Reduced Form Credit Risk Models and the Second Dimension Risk Premium – Technical Foundations, Estimation and Applications

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Zusammenfassung

Die vorliegende Dissertation beschäftigt sich mit Kreditrisiko-Modellen "reduzierter Form" (Reduced Form Credit Risk Models) zur Analyse staatlicher Kreditrisiken. In diesen Modellen wird der Insolvenzprozess dem Namen entsprechend in reduzierter Form modelliert: Erste Sprünge von Poisson-Prozessen sollen hier Kreditereignisse darstellen. Auf eine tiefergehende Abbildung der finanziellen Situation der Einheit wird verzichtet und die Modelle für verschiedene Einheiten unterscheiden sich lediglich in den Sprungintensitäten der jeweiligen Poisson-Prozesse. Die Intensitäten bzw. die Intensitätsprozesse, die die Modelle für bestimmte Einheiten charakterisieren, können entweder als deterministisch oder als stochastisch modelliert werden. Im letzteren Fall werden die Modelle in der Regel als "doppel stochastisch" (doubly stochastic[1]) bezeichnet. Dabei werden die Intensitätsprozesse als Diffusionsprozesse modelliert.

In dieser Dissertation werden technische Grundlagen und die Funktionsweise dieser Modelle erörtert. Weiterhin wird im wahrscheinlichkeitstheoretischen Rahmen dargestellt, wie man anhand dieser Modelle analysieren kann, welche Rolle eine mögliche Stochastik der Kreditausfallswahrscheinlichkeit bei der Bildung von Kreditwertpapierpreisen spielt. Eine Strategie zur Schätzung solcher Modelle unter zwei Massen anhand von Zeitreihendaten wird ebenfalls diskutiert und evaluiert. Anhand dieser Strategie werden Modelle für verschiedene europäische Länder geschätzt. Basierend darauf wird im Hinblick auf die europäische Finanzkrise analysiert, welche Rolle die Stochastik der Ausfallwahrscheinlichkeit bei der Bildung von Kreditkosten für diese Länder spielt. Weiterhin wird in diesem Zusammenhang evaluiert, wie gut die Modellierung der Kreditkosten anhand von Kreditrisiko-Modellen reduzierter Form für diese Länder funktioniert und wie gut die geschätzten Modelle zur Prognose von Kreditwertpapierpreisen geeignet sind.

[1]Es sei hierbei erwähnt, dass es keinen direkten inhaltlichen Zusammenhang zwischen der Bezeichnung einer doppelt stochastischen Matrix und eines doppelt stochastischen Kreditrisiko-Modells gibt.

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Übersichtskapitel Dissertation

1.1 Einleitung

Die vorliegende Dissertation beschäftigt sich mit Kreditrisiko-Modellen "reduzierter Form" (Reduced Form Credit Risk Models) zur Analyse staatlicher Kreditrisiken. Es werden technische Grundlagen und die Funktionsweise dieser Modelle erörtert. Weiterhin wird im wahrscheinlichkeitstheoretischen Rahmen dargestellt, wie man anhand dieser Modelle analysieren kann, welche Rolle eine mögliche Stochastik der Kreditausfallswahrscheinlichkeit bei der Bildung von Kreditwertpapierpreisen spielt. Die Risiken, die auf der Stochastik der Ausfallwahrscheinlichkeit beruhen, werden hier Risiken der "zweiten Dimension" (second dimension risks) genannt. Eine Strategie zur Schätzung solcher Modelle anhand von Zeitreihendaten wird ebenfalls diskutiert und evaluiert. Anhand dieser Strategie werden Modelle für verschiedene europäische Länder geschätzt. Basierend darauf wird im Hinblick auf die europäische Finanzkrise analysiert, welche Rolle die Stochastik der Ausfallwahrscheinlichkeit bei der Bildung von Kreditkosten für diese Länder spielt. Weiterhin wird in diesem Zusammenhang evaluiert, wie gut die Modellierung der Kreditkosten anhand von Kreditrisiko-Modellen reduzierter Form für diese Länder funktioniert und wie gut die geschätzten Modelle zur Prognose von Kreditwertpapierpreisen geeignet sind.
Das vorliegende Einführungs-Kapitel soll einen Überblick über den Inhalt der Disseration geben, der über den obigen Umriss hinausgeht. Ich skizziere hierbei zu Beginn kurz, was Kreditrisiko-Modelle reduzierter Form eigentlich sind und fasse basierend darauf den Inhalt der einzelnen eigentlichen Dissertations-Kapitel zusammen. Dabei werden eigene Beiträge und deren Zusammenhang erläutert.

1.2 Kreditrisiko-Modelle reduzierter Form

Zur Modellierung von Kreditkrisen gibt es zahlreiche Herangehensweisen. Die beiden in der Literatur verbreitetsten Plattformen sind die erwähnten Kreditrisiko-Modelle "reduzierter", sowie Kreditrisiko-Modelle "struktureller" Form (vgl. Duffie and Singleton (2008)). Bei letzterer Modellklasse wird versucht, die finanzielle Situation eines Landes oder eines Staates (im folgenden allgemeiner als Einheit bezeichnet) detailliert im Modell abzubilden. Hierbei werden mehrere stochastische Prozesse für die verschiedenen Bestandteile einer Vermögensbilanz definiert. Der Insolvenzfall tritt in dieser Modellwelt dann ein, wenn die Prozesse, die die Verschuldung abbilden, in der Summe die "Aktiva" bzw. den allgemeinen Besitz überschreiten. Diese Modelle scheinen zur Modellierung von staatlichen Kreditrisiken nicht sonderlich gut geeignet, da Informationen hinsichtlich der finanziellen Situation von Staaten nicht so klar strukturiert sind wie bei öffentlich gelisteten Firmen.

In Modellen reduzierter Form, die auf Jarrow and Turnbull (1995), Lando (1998) und Duffie and Singleton (1999) zurückgehen, wird der Insolvenzprozess dem Namen entsprechend in reduzierterer Form modelliert: Erste Sprünge von Poisson-Prozessen sollen hier Kreditereignisse darstellen. Auf eine tiefergehende Abbildung der finanziellen Situation der Einheit wird verzichtet, die Modelle für verschiedene Einheiten unterscheiden sich lediglich in den Sprungintensitäten der jeweiligen Poisson-Prozesse. Die Wahrscheinlichkeit für einen Ausfall in einer bestimmten Periode entspricht dann der Wahrscheinlichkeit eines ersten Prozessprungs in dieser Zeit. Die Intensitäten bzw. die Intensitätsprozesse, die die Modelle für bestimmte Einheiten charakterisieren, können entweder als deterministisch oder stochastisch modelliert werden.

Im letzteren Fall werden die Modelle in der Regel als "doppelt stochastisch" (doubly stochastic) bezeichnet. Dabei werden die Intensitätsprozesse in der Regel als Diffusionsprozesse modelliert. Die Änderung eines Intensitätsprozesses wird also in jedem Zeitpunkt durch eine stochastische Differentialgleichung bestimmt. Diese setzt sich aus einem deterministischem und einem stochastischem Teil zusammen. Der stochastische Teil wird dabei von einer Brownschen Bewegung getrieben. Eine Spezifikationsmöglichkeit für diese stochastische Differentialgleichung ist eine sogenannte Wurzeldiffusionsgleichung. Gleichungen diesen Typs zeichnen sich für bestimmte Parameterwerte durch ein sogenanntes Rückkopplungsniveau aus: Ist der jeweilige Prozess oberhalb dieses Niveaus, wird der deterministische Teil der Gleichung negativ und es wird eine negative Änderung erwartet. Ist der Prozess unterhalb dieses Niveaus, ist das Gegenteil der Fall.

Innerhalb des Modellrahmens können relativ einfach Preise für Finanzprodukte hergeleitet werden, welche dem jeweiligen Kreditrisiko unterliegen. Dazu berechnet man anhand der Ausfallwahrscheinlichkeit Erwartungswerte bezüglich der sich aus dem Finanzprodukt ableitenden Zahlungsströme und diskontiert diese anhand der erwarteten Rendite. Hierbei spielen in der Regel zwei unbekannte Variablen eine Rolle: Die Ausfallintensität und die erwarteten Rendite. Daher ersetzt man häufig das letztere durch die sogenannte risikofreie Rendite. Dieses eigentlich hypothetische Konstrukt ist die Rendite, die ein Investor bekommt, wenn er temporär Geld zur Verfügung stellt ohne dabei Risiken einzugehen. Die Substitution der eigentlich erwarteten Rendite wird deshalb häufig durchgeführt, da es für die risikofreie Rendite eine ganze Reihe als geeignet erachteter Approximationen gibt. Kandidaten sind hierfür Renditen auf Investitionen, die mit - aus Sicht der Investoren - vernachlässigbaren Risiken behaftet sind. Häufig werden zum Beispiel Renditen deutscher oder amerikanischer Staatsanleihen verwendet.
Wenn sich die eigentlich erwartete Rendite von der risikofreien unterscheidet, bedarf es bei der Bepreisung einer weiteren Anpassung. Typischerweise wird die eigentliche Sprungintensität durch eine hypothetische "risikoneutrale" Intensität ersetzt. Diese ist eben genau so definiert, dass sie im jeweiligen Bepreisungsrahmen für das Substituieren der erwarteten Rendite durch die risikofreie Rendite korrigiert. Ihr Name geht darauf zurück, dass diese hypothetische Intensität als tatsächlich erwartete Intensität in Preisen impliziert wäre, wenn diese in einer Welt beobachtet würden, in der Investoren für das Eingehen von Risiken keinen Anstieg der Rendite erwarteten, sondern im Rahmen ihrer Investitionsentscheidung lediglich erwartete Zahlungsströme miteinander vergleichen, würde ohne die Varianz zu berücksichtigen.
Es bedarf im Falle einer stochastischen Ausfallintensität noch einer weiteren Anpassung, falls sich die erwarteten Renditen aufgrund dieser Stochastik verän-

dern. Dies kann bei der Bepreisung berücksichtig werden, indem neben dem "wahren" Wahrscheinlichkeitsmaß, das sich auf das tatsächliche Bewegungsgesetz der "risikoneutralen" Intensität bezieht, ein hypothetisches "risikoneutrales" Wahrscheinlichkeitsmaß eingeführt wird. Die bei der Berechnung der erwarteten Zahlungsströme implizierte Erwartungswertbildung hinsichtlich der "risikoneutralen" Intensität führt unter diesem hypothetischen riskoneutralem Maß zu erwarteten Auszahlungen, die nach der Diskontierung basierend auf der risikofreien Rendite mit dem Preis des jeweiligen Finanzprodukts übereinstimmen. Dieses risikoneutrale Maß entspricht in einer hypothetischen Welt, in der Investoren keine zusätzliche Entlohnung für Risiken der zweiten Dimension erwarten, dem wahren Wahrscheinlichkeitsmaß. Die Unterschiede in den beiden Maßen spiegelt sich wider in unterschiedlichen Koeffizienten der stochastischen Differentialgleichungen unter beiden Maßen. Basierend auf dieser Unterscheidung kann anhand geschätzter Modelle reduzierter Form analysiert werden, welchen Einfluss die Stochastik der Ausfallwahrscheinlichkeit auf erwartete Renditen beziehungsweise Preise von Finanzprodukten hat.

1.3 Thematik der einzelnen Papiere

Im ersten Papier dieser Dissertation werden die wahrscheinlichkeitstheoretischen und finanzwirtschaftlichen Grundlagen vorgestellt, die zum Verständnis und zur Einordnung der anderen beiden Kapitel notwendig scheinen. Diffusionsprozesse zur Modellierung der Intensitätsprozesse werden eingeführt und bestimmte Eigenschaften affiner Prozesse diskutiert (vgl. Duffie et al. (2003)). Letzteres spielt bei der Schätzung der Modelle anhand von Zeitreihendaten eine Rolle. Weiterhin wird im Zusammenhang mit der Bepreisung in diesem Modellrahmen die Diskontierung anhand erwarteter und risikofreier Rendite erläutert. Ein besonderer Fokus liegt weiterhin auf der getrennten Einbettung der erwähnten risikoneutralen Intensität und des risikoneutralen Maßes in den wahrscheinlichkeitstheoretischen Rahmen des Modells. Zudem wird genauer erörtert, wie

basierend darauf analysiert werden kann, welche Rolle die Stochastik der Ausfallwahrscheinlichkeit bei der Preisbildung von Kreditfinanzprodukten beziehungsweise bei der Bildung erwarteter Renditen spielt.
Eine ähnliche Analyse wurde schon in den Studien von Pan and Singleton (2008) und Longstaff et al. (2011) durchgeführt. Allerdings wurde hierbei auf eine ausführliche, getrennte Darstellung des erwähnten risikoneutralen Maßes und des risikoneutralen Intensitätsprozesses im wahrscheinlichkeitstheoretischen Rahmen verzichtet. Solch eine Darstellung scheint jedoch äußerst wichtig zu sein, um diese Artikel und zukünftige Forschungsprojekte dieser Art für Leser zugänglich zu machen, und um auf der Maßunterscheidung basierende Interpretationen (wie in den Papieren von Pan and Singleton (2008) und Longstaff et al. (2011)) wissenschaftlich zu rechtfertigen. Bisherige Artikel, die Modelle reduzierter Form im Allgemeinen diskutiern, auf eine wahrscheinlichkeitstheoretische Darstellung Wert legen und die mir bekannt sind, unterscheiden bei der risikoneutralen Betrachtung nicht zwischen dem Risiko eines Ausfalls, gegeben eine bestimmte Ausfallwahrscheinlichkeit, und dem Risiko hinsichtlich einer unerwarteten Entwicklung der Ausfallwahrscheinlichkeit. Dementsprechend geben sie keinen Aufschluss darüber, wie anhand solcher Modelle analysiert werden kann, ob es zu Änderungen in den erwarteten Renditen aufgrund von Unsicherheiten bezüglich der zukünftigen Ausfallwahrscheinlichkeit kommt. Die Einbettung des risikoneutralen Maßes in den gesamten wahrscheinlichkeitstheoretischen Modellrahmen, der eine isolierte Betrachtung des Risikos zweiter Dimension erlaubt, stellt daher einen eigenen sinnvollen Beitrag dar.
Die Änderung der Preise aufgrund der Stochastik der Ausfallwahrscheinlichkeit wird in dieser Dissertation als Premium für das Risiko der zweiten Dimension bzw. "second dimension risk premium" bezeichnet. Es hat sich bisher kein fester Begriff für diese Art des Risiko etabliert und diese Bezeichnung erscheint in Anbetracht des Modellaufbaus sinnvoll: Die Stochastik der Ausfallwahrscheinlichkeit impliziert im Modellrahmen neben der Unsicherheit, ob es gegeben einer bestimmten Intenstität zum Sprung bzw. Ausfall kommt, eine zweite Unsicherheitsebene aus Sicht der Investoren.

Das zweite Papier beschäftigt sich mit der Schätzung von Kreditrisiko-Modellen reduzierter Form anhand von Kreditversicherungs-Zeitreihendaten. Hierbei muss der risikoneutrale Intensitätsprozess und die Bewegungsgesetze dieses Prozesses (d.h. die Koeffizienten der stochastischen Differentialgleichungen) unter dem tatsächlichen und dem risikoneutralen Maß geschätzt werden. Darüber hinaus ist in diesem Modell die Schätzung des Anteils einer Anleihe notwendig, der im Insolvenzfall noch zurückgezahlt wird. Ich diskutiere eine Schätzstrategie für Modelle, in denen die Diffusionsprozesse zur Modellierung von Intensitäten "affin" sind. Hierbei werden bestimmte Eigenschaften "affiner" Prozesse, welche im ersten Papier der Dissertation vorgestellt werden, bei der Erwartungswertbildung bezüglich Transformationen zukünftiger Intensitätswerte ausgenutzt. Weiterhin wird ausgenutzt, dass auf täglicher Basis Prämien für Kreditversicherungen verschiedener Versicherungshorizonte verfügbar sind.
Es werden in einem ersten Schritt die Parameter unter dem risikoneutralen Bewegungsgesetz ex-ante bestimmt und für einen einzelnen Versicherungshorizont wird eine Intensitätszeitreihe geschätzt (vgl. Longstaff et al. (2005)). Die Schätzung erfolgt anhand eines Vergleichs der echten Versicherungspreise für den bestimmten Versicherungshorizont mit den entsprechenden Modellversicherungspreisen, welche auf den zuvor ex-ante bestimmten Parameterwerten des risikoneutralen Bewegungsgesetzes basieren. Die Schätzwerte für die risikoneutrale Intensität werden so gewählt, dass sich Modellpreise und beobachtete Preise für diesen einen Versicherungshorizont zu jedem Beobachtungszeitraum entsprechen.
Die sich hieraus ergebende Intensitätszeitreihe wird dann verwendet, um die zuvor frei bestimmten Parameter neu zu schätzen. Dabei wird für die anderen verfügbaren Versicherungshorizonte die Differenz zwischen Modell- und echten Versicherungspreisen gegeben der Intensitäten und bezüglich dieser Parameter unter dem risikoneutralem Maß minimiert. Beide Schritte (d.h. Schätzung der Intensitäten und Schätzung der Parameter unter dem risikoneutralem Maß) werden so lange durchgeführt, bis in beiden Fällen Konvergenz eintritt. An-

schließend werden basierend auf der geschätzten Intensitätszeitreihe die Parameter unter dem wahren Wahrscheinlichkeitsmaß geschätzt. Mögliche für diesen Schätzschritt in Frage kommende Methoden wie Maximum Likelihood oder die Methode der kleinsten Quadrate werden hierbei diskutiert. Die Strategie zur Schätzung der Parameter unter dem risiokneutralen Maß und der Zeitreihe des Intensitätsprozesses wird ebenso wie die in Frage kommenden Methoden für die Schätzung der Parameter unter dem wahren Wahrscheinlichkeitsmaß anhand simulierter Daten überprüft.

Die Schätzmethodik basiert zwar auf Kreditversicherungsdaten. Das geschätzte Modell kann jedoch für die Bepreisung sämtlicher Finanzprodukte verwendet werden, deren Auszahlungen mit Kreditrisiken behaftet sind, da in erster Linie das Ausfallrisiko selbst und nicht der Preis für das jeweilige Finanzprodukt modelliert wird. Basierend auf der Modellierung des Kreditrisikos können dann wiederum Preise für alle möglichen Finanzprodukte hergeleitet werden, von deren Besitz sich mit dem jeweiligen Kreditrisiko behaftete Zahlungsströme ableiten. Die Erwartungswerte bezüglich dieser Zahlungsströme werden unter Berücksichtigung des modellierten Ausfallrisikos gebildet.

Mir sind Artikel bekannt, die Schätzergebnisse für solche Modelle unter beiden Massen präsentieren (nämlich die erwähnten Papiere von Pan and Singleton (2008) und Longstaff et al. (2011)). Diese Papiere gehen jedoch nicht im Detail auf die angewendete Schätzmethodik ein. Dennoch lässt sich sagen, dass sich die hier diskutierte Schätzmethodik von der in diesen Papieren angewendeten unterscheidet, da in diesen Papieren die zur Modellierung gewählten Diffusionsprozesse nicht affin sind. Die in dieser Dissertation diskutierte Schätzstrategie basiert jedoch - wie schon erwähnt - auf den Eigenschaften affiner Prozesse und orientiert sich dabei an Longstaff et al. (2005). Dabei wird ausgenutzt, dass die in Bepreisungsformeln enthaltenen Erwartungswerte über bestimmte Transformationen zukünftiger Intensitätswerte substitutiert werden können. Hierzu müssen, wenn den Modellen affine Prozesse zugrunde liegen, lediglich gewöhnliche Differentialgleichungen gelöst werden, für welche nicht nur numerische, sondern auch analytische Lösungen verfügbar sind. Dies ermöglicht im Vergleich zu einem Verfahren, das ausschließlich auf numerischem Lösen der entsprechenden gewöhnlichen Differentialgleichungen beruht, ein bezüglich der Rechenkapazität weniger anspruchsvolles Vorgehen.

Im dritten Papier dieser Dissertation wende ich die vorgestellte Schätzstrategie an und schätze Kreditrisiko-Modelle reduzierter Form für sechs europäische Länder. Die Schätzung basiert dabei auf Daten aus den Jahren 2008 bis 2012. Dieser Zeitraum war geprägt von starken Schwankungen der Kreditkosten diverser europäischer Länder. In Grafik 1.1 werden die Kreditkosten für einige Beispiel-Länder und einen auch die Jahre vor der europäischen Fiskalkrise umfassenden Zeitraum dargestellt. Grafik 4.12 zeigt die Kreditkosten für die in der Stichprobe enthaltenen Länder beziehungsweise Zeitpunkte. Die starke Schwankungen sind hierbei genau wie ähnliche Schwankungsmuster verschiedener Länder augenscheinlich. Von denen in die Studie einbezogenen Ländern sind zwei (Island und Polen) nicht Mitglieder des Euro-Währungsraums, die restlichen (Spanien, Irland, Estland und Finnland) dagegen schon. Vier dieser Länder hatten mit stark überdurchschnittlichen Kreditkosten während Teilen des Beobachtungszeitraums zu kämpfen: Irland, Spanien, Estland und Island. In den letzten beiden Fällen war dabei die Rückentwicklung der Kosten in der zweiten Hälfte des Stichprobenzeitraums zu verzeichnen, wohingegen die irischen und spanischen Kreditkosten erst in der zweiten Hälfte stark anstiegen.
Anhand der geschätzten Modelle analysiere ich den im ersten Papier präsentierten Überlegungen entsprechend, welche Rolle die Stochastik der Ausfallwahrscheinlichkeit im Stichprobenzeitraum bei der Bildung dieser Kreditpreise gespielt hat. Hintergrund für diese Untersuchung ist die europäische Finanzkrise, die ebenfalls in den Beobachtungszeitraum fällt und in deren Zusammenhang vor allem die Anstiege der Kreditkosten Spaniens und Irlands zu sehen sind. Die Determinanten staatlicher Kreditkosten sind allgemein und vor allem im Kontext der Finanzkrise noch relativ wenig erforscht. Die bisherigen Forschungsergebnisse ohne Bezug zur Finanzkrise suggerieren, dass globale Finanzmarktrisikomaße, wie der VIX Index, der sich aus impliziten Varianzen von Aktien aus

dem S&P 500 Index zusammensetzt, eine außerordentlich starke Erklärungskraft bezüglich staatlicher Kreditkosten haben. Die wenigen Untersuchungen dieser Art, deren Datenauswahl beziehungsweise deren Interpretation einen Bezug zur europäischen Finanzkrise haben, bestätigen das für die jeweiligen Länder und Jahre hingegen nicht. Allerdings ist relativ unklar, welche Treiber stattdessen relevant waren.

Das Risko zweiter Dimension könnte eine entscheidende Rolle gespielt haben. Zum einen spielt dieses Risiko bei der Kreditpreisbildung laut der Ergebnisse von Pan and Singleton (2008) und Longstaff et al. (2011) im Falle der diesen Studien zu Grunde liegenden Länder eine große Rolle. Dementsprechend könnte dies generell und eben auch bei europäischen Ländern der Fall sein. Zum anderen gibt es Grund zur Annahme, dass gerade im Euro-Raum die Unsicherheit bezüglich zukünftiger Ausfallwahrscheinlichkeiten gewachsen ist. So könnten im Zusammenhang mit der Fiskalkrise entstandene Zweifel am langfristigen Bestehen des Währungsraumes dazu geführt haben, dass die zukünftigen Determinanten der fiskalischen Lage in den jeweiligen Ländern als unsicherer beurteilt werden. Ein mögliches Auseinanderbrechen des noch jungen Währungsgebildes hätte wirtschaftlich und somit auch fiskalisch unübersehbare Folgen. Weiterhin könnte die Unsicherheit bezüglich der Qualität noch junger europäischer Institutionen, die in der kurzen Zeit ihres Bestehens viele weitreichende Entscheidungen getroffen haben, aus in Kapitel drei diskutierten Gründen gestiegen sein. Da die Handlungen dieser Institutionen wiederum direkte Auswirkungen auf die fiskalische Situation der europäischen Mitgliedsländer haben, sollte die wachsende Unsicherheit auch einen Anstieg der Unsicherheit hinsichtlich der zukünftigen fiskalischen Lage dieser Länder implizieren.

Des Weiteren könnte die Aufnahme in den Euro-Raum als positives Signal hinsichtlich der Güte fiskalischer Informationen gewertet worden sein. Die starke Korrektur griechischer Fiskaldaten im Jahr 2009 könnte der Aufnahme in den Euro-Raum in den Augen der Marktteilnehmer diese positive Signalkraft genommen haben und dadurch Unsicherheit hinsichtlich der Qualität der fiskalischen Informationen und dadurch zukünftiger Ausfallwahrscheinlichkeiten von Euro-

Ländern induziert haben.
Sowohl die Analyse, ob das Risiko zweiter Dimension bei der Bildung der staatlichen Kreditpreise im Falle dieser Länder generell eine große Rolle gespielt hat, und ob sich hierbei Euro-Länder von Ländern außerhalb des Euro-Raums unterscheiden, als auch die Analyse, ob die Schwankungen in den Kreditkosten Spaniens und Irlands stärker als Anstiege in den Kreditkosten anderer Länder mit Änderungen der Prämie auf das Risiko der zweiten Dimension assoziiert werden kann, scheint basierend auf diesen Überlegungen sinnvoll zu sein und wird durchgeführt.
Weiterhin wird überprüft, was die Korrelation der Prämien auf das Risiko der zweiten Dimension über die Korrelation der Kreditkosten zwischen zwei Ländern aussagen können. Hintergrund hierfür ist, dass die Korrelation der Spreads durch Korrelation dieser Kreditaufschläge induziert werden könnte, falls diese Kreditaufschläge tatsächlich existieren. Die Korrelation dieser Kreditaufschläge könnte zudem dann auftreten, wenn die Aufschläge verschiedener Länder durch dieselben Faktoren getrieben werden. Das könnte zum einen Risikoaversion selbst sein, aber auch Faktoren, die für die Unsicherheit bezüglich der zukünftigen Ausfallwahrscheinlichkeit verschiedener Länder eine Rolle spielen. Ein möglicher Faktor könnte (wie bereits skizziert) die Reputation von Institutionen sein, deren Entscheidungen auf die fiskalische Situation mehrerer Länder einen Einfluss haben. Die Frage ob das Premium auf Risiken zweiter Dimension Korrelationen zwischen den Kreditkosten verschiedener Länder induziert, und ob hierbei die Zugehörigkeit eines Landes zum Euro-Währungsraum eine Rolle spielt, wird dementsprechend analysiert.
Darüber hinaus evaluiere ich im Rahmen der in diesem Modell präsentierten Ergebnisse, wie gut die Modellierung des Kreditrisikos beziehungsweise der Kreditversicherungskosten innerhalb des Stichprobenzeitraums funktioniert. Zudem analysiere ich, wie gut anhand des Modells Kreditversicherungspreise außerhalb des Stichprobenzeitraums vorhergesagt werden können. Eine solche Evaluation der Vorhersagekraft von Kreditrisiko-Modellen reduzierter Form ist mir noch nicht bekannt.

1.4 Überblick

Die Papiere dieser Dissertation sind deshalb in der präsentierten Art und Weise angeordnet, da das erste Kapitel die Grundlagen vermittelt, die zum Verständnis des zweiten und dritten Kapitels notwendig sind, und die im zweiten Kapitel beschriebene Schätzmethodik im dritten Kapitel angewendet wird. Die Kapitel sind der Idee des "kumulativen" Formats entsprechend so verfasst, dass sie eigenständig gelesen werden können.

Die Beiträge der Dissertation sind: Das Zusammenstellen der wichtigsten technischen Grundlagen dieser Modelle, das Einbetten der zur Analyse der Prämie auf das Risko zweiter Dimension notwendigen Maßunterscheidung in den wahrscheinlichkeitstheoretischen Rahmen dieses Modells, das Diskutieren einer Schätzstrategie für diese Modellklasse anhand von Zeitreihendaten, die Anwendung der Schätzstrategie beziehungsweise der erörterten Interpretationsmöglichkeit für europäische Staatskreditkosten, das Herausarbeiten des möglichen Zusammenhangs zwischen dem Risiko zweiter Dimension und der europäischen Finanzkrise, die Diskussion inwieweit das Risiko zweiter Dimension im europäischen Rahmen Korrelation von staatlichen Kreditkosten verursacht haben könnte und die Evaluation der Modellgüte sowie der Vorhersagekraft dieser Modell bezüglich staatlicher Kreditkosten.

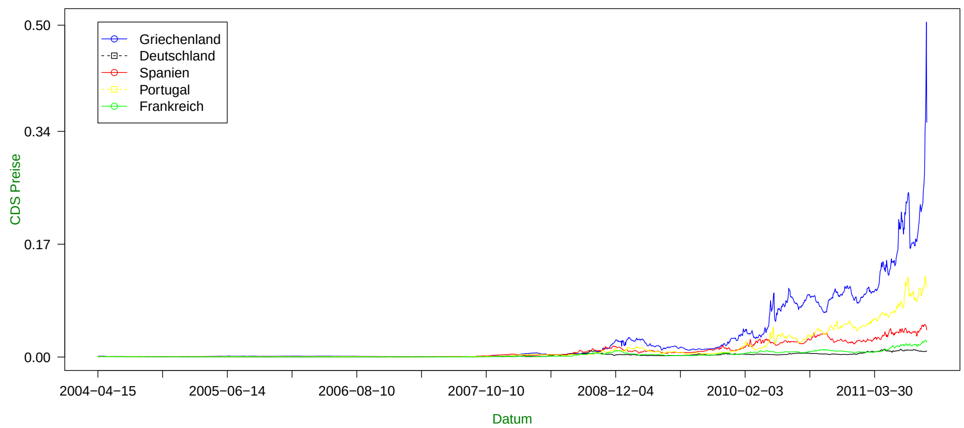

Figure 1.1: Kreditkosten diverser europäischer Länder

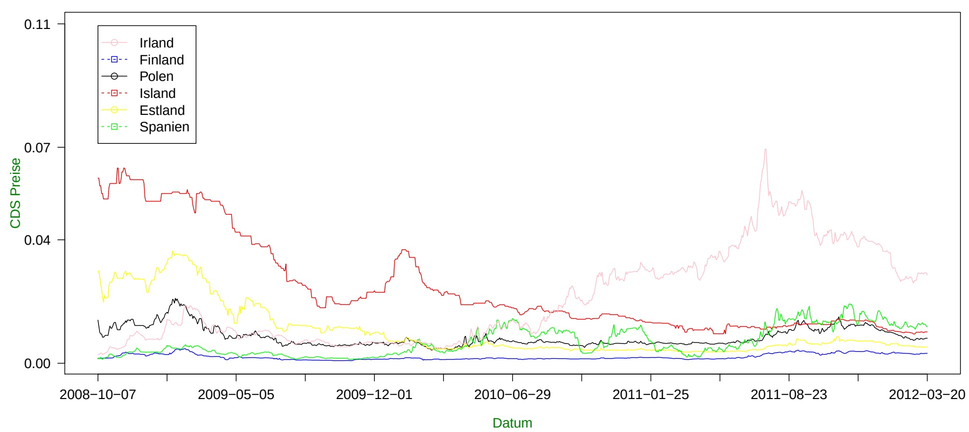

Figure 1.2: Kreditkosten der in der Stichprobe enthaltenen Länder

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Chapter 2

Doubly Stochastic Reduced Form Credit Risk Modelling, Affine Processes and Risk Neutral Measures – a Technical Toolkit for Sovereign Credit Risk Analysis

2.1 Introduction

This chapter presents the technical foundations of reduced form credit risk models. The "second dimension" risk premium, which refers to uncertainties regarding underlying default probabilities, is introduced in a reduced form framework. I show how the relevance of this risk premium can be quantified based on such models. The model setting and the "second dimension" risk premium are applied to credit default swaps (CDS) as an example for credit securities. The reason for this focus on CDS is twofold: First, the pricing of a CDS contract implies the pricing of different kinds of payment streams. Therefore, the results can be transferred to a whole range of other credit securities. Second, for several reasons CDS data is an attractive candidate for the estimation of a reduced form model (c.f. Pan and Singleton (2008) or Longstaff et al. (2005)). These reasons are discussed in the second chapter of this dissertation.

The present chapter is structured as follows: the next section presents the basic idea of reduced form credit risk modelling and introduces the probability theoretical framework. The third section discusses the Cox-Ingersoll-Ross (CIR) (c.f. Cox et al. (1985)) diffusion equations as one possible choice for modelling the default probability. The fourth section introduces credit default swaps and shows, how the established framework can be used to derive pricing formulas based on the concept of risk neutrality. The fifth section discusses the second dimension risk in the context of the established framework referring to CDS contracts. The sixth section finally introduces features of the affine process class (to which the CIR processes belongs). These features are very useful for estimating reduced form pricing models.

2.2 Basic ideas

There are various ways to statistically model the default risk of a financial unit. One strand in the literature is characterized by a detailed analysis of the credit taker's balance sheet. To precisely model single components of a unit's financial situation, authors typically distinguish between a process driving the asset side and a process driving the liabilities of the respective firm. Models of that kind are called “structural form models” and seem to be a plausible choice mainly in the non-sovereign context. The reason for this restriction is the required access to precise information on the financial situation of a unit. Such information is more likely to be available for listed companies reporting to the public.

Another very common approach to statistical modelling of default risk – which requires less detailed accounting information – is to model a default as a jump of a stochastic (Poisson) process. This implies that default time is viewed as the stopping time of that process. A helpful feature of this class of “reduced form” models is the direct link between the underlying Poisson parameter and the default probability. The basic set up is introduced in the following paragraphs. This overview builds on Duffie and Singleton (2008), Duffie (2005) and Duffie (1999).

