are standard normally distributed and t-distributed with $\nu = 3$ degrees of freedom, respectively. The parameters in the AR(1) setting are set to $\beta_1 = 1/3$ and $\beta_2 = 2/3$.

tail copula	Λ(1,1)	n = 1,000					n = 3,000				
		$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 10\%$	avg(k^*)	std(k^*)	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 10\%$	avg(k^*)	std(k^*)
i.i.d. setting, $\bar{\bar{s}} = 0.5$											
(Λ1)	0.25 to 0.50	0.0512	0.1650	0.2606	61	26	0.1360	0.3090	0.4262	113	53
	0.25 to 0.75	0.3214	0.5632	0.6944	76	30	0.6790	0.8452	0.9042	140	64
	0.50 to 0.75	0.0726	0.2112	0.3212	93	35	0.1848	0.3888	0.5072	171	71
(Λ2)	0.20 to 0.40	0.0536	0.1754	0.2788	52	22	0.1276	0.3062	0.4306	94	45
	0.20 to 0.60	0.2472	0.4850	0.6132	60	25	0.5466	0.7520	0.8302	111	52
	0.40 to 0.60	0.0438	0.1394	0.2312	71	29	0.1048	0.2576	0.3686	130	59
i.i.d. setting, $\bar{\bar{s}} = 0.25$											
(Λ1)	0.25 to 0.50	0.0186	0.0912	0.1740	66	27	0.0536	0.1734	0.2770	122	52
	0.25 to 0.75	0.1250	0.3404	0.4778	95	36	0.3942	0.6496	0.7632	174	74
	0.50 to 0.75	0.0356	0.1290	0.2166	107	40	0.0770	0.2248	0.3388	197	82
(Λ2)	0.20 to 0.40	0.0218	0.0922	0.1716	56	24	0.0504	0.1606	0.2652	104	49
	0.20 to 0.60	0.0896	0.2622	0.3952	71	29	0.2524	0.4980	0.6396	129	60
	0.40 to 0.60	0.0170	0.0792	0.1544	77	31	0.0458	0.1476	0.2384	144	63
AR(1) residuals, $\bar{\bar{s}} = 0.5$											
(Λ1)	0.25 to 0.50	0.0498	0.1636	0.2562	61	26	0.1248	0.2996	0.4202	114	53
	0.25 to 0.75	0.3294	0.5672	0.6898	76	30	0.6860	0.8492	0.9036	138	62
	0.50 to 0.75	0.0726	0.2050	0.3106	92	35	0.1902	0.3896	0.5060	169	72
(Λ2)	0.20 to 0.40	0.0518	0.1702	0.2616	52	22	0.1328	0.2952	0.4130	96	48
	0.20 to 0.60	0.2562	0.4894	0.6138	61	25	0.5646	0.7656	0.8378	111	53
	0.40 to 0.60	0.0444	0.1484	0.2416	71	29	0.1088	0.2676	0.3856	131	59
GARCH(1,1) residuals, $\bar{\bar{s}} = 0.5$											
(Λ1)	0.25 to 0.50	0.0414	0.1484	0.2464	61	26	0.1388	0.3226	0.4332	112	52
	0.25 to 0.75	0.3174	0.5558	0.6838	76	30	0.6726	0.8416	0.9030	139	61
	0.50 to 0.75	0.0694	0.1994	0.2996	91	35	0.1740	0.3728	0.4982	170	72
(Λ2)	0.20 to 0.40	0.0560	0.1650	0.2582	52	23	0.1326	0.3062	0.4192	96	47
	0.20 to 0.60	0.2460	0.4736	0.5992	60	25	0.5448	0.7460	0.8306	112	53
	0.40 to 0.60	0.0406	0.1376	0.2286	71	29	0.0994	0.2530	0.3614	131	59

Table 2: Simulated rejection probabilities of the TDC–Test 1 under various alternatives. In each of the residual scenarios, the marginals F and G are standard normally distributed and t-distributed with $\nu = 3$ degrees of freedom, respectively. The parameters in the AR(1) setting are set to $\beta_1 = 1/2$ and $\beta_2 = 1/2$.

Table 3

scenario	$\Lambda(1,1)$	$n=1{,}000$					$n=3{,}000$				
		$\alpha=1\%$	$\alpha=5\%$	$\alpha=10\%$	avg(k^*)	std(k^*)	$\alpha=1\%$	$\alpha=5\%$	$\alpha=10\%$	avg(k^*)	std(k^*)
TDC-Test 2, i.i.d. setting											
H_0^Λ	0.25	0.005	0.040	0.091	53	24	0.007	0.038	0.097	99	49
	0.50	0.011	0.041	0.088	72	29	0.010	0.046	0.086	134	58
	0.75	0.005	0.030	0.068	129	47	0.006	0.046	0.078	236	96
$H_1^\Lambda,\ \bar s=0.5$	0.25 to 0.50	0.050	0.170	0.260	61	25	0.130	0.300	0.417	115	55
	0.25 to 0.75	0.329	0.563	0.681	77	31	0.711	0.866	0.913	140	61
	0.50 to 0.75	0.080	0.225	0.332	93	35	0.194	0.395	0.505	169	71
TC-Test, i.i.d. setting											
H_0^Λ	0.25	0.010	0.070	0.111	52	22	0.011	0.049	0.101	95	46
	0.50	0.009	0.049	0.098	72	29	0.010	0.053	0.099	132	56
	0.75	0.008	0.042	0.079	127	46	0.006	0.043	0.082	236	98
$H_1^\Lambda,\ \bar s=0.5$	0.25 to 0.50	0.047	0.155	0.237	60	24	0.107	0.312	0.427	111	53
	0.25 to 0.75	0.264	0.496	0.616	77	30	0.581	0.774	0.840	135	60
	0.50 to 0.75	0.063	0.182	0.277	93	36	0.109	0.271	0.392	170	73

Table 3: Simulated rejection probabilities of the TDC-Test 2 and the TC-Test under the null hypothesis and one alternative using the Clayton copula within the i.i.d. setting.

Table 4

scenario	$\Lambda(1,1)$	$n=1{,}000$					$n=3{,}000$				
		$\alpha=1\%$	$\alpha=5\%$	$\alpha=10\%$	avg(k^*)	std(k^*)	$\alpha=1\%$	$\alpha=5\%$	$\alpha=10\%$	avg(k^*)	std(k^*)
TDC-Test 1, i.i.d. setting											
$H_0^\lambda \cap H_1^\Lambda$	0.20	0.009	0.048	0.102	45	21	0.010	0.053	0.103	95	46
	0.40	0.009	0.049	0.103	63	26	0.009	0.048	0.101	120	53
	0.60	0.010	0.053	0.107	89	35	0.012	0.050	0.101	168	72
TC-Test, i.i.d. setting											
$H_0^\lambda \cap H_1^\Lambda$	0.20	0.034	0.163	0.338	46	22	0.242	0.562	0.733	95	46
	0.40	0.045	0.222	0.421	63	26	0.263	0.634	0.801	121	56
	0.60	0.026	0.121	0.301	90	35	0.109	0.480	0.722	168	72

Table 4: Simulated rejection probabilities of the TDC-Test 1 and the TC-Test: the first half of the dataset belongs to model (Λ1), the second half to model (Λ2). The parameters are chosen such that the TDC remains constant over time while the tail copula does not.

5. Empirical applications

5.1. Energy sector In this section, we reinvestigate the bivariate dataset from Jäschke (2012) consisting of $n = 1{,}001$ daily closing quotes of WTI Cushing Crude Oil Spot and the Bloomberg European Dated Brent from October 2, 2006, to October 1, 2010, collected from Bloomberg's Financial Information Services. The analysis of the extremal dependence between the log-returns of the two time series in Jäschke (2012) is based on the implicit assumption that the tail dependence structure, more precisely their lower tail copula, remains constant over time. We are going to verify this assumption by applying the tests developed in the previous sections.

