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Christian Palmes and Jeannette H.C. Woerner

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Fakultät für Mathematik
Technische Universität Dortmund
Vogelpothsweg 87
44227 Dortmund

tu-dortmund.de/MathPreprints

The Gumbel test for jumps in stochastic volatility models

Christian Palmes and Jeannette H.C. Woerner*

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Abstract

In this paper we develop a test for jumps based on extreme value theory. We consider a continuous-time stochastic volatility model with a general continuous volatility process, allowing for long- and short-range dependence and observe it under a high-frequency sampling scheme. We show that a certain test statistics based on the maximum of increments converges to the Gumbel distribution under the null hypothesis of no additive jump component and to infinity otherwise. In contrast to most other tests based on power variation our test naturally allows to distinguish between positive and negative jumps. As a by-product of our analysis we also deduce an optimal pathwise estimator for the spot volatility process. In addition we provide a small simulation study and show that our test is more sensitive to jumps with a larger power than the Barndorff-Nielsen and Shephard test based on bipower variation. Finally we apply our results to a real data set of the world stock index.

Keywords: jump test, stochastic volatility model, spot volatility, Gumbel distribution, extreme value theory, high-frequency data

MSC Classification: 62G10, 62P05, 62G32

1 Introduction

Recently the detection of jumps in stochastic volatility models have attained much attention, since this is an important task for modeling, risk assessment and statistical inference of the integrated volatility or volatility process itself. Most tests are based on the concept of power and multipower variation and either use the different limiting behaviour of power and multipower variation in the presence of jumps, cf. Barndorff-Nielsen and Shephard [2], or use the scaling behaviour of power variation in the absence of jumps, cf. Aït-Sahalia and Jacod [1]. These methods work for quite general semimartingale stochastic volatility models observed at a high-frequency sampling scheme on a fixed time-interval and provide information if jumps are present in this time-interval, but no information concerning the direction and location of the jumps. However, especially for applications such as risk assessment the direction of jumps is crucial.

We now propose a method based on extreme value theory, which makes it possible not only to detect jumps, but also to distinguish between positive and negative jumps and locate the jumps. The idea behind is that the increments of the Brownian semimartingale in the absence of jumps, behave roughly like normally distributed random variables and hence the appropriately scaled maximum of the increments converges to the Gumbel distribution. If positive jumps are present the Gumbel statistics converges to infinity. Taking minus the log-price process analogously leads to a test for negative jumps.

*Lehrstuhl IV, Fakultät für Mathematik, Technische Universität Dortmund, D-44221 Dortmund, Germany, christian.palmes@math.tu-dortmund.de, jeannette.woerner@math.tu-dortmund.de

The idea of using extreme value theory to test for jumps has already been proposed in Lee and Mykland [9] in an econometric setting, however they did not provide the full potential of this approach. They did not separate between positive and negative jumps and furthermore their assumptions on the underlying model are restrictive, namely they have to assume that the volatility and the drift are both α-Hölder continuous for every $\alpha < 1/2$ and that the additive jumps are of finite activity and independent of the driving Brownian motion. Whereas we allow for a general drift component, a general jump component, possibly of infinity activity, an arbitrary correlation structure between continuous and jump component and a general pathwise Hölder-continuous volatility process, i.e. both long- and short-range dependent volatility processes. Furthermore, this means that we even allow to leave the setting of semimartingales for the volatility process. As a by-product of our analysis we also deduce an optimal pathwise estimator for the spot volatility process based on bipower variation.

In addition to our theoretical results we provide a small simulation study based on two types of volatility processes: fractional Brownian motions with different Hurst parameters and Ornstein-Uhlenbeck processes. For the latter we show that our test is more sensitive to jumps with a larger power than the Barndorff-Nielsen and Shephard test based on bipower variation. Furthermore, we see that the main part of the approximation error is due to the regularity of the volatility and not due to the generally slow convergence to the Gumbel distribution. Finally, we apply our results to a real data set of the world stock index.

The outline of the paper is as follows, in the next section we provide the basic definitions and notation, in section 3 we deduce an estimator of the spot volatility, in section 4 and 5 we introduce the asymptotics of the Gumbel statistics for models without and with jump component respectively and in section 6 we provide our simulation study and application to real data.

2 Definitions and Notation

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, P)$ be a filtered probability space, which we assume to fulfill the usual conditions. In the following we consider as model for log-prices the following stochastic volatility models, which are Itô semimartingales of the form

$$Y_t = \int_0^t \sigma_s \, dW_s, \quad \check{Y}_t = \int_0^t \sigma_s \, dW_s + \int_0^t d_s ds = \int_0^t \sigma_s \, dW_s + D_t, \quad 0 \leqslant t \leqslant 1,$$

where W denotes a standard Brownian motion, σ the volatility process and d the drift coefficient. All processes are $(\mathcal{F}_t)$-adapted. With $\check{Y}$ we want to emphasize that this process has a possibly non vanishing drift term. Without loss of generality we consider as time interval the unit interval $[0, 1]$ instead of an interval $[0, T]$ for some $T > 0$.

In the following we need some assumptions on the drift and the volatility process. We start with a stronger set of conditions, which makes it possible to clarify the structure of the proofs.

Assumptions 2.1. *There are three global constants $0 < V \leqslant K < \infty$ and $0 < \alpha \leqslant 1$ such that we have for every $\omega \in \Omega$*

(i) σ is pathwise bounded, i.e.

$$V \leqslant \sigma_t(\omega) \leqslant K, \quad 0 \leqslant t \leqslant 1,$$

(ii) σ is Hölder continuous of the order α, i.e.

$$|\sigma_s(\omega) - \sigma_t(\omega)| \leqslant K|t - s|^\alpha, \quad 0 \leqslant s, \, t \leqslant 1,$$

(iii) $t \mapsto d_t(\omega)$ is Lebesgue measurable and

$$|d_t(\omega)| \leqslant K, \quad 0 \leqslant t \leqslant 1.$$

Note that we allow for an arbitrary dependence structure between W, σ and d.

These assumptions have the disadvantage, that the two constants V and K are chosen globally, i.e. they are independent from the path ω. Particularly (ii) in Assumptions 2.1 is restrictive. Let us formulate here the path dependent counterpart to the Assumptions 2.1. Stopping time arguments are used in order to extend the subsequent proofs to the weakened Assumptions 2.2.

Assumptions 2.2. *Let the volatility σ be pathwise Hölder continuous, strictly positive and let the drift d be pathwise bounded. This means that there are two functions*

$$\alpha : \Omega \to (0,1] \quad \textit{and} \quad K : \Omega \to (0,\infty),$$

such that

$$|\sigma_t(\omega) - \sigma_s(\omega)| \leqslant K(\omega)|t-s|^{\alpha(\omega)}, \quad 0 \leqslant s,t \leqslant 1, \quad \omega \in \Omega \tag{1}$$

and

$$|\sigma_t(\omega)| \vee |d_t(\omega)| \leqslant K(\omega), \quad 0 \leqslant t \leqslant 1, \quad \omega \in \Omega. \tag{2}$$

Furthermore, we claim $t \mapsto d_t(\omega)$ to be Lebesgue measurable for all $\omega \in \Omega$.

Observe that even $\inf_{\omega\in\Omega} \alpha(\omega) = 0$ is possible. Moreover, since every path is assumed to be continuous and strictly positive, it follows directly that

$$V(\omega) = \inf_{0\leqslant t\leqslant 1} \sigma_t(\omega) > 0, \quad \omega \in \Omega.$$

We consider V like K as a function $V : \Omega \to (0,\infty)$.

Note that our assumptions on σ also allow for short-range and long-range dependent volatility processes, e.g. processes based on fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$ resp. $H > \frac{1}{2}$.

In the following we draw statistical inference with the sampling scheme of high frequency data or the infill asymptotics, i.e. we consider observations at the time-points $0, \frac{1}{N}, \ldots, 1$. However, for our analysis we need a two scale grid which we define as follows. We set $N = n^2$ for $n \in \mathbb{N}$ and obtain as sampling times $\frac{l}{n^2}$, $l = 0, \ldots, n^2 - 1$ with

$$\frac{l}{n^2} = \frac{kn+j}{n^2} = t_{k,j}, \quad 0 \leqslant k,j < n. \tag{3}$$

Hence the grid on the unit interval separates in two scales. The coarse one, which is indexed by k, and the finer one, which is indexed by j.

We also need some approximations of the volatility process and the price process. We define

$$\overline{\sigma}_t = \sum_{k=0}^{n^2-1} \mathbb{1}_{(\frac{k}{n^2},\frac{k+1}{n^2}]}(t)\sigma_{\frac{k}{n^2}}.$$

and hence

$$\overline{Y}_{t_{k,j}} = \int_0^{t_{k,j}} \overline{\sigma}_s \, dW_s = \sum_{l=0}^{kn+j-1} \sigma_{\frac{l}{n^2}}(W_{\frac{l+1}{n^2}} - W_{\frac{l}{n^2}}).$$

Next regarding the increments of the finer scale we define

$$\Delta W_{k,j} = W_{t_{k,j}+\frac{1}{n^2}} - W_{t_{k,j}}, \quad \Delta Y_{k,j} = Y_{t_{k,j}+\frac{1}{n^2}} - Y_{t_{k,j}}, \quad \Delta \overline{Y}_{k,j} = \overline{Y}_{t_{k,j}+\frac{1}{n^2}} - \overline{Y}_{t_{k,j}}$$

and

$$\Delta D_{k,j} = D_{t_{k,j}+\frac{1}{n^2}} - D_{t_{k,j}},$$

which yields

$$\Delta \overline{Y}_{k,j} = \sigma_{t_{k,j}} \Delta W_{k,j}.$$

Now setting $Z_{k,j} = n\Delta W_{k,j}$, $(Z_{k,j})_{0\leqslant k,j<n}$ is a family of i.i.d. $\mathrm{N}(0,1)$ (standard normal) distributed random variables, since W is a Brownian motion. Finally, we define some abbreviations concerning the volatility:

$$\sigma_{k,j} = \sigma_{t_{k,j}}, \quad \sigma_k = \sigma_{k,0}, \quad \epsilon_{k,j} = \sigma_{k,j} - \sigma_k.$$

3 Estimates of the spot volatility

For our main results we need some estimates of the spot volatility. As a first attempt to infer the spot volatility we use the concept of bipower variation (cf. Barndorff-Nielsen, Shephard [2]) which is robust to jumps. To avoid technical difficulties we work first under the Assumptions 2.1 and finally extend them in Corollary 3.5 to the weakened Assumptions 2.2 by using stopping time arguments.

Definition 3.1. *Set for* $0 \leqslant k < n$

$$\begin{aligned} \hat{\sigma}_k^2 &= \frac{\pi n^2}{2(n-1)} \sum_{j=0}^{n-2} |\Delta Y_{k,j}| \Delta Y_{k,j+1}| \quad \text{(without drift)}, \\ \check{\sigma}_k^2 &= \frac{\pi n^2}{2(n-1)} \sum_{j=0}^{n-2} |\Delta \check{Y}_{k,j}||\Delta \check{Y}_{k,j+1}| \quad \text{(with drift)}. \end{aligned}$$

The technique of the following proof of Proposition 3.2 is based on moment estimates, which follow from the Itô formula. This technique is motivated by the martingale moment inequalities of Millar [10] and Novikov [11], cf. also Karatzas and Shreve [8][Proposition 3.26].