To establish the **basic setting of a reduced form model** a measure space $(\Omega_1, \mathcal{F}_1, P_1)$ with the corresponding filtration $\mathcal{F}_{1,s}$, a measurable space $(M_1, \mathcal{M}_1)$ and an index set $S \neq \emptyset$ be defined. In addition, a Poisson process

$$\mathcal{P}oi = (Poi_s, s \in S) \tag{2.1}$$

is defined as a family of measurable mappings between probability and measure space:

$$Poi_s : (\Omega_1, \mathcal{F}_1, P_1) \to (M_1, \mathcal{M}_1) \tag{2.2}$$

$$\omega_1 \mapsto Poi_s(\omega_1) \tag{2.3}$$

with $\omega_1 \in \Omega_1$. Poi_s counts the number of *events* up to time s. In the present case, $Poi_s = 1$ means that a credit event has already occurred at time s, while $Poi_s = 0$ denotes that it has not. The increments $Poi_{s_1} - Poi_{s_0}$ are for $s_0, s_1 \in S$ and $s_1 - s_0 \geq 0$ independently Poisson distributed, the Poisson parameter depends on the length of the respective period $[s_0, s_1]$ only and Markov property is satisfied accordingly. At the first point in time, the process value be almost surely zero and the process be supported by the probability space introduced above. The intensity parameter of this Poisson process is denoted by λ_s with $s \in S$. The probability distribution $Pr^{Poi}(Poi_{s_0+t} = 0|Poi_{s_0} = 0)$ of the process value in $[s_0, s_0+t] \subset S$ conditioned on $Poi_{s_0} = 0$ is accordingly given by the poisson probability distribution $POI(j|ev)$ for $j = 0$ with ev denoting the expected value. This implies in closed form:

$$Pr^{Poi}(Poi_{s_0+t} = 0|Poi_{s_0} = 0) = POI(j = 0|ev = \lambda_{s_0,s_0+t}) = e^{-\lambda_{s_0,s_0+t}}. \quad (2.4)$$

This implies in turn (as the default time denoted as $\tau \in S$ is in this context also stopping time for Poi_s[1]) that

$$Pr^{Poi}(Poi_{s_0+t} > 0|Poi_{s_0} = 0) = 1 - e^{-\lambda_{s_0,s_0+t}}. \quad (2.5)$$

If λ_s is constant for all $s \in [0,t]$, one can rewrite $\lambda_{s_0,s_0+t} = \lambda_{\widehat{t}} \times t$ for all $\widehat{t} \in [s_0, s_0+t]$. For non constant λ_s, one rewrites

$$\lambda_{s_0,s_0+t} = \int_{s_0}^{s_0+t} \lambda_s ds. \quad (2.6)$$

The filtration $\mathcal{F}_{1,s}$ is generated by realizations of the underlying process $\mathcal{P}oi$ prior to time s:

$$\mathcal{F}_{1,s} = \sigma\{Poi_t : 0 \leq t \leq s\}. \quad (2.7)$$

So far, the intensity has been assumed to be deterministic. This does not seem to be plausible for real world applications. Therefore a **second stochastic dimension** is added and diffusion equations are introduced as stochastic drivers of the default intensities. Diffusion equations are stochastic differential equations characterized by a specific functional form, which will be introduced in

[1]It is assumed that the model holds only up to the first credit event.

detail later. A Poisson process with stochastic intensity is called "Cox" process and the framework then becomes "doubly stochastic" (c.f. Duffie and Singleton (2008)).

To introduce this "second stochastic dimension" in the model set up, a probability space $(\Omega_2, \mathcal{F}_2, P_2)$ with corresponding filtration $\mathcal{F}_{2,s}$ and a measurable space $(M_2, \mathcal{M}_2)$ with $M_2 \subseteq \mathbb{R}^n$ for $n \in \mathbb{N}^+$ denoting a multivariate state vector be defined. The index set $S \neq \emptyset$ is still the same as in the subsection before. Finally, a Brownian motion $B_s \in \mathbb{R}^n$ and the following "diffusion" process $\mathcal{Y} = (Y_s, s \in S)$ is defined as a family of measurable mappings between probability and measure space:

$$Y_s : (\Omega_2, \mathcal{F}_2, P_2) \to (M_2, \mathcal{M}_2) \tag{2.8}$$

$$\omega_2 \mapsto Y_s(\omega_2). \tag{2.9}$$

Y_s be moreover distinguished by the family of transition probability laws $Pr^{\mathcal{Y}}(Y_{s_0+t}|Y_{s_0+t-1}, .., Y_{s_0})$ and satisfies the Markov law, i.e.

$$Pr^{\mathcal{Y}}\left(Y_{s_0+t} = m_{2,s_0+t}|Y_{s_0+t-1} = m_{2,s_0+t-1}, Y_{s_0+t-2} = m_{2,s_0+t-2}, \cdots, Y_{s_0} = m_{2,s_0}\right) \tag{2.10}$$

$$= Pr^{\mathcal{Y}}\left(Y_{s_0+t} = m_{2,s_0+t}|Y_{s_0+t-1} = m_{2,s_0+t-1}\right) \tag{2.11}$$

with $s_0, s_0+1, \cdots, s_0+t \in S$, $t \geq 2$ and $m_{2_{s_0}}, m_{2,s_0+1}, \cdots, m_{2,s_0+t} \in M_2$ with $\mathcal{F}_{2,s_0} \subseteq \mathcal{F}_{2,s_0+1} \subseteq \cdots \subseteq \mathcal{F}_{2,s_0+t}$. Intuitively, one can say that the filtration $\mathcal{F}_{2,s_0}$ – containing the information provided by all realization of Y_s up to time $s_0 \in S$ – does not provide more information on the future development of Y_s than the single realization of Y_{s_0}.

The change in the process is moreover determined by a stochastic differential equation of the following form:

$$dY_s = \mu_{Y_s} ds + \sigma_{Y_s} dB_s \tag{2.12}$$

with $\mu : M_2 \to \mathbb{R}^n$ and $\sigma : M_2 \to \mathbb{R}^{n\times n}$. The change in the "diffusion" process Y_s is therefore explained by a deterministic part consisting of a so called drift parameter μ_{Y_s}, which is weighted by the respective time horizon, and a stochastic part. The stochastic component is driven by the change in the previously introduced Brownian motion B_s. The diffusion process Y_s is the solution to the stochastic differential equation of the diffusion type.

In the doubly stochastic framework, the intensity λ_s is assumed to depend on the "state vector" Y_s in affine (i.e. linear) form:

$$\lambda_s = \tilde{\rho}_0 + \tilde{\rho}_1 Y_s, \tag{2.13}$$

with $\tilde{\rho}_0 \in \mathbb{R}^1$ and $\tilde{\rho}_1 \in \mathbb{R}^n$. In the most simple and therefore most frequently applied case, the state vector is one dimensional, respectively $Y_s = \lambda_s$. λ_s itself is then the only state variable driven by the underlying diffusion equation. This implies $Y_s \in \mathbb{R}$ and one dimensionality of both the drift and the diffusion coefficients in the underlying stochastic differential equation. The present introduction includes more general, i.e. multi-variate cases.

2.3 The Cox-Ingersoll-Ross model

The set of possible specifications of a diffusion equation – i.e. the functional forms the coefficients μ_{Y_s} and σ_{Y_s} are assumed to be defined by – is rather large. There are two special specifications which are very frequently applied in Quantitative Finance: the class of Gaussian models and the square root "CIR"-model presented by Cox et al. (1985). These cases have a rather simple form and are particularly popular for short term interest rate modelling. The simplicity of these models is based on the linear functional link between the drift parameter and the current process value respectively the square product of the diffusion parameter and the current process value. One can therefore rewrite $\sigma^2_{Y_s}$ as

$$\sigma^2_{Y_s} = \sigma^2_0 + \sigma^2_1 Y^2_s \tag{2.14}$$

with $(\sigma_0, \sigma_1) \in \mathbb{R}^N \times \mathbb{R}^{N \times N}$ and μ_{Y_s} as

$$\mu_{Y_s} = \mu_0 - \mu_1 Y_s \tag{2.15}$$

with $(\mu_0, \mu_1) \in \mathbb{R}^N \times \mathbb{R}^{N \times N}$. Another popular representation of the drift coefficient is

$$\mu_{Y_t} = (\Lambda)(\Theta - Y_t)dt \tag{2.16}$$

where $(\Theta, \Lambda) \in \mathbb{R}^N \times \mathbb{R}^{N \times N}$.

The coefficient σ_1^2 is restricted to zero in the Gaussian case and σ_0^2 is restricted to zero for CIR diffusion processes. The drift parameter μ_{Y_s} is indentical for both cases and the functional form $\mu_{Y_s} = \mu_0 - \mu_1 Y_s$ brings along a reversion mechanism for certain parameter ranges. A helpful feature of both diffusion process types, is that the family of transition distributions is known in closed form.

Vasicek (1987) introduced the univariate case of the Gaussian diffusion process to model short term interest rates. However, in this model there is a positive probability of a negative realization of the underlying variable if $\sigma_0^2 \neq 1$. The CIR diffusion process is, on the other hand, not defined for negative process values because the square root of the current process value is part of the diffusion coefficient. The process is, moreover positive, for a wide range of parameter values. For intensity modelling, the CIR diffusion process seems to be the more plausible choice for intensity modelling. For this reason, the further discussion will focus on this diffusion type. Diffusion processes of the CIR type were originally presented by Feller (1951) to model demographic developments and were adopted by Cox et al. (1985) to model short term interest rates. The term "Feller diffusion" is therefore frequently used as well. However, the Quantitative Finance literature refers mostly to "CIR" diffusions.

As mentioned before, the CIR process is only defined for non-negtive process values. Moreover is the process prevented from becoming non-negative for $i)$ $\mu_0 > 0$

and *ii*) $\mu_1 > 0$. Then, the stochastic differential equation also has a "unique strong solution"[2] for every starting point Y_0 (Overbeck and Ryden (1997)) and the conditional distribution of Y_t approaches the gamma distribution for large t (Cox et al. (1985)). A CIR process satisfying the "Feller"-condition *iii*) $2\mu_0 > \sigma_1^2$ is also strictly positive (Feller (1951)). For *iv*) $0 < \mu_0 < \sigma_1^2$, the zero bound can be reached, but it is directly reflecting (Overbeck and Ryden (1997)) because the diffusion coefficient σ_1 tends to zero when the process values approaches zero. The change in the process then becomes deterministic with the mean reverting drift part being the only relevant determinant. The zero bound is, moreover, "absorbing" (Overbeck and Ryden (1997)) for $\mu_0 = 0$. For $\mu_0 < 0$, the process is "pushed" out of the defined domain ($(\mathbb{R}^+)^N$). This makes CIR diffusions with negative drift coefficients a rather abstract concept and they will not be discussed in this section.

For CIR processes, satisfying conditions *i*), *ii*) as well as condition *iv*) or condition *iii*), the probability distribution of the process value conditioned on a previous value is known in closed form (Cox et al. (1985)). The possibilities to represent and to numerically implement the probability distribution of a CIR process are presented in the following paragraphs for the univariate case.

The transition distributions of CIR processes are of the non-central χ^2-type. As described in Johnson et al. (1995a) (chapter 29), the probability density function $Pr(.|nc, dof)$ of non-central χ^2-distributed variables, with nc denoting the non-centrality parameter and dof denoting the degrees of freedom, can be expressed in the following way: an infinite sum of cumulative central-χ^2 probability distributions $Pr'(.|dof')$ is weighted by a Poisson probability distribution $POI(.|ev)$ with an expected value ev of $\frac{1}{2}$ times the noncentrality parameter nc. For a realization Y_{s_0} one can accordingly write:

$$Pr(Y_{s_0}|nc, dof) = \sum_{j=0}^{\infty} Pr'(Y_{s_0}|dof' = dof + 2j) POI(j|ev = \frac{1}{2}nc) \qquad (2.17)$$

[2]This means: $\mathbb{E}\left[\int_{s_0}^{s_0+t} |Y_s^2| ds < \infty\right]$ for all $s \in [s_0, s_0 + t]$ with $s \in S$ (c.f. Oksendahl (2003) or Iacus (2008)).

which, according to Johnson et al. (1995a), complies with

$$Pr(Y_{s_0}|nc, dof) = e^{-(nc+Y_s)/2}\frac{1}{2}\frac{Y_{s_0}}{nc}^{(dof-2)/4} Be_{\frac{dof}{2}-1}(\sqrt{Y_{s_0}nc}) \tag{2.18}$$

where $Be_{\frac{dof}{2}-1}$ denotes the modified Bessel function of the first kind of order $\frac{dof}{2} - 1$.

Using these relationships, Overbeck and Ryden (1997) and Iacus (2008) derive closed form representations for the conditional probability distribution of a CIR process realization Y_{s_0}. This representation depends on the underlying CIR coefficients and for the distribution being conditioned on a specific previous realization Y_{s_0-t} with $[s_0 - t, s_0] \subset S$ they define:

$$nc_t = \frac{2\mu_1}{\sigma_1^2\,(1 - e^{-\mu_1 t})} \tag{2.19}$$

$$dof = \frac{4\mu_0 \exp^{-\mu_1 t}}{\sigma_1^2(1 - \exp^{-\mu_1 t})} \tag{2.20}$$

and present based on that:

$$Pr(Y_{s_0}|Y_{s_0-t}, \mu_0, \mu_1, \sigma_1^2) = \frac{-0.5\frac{Y_{s_0-t}nc_t}{\exp^{-\mu_1 t}} + y_s nc_t}{2(y_s nc_t Y_{s_0-t})^{dof/4}} Be_{\frac{dof}{2}-1}(\sqrt{Y_{s_0-t}nc_t}). \tag{2.21}$$

As discussed in Zhou (2000) and Iacus (2008), the Bessel function included in the distribution function formula can be numerically difficult to handle in certain scenarios. Iacus (2008) therefore suggests a numerical approximation of the Bessel function, which is implemented in the statistical programming language R. This was done for the numerical applications in the third chapter if this dissertation, the results were compared with the non-scaled implemented version and no difference was detected.

However, simulation based maximum-likelihood estimations of the CIR processes in the second chapter of this dissertation generally turned out to be rather imprecise. The reason for this might be that the numerical approximation of the Bessel function is rather imprecise. The parameter values, which are particularly relevant in the reduced form credit risk context in combination with the small values of default intensities lead to the evaluation of the Bessel function in rather steep areas.

Therefore, an alternative representation of the χ^2-distribution without the Bessel function is often used: Zhou (2000) presents a mixing of Poisson- and Gamma-distributions as representation of a CIR probability distribution. For this representation, the fact that the central χ^2-distribution complies with a Gamma-distribution with shape parameter $v = dof/2$ and scale parameter $z = 2$ (Johnson et al. (1995b), p.437), is exploited. This leads to

$$\begin{aligned} &Pr(Y_{s_0}|Y_{s_0-t}, \mu_0, \mu_1, \sigma_1^2)) \\ &= \sum_j^{\infty} Gamma\left[cY_{s_0-t}|v = j + \frac{2\mu_0}{\sigma_1^2} - 1, Z = 1\right] POI\left[j, ev = cY_{s_0}e^{\mu_1}\right] \end{aligned} \tag{2.22}$$

with

$$c = \frac{-2\mu_1}{\sigma_1^2 - 1} \tag{2.23}$$

where $Gamma(.|v, z)$ denotes the Gamma probability distribution. Simulations in Zhou (2000), however, show that numerical applications based on this representation are particularly troublesome for certain scenarios: The resulting maximum-likelihood estimation results seem to be very imprecise, compared to results based on the representation in formula 3.17, if the underlying process has a high level of persistence. Unfortunately, the estimation results in the third chapter of this dissertation suggest a unit-root like behavior of intensity processes for different sovereign cases. Based on theses results, the first implementation approach therefore seems to be the superior choice in the reduced form model context – despite possible difficulties in numerically implementing the Bessel function.

For univariate cases, the conditional first two moments are known in closed form. The respective formulas for the conditional expectations and the conditional variance can be found in Cox et al. (1985) or in Iacus (2008):

$$\mathbb{E}\left(Y_{s+t}|Y_s\right) = \frac{\mu_0}{\mu_1} + \left(Y_s - \frac{\mu_0}{\mu_1}\right) e^{-\mu_1 t} \tag{2.24}$$

$$Var\left(Y_{s+t}|Y_s\right) = Y_s \frac{\sigma_0^2 \left(e^{-\mu_1 t} - e^{-2\mu_1 t}\right)}{\mu_1} + \frac{\mu_0 \sigma_0^2 \left(1 - e^{-2\mu_1 t}\right)}{2\mu_1^2} \tag{2.25}$$

$$Cov\left(Y_{s+t_1}, Y_{s+t_2}|Y_s\right) = Y_{s_0} \frac{\sigma_0^2}{2\mu_1} e^{-\mu_1(t_1+t_2)} \left(e^{2\mu_1 t_2} - 1\right) \tag{2.26}$$

for $t_2 \geq t_1$. The conditional expectations are linear in y_s and the coefficient multiplied with Y_s is $\exp^{-\mu_1 t}$. This reflects a stronger persistence of the process for weak mean reversion. Moreover, the level of the conditional variance is proportional to σ_0^2 and the persistence of the conditional variance increases with μ_1/σ_0^2.

2.4 Pricing formulas in the reduced form framework

For the derivation of pricing formulas, the filtration $\mathcal{F}_{2,s}$ needs to be specified in more detail, similar to $\mathcal{F}_{1,s}$. It is the σ-algebra generated by the realization of the diffusion process $\mathcal{Y}$ prior to s:

$$\mathcal{F}_{2,s} = \sigma\{Y_t : 0 \leq t \leq s\} \tag{2.27}$$

So far, two different probability spaces have been introduced: one referring to stochastic movement in the underlying intensity λ_s and one directly referring to the random jumps of the Poisson process. Both probability spaces are now combined to a single one. This is necessary for the calculation of expected values, which depend both on possible jumps given certain jump intensities, and on the future (stochastic) developments of the underlying intensity. A new sample space $\Omega = \Omega_1 \times \Omega_2$, a new sigma algebra $\mathcal{F} = \sigma\{\mathcal{F}_1 \vee \mathcal{F}_2\}$[3] and the respective filtration $\mathcal{F}_s$ are introduced. Moreover, a probability measure P is introduced which satisfies all general requirements regarding probability measures with respect to $\mathcal{F}$ and $\mathcal{F}_s$, i.e.: $P(\Omega) = 1$, $P(F) < \infty$ for all $F \in \mathcal{F}$ as well as countable additivity for disjoint collections (c.f. Davidson (1994)).

Based on this framework, pricing formulas for future payoffs, which depend on the respective credit risks, are now derived. This can be used to deduce pricing formulas for credit securities. One important input for net present values[4], which will be used for deriving pricing formulas, is still missing: the discount rate r_s. The product of discount rate and the expected payoff complies with the current value of this payoff claim. It is basically the return which investors require to get for an investment over a certain period of time.
If a unit lends the amount a for two years in a riskfree world at time $s = s_0$ and expects a single interest payment b after one year and the discounting occurs only once per year, the expected return r - assuming constancy of r_{s_0,s_0+1} on a annual basis - is given by the following equation (c.f. e.g. Dantine and Donaldson (2006)):

$$a = \frac{b}{1 + r_{s_0,s_0+1}} + \frac{a}{(1 + r_{s_0,s_0+1})^2} \tag{2.28}$$

If there is no risk, the rate r_{s_0+t} should equal the “risk free” rate $r^f_{s_0,s_0+t}$ for all $[s_0, s_0+t] \in S$ and all $t \in \mathbb{R}^+$. For now, it is assumed that the debtor cannot default during the first year, but default occurs with a 10% probability during the second year. Now, the rates r_{s_0,s_0+1} and r_{s_0,s_0+2} equating

$$a = \frac{b}{1 + r_{s_0,s_0+1}} + \frac{0.9a}{(1 + r_{s_0,s_0+2})^2} \tag{2.29}$$

would only be equal to the risk free counterpart in a so called “risk neutral” world. Risk averse investors would expect a return above the risk free rates

[3]In this context, “$\vee$” denotes the union of σ-fields.
[4]Net present value refers to the current value of future payoffs

$r^f_{s_0,s_0+1}$ and $r^f_{s_0,s_0+2}$ since they want to be remunerated for taking the risk of loosing the money invested. The difference between $r^f_{s_0,s_0+t}$ and r_{s_0,s_0+t} is called "risk premium" for the respective time frame $[s_0, s_0+t]$ (c.f. Karatzas and Shreve (1991), Singleton (2006) or Duffie (2008)).

In this example, only annual compounding is considered and the discount factor ν_{s_0,s_0+1} then becomes $\frac{1}{(1+r_{s_0+1})^t}$. If the compounding frequency increases to n times per year, the discounting factor changes to $t \leq 1$

$$\nu_{s_0,s_0+t} = \frac{1}{(1+(r_{s_0,s_0+t}/n))^{tn}}. \tag{2.30}$$

If the compounding frequency tends to infinity, i.e. with "continuous compounding", one obtains

$$\nu_{s_0,s_0+t} = e^{-r_{s_0,s_0+t}t} \tag{2.31}$$

since $lim_{n\to\infty}\left[1+\frac{x}{n}\right]^{-n} = e^{-x}$ for any $x \in \mathbb{R}$.

So far, it has been assumed that r_{s_0,s_0+t} only changes after certain periods of time (i.e. years). This assumption is, however, not plausible. Therefore, a rate that changes m times per year is assumed instead. One then obtains for a one year horizon:

$$\begin{aligned}
&\nu_{s_0,s_0+1} \\
&= e^{\left(-r_{[s_0,s_0+(1/m)]}\right)^{1/m}} e^{\left(-r_{[s_0+(1/m),s_0+(2/m)]}\right)^{1/m}} \cdots e^{\left(-r_{[s_0+1-(1/m),s_0+1]}\right)^{1/m}} \\
&= e^{-\left[r_{[s_0,s_0+(1/m)]}+r_{[s_0+(1/m),s_0+(2/m)]}+\cdots+r_{[s_0+(1/m),s_0+(2/m)]}\right](1/m)}
\end{aligned} \tag{2.32}$$

If m becomes infinitely small, one can rewrite $r_{[s,s+(1/m)]}$ as r_s for any point at time $s \in S$. Given that r_s is continuous in s, one can rewrite

$$\begin{aligned}
&lim_{m\to\infty}\left[e^{-\left[r_{[s_0,s_0+(1/m)]}+r_{[s_0+(1/m),s_0+(2/m)]}+\cdots+r_{[s_0+(1/m),s_0+(2/m)]}\right](1/m)}\right] \\
&= e^{\int_{s_0}^{s_0+1} r_s ds}.
\end{aligned} \tag{2.33}$$

The respective discount factor for any $t \in \mathcal{R}^+$ then becomes

$$\nu_{s_0,s_0+t} = e^{-\int_{s_0}^{s_0+t} r_s ds}. \tag{2.34}$$

As in the case of λ_s, one could assume that the movement of r_s is stochastic. In the presented framework, r_s will, however, always be assumed to be deterministic[5]. Risk free discount factors ν_{s_0,s_0+t} are typically approximated by zero bond[6] prices $ZB^f_{s_0,s_0+t}$ issued by AAA rated sovereigns like the United States or Germany.

The difference between the risk free rate and the interest rate expected by investors for investing in risky assets has already been established. The latter does not, however, enter the pricing formulas, which will be presented later on. Instead, the **concept of risk neutrality** is applied. This standard approach in the field of Quantitative Finance is based on the hypothetical presumption that all investors are risk neutral.

This presumption implies that the expected payoffs can be discounted by the risk free rate in order to obtain market prices. This may seem odd at first glance as real world investors are usually assumed to be risk averse and the real world market prices should ceteris paribus be inferior to the ones obtained from a model based on the risk free rate. However, it will be shown that the assumption of risk neutral investors is only a hypothetical auxiliary construct, not leading to model prices which generally are below real market prices. Instead, the pricing formulas are further adapted.

The mechanics behind this are shown based on the value of a zero bond ZB_{s_0,s_0+t} at time s_0 with an underlying default process driven by λ_s, a payment C_s summing up to the face value c at maturity $s_0 + t$, if no default has occurred. It is

[5]The possibility to buy interest rate swaps or futures enables investors to plan as if the interest rate was deterministic

[6]A zero bond is a bond which obligates to one payment only. The debtor does not pay an interest payment before the end of the contract, but pays the face value ("Nennwert") at the end of maturity only. The value of such a one time payment is per definition (with respect to the discount factor) the expected payoff - i.e. in such a case the payment sum weighted with the probability of payment - discounted by the discount factor based on the rate of return the investors expect on their investment.

assumed that the payoffs sum up to zero in the case of default. In other words, there is no recovery. The rate expected by risk averse market investors be r_s for all $s \in [s_0, s_0 + t]$. Therefore, the following equation holds :

$$
\begin{aligned}
ZB_{s_0,s_0+t} &= \mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+t} r_s ds} C_{s_0+t} | \mathcal{F}_{s_0}\right] \\
&= \mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+t} r_s ds} e^{-\int_{s_0}^{s_0+t} \lambda_s ds} c | \mathcal{F}_{s_0}\right] \\
&= e^{-\int_{s_0}^{s_0+t} r_s ds} c \mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+t} \lambda_s ds} | \mathcal{F}_{s_0}\right] \\
&= e^{-\int_{s_0}^{s_0+t} r_s ds} c \mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+t} \lambda_s ds} | \mathcal{F}_{2,s_0}\right] \qquad (2.35)
\end{aligned}
$$

The final transformation basically says that the current price of the zero bond ZB_{s_0,s_0+t} equals the discounted expected payoff. The expectation still included does not directly refer to the question whether a default occurs, but it refers to the future development of λ_s. The expectation is therefore only conditioned on the part of the filtration which refers to the development of λ_s, namely $\mathcal{F}_{2,s}$. The return is factored out because it is assumed to be deterministic. A detailed proof was presented by Lando (1998).

This equation includes several unknown variables: both λ_s and r_s are - in opposition to r_s^f - not directly observable for any $s \in S$. Just substituting r_s by r_s^f is not an appropriate approach to reduce the numbers of unknown variables to one, because the equation then should not hold anymore since

$$
\mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+t} \lambda_s + r_s ds} | \mathcal{F}_{2,s_0}\right] c < \mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+t} \lambda_s + r_s^f ds} | \mathcal{F}_{2,s_0}\right] c. \qquad (2.36)
$$

A standard trick in the context of risk neutral pricing is to adapt λ_s in a way that the expected payoffs discounted by the risk free discount rate are in accordance with the observed market prices of the respective zero bonds (Duffie and Singleton (2008)). For the presentation of this step in the present model framework, the intensity is assumed to be deterministic.

The risk premium, which is originally defined as the difference between expected return and risk free return, is roughly speaking assigned to the default intensity which is then denoted as "risk neutral"-default intensity $\lambda_s^{\mathbb{Q}}$, whereas the actual default intensity is denoted as $\lambda_s^{\mathbb{P}}$. $\lambda_s^{\mathbb{Q}}$ is the intensity process which would be implied as true intensity process in market prices of zero bonds, if these were observed in a risk neutral world. $\lambda_s^{\mathbb{Q}}$ should ceteris paribus be higher than $\lambda_s^{\mathbb{P}}$ to counterbalance the lower discount rate and one has

$$\mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+t}\lambda_s^{\mathbb{P}}+r_s ds}|\mathcal{F}_{2,s_0}\right]c=\mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+t}\lambda_s^{\mathbb{Q}}+r_s^f ds}|\mathcal{F}_{2,s_0}\right]c \tag{2.37}$$

with $\lambda_s^{\mathbb{Q}}\geq\lambda_s^{\mathbb{P}}$ and $r_s\geq r_s^f$ for all $s\in S$. The pricing formula for the zero bond is then given by

$$ZB_{s_0,s_0+t}=\mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+t}\lambda_s^{\mathbb{Q}}+r_s^f ds}|\mathcal{F}_{2,s_0}\right]c \tag{2.38}$$

So far, the difference in between the actual and the risk neutral intensity also applies in a framework with a deterministic intensity. The original framework is, however, doubly stochastic and that implies a second source of risk: this "second dimension" risk refers to the uncertainty regarding current and future default intensity levels. Risk averse investors may expect a risk premium for this kind of uncertainty in addition to a premium for the risk of a default given certain intensity levels. From the perspective of a bond buyer, it is not guaranteed – in this context – that this source of risk leads to an increase in the expected return. The respective uncertainty is also relevant for (short) sellers of credit securities or investors in credit insurances as a sudden drop in default probabilities should ceteris paribus lead to an increase the prices of bonds and to a decrease of insurance prices. The "second dimension" risk premium could – in other words – become negative. This may rather be the case for units with particular low anticipated default probabilities: Investors may – for example – rather insure people against the unlikely default of such a unit instead of insuring themselves or instead of betting on the occurrence of a credit event. The risk premium for the parties that profit from higher intensities might then dominate the risk premium from the other side. The main part of the debate in this chapter is, however, restricted to increases in returns due to the second dimension of risk respectively a positive second dimension risk premium because the empirical results in the third chapter of this dissertation, Pan and Singleton (2008) and Longstaff et al. (2011) suggest this to be the more relevant case.

It seems reasonable to consider both kinds of risk and the respective premia separately as they are indeed related, but not in 1:1 relation. It might, for example, be the case that the expected intensity levels and the respective default risk premium are particularly low, while the variance of the intensity and the respective "second dimension" risk premium are very high. On the other hand, it might be the case, that the expected intensity levels and the respective risk premium are very high, while the uncertainty regarding the intensity level respectively the second dimension risk premium is very low.

The presented approach therefore has to be further adapted to equate the expected payoff of the zero bond, discounted by the risk free rate, and the observed market prices. Consequently, two new measures with respect to $\lambda_s^{\mathbb{Q}}$ respectively two different versions of P_2 are introduced which both refer to the variation in the risk neutral intensity $\lambda_s^{\mathbb{Q}}$ but not – at least not directly – to the actual intensity $\lambda_s^{\mathbb{P}}$. The measure $\widehat{\mathbb{P}}$ refers to the actual movement of the risk neutral intensity $\lambda_s^{\mathbb{Q}}$. The measure $\widehat{\mathbb{Q}}$, on the other hand, refers to the distribution of $\lambda_s^{\mathbb{Q}}$, which the expectations in pricing equation 2.38 are built on, so the pricing formula still holds in the context of stochastic intensities. It refers, in other words, to the expectations with respect to $\lambda_s^{\mathbb{Q}}$, that would be implied by market spreads in a world that is second dimension risk neutral.

Under the new (second dimension) risk neutral measure $\widehat{\mathbb{Q}}$, the expectations with respect to future $\lambda_s^{\mathbb{Q}}$ are from now on denoted as $\mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[e^{-\int_{s_0}^{s_0+t}\lambda_s^{\mathbb{Q}}ds}|\mathcal{F}_{2,s_0}\right]$. This term differs only from $\mathbb{E}_{s_0}^{\widehat{\mathbb{P}}}\left[e^{-\int_{s_0}^{s_0+t}\lambda_s^{\mathbb{Q}}ds}|\mathcal{F}_{2,s_0}\right]$, if market participants' expected returns change due to the uncertainty regarding $\lambda_s^{\mathbb{Q}}$. If a risk premium is only demanded by investors for taking the default risk *per se* – i.e. the risk existing no matter whether the default probability is deterministic or not – there should only be a difference between $\lambda_s^{\mathbb{Q}}$ and $\lambda_s^{\mathbb{P}}$, but not between the two expec-

tations with respect to the future development of $\lambda_s^{\mathbb{Q}}$.
With a discount factor based on the risk free rate r_s^f the pricing formula of the zero bond introduced before becomes:

$$\begin{aligned} ZB_{s_0,s_0+t} &= \mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+t}\lambda_s+r_s ds}|\mathcal{F}_{2,s_0}\right]c \\ &= \mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[e^{-\int_{s_0}^{s_0+t}\lambda_s^{\mathbb{Q}}+r_s^f ds}|\mathcal{F}_{2,s_0}\right]c \\ &= \mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[e^{-\int_{s_0}^{s_0+t}\lambda_s^{\mathbb{Q}} ds}|\mathcal{F}_{2,s_0}\right]ZB_{s_0,s_0+t}^f c. \end{aligned} \tag{2.39}$$

The second dimension risk premium is further discussed in the context of reduced form risk models in the next chapter. In this section, the type of payoffs to be priced is extended first:

So far, the valuation of credit payments was based on the assumption of zero payments in the case of default, i.e. there was no recovery. This will be different now and the **pricing of recovery payments** is introduced. In this context, one has to think about the valuation of a payment that is executed in the case of default right after the default occurred. This be exemplified based on a payment obligation with payoff Z_τ. This obligation pays the amount z if the underlying unit defaults before maturity $s_0 + t$ and nothing otherwise. The payment is moreover supposed to be executed right after default time τ. The value of that default payment DP_{s_0,s_0+t} at time s_0 is

$$DP_{s_0,s_0+t} = \mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+\tau} r_s ds}Z_\tau|\mathcal{F}_{s_0}\right]. \tag{2.40}$$

The payoff of this obligation may be positive at each point in time until maturity because a default may occur in each point in time. The expectation therefore refers at each particular point in time until maturity to the question whether a default occurs just at that time and not to the question whether a default occurs anytime until maturity. This implies an expectation regarding the level of the intensity at each point conditioned on the fact that no default hast occured yet.