As pointed out in Jäschke (2012), the assumption of an i.i.d. sample is unrealistic. To account for autocorrelation and volatility clustering, it is shown that an ARMA(0,0)-EGARCH(2,3)-model including an explanatory variable (U.S. crude oil inventory) and the skewed generalized error distribution adequately describes the data generating process for the log-returns of the WTI time series. Regarding the daily Brent spot log-returns, an ARMA(1,1)-EGARCH(2,3)-model including U.S. crude oil inventory as an explanatory variable and the skewed generalized error distribution provides an adequate fit.

We calculate standardized residuals on the basis of the preceding time series models. A first view on the lower tail dependence between these residuals can be gained from the diagnostic plot in Figure 2. For various values of k such that k/n lies in the set $\{0.05, 0.06, \ldots, 0.15\}$, we depict the points in time where the pseudo-observations in both coordinates fall simultaneously below the value k/n. Note that these are exactly the joint extremal events inside the indicators in the definition of the empirical tail dependence coefficient. As the points are quite equally spaced in time, the picture suggests that the tail dependence remains rather constant.
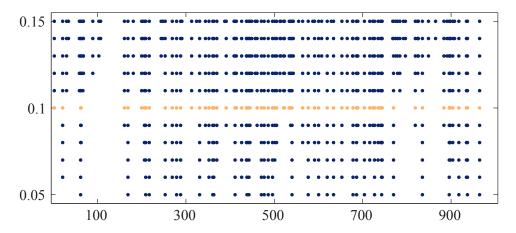

Figure 2: (WTI and Brent time series) Points in time where pseudo-observations in both coordinates fall simultaneously below the value k/n, for $k/n \in \{0.05, 0.06, \ldots, 0.15\}$. The yellow row corresponds to the plateau ratio $k^*/n = 104/1001 \approx 10\%$.

More formally, we proceed by checking the hypothesis H_0^λ of constancy of the tail dependence coefficients by an application of TDC-Test 1. First, in order to obtain a reasonable choice for the parameter k, we use the plateau algorithm from Subsection 3.5 with bandwidth $b = \lfloor 0.005n \rfloor = 5$. This yields a value of $k^* = 104$

21

(which is also depicted in yellow in Figure 2) and a plateau of length $m = 31$. Following Frahm et al. (2005), the average of the 31 empirical lower tail dependence coefficients on this plateau, given by $\hat{\lambda} = 0.732$, provides a good estimate for λ. Figure 3 shows the corresponding standardized sequential empirical tail copula process $s \mapsto \hat{\lambda}^{-1/2}\mathbb{G}_n(s, 1, 1)$ for $k^* = 104$. The graph seems to be indistinguishable from a simulated path of a one-dimensional standard Brownian bridge, which indicates that the null hypothesis cannot be rejected. In Figure 4, we depict both the value of the the Cramér-von Mises type test statistic $\mathcal{S}_n$ defined in (7) (yellow) as well as the corresponding p-valuess (blue) as a function of k. The dashed vertical line shows the outcomes for the plateau optimal $k^* = 104$, in which case we obtain $\mathcal{S}_n = 0.285$ with a resulting p-value of 0.15. Consequently, the null hypothesis H_0^λ cannot be rejected at a 5% level of significance. Moreover, Figure 4 shows that this conclusion is robust with respect to different choices of k. Results for the Kolmogorov-Smirnov-type test and for the TDC-Test 2 are very similar and are not depicted for the sake of brevity.

Finally, the assumption of a constant lower tail copula is verified by testing for the hypothesis H_0^Λ. We apply the TC-Test from Section 3.2 with $B = 2,000$ bootstrap replications using the plateau optimal $k^* = 104$. We obtain $\mathcal{T}_n = 0.068$ with a resulting p-value of 0.33. Again, the null hypothesis cannot be rejected at a 5% level of significance. Similar as for the tests for H_0^λ, this conclusion is robust with respect to different choices of k.
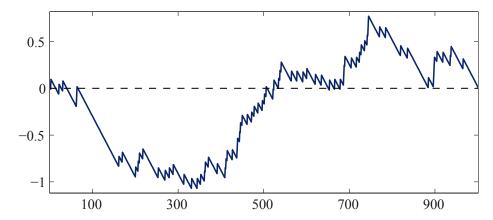

Figure 3: (WTI and Brent time series) Standardized sequential empirical tail copula process $\hat{\lambda}^{-1/2}\mathbb{G}_n(s, 1, 1)$ for $k^* = 104$ with respect to ns, $s \in [0, 1]$.

5.2. Financial markets   As an empirical application from the finance sector, we consider the Dow Jones Industrial Average and the Nasdaq Composite time series around Black Monday on 19th of October 1987. This dataset covers $n = 1,768$ log-returns from daily closing quotes between January 4, 1984, and December 31, 1990, collected from Datastream. Related studies in Wied et al. (2013) and Dehling et al. (2013) try to examine whether Black Monday constitutes a break in the dependence structure between the two time series. The outcomes of their studies do not provide a clear picture, as the answer depends on the applied test statistic. While the test for a constant Pearson correlation rejects the null hypothesis of constant correlation, the more robust (rank-based) tests for constant Spearman's
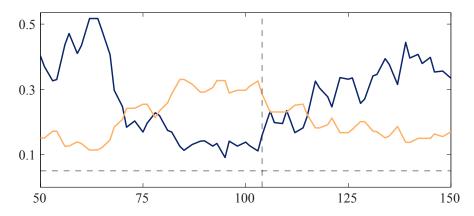
Figure 4: (WTI and Brent time series) Test statistics $\mathcal{S}_n$ (yellow) and corresponding p-values (blue) for different k. The horizontal line indicates the 5% level of significance, the vertical one the plateau $k^* = 104$.

rho and Kendall's tau yield no evidence for changes. In these papers, the contrasting result is explained by the fact that the (unfiltered) time series contain several heavy outliers around Black Monday which seriously affect the Pearson-, but not the rank-based tests for Spearman's rho and Kendall's tau.

For our analysis, we begin by an investigation of the univariate time series. Applying the model selection and verification criteria from Jäschke (2012), we find that an ARMA(0,0)-GARCH(1,1)-model with t-distribution for the Dow Jones log-returns and an ARMA(1,0)-GARCH(1,1)-model with skewed t-distribution for the Nasdaq equivalent provide the best fits. Details on the parameter estimation are given in Table 5.

parameter	Dow Jones log-returns		Nasdaq log-returns	
	estimate	std error	estimate	std error
mean equation				
μ	0.0006	0.0002	0.0005	0.0002
θ_1	-	-	0.2714	0.0234
variance equation				
ω	0.0000	0.0000	0.0000	0.0000
α_1	0.0349	0.0084	0.1407	0.0179
β_1	0.9373	0.0080	0.7914	0.0182
distribution				
ξ	—	—	0.8531	0.0294
ν	4.2016	0.4390	5.3001	0.4135

Table 5: Maximum likelihood estimates together with their corresponding standard errors for the Dow Jones ARMA(0,0)-GARCH(1,1)-model with t-distribution and the Nasdaq ARMA(1,0)-GARCH(1,1)-model including the skewed t-distribution. All estimates but the additive constant ω are significant at the 1% level.