Proposition 3.2. *Let*

$$\begin{aligned} c_{k,j} &= \sigma_{k,j}\sigma_{k,j+1} - \sigma_k^2, \quad 0 \leqslant k,j < n, \quad j < n-1, \\ F_k &= \hat{\sigma}_k^2 - \frac{\pi}{2(n-1)} \sum_{j=0}^{n-2} (\sigma_k^2 + c_{k,j})|Z_{k,j}||Z_{k,j+1}|, \quad 0 \leqslant k < n, \\ H_{k,j} &= n\Delta Y_{k,j} - (\sigma_k + \epsilon_{k,j})Z_{k,j}, \quad 0 \leqslant k,j < n. \end{aligned}$$

Then, there are two constants C_1, $C_2 > 0$ *for every fixed* $m \in \mathbb{N}$, *such that we obtain for every* $\epsilon > 0$ *and* $n \in \mathbb{N}$ *the inequalities*

$$P(|F_k| \geqslant \epsilon) \leqslant \frac{C_1}{n^{2\alpha m-1}\epsilon^m}, \quad 0 \leqslant k < n, \tag{4}$$

$$P(|H_{k,j}| \geqslant \epsilon) \leqslant \frac{C_2}{n^{4\alpha m}\epsilon^{2m}}, \quad 0 \leqslant k,j < n. \tag{5}$$

Furthermore we have the trivial relations

$$|\epsilon_{k,j}| \leqslant \frac{K}{n^\alpha}, \quad |c_{k,j}| \leqslant \frac{3K^2}{n^\alpha}, \quad 0 \leqslant k,j < n. \tag{6}$$

Proof. We separate the proof into four steps. In the first step we establish some useful moment inequalities. Next we prove in the second and third step respectively (4) and (5). Finally (6) is derived in the fourth step.

STEP 1. We have to show the following moment inequality

$$E|\Delta Y_{k,j} - \Delta \overline{Y}_{k,j}|^{2m} = E\left|\int_{t_{k,j}}^{t_{k,j}+\frac{1}{n^2}} (\sigma_s - \overline{\sigma}_s)\, dW_s\right|^{2m} \leqslant \frac{v_m K^{2m}}{n^{4\alpha m+2m}}, \quad m \geqslant 1 \tag{7}$$

with

$$v_m = m^m(2m-1)^m.$$

For proving this, we apply Itô's formula to $f : x \mapsto x^{2m}$ and

$$M_t = \int_{t_{k,j}}^{t} (\sigma_s - \overline{\sigma}_s)\, dW_s, \quad t_{k,j} \leqslant t \leqslant 1$$

which yields

$$M_{t_{k,j}+\frac{1}{n^2}}^{2m} = f\left(M_{t_{k,j}+\frac{1}{n^2}}\right) = \int_{t_{k,j}}^{t_{k,j}+\frac{1}{n^2}} f'(M_s)\,dM_s + m(2m-1)\int_{t_{k,j}}^{t_{k,j}+\frac{1}{n^2}} M_s^{2m-2}\,d\langle M\rangle_s\,.$$

Taking expectations on both sides, leads to

$$\begin{aligned} EM_{t_{k,j}+\frac{1}{n^2}}^{2m} &= m(2m-1)\int_{t_{k,j}}^{t_{k,j}+\frac{1}{n^2}} EM_s^{2m-2}|\sigma_s-\overline{\sigma}_s|^2\,ds \\ &\leqslant m(2m-1)\frac{K^2}{n^{4\alpha}}\int_{t_{k,j}}^{t_{k,j}+\frac{1}{n^2}} EM_s^{2m-2}\,ds \\ &\leqslant m(2m-1)\frac{K^2}{n^{4\alpha+2}}EM_{t_{k,j}+\frac{1}{n^2}}^{2m-2}, \quad m\geqslant 1. \end{aligned}$$

Note for the last inequality that $(M_s^{2m-2},\mathcal{F}_s)_s$ is a submartingale. Hence iteratively we get (7). Analogously we obtain

$$E|\Delta Y_{k,j}|^{2m} = E\left|\int_{t_{k,j}}^{t_{k,j}+\frac{1}{n^2}}\sigma_s\,dW_s\right|^{2m} \leqslant \frac{v_m K^{2m}}{n^{2m}}, \quad m\geqslant 1. \tag{8}$$

Finally, the Cauchy-Schwarz inequality together with (7) and (8) yields

$$E|\Delta Y_{k,j}|^m|\Delta Y_{k,j+1}-\Delta\,\overline{\mathrm{Y}}_{k,j+1}|^m \leqslant \frac{v_m K^{2m}}{n^{2\alpha m+2m}}, \quad m\geqslant 1.$$

Since

$$E|\Delta\,\overline{\mathrm{Y}}_{k,j+1}|^{2m} = \sigma_{k,j+1}^{2m}E|\Delta W_{k,j+1}|^{2m} \leqslant \frac{K^{2m}\mu_{2m}}{n^{2m}}$$

with $\mu_{2m}=E|U|^{2m}$ for $m\geqslant 1$ and $U\sim \mathrm{N}(0,1)$, we also get

$$E|\Delta\,\overline{\mathrm{Y}}_{k,j+1}|^m|\Delta Y_{k,j}-\Delta\,\overline{\mathrm{Y}}_{k,j}|^m \leqslant \frac{(v_m\mu_{2m})^{\frac{1}{2}}K^{2m}}{n^{2\alpha m+2m}}, \quad m\geqslant 1.$$

STEP 2. We proceed with the proof of (4), noting that

$$\begin{aligned} &\left|\hat{\sigma}_k^2 - \frac{\pi n^2}{2(n-1)}\sum_{j=0}^{n-2}|\Delta\,\overline{\mathrm{Y}}_{k,j}||\Delta\,\overline{\mathrm{Y}}_{k,j+1}|\right| \qquad (9)\\ \leqslant\;& \frac{\pi n^2}{2(n-1)}\sum_{j=0}^{n-2}\big||\Delta Y_{k,j}\Delta Y_{k,j+1}| - |\Delta\,\overline{\mathrm{Y}}_{k,j}\,\Delta\,\overline{\mathrm{Y}}_{k,j+1}|\big| \\ \leqslant\;& \frac{\pi n^2}{2(n-1)}\sum_{j=0}^{n-2}|\Delta Y_{k,j}||\Delta Y_{k,j+1}-\Delta\,\overline{\mathrm{Y}}_{k,j+1}| + |\Delta\,\overline{\mathrm{Y}}_{k,j+1}||\Delta Y_{k,j}-\Delta\,\overline{\mathrm{Y}}_{k,j}|. \end{aligned}$$

This together with the results of the first step and the Markov inequality yields

$$\begin{aligned} &P\left(\frac{\pi n^2}{2(n-1)}\sum_{j=0}^{n-2}|\Delta Y_{k,j}||\Delta Y_{k,j+1}-\Delta\,\overline{\mathrm{Y}}_{k,j+1}| + |\Delta\,\overline{\mathrm{Y}}_{k,j+1}||\Delta Y_{k,j}-\Delta\,\overline{\mathrm{Y}}_{k,j}| \geqslant \epsilon\right) \\ \leqslant\;& \sum_{j=0}^{n-2}P\left(|\Delta Y_{k,j}||\Delta Y_{k,j+1}-\Delta\,\overline{\mathrm{Y}}_{k,j+1}| \geqslant \frac{1}{2(n-1)}\frac{2(n-1)}{\pi n^2}\epsilon\right) \qquad (10)\\ &+P\left(|\Delta\,\overline{\mathrm{Y}}_{k,j+1}||\Delta Y_{k,j}-\Delta\,\overline{\mathrm{Y}}_{k,j}| \geqslant \frac{1}{2(n-1)}\frac{2(n-1)}{\pi n^2}\epsilon\right) \\ \leqslant\;& n\frac{v_m K^{2m}\pi^m n^{2m}}{n^{2\alpha m+2m}\epsilon^m} + n\frac{(v_m\mu_{2m})^{\frac{1}{2}}K^{2m}\pi^m n^{2m}}{n^{2\alpha m+2m}\epsilon^m} \end{aligned}$$

$$= \frac{C_1}{n^{2\alpha m-1}\epsilon^m}$$

where C_1 is defined by the last equality. Now (4) follows from the inequalities in (9) and (10) together with

$$\begin{aligned}\frac{\pi n^2}{2(n-1)}\sum_{j=0}^{n-2}|\Delta\,\overline{\mathrm{Y}}_{k,j}\,||\Delta\,\overline{\mathrm{Y}}_{k,j+1}\,| &= \frac{\pi n^2}{2(n-1)}\sum_{j=0}^{n-2}\sigma_{k,j}\sigma_{k,j+1}|\Delta W_{k,j}||\Delta W_{k,j+1}| \\ &= \frac{\pi}{2(n-1)}\sum_{j=0}^{n-2}(\sigma_k^2+c_{k,j})|Z_{k,j}||Z_{k,j+1}|\end{aligned}$$

which concludes step 2.
STEP 3. We prove (5) and consider the following decomposition

$$n\Delta Y_{k,j} = n\Delta\,\overline{\mathrm{Y}}_{k,j} + n(\Delta Y_{k,j}-\Delta\,\overline{\mathrm{Y}}_{k,j}) = (\sigma_k+\epsilon_{k,j})Z_{k,j} + n(\Delta Y_{k,j}-\Delta\,\overline{\mathrm{Y}}_{k,j}). \tag{11}$$

By the results in step 1 and the Markov inequality we obtain

$$P(|n(\Delta Y_{k,j}-\Delta\,\overline{\mathrm{Y}}_{k,j})|\geqslant\epsilon)\leqslant\frac{n^{2m}v_mK^{2m}}{\epsilon^{2m}n^{4\alpha m+2m}} = \frac{C_2}{\epsilon^{2m}n^{4\alpha m}}$$

where C_2 is defined by the last equality. This together with (11) proves step 3.
STEP 4. Finally we prove (6). The first inequality in (6) follows directly from our Assumptions 2.1. The second inequality also follows straightforward:

$$|c_{k,j}| = |\sigma_k(\epsilon_{k,j}+\epsilon_{k,j+1})+\epsilon_{k,j}\epsilon_{k,j+1}|\leqslant\frac{2K^2}{n^\alpha}+\frac{K^2}{n^{2\alpha}}\leqslant\frac{3K^2}{n^\alpha},\quad 0\leqslant k,j<n.$$

This completes our proof. □

Next we provide rates for estimates of the volatility and the log-price increments. For the proofs we need some inequalities for i.i.d. N(0,1)-distributed random variables $(Z_i)_i$ which are stated in the appendix.

We introduce the following notation

$$\max_k f_k = \max_{0\leqslant k<n} f_k,\quad \max_{k,j} g_{k,j} = \max_{0\leqslant k,j<n} g_{k,j},\quad n\in\mathbb{N}$$

for real valued functions f resp. g on the domains $\{0\leqslant k<n\}\times\Omega$ resp. $\{0\leqslant k,j<n\}\times\Omega$.