Lando (1998) shows that the discounted expectation of the payment can be rewritten as

$$\begin{aligned}\mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+\tau} r_s ds} Z_\tau | \mathcal{F}_{s_0}\right] &= \mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[\int_{s_0}^{s_0+t} \lambda_s^{\mathbb{Q}} e^{-\int_{s_0}^{s} \lambda_u^{\mathbb{Q}} + r_u^f du} z ds | \mathcal{F}_{2,s_0}\right] \\ &= z\mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[\int_{s_0}^{s_0+t} \lambda_s^{\mathbb{Q}} e^{-\int_{s_0}^{s} \lambda_u^{\mathbb{Q}} + r_u^f du} ds | \mathcal{F}_{2,s_0}\right] \qquad (2.41)\end{aligned}$$

The expectations denoted by $\mathbb{E}_s^{\widehat{\mathbb{Q}}}$ now again only refer to the future development of $\lambda_s^{\mathbb{Q}}$. Again the expectation based on the true distribution law of $\lambda_s^{\mathbb{Q}}$ would only equate this pricing formula if market participants' return expectations did not change because of the uncertainty with respect to $\lambda_s^{\mathbb{Q}}$. The proof for formula 2.41 presented in Lando (1998) is based on the following equation:

$$P(\tau \geq s_0 + t | \tau \geq s_0, \mathcal{F}_{2,s}) = e^{-\int_{s_0}^{s_0+t} \lambda_s^{\mathbb{Q}} ds} \qquad (2.42)$$

$$\frac{\partial}{\partial s} P(\tau \geq s_0 + t | \tau \geq s_0, \mathcal{F}_{2,s}) = \lambda_s^{\mathbb{Q}} e^{-\int_{s_0}^{s_0+t} \lambda_s^{\mathbb{Q}} ds} \qquad (2.43)$$

The second equation can be interpreted as the probability of default at any moment in time s, given that the default has not yet occurred. This is the probability for a payoff $Z_s = z$ in s. The value of all aggregated expected payoffs Z_s for $s \in [s_0, s_0 + t]$ is as presented in Lando (1998) given by

$$z \ \mathbb{E}_{s_0}\left[\int_{s_0}^{s_0+t} \lambda_s^{\mathbb{Q}} e^{-\int_{s_0}^{s} \lambda_u^{\mathbb{Q}} du} ds | \mathcal{F}_{2,s_0}\right]. \qquad (2.44)$$

Based on this formula, one can easily derive a risk neutral pricing formula for the value DP_{s_0,s_0+t} of a contract with maturity $s_0 + t$ paying off Z_s in all

$s \in [s_0, s_0 + t]$ with $Z_s = z$ if $s = \tau$ and $Z_s = 0$ otherwise.:

$$DP_{s_0,s_0+t} = \mathbb{E}_{s_0}\left[\left(\int_{s_0}^{s_0+t} Z_s e^{-\int_{s_0}^{s} r_u du}\right)|\mathcal{F}_{s_0}\right] \quad (2.45)$$

$$= \int_{s_0}^{s_0+t} \mathbb{E}_{s_0}\left[Z_s|\mathcal{F}_{s_0}\right] e^{-\int_{s_0}^{s} r_u du} \quad (2.46)$$

$$= z\int_{s_0}^{s_0+t} \mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[\left(\lambda_s^{\mathbb{Q}} e^{-\int_{s_0}^{s} \lambda_u^{\mathbb{Q}} + r_u du}\right)|\mathcal{F}_{2,s_0}\right] \quad (2.47)$$

$$= z\int_{s_0}^{s_0+t} ZB_{s_0,s}^{f} \mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[\left(\lambda_s^{\mathbb{Q}} e^{-\int_{s_0}^{s} \lambda_u^{\mathbb{Q}} du}\right)|\mathcal{F}_{2,s_0}\right]. \quad (2.48)$$

Now, the **pricing of credit default swaps** (CDS) is discussed as an example. Before this specific functional link between default intensity λ_s and CDS spreads is presented, the functionality of this class of credit securities is introduced.

CDS are insurance contracts between two parties with respect to the default of a third party. This basically means that the insurer or CDS seller pays a certain amount to the insurance or CDS buyer if the third party defaults. The insured party in return pays a semi- or quarter-annual payment – which is usually called "spread" payment (denoted by $SP_{s_0}(M)$ for a CDS issued in s_0 and maturity M in years) – until the contract ends. This is either the case when maturity $s_0 + M$ is reached or after a possible default of the respective third party. The spread is constant for one single CDS contract. Historical data of CDS spreads usually refer to newly issued contracts. In the following, s_0 is accordingly the index for CDS spread time series.

The amount to be paid by the insurance seller in the case of default depends on the proportion of debt which is not repaid by the third party in the context of a default. This share is called the "loss rate" LR. In the present framework, LR is defined with respect to the face value of an ordinary bond. If a third party is, for example, only able to pay back 50% of the issued bonds' face value, the seller of a CDS referring to this defaulting unit as third party has to pay 50% of the respective CDS contract's face value. This would usually lead to a payment of 50 cents per contract as the face value of an ordinary CDS contract is one.

LR is in the following assumed to be constant for the respective third party[7]. LR is identical for all CDS contracts with respect to the same third party. It is finally important to notice that the insured person does not necessarily hold a security issued by the respective third party.

For the pricing of newly issued CDS contracts, the single spread payment claims can be considered as $2 \times M$ zero bonds, with maturity $\frac{n}{2}$ and face values $SP_{s_0}(M)$, with $n \in \{1, \cdots, 2 \cdot t\}$ for a CDS maturity of $s_0 + M$, $[s_0, s_0 + M] \subset S$, $M \in \mathbb{N}^+ \cup \{0.5\}$ and semi-annually spread payment. n denotes the number of the respective spread payment. This set up implies $s_n - s_{n-1} = 0.5$ for all $n \geq 1$. The value SV_{s_0,s_n} of one single payment obligation to be paid in s_n is in s_0 based on the pricing formulas for defaultable zero bonds:

$$SV_{s_0,s_n} = \mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}} \left[e^{-\int_{s_0}^{s_n} \lambda_s^{\mathbb{Q}} ds} | \mathcal{F}_{2,s_0} \right] ZB_{s_0,s_n}^f SP_{s_0}(M). \tag{2.49}$$

The value $SV_{s_0}^{total}(M)$ of the whole set of spread payments $SP_{s_0}(s_0 + t)$ referring to a CDS contract issued in s_0 with maturity $s_0 + t$ is then in s_0:

$$SV_{s_0}^{total}(M) = SP_{s_0}(M) \sum_{n=1}^{2t} \left(\mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}} \left[e^{-\int_{s_0}^{s_n} \lambda_s^{\mathbb{Q}} ds} | \mathcal{F}_{2,s_0} \right] ZB_{s_0,s_n}^f \right). \tag{2.50}$$

For valuation of the spread payment counterpart, i.e. the insurance obligation, one can go back to the recovery payments presented in the previous section. The insurance obligation again refers to a possible payment at each point in time until maturity. This payment sums up to zero, if the respective third party has not defaulted yet and it is positive right at the point in time the default occurs. The payoff is now denoted by INS_s. The amount paid in this case of default is LR. The value of the insurance claim from the perspective of the CDS buyer is

[7] This is of course a simplifying assumption and assuming the loss rate to be stochastic and uncertain would be more realistic. An additional risk premium for uncertainty with respect to the loss rate would then be possible. This might be a field for future research.

denoted by $VINS_{s_0}(M)$ and can be obtained based on the following formula:

$$VINS_{s_0}(M) = \mathbb{E}_{s_0}\left[\int_{s_0}^{s_0+t} e^{-\int_{s_0}^{s} r_s ds} INS_s|\mathcal{F}_{s_0}\right] \tag{2.51}$$

$$= LR\left[\int_{s_0}^{s_0+t} ZB^f_{s_0,s}\mathbb{E}^{\widehat{\mathbb{Q}}}_{s_0}\left[\lambda^{\mathbb{Q}}_s e^{-\int_{s_0}^{s}\lambda^{\mathbb{Q}}_u du}|\mathcal{F}_{2,s_0}\right] ds\right]. \tag{2.52}$$

The "market" spread $SP_{s_0}(M)$ is then the one that equates the values of both payment sides, namely the value of total spread payments $SV^{total}_{s_0}(M)$, and the value of the insurance claim $VINS_{s_0}(M)$. The following equation is supposed to hold accordingly (c.f. Duffie (1999)):

$$SP_{s_0}(M)\sum_{n=1}^{2M}\left(\mathbb{E}^{\widehat{\mathbb{Q}}}_{s_0}\left[e^{-\int_{s_0}^{s_0+0.5n}\lambda^{\mathbb{Q}}_s ds}|\mathcal{F}_{2,s_0}\right] ZB^f_{s_0,s_0+0.5n}\right)$$

$$= LR\left[\int_{s_0}^{s_0+M} ZB^f_{s_0,s}\mathbb{E}^{\widehat{\mathbb{Q}}}_{s_0}\left[\lambda^{\mathbb{Q}}_s e^{-\int_{s_0}^{s}\lambda^{\mathbb{Q}}_u du}|\mathcal{F}_{2,s_0}\right] ds\right]. \tag{2.53}$$

So far, two versions of P_2 have been introduced: $\widehat{\mathbb{Q}}$ and $\widehat{\mathbb{P}}$. Now, the notation of the CIR diffusions, which determine the distribution law of $\lambda^{\mathbb{Q}}_s$, is extended to distinguish between the diffusion equations under both measures (c.f. Pan and Singleton (2008)). This is done referring to the CDS pricing formula. Then, it is shown in the context of the CDS pricing formula 2.53, how the coefficients of the respective stochastic differential equation can be interpreted with respect to the second dimension risk premium.

2.5 Diffusion equations under both measures and the second dimension risk premium

In the previous section, the difference between $\widehat{\mathbb{Q}}$ and $\widehat{\mathbb{P}}$ has already been discussed. The difference between both measures refers to the distribution law of $\lambda^{\mathbb{Q}}_s$. The distribution law of the diffusion process $\lambda^{\mathbb{Q}}_s$ is generally determined by an underlying stochastic differential equation like the CIR diffusion. Considering these two ingredients of the model set up, it seems to be reasonable to adjust the notation of the respective diffusion equation accordingly. The diffusion equation determining the distribution law under $\widehat{\mathbb{Q}}$ is denoted in the following way:

$$d\lambda_s^{\mathbb{Q}} = \left(\mu_0^{\widehat{\mathbb{Q}}} - \mu_1^{\widehat{\mathbb{Q}}}\lambda_s^{\mathbb{Q}}\right) ds + \sigma_1\sqrt{\lambda_s^{\mathbb{Q}}}dB_s^{\widehat{\mathbb{Q}}}. \tag{2.54}$$

The true distribution law of $\lambda_s^{\mathbb{Q}}$ is given by:

$$d\lambda_s^{\mathbb{Q}} = \left(\mu_0^{\widehat{\mathbb{P}}} - \mu_1^{\widehat{\mathbb{P}}}\lambda_s^{\mathbb{Q}}\right) ds + \sigma_1\sqrt{\lambda_s^{\mathbb{Q}}}dB_s^{\widehat{\mathbb{P}}}. \tag{2.55}$$

Drift coefficients and Brownian motion differ in both equations, while the diffusion coefficient is identical. The reason for that lies in equation 2.24: only the drift coefficient and the respective value of the process itself go into the formula for the conditional expectation. And the expectations regarding the intensities are what matters in the "second dimension" risk premium context. This is shown based on the CDS pricing formula 2.53 and the idea of a positive second dimension risk premium introduced before:

The "first dimension" risk premium, i.e. the premium with respect to the default risk per se (i.e. given a specific deterministic series of intensities), is already taken into account by substituting $\lambda_s^{\mathbb{P}}$ by $\lambda_s^{\mathbb{Q}}$. Because of the uncertainty with respect to $\lambda_s^{\mathbb{Q}}$, the discount factor $ZB_{s,s+t}^f$ may, however, still be larger (or smaller) than the discount factor based on the expected return, even after this substitution. In other words, the discount factor $ZB_{s,s+t}^f$ might only be the appropriate one without any further adjustments, if there is no "second dimension" risk premium in this model. In the following, this is shown referring to the case of positive second dimension risk premia. To adjust for the effect of the lower discount factor respectively the higher discount rate, positive payoffs have to get lower weights and negative payoffs have to get higher weights[8]. This is the case, if the expectations regarding future intensities, which are conditioned on

[8] In a risk neutral world, the observed spreads and loss rates would only be reasonable from a no arbitrage pricing point of view, if the expected values of $\lambda_s^{\mathbb{Q}}$ respectively the expected default probabilities were higher (than they actually are). The actual expectations regarding future intensities would be as pessimistic as they are when based under the diffusion equation referring to $\widehat{\mathbb{Q}}$.

the current intensity levels, tend to be higher. Then, the negative payoff in the default case is more likely and the actual payment of all single spreads is more unlikely. The reasoning for a negative second dimension risk premium works accordingly.

This can be shown based on the expectations with respect to *transforms* of the intensity process, which are included in formula 2.53 as well. The expectation with respect to the first transform ($e^{-\int_{s_0}^{s_0+0.5n} \lambda_s^{\mathbb{Q}} ds}$) refers to the probability that a default has not occurred yet at the point in time chosen as higher boundary of the included integral. This figure is lower if expected future intensities are higher – both intuitively and based on mathematical reasoning[9]. Accordingly, single positive payoffs are weighted by lower weights if the expected future intensities are higher – which is in accordance with the presented economic reasoning.

The relation between future intensities and the level of the second transform ($\lambda_s^{\mathbb{Q}} e^{-\int_{s_0}^{s} \lambda_u^{\mathbb{Q}} du}$) is not directly clear. The intensities' expected values enter this transform in two ways: the transform decreases in the intensity, which goes into the exponential function negatively, and it increases with the intensity, by which the exponential function is multiplied. Considering the economic meaning of this transform, this is reasonable: As discussed before, the transform refers to the probability that the default has not yet occurred at the point in time chosen as upper border in the included integral, but occurs just right then. There is, moreover, an integral built over that transform. This integral over the transform refers to the probability that the default occurs at any point in time between the time chosen as lower boundary of the outer integral and the time chosen as higher boundary of the outer integral. The insurance payment is, in other words, weighted higher if the expectations of the future default intensity tend to be higher. This is again in accordance with the presented economic reasoning. The risk neutral expectations regarding the future values of the intensities therefore have to be higher (compared to expectations based on the

[9]The intensity goes into the exponential function negatively.

true distribution law), the stronger the expected return (after taking into account the "first dimension" risk premium) exceeds the risk free return[10].

The established positive relation between the second dimension risk premium and the expected values of the intensities can also be explained in a less complicated fashion based on the temporary assumption that there is no first dimension risk premium (i.e. $\lambda_s^{\mathbb{P}} = \lambda_s^{\mathbb{Q}}$) and the zero bond pricing formula 2.39, which refers to the price of a zero bond without recovery. If the second dimension risk premium is zero as well, the following version of the pricing equation 2.39 holds:

$$\begin{aligned} ZB_{s_0,s_0+t} &= \mathbb{E}_{s_0}^{\widehat{\mathbb{P}}}\left[e^{-\int_{s_0}^{s_0+t}\lambda_s^{\mathbb{Q}}ds}|\mathcal{F}_{2,s_0}\right]ce^{-\int_{s_0}^{s_0+t}r_s ds} \\ &= \mathbb{E}_{s_0}^{\widehat{\mathbb{P}}}\left[e^{-\int_{s_0}^{s_0+t}\lambda_s^{\mathbb{Q}}ds}|\mathcal{F}_{2,s_0}\right]ce^{-\int_{s_0}^{s_0+t}r_s^f ds} \end{aligned} \tag{2.56}$$

If there exists a positive second dimension risk premium, the risk-free rate is not equal to the expected return ($r_s > r_s^f$) and the equation 2.56 does not hold anymore. As described before, one can adjust for the difference between the discount factors resulting from r_s^f respectively r_s by introducing the risk-free measure $\widehat{\mathbb{Q}}$:

$$ZB_{s_0,s_0+t} = \mathbb{E}_{s_0}^{\widehat{\mathbb{P}}}\left[e^{-\int_{s_0}^{s_0+t}\lambda_s^{\mathbb{Q}}ds}|\mathcal{F}_{2,s_0}\right]ce^{-\int_{s_0}^{s_0+t}r_s ds} \tag{2.57}$$

$$= \mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[e^{-\int_{s_0}^{s_0+t}\lambda_s^{\mathbb{Q}}ds}|\mathcal{F}_{2,s_0}\right]ce^{-\int_{s_0}^{s_0+t}r_s^f ds}. \tag{2.58}$$

If the expected return is higher (lower) than the risk-free return because of a positive (negative) second dimension risk premium, the intensity values which are expected under the measure $\widehat{\mathbb{Q}}$ should exceed (be inferior to)[11] the values expected under $\widehat{\mathbb{P}}$.

Accordingly, the difference between the conditional expectations of the intensity under both measures directly measures the "second dimension" risk premium. Formula 2.24 shows how the drift coefficients impact the conditional expecta-

[10]The opposite is the case if the second dimension risk premium is negative.
[11]The intensity goes into the exponential function negatively.

tions. If the ratio $\frac{\mu_0}{\mu_1}$ (i.e. the mean reversion) is the same under both measures, a comparison of the drift parameter μ_1 is sufficient to evaluate the difference in the conditional expectations. A larger value for μ_1 implies a larger conditional expectation (closer to the mean reversion level), if the value of the intensity, which the expectation is conditioned on, is below the mean reversion level. The opposite holds if the value of the intensity is above the mean reversion level. General statements are however for all other cases (e.g. $\frac{\mu_0}{\mu_1}$ is higher and μ_1 is smaller under one measure) rather difficult to make. The difference between both measures with respect to the "second dimension" risk premium is therefore optimally evaluated with reference to the actual time series of $\lambda_s^{\mathbb{Q}}$.

Based on the CIR coefficients under both measures and this time series, conditional expectations can be calculated for all horizons. The difference between the resulting conditional expected values can then be evaluated. Another reasonable approach to evaluate the relevance of the "second dimension" risk premium is the following: the model implied CDS spreads can be calculated based on the respective time series of $\lambda_s^{\mathbb{Q}}$. The expectations can be calculated based on both $\widehat{\mathbb{Q}}$ leading to "true" model spreads $\widehat{SP}_{s_0}$ and $\widehat{\mathbb{P}}$ leading to "wrong" model spreads $\widehat{SP}_{s_0}^{\widehat{\mathbb{P}}}$. The latter is calculated based on this formula:

$$\widehat{SP}_{s_0}(M) = \frac{\widehat{LR}\left[\int_{s_0}^{s_0+M} ZB_{s_0,s}^f \mathbb{E}^{\widehat{\mathbb{P}}}_{s_0,\widehat{\mu}_0^{\widehat{\mathbb{P}}},\widehat{\mu}_1^{\widehat{\mathbb{P}}},\widehat{\sigma}_1}\left[\widehat{\lambda}_s^{\mathbb{Q}} e^{-\int_{s_0}^{s}\widehat{\lambda}_u^{\mathbb{Q}} du} | \mathcal{F}_{2,s_0}\right] ds\right]}{\sum_{n=1}^{2M}\left(\mathbb{E}^{\widehat{\mathbb{P}}}_{s_0,\widehat{\mu}_0^{\widehat{\mathbb{P}}},\widehat{\mu}_1^{\widehat{\mathbb{P}}},\widehat{\sigma}_1}\left[e^{-\int_{s_0}^{s_0+0.5n}\widehat{\lambda}_s^{\mathbb{Q}} ds} | \mathcal{F}_{2,s_0}\right] ZB_{s_0,s_0+0.5n}^f\right)} \quad (2.59)$$

with $\widehat{LR}, \widehat{\mu}_0^{\widehat{\mathbb{P}}}, \widehat{\mu}_1^{\widehat{\mathbb{P}}}, \widehat{\sigma}_1$ denoting estimated coefficients, $\mathbb{E}^{\widehat{\mathbb{P}}}_{s_0,\widehat{\mu}_0^{\widehat{\mathbb{P}}},\widehat{\mu}_1^{\widehat{\mathbb{P}}},\widehat{\sigma}_1}$ denoting the resulting expectation and $\widehat{\lambda}_s$ denoting the estimated intensity process. The true model spreads are accordingly calculated as

$$\widehat{SP}_{s_0}^{\widehat{\mathbb{Q}}}(M) = \frac{\widehat{LR}\left[\int_{s_0}^{s_0+M} ZB_{s_0,s}^f \mathbb{E}^{\widehat{\mathbb{Q}}}_{s_0,\widehat{\mu}_0^{\widehat{\mathbb{Q}}},\widehat{\mu}_1^{\widehat{\mathbb{Q}}},\widehat{\sigma}_1}\left[\widehat{\lambda}_s^{\mathbb{Q}} e^{-\int_{s_0}^{s}\widehat{\lambda}_u^{\mathbb{Q}} du} | \mathcal{F}_{2,s_0}\right] ds\right]}{\sum_{n=1}^{2M}\left(\mathbb{E}^{\widehat{\mathbb{Q}}}_{s_0,\widehat{\mu}_0^{\widehat{\mathbb{Q}}},\widehat{\mu}_1^{\widehat{\mathbb{Q}}},\widehat{\sigma}_1}\left[e^{-\int_{s_0}^{s_0+0.5n}\widehat{\lambda}_s^{\mathbb{Q}} ds} | \mathcal{F}_{2,s_0}\right] ZB_{s_0,s_0+0.5n}^f\right)}. \quad (2.60)$$

A significant difference between the true and wrong model spreads implies that the "second dimension" risk premium is an important driver of credit spreads. Finally, the difference in the CIR diffusions under both measures can – as for the standard Quantitative Finance stock price or short term rate models – be evaluated based on the **Girsanov theorem**. This standard theorem is introduced in the next paragraphs:
consider a measure space $\left(\widehat{\Omega}, \widehat{\mathcal{P}}, \mathcal{F}\right)$. $\widehat{B}_s$ be a Brownian motion under probability measure $\widehat{\mathcal{P}}$, Θ_t be an adapted process to the resulting filtration $\mathcal{F}_s$, the index set S be the same as before and a process Z_s be defined as

$$Z_s = e^{\left[-\int_0^s \Theta_t d\widehat{B}_t - \frac{1}{2}\int_0^s \Theta_t^2 dt\right]} \tag{2.61}$$

for $s \in S$. $\widehat{\mathcal{P}}$ be, moreover, related to the second probability measure $\tilde{\mathcal{P}}$ with Z_s being a Radon-Nykodin derivative linking these two measures:

$$\frac{d\tilde{\mathcal{P}}}{d\widehat{\mathcal{P}}} = Z_s \tag{2.62}$$

According to the Girsanov theorem, under mild technical conditions, $\tilde{B}_s$ defined as $\tilde{B}_s = \widehat{B}_s + \int_0^s \Theta_t dt$ is a Brownian motion under the measure $\tilde{\mathcal{P}}$. In equity modelling, the variable Θ_s is frequently considered to be the market price of risk. Applying this approach to the presented framework is supposed to show its reasonability[12]. Θ_s be in this context denoted by η_s and the Radon-Nykodin derivative relating $\widehat{\mathbb{Q}}$ and $\widehat{\mathbb{P}}$ be defined by

$$\widehat{Z}_s = e^{\left[-\int_0^s \eta_t dB_t^{\widehat{\mathbb{Q}}} - \frac{1}{2}\int_0^s \eta_t^2 dt\right]} \tag{2.63}$$

for $t \in S$ and $s \geq t$ so that

$$\frac{d\widehat{\mathbb{P}}}{d\widehat{\mathbb{Q}}} = Z_s. \tag{2.64}$$

[12] This approach was applied before in the reduced form credit risk context by Pan and Singleton (2008).

This implies that

$$d\lambda_s^{\mathbb{Q}} = \left(\mu_0^{\widehat{\mathbb{P}}} - \mu_1^{\widehat{\mathbb{P}}}\lambda_s^{\mathbb{Q}}\right) ds + \sigma_1\sqrt{\lambda_s^{\mathbb{Q}}}\left(dB_s^{\widehat{\mathbb{Q}}} + \eta_s ds\right). \tag{2.65}$$

$\sigma_1\sqrt{\lambda_s^{\mathbb{Q}}}\eta_s$ accordingly gives the difference in change in $\lambda_s^{\mathbb{Q}}$ between $\widehat{\mathbb{P}}$ and $\widehat{\mathbb{Q}}$. The greater η_s, the greater is the increase of $\lambda_s^{\mathbb{Q}}$ under $\widehat{\mathbb{Q}}$ compared to the increase under $\widehat{\mathbb{P}}$. η_s is therefore another reasonable measure for the size of the "second dimension" risk premium. A negative value for η_s would refer to situations in which the insurance buyer expects a price reduction for the possibility of changes in the default intensity as the insurance may be worthless in the case of a sudden decrease in default intensities.

η_s is in the following assumed to depend on $\lambda_s^{\mathbb{Q}}$ in a specific functional form. This step is line with the literature on quantitative equity modelling (c.f. Karatzas and Shreve (1991), Duffie (2008), Singleton (2001)). An according adaption to the reduced form model context for a model with another diffusion equation has been presented by Pan and Singleton (2008). The technical context is, however, not discussed in the respective application paper. The specific form is chosen based on the plausible assumption that the difference in change should increase linearly in the level of the underlying intensity (c.f. Cheridito et al. (2007) and Duffee (2002)). η_s already goes into the change of $\lambda_s^{\mathbb{Q}}$ as a factor multiplied by $\sigma_1\sqrt{\lambda_s^{\mathbb{Q}}}$. To obtain a linear form, it is accordingly assumed that η_s depends on $\lambda_s^{\mathbb{Q}}$ in the following way:

$$\eta_s = \frac{\rho_0}{\sqrt{\lambda_s^{\mathbb{Q}}}} + \rho_1\sqrt{\lambda_s^{\mathbb{Q}}}. \tag{2.66}$$

This results in the actual difference in change of $\lambda_s^{\mathbb{Q}}$ being given by

$$\sigma_1\left(\rho_0 + \rho_1\lambda_s^{\mathbb{Q}}\right) \tag{2.67}$$

which is a linear function in $\lambda_s^{\mathbb{Q}}$ as it is supposed to be. This implies the following link between ρ_0, ρ_1 and the CIR coefficients under both measures:

$$\rho_0 = \frac{\mu_0^{\widehat{\mathbb{Q}}} - \mu_0^{\widehat{\mathbb{P}}}}{\sigma_1} \tag{2.68}$$

$$\rho_1 = \frac{\mu_1^{\widehat{\mathbb{P}}} - \mu_1^{\widehat{\mathbb{Q}}}}{\sigma_1}. \tag{2.69}$$

2.6 Affine processes in the credit risk context

Roughly speaking an affine process is a Markov process, the characteristic function of which depends on the state vector in an exponential affine form. Under technical conditions, the respective coefficients of this affine function can, furthermore, be described as solution to specific ordinary differential equations (ODEs). The following introduction strongly builds up on Singleton (2006), as well as Duffie et al. (2000) and Duffie et al. (2003). A multivariate state space $K \subseteq \mathbb{R}^n$ for $n \in \mathbb{N}^+$ and the Markov process $(X_s, (\mathbb{P}_x)_{x\in K})$ characterized by the family probability laws $(\mathbb{P}_x)_{x\in K}$ be defined. This process is called affine, if the conditional characteristic function (CCF) $\phi_{s_1}(s_2 - s_1, iu)$[13] of the transition distributions depends on x in "exponentially affine form" (c.f. Singleton (2006)):

$$\begin{aligned} \phi_{s_1}(s_2 - s_1, iu) &= \mathbb{E}\left[e^{iu\cdot X_{s_2}} | X_{s_1}\right] && (2.70) \\ &= \int_K e^{iu\cdot X_{s_2}} p_s(X_{s_1}, X_{s_2}) && (2.71) \\ &= e^{\psi(s_1 - s_2, iu) + \zeta(s_1 - s_2, iu)\cdot X_{s_1}} && (2.72) \end{aligned}$$

with $u \in \mathbb{R}^n$, $i = \sqrt{-1}$, $\psi(s_1 - s_2, iu)$, $\zeta(s_1 - s_2, iu)$ being complex coefficients, $s_1 \in S$ and $s_2 \in S$. Important examples, introduced above, for the class of affine processes are solutions to Gaussian and square root diffusions (c.f. Piazzesi (2010)). The coefficients $\psi(s_1 - s_2, iu)$ and $\zeta(s_1 - s_2, iu)$ are solutions to ordinary differential equations, which depend in these two cases only on the actual value of the respective underlying process and the coefficients of the respective diffusion

[13]for $s_2 \geq s_1, s_1 \in S, s_2 \in S$

equation (c.f. Duffie et al. (2003)). The presentation of the results in the next paragraphs is accordingly based on these two cases. The presentation is, moreover, for the sake of a simpler notation restricted to one dimensional cases (i.e. $n = 1$), even though all results hold for multivariate cases, too. The process Y_s is given by formula (2.12) ($dY_s = \mu_{Y_s} ds + \sigma_{Y_s} dB_s$) with drift and diffusion part being given by equation 2.15 respectively 2.14.

To derive formulas linking the value of an affine process and expectations with respect to certain functions of this process, it has been exploited that the CCF of an affine process is known under certain technical conditions. For example, Duffie and Kan (1996) present closed form solutions to expectations with respect to functions of process values frequently seen in credit pricing formulas:

$$E_{s_0}\left[e^{-\int_{s_0}^{s_0+t} X_s ds} e^{uX_{s_0+t}} | Y_{s_0}\right] = f(t, X_{s_0}) \tag{2.73}$$

with u being a one dimensional real valued coefficient. X_s is not even necessarily the respective affine process itself, but it depends in linear form on an affine state vector Y_s:

$$X_s = \tilde{\rho_0} + \tilde{\rho_1} Y_s \tag{2.74}$$

with $\tilde{\rho_0}$ and $\tilde{\rho_1}$ defined as introduced in section three. This includes the case of X_s being a one dimensional affine process. Duffie and Kan (1996) show that

$$f(t, X_{s_0}) = e^{A(t)+B(t)X_{s_0}} \tag{2.75}$$

holds with $A(t)$ and $B(t)$ being solutions to complex valued ODEs depending on μ_0, μ_1, σ_0 and σ_1. Equation 2.73 is basically the pricing formula for a zero bond with maturity t and face value 1. Pricing formulas for some other credit securities include however - as seen in equation 2.53 - expectations with respect to transforms of the underlying state process that go beyond transform 2.73. Luckily, Duffie et al. (2000) built on Duffie and Kan (1996) and present close form solutions to expectations for a wider range of transforms represented by

$$E_{s_0}\left[e^{-\int_{s_0}^{s_0+t} r_s X_s ds} v X_s e^{uX_{s_0+t}}\right] = F(t, X_{s_0}) \tag{2.76}$$

with v being in this context univariate and real values coefficients. Their results are presented following their own notation[14]. Duffie and Kan (1996) show that

$$F(t, X_{s_0}) = e^{A(t)+B(t)X_{s_0}}(\alpha(t) + \beta(t)X_{s_0}) \tag{2.77}$$

with $A(t)$ and $B(t)$ being the same coefficients as in equation 2.75 as well as $\alpha(t)$ and $\beta(t)$ again being solutions to complex valued ODEs which depend on μ_0, μ_1, σ_0 and σ_1.

Based on Duffie et al. (2000) and Singleton (2006), the ODEs determining the coefficients in equation 2.75 have the following form:

$$\dot{B}(s) = \tilde{\rho_1} - \mu_1 - \frac{1}{2}\sigma_1^2 B(s)^2 \tag{2.78}$$

$$\dot{A}(s) = \tilde{\rho_0} - \mu_0 B(s) - \frac{1}{2}\sigma_0^2 B(s)A(s) \tag{2.79}$$

with boundary conditions $B(t) = u$ and $A(t) = 0$. The additional coefficients in equation 2.77 are given by the following ODEs:

$$-\dot{\beta}_s = \mu_1\beta_s + \sigma^2\beta_s \tag{2.80}$$

$$-\dot{\alpha}_s = \mu_0\beta_s + \sigma_0^2 \tag{2.81}$$

with boundary conditions $\beta(t) = v$ and $\alpha(t) = 0$.

In the credit risk context, the values of the expectations regarding these transforms, which are conditioned on the current value of the intensity process, can be calculated if the coefficients of the underlying diffusion equations are known. The only step that has to be taken, is solving the respective ODEs (2.78)-(2.81). This can, of course, be done numerically. It may according to Huang

[14] This notation has also been adopted from Singleton (2006).

and Yu (2007), be reasonable to chose an implicit solution method to avoid wrong convergence due to stiffness. For the standard case (i.e. $\tilde{\rho}_0 = 0$, $\tilde{\rho}_1 = 1$ and $\sigma_0 = 0$), there are however analytical solutions available. Duffie and Garleanu (2001) presented solutions for the equations (2.78) and (2.79) for the case that the underlying stochastic differential equation is a CIR diffusion with an additional jump component. Longstaff et al. (2005) present closed form representations for more complex transforms for the simple CIR case. Combining these results in the presented case (i.e. a simple CIR diffusion without jump) leads to the following closed form solutions:

$$A(t) = \frac{\mu_0\mu_1 + \mu_0\xi}{\sigma_1^2}t + \frac{2\mu 0}{\sigma_1^2}\left[ln\left[1 - \frac{\mu_1 + \xi}{\mu_1 - \xi}\right] - ln\left[1 - \frac{\mu_1 + \xi}{\mu_1 - \xi}e^{t\xi}\right]\right] \tag{2.82}$$

$$B(t) = \frac{2\left(1 - e^{\xi t}\right)}{(\xi + \mu_1)\left(e^{\xi t} - 1\right) + 2\xi} \tag{2.83}$$

$$\alpha(t) = \frac{\nu_0}{\xi}\left(e^{\xi t} - 1\right) \tag{2.84}$$

$$\beta(t) = e^{\xi t} \tag{2.85}$$

$$\xi = \sqrt{2\sigma_1^2 + \mu_1^2}. \tag{2.86}$$

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Chapter 3

Estimation of Reduced Form Credit Risk Models

3.1 Introduction

Modelling and analyzing credit spreads are very important issues for both economic researchers and finance practitioners. A very versatile platforms for challenges in this context are reduced form credit risk models. These models were introduced by Jarrow and Turnbull (1995), Lando (1998) respectively Duffie and Singleton (1999) and basically assume the default of a unit being representable by a first jump of an underlying Poisson process. The intensity of this Poisson process is typically driven by a stochastic differential equation. Applications of such models in financial market research can, for example, be found in Pan and Singleton (2008) and Longstaff et al. (2011). These studies analyze the structure of credit spreads by isolating the "second dimension" risk premium from the rest of the spread. This risk premium refers to the additional payoff which investors expect if the present and future default probabilities are not deterministic.