23

Along the lines of Dehling et al. (2013) we first seek to answer the question whether Black Monday constitutes a break in the tail dependence between the two time series. A positive answer would indicate that the market conditions have substantially changed after this date. For the ease of a clear exposition, we restrict ourselves to an investigation of the lower tail dependence coefficient. A first visual description of the joint tail behavior similar to the one in Figure 2 can be found in Figure 5, which, however, does not provide a clear picture: even though there seems to be a tendency of stronger tail dependence for later dates in the time series, it is unclear whether this is due to a change on Black Monday (second dashed vertical line).
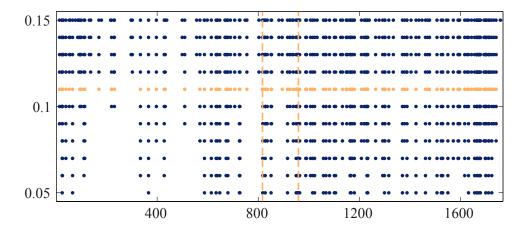

Figure 5: (Dow Jones and Nasdaq time series) Points in time where pseudo-observations in both coordinates fall simultaneously below the value k/n, for $k/n \in \{0.05, 0.06, \ldots, 0.15\}$. The yellow row corresponds to the plateau ratio $k^*/n \approx 11\%$. The first yellow vertical line reflects the argmax-estimator $\lfloor n\hat{s}^\lambda \rfloor = 817$, the second equivalent indicates Black Monday $\lfloor n\bar{s}_{\mathrm{BM}} \rfloor = 959$.

In the following, we examine this formally by applying the tests from Section 3, in particular the test from Section 3.4 for a specific break-point. First, a careful inspection of the plot $k \mapsto \mathrm{TDC}(k)$ and the statistics defining the plateau algorithm (which are not depicted for the sake of brevity) suggests that $k^* = 191$ is a reasonable choice for the parameter k, with a corresponding length of the plateau of $m = 41$. The average of the empirical lower tail dependence coefficients over the corresponding values $k \in \{171, \ldots, 211\}$ is given by $\hat{\lambda} = 0.620$.

Now, we apply the test from Section 3.4 for a specific break-point at $\lfloor n\bar{s}_{\mathrm{BM}} \rfloor = 959$, the date of Black Monday. The results are depicted in Figure 6, where we plot the p-values of the test against the parameter k. For $k^* = 191$, the resulting p-value of 0.082 does not allow for a clear rejection of the null-hypothesis. In contrast to this, slightly lower values of k yield to a rejection at the 5%-significance level, whence, as a summary, there seems to be some light, but disputable evidence against H_0. However, the rejection of the null hypothesis might be due to different reasons than a break precisely on Black Monday. To conclude upon the latter, one would have to accept the hypothesis of constancy of the lower tail dependence coefficient in the subsamples before and after Black Monday. Therefore, we perform the corresponding TDC-Test 1 in the subsamples, whose results are depicted in Figures 7 and 8 in a similar manner as before; in particular, they are based on new
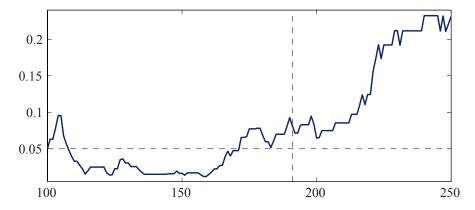
24

Figure 6: (Dow Jones and Nasdaq time series) Chi-squared test for a break at $\lfloor n\bar{s}_{\mathrm{BM}} \rfloor = 959$: p-values for different k. The horizontal line indicates the 5% level of significance, the vertical one the plateau $k^* = 191$.

Figure 7: (Dow Jones and Nasdaq time series) TDC-Test 1 for the subsample before $\lfloor n\bar{s}_{\mathrm{BM}} \rfloor = 959$: p-values for different k. The horizontal line indicates the 5% level of significance, the vertical one the plateau $k^* = 48$.
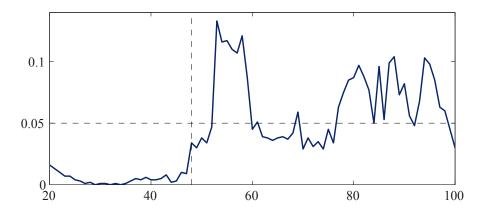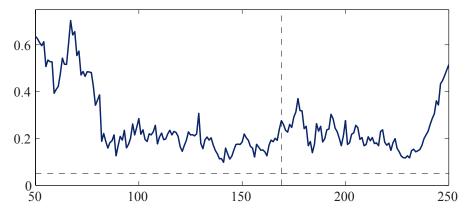

Figure 8: (Dow Jones and Nasdaq time series) TDC-Test 1 for the subsample after $\lfloor n\bar{s}_{\mathrm{BM}} \rfloor = 959$ (including Black Monday): p-values for different k. The horizontal line indicates the 5% level of significance, the vertical one the plateau $k^* = 169$.

parameter	before Black Monday	after Black Monday	full sample size
n	958	810	1768
k^*	48	169	191
m	30	28	41
$\hat{\lambda}$	0.449	0.678	0.620
$\mathcal{S}_n$	0.546	0.211	1.064
p-value	0.028	0.244	0.003

Table 6: Summary of results for TDC-Test 1 applied to the subsample before Black Monday, to the subsample after Black Monday and to the full sample.

(plateau-based) choices of k for the reduced samples. We can accept constancy after Black Monday, but have to reject it for the subsample before Black Monday. A summary of the results can also be found in the first two columns of Table 6.

In principal, one could now proceed by a refined analysis of the subsample before Black Monday in order to identify potential additional break-points. Motivated by the diagnostic plot in Figure 5, we prefer an application of the TDC-Test 1 to the whole sample since this might reveal that a model with at most one break-point is also appropriate. In other words, we dismiss the initial guess of a break precisely on Black Monday and rather split the sample at an estimated break-point, hoping that the latter yields to a simpler model with at most one break-point.

We do not depict the results of the corresponding test, since it clearly rejects the null-hypothesis H_0^λ at the 1%-significance level for almost all choices of k. A short summary can be found in the last column of Table 6. More enlightening conclusions can be drawn from the plot of the the function $ns \mapsto |\hat{\lambda}^{-1/2}\mathbb{G}_n(s, 1, 1)|$ in Figure 9, for $k^* = 191$. The dashed vertical lines denote Black Monday $\lfloor n\bar{s}_{\text{BM}} \rfloor = 959$ (blue) and the value $\lfloor n\hat{s}^\lambda \rfloor = 817$ where the graph attains its maximum (yellow). The latter corresponds to the 27th of March 1987 and appears to be the argmax for most choices of k in a neighborhood of $k^* = 191$. We split the sample at this estimated break-point and conduct a refined analysis in the respective subsamples. The procedure is similar to what we have done before, whence we restrict ourselves to a brief summary of the results: in both subsamples, we cannot reject the null hypothesis for all reasonable choices of k, including the values obtained from the plateau algorithm, with p-values lying between 0.2 and 0.5. Similar to the reported values in Table 6 we find $\hat{\lambda} = 0.430$ for the first subsample ($k^* = 43$) and $\hat{\lambda} = 0.656$ for the second one ($k^* = 57$), respectively.

We conclude this application with a short summary of the main findings:

(i) The test for a break on Black Monday does not yield entirely unambiguous results; in particular, we have to reject the null hypothesis of constant tail dependence in the subsample before Black Monday resulting in an overall model with more than one break-point.

(ii) Testing against the existence of *some* unspecified break-point in the full sample clearly rejects the null, with an estimated break-point at $\lfloor n\hat{s}^\lambda \rfloor = 817$. Since we cannot reject the null hypothesis in the corresponding subsamples, an overall model with only one break-point can be accepted.
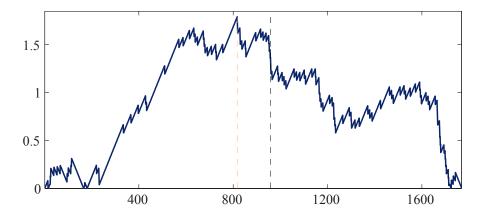
Figure 9: (Dow Jones and Nasdaq time series) Absolute standardized sequential empirical tail copula process $|\hat{\lambda}^{-1/2}\mathbb{G}_n(s,1,1)|$ for $k^* = 191$ with respect to ns, $s \in [0,1]$. The yellow vertical line indicates the argmax estimator $\lfloor n\hat{s}^\lambda \rfloor = 817$, the black one shows Black Monday $\lfloor n\bar{s}_{\mathrm{BM}} \rfloor = 959$.