Proposition 3.3. *For $\gamma<\alpha$ we obtain*

$$n^\gamma\max_{k,j}|n\Delta Y_{k,j}-\sigma_kZ_{k,j}|\to 0,\quad P\text{-a.s.} \tag{12}$$

and for $\gamma<\alpha\wedge\frac{1}{2}$

$$n^\gamma\max_k|\widehat{\sigma}_k^2-\sigma_k^2|\to 0,\quad P\text{-a.s.} \tag{13}$$

Proof. We start with the proof of (12). Using the notation of Proposition 3.2 we write

$$n^\gamma\max_{k,j}|n\Delta Y_{k,j}-\sigma_kZ_{k,j}| = n^\gamma\max_{k,j}|\epsilon_{k,j}Z_{k,j}+H_{k,j}|\leqslant Kn^{\gamma-\alpha}\max_{k,j}|Z_{k,j}|+n^\gamma\max_{k,j}|H_{k,j}|.$$

Using (A.3), (5) and the Markov inequality this yields

$$\begin{aligned}&P(n^\gamma\max_{k,j}|n\Delta Y_{k,j}-\sigma_kZ_{k,j}|\geqslant\epsilon)\\ \leqslant\;& P\left(Kn^{\gamma-\alpha}\max_{k,j}|Z_{k,j}|\geqslant\frac{\epsilon}{2}\right)+P\left(n^\gamma\max_{k,j}|H_{k,j}|\geqslant\frac{\epsilon}{2}\right)\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{2^m K^m n^{m(\gamma-\alpha)}}{\epsilon^m} E(\max_{k,j} |Z_{k,j}|)^m + \sum_{0 \leqslant k,j < n} P\left(|H_{k,j}| \geqslant \frac{\epsilon n^{-\gamma}}{2} \right) \\
&\leqslant n^{m(\gamma-\alpha)} \frac{2^m K^m}{\epsilon^m} \left(2^m (\log n^2)^{\frac{m}{2}} + 2m!\right) + n^2 C_2 \frac{2^{2m} n^{2\gamma m}}{\epsilon^{2m}} n^{-4\alpha m} \\
&= O\left(\frac{1}{n^2}\right), \qquad (14)
\end{aligned}
$$

for any fixed $\epsilon > 0$ and m large enough, keeping in mind that $\gamma < \alpha$. Then applying Borel-Cantelli to

$$
A_l = \limsup_n \left\{ n^\gamma \max_{k,j} |n\Delta Y_{k,j} - \sigma_k Z_{k,j}| \geqslant \frac{1}{l} \right\}, \quad l \in \mathbb{N}
$$

yields the desired result.

STEP 2. To prove (13) we write with the use of Proposition 3.2 for any $0 \leqslant k < n$

$$
\begin{aligned}
& \max_k |\hat{\sigma}_k^2 - \sigma_k^2| \\
&= \max_k \left| \left(\frac{\pi}{2(n-1)} \sum_{j=0}^{n-2} |Z_{k,j}||Z_{k,j+1}| - 1 \right) \sigma_k^2 + \frac{\pi}{2(n-1)} \sum_{j=0}^{n-2} c_{k,j} |Z_{k,j}||Z_{k,j+1}| + F_k \right| \\
&\leqslant K^2 \max_k \left| \frac{\pi}{2(n-1)} \sum_{j=0}^{n-2} |Z_{k,j}||Z_{k,j+1}| - 1 \right| + \frac{3\pi K^2}{2(n-1)n^\alpha} \sum_{j=0}^{n-2} \max_k |Z_{k,j}| \max_k |Z_{k,j+1}| \\
&\quad + \max_k |F_k| \\
&= K^2 \max_k \eta_1^{(k)} + \eta_2 + \max_k |F_k|. \qquad (15)
\end{aligned}
$$

For the first term we use Corollary A.2 and get

$$
P\left(n^\gamma \max_k \eta_1^{(k)} > \epsilon \right) \leqslant \sum_{k=0}^{n} P\left(\eta_1^{(k)} > \epsilon n^{-\gamma} \right) \leqslant nC\epsilon^{-2m} n^{2m\gamma} n^{-m}.
$$

Noting that $\gamma < \frac{1}{2}$, we obtain analogously as in the first step with Borel-Cantelli

$$
n^\gamma K^2 \max_k \eta_1^k \to 0 \quad P\text{-a.s.} \qquad (16)
$$

Similarly we can show the convergence of the other two terms, by establishing suitable estimates. By Proposition A.3 and the fact that the random variables $\max_k |Z_{k,j}|$ and $\max_k |Z_{k,j+1}|$ are independent for every $j = 0, \ldots, n-2$ we obtain

$$
\begin{aligned}
& P\left(n^{\gamma-\alpha} \sum_{j=0}^{n-2} \max_k |Z_{k,j}| \max_k |Z_{k,j+1}| \geqslant \epsilon(n-1) \right) \\
&\leqslant \sum_{j=0}^{n-2} P\left(\max_k |Z_{k,j}| \max_k |Z_{k,j+1}| \geqslant \epsilon n^{\alpha-\gamma} \right) \\
&\leqslant (n-1)\left(2^m (\log n)^{\frac{m}{2}} + 2m!\right)^2 \epsilon^{-m} n^{m(\gamma-\alpha)}.
\end{aligned}
$$

and similarly for the last term

$$
P\left(n^\gamma \max_k |F_k| > \epsilon \right) \leqslant \sum_{k=0}^{n-1} P\left(|F_k| > \epsilon n^{-\gamma}\right) \leqslant n\epsilon^{-m} n^{\gamma m} C_1 n^{1-2\alpha m}.
$$

Recall that $\gamma < \alpha$. Piecing together the three estimates lead to the desired result. □

The next proposition is a generalization of the previous one to a model with drift term and provides the result for the spot volatility and not only the squares of the spot volatility. Note that this proposition is interesting on its own, because it states a possibility to estimate the spot volatility pathwise and uniformly on a grid.

Proposition 3.4. *For all $\gamma < \frac{1}{2} \wedge \alpha$ we obtain*

$$n^{\gamma} \max_{k,j} |\breve{\sigma}_k - \sigma_k| \to 0 \quad P\text{-a.s.} \tag{17}$$

Proof. First consider the difference between the estimates with and without drift

$$\begin{aligned}
&|\breve{\sigma}_k^2 - \hat{\sigma}_k^2| \qquad (18)\\
= \;& \left| \frac{\pi n^2}{2(n-1)} \sum_{j=0}^{n-2} |\Delta Y_{k,j}||\Delta Y_{k,j+1}| - |\Delta \check{Y}_{k,j}||\Delta \check{Y}_{k,j+1}| \right| \\
\leqslant \;& \frac{\pi n^2}{2(n-1)} \sum_{j=0}^{n-2} |\Delta Y_{k,j}(\Delta Y_{k,j+1} - \Delta \check{Y}_{k,j+1}) - \Delta \check{Y}_{k,j+1}(\Delta \check{Y}_{k,j} - \Delta Y_{k,j})| \\
\leqslant \;& \frac{\pi n^2}{2(n-1)} \sum_{j=0}^{n-2} |\Delta Y_{k,j}||\Delta D_{k,j+1}| + (|\Delta Y_{k,j+1}| + |\Delta D_{k,j+1}|)|\Delta D_{k,j}| \\
\leqslant \;& \frac{\pi n^2}{2(n-1)} \sum_{j=0}^{n-2} (|\Delta Y_{k,j} - \Delta \overline{\mathrm{Y}}_{k,j}| + |\Delta D_{k,j}|)|\Delta D_{k,j+1}| + |\Delta Y_{k,j+1} - \Delta \overline{\mathrm{Y}}_{k,j+1}||\Delta D_{k,j}| \\
&+ \frac{\pi n^2}{2(n-1)} \sum_{j=0}^{n-2} |\Delta \overline{\mathrm{Y}}_{k,j}||\Delta D_{k,j+1}| + |\Delta \overline{\mathrm{Y}}_{k,j+1}||\Delta D_{k,j}| \\
\leqslant \;& \frac{\pi K}{2(n-1)} \sum_{j=0}^{n-2} |\Delta Y_{k,j} - \Delta \overline{\mathrm{Y}}_{k,j}| + \frac{K}{n^2} + |\Delta Y_{k,j+1} - \Delta \overline{\mathrm{Y}}_{k,j+1}| \\
&+ \frac{\pi K^2}{2(n-1)} \sum_{j=0}^{n-2} |\Delta W_{k,j}| + |\Delta W_{k,j+1}|.
\end{aligned}$$

With (7) we obtain

$$\begin{aligned}
&P\left(n^{\gamma} \max_k \left(\frac{1}{n-1} \sum_{j=0}^{n-2} |\Delta Y_{k,j} - \Delta \overline{\mathrm{Y}}_{k,j}| + |\Delta Y_{k,j+1} - \Delta \overline{\mathrm{Y}}_{k,j+1}| \right) \geqslant \epsilon \right) \qquad (19)\\
\leqslant \;& \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} P\left(|\Delta Y_{k,j} - \Delta \overline{\mathrm{Y}}_{k,j}| \geqslant \frac{\epsilon}{2} n^{-\gamma} \right) \\
\leqslant \;& n^2 \frac{2^{2m} v_m K^{2m} n^{2m\gamma}}{n^{4\alpha m + 2m} \epsilon^{2m}}.
\end{aligned}$$

and for the last term

$$n^{\gamma} \max_k \frac{1}{2(n-1)} \sum_{j=0}^{n-2} (|\Delta W_{k,j}| + |\Delta W_{k,j+1}|) \leqslant n^{\gamma - 1} \max_{k,j} |Z_{k,j}|.$$

This implies together with Proposition A.3 the inequality

$$\begin{aligned}
P\left(n^{\gamma} \max_k \frac{1}{2(n-1)} \sum_{j=0}^{n-2} (|\Delta W_{k,j}| + |\Delta W_{k,j+1}|) \geqslant \epsilon \right) &\leqslant P(\max_{k,j} |Z_{k,j}| \geqslant \epsilon n^{1-\gamma}) \qquad (20)\\
&\leqslant \frac{n^{m(\gamma-1)}}{\epsilon^m} \left(2^m (\log n^2)^{\frac{m}{2}} + 2m! \right)
\end{aligned}$$

Finally (19) and (20) yield together with (18) and the same Borel-Cantelli argument as in Proposition 3.3 for all $\gamma < 1$ the convergence

$$n^{\gamma} \max_k |\breve{\sigma}_k^2 - \hat{\sigma}_k^2| \to 0 \quad P\text{-a.s.} \tag{21}$$

Now (17) follows from (21) together with (13) and

$$n^{\gamma}\max_k|\breve{\sigma}_k-\sigma_k| = n^{\gamma}\max_k\frac{|\breve{\sigma}_k^2-\sigma_k^2|}{\breve{\sigma}_k+\sigma_k} \leqslant V^{-1}n^{\gamma}\left(\max_k|\breve{\sigma}_k^2-\widehat{\sigma}_k^2| + \max_k|\widehat{\sigma}_k^2-\sigma_k^2|\right),$$

where V is defined in Assumptions 2.1 (i). □

Corollary 3.5. *The statement of Proposition 3.4 still holds under the weakened Assumptions 2.2 if the function $\alpha : \Omega \to [0,1)$ is constant, i.e. all paths of the volatility are α-Hölder continuous for some constant $0 < \alpha \leqslant 1$.*

Proof. Define for each $m \in \mathbb{N}$

$$S_m^{(1)} = \inf\left\{t \geqslant 0 \,:\, \sigma_t \notin \left[\frac{1}{m}, m\right]\right\}, \quad S_m^{(2)} = \inf\{t \geqslant 0 \,:\, |d_t| > m\}$$

and

$$S_m^{(3)} = \inf\{t \geqslant 0 \,:\, \exists s < t \,:\, |\sigma_t - \sigma_s| > m|t-s|^{\alpha}\}$$

with $\inf\varnothing = 1$. Clearly $S_m^{(j)}$, $j = 1,2,3$ are stopping times. Set

$$A_m = \bigcap_{j=1}^{3}\left\{S_m^{(j)} = 1\right\} \cap \left\{\frac{1}{m} \leqslant \sigma_0 \leqslant m\right\}.$$