The present paper discusses a convenient estimation strategy for such a reduced form credit risk model under both measures based on credit default swap (CDS) data and tests this estimation strategy based on simulated data. The discussed estimation strategy – which is oriented towards the strategy employed by Longstaff et al. (2005)[1] – is not the first strategy applied for reduced form credit risk models under both measures in published studies. Estimation results for such models are, for example, presented in the mentioned studies by Pan and Singleton (2008) and Longstaff et al. (2011). The strategy discussed in the present article is, however, particularly convenient and practicable and the characteristics of the affine process class is exploited. The strategy is consequently only applicable in the context of models driven by affine diffusion processes like the "Cox-Ingersoll-Ross" process (CIR) introduced in the finance literature by Cox et al. (1985). The estimation procedures applied by Pan and Singleton (2008) and Longstaff et al. (2011) are in these respective articles not documented in length but they definitely differ from the one presented in the present paper as their specification choice for the underlying stochastic differen-

[1]They estimate their model however not exclusively based on CDS and only under the risk-neutral measure

tial equation does not allow to exploit the affine process class's characteristics. It seems very likely that the estimation strategy discussed in the present paper is a particularly practicable one since – although still being demanding with respect to computational capacities – it does not require numerical solving of respective Feynman-Kac differential equations.

3.2 Model set up

A measure space $(\Omega_1, \mathcal{F}_1, P_1)$ with the corresponding filtration $\mathcal{F}_{1,s}$, a measurable space $(M_1, \mathcal{M}_1)$ and an index set $S \neq \emptyset$ is defined. Furthermore, a Poisson process

$$\mathcal{P}oi = (Poi_s, s \in S) \tag{3.1}$$

is defined as a family of measurable mappings between probability and measure space:

$$Poi_s : (\Omega_1, \mathcal{F}_1, P_1) \to (M_1, \mathcal{M}_1) \tag{3.2}$$

$$\omega_1 \mapsto Poi_s(\omega_1) \tag{3.3}$$

with $\omega_1 \in \Omega_1$. Poi_s counts the number of *events* up to time s. In this model now, the default of a unit is depicted as first jump of this Poisson process and the time of the first jump denoted as $\tau \in S$ is therefore stopping time for this process as well[2]. The increments $Poi_{s_1} - Poi_{s_0}$ are for $s_0, s_1 \in S$ and $s_1 - s_0 \geq 0$ independently Poisson distributed, the Poisson parameter depends on the length of the respective period $[s_0, s_1]$ only and Markov property is satisfied accordingly. At the first point in time, the process value be almost surely zero and the process be supported by the probability space introduced above. The intensity parameter of this Poisson process is denoted by λ_s with $s \in S$. The time period between a starting time $s_0 \in S$ and the first jump of the underlying Poisson

[2] I.e. the model stops after the first credit event to avoid the assumption that a model's structure is still the same after such an event.

process is exponentially distributed with the parameter process λ_s, which is first assumed to be deterministic.

Pricing formulas for all kinds of credit risk related securities have been derived based on that. This usually implies the application of the risk neutrality concept: as returns r_s (with $s \in S$) expected by the investors are unknown, they are in these pricing formulas usually substituted by the risk free rate r_s^f, with $r_s \geq r_s^f$. The original discount factor $ZB_{s_0,s}$ for a respective discount horizon $s - s_0 \geq 0$ (and $[s_0, s] \subset S$) is now below the resulting risk neutral discount factor $ZB^f_{s_0,s}$. The true default intensity λ_s is then usually substituted by a "risk neutral" counterpart $\lambda_s^{\mathbb{Q}}$ to adjust for this effect. The "true" intensity is typically denoted by $\lambda_s^{\mathbb{P}}$. $\lambda_s^{\mathbb{Q}}$ should contain a fraction referring to the default risk premium which investors in credit securities usually expect. The intensities $\lambda_s^{\mathbb{Q}}$ equating the non-arbitrage pricing formulas are therefore supposed to be higher than "real" intensities $\lambda_s^{\mathbb{P}}$[3].

So far, this framework holds for deterministic intensities. This limitation is however rather implausible for real world applications. Therefore, in the following the framework is extended and a second dimension of randomness is introduced: a probability space $(\Omega_2, \mathcal{F}_2, P_2)$ with corresponding filtration $\mathcal{F}_{2,s}$ and a measurable space $(M_2, \mathcal{M}_2)$ with $M_2 \subseteq \mathbb{R}^n$ for $n \in \mathbb{N}^+$ denoting a multivariate state vector be defined. The index set $S \neq \emptyset$ is still the same as in the subsection before. Finally, a Brownian motion $B_s \in \mathbb{R}^n$ and the following "diffusion" process $\Lambda^{\mathbb{Q}} = \left(\lambda_s^{\mathbb{Q}}, s \in S\right)$ are defined as a family of measurable mappings between probability and measure space:

$$\lambda_s^{\mathbb{Q}} : (\Omega_2, \mathcal{F}_2, P_2) \to (M_2, \mathcal{M}_2) \tag{3.4}$$

$$\omega_2 \mapsto \lambda_s^{\mathbb{Q}}(\omega_2). \tag{3.5}$$

[3] $\lambda_s^{\mathbb{Q}}$ would be the actual intensities implied by credit prices if these prices would have been observed in a hypothetical world of risk neutral investors who do not expect an additional remuneration for any kind of risk.

It is moreover assumed that the process $\lambda_s^{\mathbb{Q}}$ is driven by a Cox-Ingersoll-Ross (CIR) diffusion:

$$d\lambda_s^{\mathbb{Q}} = \left(\mu_0 - \mu_1 \lambda_s^{\mathbb{Q}}\right) + \sigma_1 \sqrt{\lambda_s^{\mathbb{Q}}} dB_s \tag{3.6}$$

with B_s denoting a Brownian motion and μ_0, μ_1 and σ_1 being constant coefficients. This process accordingly reverts to the mean for $\mu_0 > 0$ and $\mu_1 > 0$. Both introduced measure spaces can be combined resulting in a new sample space $\Omega = \Omega_1 \times \Omega_2$, a new sigma algebra $\mathcal{F} = \sigma\{\mathcal{F}_1 \vee \mathcal{F}_2\}$[4] and the respective filtration $\mathcal{F}_s$. Moreover, the probability measure P is introduced, which just satisfies all general requirements regarding probability measures with respect to $\mathcal{F}$ respectively $\mathcal{F}_s$. This means: $P(\Omega) = 1$, $P(F) < \infty$ for all $F \in \mathcal{F}$ as well as countable additivity for disjoint collections (c.f. Davidson (1994)).

The establishment of a CDS pricing formula based on this extended "doubly stochastic" framework now only requires the introduction of a loss rate LR and the spread processes $SP_{s_0}(M)$ referring to newly issued CDS contracts. M denotes the maturity of CDS contracts issued in s_0 with $[s_0, s_0 + M] \subset S$. Accordingly s_0 denotes the time index for historical CDS spread time series. LR refers to the fraction of the face value[5] of a zero bond issued by the third party that is not paid back when a default event occurs. This typically complies with the amount which the CDS seller is expected to pay in the case of default, as CDS usually have a face value of one. The term "default" refers in this context to all kinds of credit events and the default probability accordingly refers to the aggregated probabilities of all kinds of credit events. LR is then the average loss rate over all credit events weighted by the relative probabilities of the respective credit event. The resulting pricing formula for a newly issued CDS is then as

[4] "$\vee$" denotes in this context the union of σ-fields.

[5] "Face value" of a zero bond refers to the amount of money the borrower is supposed to receive when maturity is reached.

presented by Duffie (1999) based on Lando (1998):

$$
\begin{aligned}
&SP_{s_0}(M)\sum_{n=1}^{2M}\left(\mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[e^{-\int_{s_0}^{s_0+0.5n}\lambda_s^{\mathbb{Q}}ds}|\mathcal{F}_{2,s_0}\right]ZB_{s_0,s_0+0.5n}^{f}\right)\\
&=LR\left[\int_{s_0}^{s_0+M}ZB_{s_0,s}^{f}\mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[\lambda_s^{\mathbb{Q}}e^{-\int_{s_0}^{s}\lambda_u^{\mathbb{Q}}du}|\mathcal{F}_{2,s_0}\right]ds\right].
\end{aligned}
\tag{3.7}
$$

The expectations in this pricing formula now all refer to the development of $\lambda_s^{\mathbb{Q}}$, i.e. they do not refer to the true default intensities $\lambda_s^{\mathbb{P}}$. This differentiation between $\lambda_s^{\mathbb{P}}$ and $\lambda_s^{\mathbb{Q}}$ introduced before refers to the credit risk *per se*, i.e. the uncertainty regarding the current and future levels is ignored here. It may, however, be that investors' expected return changes because of the additional uncertainty (c.f. Pan and Singleton (2008)). The expectations equating the arbitrage pricing formula would then differ from the expectations built based on the "true" distribution law of $\lambda_s^{\mathbb{Q}}$. Consequently, two variations of P_2 are introduced: $\widehat{\mathbb{Q}}$, which refers to the hypothetical risk neutral probability measure that leads to expectations equating the no-arbitrage pricing formula and $\widehat{\mathbb{P}}$ referring to the "real" distribution law of $\lambda_s^{\mathbb{Q}}$.
This differentiation also has to be reflected by the notation of the diffusion equations driving $\lambda^{\mathbb{Q}}$ under the respective probability measure. The one referring to $\widehat{\mathbb{Q}}$ is denoted by:

$$
d\lambda_s^{\mathbb{Q}}=\left(\mu_0^{\widehat{\mathbb{Q}}}-\mu_1^{\widehat{\mathbb{Q}}}\lambda_s^{\mathbb{Q}}\right)ds+\sigma_1\sqrt{\lambda_s^{\mathbb{Q}}}dB_s^{\widehat{\mathbb{Q}}}. \tag{3.8}
$$

The true variation of $\lambda_s^{\mathbb{Q}}$ is given by:

$$
d\lambda_s^{\mathbb{Q}}=\left(\mu_0^{\widehat{\mathbb{P}}}-\mu_1^{\widehat{\mathbb{P}}}\lambda_s^{\mathbb{Q}}\right)ds+\sigma_1\sqrt{\lambda_s^{\mathbb{Q}}}dB_s^{\widehat{\mathbb{P}}}, \tag{3.9}
$$

(c.f. Pan and Singleton (2008) and Longstaff et al. (2011)). The difference between these two equations can be found in the superscript of the drift parameters and the Brownian motion. The diffusion parameter is identical in both cases. This complies with the Quantitative Finance literature on stock developments respectively the Black-Scholes formula. The diffusion coefficient σ is – in opposition to the drift coefficients μ_0 and μ_1 – not required to calculate the conditional expectation of a CIR process. The stronger the difference in the drift coefficients μ_0 respectively μ_1, the stronger is the difference in conditional expectations with respect to $\lambda_s^{\mathbb{Q}}$ (this is discussed in the first chapter of this dissertation and formulas are presented in the third section of the present chapter as well.). A comparison of drift components therefore measures the relevance of that "second dimension" risk premium[6]. The estimation of the CIR coefficients under both measures is, moreover, necessary to forecast future CDS spreads by forecasting $\lambda_s^{\mathbb{Q}}$ and to calculate the respective expectations respectively the CDS spreads in the resulting pricing formula.

3.3 Estimation procedure

This section discusses the estimation procedure with respect to the coefficients under both $\widehat{\mathbb{Q}}$ and $\widehat{\mathbb{P}}$. The main focus of this chapter is clearly on the estimation strategy under $\widehat{\mathbb{Q}}$ which is discussed first.

3.3.1 Estimation of the diffusion parameters under $\widehat{\mathbb{Q}}$

To estimate the distribution law of $\lambda_s^{\mathbb{Q}}$ under the risk neutral measure $\widehat{\mathbb{Q}}$ is a challenging task since only a set of spread time series $SP_{s_0}(M)$ and the risk neutral discount factors $ZB^f_{s_0,s_0+s}$ are directly observable. A loss rate LR is frequently assumed ex-ante as well. Pan and Singleton (2008) suggest, however, to estimate LR simultaneously with the other coefficients by taking advantage of the broad set of contracts with different maturities issued on a daily basis. This idea is adopted for this project since the empirical results in Pan and Singleton (2008) show that the typically assumed loss rate of 70 percent is sometimes far from the loss rate equating the pricing formula in their model.[7]

[6] This set up allows for the application of the Girsanov theorem to compare the drift under both measures, c.f.Pan and Singleton (2008) or the third chapter of this dissertation.

[7] Considering for example sovereign data shows that loss rates can strongly vary. Historical data as published by Moody's (2008) reflect a wide range of loss rates, ranging from 1.9% in

This demonstrates, moreover, that the loss rate implied in CDS spreads can be exactly identified because the loss rate is the same for all maturities, but it has a different impact on the observed market spreads for different maturities.

The discussed procedure is – as mentioned before – oriented towards Longstaff et al. (2005) and restricted to models driven by affine diffusion processes. The theory on affine processes is exploited to substitute the expectations included in formula 4.16.

An affine process, as defined by Duffie et al. (2003a), is roughly speaking a Markov process the characteristic function of which depends in "exponentially affine form" (c.f. Singleton (2006), Duffie et al. (2003b)) on the current process value (c.f. the first chapter of this dissertation). Furthermore, the respective coefficients of this affine function can under technical conditions be described as solutions to specific ordinary differential equations (ODEs). If the affine process is a diffusion process, the ODEs are fully determined by the process parameters of that diffusion[8].

Duffie and Singleton (1999) moreover show that expectations with respect to transforms of such affine processes can be depicted in exponential linear form depending on the value of the state process at the point in time when the expectation is built in. The coefficients of this function can again be obtained as solutions to given ODEs that depend on the parameters of the underlying diffusions[9].

Adapting the results in Duffie et al. (2000) to the expectations included in the CDS pricing formula, one yields

$$E_{s_0}\left[e^{\int_{s_0}^{s_1}\lambda_s^{\mathbb{Q}}ds}\right] = e^{\alpha_{s_1-s_0}+\beta_{s_1-s_0}\lambda_{s_0}^{\mathbb{Q}}} \tag{3.10}$$

$$E_{s_0}\left[\lambda_s e^{\int_{s_0}^{s}\lambda_u^{\mathbb{Q}}ds}\right] = e^{\alpha_{s_1-s_0}+\beta_{s_1-s_0}\lambda_{s_0}^{\mathbb{Q}}}(A_{s_1-s_0}+B_{s_1-s_0})\lambda_{s_0}^{\mathbb{Q}} \tag{3.11}$$

the case of Belize in 2006, to 82 % in the Russian case.

[8]As the ones introduced in the following paragraphs, these ODEs can be found in the first chapter of this dissertation.

[9]Details are presented in the first chapter of this dissertation.

with $\alpha_{s_1-s_0}, \beta_{s_1-s_0}, A_{s_1-s_0}$ and $B_{s_1-s_0}$ being solutions to ODEs. The coefficients depend on the parameter of the diffusion equation driving $\lambda_s^{\mathbb{Q}}$ under the respective measure.

Knowledge regarding the diffusion coefficients would therefore allow to substitute the expectations in the CDS pricing formula by the exponential linear functions depending on the current realization $\lambda_{s_0}^{\mathbb{Q}}$ of $\lambda_s^{\mathbb{Q}}$ only. The coefficients of this exponential linear form are, however, still unknown as the diffusion coefficients are not known either. The set of coefficients $\{\widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1, \widehat{LR}\}$ is therefore – following Longstaff et al. (2005) – assumed ex-ante and the resulting ODEs are solved to get a series of coefficients for the exponential linear form. The expectations in the pricing formula are then substituted by the respective exponential linear functions depending on the realization of $\lambda_{s_0}^{\mathbb{Q}}$ and an estimation $\widehat{\lambda}_{s_{0_i}}^{\mathbb{Q}}$ can then be obtained for each observation $s_{0_i} \in [s_{0_1}, s_{0_2}.., s_{0_N}]$ with N denoting the respective sample size: define

$$\begin{aligned}
& f(\lambda_{s_0}^{\mathbb{Q}} | \widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1^{\widehat{\mathbb{Q}}}, \widehat{LR}) \\
& = SP_{s_0}(M) \sum_{n=1}^{2M} \left(\mathbb{E}^{\widehat{\mathbb{Q}}}_{s_0, \widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1^{\widehat{\mathbb{Q}}}} \left[e^{-\int_{s_0}^{s_0+0.5n} \lambda_s^{\mathbb{Q}} ds} | \mathcal{F}_{2,s_0} \right] ZB^f_{s_0, s_0+0.5n} \right) \\
& - \widehat{LR} \left[\int_{s_0}^{s_0+M} ZB^f_{s_0,s} \mathbb{E}^{\widehat{\mathbb{Q}}}_{s_0, \widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1^{\widehat{\mathbb{Q}}}} \left[\lambda_s^{\mathbb{Q}} e^{-\int_{s_0}^{s} \lambda_u^{\mathbb{Q}} du} | \mathcal{F}_{2,s_0} \right] ds \right]. \qquad (3.12)
\end{aligned}$$

$\mathbb{E}^{\widehat{\mathbb{Q}}}_{s_0, \widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1^{\widehat{\mathbb{Q}}}}$ denotes expectations built in s_0 under $\widehat{\mathbb{Q}}$ depending on the set of coefficients $\{\widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1^{\widehat{\mathbb{Q}}}\}$. For each time step $s_{0_i} \in [s_{0_1}, s_{0_2}.., s_{0_N}]$, one searches for $\widehat{\lambda}_{s_{0_i}}^{\mathbb{Q}}$ which satisfies $f(\widehat{\lambda}_{s_{0_i}}^{\mathbb{Q}} | \widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1, \widehat{LR}) = 0$. The extracted time series $\widehat{\lambda}_{s_{0_i}}^{\mathbb{Q}}$ is then however depending on the ex-ante determined coefficient set and it is therefore probably biased. This bias is, however, following Longstaff et al. (2005) still going to be corrected:

spreads from contracts with other maturities (i.e. in the present case 1,3,7 and 10 years) are taken and the sum of squared distances between these observed spreads $SP_{s_{0_i}}(M)$ and the model spreads $\widehat{SP}_{s_{0_i}}(M)$ based on the time series of intensities estimated in our first step is minimized by choosing a new set of coefficients. Model spreads can in this context be calculated based on this formula:

$$\widehat{SP}_{s_{0_i}}(M) = \frac{\widehat{LR}\left[\int_{s_{0_i}}^{s_{0_i}+M} ZB^f_{s_{0_i},s}\mathbb{E}^{\widehat{\mathbb{Q}}}_{s_{0_i},\widehat{\mu}_0^{\widehat{\mathbb{Q}}},\widehat{\mu}_1^{\widehat{\mathbb{Q}}},\widehat{\sigma}_1^{\widehat{\mathbb{Q}}}}\left[\widehat{\lambda}_s^{\mathbb{Q}} e^{-\int_{s_{0_i}}^{s}\widehat{\lambda}_u^{\mathbb{Q}}du}|\mathcal{F}_{2,s_{0_i}}\right]ds\right]}{\sum_{n=1}^{2M}\left(\mathbb{E}^{\widehat{\mathbb{Q}}}_{s_{0_i},\widehat{\mu}_0^{\widehat{\mathbb{Q}}},\widehat{\mu}_1^{\widehat{\mathbb{Q}}},\widehat{\sigma}_1^{\widehat{\mathbb{Q}}}}\left[e^{-\int_{s_{0_i}}^{s_{0_i}+0.5n}\widehat{\lambda}_s^{\mathbb{Q}}ds}|\mathcal{F}_{2,s_{0_i}}\right]ZB^f_{s_{0_i},s_{0_i}+0.5n}\right)} \tag{3.13}$$

and the minimization problem is accordingly given by

$$\underbrace{min}_{\{\widehat{\mu}_0^{\widehat{\mathbb{Q}}},\widehat{\mu}_1^{\widehat{\mathbb{Q}}},\widehat{\sigma}_1,\widehat{LR}\}} \sum_{M\in\{1,3,7,10\}}\sum_{s_{0_i}\in\{s_{0_1},..,s_{0_N}\}}\left[\widehat{SP}_{s_{0_i}}(M) - SP_{s_{0_i}}(M)\right]^2. \tag{3.14}$$

This new set of coefficients $\{\widehat{\mu}_0^{\widehat{\mathbb{Q}}},\widehat{\mu}_1^{\widehat{\mathbb{Q}}},\widehat{\sigma}_1,\widehat{LR}\}$ is, however, again biased as it depends in turn on the time series of intensities estimated based on the coefficient values, which were chosen ex-ante. The estimation has therefore not been completed yet. The new set of coefficients is subsequently used for estimating a times series $\widehat{\lambda}_{s_{0_i}}^{\mathbb{Q}}$ which is again based on the time series of $SP_{s_{0_i}}(5)$. The estimated time series $\widehat{\lambda}_{s_{0_i}}^{\mathbb{Q}}$ is in turn used for the estimation of a new coefficient set by comparing model spreads $\widehat{SP}_{s_{0_i}}(M)$ with the actual spreads $SP_{s_{0_i}}(M)$ for $M \in [1,3,7,10]$. Both steps are afterwards repeated until the estimates of the coefficients and the intensities converge (c.f. Longstaff et al. (2005)). All variables are identified (c.f. Pan and Singleton (2008)).

The ODEs resulting in the coefficients of the exponential linear form for the conditional expectations have thereby of course to be solved over and over again. On the one hand, this can be done numerically but there are on the other hand fortunately also analytical solutions available that were presented by Longstaff et al. (2005) (c.f. the first chapter of this dissertation). Both the numerical and the analytical approaches to find solutions to the respective ODEs have been implemented for the present study. For the numerical approach implicit solution methods[10] were applied instead of explicit ones to avoid stiffness problems

[10] The method applied was following Huang and Yu (2007): the implicit Runge-Kutta procedure.

(c.f Huang and Yu (2007)). For the numerical solution approach, the ODEs were solved every $\frac{1}{3600}$th time step. The numerical solution procedure, however, turns out to be computationally extremely demanding.[11]

3.3.2 Estimation of the diffusion parameter under $\widehat{\mathbb{P}}$

After having estimated $\{\widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1, \widehat{LR}\}$ as well as a times series of intensities $\lambda_{s_{0_i}}^{\widehat{\mathbb{Q}}}$, the set of CIR drift coefficients under the historical measure $\widehat{\mathbb{Q}}$ can be estimated. The diffusion coefficient σ under $\widehat{\mathbb{P}}$ is the same as under $\widehat{\mathbb{Q}}$ and therefore only $\{\mu_0^{\widehat{\mathbb{P}}}, \mu_1^{\widehat{\mathbb{P}}}\}$ are left for estimation under $\widehat{\mathbb{Q}}$. The transition probability distribution of the CIR process is luckily known to be a non-central χ^2-distribution with non-centrality parameter nc_t and degrees of freedom dof given by

$$nc_t = \frac{2\mu_1}{\sigma_1^2\left(1 - e^{-\mu_1 t}\right)} \tag{3.15}$$

$$dof = \frac{4\mu_0 \exp^{-\mu_1 t}}{\sigma_1^2(1 - \exp^{-\mu_1 t})}. \tag{3.16}$$

Overbeck and Ryden (1997) and Iacus (2008) present closed form representations for probability distributions of diffusion process realization Y_{s_0+t} based on the underlying CIR coefficient and conditioned on a specific previous realization Y_{s_0}:

$$Pr(Y_{s_0}|Y_{s_0-t}, \mu_0, \mu_1, \sigma_1^2) = \frac{-0.5\frac{Y_{s-t}nc_t}{\exp^{-\mu_1 t}} + y_s nc_t}{2(y_s nc_t Y_{s-t})^{dof/4}} Be_{\frac{dof}{2}-1}(\sqrt{Y_{s-t}nc_t}) \tag{3.17}$$

where $Be_{\frac{dof}{2}-1}$ denotes the modified Bessel function of the first kind of order $\frac{dof}{2} - 1$. The evaluation of the Bessel function included in this distribution function formula can, as discussed in Zhou (2000) or Iacus (2008), be numerically difficult in certain scenarios. Alternative representations are, however, numerically troublesome for typical reduced form model parameter ranges. (c.f. Zhou

[11]The approach was run for several data sets and did not lead to converged results. The finally presented results are therefore the ones based on the analytical solution.

(2000) respectively the first chapter of this doctoral thesis), so the estimates in the context of this study are based on the numerical approximation of the Bessel function implemented in R.

The closed form formulas for the conditional expectations and the conditional variance are as presented in Cox et al. (1985) or Iacus (2008):

$$\mathbb{E}\left(Y_{s+t}|Y_s\right) = \frac{\mu_0}{\mu_1} + \left(Y_s - \frac{\mu_0}{\mu_1}\right) e^{-\mu_1 t}, \tag{3.18}$$

$$Var\left(Y_{s+t}|Y_s\right) = Y_s \frac{\sigma_0^2\left(e^{-\mu_1 t} - e^{-2\mu_1 t}\right)}{\mu_1} + \frac{\mu_0 \sigma_0^2\left(1 - e^{-2\mu_1 t}\right)}{2\mu_1^2}. \tag{3.19}$$

The set of possible approaches to estimate the drift parameters under the historical measure based on the times series of extracted risk neutral intensities is accordingly wide, including maximum-likelihood estimators (MLE), quasi-maximum-likelihood estimators (QML) or methods-of-moments estimators (MoM). Most publications dealing with the estimation of square root diffusions based on time series data focus on the short term model context. Most of these publications, moreover, show that this estimation of a CIR process based on time series can be troublesome despite the transition distribution being known in closed form (c.f. Gourieroux and Monfort (2007) or Faff and Gray (2006)). The results of such studies can, furthermore, not be directly transferred to the estimation in the credit risk context as there are two important differences between both estimation situations. The first difference refers to the number of coefficients which have to be estimated from time series data. In the reduced form context, the diffusion parameter σ is usually assumed to be identical under both the risk neutral and the historical measure ($\widehat{\mathbb{Q}}$ respectively $\widehat{\mathbb{P}}$). Since the parameters under $\widehat{\mathbb{Q}}$ are estimated simultaneously with the estimation of the time series of intensities and the estimation of the coefficients under $\widehat{\mathbb{Q}}$ necessarily have to be estimated first, σ is usually already known when drift parameters μ_0 and μ_1 are estimated under $\widehat{\mathbb{P}}$. In the short term model context on the other hand, the diffusion parameter has to be estimated simultaneously with the drift parameters based on the time series of state process realizations.

The second difference between the estimation in the reduced form model context and the short term model context is the relevant parameter range. Empirical results in the short term literature are often restricted to cases where the Feller-condition $2\mu_0/\sigma_1^2 > 1$ is satisfied (c.f. Overbeck and Ryden (1997)). The estimations in the context of the model set up used for this study suggest that this does not necessarily seem to be the case for reduced form credit risk applications (c.f. estimation results in the third chapter of this dissertation). The estimation of the drift parameters based on the parameter estimates regarding the diffusion coefficient is therefore discussed in the following paragraph without excluding the case that $2\mu_0/\sigma_1^2 \leq 1$.

Putting aside these differences, MLE and QML are, for example, discussed in Overbeck and Ryden (1997) and in Gourieroux and Monfort (2007). Overbeck and Ryden (1997) show, for situations with three free parameters, that MLE leads to consistent results without the Feller-condition. These MLE results seem, however, to be not necessarily very precise for moderate sample sizes (e.g. 1000 observations). This corresponds to a time period of roughly three years in the reduced form credit risk context. Three years seem to be a plausible time frame for real world applications [12]. This trade-off between the preciseness of the estimation and the validity of the results with respect to structural model changes may be problematic.

Overbeck and Ryden (1997) moreover refer to the simultaneous estimation of all three CIR parameters. As discussed before, the simulation in the present study now only includes the situation with two free parameters. For the present analysis, other than the study in Overbeck and Ryden (1997) the empirical mean of the underlying time series is considered as a non-parametric estimate for μ_0/μ_1:

$$\widehat{\left[\frac{\mu_0^{\widehat{\mathbb{P}}}}{\mu_1^{\widehat{\mathbb{P}}}}\right]}^{np} = \frac{1}{N} \sum_{s_{0_i} \in \{s_{0_1},..,s_{0_N}\}} \lambda_{s_i}^{\mathbb{Q}}. \tag{3.20}$$

This seems intuitively reasonable because of the mean reversion characteris-

[12]Longer time horizons might be questionable because of possible structural changes leading to different models after more than three years.

tic of the drift coefficient. Then there is only one free parameter left for the next (parametric) estimation step (μ_1). The ML estimator based on the non-parametric estimation of μ_0/μ_1 is called “mixed” MLE in the following. The “normal” MLE is considered as well.

Another estimator considered in this study is the MoM respectively LS estimator which minimizes the sum of squared deviations from the model implied conditional expected values (conditioned on the previous process value) to the actual process values. Based on formula 3.18, this implies for a sampling frequency of $\frac{1}{250}$ that the MoM estimator $\widehat{\mu}_1^{\widehat{\mathbb{P}},MoM}$ is the solution to

$$\underbrace{min}_{\mu_1^{\widehat{\mathbb{P}}}} \sum_{i\in\{2,3,..,N\}} |\widehat{\lambda}^{\mathbb{Q}}_{s_{0_i}} - \widehat{\left[\frac{\mu_0^{\widehat{\mathbb{P}}}}{\mu_1^{\widehat{\mathbb{P}}}}\right]}^{np} - \left(\widehat{\lambda}^{\mathbb{Q}}_{s_{0_{i-1}}} - \widehat{\left[\frac{\mu_0^{\widehat{\mathbb{P}}}}{\mu_1^{\widehat{\mathbb{P}}}}\right]}^{np}\right) e^{-\mu_1^{\widehat{\mathbb{P}}}(1/250)}|. \tag{3.21}$$

In this context, only one parameter is freely estimated and the empirical mean is again used as a non-parametric estimate of μ_0/μ_1. Third, a QML estimator has been implemented, which assumes the difference between the values to be normally distributed. Results in Overbeck and Ryden (1997) and Gourieroux and Monfort (2007) suggest, however, that this procedure might not be very promising and the results in the context of the present study endorse that view. Accordingly, the procedure is not discussed any further.

3.4 Simulation

The following subsections discuss the simulation strategies. The estimates are presented as well. Even though the main contribution of this study is the estimation of parameters under $\widehat{\mathbb{Q}}$, the simulation of the data set to evaluate the estimators under $\widehat{\mathbb{P}}$ is rather lengthy because of the difficulties related to this simulation . The parameters chosen for the simulations were the ones estimated for Finish and Polish data.

3.4.1 Simulation of the spread time series $SP_{s_0}(M)$ for all maturities M

This paragraph deals with the simulation of CDS spreads for several maturities. The discussed estimation procedure for the parameters under the risk neutral measure is then evaluated based on the simulated time series of spreads. The simulation is based on the assumption that the difference between model spreads $\widehat{SP}_{s_0}(M)$ and the real spreads $SP_{s_0}(M)$ is normally distributed with an expected value of zero. In the following, these model errors are denoted by $\epsilon_{s_0}(M)$ respectively $\widehat{\epsilon}_{s_0}$ for the empirical counterpart. One can write accordingly:

$$\epsilon(M) \sim \mathcal{N}(0, \sigma_{\epsilon(M)}). \tag{3.22}$$

In this context, the empirical variance of the model errors for the respective country is chosen as estimate for the variance of the normal-distribution which the model error simulation is based on. The spreads are then simulated by adding the simulated errors to the model spreads which are calculated based on the estimated default intensities. Spreads of the following maturities are simulated: one, three, five, seven and ten years.

The normality assumption is, of course, a simplification to exploit that the normal-distribution is fully characterized by its first two moments. The basic structure of the model errors is therefore reproduced in a satisfying manner based on the described moment-matching approximation as the normal-distribution is fully characterized by its first two moments.

Another simplification, which has, for example, been made by Pan and Singleton (2008), is an independence assumption regarding the model errors. The values of the empirical autocorrelation function[13] (ACF) do, however, clearly show that there exists (at least in the case of the present model) a strong dependence between the model errors. Figure 3.1 depicts, for example, values of the empirical ACF for the Finish 1 year case. This dependence is factored in by the estimation of an AR process based on the error realizations. For com-

[13]The depicted ACF is again calculated based on the Finish data.

parison, two simulations are run in the Finish case: one based on independent draws from the normal-distribution, with a variance complying to the empirical variance of the error terms and one based on the estimated AR(3) process.