6. Conclusion and Outlook

In this paper, we developed new tests for detecting structural breaks in the tail dependence of multivariate time series, derived several theoretical properties (asymptotic null distribution, behavior under alternatives), investigated the finite-sample performance and applied them to datasets from energy and financial markets.

Our work hints at interesting directions for further research. First of all, we did not give a formal proof for the conjecture derived from the simulation study, that the test statistics based on estimated residuals show the same asymptotic behavior as the ones based on i.i.d. samples. To the best of our knowledge, this problem is also unsolved for the estimation techniques described in Section 2: under what conditions does (or does not) the additional estimation step of forming *almost i.i.d.* residuals influence the asymptotic behavior of the nonparametric estimators for the tail dependence? Second, extensions to the case of serially dependent datasets (e.g., to mixing sequences) would allow to check for constant tail dependence of the raw data which might also be of interest for practitioners. In particular with a view on the necessary (block) bootstrap procedure this could be a quite challenging task.

A. Appendix

A.1. Proof of the results in the main text   For all proofs, by asymptotic equicontinuity, we may redefine $\hat{U}_i = F_n(X_i)$ and $\hat{V}_i = G_n(Y_i)$. For any $s \in [0,1]$ and $(x,y) \in \mathbb{E}$, let

$$\tilde{\Lambda}_n^\circ(s,x,y) = \frac{1}{k} \sum_{i=1}^{\lfloor ns \rfloor} \mathbb{1}(U_i \leq kx/n, V_i \leq ky/n).$$

Under H_0^Λ, this is a sequential (oracle) estimator for $\Lambda^\circ(s,x,y) = s\Lambda(x,y)$. To begin with, we investigate the associated sequential empirical process, defined as

$$\mathbb{B}_n(s,x,y) = \sqrt{k}\left\{ \tilde{\Lambda}_n^\circ(s,x,y) - \Lambda^\circ(s,x,y) \right\}.$$

27

The proof of the following lemma is given in Appendix A.2.

Lemma A.1. *Suppose that Assumptions 3.1 and 3.2 hold. Then, under H_0^Λ,*

$$\mathbb{B}_n \rightsquigarrow \mathbb{B}_\Lambda \quad in \ (\mathcal{B}_\infty([0,1] \times \mathbb{E}), d),$$

where $\mathbb{B}_\Lambda$ is given as in Proposition 3.3.

Proof of Proposition 3.3. Since the rank of X_i among $X_1, \ldots, X_n$ is the same as the rank of U_i among $U_1, \ldots, U_n$ (similar for the second coordinate) we may assume without loss of generality that (X_i, Y_i) is distributed according to C_i, i.e., $F(x) = G(x) = x$ for all $x \in [0,1]$. Some thoughts reveal that

$$|\hat{\Lambda}_n^\circ(s, x, y) - \bar{\Lambda}_n^\circ(s, x, y)| \le 2/k,$$

uniformly in $(s, x, y) \in S_m$, where

$$\bar{\Lambda}_n^\circ(s, x, y) = \frac{1}{k} \sum_{i=1}^{\lfloor ns \rfloor} \mathbb{1}\left\{ X_i \le F_n^-(kx/n), Y_i \le G_n^-(ky/n) \right\}$$

and where F_n^- and G_n^- denote the generalized inverse functions of F_n and G_n, respectively. Note that $\bar{\Lambda}_n^\circ$ can be expressed in terms of $\tilde{\Lambda}_n^\circ$ as

$$\bar{\Lambda}_n^\circ(s, x, y) = \tilde{\Lambda}_n^\circ\left\{ s, \frac{n}{k} F_n^-\left(\frac{kx}{n}\right), \frac{n}{k} G_n^-\left(\frac{ky}{n}\right) \right\}.$$

Now, we have $n/kF_n(kx/n) = \tilde{\Lambda}_n^\circ(1, x, \infty)$ and $n/kG_n(kx/n) = \tilde{\Lambda}_n^\circ(1, \infty, y)$, whence, by Hadamard-differentiability of the inverse mapping as stated in Bücher and Dette (2011),

$$\sup_{x \in [0,M]} \left| \frac{n}{k} F_n^-\left(\frac{kx}{n}\right) - x \right| = o_P(1), \quad \sup_{y \in [0,M]} \left| \frac{n}{k} G_n^-\left(\frac{ky}{n}\right) - y \right| = o_P(1) \qquad (13)$$

for any $M > 0$ (this result can also be obtained by deducing weak convergence of $x \mapsto \mathbb{B}_n(1, x, \infty)$ as an element of the càdlàg space $D([0, M])$ with the Skorohod topology (from Lemma A.1), invoking a Skorohod construction and applying Vervaat's Lemma, see Vervaat (1972) or Lemma A.0.2 in de Haan and Ferreira (2006)). Therefore, by asymptotic equicontinuity of $\mathbb{B}_n$,

$$\mathbb{G}_n(s, x, y) = \sqrt{k}\left\{ \bar{\Lambda}_n^\circ(s, x, y) - s\bar{\Lambda}_n^\circ(1, x, y) \right\} + O(k^{-1/2})$$

$$= \mathbb{B}_n\left\{ s, \frac{n}{k} F_n^-\left(\frac{kx}{n}\right), \frac{n}{k} G_n^-\left(\frac{ky}{n}\right) \right\}$$

$$- s\mathbb{B}_n\left\{ 1, \frac{n}{k} F_n^-\left(\frac{kx}{n}\right), \frac{n}{k} G_n^-\left(\frac{ky}{n}\right) \right\} + O(k^{-1/2}) \quad (14)$$

weakly converges to $\mathbb{G}_\Lambda(s, x, y) = \mathbb{B}_\Lambda(s, x, y) - s\mathbb{B}_\Lambda(1, x, y)$ on $(S_m, \|\cdot\|_{S_m})$, for any $m \in \mathbb{N}$. The Proposition is proved. $\qquad \square$

Remark A.2. A crucial argument in the preceding proof is the decomposition (14) of $\mathbb{G}_n$ into a sum involving $\mathbb{B}_n$ from Lemma A.1. A similar decomposition is possible with $\mathbb{B}_n$ replaced by $\mathbb{B}_n'$ from the proof of Lemma A.1, and weak convergence of

the latter holds without imposing Assumption 3.2 (b). Therefore, a relaxation of the assumptions for Proposition 3.3 seems to be possible. Indeed, a sufficient condition that makes occurring bias terms negligible and allows to dispense with Assumption 3.2 (b) is given by

$$\sup_{(s,x,y)\in S_m} \frac{\sqrt{k}}{n}\left|\sum_{i=1}^{\lfloor ns\rfloor} R_i(x,y,k,n) - s\sum_{i=1}^{n} R_i(x,y,k,n)\right| = o(1),$$

as $n \to \infty$, where

$$R_i(x,y,n,k) = \frac{n}{k}C_i(kx/n, ky/n) - \Lambda(x,y).$$

In case $C_i \equiv C$ is constant over time, this condition reduces to $\sqrt{k}/n = o(1)$, which is satisfied anyway since $k = o(n)$.

