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The Return of Homothety in Mathematical Contests

1. Situation in Schools and Advanced Contests

The school curricula have a tendency to emphasize the practical application. Geometry is a training ground for deductive thinking but geometrical facts themselves are not obviously practical. Separate subjects "geometry" and "algebra" are merged into one subject "mathematics" in many countries. The standards for geometry teaching are being lowered.

Standards are not being lowered in mathematical contests and olympiads. Teachers use this to get students to learn something more than the curricula require. That way the importance of such activities increases.

2. Two Approaches towards Geometry Fact Introducing

One approach is to split a problem into small parts and to analyze each part separately, after that combining them back together. E.g., many shapes can be reduced to triangles, and many problems can be solved using a chain of triangles' congruences.

Another approach is to consider object under interest as a part of a general system and to deduce its properties from those of the system. This is how geometrical transformations mostly work.

In schools the first approach dominates, because it is generally easier for average students.

In mathematical olympiads geometrical transformations were sometimes very popular, especially in connection with construction tasks. Then during some period of time they appeared less and less often, but now they become popular again.

3. Main Applications of Homothety

There are two main classes of problems to which homothety can be successfully applied:

a) incidence problems,

b) geometrical construction problems.

The properties of homothety can be naturally divided into two classes:

a) properties that are common for all isometries and transformations of similitude:

a1) each shape transforms into a similar shape, a2) elements with any geometrical meaning transform into elements with the same meaning;

b) properties specific for homothety:

b1) any line transforms into line parallel to it,

b2) all circles transform through homothety into one another,

b3) all triangles with sides pairwise parallel transform into one another, except equally positioned equal triangles,

b4) if three figures are pairwise homothetical then the centers of homotheties are collinear.

The applications of homothety usually follow a general schema: at first the existence of homothety is established from the properties of **some** elements of the configuration, and after that new properties of **other** elements are deduced from the general properties of homothety.

Comment. The informal formulation of the property a2 above appears to be especially useful for achieving better understanding by students, because it is intuitively clear and does not require long explanations.

4. Characteristic Examples

The following example is a nice proof of collinearity.

Example 1 (I.Sharygin). Three equal circumferences pass through the point S. Common tangents are drawn as shown in Fig. 1. Prove that O, the circumcenter of triangle ABC, I, the incenter of triangle ABC, and S are collinear.

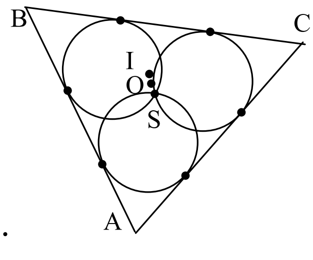

Fig. 1.

To solve this we connect the centers of the equal circumferences and acquire a triangle homothetical to ABC. The point I is the center of homothety between the triangles, while point S is the circumcenter of the new triangle. Therefore O and S transform into each other. The conclusion follows, because each two mutually corresponding points are collinear with the center of homothety.

The next example shows a proof that some points are concyclic.

Example 2 (L.Euler). Let's consider a triangle and it's altitudes. Prove that the feet of the altitudes and the midpoints of segments connecting the orthocenter with vertices are concyclic.

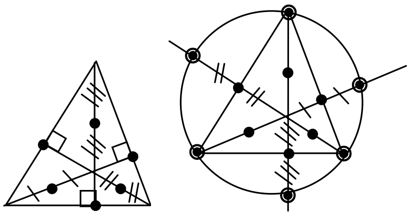

Fig. 2.

To prove it draw a circumcircle of the triangle (see Fig. 2.). Consider a homothety centered at the orthocenter with a coefficient 1/2.

Now we show how the homothety can be used to prove that some lines are concurrent.

Example 3 (Polish competition). A point is given inside an acute triangle. Perpendiculars are dropped from this point to the sides. Through the bases of these perpendiculars a circumference is drawn. It intersects each side a second time. Through these new intersection points perpendiculars are drawn to the corresponding sides. Prove that these perpendiculars intersect at the same point.

The problem exploits central symmetry or homothety with coefficient k=-1 and center coinciding with the center of the given circle.

The next example shows how homothety is used to prove the parallelity of two lines.

Example 4 (V.Prasolov). A circumference is touched on the inside by two smaller equal circumferences. For each point M on the outer circumference prove that AB is parallel to A_1B_1 (see Fig. 3.).

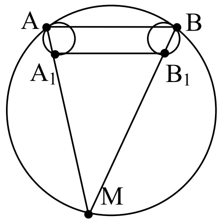

Fig. 3.

In the solution two homotheties with centers A and B and coefficients equal to the ratio of radiuses are used.

As the last example of incidence problems we consider a proof that two circumferences are touching each other.

Example 5 (Russian competition). Two circumferences touch each other in an inside manner at the point A (see Fig. 4.). Lines passing through A intersect the circuferences at points B, C and D, E correspondingly.

The segments BE and CD intersect at F; it is given that F is on the inner circumference. Prove that the circumcircle of CEF touches the inner circumference.

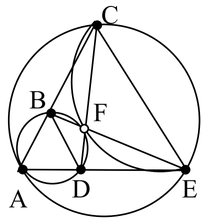

Fig. 4.

In the solution, two homotheties with centers A and F are considered.

At the end we give an example of geometrical construction.

Example 6 (Latvian competition). An angle and an inner point of it are given. Construct a circumference that touches the sides of the angle and passes through the given point.

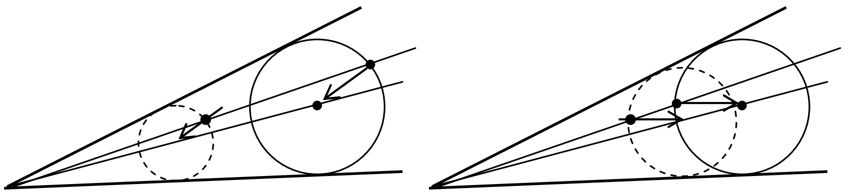
Fig. 5.

First we draw any circumference that touches the sides of the angle. Using this circumference's center we find centers of the homothetical circumferences. Notice that there are two solutions (see Fig. 5.).

There are also many problems combining homothety with some classical geometrical fact, with other transformations etc.

5. Why the Return?

The extended use of geometrical transformations fits good with the general „functional approach" to mathematics: they are functions whose arguments and values are points. The idea to consider separate elements as parts of greater system is characteristic for today's mathematics. The class of problems solvable by students becomes broader and more advanced.

Literature

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