The variances respectively the AR coefficients are estimated separately for each maturity as there are reasons to presume that the errors behave differently for different maturities. The five year spread was, for example, chosen as reference value for the estimation of the intensities $\widehat{\lambda}^{\mathbb{Q}}_{s_0,i}$ because the CDS with that maturity are characterized by the highest level of liquidity (c.f. Pan and Singleton (2008)). This is in turn reflected by the low model errors: the respective mean, the relative mean[14], the respective empirical variance, as well as the variance of the AR(3)-innovations (denoted by $\sigma^2_{\epsilon_{s_0}(M),AR(p)}$ with p referring to the number of AR-lags) are the lowest.

Simulation results spreads

The estimates of the parameters under $\widehat{\mathbb{Q}}$ based on the simulated data sets were obtained based on the following details: for the first simulated data set, the starting value for the estimation was at the average value of the estimates for six countries, which are presented in the third chapter of this dissertation[15]. For the further data sets, the final estimation results based on the respective previous data set were chosen as starting value. The starting value was, moreover, changed randomly, if the number of iterations was above 40. In this case, the values were increased or decreased by 50% with a probability of 50%. The estimator is finally assumed not to be converging for one data set, if the number of iterations exceeds 120. The estimates for the error term variance and the estimated variance of the AR-innovations are presented in tables 3.1 and 3.2. The estimation results referring to the larger samples sizes (i.e. numbers of observations N=901) can be found in tables 3.3-3.5. The number of simulated data sets is denoted by n. The results suggest that the estimation strategy leads to precise results for a sample sizes of N=901. For the Finish case, the estimates

[14]This refers to the mean of the *relative* deviation $\frac{Error}{Spread}$.

[15]This complies with $\{-4.17e^{-10}; -1.493.9 \times 10^{-5}; 0.5\}$.

based on the AR-simulation are, moreover, on average more precise than the estimates based on the independently simulated errors. This suggests this modelling that auto-regressive structure, is an important detail for the simulation of spreads. The simulation for Poland was accordingly only run based on the AR estimation. The 5-year maturity is included for all cases. The respective results must, however, be interpreted cautiously as the intensities have been estimated based on this maturity. The Polish results for all estimations based on the smaller samples (i.e. N=100) can be found in table 3.6[16]. These results are very imprecise. In the Finish case, the estimations based on the short samples do not even converge in most cases. To sum up, one can say that the estimation strategy seems to perform very well for larger samples, whereas the results for smaller samples seem to be rather problematic.

3.4.2 Simulation of a time series of intensities $\lambda_s^{\mathbb{Q}}$

Simulation procedure

In the presented model set up $\lambda_s^{\mathbb{Q}}$ is driven by a CIR diffusion. A CIR diffusion is generally not defined for negative process values as the square root of the current process value goes into the model's diffusion coefficient. The process is, moreover, prevented from taking values outside the defined domain if the conditions $\mu_1 > 0$ and $\mu_2 > 0$ are satisfied. In such cases, the impact of the diffusion part on the change of such a process becomes indefinitely small as the process value gets close to zero and the drift part of the stochastic differential equation dominates. This means that the change in $\lambda_s^{\mathbb{Q}}$ becomes approximately deterministic and positive.

The process does moreover not even touch the zero bound if the so called Feller-condition $2\mu_1\mu_2 > \sigma^2$ is satisfied. If the parameters of the drift coefficient are both negative, the drift part still dominates for indefinitely small process values. The drift is, however, negative if the process value is below the level

[16]The 5-year case is included as well, the respective results must be evaluated very cautiously as the intensities were estimated based on this maturity.

$\frac{\mu_1}{\mu_2}$. The values are then directly pushed out of the defined domain which makes this parameter range irrelevant for credit risk applications[17]. The focus of the subsequent discussion is therefore restricted to positive drift parameters.

The simulation of CIR processes is, for example, discussed in Andersen et al. (2010) or Iacus (2008). The most obvious simulation strategy exploits that the transition distribution of a CIR process is – as mentioned before – a non-central χ^2-distribution with non-centrality parameter nc_t and degrees of freedom dof depending on the current value of the process in known functional form.

Simulations based on the non-central χ^2-distribution exploit that the non-central χ^2-distributed variable complies with a central χ^2-distributed variable with degrees of freedom being Poisson distributed. The central χ^2-distribution is in turn a special case of the Gamma-distribution (c.f. Kahl and Jaeckel (2006a) and the first chapter of this dissertation). Draws from the Gamma-distribution in R are executed as implemented in R based on the algorithm by Loader (2000).

Andersen et al. (2010) point out that despite the transition distribution being known in closed form, the described simulation approach might not be the best choice because the simulation of non-central χ^2-distributed random variables can be numerical challenging according to them. The computational time be particularly high and the draws from a non-central χ^2-distribution may be rather imprecise. The latter issue might be particularly important in the reduced form credit risk context: the resulting non-central χ^2-distributions are in this context characterized by low degrees of freedom and very low average values. The χ^2-distribution becomes moreover very steep for values close to zero, if the degrees of freedom are one or smaller.

The simulation was therefore not only run based on the approach discussed so far but additional simulation approaches were implemented so that the results can be compared. The other approaches are based on approximation algorithms presented in Andersen et al. (2010): the first approximation is based on a "full

[17] This refers of course to the actual distribution law of $\lambda_s^{\mathbb{Q}}$. The respective parameter can be relevant under the risk neutral measure $\widehat{\mathbb{Q}}$ since this distribution law is just a hypothetical construct.

truncation" Euler scheme. This scheme is – as discussed in Andersen et al. (2010) – based on the Euler algorithm which approximates the difference in the state process in the following way:

$$\tilde{\lambda}_{s+\Delta} = \tilde{\lambda}_s + \left(\mu_0 - \mu_1 \tilde{\lambda}_{s+\Delta}\right) \Delta + \sigma \sqrt{\tilde{\lambda}_{s+\Delta}} \mathcal{N} \sqrt{\Delta}. \tag{3.23}$$

Symbol Δ refers to the change in time and $\mathcal{N}$ denotes a standard-normally distributed random variable. An important issue in the context of such discrete simulation algorithms is how to prevent the simulated process from becoming negative. The "true" process is – as discussed before – always non-negative if the drift coefficients are positive (which is the case in all scenarios analyzed in this study). The mechanism which prevents the process from becoming negative in the continuous original process is, however, difficult to implement in such discrete algorithms. The stochastic component of the change in the algorithm above can dominate the deterministic drift component for all current process values. The simulated process therefore can take negative values which is not desirable since the original process is defined for positive values only. The next algorithm step cannot be executed either.

The "*full-truncation*" algorithm tries (as discussed in Andersen (2008)) to minimize this issue by setting the current process value in the drift part and the stochastic part to zero, if the simulated values are negative. The change in the process then becomes deterministic and positive:

$$\tilde{\lambda}_{s+\Delta} = \tilde{\lambda}_s + \left(\mu_0 - \mu_1 \tilde{\lambda}^{+}_{s+\Delta}\right) \Delta + \sigma \sqrt{\tilde{\lambda}^{+}_{s+\Delta}} \mathcal{N} \sqrt{\Delta} \tag{3.24}$$

with $\lambda_s^+ = max[\lambda_s, 0]$. The simulated process can, however, still become negative. The probability of negative values increases, if the mean reversion level is very low, and the diffusion coefficient is in relation to the drift coefficient rather high. Unfortunately, this is both the case in the reduced form credit risk context compared to applications in the short term context. The best way to minimize the problem of negative simulation values seems to set the simulation steps particularly small. For the present study, the time steps were set to 1×10^{-6}, which refers roughly to half a minute in the application context.

Another problem with the "fully truncated" Euler scheme is weak convergence (c.f. Andersen et al. (2010)). A possible alternative is the "modified implicit Milstein scheme":

$$\tilde{\lambda}_{s+\Delta} = \frac{\tilde{\lambda}_s + \mu_0 \Delta + \epsilon \sqrt{\tilde{\lambda}_s} \mathcal{N} \sqrt{\Delta} + \frac{1}{4}\sigma^2 \Delta (\mathcal{N}^2 - 1)}{1 + \mu_1 \Delta}. \tag{3.25}$$

This scheme is discussed in Andersen et al. (2010) and has been presented and numerically evaluated by Kahl and Jaeckel (2006b) (c.f. equation 3.28). This algorithm produces only strictly positive values if the Feller-condition is satisfied. As mentioned before, this is not generally the case in the reduced form credit risk context. For simulations run in the context of the present study, following Andersen et al. (2010) it was switched to algorithm 3.24 if a negative values had been simulated for the previous time step.

Finally, a moment matching scheme presented in Andersen (2008) and discussed in Andersen et al. (2010) has been implemented for the present study: the "$Quadratic - Exponential$" scheme. This simulation scheme is suggested in the mentioned articles to overcome the non-negativity issue described for the other algorithms. The basic idea is to draw values from other distributions, which are calibrated so that the distributions' moments comply with the moments of the respective CIR process. The chosen scheme is "based on a combination of squared Gaussian and an exponential distribution" (c.f. Andersen et al. (2010):

$$\tilde{\lambda}_{s+\Delta} = a(b + Z)^2 \tag{3.26}$$

with a and b being constants and Z being a standard normal random variable. For small process values, the density is following Andersen (2008) and Kahl and Jaeckel (2006a) approximated by:

$$\phi(\lambda_s) = p + (1-p)(1 - \exp^{-\beta \lambda_s}), \lambda_s \geq 0. \tag{3.27}$$

Details regarding the numerical implementation can be found in proposition 5 and 6 in Andersen (2008) or in proposition 3.1 in Andersen et al. (2010). The suggested condition for switching from the first to the second approximation scheme is the ratio of the conditional expectation and the conditional variance for the respective simulation step being beyond a certain level. According to Andersen (2008), this level must be between one and two. The exact choice not critical is according to this study.

The simulation was executed based on both the modified implicit Milstein scheme and the quadratic exponential scheme.

Simulation results intensities

For two parameter sets, the results with respect to the direct simulation scheme are presented in tables 3.7 and 3.8. The results based on the approximation schemes are only presented for one parameter set in tables 3.9-3.12. N denotes the length of the simulated time series and n denotes the number of simulated time series. The "full truncation" and the "implicit Milstein" approach resulted in a high number of negative values. The only way to test the described estimation approaches based on this sample is to substitute these negative values by zeros or alternatively by the minimum positive draws. The estimates based on these simulations turned however out to be highly imprecise. The significant superiority of the mixed MLE estimator based on the direct simulation (the respective results will be discussed afterwards) seems to show that the final simulations based on the other two schemes are very imprecise. The substitution of negative values has probably changed the structure of the time series too strongly. The results based on the quadratic exponential scheme do in opposition to that not include negative values. The very bad estimation results based on that simulation (compared to the estimate based on the direct simulation approach) do, however, again suggest that this approximation might be quite imprecise and that the approximation does not work very well for the respective parameter values. The results do, moreover, not strongly improve after chang-

ing the step size from one day to $\frac{1}{1000}$ day.
The estimates based on these approximation schemes are therefore assumed to be not applicable for the evaluation of the estimation techniques. That is also the reason why only the results for one parameter set are presented in this chapter. The estimates based on the data set simulated following the direct approach are - as mentioned before - at least in the mix MLE case not as bad as the estimates based on the data sets simulated following the approximating schemes. The results are still highly imprecise. The estimates are, however, at least spread around the true parameter values (c.f. table 3.3 and table 3.4).
It is not possible to empirically evaluate to which extent the impreciseness is caused by numerical impreciseness of the simulation implementation. The results do, however, suggest that estimation results based on the mix MLE give at least a rough idea about where the levels of the true parameter are at. The results based on the LS estimator are even less precise than the mix MLE but still spread around the true parameter. The estimates of the μ_1-parameter based on the normal ML approach are on the other hand completely wrong.

3.5 Conclusion

This chapter discusses a practicable estimation strategy for reduced form credit risk models under two measures based on CDS data. The focus lies on the estimation of the coefficients under the risk neutral measure $\widehat{\mathbb{Q}}$. The presented simulation results suggests that this estimation strategy performs quite well for longer time series (N=901). For short series, the estimates seem to be quite imprecise. The estimation of the parameters under $\widehat{\mathbb{P}}$ seems to be quite troublesome. It is not clear whether these difficulties result from the numerically demanding direct simulation approach or from general impreciseness of the implemented estimation procedures. Approximating simulation schemes however seem to be even less suited for the evaluation of the discussed estimation procedures. The results suggest these schemes to lead to simulations that are even less precise than the ones based on the direct approach.

Maturity	1Y	3Y	5Y	7Y	10Y
Relative Mean	0.12	-0.029	$-7.9 \times 10{-}16$	0.07	-0.059
Variance	2.33×10^{-7}	8.17×10^{-8}	1.27×10^{-8}	2.34×10^{-8}	5.27×10^{-8}
$\sigma^2_{\epsilon_{AR(3)}}$	5.14×10^{-9}	1.45×10^{-9}	1.27×10^{-8}	5.01×10^{-10}	4.75×10^{-10}

Table 3.1: Model Error Variance, Finland

Maturity	1Y	3Y	5Y	7Y	10Y
Relative Mean	-0.39	0.25	-4.81×10^{-18}	-0.09	-0.06
Variance	1.25×10^{-3}	0.71×10^{-3}	0	0.92×10^{-4}	1.95×10^{-3}
$\sigma^2_{\epsilon_{AR(3)}}$	9.37×10^{-9}	2.37×10^{-8}	0	0.56×10^{-7}	2.94×10^{-7}

Table 3.2: Model Error Variance, Poland

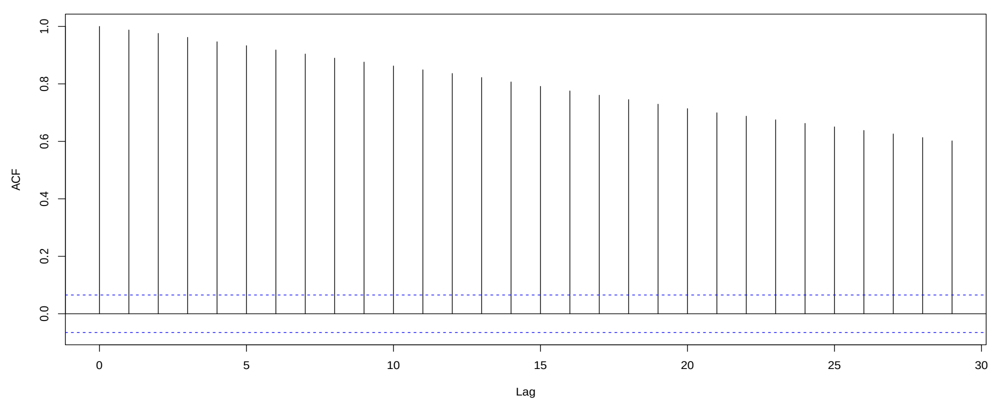

Figure 3.1: Error-ACF for the 1Y-case

Coefficient	$\mu_0^{\widehat{\mathbb{Q}}}$	$\mu_1^{\widehat{\mathbb{Q}}}$	1-LR	σ
Mean, Estimates	-1.59×10^{-10}	-0.461	3.55×10^{-10}	0.169
True value	$-2.63e^{-12}$	-0.482	3.40×10^{-10}	0.17
St. Dev., Estimates	$1.08e^{-9}$	0.028	6.6×10^{-10}	2.69×10^{-3}

Table 3.3: Finland, N=901, estimates under $\widehat{\mathbb{Q}}$ without AR; n=100; non-converging: 5

Coefficient	$\mu_0^{\widehat{\mathbb{Q}}}$	$\mu_0^{\widehat{\mathbb{Q}}}$	1-LR	σ
Mean, Estimates	$-2.66e^{-12}$	-0.49	3.52×10^{-19}	0.169
True value	$-2.63e^{-12}$	-0.482	3.40×10^{-10}	0.17
St. Dev., Estimates	$2.06e^{-14}$	2.5×10^{-14}	1.31×10^{-15}	3.13×10^{-15}

Table 3.4: Finland, N=901, estimates under $\widehat{\mathbb{Q}}$ with AR; n=300; non-converging: 4

Coefficient	$\mu_0^{\widehat{\mathbb{Q}}}$	$\mu_1^{\widehat{\mathbb{Q}}}$	1-LR	σ
Mean, Estimates	$-3.17e^{-13}$	-5.36	0.99	0.12
True value	$-1.89e^{-13}$	-5.36	0.99	0.13
St. Dev., Estimates	$1.53e^{-11}$	$3.32e^{-3}$	7.81×10^{-5}	4.52×10^{-4}

Table 3.5: Poland, N=901, estimates under $\widehat{\mathbb{Q}}$ with AR, n=300, non-converging: 0

Coefficient	$\mu_0^{\widehat{\mathbb{Q}}}$	$\mu_1^{\widehat{\mathbb{Q}}}$	LR	σ
Mean, Estimates	$-1.89e^{-11}$	-0.16	0.84	0.18
True value	$-1.89e^{-13}$	-5.36	0.99	0.13
St. Dev., Estimates	$1.86e^{-11}$	2.34×10^{-3}	4.01×10^{-4}	2.8×10^{-3}

Table 3.6: Poland, N=100, estimates under $\widehat{\mathbb{Q}}$ with AR; n=150; non-converging: 0

Quantile	0.01	0.05	0.25	0.5	0.75	0.95	0.99
Mixed MLE μ_1	10.18538	11.99	14.27	**16.21**	18.49	22.11	25.02
MLE μ_1	-8.83	-7.15	-5.29	**-4.13**	-2.99	0.96	0.98
MoM μ_1	11.36	13.97	19.75	**24.02**	28.84	35.92	42.70
Mixed MLE μ_0	0.009	0.01	0.011	**0.012**	0.012	0.013	0.013
MLE μ_0	0.0098	0.0103	0.0110	**0.0115**	0.0120	0.012	0.014
MoM μ_0	0.009	0.011	0.014	**0.0169**	0.019	0.024	0.027

Table 3.7: Estimation under $\widehat{\mathbb{P}}$ based on simulations with μ_0 : 0.0149 and μ_1 : 20.98 and σ: 0.17; Simulated time series: 1000; observations per series: 901.

Quantile	0.01	0.05	0.25	0.5	0.75	0.95	0.99
Mixed MLE μ_1	0.001	0.0011	0.011	**0.248**	0.545	1.096	2.231
MLE μ_1	-0.83	-0.56	-0.11	**0.05**	0.33	0.96	1.87
MoM μ_1	-0.072	0.57	1.636	**2.605**	3.857	6.72	8.687
Mixed MLE μ_0	3×10^{-4}	4×10^{-4}	0.06	**0.14**	0.32	0.69	1.26
MLE μ_0	$5.54e^{-6}$	$8.56e^{-5}$	5.77×10^{-2}	**0.15**	0.34	0.7	1.19
MoM μ_0	-0.04	0.32	0.93	**1.52**	2.34	4.21	6.13

Table 3.8: Estimation under $\widehat{\mathbb{P}}$ based on simulations with μ_0 : 0.5 and μ_1 : 0.8 and σ: 0.5; Simulated time series: 1000; observations per series: 901.

Quantile	0.01	0.05	0.25	0.5	0.75	0.95	0.99
Mixed MLE μ_1	0.02	0.02	1.4	1.42	418.17	1459.93	2415.73
MLE μ_1	0.02	0.02	0.02	0.02	0.02	0.02	0.02
MoM μ_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01
Mixed MLE μ_0	0	1×10^{-3}	1×10^{-3}	0.05	0.56	24.66	25.67
MLE μ_0	0.96	0.97	0.97	0.97	0.97	0.98	0.98

Table 3.9: Estimation under $\widehat{\mathbb{P}}$ based on simulation approximation scheme with μ_0 : 0.0149 and μ_1 : 20.98 and σ: 0.17; Simulated time series: 1000; observations per series: 901.

Quantile	0.01	0.05	0.25	0.5	0.75	0.95	0.99
Mixed MLE μ_0	0.0098	0.0103	0.011	0.0114	0.0119	0.0127	0.0133
MLE μ_0	0.0098	0.0103	0.011	0.0115	0.012	0.0128	0.0135
MoM μ_0	0.0091	0.0109	0.0144	0.0169	0.0193	0.0236	0.0273
Mixed MLE μ_1	10.19	11.99	14.27	16.21	18.49	22.11	25.02
MLE μ_1	-8.83	-7.15	-5.29	-4.13	-2.99	0.96	0.98
MoM μ_1	11.37	13.97	19.75	24.02	28.84	35.92	42.7

Table 3.10: Estimation under $\widehat{\mathbb{P}}$ based on the fully truncation approximation scheme with μ_0 : 0.0149 and μ_1 : 20.98 and σ: 0.17; Simulated time series: 1000; observations per series: 901.

Quantile	0.01	0.05	0.25	0.5	0.75	0.95	0.99
Mixed MLE μ_0	0	2.1×10^{-5}	1.4	1.42	418.17	1459.93	2415.73
MLE μ_0	0	0.54×10^{-6}	4.14×10^{-6}	3.03×10^{-5}	0.64×10^{-3}	1.96×10^{-3}	2.59×10^{-3}
MoM μ_0	0	-3.15×10^{-3}	-0.97×10^{-3}	0.66×10^{-3}	2.26×10^{-3}	4.66×10^{-3}	0.01
Mixed MLE μ_1	0	1×10^{-3}	1×10^{-3}	1.03×10^{-3}	0.3	1.04	1.72
MLE μ_1	-0.16	-0.1	-0.02	0.02	0.08	0.15	0.22
MoM μ_1	-0.27	-0.16	-0.05	0.03	0.12	0.25	0.34

Table 3.11: Estimation under $\widehat{\mathbb{P}}$ based on quadratic exponential approximation scheme with μ_0 : 0.0149 and μ_1 : 20.98 and σ: 0.17; Simulated time series: 1000; observations per series: 901.

Quantile	0.01	0.05	0.25	0.5	0.75	0.95	0.99
Mixed MLE μ_0	0.02	0.02	1.4	1.42	418.17	1459.93	2415.73
MLE μ_0	0.02	0.02	0.02	0.02	0.02	0.02	0.02
MoM μ_0	0.008	0.009	0.011	0.011	0.014	0.021	0.022
Mixed MLE μ_1	0	1×10^{-3}	1×10^{-3}	0.05	0.56	24.66	25.67
MLE μ_1	0.96	0.97	0.97	0.97	0.97	0.98	0.98
MoM μ_1		0	0	0	0	0.01	0.013

Table 3.12: Estimation under $\widehat{\mathbb{P}}$ based on quadratic exponential simulation approximation scheme with μ_0 : 0.0149 and μ_1 : 20.98 and σ: 0.17; Simulated time series: 1000; observations per series: 901.

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Chapter 4

European Sovereigns' Credit Spreads and the Second Dimension Risk – A reduced form Analysis

4.1 Introduction

Sovereign credit spreads became a very frequently cited economic figure during the last months and years. The strong increase in certain European countries' credit costs triggered the current European financial crisis by exposing massive problems of these countries to refinance themselves at costs they can afford in the long run. The evaluation of policy actions to overcome that problem turned out to be very difficult since there seems to be only very little knowledge about the factors driving European sovereigns' credit spreads.

This chapter analyzes one possible driver of credit spreads during the European fiscal crisis: the risk premium that market participants expect because of uncertainties with respect to the prospective and current default probabilities. This risk premium does not directly refer to the possibility of a default *per se* at a given default probability, but it refers to the possibility of unfavorable corrections regarding the default probability. This premium should consequently be irrelevant by definition, if the default probability were deterministic and directly observable – even if this default probability was characterized by very high or increasing levels. An analytical model based definition of this premium will be provided later. It is in this chapter empirically tested whether such a risk premium has had an impact on both Euro and non-Euro sovereigns' credit costs during the European fiscal crisis. Moreover, the impact of such a risk premium on correlations between sovereigns' credit costs is analyzed in this context.
The risk premium for uncertainties regarding default probabilities has not been widely studied in the sovereign context so far and no specific term exists, which is always used to refer to the risk of unfavorable changes in the default probability[1]. Longstaff et al. (2011) refer to the "distress risk", but this term does not seem to capture the exact meaning of the respective risk dimension. Possible unexpected changes in default probabilities may be relevant for credit spreads

[1] In the presented framework, the stochastic development of a Poisson intensity – and not the default probability per se – is modelled. This does however – as will be elucidated later on – directly imply a stochastic behavior of the default probability itself.

even if the concerned unit is not in actual distress. The term “distress” is, moreover, often used in a different context.

Consequently, another term is introduced and used in this chapter to refer to the respective risk: the “second dimension risk”. This terminology seems to be appropriate as the respective risk does not refer to the default possibility per se, but to a second risk dimension caused by the plausibly assumed non-observability and non-deterministic development of the underlying default probability.

The third section of the present chapter shows why this second dimension risk might be highly relevant for the development of sovereigns’ credit spreads during the European fiscal crisis. In the centre of this argument are surprising insights into member countries’ true fiscal situations as well as sudden changes in legal determinants of the fiscal policy in member countries. The empirical analysis is based on a doubly stochastic reduced form credit risk model introduced by Jarrow and Turnbull (1995), Lando (1998) respectively Duffie and Singleton (1999). In the doubly stochastic reduced form framework a unit’s default is modelled as a first jump of a Poisson process with the underlying intensity being driven by a stochastic differential equation. The model estimation is based on European sovereign “credit default swap” (CDS) data for the years 2008-2012. The modelling framework is introduced in section four and the estimation strategy and results are discussed in section five. In this context, the fit of the model is studied, the results are analyzed with respect to the relevance of the second dimension risk premium during the fiscal crisis and forecasting capacities of the respective models are studied as well. The next section gives an overview of related research projects.

4.2 Literature review

The following overview considers two kinds of articles: the ones analyzing possible determinants of sovereign credit spreads in general and articles focussing on the relevance of risk premia similar to the one defined above in both the corporate and sovereign context. The following paragraphs focus on the numerous studies of the first kind.

Most of these studies do not consider European sovereigns during the fiscal crisis but other geographic areas or other periods. These studies mostly suggest that sovereign credit spreads are mainly driven by global financial market risk factors approximated by measures like the implied volatility index VIX (see e.g. Kaminsky and Reinhart (2002), Pan and Singleton (2008), respectively Longstaff et al. (2011), Favero et al. (2010), Hao (2011), Baek et al. (2005), Eichengreen and Mody (2000), Mauro et al. (2002), Remolona et al. (2008), Geyer et al. (2004)). In opposition to that, country specific economic figures did not seem to be very important (see e.g. Alper et al. (2012)).

Most of these studies describe the correlation of sovereign credit spreads as rather strong (see e.g. Kaminsky and Reinhart (2002)), which is often assumed to be mainly caused by global financial market risk measures being important drivers of sovereign spreads. The described findings are also supported in the European sovereign context for the years before 2008 (De Santis (2012)). The explanatory power of variables like the VIX index with respect to European sovereign credit spreads decreased strongly during the past few years. The co-movement between spreads of specific countries stays high for this period (c.f. De Santis (2012)). De Santis (2012) suggests that – in the cases of sovereigns like Portugal, Ireland or Spain – in these years the spread development is instead largely affected by contagion effects going back to the Greek crisis.

This contagion effect may have been enforced by the bank rescue packages and the related risk transfer from banks to sovereigns (c.f. Ejsing and Lemke (2011)). A detailed understanding of how that contagion could have worked technically in the context of the European fiscal crisis is, however, still missing. The present chapter provides evidence on the relevance of the mentioned risk premium and argues that this risk premium might have been an important driver of these contagion effects.

Among the second group of studies (i.e. the studies that analyze similar kinds of risk premia in both the sovereign and the corporate context) only very few are taking exactly the same type of risk into consideration as the present study. There are, however, (mainly theoretical) studies dealing with related kinds of risk as possible drivers of credit spreads. Pouzo and Presno (2011) replicate, for example, the surprisingly high credit spreads during the Latin fiscal crisis in a general equilibrium model. They do so by factoring in investors' uncertainty regarding the true probability of borrowers' future endowment states, while assuming default probabilities to be in accordance with historical levels. They talk about an "additional uncertainty risk premium". Hao (2011) calculates the difference between implied realized volatility in the corporate context. His study is based on equity options and the realized variance of equity returns for several countries. He analyzes the impact of this difference on the respective corporates' credit spreads and concludes that the "variance risk premium" is indeed an important factor for a corporate's credit spreads. These results may explain the mentioned findings that sovereign credit spreads are mainly driven by global risk factors.

Longstaff et al. (2011) and Pan and Singleton (2008) empirically analyze the relevance of the risk premium as it is defined in the present chapter and – unlike Pouzo and Presno (2011) and Hao (2011) – refer directly to the risk of corrections regarding current and future default probabilities. Like the present study, their analysis is like the present study executed in a doubly stochastic reduced form credit risk model framework. Their results suggest that the analyzed risk premium is highly relevant for the included sovereigns' credit spreads during the respective years.

The present analysis is oriented towards the methodology in Pan and Singleton (2008) and Longstaff et al. (2011). In opposition to the estimation in the present chapter, the respective estimations in these studies are not based on sovereign credit data from the years of the European fiscal crisis. The possible interplay between the events during the European fiscal crisis and the examined risk premium is consequently not analyzed and discussed either. In the present chapter a convenient and practicable estimation procedure for such a reduced form model driven by an affine process based on credit default swap (CDS) is applied. This estimation strategy is based on the modelling choice for the underlying stochastic differential equation. Unlike Pan and Singleton (2008) and Longstaff et al. (2011), an affine process was chosen for the present study: the "CIR" diffusion introduced in the finance literature by Cox et al. (1985). The estimation strategy applied in the present chapter uses the characteristics of the affine process class.

The next section discusses why the second dimension risk premium might indeed have been an important factor for sovereign credit spreads during the European fiscal crisis. First of all, it is argued that this premium may have had an important impact on European sovereign credit spreads during the last years from a single country perspective and it is moreover argued that this risk premium may also have induced higher correlation between sovereign credit spreads during these years. This discussion is followed by an introduction to the modelling framework. The estimation procedure is described afterwards. Then the technical interpretation of the estimated model is discussed. Finally, the estimation results are interpreted and the forecast capacities of the estimated models are evaluated.

4.3 The "second dimension risk premium" and the European fiscal crisis

The second dimension risk might be highly relevant for spreads of both European countries actually struggling during the fiscal crisis and countries which have not been in acute distress. Revealed uncertainty regarding the current and future fiscal situations are an important aspect of the fiscal crisis. The Greek government corrected previously published fiscal information significantly[2]. Moreover, Germany and France violated the putative legally binding upper household deficit limit and have not been punished for that.

Both the correction of Greek fiscal figures and the high fiscal deficits of Ger-

[2]In November 2009, "the Greek government revealed a revised budget deficit of -12.7% of GDP for 2009, which was the double of the previous estimate" (c.f. De Santis (2012))

many and France could have lead to a twofold increase in these countries' credit spreads: On the one hand, the negative correction of Greek fiscal figures could have lead to an increase in credit spreads due to an actual increase in the default probability related to the actual deterioration of the observed fiscal situation. The fact that the presumptions regarding the country's fiscal situation are based on information which turned out to be not very robust could have on the other hand lead to an increase in the second dimension risk premium as well.

The same line of reasoning holds for the unpunished violation of alleged legal fiscal policy determinants: the fact that the net indebtedness of Germany and France were higher than originally postulated in the European Stability and Growth pact should ceteris paribus already have lead to an increase in actual default probabilities for these countries. The fact, that alleged legal determinants of fiscal policies actually do not exist, also leads to higher uncertainty regarding these countries' future fiscal policies and therefore regarding future default probabilities. This could in turn have lead to another increase in the second dimension risk premium component.

Events like the ones just described could, moreover, not only explain spread developments of single countries, but they may also cause or catalyse correlations between different European countries' credit spreads due to the second dimension risk premium. This is a very important aspect with respect to the events during the European fiscal crisis, since the structure of credit cost time series were perceived as surprisingly similar for several European sovereigns by many financial market commentators, because "the transmission of the initial instability goes beyond what could be expected from the normal relationships between markets or intermediaries, for example in terms of its speed, strength or scope." (Constancio (2011)).

A comovement between two countries' credit spreads might be induced by the existence of a second dimension risk premium if these countries' risk premium components are driven by common factors. Such a factor might of course be the market participants' risk appetite itself, but it could as well be a common source driving the market participants' uncertainty regarding current and future default probabilities of two countries. The second dimension risk premium might therefore have been an important driver of the mentioned "contagion" effects during the European fiscal crisis.

A factor driving the uncertainty regarding the default probabilities of several countries at the same time could, for example, be the reputation of certain institutions. By accepting countries as members of the Euro zone, European institutions likewise implicitly rate both their fiscal information and their fiscal stability as sufficient. If it turns out that fiscal information published by Euro zone members needs to be significantly corrected or that the fiscal situation of one country was rated overly optimistic, being accepted as member in the Euro zone may loose its characteristic as positive signal to market participants. Market participants' uncertainty regarding the assessment of the other countries' financial situations could then – due to the induced uncertainty regarding the countries' true fiscal situation – increase, even if the level of other countries' default probabilities may not be impacted directly by a change in information with respect to the situation of the first country.

The exemplarily described situation could be adapted to the Greek crisis. Market participants' perception of the membership in the European monetary union as a signal for sufficient fiscal information quality might have suffered. Market participants could have felt more insecure with respect to their anticipation of European sovereigns' default probabilities and the Greek balance sheet corrections might have lead to a twofold increase in other European sovereigns' credit spreads: first due to real economic effects respectively to a related direct actual increase in the default probabilities in other countries and secondly due to the second dimension risk respectively higher uncertainty regarding the actual default probability.