Proof of Proposition 3.4. It follows from Proposition 3.3 that

$$s \mapsto \{\hat{\Lambda}_n^\circ(1,1,1)\}^{-1/2}\mathbb{G}_n(s,1,1)$$

converges to a standard Brownian bridge. Therefore, both assertions are simple consequences of the continuous mapping theorem. $\qquad\square$

Proof of Proposition 3.6. Let us first fix a $b \in \{1,\dots,B\}$ and show that $\mathbb{G}_{n,\xi^{(b)}}$ weakly converges to $\mathbb{G}_{\Lambda^{(b)}}$. For the sake of a clear notation, we omit the index b for the proof of this result. In light of the continuous mapping theorem, it is sufficient to prove that $\mathbb{B}_{n,\xi}$ weakly converges to $\mathbb{B}_\Lambda$. As in the proof of Proposition 3.3, we may assume that the marginal distributions are standard uniform. Let us suppose that we have proved $\tilde{\mathbb{B}}_{n,\xi} \rightsquigarrow \mathbb{B}_\Lambda$, where

$$\tilde{\mathbb{B}}_{n,\xi}(s,x,y) = \sqrt{k}\{\tilde{\Lambda}_{n,\xi}^\circ(s,x,y) - \tilde{\Lambda}_n^\circ(s,x,y)\},$$

and

$$\tilde{\Lambda}_{n,\xi}^\circ(s,x,y) = \frac{1}{k}\sum_{i=1}^{\lfloor ns\rfloor}\frac{\xi_i}{\bar{\xi}_n}\mathbb{1}(U_i \le kx/n, V_i \le ky/n).$$

Then, by a similar reasoning as in the proof of Proposition 3.3,

$$\mathbb{B}_{n,\xi}(s,x,y) = \tilde{\mathbb{B}}_{n,\xi}\left\{s, \frac{n}{k}F_n^-\left(\frac{kx}{n}\right), \frac{n}{k}G_n^-\left(\frac{ky}{n}\right)\right\}$$
$$+ O\left(k^{-1/2} + k^{-1/2}\max_{i=1}^{n}\xi_i/\bar{\xi}_n\right). \quad (15)$$

By (13) and asymptotic equicontinuity of $\tilde{\mathbb{B}}_{n,\xi}$, the first expression on the right-hand side weakly converges to $\mathbb{B}_\Lambda$ in $\ell^\infty(S_m)$, for any fixed S_m. In light of the fact that ξ_1 has finite moments of any order we have $\mathbb{P}(\xi_1 > x) = O(x^{-q})$ for any $q \in \mathbb{N}$. Therefore, the estimation

$$\mathbb{P}(k^{-1/2}\max_{i=1}^{n}\xi_i/\bar{\xi}_n > \varepsilon) \le \mathbb{P}(\bar{\xi}_n \le 1/2) + n\mathbb{P}(\xi_1 > \varepsilon\sqrt{k}/2) = o(1) + nO(k^{-q/2})$$

shows that the O-term in (15) converges to 0 in probability, by choosing q sufficiently large. This proves that $\mathbb{G}_{n,\xi^{(b)}}$ weakly converges to $\mathbb{G}_{\Lambda^{(b)}}$.

It remains to be shown that $\tilde{\mathbb{B}}_{n,\xi} \rightsquigarrow \mathbb{B}_\Lambda$ in $\ell^\infty(S_m)$, for any fixed S_m. We have

$$\tilde{\mathbb{B}}_{n,\xi}(s,x,y) = \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor ns \rfloor} \left(\frac{\xi_i}{\bar{\xi}_n} - 1 \right) \mathbb{1}(U_i \le kx/n, V_i \le ky/n)$$

$$= \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor ns \rfloor} \left(\frac{\xi_i}{\bar{\xi}_n} - 1 \right) \{ \mathbb{1}(U_i \le kx/n, V_i \le ky/n) - \Lambda(kx/n, ky/n) \}$$

$$= A_n(s,x,y) + B_n(s,x,y) + C_n(s,x,y),$$

where

$$A_n = \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor ns \rfloor} (\xi_i - 1) \{ \mathbb{1}(U_i \le kx/n, V_i \le ky/n) - \Lambda(kx/n, ky/n) \},$$

$$B_n = \left(\frac{1}{\bar{\xi}_n} - 1 \right) \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor ns \rfloor} (\xi_i - 1) \{ \mathbb{1}(U_i \le kx/n, V_i \le ky/n) - \Lambda(kx/n, ky/n) \},$$

$$C_n = \left(\frac{1}{\bar{\xi}_n} - 1 \right) \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor ns \rfloor} \{ \mathbb{1}(U_i \le kx/n, V_i \le ky/n) - \Lambda(kx/n, ky/n) \}.$$

It follows from Lemma A.1 and from the fact that $\bar{\xi}_n$ converges to 1, almost surely, that $C_n = o_P(1)$, uniformly on each S_m. Hence, observing that $B_n = A_n \times o_P(1)$, the proposition is proved if we show that A_n converges weakly to $\mathbb{B}_\Lambda$ in $\ell^\infty(T_m)$. By similar arguments as in proof of Lemma A.1 (Ottaviani's inequality; the details are omitted for the sake of brevity), it suffices to establish the (non-sequential) weak convergence of $\tilde{A}_n(x,y) = A_n(1,x,y)$ to $\mathbb{D}_\Lambda(x,y) = \mathbb{B}_\Lambda(1,x,y)$ in $\ell^\infty(T_m)$. To show this, we decompose $\tilde{A}_n = A_{n1} + A_{n2}$, where

$$A_{n1} = \frac{1}{\sqrt{k}} \sum_{i=1}^{n} (\xi_i - 1) \{ \mathbb{1}(U_i \le kx/n, V_i \le ky/n) - C_i(kx/n, ky/n) \},$$

$$A_{n2} = \frac{1}{\sqrt{k}} \sum_{i=1}^{n} (\xi_i - 1) \{ C_i(kx/n, ky/n) - \Lambda(kx/n, ky/n) \}.$$

Now, $C_i(kx/n, ky/n) - \Lambda(kx/n, ky/n) = k/n \times O(B(k/n))$ by Assumption 3.1, uniformly in i and uniformly on T_m, whence

$$A_{n2} = \frac{1}{n} \sum_{i=1}^{n} (\xi_i - 1) \times \sqrt{k} O(B(k/n)) = o(1),$$

uniformly on T_m, almost surely. Finally, we obtain weak convergence of A_{n1} to $\mathbb{D}_\Lambda$ from Theorem 11.19 in Kosorok (2008) and the fact that conditional weak convergence as considered by the last named author implies unconditional weak convergence.

Now, let us give the proof of the proposition. On each S_m, the sequence $(\mathbb{G}_n, \mathbb{G}_{n,\xi^{(1)}}, \ldots, \mathbb{G}_{n,\xi^{(B)}})$ is jointly asymptotically tight by Lemma 1.4.3 in Van der Vaart and Wellner (1996). Hence, it remains to consider weak convergence of the finite-dimensional distributions. It suffices to consider the finite-dimensional distributions of $(\mathbb{B}_n, \mathbb{B}_{n,\xi^{(1)}}, \ldots, \mathbb{B}_{n,\xi^{(B)}})$. By a similar argumentation as above in the

case of a fixed $b \in \{1, \ldots, B\}$, we may replace each coordinate $\mathbb{B}_{n,\xi^{(b)}}$ by

$$\frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor ns \rfloor} \left(\xi_i^{(b)} - 1\right) \{\mathbb{1}(U_i \leq kx/n, V_i \leq ky/n) - C_i(kx/n, ky/n)\}.$$

Then, the coordinates are uncorrelated and row-wise independent, whence the finite-dimensional distributions weakly converge to those of $(\mathbb{B}_\Lambda, \mathbb{B}_\Lambda^{(1)}, \ldots, \mathbb{B}_\Lambda^{(B)})$ by the central limit theorem for row-wise independent triangular arrays.