Then (1) and (2) yield $A_m \uparrow \Omega$. Note that $\sigma^{S_m^{(3)}}$ is Hölder continuous with exponent α and coefficient m. Next set $S_m = S_m^{(1)} \wedge S_m^{(2)} \wedge S_m^{(3)}$ and

$$\begin{aligned}\sigma_t^{(m)} &= \mathbb{1}_{[\frac{1}{m},m]}(\sigma_0)\sigma_t^{S_m} + \frac{1}{m}\mathbb{1}_{[\frac{1}{m},m]^c}(\sigma_0)\\ d_t^{(m)} &= \mathbb{1}_{[0,S_m)}(t)d_t, \quad 0 \leqslant t \leqslant 1, \quad m \in \mathbb{N}.\end{aligned}$$

Then $\sigma^{(m)}$ and $d^{(m)}$ fulfill the Assumptions 2.1 with

$$V = \frac{1}{m}, \quad K = m. \tag{22}$$

As $\sigma^{(m)}$ is $(\mathcal{F}_t)_t$ adapted and $t \mapsto d_t^{(m)}(\omega)$ is measurable, we can define

$$\breve{Y}_t^{(m)} = \int_0^t \sigma_s^{(m)}\,dW_s + \int_0^t d_s^{(m)}\,d\lambda(s), \quad m \in \mathbb{N}.$$

Using results of Jacod and Shiryaev [7][p. 46 ff.] and the fact that S_m is a stopping time and $\mathbb{1}_{[\frac{1}{m},m]}(\sigma_0) \in \mathcal{F}_0$, we obtain for every $m \in \mathbb{N}$ and $0 \leqslant t \leqslant 1$ the P-a.s. equality

$$\int_0^t \sigma_s^{(m)}\,dW_s = \mathbb{1}_{[\frac{1}{m},m]}(\sigma_0)\left(\breve{Y}_t^{S_m} + \mathbb{1}_{(S_m,1]}(t)(W_t - W_{S_m})\sigma_{S_m}\right) + \mathbb{1}_{[\frac{1}{m},m]^c}(\sigma_0)\frac{1}{m}W_t.$$

This yields

$$\breve{Y}_t^{(m)}(\omega) = \breve{Y}_t(\omega), \quad 0 \leqslant t \leqslant 1, \quad \omega \in A_m \cap N_m^c, \tag{23}$$

for a family of negligible sets (N_m).

Next define $(\breve{\sigma}_k^{(m)})_{k=0,\ldots,n-1}$ as a function of $(\Delta\breve{Y}_{k,j}^{(m)})_{0\leqslant k,j<n}$ instead of $(\Delta\breve{Y}_{k,j})_{0\leqslant k,j<n}$, i.e.

$$(\breve{\sigma}_k^{(m)})^2 = \frac{\pi n^2}{2(n-1)}\sum_{j=0}^{n-2}|\Delta\breve{Y}_{k,j}^{(m)}||\Delta\breve{Y}_{k,j+1}^{(m)}|, \quad 0 \leqslant k < n, \quad m \in \mathbb{N}.$$

Then, (17) states that we have for every $m \in \mathbb{N}$, the convergence

$$n^{\gamma} \max_{k,j} |\breve{\sigma}_k^{(m)} - \sigma_k^{(m)}|(\omega) \to 0, \quad n \to \infty, \quad \omega \in M_m^c \tag{24}$$

for a negligible set M_m. Crucial for this proof is, that we have

$$\breve{\sigma}_k^{(m)}(\omega) = \breve{\sigma}_k(\omega), \quad \sigma_k^{(m)}(\omega) = \sigma_k(\omega), \quad 0 \leqslant k < n, \quad \omega \in A_m \cap N_m^c, \quad m, n \in \mathbb{N}. \tag{25}$$

Now fix any

$$\omega \in A = \left(\bigcup_{m \in \mathbb{N}} A_m \cap N_m^c \right) \cap \left(\bigcup_{m \in \mathbb{N}} M_m \right)^c .$$

Then there is a number $m \in \mathbb{N}$ with $\omega \in A_m \cap N_m^c \cap M_m^c$. Hence, (24) and (25) yield

$$n^{\gamma} \max_{k,j} |\breve{\sigma}_k - \sigma_k|(\omega) \overset{(25)}{=} n^{\gamma} \max_{k,j} |\breve{\sigma}_k^{(m)} - \sigma_k^{(m)}|(\omega) \overset{(24)}{\to} 0, \quad n \to \infty.$$

Since $A_m \uparrow \Omega$, we have $P(A) = 1$ and our claim is proven. □

Next we aim to show that the rate in the pathwise, uniform convergences setting of our previous Corollary 3.5 is indeed optimal. We assume the Assumptions 2.2 with some constant $0 < \alpha \leqslant 1$. Our estimator is a linear interpolation of the realized bipower variation estimator in Definition 3.1 on the grid. So its calculation can be done very easily and quickly. A similar approach with a kernel type estimator has also been investigated in Fan and Wang [3]. They reach similar convergence rates as our test in the case $\alpha > \frac{1}{2}$, whereas the case $\alpha < \frac{1}{2}$ is not covered by their results. Note also an interesting alternative approach by Hoffman, Munk and Schmidt-Hieber [5]. They use a wavelet type estimator and consider the L^p error, $p < \infty$.

In order to avoid confusion with the notation, the time argument of all processes in this section is denoted in brackets and the grid fineness is denoted with the subindex n. For (26), this means, for instance, $\breve{\sigma}_k$ is denoted as $\breve{\sigma}_n\left(\frac{k}{n}\right)$. Set

$$\breve{\sigma}_n\left(\frac{n}{n}\right) = \breve{\sigma}_n\left(\frac{n-1}{n}\right).$$

Then the statement (17) in Corollary 3.5 yields, together with

$$n^{\gamma} \left| \breve{\sigma}_n\left(\frac{n-1}{n}\right) - \sigma\left(\frac{n}{n}\right) \right| \leqslant n^{\gamma} \left| \breve{\sigma}_n\left(\frac{n-1}{n}\right) - \sigma\left(\frac{n-1}{n}\right) \right| + n^{\gamma-\alpha} K(\omega),$$

the convergence

$$n^{\gamma} \max_{0 \leqslant k \leqslant n} \left| \breve{\sigma}_n\left(\frac{k}{n}\right) - \sigma\left(\frac{k}{n}\right) \right| \to 0, \quad P\text{-a.s.} \tag{26}$$

for all $\gamma < \alpha \wedge \frac{1}{2}$. Equation (26) is our starting point. Next, for every $n \in \mathbb{N}$, a natural estimator $\breve{\tau}_n$ for the spotvolatility σ is defined, which is based on $n^2 + 1$ equidistant high-frequency observations at the time points $0, \frac{1}{n^2}, \frac{2}{n^2}, \dots, 1$ of the underlying process $\breve{Y}$, i.e. we have

$$\breve{\tau}_n : \Omega \times [0,1] \to \mathbb{R}_+, \quad n \in \mathbb{N}.$$

For this purpose $\breve{\tau}_n$ is defined to be the linear interpolation of the points $\left(\breve{\sigma}_n\left(\frac{k}{n}\right)\right)_{k=0,1,\dots,n}$. This formally means

$$\breve{\tau}_n(t) = r(t)\breve{\sigma}_n(k(t)) + (1 - r(t))\breve{\sigma}_n\left(k(t) + \frac{1}{n}\right), \quad 0 \leqslant t \leqslant 1, \quad n \in \mathbb{N}$$

with the two deterministic functions $k, r : [0,1] \mapsto \mathbb{R}_+$ defined via

$$k(t) = \frac{\lfloor nt \rfloor}{n}, \quad 0 \leqslant t < 1, \quad k(1) = \frac{n-1}{n}$$

and

$$r(t) = \begin{cases} 1, & nt \in \{0, 1, \ldots, n-1\} \\ \lceil nt \rceil - nt, & \text{else} \end{cases}, \quad 0 \leqslant t \leqslant 1.$$

An estimator $\widehat{\tau}_n$ in the case of a vanishing drift term is defined in an analogous way. Note that our interpolation-based estimators coincide with the estimators in Definition 3.1 on the grid points. Next we define for every $0 < \alpha \leqslant 1$ the following sets

$$\mathcal{V}_\alpha = \{(\sigma, d) : (\sigma, d) \text{ satisfies Assumptions 2.2 with constant } \alpha\}$$

and denote for any continuous function $f : [0, 1] \to \mathbb{R}$ with

$$\|f\|_\infty = \sup\{|f(t)| : 0 \leqslant t \leqslant 1\}$$

its supremum norm.

Now, a theorem concerning the convergence of the spot volatility estimator $\check{\tau}_n$ can be formulated.

Theorem 3.6. *Let the Assumptions 2.2 hold with a constant* $0 < \alpha \leqslant 1$. *Then, we have for all* $\gamma < \alpha \wedge \frac{1}{2}$ *the convergence*

$$n^\gamma \|\check{\tau}_n - \sigma\|_\infty \to 0, \quad n \to \infty \quad P\text{-a.s.} \tag{27}$$

and the upper bound $\alpha \wedge \frac{1}{2}$ *is sharp in the sense that*

$$\left\{\beta \in \mathbb{R} : n^\beta \|\check{\tau}_n - \sigma\|_\infty \overset{P\text{-a.s.}}{\to} 0 \text{ for all } (\sigma, d) \in \mathcal{V}_\alpha\right\} \subseteq \left(-\infty, \frac{1}{2} \wedge \alpha\right]. \tag{28}$$

Hence, the optimal convergence rate is $n^{\alpha \wedge \frac{1}{2}}$.

Proof. The proof is divided into two steps. The first one is devoted to the convergence result (27) and the second one to the rate result (28).

STEP 1. *Proof of (27).* Using the linear interpolation approach, we obtain for $0 \leqslant t \leqslant 1$

$$\begin{aligned} |\check{\tau}_n(t) - \sigma(t)| &\leqslant |\check{\tau}_n(t) - \check{\tau}_n(k(t))| + |\check{\tau}_n(k(t)) - \sigma(k(t))| + |\sigma(k(t)) - \sigma(t)| \\ &\leqslant \frac{1}{n} \max_{0 \leqslant k \leqslant n} \check{\sigma}_n\left(\frac{k}{n}\right) + |\check{\sigma}_n(k(t)) - \sigma(k(t))| + \frac{K(\omega)}{n^\alpha} \\ &\leqslant \frac{1}{n}\left(\max_{0 \leqslant k \leqslant n} \left|\check{\sigma}_n\left(\frac{k}{n}\right) - \sigma\left(\frac{k}{n}\right)\right| + K(\omega)\right) + |\check{\sigma}_n(k(t)) - \sigma(k(t))| + \frac{K(\omega)}{n^\alpha} \end{aligned}$$

which implies together with (26) the convergence (27).

STEP 2. *Proof of (28).* Set for any $0 < \alpha < 1$

$$\sigma(s) = 1 + s^\alpha, \quad 0 \leqslant s \leqslant 1,$$

i.e. σ is positive, deterministic and the prototype of an α-Hölder continuous function. Furthermore, set $d = 0$ and note that $(\sigma, d) \in \mathcal{V}_\alpha$. Fix any $\beta > \alpha$. Then, we obtain the pointwise divergence

$$n^\beta \|\widehat{\tau}_n - \sigma\|_\infty(\omega) \to \infty, \quad n \to \infty, \quad \omega \in \Omega. \tag{29}$$

This divergence to infinity is due to our interpolation approach. The proof of (29) is straightforward and therefore omitted.