Another example for a factor possibly driving the uncertainty regarding several countries' fiscal information and stability might be the European Stability and Growth pact. This arrangement aims to assure the fiscal stability of the Euro zone member countries. The violation of such a legal arrangement by one country may not only affect the investors' uncertainty regarding the true cur-

rent and future level of default probability with respect to countries violating the agreement. If the expected consequences for the violating countries are not put through, the uncertainty whether other countries will be tempted to ignore the rules increases as well. The market participants' uncertainty with respect to the other countries' future fiscal policies respectively default probabilities should therefore increase likewise.
The latter situation has already been described before. It is, however, important to point out that the second dimension risk premium might not only lead to a disproportionate increase in the spreads of the countries violating the respective agreement, but it might also lead to a twofold increase in other countries' spreads: First, the anticipated default probability of these countries may directly be reassessed as unstable fiscal strategies seem to be more likely after the edge was taken off an important rule intended to fiscally discipline the European sovereigns. Secondly, the revealed general uncertainty about the fiscal policies may have lead to an increase in the second dimension risk premium – both in the case of the particular country as well as in the case of the other member countries. The question whether the unpunished violation of the European Stability and Growth induced credit spreads correlation between European sovereigns can, however, not be empirically analyzed in the presented framework as – from a research perspective unfortunately – there has not been a similar incident since CDS contracts are frequently enough traded to be the basis of a sufficient data supply.

Summing up, the second dimension risk premium might have been an important driver of sovereign credit spreads in Europe. Moreover, it might have been an important driver of the observed comovement between sovereign credit spreads respectively the contagion during the European fiscal crisis as well.

4.4 The modelling framework

We consider a measure space $(\Omega, \mathcal{F}_1, P)$, an index set $S \neq \emptyset$[3] and the Poisson process

$$\mathcal{P}oi = (Poi_s, s \in S) \tag{4.1}$$

driven by the intensity λ_s. This Poisson process generates a filtration $\mathcal{F}_{1,s}$: $\mathcal{F}_{1,s} = \sigma\{Poi_t : 0 \leq t \leq s\}$ with $t \in S$. The default of a unit is in this model defined as a first jump of this Poisson process and the time of the first jump denoted as $\tau \in S$ is therefore the stopping time for this process as well.

The period of time between a starting time $s_0 \in S$ and the first jump of the underlying Poisson process are exponentially distributed with the parameter process λ_s. No-arbitrage pricing formulas for all kinds of credit risk related securities have been derived based on that and Lando (1998) presents for example pricing formulas for simple zero bonds.

For this example, a zero bond be defined with face value one, issued at time $s_0 \in S$, with a recovery rate $1 - LR$ (denoting the fraction of the face value which is paid in the case of default right after the default occurred), maturity M (denoting the number of years until the principal is paid back) with $[s_0, s_0 + M] \subset S$, and payoff Z_s for $s \in S$, with $Z_{s_0+M} = 1$ and $Z_{s'} = 0$ for all $s' \neq s_0 + M$ if $\tau \notin [s_0, s_0 + M]$ as well as $Z_\tau = 1 - LR$ and $Z_{s''}$=0 for all $s'' \neq \tau$ if $\tau \in [s_0, s_0 + M]$. Lando (1998) shows that the market price ZB_{s_0, s_0+M} of

[3] One time unit refers in the context of the estimation, which is discussed later on, to one year (and not one day).

this bond in s_0 is for deterministic intensities given by:

$$
\begin{aligned}
& ZB_{s_0,s_0+M} \\
&= \mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+M} r_s ds} Z_s | \mathcal{F}_{1,s_0}\right] && (4.2) \\
&= \mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+M} \lambda_s + r_s ds} | \mathcal{F}_{1,s_0}\right] + (1-LR)\left[\int_{s_0}^{s_0+M} \mathbb{E}_{s_0}\left[\lambda_s e^{-\int_{s_0}^{s} \lambda_u + r_u du} | \mathcal{F}_{1,s_0}\right] ds\right] && (4.3) \\
&= \mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+M} \lambda_s^{\mathbb{Q}} + r_s^f ds} | \mathcal{F}_{1,s_0}\right] + (1-LR)\left[\int_{s_0}^{s_0+M} \mathbb{E}_{s_0}\left[\lambda_s^{\mathbb{Q}} e^{-\int_{s_0}^{s} \lambda_u^{\mathbb{Q}} + r_u^f du} | \mathcal{F}_{1,s_0}\right] ds\right] && (4.4)
\end{aligned}
$$

r_s^f denotes the risk free rate and the resulting discount factor complies with $ZB^f_{s_0,s_0+m}$ denoting the price of a risk free zero bond issued in s_0 with maturity m. r_s denotes the return expected by the investors in this zero bond and $\lambda_s^{\mathbb{Q}}$ denotes the risk neutral default intensity that allows to switch from r_s to r_s^f. The first component of the sum refers to the actual repayment that is executed after time M if the counterparty has not defaulted yet. The payment is therefore weighted by the respective probability that the default has not occurred yet after M periods. The second component refers to the payment that is executed if the counterparty defaults right after the default occurred. The respective amount is therefore in each point in time weighted by the joint probability that the default has not occurred yet but occurs right then (c.f. Lando (1998) respectively the first chapter of this dissertation).

This framework is now extended for allowing more general intensities. These are from now on assumed to be stochastic and to follow a Cox-Ingersoll-Ross (CIR) diffusion:

$$d\lambda_s = (\mu_0 - \mu_1\lambda_s) + \sigma_1\sqrt{\lambda_s}dB_s \qquad (4.5)$$

with B_s denoting a Brownian motion and μ_0, μ_1 and σ_1 being constant coefficients. The intensity process generates a filtration $\mathcal{F}_{2,s} = \sigma\{\lambda_t : 0 \leq t \leq s\}$ as well with $t, s \in S$. Finally, a filtration $\mathcal{F}_s$ is defined as

$$\mathcal{F}_s = \sigma\{\mathcal{F}_{1,s} \vee \mathcal{F}_{2,s}\}, \; for \; all \; s \in S \tag{4.6}$$

with "$\vee$" in this context denoting the union of σ-fields respectively filtrations.

After the introduction of this second uncertainty dimension, the equality of equation 4.3 and equation 4.4 does not necessarily hold anymore. It may be the case, that investors' expected returns change because of this particular source of uncertainty. Switching from $\lambda^{\mathbb{P}}$ to $\lambda^{\mathbb{Q}}$ would then not be sufficient to adjust for switching from r_s to r_s^f[4]. Accordingly, two further variants of measures with respect to $\lambda_s^{\mathbb{Q}}$ are introduced: $\widehat{\mathbb{Q}}$ and $\widehat{\mathbb{P}}$ are introduced. The latter refers to the actual distribution law of $\lambda_s^{\mathbb{Q}}$ and $\widehat{\mathbb{Q}}$ refers to expectations with respect to (transforms of) $\lambda_s^{\mathbb{Q}}$ so that the pricing formula including a discount rate based on r_s^f holds despite of the possible existence of the respective "second dimension" risk premium. The expectations based on these distribution laws are denoted by $\mathbb{E}_s^{\widehat{\mathbb{P}}}$ respectively $\mathbb{E}_s^{\widehat{\mathbb{Q}}}$ in the following and one rewrites – following Pan and Singleton (2008) and Longstaff et al. (2011) – for formulas 4.11 and 4.12:

$$\mathbb{E}_{s_0}\left[e^{-\int_{s_0}^{s_0+M}\lambda_s + r_s ds}|\mathcal{F}_{s_0}\right] + (1-LR)\left[\int_{s_0}^{s_0+M}\mathbb{E}_{s_0}\left[\lambda_s e^{-\int_{s_0}^{s}\lambda_u + r_u du}|\mathcal{F}_{s_0}\right]ds\right]$$
$$= \mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[e^{-\int_{s_0}^{s_0+M}\lambda_s^{\mathbb{Q}} + r_s^f ds}|\mathcal{F}_{2,s_0}\right] + (1-LR)\left[\int_{s_0}^{s_0+M}\mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}}\left[\lambda_s^{\mathbb{Q}} e^{-\int_{s_0}^{s}\lambda_u^{\mathbb{Q}} + r_u^f du}|\mathcal{F}_{2,s_0}\right]ds\right] \tag{4.7}$$

The distinction between the two distribution laws of $\lambda_s^{\mathbb{Q}}$ requires another notation of the diffusion equations driving $\lambda_s^{\mathbb{Q}}$ under both measures. Following the literature standard (see e.g. Longstaff et al. (2005), Pan and Singleton (2008),

[4] It is not guaranteed that this source of risk leads to an increase in the aggregated expected return. It could be the case that investors profiting from increases in default intensities (e.g. as they sell bonds short or as they are betting on a default by buying credit insurance contracts) dominate the price setting process. Such outcomes are in the following denoted by the term "negative risk premia". The main part of the debate in this chapter is however restricted to positive market prices of second dimension risk as it seems to be the more relevant scenario.

Longstaff et al. (2011)), one rewrites the underlying diffusion equations under the risk neutral measure $\widehat{\mathbb{Q}}$ as

$$d\lambda_s^{\mathbb{Q}} = \left(\mu_0^{\widehat{\mathbb{Q}}} - \mu_1^{\widehat{\mathbb{Q}}}\lambda_s^{\mathbb{Q}}\right) ds + \sigma_1\sqrt{\lambda_s^{\mathbb{Q}}}dB_s^{\widehat{\mathbb{Q}}} \tag{4.8}$$

respectively under the actual measure $\widehat{\mathbb{P}}$

$$d\lambda_s^{\mathbb{Q}} = \left(\mu_0^{\widehat{\mathbb{P}}} - \mu_1^{\widehat{\mathbb{P}}}\lambda_s^{\mathbb{Q}}\right) ds + \sigma_1\sqrt{\lambda_s^{\mathbb{Q}}}dB_s^{\widehat{\mathbb{P}}}. \tag{4.9}$$

If market participants do not expect a specific remuneration for taking the uncertainty regarding $\lambda_s^{\mathbb{Q}}$, the expectations under $\widehat{\mathbb{P}}$ and $\widehat{\mathbb{Q}}$ with respect to this risk neutral intensities respectively the transforms included in these pricing formulas should not differ. The opposite is the case, if the change in expected returns due to this uncertainty is high. The relevance of the "second dimension" risk premium can be analyzed accordingly based on the coefficients of these diffusion equations under both measures.

There are various ways to do that. First, the coefficients can be evaluated directly. Alternatively, the coefficients can be analyzed with reference to the underlying intensity process. The expectations regarding $\lambda_s^{\mathbb{Q}}$ can be calculated for various time horizons conditioned on the current value of the underlying process based on both distribution laws and the difference between these expectations can be evaluated. Moreover, the prices of credit securities can be calculated based on no-arbitrage pricing formulas like the one introduced with the respective expectations being based on $\widehat{\mathbb{P}}$ and $\widehat{\mathbb{Q}}$. In the presented framework, the difference in the respective model prices refers to the second dimension risk premium.

Finally, one can also evaluate the coefficients based on the Girsanov theorem. This theorem is often applied in the field of quantitative equity modelling and has been applied in the credit risk context by Pan and Singleton (2008). In this context, the variable, which – after being transformed based on the Girsanov theorem – is Radon-Nikodym derivative (to describe the change from the actual measure $\widehat{\mathbb{P}}$ to the risk neutral measure $\widehat{\mathbb{Q}}$) is called "market price of risk". This leads to the link between $B_s^{\widehat{\mathbb{P}}}$ and $B_s^{\widehat{\mathbb{Q}}}$ in dependence on this market price of risk denoted by η_s:

$$B_s^{\widehat{\mathbb{P}}} = B_s^{\widehat{\mathbb{Q}}} + \int_{s_0}^{s} \eta_u du. \tag{4.10}$$

Substituting "$dB_s^{\widehat{\mathbb{P}}}$" by "$dB_s^{\widehat{\mathbb{Q}}} + \eta_s ds$" in the diffusion equations under $\widehat{\mathbb{P}}$ results in

$$d\lambda_s^{\mathbb{Q}} = \left(\mu_0^{\widehat{\mathbb{P}}} - \mu_1^{\widehat{\mathbb{P}}}\lambda_s^{\mathbb{Q}}\right) ds + \sigma_1\sqrt{\lambda_s^{\mathbb{Q}}}\left(dB_s^{\widehat{\mathbb{Q}}} + \eta_s ds\right). \tag{4.11}$$

Moreover, a specific functional form linking η_s and $\lambda_s^{\mathbb{Q}}$ is assumed. The specific form is chosen based on the plausible assumption that the increase in change in the respective intensity should increase linearly in this intensity's levels (c.f. Cheridito et al. (2007) and Duffee (2002)). η_s already goes into the change of $\lambda_s^{\mathbb{Q}}$ as a factor multiplied by $\sigma_1\sqrt{\lambda_s^{\mathbb{Q}}}$. To obtain a linear form, it is accordingly assumed that η_s depends on $\lambda_s^{\mathbb{Q}}$ in the following way:

$$\eta_s = \frac{\rho_0}{\sqrt{\lambda_s^{\mathbb{Q}}}} + \rho_1\sqrt{\lambda_s^{\mathbb{Q}}}. \tag{4.12}$$

This results in the actual difference in change of $\lambda_s^{\mathbb{Q}}$ being given by

$$\sigma_1\left(\rho_0 + \rho_1\lambda_s^{\mathbb{Q}}\right) \tag{4.13}$$

which is a linear function in $\lambda_s^{\mathbb{Q}}$. This implies the following link between ρ_0, ρ_1 and the CIR coefficients under both measures:

$$\rho_0 = \frac{\mu_0^{\widehat{\mathbb{Q}}} - \mu_0^{\widehat{\mathbb{P}}}}{\sigma_1} \tag{4.14}$$

$$\rho_1 = \frac{\mu_1^{\widehat{\mathbb{P}}} - \mu_1^{\widehat{\mathbb{Q}}}}{\sigma_1}. \tag{4.15}$$

Accordingly, η_s refers to the change in the deterministic drift induced by a change from the historical to the risk neutral measure. A positive value for η_s means that the short term expectations equating the no-arbitrage pricing for-

mula (4.7) are "pessimistic" compared to the expectations one would get under the historical measure. This means, for example, in the context of credit insurance, that the insurer expects and insurance buyer agrees to pay an additional compensation for taking the risk of changes in $\lambda_s^{\mathbb{Q}}$ on the short run. A negative value would on the other hand, for example, refer to situations where investors in credit insurance contracts "betting" on the default of a unit on the short run, are only willing to pay a price below the risk neutral level and this reluctance dominates the respective risk aversion of the insurance seller side. It is in this context important to point out, that a positive value of η_s does not necessarily lead in a complementary final risk premium. The difference in deterministic drift only refers to a change in the underlying process for a specific point in time. It is possible that for a wider horizon expected process values are higher under $\widehat{\mathbb{Q}}$ than under $\widehat{\mathbb{P}}$ even though η_s is negative and vice versa.
In the following, the estimation of such a model is discussed. Credit insurance securities are also introduced in the presented framework, because insurance data is a possible basis for this estimation.

4.5 Data and related transformations of the model representation

The estimation of such a doubly stochastic reduced form credit risk model can be acchieved based on any kind of credit related security, but there are two particularly relevant candidates: historical bond prices issued by a particular sovereign and historical spreads of "credit default swaps" (CDS) that insure the buyer against a default of the particular sovereign. The information based on the latter choice might be affected by counterparty risk. A frequently used argument in this context is that the large number of counterparties enables insurance buyers to diversify the counterparty risk to a neglectable level. Fontana and Scheicher (2010) argue that this is less convincing in the sovereign context due to the strong linkage between the financial industries' and the particular sovereigns' financial stability. The spreads published are, however, based on a wide range of institutions located in many different global areas. Here, this should lead to a strong diversification effect despite the strong connection of financial institutions' stability and sovereigns' ability to serve their debt.

The advantage of CDS spreads is the very comfortable data situation. New contracts referring to one sovereign's default risk are issued on a daily basis for a whole range of maturities. Bond prices moreover seem to be rather affected by the flight to liquidity phenomenon as discussed in Longstaff et al. (2005), Beber et al. (2009) or Fontana and Scheicher (2010). This is in accordance with the results presented by Alper et al. (2012). They analyze the difference between advanced economies' CDS and "relative asset spreads"[5] (RAS) of advanced economies during the European fiscal crisis and find that CDS quotes seem to lead the pricing process of sovereign credit risk. The estimation for this project is therefore going to be based on CDS data.

The CDS pricing formula can be easily derived in the doubly stochastic reduced form framework. Considering a contract issued in $s_0 \in S$, with maturity M ($[s_0, s_0 + M] \subset S$), with spread $SP_{s_0}(M)$ to be paid semi-annually and the loss rate LR defined as before. The loss rate complies for CDS with a face value[6] of one with the amount the insurer has to pay in the case of default[7]. The pricing formula is, as described in Duffie (1999), given by

$$\begin{aligned} SP_{s_0}(M) \sum_{n=1}^{2M} \left(\mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}} \left[e^{-\int_{s_0}^{s_0+0.5n} \lambda_s^{\mathbb{Q}} ds} | \mathcal{F}_{2,s_0} \right] ZB_{s_0,s_0+0.5n}^f \right) \\ = LR \left[\int_{s_0}^{s_0+M} ZB_{s_0,s}^f \mathbb{E}_{s_0}^{\widehat{\mathbb{Q}}} \left[\lambda_s^{\mathbb{Q}} e^{-\int_{s_0}^{s} \lambda_u^{\mathbb{Q}} du} | \mathcal{F}_{2,s_0} \right] ds \right]. \end{aligned} \tag{4.16}$$

[5]Denoting the gap between a particular government yield and the fixed rate arm of an interest rate swap in the domestic currency with the same maturity.

[6]Face value refers in the CDS context to the insured amount of money.

[7]Both the term recovery rate and the term loss rate refer to a fraction with respect to the face value and not the market value. The ratio of two CDS spreads should therefore not depend on the loss rate as described in Pan and Singleton (2008). The model, moreover, implicitly assumes that the loss rate in the pricing formula does not include a risk premium, in opposition to the risk neutral intensity.

The left part of the equation gives the expected discounted payoff from the perspective of the CDS seller, while the counterpart from the perspective of the CDS buyer can be seen on the right hand side. The single payment obligations to be paid by the buyer can here be considered to be zero bonds depending on the survival of the third party whereas the right hand side refers to a payment that is executed if the third party defaults and the particular default probability is only positive if the third party has not defaulted yet. Index s_0 refers in this context not only to the point in time when a single contract is issued, but it is in turn index for the historical time series of CDS spreads used for the estimation. The formula is designed for pricing of both sovereign and corporate CDS. Following Duffie et al. (2003), it is assumed that the total default intensity λ_s combines the probabilities of different kinds of credit events like liquidation events or restructuring with λ_s being the sum of intensities referring to one particular default event. The loss rate is then correspondingly the average of the loss rates for all the different credit events, weighted by the particular probabilities.

The analysis is executed for several European countries: Spain, Ireland, Iceland, Estonia, Finland, Poland. These countries can be classified by different criteria: membership in the Euro area (this excludes Iceland and Poland) and countries that have been in acute stress during the crisis (this excludes Finland and Poland). The spreads for Estonia and Iceland have, moreover, decreased significantly from the first part of the sample to the second part, while the opposite can be said about Ireland and Spain. The period which the analysis is from October 2008 to march 2012. The reason for not taking earlier data is that historical CDS spread data does not reach back very far as CDS is rather a new security type. The historical CDS spread time series were supplied by Thomson-Reuters. The spreads for the particular period are depicted in figure 4.1. Both the spreads' strong increase during this period and the similarity in time series patterns is striking.

The prices for risk free zero bonds are approximated by using prices of zero bonds issued by AAA rated units. These prices are calculated based on the spot rate curve published by the ECB. The published data points (every three months with a range from three months to 30 years) are linked by linear interpolation.

4.6 Estimation and results

The key of the empirical analysis is the estimation and the comparison of the CIR coefficients under measure $\widehat{\mathbb{Q}}$ and measure $\widehat{\mathbb{P}}$. Estimating the distribution law of $\lambda_s^{\mathbb{Q}}$ under the risk neutral measure $\widehat{\mathbb{Q}}$ is a challenging task as a set of spread time series $SP_{s_{0_i}}(M)$ (with s_0 denoting the index variable for spreads of newly issued CDS contracts and $i \in [1, .., N]$ denoting the index variable referring to the actual observations in a data-set with N observations) and the risk neutral discount factors $ZB^f_{s_{0_i}, s_{0_i}+m}$ are the only observable data. The coefficients of the CIR diffusion, a time series $\lambda^{\mathbb{Q}}_{s_{0_i}}$ as well as the loss rate LR have to be estimated. The estimation procedure is presented in the second chapter of this dissertation in detail. It is quickly summarized in the following paragraph: If the underlying CIR diffusion was already well known, one could easily estimate a time series of the intensity process $\lambda_s^{\mathbb{Q}}$ by numerically solving the pricing formula. The expectations with respect to transforms of future intensity values could be substituted by formulas which only depend on the intensity's value on the day the expectation has been built. This could be done by exploiting that the CIR diffusion process belongs to the class of affine processes (c.f. Duffie et al. (2000)[8]).

However, the set of CIR coefficients has still to be estimated. To overcome this difficulty, it is exploited that CDS spreads are published on a daily basis for a wide range of maturities. The CDS of different maturities with respect to the particular unit moreover refer all to the same underlying Poisson process. The iterative procedure chosen for the present study is based on this wide range of

[8]Duffie et al. (2000) show that an expectation with respect to certain transforms of affine processes can be substituted by an exponential affine function only depending on the value of the process at the time the expectation is built in. The coefficients of this exponential affine function are, moreover, solutions to specific ordinary differential equations (ODEs) only depending on the coefficients of the underlying diffusion equation. The analytical solutions to these ODEs were presented by Longstaff et al. (2005)

daily issued CDS in the following way:

First, a range of parameters $\{\widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1, \widehat{LR}\}$ is assumed ex-ante (c.f. Longstaff et al. (2005)). A time series of intensities $\widehat{\lambda}_{s_{0_i}}^{\mathbb{Q}}$ is then estimated based on this set of assumed coefficients, the time series of 5-year CDS spreads $SP_{s_{0_i}}(5)$ and the time series of risk neutral discount factors $ZB_{s_{0_i},s}^f$ for all maturities $[s_{0_i}, s] \subset S$. The estimated time series is then taken and the sum of squared differences between the model implied CDS spreads (based on these previously estimated intensities) denoted by $\widehat{SP}_{s_{0_i}}(M)$ and the observed CDS spreads $SP_{s_{0_i}}(M)$ is minimized for other maturities (1,3,7,10 years) with respect to the set of coefficients $\{\mu_0^{\widehat{\mathbb{Q}}}, \mu_1^{\widehat{\mathbb{Q}}}, \sigma_1^{\widehat{\mathbb{Q}}}, LR\}$. The particular model spreads are in this context given by

$$\widehat{SP}_{s_{0_i}}(M) = \frac{\widehat{LR}\left[\int_{s_{0_i}}^{s_{0_i}+M} ZB_{s_{0_i},s}^f \mathbb{E}_{s_{0_i},\widehat{mu}_0^{\widehat{\mathbb{Q}}},\widehat{mu}_1^{\widehat{\mathbb{Q}}},\widehat{\sigma}_1}^{\widehat{\mathbb{Q}}}\left[\widehat{\lambda}_s^{\mathbb{Q}} e^{-\int_{s_{0_i}}^{s} \widehat{\lambda}_u^{\mathbb{Q}} du} | \mathcal{F}_{2,s_{0_i}}\right] ds\right]}{\sum_{n=1}^{2M}\left(\mathbb{E}_{s_{0_i},\widehat{\mu}_0^{\widehat{\mathbb{Q}}},\widehat{\mu}_1^{\widehat{\mathbb{Q}}},\widehat{\sigma}_1}^{\widehat{\mathbb{Q}}}\left[e^{-\int_{s_{0_i}}^{s_{0_i}+0.5n} \widehat{\lambda}_s^{\mathbb{Q}} ds} | \mathcal{F}_{2,s_{0_i}}\right] ZB_{s_{0_i},s_{0_i}+0.5n}^f\right)} \tag{4.17}$$

where $\mathbb{E}_{s_{0_i},\widehat{\mu}_0^{\widehat{\mathbb{Q}}},\widehat{\mu}_1^{\widehat{\mathbb{Q}}},\widehat{\sigma}_1}^{\widehat{\mathbb{Q}}}$ denotes an expectation, which is built in s_{0_i} under $\widehat{\mathbb{Q}}$ and depends on the set of coefficients $\{\widehat{\mu}_0^{\widehat{\mathbb{Q}}}, \widehat{\mu}_1^{\widehat{\mathbb{Q}}}, \widehat{\sigma}_1\}$. The newly obtained set of coefficients is afterwards used to re-estimate the time series of intensities based on the 5-year CDS spread like in the first step. This time series is then used again to re-estimate the set of coefficients. Both steps are repeated over and over again until the estimates for coefficients and intensities converge. Figure 4.2 depicts the Finish estimates for for each single iterations.

The coefficients under the measure $\widehat{\mathbb{P}}$ can then be estimated based on the previously estimated intensity time series. It can be in this context exploited that the transition distribution the CIR process is known in closed form. For this study, the average of the intensities has been chosen as non-parametric estimate for μ_0/μ_1. This is reasonable as μ_0/μ_1 complies with the mean reversion level of the particular CIR process. μ_1 is estimated afterwards via maximum-likelihood estimation (MLE) based on the previously obtained estimate for μ_0/μ_1.

Table 4.1 presents the average model errors for all maturities. "Mean in relative difference" denotes the average in differences between model spreads and observed spreads:

$$\frac{1}{N}\sum_{i\in[1,..,N]}\frac{\widehat{SP}_{s_{0_i}}(M)-SP_{s_{0_i}}(M)}{SP_{s_{0_i}}(M)}. \tag{4.18}$$

The models seem to work very well for the 5-year horizon with the average relative model error being particular small in all six cases. This result has however to be interpreted cautiously as the intensity has been estimated based on that maturity. For Iceland, Ireland, and Finland the relative model error is modest (17% being the highest) for all maturities. In the Estonian and Polish cases, the errors are in a modest range for all maturities except 1-year. The fit for spreads with respect to maturities being higher than the 1-year case is only in the Spanish case rather disappointing.

It is eye catching that the results for the 1-year case are rather bad in three from six cases. In the Estonian case, the model even completely fails to replicate the 1-year spread. Summing up one can say that the model has a quite satisfactory fit for the 3-, 5-, 7- and 10-year maturities. Spain is the only country with rather disappointing average relative errors (more than 25%) for these maturities [9]. The model does, however, not work very well for the 1-year maturity in three cases. The relative error is finally in all six cases particularly small for the 5-year maturity[10]. The standard deviation of the model errors is moreover rather small. This indicates that the model spreads either systematically exceed the true spreads or that they are systematically below them, instead of fluctuating around them. This could again indicate that the model has difficulties to replicate the term structure of CDS spreads. The overall fit is however, as said before, satisfying.

The estimation results for all countries can be found in table 4.2. The number of

[9] A reason, why the model fit is rather bad in the Spanish case compared to the other countries rather bad has not been detected. It may, however, be a sign for a structural break. The detection of such breaks might be a topic for further research.

[10] This must, however, be interpreted cautiously as the intensities have been estimated based on this maturity.

iterations refers to the number of times the model had to be estimated until both intensities and coefficients converged. The estimated loss rates differ strongly from 0.75 which, according to Pan and Singleton (2008), is the typical assumption in the literature if the loss rate is not estimated itself. This result supports the suggestion by Pan and Singleton (2008) to estimate the loss rate within the model framework. They demonstrate in their article that the identification is assured in this context. In five out of six cases (all sovereigns besides Poland) the loss rates are particularly high, which suggests that – in opposition to total bankruptcy – migration or rating risks (see e.g. Giesecke et al. (2010)) are not particularly relevant for the pricing of sovereign CDS. These estimation results may suggest a border value issue. The values of the objective functions for loss rates above one or below zero suggest however that the estimation results are the actual optimal in this model context.

The estimates of $\mu_1^{\widehat{\mathbb{Q}}}$ strongly differ from the estimates of $\mu_1^{\widehat{\mathbb{P}}}$: the estimated system is in all six cases mean reverting under $\widehat{\mathbb{P}}$ but it is only non-explosive under $\widehat{\mathbb{Q}}$ for Ireland. The estimate of $\mu_1^{\widehat{\mathbb{P}}}$ is in the latter case still higher than its counterpart under $\widehat{\mathbb{Q}}$. Moreover, $\frac{\mu_0^{\widehat{\mathbb{P}}}}{\mu_1^{\widehat{\mathbb{P}}}}$ is higher than $\frac{\mu_0^{\widehat{\mathbb{Q}}}}{\mu_1^{\widehat{\mathbb{Q}}}}$ in all cases besides the Irish one.
It is difficult to make general statements about conditional expectations regarding $\lambda_s^{\mathbb{Q}}$ for all possible horizons: the expectations are only for Ireland higher under $\widehat{\mathbb{Q}}$, no matter which value the expectations are conditioned on and which horizon is chosen. The comparison of $\mu_1^{\widehat{\mathbb{Q}}}$ and $\mu_1^{\widehat{\mathbb{P}}}$ suggests that the expectations regarding $\lambda_s^{\mathbb{Q}}$ are lower under $\widehat{\mathbb{P}}$ than under $\widehat{\mathbb{Q}}$ for most – and not all – values which the expectation is conditioned on.
The coefficient estimates ρ_0 and ρ_1 (implied by the estimates for the CIR coefficients) can be found in table 4.2 as well. The estimate for ρ_0 is in some cases negative and in some cases positive, whereas the estimate for ρ_1 is always positive. Both coefficients being positive implies positive “market prices” of risk η_s respectively a positive change in deterministic drift for a change from measure $\widehat{\mathbb{P}}$ to $\widehat{\mathbb{Q}}$ for all values of $\lambda_s^{\mathbb{Q}}$ ($\sigma_1\sqrt{\lambda_s^{\mathbb{Q}}}\eta_s$). In opposition to that, the market price

of risk can be negative for small values of $\lambda_s^{\mathbb{Q}}$ when ρ_1 is negative. This is not implausible per se. As discussed before, a positive market price of risk refers to the additional remuneration the insurance buyer agrees on paying for the insurer to take the risk of an increase in $\lambda_s^{\mathbb{Q}}$. A decrease in the default intensities, however, lowers the value of the CDS contract from the perspective of the insurance buyer. There is – in other words – a second dimension risk for the insurance buyer as well. In the presented framework, negative market prices of second dimension risk basically suggest that this reversed second dimension risk premium dominates the "usual" second dimension risk premium. This may in fact be particularly relevant in scenarios of particularly low default intensities, when a decrease in the intensity would make the respectve insurance almost worthless whereas actual default events are particularly unlikely.

The market price average, which is calculated based on the whole sample period, respectively the average of the difference in the deterministic drift, which is calculated based on the whole sample period is positive in all six cases. This result is also reflected by the whole sample average of the difference in conditional expectations for 1-day and 1-year horizons under both measures i.e.

$$\frac{\mathbb{E}^{\widehat{\mathbb{Q}}}_{s_{0_i},\widehat{\mu}_0^{\widehat{\mathbb{Q}}},\widehat{\mu}_1^{\widehat{\mathbb{Q}}},\widehat{\sigma}_1}[\lambda^{\mathbb{Q}}_{s_{0_i}+1/360}|\mathcal{F}_{2,s_{0_i}}] - \mathbb{E}^{\widehat{\mathbb{P}}}_{s_{0_i},\widehat{\mu}_0^{\widehat{\mathbb{P}}},\widehat{\mu}_1^{\widehat{\mathbb{P}}},\widehat{\sigma}_1}[\lambda^{\mathbb{Q}}_{s_{0_i}+1/360}|\mathcal{F}_{2,s_{0_i}}]}{\mathbb{E}^{\widehat{\mathbb{P}}}_{s_{0_i},\widehat{\mu}_0^{\widehat{\mathbb{P}}},\widehat{\mu}_1^{\widehat{\mathbb{P}}},\widehat{\sigma}_1}[\lambda^{\mathbb{Q}}_{s_{0_i}+1/360}|\mathcal{F}_{2,s_{0_i}}]} \tag{4.19}$$

respectively

$$\frac{\mathbb{E}^{\widehat{\mathbb{Q}}}_{s_{0_i},\widehat{\mu}_0^{\widehat{\mathbb{Q}}},\widehat{\mu}_1^{\widehat{\mathbb{Q}}},\widehat{\sigma}_1}[\lambda^{\mathbb{Q}}_{s_{0_i}+1}|\mathcal{F}_{2,s_{0_i}}] - \mathbb{E}^{\widehat{\mathbb{P}}}_{s_{0_i},\widehat{\mu}_0^{\widehat{\mathbb{P}}},\widehat{\mu}_1^{\widehat{\mathbb{P}}},\widehat{\sigma}_1}[\lambda^{\mathbb{Q}}_{s_{0_i}+1}|\mathcal{F}_{2,s_{0_i}}]}{\mathbb{E}^{\widehat{\mathbb{P}}}_{s_{0_i},\widehat{\mu}_0^{\widehat{\mathbb{P}}},\widehat{\mu}_1^{\widehat{\mathbb{P}}},\widehat{\sigma}_1}[\lambda^{\mathbb{Q}}_{s_{0_i}+1}|\mathcal{F}_{2,s_{0_i}}]}. \tag{4.20}$$

The average conditional expectations are in all cases higher under $\widehat{\mathbb{Q}}$ than under $\widehat{\mathbb{P}}$. The relative differences are still rather small for 1-day horizons but increase tremendously for the 1-year cases. Figures 4.3-4.6 plot the expected Spanish risk neutral intensities conditioned on the estimated current realization for several horizons. The short term expectations are not higher under $\widehat{\mathbb{Q}}$ for all intensity realizations which the expectations are conditioned on. Moreover, they do not differ strongly for the 3-day horizon, but there is a significant difference for the 10-, 20-, and especially the 360-day horizon. For these horizons, the values vary significantly less under $\widehat{\mathbb{P}}$ than under $\widehat{\mathbb{Q}}$. The expectations under $\widehat{\mathbb{Q}}$ are for each date higher than the ones under $\widehat{\mathbb{P}}$.