$\square$

Proof of Corollary 3.7. For TDC-Test 1, this is a direct consequence of Proposition 3.4 (*i*). The proofs of TDC-Test 2 and TC-Test being essentially the same, we restrict ourselves to the proof of TDC-Test 2. For monotonicity reasons it suffices to consider $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Let K denote the c.d.f. of $\mathcal{S}$ and define

$$K_{n,B}(x) = B^{-1} \sum_{b=1}^{B} \mathbb{1}(\mathcal{S}_{n,\xi^{(b)}} \leq x), \qquad K_B(x) = B^{-1} \sum_{b=1}^{B} \mathbb{1}(\mathcal{S}^{(b)} \leq x),$$

where $\mathcal{S}^{(1)}, \ldots, \mathcal{S}^{(B)}$ denote independent copies of $\mathcal{S}$. Then we can write $\Pr(\mathcal{S}_n \geq \hat{q}_{\mathcal{S}_n, 1-\alpha}) = \Pr\{K_{n,B}(\mathcal{S}_n) \geq 1 - \alpha\}$. Let us first show that, for any $B \in \mathbb{N}$ fixed, we have

$$\lim_{n \to \infty} \Pr\{K_{n,B}(\mathcal{S}_n) \geq 1 - \alpha\} = \Pr\{K_B(S) \geq 1 - \alpha\}. \tag{16}$$

For that purpose, let $\varepsilon > 0$ be given. Define a map $\Psi : \mathbb{R}^{B+1} \to \mathbb{R}$ by $\Psi(t_0, \ldots, t_B) = B^{-1} \sum_{b=1}^{B} \mathbb{1}(t_b \leq t_0)$ and note that Ψ is continuous at any point $(t_0, \ldots, t_B)$ with pairwise different coordinates (i.e., $t_i \neq t_j$ for $i \neq j$). Then, observing that $(\mathcal{S}_n, \mathcal{S}_{n,\xi^{(1)}}, \ldots, \mathcal{S}_{n,\xi^{(B)}}) \rightsquigarrow (\mathcal{S}, \mathcal{S}^{(1)}, \ldots, \mathcal{S}^{(B)})$ with the limit having pairwise different coordinates, almost surely, the continuous mapping theorem implies that $K_{n,B}(\mathcal{S}_n) \rightsquigarrow K_B(\mathcal{S})$, for $n \to \infty$. The Portmanteau-Theorem implies that there exists some $n_0 = n_0(\varepsilon, B)$ such that

$$|\Pr\{K_{n,B}(\mathcal{S}_n) \geq 1 - \alpha\} - \Pr\{K_B(\mathcal{S}) \geq 1 - \alpha\}| < \varepsilon$$

(note that $\Pr(K_B(\mathcal{S}) = 1 - \alpha) = 0$ since $\alpha \in \mathbb{R} \setminus \mathbb{Q}$), which proves (16).

It remains to be shown that

$$\lim_{B \to \infty} \Pr\{K_B(\mathcal{S}) \geq 1 - \alpha\} = \alpha. \tag{17}$$

By the Glivenko-Cantelli Theorem, we may choose $B_0 = B_0(\varepsilon) \in \mathbb{N}$ such that

$$\Pr\left\{\sup_{x \in \mathbb{R}} |K_B(x) - K(x)| > \varepsilon\right\} \leq \varepsilon.$$

for all $B \geq B_0$. For all such B,

$$\Pr\{K_B(\mathcal{S}) \geq 1 - \alpha\} \leq \Pr\{K(\mathcal{S}) \geq 1 - \alpha - \varepsilon) + \varepsilon = \alpha + 2\varepsilon,$$

and similarly,

$$\Pr\{K_B(\mathcal{S}) \geq 1 - \alpha\} \geq \Pr\{K(\mathcal{S}) \geq 1 - \alpha + \varepsilon\} = \alpha - \varepsilon,$$

which implies that

$$|\Pr\{K_B(\mathcal{S}) \geq 1 - \alpha\} - \alpha| \leq 2\varepsilon.$$

This proves (17) and hence the Corollary. □

Proof of Proposition 3.8. As in the proof of Proposition 3.3 we may assume without loss of generality that the marginals are standard uniform.

We begin with the proof of (i). It suffices to derive the p-limit of

$$\hat{\Lambda}_n^\circ(s, x, y) = \frac{1}{k} \sum_{i=1}^{\lfloor ns \rfloor} \mathbb{1}\left(\hat{U}_i \leq kx/n, \hat{V}_i \leq ky/n\right)$$

for, on the one hand, $s \leq \bar{s}$, and, on the other hand, $s > \bar{s}$. The rest are simple calculations and applications of the continuous mapping theorem.

For $s \leq \bar{s}$, we show that

$$\sup_{s \in [0, \bar{s}]} \left| \frac{1}{k_n} \sum_{i=1}^{\lfloor ns \rfloor} \mathbb{1}\left(\hat{U}_i \leq k_n/n, \hat{V}_i \leq k_n/n\right) - s\lambda^{(1)} \right| = o_P(1).$$

As in the proof of Proposition 3.3, we may replace the indicators in the previous expression by $\mathbb{1}\{U_i \leq F_n^-(k_n/n), V_i \leq G_n^-(k_n/n)\}$, whence we need to show that

$$\sup_{s \in [0, \bar{s}]} \left| \frac{1}{k_n} \sum_{i=1}^{\lfloor ns \rfloor} \mathbb{1}\left\{U_i \leq F_n^-(k_n/n), V_i \leq G_n^-(k_n/n)\right\} - s\lambda^{(1)} \right| = o_P(1).$$

The latter result follows from Lemma A.1 , uniform continuity of $\Lambda^{(1)}$ and (13).

For $s > \bar{s}$, we write

$$\frac{1}{k_n} \sum_{i=1}^{\lfloor ns \rfloor} \mathbb{1}\left(\hat{U}_i \leq k_n/n, \hat{V}_i \leq k_n/n\right)$$

$$= \frac{1}{k_n} \sum_{i=1}^{\lfloor n\bar{s} \rfloor} \mathbb{1}\left(\hat{U}_i \leq k_n/n, \hat{V}_i \leq k_n/n\right) + \frac{1}{k_n} \sum_{i=\lfloor n\bar{s} \rfloor + 1}^{\lfloor ns \rfloor} \mathbb{1}\left(\hat{U}_i \leq k_n/n, \hat{V}_i \leq k_n/n\right).$$

Then, by similar arguments as above, we can conclude that

$$\sup_{s \in [\bar{s}, 1]} \left| \frac{1}{k_n} \sum_{i=1}^{\lfloor ns \rfloor} \mathbb{1}\left(\hat{U}_i \leq k_n/n, \hat{V}_i \leq k_n/n\right) - \left\{\bar{s}\lambda^{(1)} + (s - \bar{s})\lambda^{(2)}\right\} \right| = o_P(1).$$

□

Proof of Corollary 3.9. For TDC-Test 1, this is a direct consequence of Proposition 3.8 (*i*). The proofs for TDC-Test 2 and TC-Test being essentially the same, we only consider the TC-Test.