With (29) in mind, it obviously suffices to verify

$$P\left(\limsup_n n^{\frac{1}{2}} \|\check{\tau}_n - \sigma\|_\infty > 0\right) > 0 \tag{30}$$

for a pair $(\sigma, d) \in \mathcal{V}_1$ in order to establish (28). For this purpose, choose $\sigma = 1$ and $d = 0$, i.e. Y is a standard Brownian motion and obviously $(\sigma, d) \in \mathcal{V}_1$. Now, (30) holds because of the classical central limit theorem (CLT) for i.i.d random variables. □

4 The Gumbel test under the null hypothesis of continuous sample paths

Now we have all technical background to construct a test statistics and derive their asymptotic behaviour under the null hypotheses of no jumps. The idea behind is that the increments of our Itô semimartingale normed with a volatility estimator are close enough to i.i.d. normal random variables to use the classical extreme value theory.

We start with the approximation of our increments by normal random variables.

Proposition 4.1. *Let Assumptions 2.1 hold and let $\gamma < \frac{1}{2} \wedge \alpha$ be a constant, then we obtain*

$$n^{\gamma} \max_{k,j} \left| \frac{n\Delta\check{Y}_{k,j}}{\check{\sigma}_k} - Z_{k,j} \right| \to 0 \quad P\text{-a.s.} \tag{31}$$

Proof. We consider

$$\max_{k,j} \left| \frac{n\Delta\check{Y}_{k,j}}{\check{\sigma}_k} - Z_{k,j} \right| \leqslant \max_{k,j} \left| \frac{n\Delta\check{Y}_{k,j}}{\check{\sigma}_k} - \frac{n\Delta Y_{k,j}}{\sigma_k} \right| + V^{-1} \max_{k,j} |n\Delta Y_{k,j} - \sigma_k Z_{k,j}|.$$

By (12) it suffices to prove for $\gamma < \frac{1}{2} \wedge \alpha$ the convergence

$$n^{\gamma} \max_{k,j} \left| \frac{n\Delta\check{Y}_{k,j}}{\check{\sigma}_k} - \frac{n\Delta Y_{k,j}}{\sigma_k} \right| \to 0 \quad P\text{-a.s.}$$

First note that for every $\delta > 0$

$$n^{1-\delta} \max_{k,j} |\Delta Y_{k,j}| \to 0 \quad P\text{-a.s.} \tag{32}$$

namely

$$n^{1-\delta} \max_{k,j} |\Delta Y_{k,j}| \leqslant n^{-\delta} \max_{k,j} |n\Delta Y_{k,j} - \sigma_k Z_{k,j}| + Kn^{-\delta} \max_{k,j} Z_{k,j} \tag{33}$$

together with (12), Proposition A.3 and a Borel-Cantelli argument yields the result. Next use a further decomposition

$$\begin{aligned} \left| \frac{n\Delta\check{Y}_{k,j}}{\check{\sigma}_k} - \frac{n\Delta Y_{k,j}}{\sigma_k} \right| &= n\frac{|\Delta\check{Y}_{k,j}\sigma_k - \Delta Y_{k,j}\check{\sigma}_k|}{\check{\sigma}_k\sigma_k} \\ &= n\frac{|(\Delta\check{Y}_{k,j} - \Delta Y_{k,j})\sigma_k + \Delta Y_{k,j}(\sigma_k - \check{\sigma}_k)|}{\check{\sigma}_k\sigma_k} \\ &\leqslant \frac{\frac{K^2}{n} + n|\Delta Y_{k,j}||\sigma_k - \check{\sigma}_k|}{\check{\sigma}_k\sigma_k}. \end{aligned} \tag{34}$$

By (17) we obtain the convergence

$$\max_{k} |\check{\sigma}_k\sigma_k - \sigma_k^2| \leqslant K \max_{k} |\check{\sigma}_k - \sigma_k| \to 0 \quad \text{P-a.s.}$$

which yields

$$\min_{k} |\check{\sigma}_k\sigma_k| \geqslant \min_{k}(\sigma_k^2 - |\check{\sigma}_k\sigma_k - \sigma_k^2|) \geqslant V^2 - \max_{k} |\check{\sigma}_k\sigma_k - \sigma_k^2| \to V^2 > 0 \quad \text{P-a.s.} \tag{35}$$

Now we choose $\delta > 0$ such that $\gamma + \delta < \frac{1}{2} \wedge \alpha$ and get

$$\begin{aligned} n^{\gamma} \max_{k,j} \left(\frac{K^2}{n} + n|\Delta Y_{k,j}||\sigma_k - \check{\sigma}_k| \right) &\leqslant K^2 n^{\gamma-1} + n^{1-\delta} \max_{k,j} |\Delta Y_{k,j}| n^{\gamma+\delta} \max_{k} |\sigma_k - \check{\sigma}_k| \\ &\to 0 \quad P\text{-a.s.}, \end{aligned} \tag{36}$$

by (17) and (32). Finally (34), (35) and (36) prove this proposition. ☐

Now our main result follows straightforward

Theorem 4.2. *Set*

$$a_N = \sqrt{2\log N}, \quad b_N = a_N - \frac{\log(\log N) + \log(4\pi)}{2\sqrt{2\log N}}, \quad N \in \mathbb{N} \tag{37}$$

and define a statistics T_n *by*

$$T_n = n \max_{k,j} \left(\frac{\Delta \check{Y}_{k,j}}{\check{\sigma}_k} \right), \quad n \in \mathbb{N}. \tag{38}$$

Then under the Assumptions 2.2 we obtain

$$a_{n^2}(T_n - b_{n^2}) \xrightarrow{d} \mathcal{G}, \quad n \to \infty,$$

where $\mathcal{G}$ *denotes the Gumbel distribution with the cumulative distribution function* $x \mapsto e^{-e^{-x}}$, $x \in \mathbb{R}$.

Proof. First we prove the result under the stronger Assumptions 2.1. From extreme value theory we know that

$$a_{n^2}(\max_{k,j} Z_{k,j} - b_{n^2}) \xrightarrow{\mathrm{d}} \mathcal{G}, \quad n \to \infty,$$

compare e.g. Lemma 1.1.7 in Haan and Ferreira [4]. This together with Proposition 4.1, Slutsky's Lemma and Lemma A.4 in the appendix proves the desired result under the stronger Assumptions 2.1.

Now we proceed to an extension of this result to the Assumptions 2.2 by defining appropriate stopping times. Set $S_m^{(1)}, S_m^{(2)}$ as in Corollary 3.5 and redefine

$$S_m^{(3)} = \inf\left\{ t \geqslant 0 \,:\, \exists s < t \,:\, |\sigma_t - \sigma_s| > m|t-s|^{\frac{1}{m}} \right\}. \tag{39}$$

We need the above modification of $S_m^{(3)}$ since we do not have a fixed Hölder exponent $\alpha > 0$ in contrast to Corollary 3.5. Nevertheless, (23) in Corollary 3.5 still holds with our redefinition in (39) and $t \mapsto \sigma_t^{(m)}$ is Hölder continuous with the Hölder exponent $\frac{1}{m}$. Hence, for every $r \in \mathbb{R}$ we have

$$P(a_n(T_n^{(m)} - b_n) \leqslant r) \leqslant P(a_n(T_n - b_n) \leqslant r) + P(A_m^c), \quad m, n \in \mathbb{N}, \tag{40}$$

where $T_n^{(m)}$ denotes the statistics T_n in (38) with $\check{Y}^{(m)}$ instead of $\check{Y}$, compare the notation in Corollary 3.5. Then our result of the first part of this proof yields

$$a_n(T_n^{(m)} - b_n) \xrightarrow{\mathrm{d}} \mathcal{G}, \quad n \to \infty \tag{41}$$

for all $m \in \mathbb{N}$. From (40) and (41) we obtain

$$\liminf_n P(a_n(T_n - b_n) \leqslant r) \geqslant \mathcal{G}((-\infty, r]) - P(A_m^c), \quad m \in \mathbb{N},$$

hence $A_m^c \downarrow \varnothing$ implies

$$\liminf_n P(a_n(T_n - b_n) \leqslant r) \geqslant \mathcal{G}((-\infty, r]), \quad r \in \mathbb{R}.$$

Similar considerations yield

$$\limsup_n P(a_n(T_n - b_n) \leqslant r) \leqslant \mathcal{G}((-\infty, r]), \quad r \in \mathbb{R}$$

which completes our proof. $\square$

5 The Gumbel test under the alternative of an additive jump component

In the previous section we established that our test statistics T_n appropriately normed converges for quite general continuous stochastic volatility models to a Gumbel distribution. Now our aim is to derive the behaviour of the normed test statistics when a jump component is added to our continuous semimartingale model. We will show that when positive jumps are present the normed test statistics converges to infinity at a certain rate, which makes it possible to distinguish between purely continuous models and models with jumps. By the construction of our test statistics in terms of maxima we test for positive jumps. However, negative jumps may be covered analogously by considering $-Y$ instead of Y. Note that in contrast to other jump test based on power and bipower variation the Gumbel test makes it possible to test separately for positive and negative jumps, which might be of considerable interest e.g. for prediction of turbulent periods possibly leading to financial crisis.

We start with a general result for an additive jump semimartingle which leads to a rate of $\sqrt{n}$.

Theorem 5.1. *Let $(X_t, \mathcal{F}_t)_{t\in[0,1]}$ be a semimartingale and*

$$\Lambda = \{\omega \in \Omega \,:\, \exists t_0 \in (0,1] \,:\, X_{t_0}(\omega) - X_{t_0-}(\omega) > 0\}. \tag{42}$$

For $\gamma < \frac{1}{2}$, we obtain

$$n^{-\gamma} a_{n^2}(T_n - b_{n^2}) \to \infty \quad P\text{-stoch. on } \Lambda, \tag{43}$$

where a_n, b_n are defined in (37), and T_n is defined in (38) with $(\check{Y}_t)_t$ replaced by $(X_t)_t$.

Proof. First we have to show that $\Lambda \in \mathcal{F}$. Choose a sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ that exhausts the jumps of X, cf. Proposition 1.32 in Jacod and Shiryaev [7], then

$$\Lambda = \bigcup_{n,m\in\mathbb{N}} \left\{\Delta X_{T_n} > \frac{1}{m}\right\} \in \mathcal{F}_1 \subseteq \mathcal{F}.$$

Define two functions $k, j \,:\, (0,1] \to \{0, \ldots, n-1\}$ by

$$k(t) = (\lceil nt \rceil - 1), \quad j(t) = (\lceil n^2 t \rceil - 1) - nM(t), \quad t \in (0,1]$$

and set as usual

$$\Delta X_{k,j} = X_{t_{k,j}+\frac{1}{n^2}} - X_{t_{k,j}}, \quad 0 \leqslant k, j < n.$$

Choose $\hat{\omega} \in \Lambda$ and let $\eta \in (0,1]$ such that $\Delta X_\eta(\hat{\omega}) > 0$. Then we have

$$\liminf_n \max_{k,j} \Delta X_{k,j}(\hat{\omega}) \geqslant \liminf_n \Delta X_{k(\eta),j(\eta)}(\hat{\omega}) = \Delta X_\eta(\hat{\omega}) > 0, \tag{44}$$

where the equality sign above holds since the paths of $(X_t)_{t\in[0,1]}$ are càdlàg.