The true model spreads $\widehat{SP}_{s_{0_i}}(M)$ are on average significantly higher than model spreads $\widehat{SP}^{\widehat{\mathbb{P}}}_{s_{0_i}}(M)$ with the expectations calculated based on $\widehat{\mathbb{P}}$:

$$\widehat{SP}^{\widehat{\mathbb{P}}}_{s_{0_i}}(M) = \frac{\widehat{LR}\left[\int_{s_{0_i}}^{s_{0_i}+M} ZB^f_{s_{0_i},s}\mathbb{E}^{\widehat{\mathbb{P}}}_{s_{0_i},\widehat{\mu}_0^{\widehat{\mathbb{P}}},\widehat{\mu}_1^{\widehat{\mathbb{P}}},\widehat{\sigma}_1}\left[\widehat{\lambda}^{\mathbb{Q}}_s e^{-\int_{s_{0_i}}^{s}\widehat{\lambda}^{\mathbb{Q}}_u du}|\mathcal{F}_{2,s_{0_i}}\right]ds\right]}{\sum_{n=1}^{2M}\left(\mathbb{E}^{\widehat{\mathbb{P}}}_{s_{0_i},\widehat{\mu}_0^{\widehat{\mathbb{P}}},\widehat{\mu}_1^{\widehat{\mathbb{P}}},\widehat{\sigma}_1}\left[e^{-\int_{s_{0_i}}^{s_{0_i}+0.5n}\widehat{\lambda}^{\mathbb{Q}}_s ds}|\mathcal{F}_{2,s_{0_i}}\right]ZB^f_{s_{0_i},s_{0_i}+0.5n}\right)}. \tag{4.21}$$

$\widehat{SP}^{\widehat{\mathbb{P}}}_{s_{0_i}}(M)$ is the hypothetical insurance model price, which would be valid as actual model price, if the uncertainty regarding future default probabilities had no impact on expected returns. In the following, this figure is denoted by "hypothetical model spread". Table 4.2 contains the based ont the complete sample averaged values for

$$\frac{\widehat{SP}_{s_{0_i}}(M) - \widehat{SP}^{\widehat{\mathbb{P}}}_{s_{0_i}}(M)}{\widehat{SP}_{s_{0_i}}(M)} \tag{4.22}$$

as well as the estimated standard deviations of this figure. The average values of figure 4.22 are around 0.9 for four of six cases. The only country with a rather modest averaged relative deviation of the wrong model spreads from the true model spreads is Ireland. Ireland is also the only country for which the hypothetical model spread is at some dates smaller than the actual model spread. These results suggest accordingly that the second dimension risk premium has been positive for the other five sovereigns during the complete sample period. Figures 4.7 and 4.8 show the actual and the hypothetical 5-year model spreads for the Irish and the Polish case. Figures 4.9 and 4.10 show the relative difference between the actual and the hypothetical 5-year model spreads for the Irish and the Spanish case. Summing up the results referring to the complete sample, one can say, that the “second dimension” risk premium seems to be a very important driver of the included countries’ CDS spreads. Based on these results, it can, however, not be concluded that the second dimension risk seems to be particularly important in the European currency union: for Poland and Iceland – i.e. the two non-member countries – the second dimension risk premium seems to be important as well.

Table 4.2 moreover includes results for the averaged difference in the deterministic drift $\sigma_1\sqrt{\lambda_s^{\mathbb{Q}}}\eta_s$ for two sub-samples. The sample is divided by the last day of November 2009. This was the day when significant corrections of Greek fiscal data were announced (c.f. De Santis (2012): in November 2009, “the Greek government revealed a revised budget deficit of -12.7% of GDP for 2009, which was double of the previous estimate”). The results can be subdivided into three cases: for Ireland, Spain and Finland, the average difference changed from being negative to being positive, for Iceland and Estonia, the opposite is the case and both values are positive but decreasing for Poland. This reflects the fact that the Spanish and Irish spreads are on average significantly higher in the second sub-sample compared to the values in the first sub-sample, whereas the opposite is the case for Iceland and Estonia.
The strongest relative change in averaged $\sigma_1\sqrt{\lambda_s^{\mathbb{Q}}}\eta_s$ is detected for Ireland and Spain, the strongest absolute change has been detected for Spain, Ireland and Poland. This suggests that the changes of spreads, which led to the Spanish and Irish crisis, were strongly induced by changes in the market price of risk. This supports the hypothesis that the contagion from Greek to Spain and Ireland may have indeed catalyzed by the second dimension risk premium. This may also explain the strong increase in the relative difference between actual and hypothetical spreads (defined in equation 4.22) for these two countries (shown in figures 4.9 and 4.10), as well as the rather high standard deviations of the relative differences. The estimate for the latter can be found in table 4.2. In opposition to the Irish and Spanish cases, changes in Icelandic and Estonian spreads may have rather been driven by other factors, namely problems in the

Icelandic banking sector and actual fiscal difficulties in Estonia.

In addition, correlations between 5-year spreads for all countries as well as correlations between all countries' $\sigma_1\sqrt{\lambda_s^{\mathbb{Q}}}\eta_s$ values are presented in tables 4.3 and 4.5. The correlations of the Euro sovereigns' spreads are not always positive. The correlations between Irish and Estonian spreads are, for example, distinctly negative. The correlations between Finland and the non-Euro country Poland or between Estonia and non-Euro country Iceland are in opposition to that the highest positive ones. Two further pairs which show a distinct positive correlation are Estonia and Poland as well as Spain and Finland. These results do not suggest that membership in the Euro currency area leads to stronger correlations between spreads per se and comply with the correlations between the changes in drift $\sigma_1\sqrt{\lambda_s^{\mathbb{Q}}}\eta_s$.

The correlations of both figures have been calculated for both sub-samples. The difference between the resptive correlations can be found in tables 4.4 and 4.6. The differences show, that correlations between both spreads as well as the changes in drift decreased in all but two cases between the first period and the second period. Only both figures' correlations between Ireland and Finland respectively Poland increased slightly. The strongest decreases in both figures' correlations were found for non-Euro country Iceland. The correlations between Spain and Ireland also decreased, but not as significantly as for country pairs including Iceland. The difference in correlation between changes in drift for the Spain and Ireland is particularly low.

The results for the change in spread correlations contradict the hypothesis that the outbreak of the Greek crisis lead to higher correlations between other European sovereigns' credit costs. The results regarding the change in the market price of second dimension risk contradict the hypothesis that the corrections of Greek fiscal balances lead to a stronger relation between the uncertainties regarding other European sovereigns' future default probabilities. The correlation in Spanish and Irish changes in drift decreased, for example, slightly.

Graphs 4.11 and 4.12 show the correlations between the Spanish and Irish

$\sigma_1 \sqrt{\lambda_s^{\mathbb{Q}}}\eta_s$ values for a rolling window with widths of 40 respectively 100 days. These plots do also not support the hypothesis of changes in correlations between sovereigns' second dimension risk premiums due to the Greek fiscal information correction. It is instead eye catching that these correlations vary strongly over time and that there is no stable linear dependency between these two countries' market prices of second dimension risk. Summing up, one can say that these correlation results do not support the hypothesis that the Greek fiscal balance corrections have affected the interplay between second dimension risk premia for different countries.

Moreover, the spread values $SP_s(5)$ are associated with data for the CBOE volatility index "VIX", measuring implied volatility for the S&P 500 stock index. The VIX index is often used as an approximation for global financial market "nervousness". Table 4.9 simply contains the adjusted R^2 values for the regressions of the 5 year CDS spread on the VIX index VIX_s:

$$SP_s(5) = \beta_0 + \beta_1 VIX_s + \epsilon_s^{SP,VIX} \qquad (4.23)$$

The adjusted R^2 value for Iceland decreases strongly from the first part of the sample to the second part. In other words, the linear relation between the global financial market nervousness indicator and the spreads has been significantly stronger during the times of distress. This result seems to reflect that the fiscal crisis in Iceland has mainly been induced by problems of Icelandic banks. The adjusted R^2 values for Ireland and Spain are rather modest for both sub-samples compared to the Icelandic value for the first sample part, suggesting a relatively weak linear relation between the VIX index and the respective market price of risk. The change in this value from the first to the second sample is, moreover, relatively small. In combination with the finding that the average difference in drift changes more strongly between the two sub-samples for these two countries, this suggests that the global financial market nervousness may not have been a very important factor for the increases in Spanish and Irish spreads. These increases rather seem to be induced by an increase in the market price of second dimension risk. Moreover, the residuals from the regression of the difference of change in drift $\sigma_1\sqrt{\lambda_s^{\mathbb{Q}}}\eta_s$ on the VIX index are calculated:

$$\sigma_1\sqrt{\lambda_s^{\mathbb{Q}}}\eta_s = \beta_0 + \beta_1 VIX_s + \epsilon_s^{\sigma_1(\rho_0+\rho_1\lambda_s^{\mathbb{Q}}),VIX} \tag{4.24}$$

The residuals' correlations for the whole sample respectively the difference in correlations between both sub-samples are displayed in tables 4.7 and 4.8.
This correlation of the Irish and Spanish change in drift is still high after filtering the variation, which can be linearly explained by the VIX index. This suggests that the correlation between the market prices of risk is not induced by the simultaneous impact of the general global financial market nervousness. The correlation induced by changes in the market price of risk might instead go back to simultaneous changes in the actual uncertainty regarding default intensities.
Graphs 4.13 and 4.14 show the correlations between the Spanish and Irish residuals for a rolling window with widths of 40 respectively 100 days. These figures do also not support the hypothesis that the Greek fiscal information correction has lead to changes in the linear dependency of market participants' second dimension risk perception for all other European sovereigns after the impact of global market nervousness is filtered out. It is, however, eye catching that the variation of these correlations is much weaker than the variation of the correlations between changes in drift, which are plotted in figures 4.11 and 4.12. This suggests that there might be – independently from the Greek fiscal crisis – a stably strong linear dependency between the actual perception of these two countries' second dimension risk.
Finally, the forecasting capacities of the estimated models are analyzed. Forecasts are calculated with respect to the five horizons: 1, 5, 20, 50 and 100 days. The forecasts are then compared with actual observed spreads lying outside of the sample period which the model estimates are based on. The sample horizons for the model estimation reach from October 17, 2008 until April 9, 2012. The comparison sample reaches from the April 18, 2012 until January 1, 2013.

The comparison for the 1-day horizon is accordingly based on 190 observations, while the comparison for the 100-day-horizon is only based on 90 observations per country.

The estimated models are considered as a possible vehicle for forecasting since their framework allows to forecast intensities based on its actual distribution law, separately from the calculation of the spreads. This step is taken afterwards based on the risk neutral distribution law $\widehat{\mathbb{Q}}$. Table 4.10 contains six figures per country and per forecast horizon: the average forecast errors, the average relative forecast errors and the standard deviation of the forecast error as well as the three counterparts for a naive forecast which is simply based on the spread value in the respective forecast moment. The results do not support the idea of using reduced form credit risk models as forecast vehicle. The forecasts based on the model are only in two out of six cases better than the naive approach. The naive forecast performs in these cases better with respect to both average error, relative average error and error standard deviation.

4.7 Concluding remarks

This chapter analyzed the relevance of the "second dimension" risk premium in the context of the European fiscal crisis. It is argued that second dimension risk may have been a crucial aspect for sovereign credit spreads in the context of this crisis and a reduced form credit risk model has been estimated to analyze the relevance of the second dimension risk premium in this context. The empirical results suggest that the second dimension risk premium is indeed an important driver for the credit spreads of the included Euro countries – this is however also the case for the countries, which are not members of the Euro currency area and are included in the sample. The results support moreover the hypothesis that – compared to the credit cost variations during the Icelandic and Estonian crises –the increase of the credit spreads of Spain and Ireland after the beginnings of the Greek crisis has been rather induced by the second dimension risk premium. A strong increase in the average market price of risk after the corrections of the Greek fiscal balances in both the Spanish and the Irish case suggests that the second dimension risk premium might have been in opposition to the other country pairs contagion catalysing for these two particularly troubled countries. The linear dependency between the uncertainty regarding both sovereigns' future default probability seems, moreover, to be strong and time consistent. The empirical results do not support the hypothesis that the second dimension risk premium induced contagion among Euro countries in general or that the Greek fiscal balance corrections lead to stronger correlation among other European sovereigns' second dimension risk premia. Finally, the forecasting capacity of the model has been evaluated and the results do not support reduced form credit risk models as forecasting vehicles.

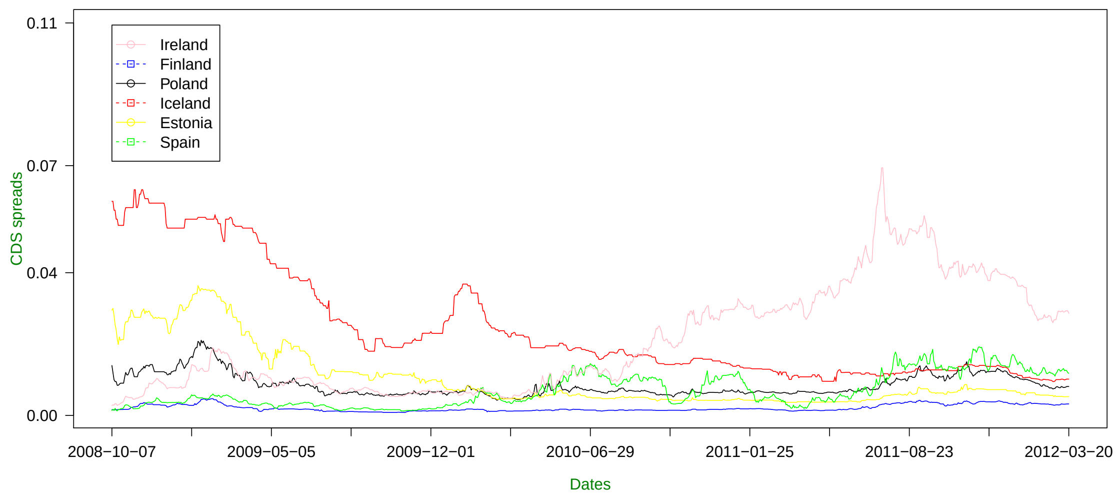

Figure 4.1: CDS spreads for all sovereigns

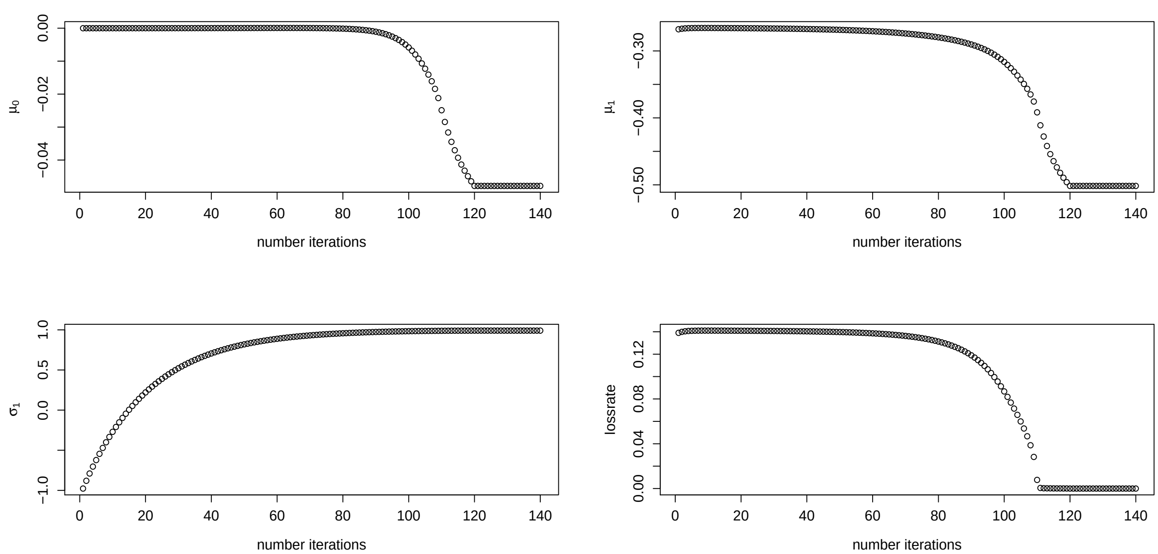

Figure 4.2: Convergence of the estimates - Finland

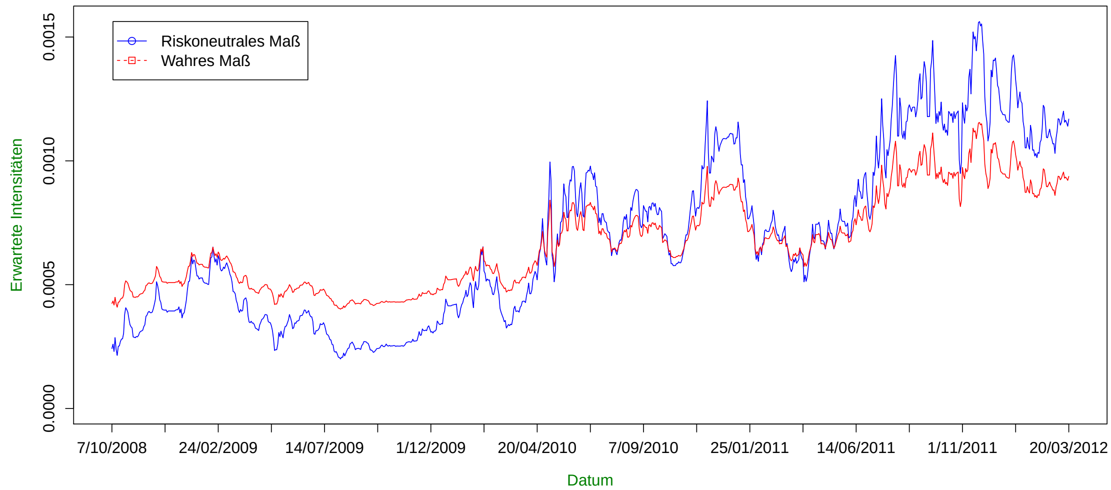

Figure 4.3: Expected intensities, conditioned on the actual estimated intensity realizations, Spain, horizon: 3 days

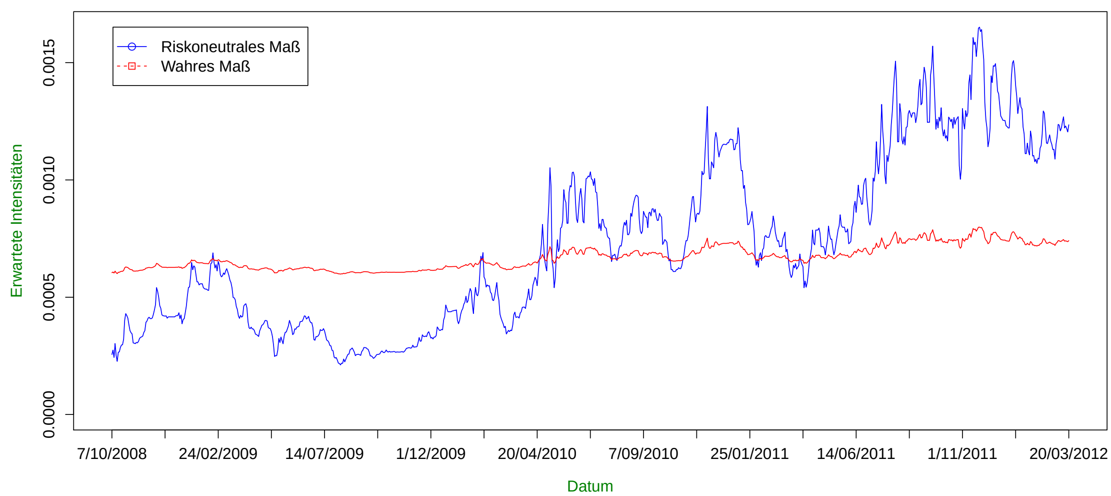

Figure 4.4: Expected intensities, conditioned on the actual estimated intensity realizations, Spain, horizon: 10 days

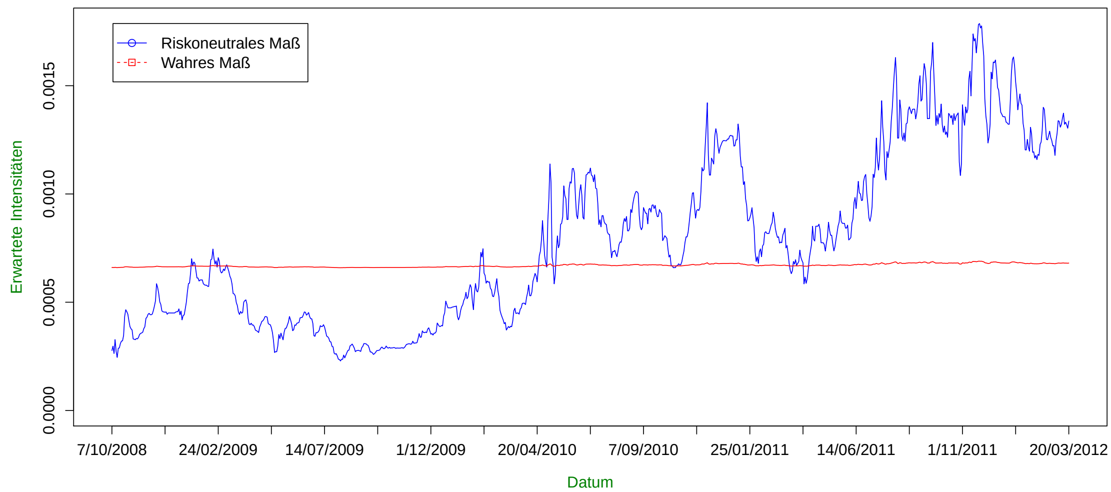

Figure 4.5: Expected intensities, conditioned on the actual estimated intensity realizations, Spain, horizon: 20 days

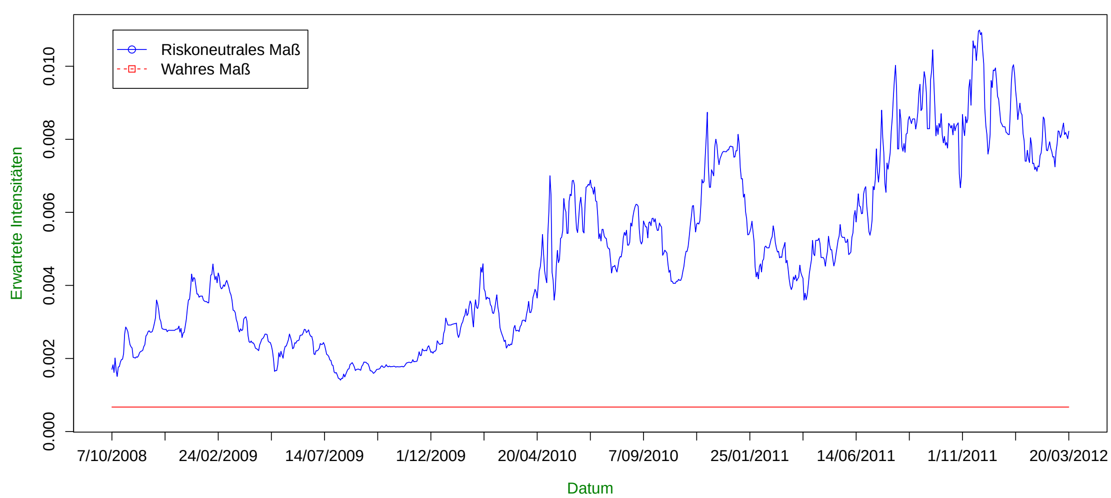

Figure 4.6: Expected intensities, conditioned on the actual estimated intensity realizations, Spain, horizon: 360 days

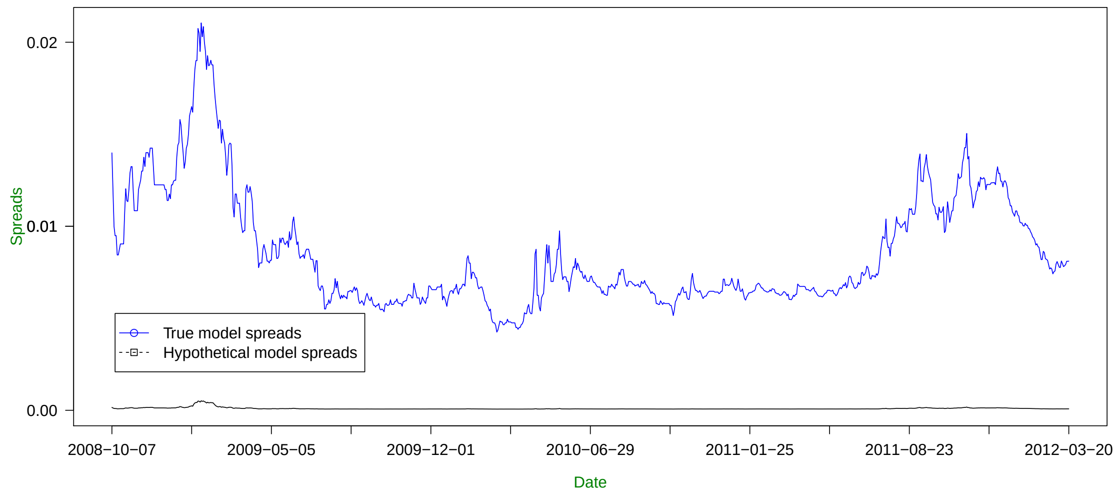

Figure 4.7: Actual and hypothetical 5-year model spreads, Poland

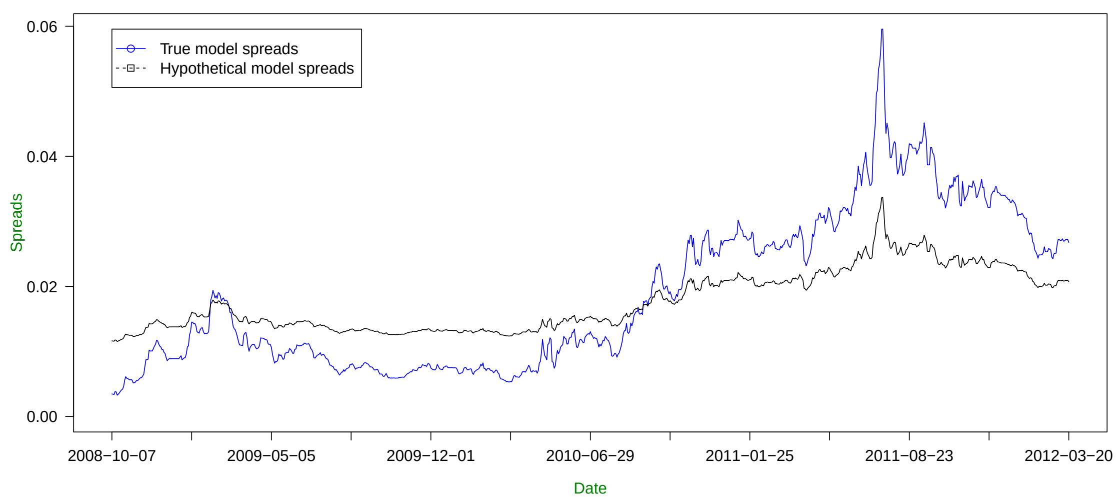

Figure 4.8: Actual and hypothetical 5-year model spreads, Ireland

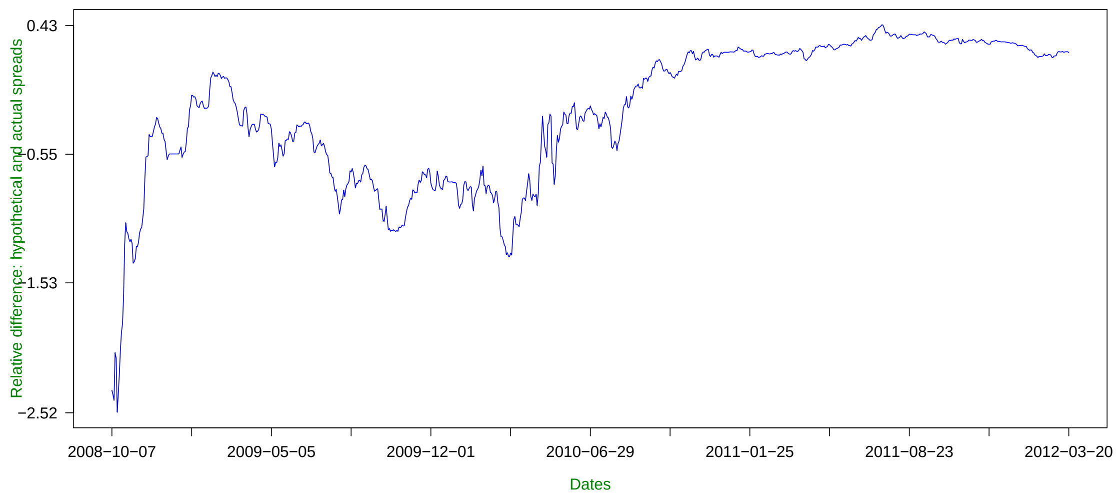

Figure 4.9: Relative difference in actual and hypothetical model spreads, Ireland

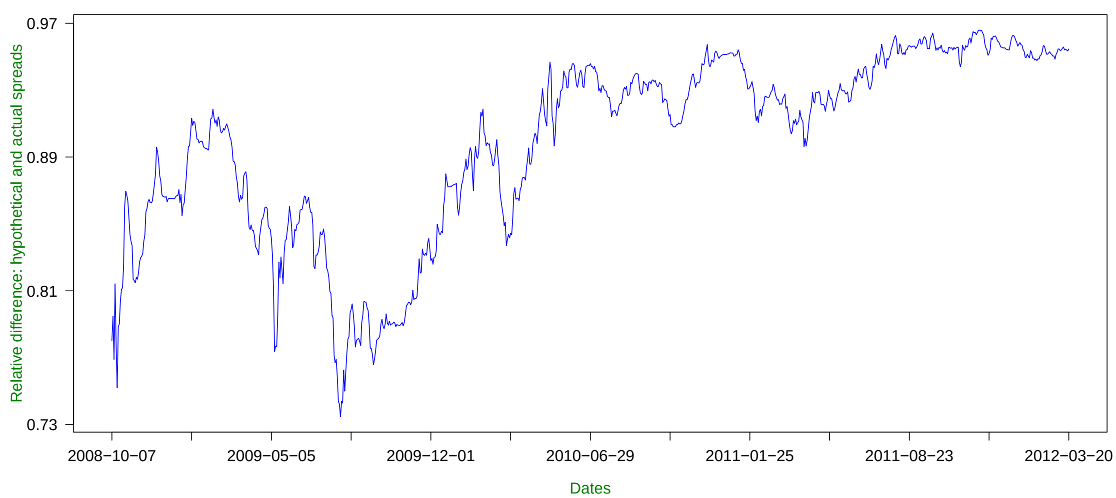

Figure 4.10: Relative difference in actual and hypothetical model spreads, Spain

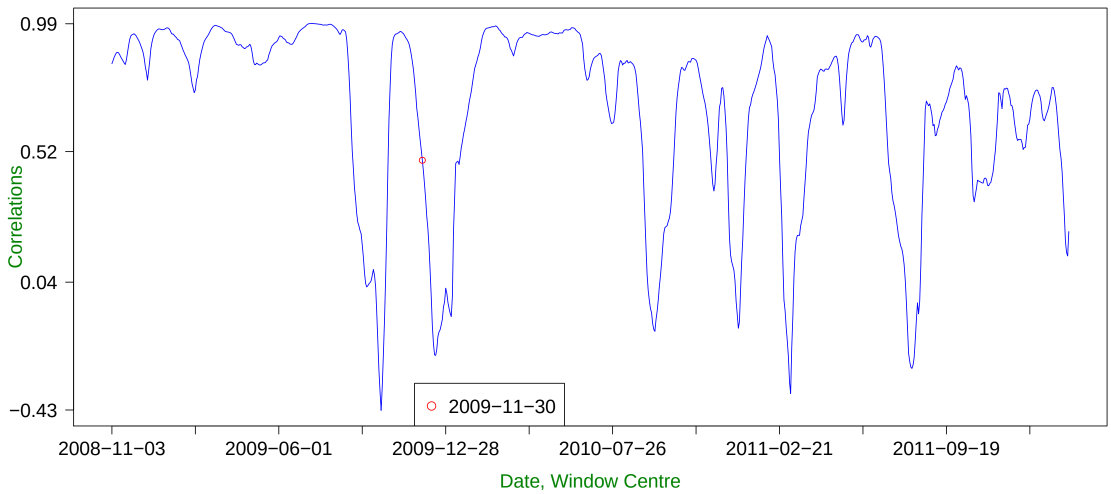

Figure 4.11: Correlations of Irish and Spanish changes in drift, rolling window, width: 40 days