Let us first show that $\mathcal{T}_{n,\xi}$ is stochastically bounded. This follows if we prove that $\sup_{(s,x,y)\in S_m} |\mathbb{B}_{n,\xi}(s,x,y)| = O_P(1)$, for $n \to \infty$. By a similar reasoning as in (15) and the subsequent paragraph, it suffices to show the same for $\tilde{\mathbb{B}}_{n,\xi}(s,x,y)$. Since

$$\sup_{(s,x,y)\in S_m} |\tilde{\mathbb{B}}_{n,\xi}(s,x,y)| = \max\left\{ \sup_{s\leq\bar{s},(x,y)\in T_m} |\tilde{\mathbb{B}}_{n,\xi}(s,x,y)|, \sup_{s\geq\bar{s},(x,y)\in T_m} |\tilde{\mathbb{B}}_{n,\xi}(s,x,y)| \right\},$$

we can verify the claim for each of the suprema in the maximum. The first term converges weakly due to the proof of Proposition 3.6 and is therefore bounded in probability. For the second term, we decompose

$$\tilde{\mathbb{B}}_{n,\xi}(s,x,y) = \tilde{\mathbb{B}}_{n,\xi}(\bar{s},x,y) + k^{-1/2} \sum_{i=\lfloor n\bar{s}\rfloor+1}^{\lfloor ns\rfloor} \left(\frac{\xi_i}{\bar{\xi}_n} - 1 \right) \mathbb{1}(U_i \leq kx/n, V_i \leq ky/n).$$

$$(18)$$

Again by the proof of Proposition 3.6, the first summand on the right-hand side converges weakly in $\ell^\infty(T_m)$ (with respect to (x,y)) to $\mathbb{B}_{\Lambda^{(1)}}(\bar{s},x,y)$, whence the supremum of it absolute value is bounded in probability. The second summand is equal in law to $\bar{\mathbb{B}}_{n,\xi}(s,x,y) - \bar{\mathbb{B}}_{n,\xi}(\bar{s},x,y)$, where $\bar{\mathbb{B}}_{n,\xi}$ is defined analogously as $\tilde{\mathbb{B}}_{n,\xi}$, with (U_i,V_i) replaced by a sequence $(U_i^\star,V_i^\star)$ of i.i.d. random vectors such that $(U_i^\star,V_i^\star) \sim C_i$ for $i > \lfloor n\bar{s}\rfloor$ and $(U_i^\star,V_i^\star) \sim C_n$ for $i \leq \lfloor n\bar{s}\rfloor$. This sequence meeting the assumptions of Proposition 3.6, we can conclude that the second summand on the right hand side of (18) weakly converges to $\mathbb{B}_{\Lambda^{(2)}}(s,x,y) - \mathbb{B}_{\Lambda^{(2)}}(\bar{s},x,y)$. Hence, its supremum is bounded in probability.

Now, fix $B \in \mathbb{N}$ and let $\varepsilon > 0$ be given. Then, since $\mathcal{T}_{n,\xi^{(b)}} = O_P(1)$ for each $b = 1, \ldots, B$, we may choose $K = K(\varepsilon, B) > 0$ such that

$$\sup_{n\in\mathbb{N}} \Pr\left(\max_{b=1}^{B} |\mathcal{T}_{n,\xi^{(b)}}| > K \right) \leq \varepsilon.$$

Therefore, $\hat{q}_{\mathcal{T}_n,1-\alpha} \leq K$ with probability $1 - \varepsilon$, and since $\mathcal{T}_n \to \infty$ in probability, we get that

$$\liminf_{n\to\infty} \Pr(\mathcal{T}_n \geq \hat{q}_{\mathcal{T}_n,1-\alpha}) \geq \liminf_{n\to\infty}\{\Pr(\mathcal{T}_n \geq K) - \Pr(\hat{q}_{\mathcal{T}_n,1-\alpha} > K)\} \geq 1 - \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, the assertion is proved. $\qquad\square$

Proof of Proposition 3.10. It follows from Proposition 3.8, that the random functions $s \mapsto k_n^{-1/2}|\mathbb{G}_n(s,1,1)|$ and $s \mapsto k_n^{-1/2}\sup_{t\in[0,1]}|\mathbb{G}_n(s,2-2t,2t)|$ uniformly converge to deterministic functions in probability, respectively, which have a unique maximum in the point $\bar{s}$ (the graph is a triangle with a peak in $\bar{s}$). Consistency of the change-point estimators then follows from the argmax-continuous mapping theorem, see for instance Kim and Pollard (1990), Theorem 2.7. $\qquad\square$

A.2. Proofs of additional results

Proof of Lemma A.1. Define a centered version of $\mathbb{B}_n$ by

$$\mathbb{B}'_n(s,x,y) = \frac{1}{\sqrt{k_n}} \sum_{i=1}^{\lfloor ns \rfloor} \left\{ \mathbb{1}\left(U_i \leq k_n x/n, V_i \leq k_n y/n \right) - C_i(k_n x/n, k_n y/n) \right\}$$

and let us show that $d(\mathbb{B}'_n, \mathbb{B}_n) = o(1)$. We have

$$|\mathbb{B}'_n(s,x,y) - \mathbb{B}_n(s,x,y)| = \sqrt{k_n} \left| \frac{1}{k_n} \sum_{i=1}^{\lfloor ns \rfloor} C_i(k_n x/n, k_n y/n) - s\Lambda(x,y) \right|$$

$$\leq \sqrt{k_n} \left| \left(\frac{\lfloor ns \rfloor}{n} - s \right) \Lambda(x,y) \right| + \sqrt{k_n} \frac{\lfloor ns \rfloor}{n} \max_{i=1}^{n} \left| \frac{n}{k_n} C_i(k_n x/n, k_n y/n) - \Lambda(x,y) \right|$$

The first term on the right-hand side of this expression is of order $O(\sqrt{k_n}/n) = o(1)$, uniformly on each S_m. By (8), the second term is of order $O(\sqrt{k_n}B(n/k_n)) = o(1)$, uniformly on each T_m.

Now, let us show weak convergence of $\mathbb{B}'_n$. It suffices to fix one set S_m. Finite-dimensional convergence follows from the classical central limit Theorem for row-wise independent triangular arrays. To show tightness, we proceed similar as in the proof of Theorem 2.12.1 in Van der Vaart and Wellner (1996). For $\ell \in \{1, \ldots, n\}$, define

$$\mathbb{D}'_{\ell,n}(x,y) = \frac{1}{\sqrt{k_n}} \sum_{i=1}^{\ell} \left\{ \mathbb{1}\left(U_i \leq k_n x/n, V_i \leq k_n y/n \right) - C_i(k_n x/n, k_n y/n) \right\}$$

and note that $\mathbb{D}'_{n,n}(x,y) = \mathbb{B}'_n(1,x,y)$. For any $\delta > 0$, set

$$w_\delta(\mathbb{D}'_{\ell,n}) = \sup\{ |\mathbb{D}'_{\ell,n}(x_1,y_1) - \mathbb{D}'_{\ell,n}(x_2,y_2)| :$$
$$(x_j,y_j) \in T_M \text{ for } j = 1,2, |x_1 - x_2| + |y_1 - y_2| \leq \delta\}.$$

Invoking Theorem 11.16 in Kosorok (2008), it follows as in Lemma 4.1 in Bücher and Dette (2011) that $\mathbb{D}'_{n,n}(x,y)$ weakly converges to $\mathbb{B}_\Lambda(1,x,y)$ in $\ell^\infty(T_m)$ (by invoking Theorem 11.16 rather than Theorem 11.20 in Kosorok (2008), the *i.i.d.* result in Bücher and Dette (2011) also holds under the slightly more general situation of a changing copula but a constant tail copula). Hence, observing that $\mathbb{B}_\Lambda(1, \cdot, \cdot)$ has continuous trajectories, Theorem 1.5.7 and its addendum in Van der Vaart and Wellner (1996) imply that

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}^*(w_\delta(\mathbb{D}'_{n,n}) > \varepsilon) = 0. \tag{19}$$

Similar as for $\mathbb{D}'_{\ell,n}$, set

$$w_\delta(\mathbb{B}'_n) = \sup\{ |\mathbb{B}'_n(s_1,x_1,y_1) - \mathbb{B}'_n(s_2,x_2,y_2)| :$$
$$(s_j,x_j,y_j) \in S_m \text{ for } j = 1,2, |s_1 - s_2| + |x_1 - x_2| + |y_1 - y_2| \leq \delta\}.$$

In order to establish tightness, again by Theorem 1.5.7 in Van der Vaart and Wellner (1996), it suffices to show that (19) holds with $\mathbb{D}'_{n,n}$ replaced by $\mathbb{B}'_n$.