Next we have establish for arbitrary $L \in \mathbb{R}$ and $\gamma < \frac{1}{2}$ the convergence

$$P(\{n^{-\gamma} a_{n^2}(T_n - b_{n^2}) \leqslant L\} \cap \Lambda) \to 0, \quad n \to \infty.$$

We will prove this by contradiction, hence we assume the existence of a subsequence $(n_r)_r$ such that

$$P(\{n_r^{-\gamma} a_{n_r^2}(T_{n_r} - b_{n_r^2}) \leqslant L\} \cap \Lambda) \to \eta > 0, \quad r \to \infty. \tag{45}$$

By

$$\sum_{k=0}^{n-1} \sum_{j=0}^{n-1} (\Delta X_{k,j})^2 \to [X,X]_1, \quad n \to \infty, \quad P\text{-stoch.},$$

cf. Chapter 1, Theorem 4.47 in Jacod and Shiryaev [7], there exists a subsequence $(m_l)_l$ of $(n_r)_r$ such that

$$\sum_{k=0}^{m_l-1} \sum_{j=0}^{m_l-1} (\Delta X_{k,j})^2 \to [X,X]_1, \quad s \to \infty, \quad P\text{-a.s.}$$

Together with

$$|\Delta X_{k,j}||\Delta X_{k,j+1}| \leqslant (\Delta X_{k,j})^2 + (\Delta X_{k,j+1})^2$$

this yields

$$P\left(\limsup_l \sum_{k=0}^{m_l-1}\sum_{j=0}^{m_l-2} |\Delta X_{k,j}||\Delta X_{k,j+1}| < \infty\right) = 1. \tag{46}$$

Next consider the inclusions

$$\begin{aligned}
& \limsup_l \{m_l{}^{-\gamma} a_{m_l^2}(T_{m_l} - b_{m_l^2}) \leqslant L\} \cap \Lambda \\
= \ & \limsup_l \left\{T_{m_l} \leqslant \frac{L m_l{}^{\gamma}}{a_{m_l^2}} + b_{m_l^2}\right\} \cap \Lambda \\
\subseteq \ & \limsup_l \{T_{m_l} \leqslant m_l{}^{\gamma}\} \cap \Lambda \\
= \ & \limsup_l \left\{\frac{\sqrt{2}\sqrt{m_l-1}}{\sqrt{\pi}} \max_{k,j} \frac{\Delta X_{k,j}}{(\sum_{i=0}^{m_l-2}|\Delta X_{k,i}||\Delta X_{k,i+1}|)^{\frac{1}{2}}} \leqslant m_l{}^{\gamma}\right\} \cap \Lambda \\
\subseteq \ & \limsup_l \left\{\max_{k,j} \frac{\Delta X_{k,j}}{(\sum_{i=0}^{m_l-2}|\Delta X_{k,i}||\Delta X_{k,i+1}|)^{\frac{1}{2}}} \leqslant 2 m_l{}^{\gamma-\frac{1}{2}}\right\} \cap \Lambda \\
\subseteq \ & \limsup_l \left\{\max_{k,j} \Delta X_{k,j} \leqslant 2 m_l{}^{\gamma-\frac{1}{2}} \left(\sum_{k=0}^{m_l-1}\sum_{j=0}^{m_l-2}|\Delta X_{k,j}||\Delta X_{k,j+1}|\right)^{\frac{1}{2}}\right\} \cap \Lambda.
\end{aligned} \tag{47}$$

The P-measure of the set in (47) is zero by (44) and (46) since $\gamma - \frac{1}{2} < 0$. Hence the above inclusions together with Fatou's Lemma yield

$$\limsup_l P(\{m_l{}^{-\gamma} a_{m_l^2}(T_{m_l} - b_{m_l^2}) \leqslant L\} \cap \Lambda) \leqslant P(\limsup_l \{m_l{}^{-\gamma} a_{m_l}(T_{m_l} - b_{m_l}) \leqslant L\} \cap \Lambda) = 0,$$

which contradicts (45), since $(m_l)_l \subseteq (n_r)_r$. □

Note that (46) is the crucial property we need for the proof of the above theorem. This is a quite general assumption, which is for example fulfilled by a semimartingale as above. It is a natural question, whether we can improve the convergence rate $\sqrt{n}$ in (43) under stronger assumptions. Assume for this purpose that we observe a process

$$\tilde{Y} = \check{Y} + J$$

instead of a general semimartingale X. Here J denotes an additive, finite activity jump process. Then we get the heuristics

$$n\sum_{j=0}^{n-2}|\Delta\tilde{Y}_{k,j}||\Delta\tilde{Y}_{k,j+1}| = n\sum_{j=0}^{n-2}|\Delta\check{Y}_{k,j} + J_{k,j}||\Delta\check{Y}_{k,j+1} + J_{k,j+1}| = O_P(1). \tag{48}$$

Note that each pair of neighboring increments possesses at most one jump, if the increment size is small enough, i.e. n is large enough. This is due to the fact that we have only finite many jumps in each path. Using (48) we get easily the heuristics

$$n\max_{k,j}\frac{\Delta\tilde{Y}_{k,j}}{\tilde{\sigma}_k} = O_P(n) \tag{49}$$

with the corresponding notation $(\tilde{\sigma}_k)_{0\leqslant k<n}$ for the volatility. Hence (43) should hold for each $\gamma < 1$ instead of $\gamma < \frac{1}{2}$, i.e. we have the convergence rate n.

Note that, on the other hand, for the quadratic variation estimator

$$\tilde{\sigma}_k^2 = n\sum_{j=0}^{n-1}(\Delta\check{Y}_{k,j})^2,$$

we have the heuristic

$$\widetilde{\sigma}_k^2 = n \sum_{j=0}^{n-1} (\Delta \check{Y}_{k,j} + \Delta J_{k,j})^2 = O_P(n),$$

so that only a convergence rate of $\sqrt{n}$ can be expected, compare (49). Observe that the proof of Theorem 5.1 also works with the quadratic variation estimator. Thus, the above heuristic implies that Theorem 5.1 with the quadratic variation estimator yields the optimal convergence rate $\sqrt{n}$, despite we have in general infinite activity jump paths.

The following proposition makes the above heuristic rigorous.

Proposition 5.2. *Let Assumptions 2.2 hold. Furthermore, assume that there are two families $(H_l)_{l\in\mathbb{N}}$ and $(R_l)_{l\in\mathbb{N}}$ of random variables on the same probability space with*

$$R_l(\omega) \uparrow \infty, \quad R_l(\omega) > 0, \quad R_l(\omega) < R_{l+1}(\omega), \quad l \in \mathbb{N}, \quad \omega \in \Omega$$

and (H_l) arbitrary and define a finite activity, pure jump process

$$J_t = \sum_{l=1}^{\infty} H_l \, \mathbb{1}_{[R_l,\infty)}(t), \quad 0 \leqslant t \leqslant 1.$$

Set

$$\widetilde{Y}_t = \check{Y}_t + J_t, \quad 0 \leqslant t \leqslant 1$$

and define (T_n) as a function of $\widetilde{Y}$. For any fixed $\gamma < 1$ we obtain

$$n^{-\gamma} a_{n^2}(T_n(\omega) - b_{n^2}) \to \infty, \quad n \to \infty, \quad \omega \in \Lambda \cap N \tag{50}$$

for a negligible set N, where Λ is defined in (42).

Proof. The proof is omitted since the arguments are straightforward. □

6 Simulation results

In this section, the performance of the test statistics (38) concerning a jump detection test is investigated using numerical simulations. The resulting test is called the *Gumbel test*. Furthermore, the latter test is compared with a test developed by Barndorff-Nielsen and Shephard in [2], and both tests are applied to a real dataset. For this purpose, we implemented the underlying processes and tests in MATLAB.

Note, that the difference between the distribution of our scaled test statistics (38) and the Gumbel distribution results from two different error sources. First, the difference

$$n \frac{\Delta \check{Y}_{k,j}}{\check{\sigma}_k} - Z_{k,j}, \quad 0 \leqslant k, j < n, \tag{51}$$

second, the difference between the Gumbel distribution $\mathcal{G}$ and the distribution of

$$M_n = a_{n^2}(\max_{k,j} Z_{k,j} - b_{n^2}). \tag{52}$$

Concerning (51), consider the following Figure 1. Here, we have chosen the step size $n = 50$ and the volatility process

$$\sigma_t = 1 + |W_t^H|, \quad 0 \leqslant t \leqslant 1$$

where W^H denotes a fractional Brownian motion with Hurst parameter H. Further, we set the drift term $d = 0$ and assume that all processes are independent of each other. Figure 1 presents the empirical cumulative distribution functions of M_{50} and our scaled test statistics T_{50} in (38) with $H = 0.05$, 0.2, 0.5 and 0.95. The figure clearly shows that the difference depends on the Hurst parameter. It is remarkable

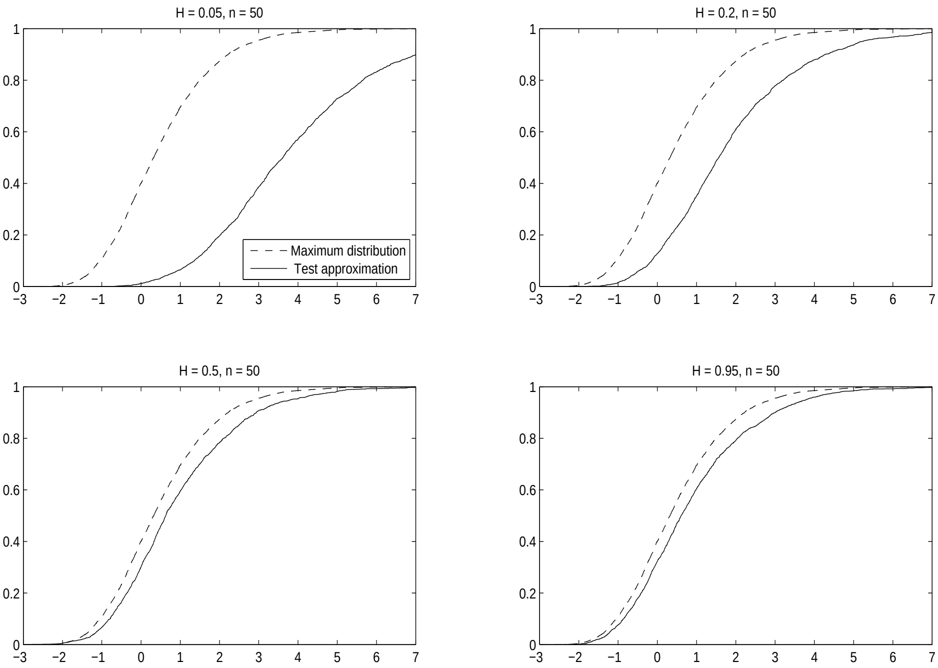

Figure 1: Approximation to the distribution of M_{50}

that there is no improvement between $H = 0.5$ and $H = 0.95$. This attitude is explained by the restriction $\gamma < \frac{1}{2} \wedge \alpha$ in Proposition 4.1 together with the well known fact that every path of a fractional Brownian motion with Hurst parameter H is Hölder continuous with every Hölder exponent $0 < h < H$.

Concerning (52), consider the next Figure 2. This figure demonstrates the weak convergence of M_n to the Gumbel distribution $\mathcal{G}$ if n tends to infinity. It is well-known that this is a rather slow convergence of logarithmic rate. Hence, it is no surprise that there is not much improvement between $n = 50$ and $n = 1000$. However, if we consider the two error sources in this setting, the error in (51), visualized by Figure 1, is the crucial one. Therefore, it is reasonable to use for our following statistical tests the quantiles of the Gumbel distribution rather than the empirical quantiles of the M_n distributions.

