MULTIPLICATION OF NON DEGENERATED CURVES OF SECOND ORDER

G. STANILOV, G. PANAYOTOVA, SL. SLAVOVA

*In this paper we introduce two operations in the set of non degenerated curves of second order:**a**) multiplication of curves and **b**) inversion of a curve. The main result is that this set with these two operations is a commutative and associative group. We prove also the multiplication of the curves is independent of the rotation of the coordinate system. All calculation are done using Computer System **Maple**. So we demonstrate the power of this system.*

Product of curves of second order. If $A = (a_{ij})$ is a matrix of order 3 the corresponding curve of second order has equation

$$a := a_{11}x^2 + a_{22}y^2 + a_{33} + (a_{12} + a_{21})xy + (a_{13} + a_{31})x + (a_{23} + a_{32})y = 0 .$$

If $B = (b_{ij})$ is a matrix of order 3 the corresponding curve of second order has equation

$$b := b_{11}x^2 + b_{22}y^2 + b_{33} + (b_{12} + b_{21})xy + (b_{13} + b_{31})x + (b_{23} + b_{32})y = 0$$

Let $F = A.B$ with elements $F = (f_{ij})$. We define the curve of second order by the equation:

$$f := f_{11}x^2 + f_{22}y^2 + f_{33} + (f_{12} + f_{21})xy + (f_{13} + f_{31})x + (f_{23} + f_{32})y = 0$$

This curve we call a **product of the curves** a and b we write $f = a.b$.
Theorem 1.The product of curves with symmetric matrices is commutative:

$$a.b = b.a .$$

Proof. We apply the computer system of MAPLE. Let

```
> A:=Matrix([[a11,a12,a13],[a21,a22,a23],[a31,a32,a33]]);
```

$$A := \begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix}$$

```
> B:=Matrix([[b11,b12,b13],[b21,b22,b23],[b31,b32,b33]]);
```

$$B := \begin{bmatrix} b11 & b12 & b13 \\ b21 & b22 & b23 \\ b31 & b32 & b33 \end{bmatrix}$$

```
> C:=Matrix([[c11,c12,c13],[c21,c22,c23],[c31,c32,c33]]);
```

$$C := \begin{bmatrix} c11 & c12 & c13 \\ c21 & c22 & c23 \\ c31 & c32 & c33 \end{bmatrix}$$

The elements of

```
> F:=A.B;
```

$F :=$

$[a11\, b11 + a12\, b21 + a13\, b31,\ a11\, b12 + a12\, b22 + a13\, b32,$
$a11\, b13 + a12\, b23 + a13\, b33]$
$[a21\, b11 + a22\, b21 + a23\, b31,\ a21\, b12 + a22\, b22 + a23\, b32,$
$a21\, b13 + a22\, b23 + a23\, b33]$

$[a31\, b11 + a32\, b21 + a33\, b31,\ a31\, b12 + a32\, b22 + a33\, b32,$
$a31\, b13 + a32\, b23 + a33\, b33]$

are:>

```
f11:=F[1,1];f12:=F[1,2];f13:=F[1,3];f21:=F[2,1];f22:=F[2,2]
;f23:=F[2,3];f31:=F[3,1];f32:=F[3,2];f33:=F[3,3];
```

$$f11 := a11\, b11 + a12\, b21 + a13\, b31$$

$$f12 := a11\, b12 + a12\, b22 + a13\, b32$$

$$f13 := a11\, b13 + a12\, b23 + a13\, b33$$

$$f21 := a21\, b11 + a22\, b21 + a23\, b31$$

$$f22 := a21\, b12 + a22\, b22 + a23\, b32$$

$$f23 := a21\, b13 + a22\, b23 + a23\, b33$$

$$f31 := a31\, b11 + a32\, b21 + a33\, b31$$

$$f32 := a31\, b12 + a32\, b22 + a33\, b32$$

$$f33 := a31\, b13 + a32\, b23 + a33\, b33$$

Analogously:

```
> K:=B.A;
```

$K :=$

$[a11\, b11 + a21\, b12 + a31\, b13,\ b11\, a12 + b12\, a22 + b13\, a32,$
$b11\, a13 + b12\, a23 + b13\, a33]$
$[b21\, a11 + b22\, a21 + b23\, a31,\ a12\, b21 + a22\, b22 + a32\, b23,$
$b21\, a13 + b22\, a23 + b23\, a33]$

$[b31\, a11 + b32\, a21 + b33\, a31,\ b31\, a12 + b32\, a22 + b33\, a32,$
$a13\, b31 + a23\, b32 + a33\, b33]$

```
> k11:=K[1,1];k12:=K[1,2];k13:=K[1,3];k21:=K[2,1];k22:=K[2,2];k23:=K[2,3];k31:=K[3,1];k32:=K[3,2];k33:=K[3,3];
```