First of all, by the triangular inequality,

$$w_\delta(\mathbb{B}'_n) \leq \sup_{|s_1-s_2|\leq\delta} \sup_{(x,y)\in T_m} |\mathbb{B}'_n(s_1,x,y) - \mathbb{B}'_n(s_2,x,y)|$$
$$+ \sup_{0\leq s\leq 1} \sup_{|x_1-x_2|+|y_1-y_2|\leq\delta} |\mathbb{B}'_n(s,x_1,y_1) - \mathbb{B}'_n(s,x_2,y_2)|. \quad (20)$$

The second summand on the right-hand side can be written as

$$\max_{\ell=1}^{n} \sup_{|x_1-x_2|+|y_1-y_2|\leq\delta} |\mathbb{B}'_n(\ell/n,x_1,y_1) - \mathbb{B}'_n(\ell/n,x_2,y_2)| = \max_{\ell=1}^{n} w_\delta(\mathbb{D}'_{l,n})$$

By Ottaviani's inequality, see Lemma A.1.1 in Van der Vaart and Wellner (1996), we have

$$\mathbb{P}^*(\max_{\ell=1}^{n} w_\delta(\mathbb{D}'_{\ell,n}) > 2\varepsilon) \leq \frac{\mathbb{P}^*(w_\delta(\mathbb{D}'_{n,n}) > \varepsilon)}{1 - \max_{\ell=1}^{n} \mathbb{P}^*(w_\delta(\mathbb{D}'_{\ell,n}) > \varepsilon)}.$$

From (19), we know that the numerator converges to 0 for $n \to \infty$ followed by $\delta \to 0$. Let us show that the denominator is bounded away from 0. Observing that $\mathbb{D}'_{\ell,n}(x,y) = \sqrt{k_\ell/k_n}\mathbb{D}'_{\ell,\ell}(k_n\ell x/(k_\ell n), k_n\ell y/(k_\ell n))$ and that $k_n\ell/(k_\ell n) \leq 1$ for all $\ell \leq n$, we have $w_\delta(\mathbb{D}'_{\ell,n}) \leq \sqrt{k_\ell/k_n}w_\delta(\mathbb{D}'_{\ell,\ell})$. Asymptotic equicontinuity of $\mathbb{D}'_{n,n}$ allows to choose some $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ and for some fixed $\delta_0 > 0$ (and hence for all $\delta \leq \delta_0$),

$$\max_{\ell=n_0}^{n} \mathbb{P}^*(\sqrt{k_\ell/k_n}w_\delta(\mathbb{D}'_{\ell,\ell}) > \varepsilon) \leq \max_{\ell=n_0}^{n} \mathbb{P}^*(w_\delta(\mathbb{D}'_{\ell,\ell}) > \varepsilon) \leq 1/2.$$

On the other hand, for $\ell < n_0$, we have $\sqrt{k_\ell}\, w_\delta(\mathbb{D}'_{\ell,\ell}) \leq 4n_0 = o(\sqrt{k_n})$, which shows that the denominator is bounded away from 0. Hence, the second summand in (20) converges to 0 in probability for $n \to \infty$ followed by $\delta \to 0$.

To show the same for the first summand in (20) it suffices to show that

$$\mathbb{P}^*\left(\max_{\substack{0\leq j\delta\leq 1 \\ j\in\mathbb{N}}} \sup_{j\delta\leq s\leq(j+1)\delta} \sup_{(x,y)\in T_m} |\mathbb{B}'_n(s,x,y) - \mathbb{B}_n(j\delta,x,y)| > 2\varepsilon \right)$$

converges to 0. The at most $\lceil 1/\delta \rceil$ terms in the first maximum are identically distributed whence we can estimate the probability by

$$\lceil 1/\delta \rceil \mathbb{P}^*\left(\sup_{0\leq s\leq\delta} \sup_{(x,y)\in T_m} |\mathbb{B}'_n(s,x,y)| > 2\varepsilon \right)$$
$$\leq \lceil 1/\delta \rceil \mathbb{P}^*\left(\max_{\ell\leq\lfloor n\delta\rfloor} \|\mathbb{D}'_{\ell,n}\|_{T_m} > 2\varepsilon \right)$$
$$\leq \frac{\lceil 1/\delta \rceil \mathbb{P}^*\left(\|\mathbb{D}'_{\lfloor n\delta\rfloor,n}\|_{T_m} > \varepsilon \right)}{1 - \max_{\ell\leq\lfloor n\delta\rfloor} \mathbb{P}^*\left(\|\mathbb{D}'_{\ell,n}\|_{T_m} > \varepsilon \right)}, \quad (21)$$

where, in the last step, we invoked Ottaviani's inequality again. Regarding the

numerator, exploiting that $k_n\ell/(k_\ell n) \le 1$ for all $\ell \le n$ again, we have

$$\limsup_{n\to\infty} \mathbb{P}^*\big(\big\|\mathbb{D}'_{\lfloor n\delta\rfloor,n}\big\|_{T_m} > \varepsilon\big) \le \limsup_{n\to\infty} \mathbb{P}^*\big(\sqrt{k_{\lfloor n\delta\rfloor}/k_n}\big\|\mathbb{D}'_{\lfloor n\delta\rfloor,\lfloor n\delta\rfloor}\big\|_{T_m} > \varepsilon\big)$$
$$\le \mathbb{P}\big(\big\|\mathbb{D}_\Lambda\big\|_{T_m} > \varepsilon\delta^{-\alpha/2}\big)$$

by the Portmanteau Theorem and Assumption 3.2(c), where $\mathbb{D}_\Lambda = \mathbb{B}_\Lambda(1,\cdot,\cdot)$ denotes the weak limit of $\mathbb{D}'_{n,n}$. The fact that suprema of Gaussian processes possess finite moments of any order implies that

$$\lceil 1/\delta\rceil\mathbb{P}\big(\big\|\mathbb{D}_\Lambda\big\|_{T_m} > \varepsilon\delta^{-\alpha/2}\big) = o(1)$$

as $\delta \to 0$. Regarding the denominator in (21), we can proceed as before: since $\eta = 1 - \mathbb{P}(\|\mathbb{D}_\Lambda\|_{T_m} > \varepsilon)$ is larger than 0, we may choose $n_0 \in \mathbb{N}$ such that

$$\max_{\ell=n_0}^{\lfloor n\delta\rfloor} \mathbb{P}^*\big(\|\mathbb{D}'_{\ell,n}\|_{T_m} > \varepsilon\big) \le \max_{\ell=n_0}^{\lfloor n\delta\rfloor} \mathbb{P}^*\big(\sqrt{k_\ell/k_n}\|\mathbb{D}'_{\ell,\ell}\|_{T_m} > \varepsilon\big)$$
$$\le \max_{\ell=n_0}^{\lfloor n\delta\rfloor} \mathbb{P}\big(\|\mathbb{D}'_{\ell,\ell}\|_{T_m} > \varepsilon\big) \le 1 - \eta/2.$$

For $\ell < n_0$, we have $\sqrt{k_\ell}\|\mathbb{D}'_{\ell,\ell}\|_{T_m} \le 2n_0 = o(\sqrt{k_n})$. Thus, the denominator in (21) is bounded away from 0 and the Proposition is proved. □

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