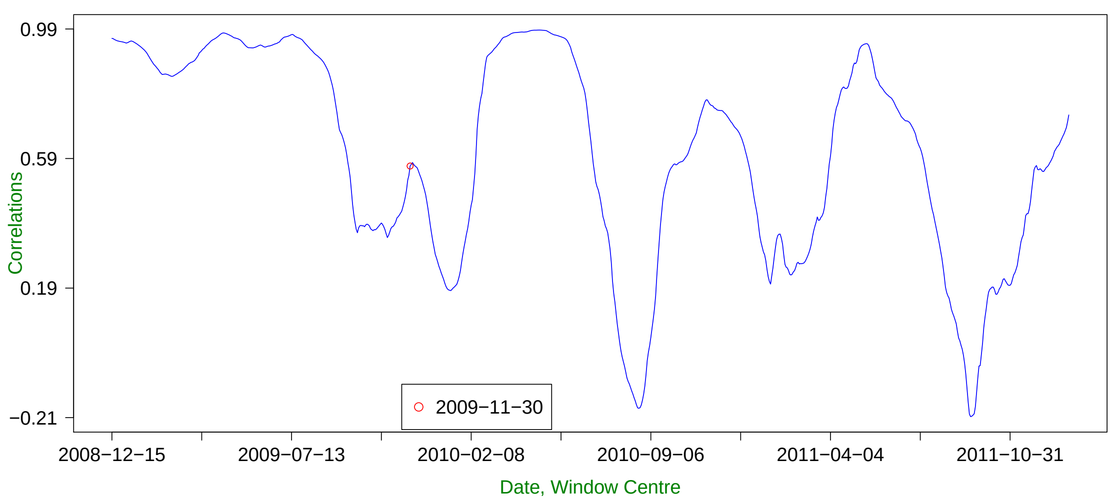

Figure 4.12: Correlations of Irish and Spanish changes in drift, rolling window, width: 100 days

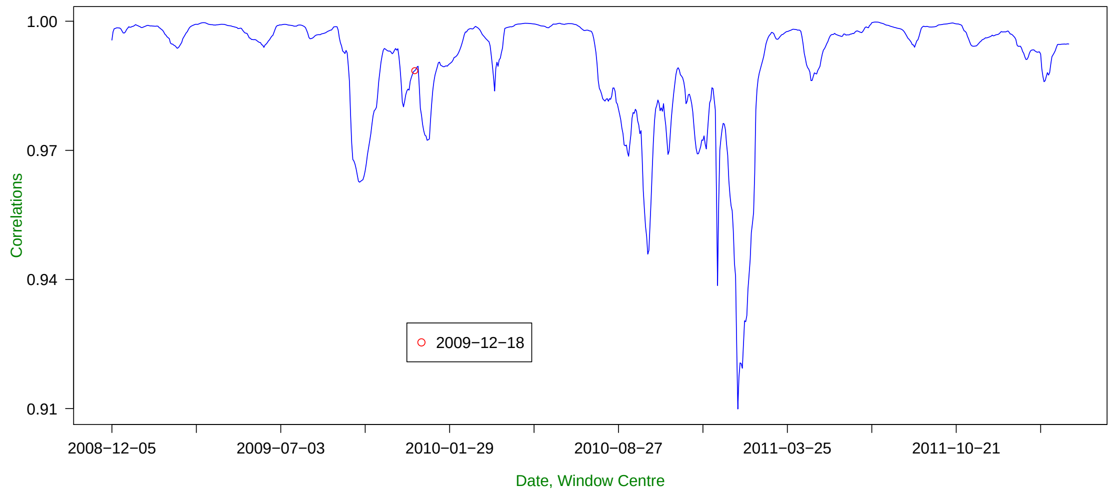

Figure 4.13: Correlations of Irish and Spanish residuals, regression: change of drift on VIX (equation 4.24, rolling window, width: 40 days

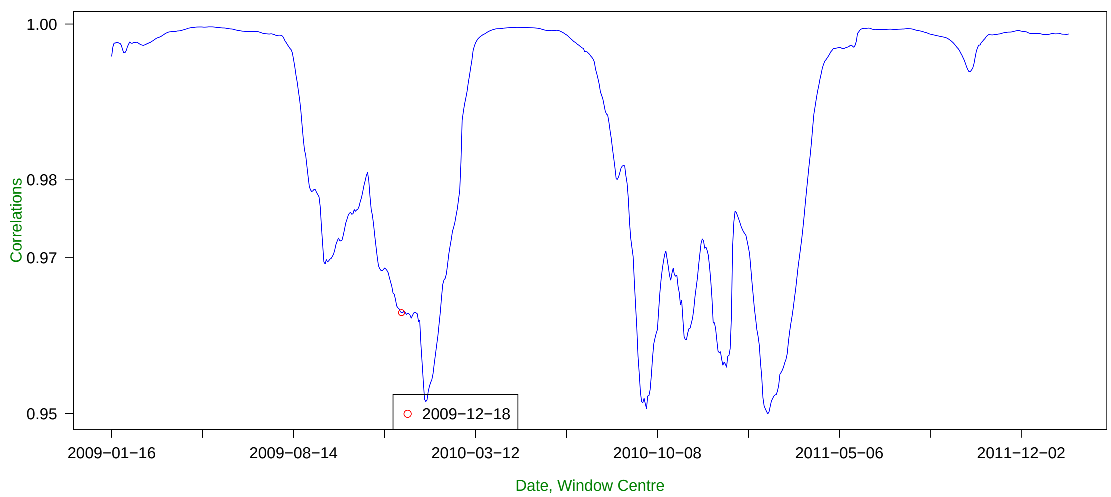

Figure 4.14: Correlations of Irish and Spanish residuals, regression: change of drift on VIX (equation 4.24, rolling window, width: 100 days

	1Y	3Y	5Y	7Y	10Y
Finland					
Mean in rel diff.	0.12	-0.03	-0.69×10^{-16}	0.07	-0.06
St.dev. difference	4.82×10^{-4}	2.86×10^{-4}	4.35×10^{-19}	1.53×10^{-4}	2.3×10^{-4}
Mean in difference	-2.28×10^{-4}	-1.29×10^{-4}	-1.46×10^{-19}	1.6×10^{-4}	-1.48×10^{-4}
Iceland					
Mean in rel diff.	-0.098	0.11	-1.1×10^{-17}	-0.07	-0.13
St.dev. difference	5.42×10^{-3}	3.32×10^{-3}	4.31×10^{-18}	2.72×10^{-3}	2.18×10^{-3}
Mean in difference	0.83×10^{-3}	0.54×10^{-3}	-1.93×10^{-19}	-3.17×10^{-4}	-1.61×10^{-3}
Poland					
Mean in rel diff.	-0.39	0.25	-0.99×10^{-17}	-0.09	-0.06
St.dev. difference	1.25×10^{-3}	0.71×10^{-3}	1.16×10^{-18}	0.92×10^{-3}	1.95×10^{-3}
Mean in difference	-1.65×10^{-3}	1.46×10^{-3}	-4.81×10^{-20}	-0.76×10^{-3}	-0.67×10^{-3}
—- Estonia					
Mean in rel diff.	-0.97	0.078	-1×10^{-16}	-0.21	-0.22
St.dev. difference	0.01	1.28×10^{-3}	2.31×10^{-6}	2.36×10^{-3}	1.3×10^{-3}
Mean in difference	-0.009	-1.75×10^{-4}	-1.05×10^{-18}	-2.56×10^{-5}	-1.54×10^{-3}
Spain					
Mean in rel diff.	-0.66	0.36	-3.46×10^{-17}	-0.25	-0.43
St.dev. difference	2.60×10^{-3}	0.95×10^{-3}	1.58×10^{-18}	0.96×10^{-3}	1.6×10^{-3}
Mean in difference	-1.34×10^{-3}	2.53×10^{-3}	-2.96×10^{-19}	-2.13×10^{-3}	-3.66×10^{-3}
Ireland					
Mean in rel diff.	-0.04	-0.05	-4.35×10^{-19}	0.03	0.06
St.dev. difference	2.28×10^{-3}	2.28×10^{-3}	3.2×10^{-18}	4.27×10^{-4}	0.94×10^{-3}
Mean in difference	-1.1×10^{-3}	-1.55×10^{-3}	-1.24×10^{-19}	3.23×10^{-4}	0.52×10^{-3}

Table 4.1: Model Errors

Country	Spain	Ireland	Iceland
$\mu_0^{\hat{\mathcal{Q}}}$	-1.3×10^{-12}	1.46×10^{-3}	$-3.32\times^{-17}$
$\mu_1^{\hat{\mathcal{Q}}}$	-1.97	3.6×10^{-3}	-6.28
$\mu_0^{\hat{\mathcal{P}}}$	0.032	0.01	$5.87e^{-8}$
$\mu_1^{\hat{\mathcal{P}}}$	47.34	0.57	50.87
σ	0.25	0.21	0.00085
LR	1	1	0.99
ρ_0	-0.13	-0.04	-0.55×10^{-3}
ρ_1	198	2.68	53356
Avg. η_s	0.21	0.05	0.63
Avg. diff. in drift	1.32×10^{-3}	1.4×10^{-3}	0.73×10^{-7}
Pre 11/2009 avg. diff. in drift	-1.47×10^{-2}	-4.72×10^{-3}	0.54×10^{-6}
Post 11/2009 avg. diff. in drift	9.33×10^{-3}	4.45×10^{-3}	-1.62×10^{-7}
Avg. diff. in cond. expec. (1D)	$5.47e^{-3}$	$2.2e^{-4}$	$1.73e^{-3}$
Avg. diff. in cond. expec. (1Y)	0.86	0.07	0.98
Avg. rel. diff. in model spreads[11]	0.92	0.06	0.98
St. Dev. Avg. rel. diff. in model spreads	0.06	0.53	3×10^{-7}
Iterations	48	41	185

Country	Finland	Poland	Estonia
$\mu_0^{\hat{\mathcal{Q}}}$	-2.64×10^{-12}	-1.9×10^{-13}	$-7.52\times^{-15}$
$\mu_1^{\hat{\mathcal{Q}}}$	-0.48	-5.35	-5.35
$\mu_0^{\hat{\mathcal{P}}}$	0.015	0.0048	4.62×10^{-6}
$\mu_1^{\hat{\mathcal{P}}}$	20.98	0.42	6.79
σ	0.17	0.13	$3.03e^{-3}$
LR	0.99	0.03	0.91
ρ_0	-0.087	-0.04	-1.48×10^{-3}
ρ_1	126	45.32	3880
Avg. η_s	0.076	4.5	1.41
Avg. diff. in drift	3.43×10^{-4}	0.06	3.63×10^{-6}
Pre 11/2009 mean diff. in drift	-1.14×10^{-4}	0.1	1.34×10^{-5}
Post 11/2009 mean diff. in drift	5.71×10^{-4}	0.04	-1.26×10^{-6}
Avg. rel. diff. in cond. expec. (1D)	$1.34e^{-3}$	$1.48e^{-2}$	$1.47e^{-2}$
Avg. rel. diff. in cond. expec. (1Y)	0.38	0.98	0.97
Avg. rel. diff. in model spreads	0.65	0.97	0.9
St. Dev. Avg. rel. diff. in model spreads	0.16	0.002	4.5×10^{-6}
Iterations	18	201	31

Table 4.2: Estimation results under both measures

Notation:

- Avg. η_s: Refers to the average value for η_s over the complete sample.

- Avg. diff. in drift: Average of $\sigma_1\sqrt{\lambda^{\mathbb{Q}}_{s_{0_i}}}\eta_s$. This refers to the difference in the deterministic drift under $\widehat{\mathbb{P}}$ compared to $\widehat{\mathbb{Q}}$, i.e. a negative value characterizes a higher (i.e. more positive) deterministic drift under $\widehat{\mathbb{Q}}$.

- Avg. rel. diff. in cond. exp. refers to the average relative difference in expectations of the intensity conditioned on the respective current value with a one day (1D) respectively (1Y) horizon (i.e. $\frac{\mathbb{E}^{\widehat{\mathbb{Q}}}_{s_{0_i},\widehat{\mu}^{\widehat{\mathbb{Q}}}_0,\widehat{\mu}^{\widehat{\mathbb{Q}}}_1,\widehat{\sigma}_1}[\lambda^{\mathbb{Q}}_{s_{0_i}+1/360}]-\mathbb{E}^{\widehat{\mathbb{P}}}_{s_{0_i},\widehat{\mu}^{\widehat{\mathbb{P}}}_0,\widehat{\mu}^{\widehat{\mathbb{P}}}_1,\widehat{\sigma}_1}[\lambda^{\mathbb{Q}}_{s_{0_i}+1/360}]}{\mathbb{E}^{\widehat{\mathbb{P}}}_{s_{0_i},\widehat{\mu}^{\widehat{\mathbb{P}}}_0,\widehat{\mu}^{\widehat{\mathbb{P}}}_1,\widehat{\sigma}_1}[\lambda^{\mathbb{Q}}_{s_{0_i}+1/360}]}$ respectively $\frac{\mathbb{E}^{\widehat{\mathbb{Q}}}_{s_{0_i},\widehat{\mu}^{\widehat{\mathbb{Q}}}_0,\widehat{\mu}^{\widehat{\mathbb{Q}}}_1,\widehat{\sigma}_1}[\lambda^{\mathbb{Q}}_{s_{0_i}+1}]-\mathbb{E}^{\widehat{\mathbb{P}}}_{s_{0_i},\widehat{\mu}^{\widehat{\mathbb{P}}}_0,\widehat{\mu}^{\widehat{\mathbb{P}}}_1,\widehat{\sigma}_1}[\lambda^{\mathbb{Q}}_{s_{0_i}+1}]}{\mathbb{E}^{\widehat{\mathbb{P}}}_{s_{0_i},\widehat{\mu}^{\widehat{\mathbb{P}}}_0,\widehat{\mu}^{\widehat{\mathbb{P}}}_1,\widehat{\sigma}_1}[\lambda^{\mathbb{Q}}_{s_{0_i}+1}]}$).

- Rel. diff. in model spreads refers to the relative deviation of the 5-year model spread with expectations calculated based on $\widehat{\mathbb{Q}}$ (i.e.$\widehat{SP}_{s_{0_i}}(5)$) from the 5-year model spread with expectations calculated based on $\widehat{\mathbb{P}}$ (i.e.$\widehat{SP}^{\widehat{\mathbb{P}}}_{s_{0_i}}(5)$) - this means: $\frac{average(\widehat{SP}_{s_{0_i}}(5))-average(\widehat{SP}^{\widehat{\mathbb{P}}}_{s_{0_i}}(M))}{average(\widehat{SP}_{s_{0_i}}(5))}$

	Ireland	Finland	Poland	Iceland	Estonia	Spain
Ireland	1	0.55	0.18	-0.59	-0.42	0.68
Finland	0.55	1	0.83	0.13	0.32	0.58
Poland	0.18	0.83	1	0.54	0.72	0.24
Iceland	-0.59	0.13	0.54	1	0.93	-0.49
Estonia	-0.42	0.32	0.72	0.93	1	-0.39
Spain	0.68	0.58	0.24	-0.49	-0.39	1

Table 4.3: Correlations of spreads

	Ireland	Finland	Poland	Iceland	Estonia	Spain
Ireland	0	-0.11	-0.04	1.13	0.57	0.38
Finland	-0.11	0	0.04	1.23	0.43	0.05
Poland	-0.04	0.04	0	1.14	0.35	0.07
Iceland	1.13	1.23	1.14	0	0.51	1.06
Estonia	0.57	0.43	0.35	0.51	0	0.45
Spain	0.38	0.05	0.07	1.06	0.45	0

Table 4.4: Difference in correlations of spreads pre 11/2009 vs post 11/2009

	Ireland	Finland	Poland	Iceland	Estonia	Spain
Ireland	1	0.54	0.04	-0.58	-0.33	0.84
Finland	0.54	1	0.65	0.14	0.39	0.59
Poland	0.04	0.65	1	0.49	0.8	0.05
Iceland	-0.58	0.14	0.49	1	0.87	-0.59
Estonia	-0.33	0.39	0.8	0.87	1	-0.37
Spain	0.84	0.59	0.05	-0.59	-0.37	1

Table 4.5: Correlations $\sigma_1 \sqrt{\lambda_s^{\mathbb{Q}}} \eta_s$

	Ireland	Finland	Poland	Iceland	Estonia	Spain
Ireland	0	-0.07	-0.01	1.17	0.68	0.13
Finland	-0.07	0	-0.06	1.22	0.45	-0.01
Poland	-0.01	-0.06	0	0.81	0.3	0.01
Iceland	1.17	1.22	0.81	0	0.39	1.25
Estonia	0.68	0.45	0.3	0.39	0	0.64
Spain	0.13	-0.01	0.01	1.25	0.64	0

Table 4.6: Difference in correlations of $\sigma_1 \sqrt{\lambda_s^{\mathbb{Q}}} \eta_s$ pre 11/2009 vs post 11/2009

	Ireland	Finland	Poland	Iceland	Estonia	Spain
Ireland	1	0.7	0.19	-0.65	-0.27	0.83
Finland	0.7	1	0.56	-0.28	0.15	0.75
Poland	0.19	0.56	1	0.15	0.71	0.2
Iceland	-0.65	-0.28	0.15	1	0.69	-0.66
Estonia	-0.27	0.15	0.71	0.69	1	-0.33
Spain	0.83	0.75	0.2	-0.66	-0.33	1

Table 4.7: Correlations $\epsilon_s^{\sigma_1(\rho_0+\rho_1\lambda_s^{\mathbb{Q}}),VIX}$

	Ireland	Finland	Poland	Iceland	Estonia	Spain
Ireland	0	0.01	0.13	1.14	0.95	0.19
Finland	0.01	0	0.06	1.04	0.99	-0.15
Poland	0.13	0.06	0	0.2	0.46	0.05
Iceland	1.14	1.04	0.2	0	-0.31	1.01
Estonia	0.95	0.99	0.46	-0.31	0	0.94
Spain	0.19	-0.15	0.05	1.01	0.94	0

Table 4.8: Correlations $\epsilon_s^{\sigma_1(\rho_0+\rho_1\lambda_s^{\mathbb{Q}}),VIX}$ Period 1 - Period 2

	complete sample	first sample	snd. sample
Ireland	0.04	-0.78×10^{-3}	0.08
Finland	0.16	0.4	0.19
Poland	0.49	0.49	0.33
Iceland	0.56	0.7	-1.64×10^{-3}
Estonia	0.62	0.63	0.18
Spain	0.01	0.15	0.21

Table 4.9: adjusted R^2 regression 4.23

	1D	5D	20D	50D	100D
Finland					
Mean error	0	1.36×10^{-4}	0.82×10^{-3}	1.58×10^{-3}	1.35×10^{-3}
Rel. mean err.	0	0.04	0.23	0.39	0.39
Mean err. st.dv.	0	0.54×10^{-4}	2.42×10^{-4}	3.74×10^{-4}	0.68×10^{-3}
Mean error naive	0	4.89×10^{-5}	2.47×10^{-4}	0.75×10^{-3}	1.28×10^{-3}
Rel. mean err. naive	0	0.01	0.07	0.19	0.36
Mean err.naiv st.dv.	0	1.7×10^{-5}	0.78×10^{-4}	1.33×10^{-4}	3.26×10^{-4}
Iceland					
Mean error	0	0.99×10^{-4}	-4.93×10^{-3}	-0.01	-0.01
Rel. mean err.	0.19	0.01	-0.39	-0.73	-0.81
Mean err. st.dv.	0	1.01×10^{-3}	0.56×10^{-3}	1.22×10^{-3}	0.96×10^{-3}
Mean error naive	0	-1.95×10^{-3}	-0.01	-0.01	-0.01
Rel. mean err. naive	0.19	-0.16	-0.6	-0.98	-0.93
Mean err.naiv st.dv.	0	0.94×10^{-4}	1.42×10^{-4}	4.9×10^{-4}	1.8×10^{-3}
Poland					
Mean error	0	0.86×10^{-3}	1.16×10^{-3}	1.35×10^{-3}	-1.35×10^{-3}
Rel. mean err.	0	3.7×10^{-4}	3.64×10^{-4}	2.07×10^{-3}	1.32×10^{-3}
Mean err. st.dv.	0	3.7×10^{-4}	3.64×10^{-4}	2.07×10^{-3}	1.32×10^{-3}
Mean error naive	0	-0.63×10^{-4}	-1.53×10^{-5}	2.15×10^{-5}	0.84×10^{-4}
Rel. mean err. naive	0.1	-0.01	-1.69×10^{-3}	2.11×10^{-3}	0.01
Mean err.naiv st.dv.	0	0.71×10^{-4}	1.8×10^{-4}	1.79×10^{-4}	3.45×10^{-4}
Estonia					
Mean error	0	-1.31×10^{-4}	-1.01×10^{-3}	-2.76×10^{-3}	-0.01
Rel. mean err.	0	-0.02	-0.2	-0.52	-0.96
Mean err. st.dv.	0	1.58×10^{-5}	1×10^{-4}	4.69×10^{-4}	3.53×10^{-4}
Mean error naive	0	-1.53×10^{-4}	-0.67×10^{-3}	-2.22×10^{-3}	-3.75×10^{-3}
Rel. mean err. naive	0	-0.03	-0.13	-0.42	-0.71
Mean err.naiv st.dv.	0	1.29×10^{-5}	1.06×10^{-4}	2.13×10^{-4}	0.59×10^{-3}
Spain					
Mean error	0	3.84×10^{-3}	0.01	0.01	0.01
Rel. mean err.	0.18	0.21	0.35	0.54	0.57
Mean err. st.dv.	0	1.95×10^{-3}	1.14×10^{-3}	1.99×10^{-3}	1.95×10^{-3}
Mean error naive	0	0.52×10^{-3}	2.49×10^{-3}	0.01	0.01
Rel. mean err. naive	0.18	0.03	0.14	0.26	0.34
Mean err.naiv st.dv.	0	3.42×10^{-4}	3.52×10^{-4}	0.93×10^{-3}	1.94×10^{-3}
Ireland					
Mean error	0	-1.9×10^{-3}	-1.6×10^{-3}	2.92×10^{-4}	-4.97×10^{-3}
Rel. mean err.	-0.07	-0.08	-0.07	0.01	-0.2
Mean err. st.dv.	0	0.97×10^{-3}	1.36×10^{-3}	4.02×10^{-3}	2.89×10^{-3}
Mean error naive	0	0.54×10^{-4}	-1.12×10^{-5}	3.77×10^{-4}	1×10^{-3}
Rel. mean err. naive	-0.07	2.15×10^{-3}	-4.54×10^{-4}	0.01	0.04
Mean err.naiv st.dv.	0	2.5×10^{-4}	0.5×10^{-3}	0.54×10^{-3}	0.67×10^{-3}

Table 4.10: Forecast Errors

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Appendix A

R-Code – Estimation of the intensity process and the parameter set under the risk neutral measure

```
library(pracma)
library(deSolve)
library(base)

# load interest rate and CDS data
IR <- read.table("IR.txt")
sp <- read.table("countryXY.spreads.txt", header=TRUE)

# calculate the semi annually paid spread payments (original data is annualized)
sp[,2:dim(sp)[2]] <- sp[,2:dim(sp)[2]]*0.5*0.0001
```

```
IR2 <- IR2*0.01
IR <- IR2

# interest rate interpolation
zins <- function(g) {
if (g < 1/12) {
z <- 1/12
}
else  {
z <- g
}
dip <- ( 1/   ((1+ ( (IR2[teim,floor(z*12)])* ( (z*12)-floor(z*12) )  ) + (
(IR2[teim,(floor(12*z)+1)] ) * (1- ( (z*12)-floor(12*z) ) ) ) )^g ) )
return(dip)
}
R <- IR[,1]

# create matrix to save risk free discount factors required to calculate the left
#hand side of the CDS pricing formula
rfreeleft <- matrix(data = NA, nrow = length(R), ncol = 20, byrow = FALSE)

# calculate the particular discount factors
for (i in 1:length(R)) {
for (j in 1:20) {
teim <- i
rfreeleft[i,j] <- zins(j*0.5)
}
}

# create vector to save estimated intensities
wlichkeit <- c(1:(dim(sp)[1]-1))
```

```
##################################

# determine the period the estimation is based on
timeframe <- c(1, 901)

#################################

# adjust the length of all vectors / dimension of all matrices to the determined
# timeframe
rfreeleft <- rfreeleft[timeframe[1]:timeframe[2],]
wlichkeit <- wlichkeit[timeframe[1]:timeframe[2]]
sp <- sp[timeframe[1]:timeframe[2],]
R <- R[timeframe[1]:timeframe[2]]
IR2 <- IR2[timeframe[1]:timeframe[2],]
IR <- IR2

# the interest function shall be defined for horizon "zero". The interest rate
# matrix is adjusted accordingly
nullgeschichte <- matrix(data=NA,nrow=nrow(IR),ncol=ncol(IR)+1)
nullgeschichte[,1] <- IR[,1]
nullgeschichte[,2:ncol(nullgeschichte)] <- IR
IR <- nullgeschichte
IR2 <- IR

# Generate a vector / matrix to save the estimated intensities and coefficients
# for each estimation step
parametermatrix <- matrix(data = NA, nrow = 250, ncol = 4, byrow = FALSE)
wlichkeitmatrix <- matrix(data = NA, nrow =timeframe[2]-timeframe[1]+1 , ncol =
250, byrow = FALSE)
```

```
# determine starting values
sig1 <- -0.0001
sig2 <- -0.5
loss <- 0.05
lossr <- 0.05
diffpa <- 0.1

# generate parameter vector
sigma <- c(sig1,sig2,loss,diffpa)

# start the estimation procedure
for (z in 1:250) {

# generate parameter vector for each iteration step
parameters <- c(sigma[1], sigma[2], sigma[3], sigma[4])

# generate functions which the substitutes for the expectations in the pricing
#formula depend on
dingsb <- ((2*((parameters[4])^2)+((parameters[2])^2))^0.5)
kapdings <- (parameters[2]+dingsb)/(parameters[2]-dingsb)
A <- function(te) {
exp(parameters[1]*te*(parameters[2]+dingsb)/(parameters[4]^2)) * ((1-kapdings)/
(1-(kapdings*exp(dingsb*te))))^(2*parameters[1]/(parameters[4]^2))
}

B <- function(te) {
(((parameters[2]-dingsb)/(parameters[4]^2))+(2*dingsb/((parameters[4]^2)*(1-
(kapdings*exp(te*dingsb))))))
}

G <- function(te) {
```

```
(parameters[1]/dingsb)*(exp(dingsb*te)-1)*exp(parameters[1]*(parameters[2]+dingsb)*
te/(parameters[4]^2))*(((1-kapdings)/(1-kapdings*exp(dingsb*te)))^((2*parameters[1]
/(parameters[4]^2)) + 1)  )
}

H <- function(te) {
exp((parameters[1]*(parameters[2]+dingsb)+(dingsb*(parameters[4]^2)))*te/(parameters[4]^2))
*((1-kapdings)/(1-kapdings*exp(dingsb*te)))^((2*parameters[1]/(parameters[4]^2)) + 2)
}

# determine the maturity for the 5-year contract
frist <- 5

# calculate the corresponding number of days
fristtage <- frist*364

# calculate for each sample day the intensity based on the 5-year CDS spread and
#the respective CDS spreads
for (p in 1:(timeframe[2]-timeframe[1]+1))  {

# choose the 5-year CDS spread for the particular day
spread <- sp[p,4]

# calculate the vector of discount factors required for the calculation of the left
#hand side of the CDS pricing formula in the 5-year case
linksr <- rfreeleft[p,1:(2*frist)]

# choose the row of the interest-rate matrix for the respective date "p"
reit <- IR2[p,]

# generate the discount function that interpolates the monthly available interest-
```

```
#rates for the respective day
zins <- function(g) {
if (g < (1/12)) {
z <- 1/12
}
else  {
z <- g
}
dip <- ( 1/   ((1+ ( (reit[floor(z*12)])* ( (z*12)-floor(z*12) )  ) + (
(reit[(floor(12*z)+1)] ) * (1- ( (z*12)-floor(12*z) ) ) ) )^g ) )
return(dip)
}

# generate a vector for the singly payments associated to the left hand side of
#the pricing formula
links1 <- c(1:(2*frist))

# generate a function that gives the deviation from the model spread and the actual
#spread for the particular day "p"
checks <- function(x) {

# calculate the left hand side of the left hand side of the pricing value
for (i in 1:(2*frist)) {
links1[i] <- zins(i*0.5)*A(i*0.5)*exp(B(i*0.5)*x)
}
links <- sum(links1)*0.5*spread

# calculate the left hand side of the right hand side of the pricing value
rechts1 <- function(tet) {

zins(tet)*exp(B(tet)*x)*(G(tet)+H(tet)*x)
```

```
}


rechts <- ((1-parameters[3])*(gauss_kronrod(Vectorize(rechts1),0,frist)$value))

# return deviation from the right hand side and the left hand side of the pricing
# equation
return(links - rechts)

}

# calculate the intensity value equating the pricing formula
# negative intensity values are not logical. The estimated intensity values are at
#these days substituted by zero
if (checks(0) <=0 ) {
uni <- 0
}   else {
uni <-  uniroot(checks, c(0,1),tol=.Machine$double.eps^4)$root
}
wlichkeit[p] <- uni

}

# save intensity vector in intensity matrix for the particular estimation step
wlichkeitmatrix[,z] <- wlichkeit



# choose the maturities which the estimation of the new set of CIR-parameters is
#based on
```

```
vergleichfristen <- c(1,3,7,10)

# create a matrix that only contains the spreads the estimation of the spreads
# the estimation is based on
spreadmatrix <- matrix(data = NA, nrow = (timeframe[2]-timeframe[1]+1), ncol =
length(vergleichfristen), byrow = FALSE)
spreadmatrix[,1] <- sp[1:(timeframe[2]-timeframe[1]+1),2]
spreadmatrix[,2] <- sp[1:(timeframe[2]-timeframe[1]+1),3]
spreadmatrix[,3] <- sp[1:(timeframe[2]-timeframe[1]+1),5]
spreadmatrix[,4] <- sp[1:(timeframe[2]-timeframe[1]+1),6]

sigma <- c(sig1,sig2,loss,diffpa)

rm('frist')
rm('fristtage')
rm('sig1')
rm('sig2')
rm('lossr')
rm('between')
rm('spread')
rm('epsilon')

# create scalar to save the deviation of the model spread from the actual spread
epsilon <- 0
buff <- 0

# create functions that gives the deviation in dependence on the set of CIR-parameters
checks2 <- function(x) {

parameters[1] <- x[1]*x[2]
parameters[2] <- x[1]
```

```
parameters[3] <- x[3]
parameters[4] <- x[4]

# create new set of functions for the substitutions of the expectations contained
# in the pricing formula (based on the undetermined set of CIR-parameters
dingsb <- ((2*((parameters[4])^2)+((parameters[2])^2))^0.5)
kapdings <- (parameters[2]+dingsb)/(parameters[2]-dingsb)
A <- function(te) {
exp(parameters[1]*te*(parameters[2]+dingsb)/(parameters[4]^2)) * ((1-kapdings)/
(1-(kapdings*exp(dingsb*te))))^(2*parameters[1]/(parameters[4]^2))
}

B <- function(te) {
(((parameters[2]-dingsb)/(parameters[4]^2))+(2*dingsb/((parameters[4]^2)*(1-
(kapdings*exp(te*dingsb))))))
}
G <- function(te) {
(parameters[1]/dingsb)*(exp(dingsb*te)-1)*exp(parameters[1]*(parameters[2]+dingsb)*
te/(parameters[4]^2))*(((1-kapdings)/(1-kapdings*exp(dingsb*te)))^((2*parameters[1]/
(parameters[4]^2)) + 1)  )
}

H <- function(te) {
exp((parameters[1]*(parameters[2]+dingsb)+(dingsb*(parameters[4]^2)))*te /(parameters[4]^2))
*((1-kapdings)/(1-kapdings*exp(dingsb*te)))^((2*parameters[1]/(parameters[4]^2)) + 2)
}


# calculate the aggregated deviations of the model spreads from the observed spreads
# based on the undetermined coefficients
  for (n in 1:length(vergleichfristen)) {
```

```
frist <- vergleichfristen[n]
links1 <- c(1:(2*frist))


for (p in 1:(timeframe[2]-timeframe[1]+1))  {

spread <- spreadmatrix[p,n]


intens <- wlichkeit[p]
reit <- IR2[p,]


zins <- function(g) {
if (g < 1/12) {
z <- 1/12
}
else  {
z <- g
}
dip <- ( 1/   ((1+ ( (reit[floor(z*12)])* ( (z*12)-floor(z*12) )  ) + (
(reit[(floor(12*z)+1)] ) * (1- ( (z*12)-floor(12*z) ) ) ) )^g ) )
return(dip)
}


for (i in 1:(2*frist)) {
links1[i] <- zins(i*0.5)*A(i*0.5)*exp(B(i*0.5)*intens)
}

links <- sum(links1)*0.5
```

```
  rechts1 <- function(tet) {

  zins(tet)*exp(B(tet)*intens)*(G(tet)+H(tet)*intens)

  }

  rechts <- (1-parameters[3])* (gauss_kronrod(Vectorize(rechts1),0,frist)$value)

  epsilon <- epsilon + ((spread-(rechts/links))^2)

  }
  }

# return the aggregated deviations
return(epsilon)
}

# minimze the function that gives the deviation from model spreads from the actual
#spreads with respect to the set of parameters
a <- matrix(c(0,1,0,0,0,0,1,0,0,0,-1,0,0,0,0,1),nrow=4,ncol=4,byrow=TRUE)
b <- c(0,0,-1,0)
res2gmm <- constrOptim(c(sigma[2],sigma[1]/sigma[2],sigma[3],sigma[4]), checks2,
NULL , ui = a, ci = b)

# save estimation results and use them to estimate a new time series of intensities
sig1 <- res2gmm$par[1]*res2gmm$par[2]
sig2 <- res2gmm$par[1]
loss <- res2gmm$par[3]
lossr <- loss
diffpa <- res2gmm$par[4]
```

```
sigma <- c(sig1,sig2,lossr,diffpa)

# save estmiation results of the respective iteration in the estmiation result matrix
parametermatrix[z,] <- sigma

}
```

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