The fractional Brownian motions were simulated by use of Cholesky decompositions and the above empirical cumulative distribution functions were calculated based on 2000 simulated paths.

For the rest of this section, we assume that the drift term is set zero and a classical Ornstein-Uhlenbeck process is chosen as the volatility process with initial value $a = 1$, mean reversion $\mu = 1$, mean reversion speed $\theta = 0.5$ and diffusion $\overline{\sigma} = 0.2$ which is bounded away from zero, i.e.

$$dZ_t = \theta(\mu - Z_t)\,dt + \overline{\sigma}\,dB_t, \quad Z_0 = a, \quad 0 \leqslant t \leqslant 1, \tag{53}$$

$$\sigma_t = \max(0.1, Z_t). \tag{54}$$

Here, B denotes a Brownian motion that is independent of W, i.e. σ and W are independent.

For what follows, the approximation to the Gumbel distribution in Theorem 4.2 is investigated. For this purpose, in Figure 3, the grid sizes $\frac{1}{n^2}$, $n = 20, 50, 200, 1000$ are used and the respective empirical distribution functions using 10000 paths for each n are calculated.

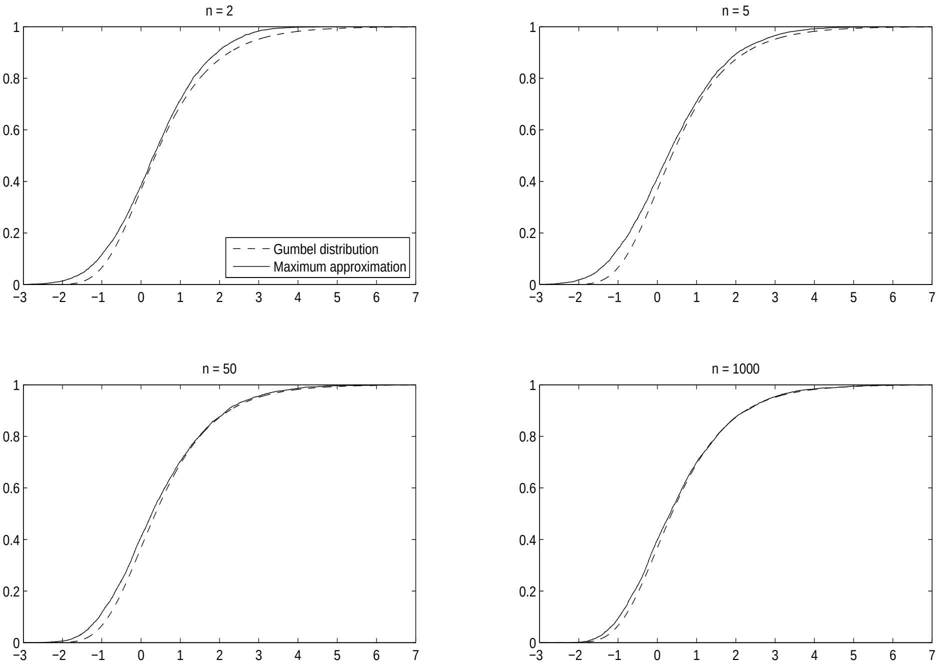

Figure 2: Convergence of the scaled maximum of i.i.d. N(0,1) random variables

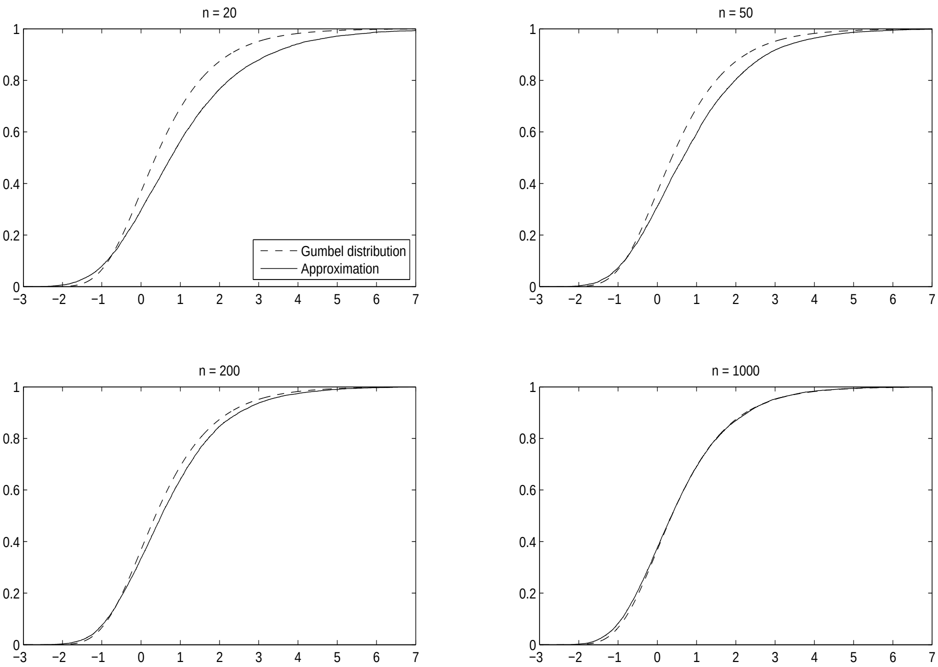

Figure 3: Approximation to the Gumbel distribution

Next, set $n = 50$ and consider the additional jump process

$$J_t = \sum_{i=0}^{N_t} U_i, \quad 0 \leqslant t \leqslant 1 \tag{55}$$

where N, U, σ, W are independent, N is a Poisson process with intensity λ and $(U_i)_i$ are i.i.d. Γ distributed random variables with shape parameter k and scale parameter θ, i.e. J is a compound Poisson process. Figure 4 presents four simulation plots with the respective settings $\lambda = 5$, $k = 10L^2$ and $\theta = \frac{1}{500L}$, $L = 1, 2, 3, 4$. This results in

$$E(U_i) = \frac{L}{50}, \quad \mathrm{Var}(U_i) = \frac{1}{10 \cdot 50^2}.$$

Note that by the Chebyshev inequality

$$P\left(\left|U_i - \frac{L}{50}\right| \geqslant \frac{1}{50}\right) \leqslant \frac{1}{10}$$

and that, in the case of a constant, deterministic volatility $\sigma = \sigma_0 > 0$ and no external jumps ($J = 0$), the process $Y_t = \sigma_0 W_t$ is obtained. In this case, it follows that

$$E|Y_{\frac{i+1}{n^2}} - Y_{\frac{i}{n^2}}| = \sigma_0 E|W_{\frac{1}{n^2}}| = \frac{\sigma_0\sqrt{2}}{n\sqrt{\pi}} = \frac{\sigma_0\sqrt{2}}{\sqrt{\pi}} \cdot \frac{1}{50}.$$

Hence, the jumps have a critical size in the sense that it is not clear whether they can be detected or not. (recall $a = 1$ in (53))

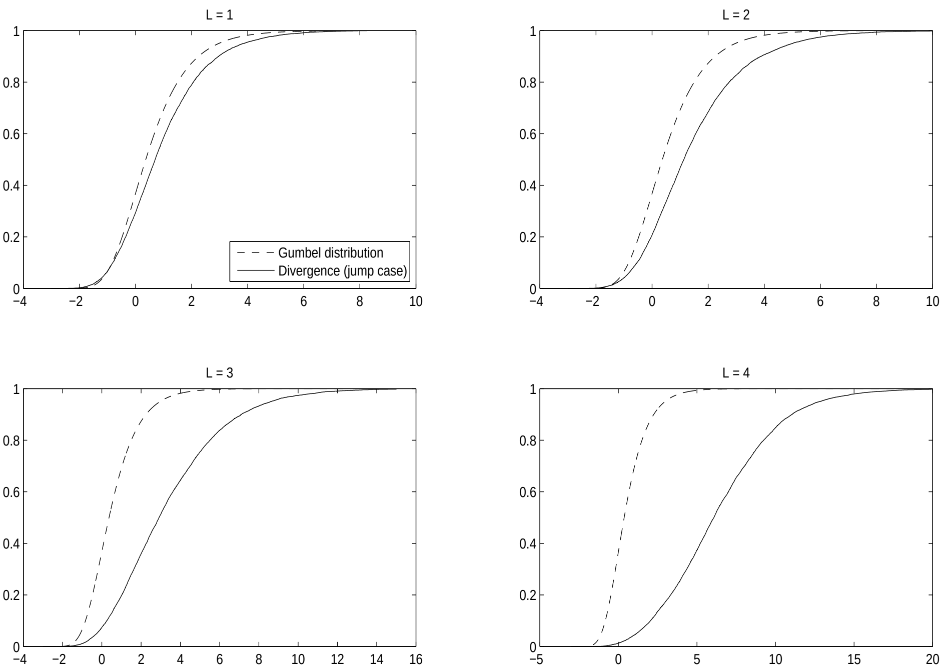

Figure 4: Divergence from the Gumbel distribution

Next, the test proposed by Barndorff-Nielsen and Shephard in [2] is compared with the Gumbel test. Barndorff-Nielsen and Shephard use the statistic

$$S_n = \frac{\mu^{-2} Y_n^{[1,1]} - Y_n^{[2]}}{\sqrt{\vartheta \mu^{-4} Y_n^{[1,1,1,1]}}}$$

with

$$\mu = \sqrt{\frac{2}{\pi}}, \quad \vartheta = \frac{\pi^2}{4} + \pi - 5$$

and

$$\begin{aligned} Y_n^{[2]} &= \sum_{i=0}^{n^2-1} |\Delta Y_i|^2, \quad \Delta Y_i = Y_{\frac{i+1}{n^2}} - Y_{\frac{i}{n}}, \\ Y_n^{[1,1]} &= \sum_{i=0}^{n^2-2} |\Delta Y_i||\Delta Y_{i+1}|, \\ Y_n^{[1,1,1,1]} &= \sum_{i=0}^{n^2-4} |\Delta Y_i||\Delta Y_{i+1}||\Delta Y_{i+2}||\Delta Y_{i+3}|. \end{aligned}$$

By Theorem 1 in [2] we obtain

$$S_n \xrightarrow{\mathrm{d}} \mathrm{N}(0,1), \quad n \to \infty$$

under certain conditions on σ which are fulfilled by the choice in (54) and

$$S_n \to -\infty, \quad n \to \infty \quad P\text{-stoch.},$$

if there is an additional external jump term J as in (55). Define the null hypothesis

$$H_0 \; : \text{there are no jumps}$$

and the alternative hypothesis

$$H_1 \; : \text{there are jumps (i.e. J in (55) is added).}$$

We have the two errors types

$$\begin{aligned} &\text{Type I error} &:& \quad \text{rejecting a true null hypothesis,} \\ &\text{Type II error} &:& \quad \text{failing to reject a false null hypothesis} \end{aligned}$$

and decide that a path $\hat{\omega}$ possesses a jump on the significance level $\alpha \in (0,1)$, iff

$$G(a_{n^2}(T_n(\hat{\omega}) - b_{n^2})) \geqslant 1 - \alpha \quad \text{resp.} \quad \phi(S_n(\hat{\omega})) \leqslant \alpha \tag{56}$$

where G denotes the distribution function of the Gumbel distribution and ϕ of the standard normal distribution.

