Ildar Safuanov, Moskau

Design of a system of teaching elements of group theory

According to principles described in (Safuanov, 2005), we present here the design of a system for the teaching of the concepts of group theory.

The preliminary analysis.

1) Genetic development of a material.

a) Historical analysis.

F.Klein, who had brought in the essential contribution to development of the group theory due to “Erlangen program” of the study of geometry through the study of groups of geometrical transformations, argued that “the concept of a group was originally developed in the theory of algebraic equations”. Sources of the concept of a group are in the theory of solving algebraic equations as well as in geometry, where groups of geometrical transformations have been investigated since the middle of the 19-th century by A. Cayley. However abstract groups were introduced by S.Lie only at the end of the 19-th century.

b) Logical and epistemological analysis.

For the introduction of the concept of a group, the preliminary possession of a lot of set-theoretical and logical concepts and constructions is necessary which can be seen from the detailed logical and epistemological analysis of the homomorphism theorem.

In turn, the group-theoretical concepts are used in the subsequent sections..

c) Psychological analysis.

School graduates are not actually prepared for mastering so abstract concept as a group. They can not operate with general concepts of algebraic operations and even mappings. Therefore, in particular, they can not freely investigate geometrical transformations and their compositions.

On the initial stage, in our view, it is inexpedient to motivate the introduction of the concept of a group by examples of sets of transformations (for example, translations or rotations), because, as the experience of teaching geometry to the first year students of pedagogical universities shows, the geometrical imagination of many students (and spatial imagination in general) is very poorly developed.

d) Analysis from the point of view of possible applocations.

The concept of a group since several decades became rather popular part of the cultural property of mankind. For example, the experts in the quantum mechanics believed that the group theory can be used in the solution of any problem. The group theory turned out to be extremely useful in the search of elementary particles an in the study of the structure of chemical molecules. Of great interest are the consideration of symmetry groups of geometrical figures. Good examples of the applications of the group theory are graceful group-theoretical proofs of the number-numerical theorems of L.Euler and P. Fermat.

2) Analysis from the point of view of the arrangement of a subject matter, of the opportunities of use of various means of representation of objects, concepts and ideas and of the influence on students.

Using results of the genetic development, it is possible to offer the following version of the arrangement of a subject matter and of the use of means of influence.

As the theory of groups has grown out of generalizations of diverse ideas and constructions, we offer also to use some lines leading to group-theoretical concepts from the different perspectives: numbers, cosets, bijective transformations and permutations.

The first stage: already at introductory lecture it is possible to offer to the students to consider systems of integers with addition and non-zero rational numbers with multiplication, to recollect properties of these arithmetic actions.

The second stage: after the addition of cosets and the addition tables for small modules (for example, 2, 3, 4) are considered, it is possible to raise the question about the performance of addition in a set of cosets modulo arbitrary $n>1$. Properties will be similar to properties of the addition of numbers. The students can guess the fulfilment of laws of associativity and commutativity, the existence of neutral and inverse elements, and even in some extent to participate in proving these properties. After that it is possible to introduce a stricter definition of a group, beginning with the definition of ordered pairs and binary algebraic operations

The third stage: preliminary, but already quite strict statement of elements of the theory of groups after the consideration of elements of the theory of sets, direct products, mappings, including bijective ones, and permutations. At this stage all formal definitions of concepts necessary for the strict introduction of group-theoretical concepts, and sufficient amount of motivating and illustrating properties and examples are available. At this stage the examples of non-commutative groups (symmetry groups and groups of permutations) are considered.

The fourth stage: systematic study of elements of the theory of groups (including generalized associativity, cosets, normal subgroups, Lagrange's and homomorphism theorems).

As to means of influence on students, in the teaching of elements of the theory of groups it is possible to use various evident ways of representation of a subject matter, considering, for example, permutations, symmetry of geometrical figures, geometrical transformations. Among ways of representation of groups it is possible to employ, in case of finite groups, lists of elements, the multiplication tables. Among other means of influence one can mention the contrast (examples of groups and semigroups which are not groups, normal subgroups and subgroups that are not normal), variation (abelian and non-abelian groups, additive and multiplicative ones etc.).

Design of the process of study of group-theoretical concepts.

1) Construction of a problem situation.

As is already shown, for the successful construction of a problem situation it is necessary to organize it (including new questions, naturally arising from it) so that in a certain time there would occur the "moment of truth" when the students independently or with the minimal help of the teacher would open for the new concept for themselves.

For the first time such moment of truth arises already during the introductory lecture, when the preliminary wersion of the concept of a group arises as a generalization of properties of arithmetic actions in sets of integers (addition) and non-zero rational numbers (multiplication). At further stages this preliminary version of the definition forms the basis for the motivation of the consideration of the concept of a group, basis for its stricter study. So, for example, studying properties of the addition of cosets or multiplication of bijections of a set, permutations of a finite set, symmetries of a geometrical figure, the students already can find out that each time they deal with groups – and thus new moments of truth arise.

2) Statement of new naturally arising questions.

For example, constructing a problem situation at the third stage (when passing to types and elementary properties of groups) one can use questions of the following kind: whether are groups under consideration commutative? Whether there exists an infinite non-commutative group? Is the neutral element of a group unique? For a given element of a group, is an inverse element unique? Is it possible to solve equations in groups? At the fourth stage (systematic study of more complicated group-theoretical concepts) the questions are pertinent: do the right and left cosets coincide? Do cosets of a normal subgroup form a group under multiplication? etc.

3) Conceptual and structural analysis and logical organization of educational material.

Conceptual and structural analysis and logical organization of group-theoretical concepts is rather complicated, as is seen, e.g., from the genetic decomposition of the homomorphism theorem. This process is not straightforward, but rather long and, moreover, often occurs in several stages divided in time. From group axioms the properties of groups are deduced, and at final stages of study of groups a number of rather difficult theorems is proved.

4) Development of applications and algorithms.

It is important to consider such simple and interesting examples of applcations as the fifteen puzzle, group-theoretical proofs of number-theoretical theorems of L.Euler and P.Fermat, symmetry groups of geometrical figures etc.

The students also should learn such procedures as construction of the multipliction table of a finite group, finding cosets of a normal subgroup (i.e. construction of a quotient group) etc.

References.

Safuanov, I. (2005). The genetic approach to the teaching of algebra at universities. *International Journal of Mathematical Education in Science and Technology* 36 (2-3), 257-270.