Figure 5 is based on a setting with the gridsize $\frac{1}{n^2}, n = 50$, the volatility process in (54) and $\lambda = 10$, $L = 4$ for the jump process. Here, 2000 sample paths for each $\alpha = \frac{i}{1000}$, $i = 0, 1, \ldots, 100$ were simulated and it is verified that the Gumbel test is more sensitive than the test proposed in [2].

In order to check whether the Gumbel test has a larger power, both tests were recalibrated so that the type I error is exactly α for both tests. Hence G and ϕ are replaced by G_n and ϕ_n in (56) with

$$G_n(x) = P(a_{n^2}(T_n - b_{n^2}) \leqslant x), \quad \phi_n(x) = P(S_n \leqslant x), \quad x \in \mathbb{R} \quad (n = 50).$$

G_n and ϕ_n were approximated with the empirical distribution functions based on 10000 paths. Figure 6 shows the type I error calculated with 2000 paths per α. Observe the desired approximation to the diagonal.

As mentioned above, the power of the tests was calculated. It is defined by

$$\text{power}_\alpha = 1 - \text{type II error}_\alpha$$

with G_n resp. ϕ_n. For this purpose, we set $\lambda = 2, 15$ and $L = 4$.

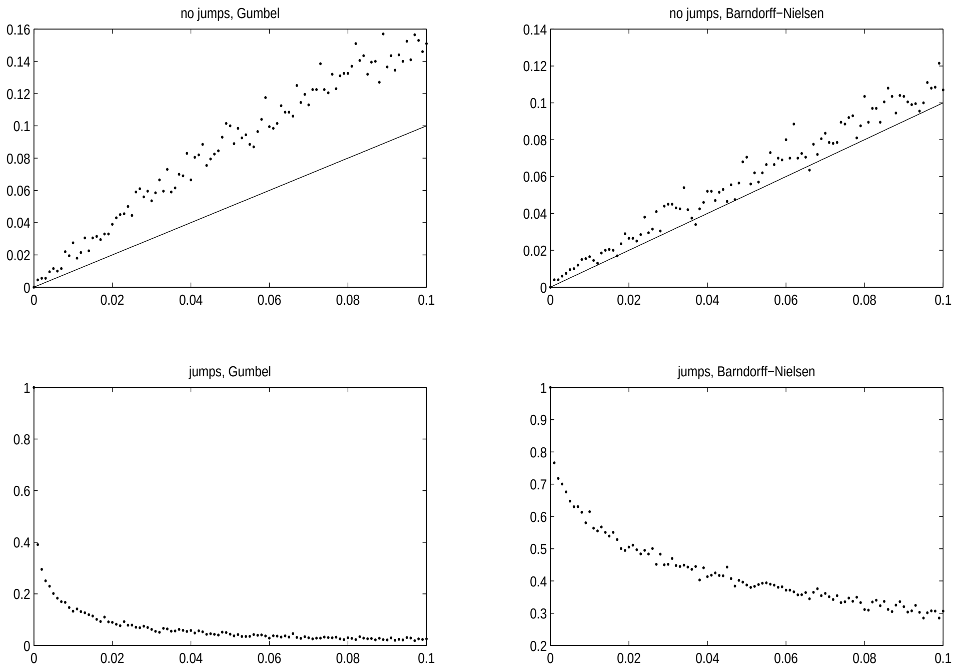

Figure 5: Error types

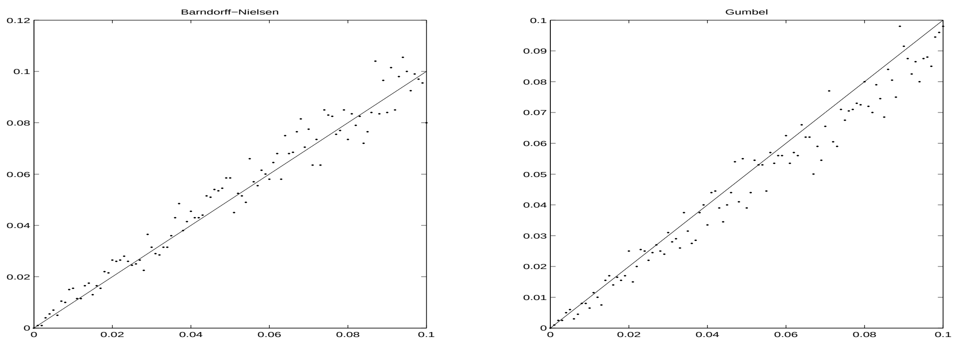

Figure 6: Recalibrated type I error

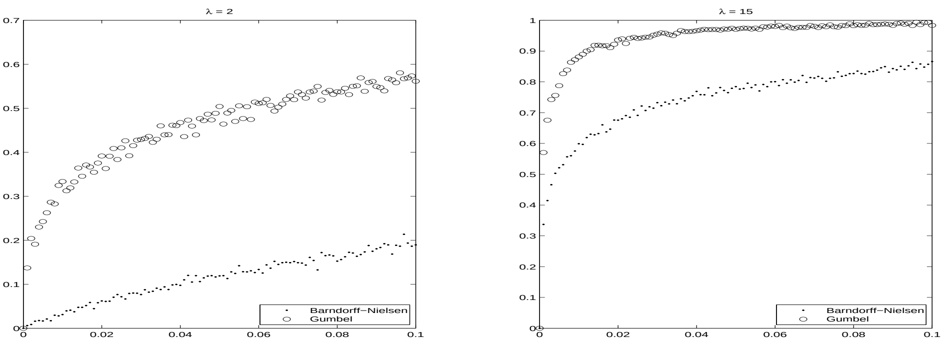

Figure 7: Power of tests

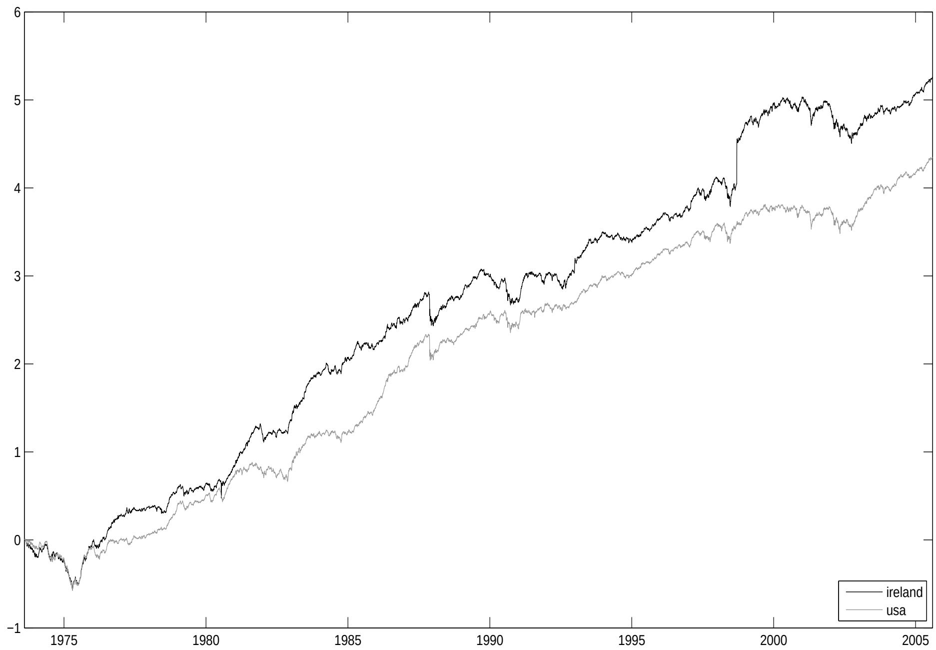

Figure 8: Worldstock index

Hence, in this setting the Gumbel test clearly has more power than the test proposed by Barndorff-Nielsen and Shephard in [2].

Finally, both tests are applied to a real dataset, i.e. the world stock indices constructed according to the best performing portfolio in Platen et al. [6, 13] of the USA and Ireland with daily observations. With

$$\widetilde{T} = \widetilde{T}_n = a_{n^2}(T_n - b_{n^2})$$

the results

$$G(\widetilde{T}^{\text{up}}_{\text{Ireland}}) = 1,\ G(\widetilde{T}^{\text{down}}_{\text{Ireland}}) = 1, \quad G(\widetilde{T}^{\text{up}}_{\text{USA}}) = 0.99268,\ G(\widetilde{T}^{\text{down}}_{\text{USA}}) = 1$$

and

$$\phi(S_{\text{Ireland}}) = 0, \quad \phi(S_{\text{USA}}) = 0.81401$$

are achieved. Note that the Gumbel test not only indicates a jump but also states the position and direction of the jump, i.e. whether we have an upwards or downwards jump. In order to detect downwards jumps, we simply have to switch from Y to $-Y$ and the maximum in the Gumbel test statistic turns to a minimum, this, of course, is a big advantage over the Barndorff-Nielsen and Shephard test.

The latter results indicate the following: Using the Barndorff-Nielsen and Shephard test the world stock index for Ireland possesses a jump but that of the USA does not. Using the Gumbel test, the index for Ireland possesses an upwards and downwards jump and that for the USA has a downwards jump for sure and an upwards jump with high probability. The different results for the indices of the two countries are not surprising considering that the Gumbel test is more sensitive as discussed above.

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A Auxiliary results

In what follows some rather elementary results concerning i.i.d. standard normal distributed random variables are stated. All proofs are omitted but can be found, for example, in Palmes [12].

Proposition A.1. *Let* $(H_i)_{i\in\mathbb{N}}$ *be a family of i.i.d. random variables with the property*

$$\exists r > 0 \,:\, Ee^{tH_1} < \infty, \quad t \in (-r, r).$$

Then we have for every $N \in \mathbb{N}$

$$E\left[\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(H_i - EH_1)\right)^{2N}\right] \to E[N(0, \text{Var } H_1)]^{2N}, \quad n \to \infty \tag{57}$$

which implies

$$E\left(\frac{1}{n}\sum_{i=1}^{n}H_i - EH_1\right)^{2N} = O\left(\frac{1}{n^N}\right), \quad n \to \infty. \tag{58}$$

Corollary A.2. *Let* $(Z_i)_{i\in\mathbb{N}}$ *be a family of* $N(0,1)$ *i.i.d. random variables and* $N \in \mathbb{N}$*. Then there is a constant* $C = C(N) > 0$ *such that*

$$P\left(\left|\frac{\pi}{2(n-1)}\sum_{i=0}^{n-2}|Z_i||Z_{i+1}| - 1\right| \geqslant \epsilon\right) \leqslant \frac{C}{\epsilon^{2N}n^N} \tag{59}$$

holds for every $n \in \mathbb{N}$ *and* $\epsilon > 0$.

Proposition A.3. *Let $(Z_i)_{i\in\mathbb{N}}$ be a family of i.i.d. random variables with $Z_i \sim N(0,1)$ and $N \in \mathbb{N}$. Then we have*

$$E\left(\max_{1\leqslant i\leqslant n}|Z_i|\right)^N \leqslant 2^N(\log n)^{\frac{N}{2}} + 2N! \tag{60}$$

for all sufficient large $n \in \mathbb{N}$.

Finally, let us state an easy, but useful lemma.

Lemma A.4. *Fix $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$. Then, the following inequalities hold:*

(i)

$$\min(b_1,\ldots,b_n) \leqslant \max(a_1+b_1,\ldots,a_n+b_n) - \max(a_1,\ldots,a_n) \leqslant \max(b_1,\ldots,b_n)$$

(ii)

$$|\max(a_1+b_1,\ldots,a_n+b_n) - \max(a_1,\ldots,a_n)| \leqslant \max(|b_1|,\ldots,|b_n|).$$

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