$$k11 := a11\, b11 + a21\, b12 + a31\, b13$$

$$k12 := b11\, a12 + b12\, a22 + b13\, a32$$

$$k13 := b11\, a13 + b12\, a23 + b13\, a33$$

$$k21 := b21\, a11 + b22\, a21 + b23\, a31$$

$$k22 := a12\, b21 + a22\, b22 + a32\, b23$$

$$k23 := b21\, a13 + b22\, a23 + b23\, a33$$

$$k31 := b31\, a11 + b32\, a21 + b33\, a31$$

$$k32 := b31\, a12 + b32\, a22 + b33\, a32$$

$$k33 := a13\, b31 + a23\, b32 + a33\, b33$$

The corresponding curve for the matrix **F** is (we write only the left side of the equation)

```
> f:=f11*x^2+f22*y^2+f33+(f12+f21)*x*y+(f13+f31)*x+(f23+f32)*y;
```

$$\begin{aligned} f := &(a11\, b11 + a12\, b21 + a13\, b31)\, x^2 \\ &+ (a21\, b12 + a22\, b22 + a23\, b32)\, y^2 + a31\, b13 + a32\, b23 + a33\, b33 \\ &+ (a11\, b12 + a12\, b22 + a13\, b32 + a21\, b11 + a22\, b21 + a23\, b31)\, x\, y \\ &+ (a11\, b13 + a12\, b23 + a13\, b33 + a31\, b11 + a32\, b21 + a33\, b31)\, x \\ &+ (a21\, b13 + a22\, b23 + a23\, b33 + a31\, b12 + a32\, b22 + a33\, b32)\, y \end{aligned}$$

and for the matrix **K** -

```
k:=k11*x^2+k22*y^2+k33+(k12+k21)*x*y+(k13+k31)*x+(k23+k32)*y;
```

$$\begin{aligned} k := &(a11\, b11 + a21\, b12 + a31\, b13)\, x^2 \\ &+ (a12\, b21 + a22\, b22 + a32\, b23)\, y^2 + a13\, b31 + a23\, b32 + a33\, b33 \\ &+ (b11\, a12 + b12\, a22 + b13\, a32 + b21\, a11 + b22\, a21 + b23\, a31)\, x\, y \\ &+ (b11\, a13 + b12\, a23 + b13\, a33 + b31\, a11 + b32\, a21 + b33\, a31)\, x \\ &+ (b21\, a13 + b22\, a23 + b23\, a33 + b31\, a12 + b32\, a22 + b33\, a32)\, y \end{aligned}$$

Then we calculate

```
> simplify(eval((f-k),[a12=a21,a13=a31,a23=a32,b12=b21,b13=b31,b23=b32]));
```

$$0$$

In this way the theorem 1 is proved.>

Theorem 2.The product of curves is associative:

$$(a.b).c = a.(b.c)$$

where the curve

$$c := c11\,x^2 + c22\,y^2 + c33 + (c12 + c21)\,x\,y + (c13 + c31)\,x + 2\,c32\,y = 0$$

Proof. We have:

```
> H:=F.C;
```

$H :=$

$[(a11\,b11 + a12\,b21 + a13\,b31)\,c11$
$+ (a11\,b12 + a12\,b22 + a13\,b32)\,c21$
$+ (a11\,b13 + a12\,b23 + a13\,b33)\,c31,$
$(a11\,b11 + a12\,b21 + a13\,b31)\,c12$

$+ (a11\,b12 + a12\,b22 + a13\,b32)\,c22$
$+ (a11\,b13 + a12\,b23 + a13\,b33)\,c32,$
$(a11\,b11 + a12\,b21 + a13\,b31)\,c13$
$+ (a11\,b12 + a12\,b22 + a13\,b32)\,c23$
$+ (a11\,b13 + a12\,b23 + a13\,b33)\,c33]$

$[(a21\,b11 + a22\,b21 + a23\,b31)\,c11$
$+ (a21\,b12 + a22\,b22 + a23\,b32)\,c21$
$+ (a21\,b13 + a22\,b23 + a23\,b33)\,c31,$
$(a21\,b11 + a22\,b21 + a23\,b31)\,c12$
$+ (a21\,b12 + a22\,b22 + a23\,b32)\,c22$

$+ (a21\,b13 + a22\,b23 + a23\,b33)\,c32,$
$(a21\,b11 + a22\,b21 + a23\,b31)\,c13$
$+ (a21\,b12 + a22\,b22 + a23\,b32)\,c23$
$+ (a21\,b13 + a22\,b23 + a23\,b33)\,c33]$
$[(a31\,b11 + a32\,b21 + a33\,b31)\,c11$

$+ (a31\ b12 + a32\ b22 + a33\ b32)\ c21$
$+ (a31\ b13 + a32\ b23 + a33\ b33)\ c31$,
$(a31\ b11 + a32\ b21 + a33\ b31)\ c12$
$+ (a31\ b12 + a32\ b22 + a33\ b32)\ c22$
$+ (a31\ b13 + a32\ b23 + a33\ b33)\ c32$,

$(a31\ b11 + a32\ b21 + a33\ b31)\ c13$
$+ (a31\ b12 + a32\ b22 + a33\ b32)\ c23$
$+ (a31\ b13 + a32\ b23 + a33\ b33)\ c33$]

```
> h11:=H[1,1];h12:=H[1,2];h13:=H[1,3];h21:=H[2,1];h22:=H[2,2]
;h23:=H[2,3];h31:=H[3,1];h32:=H[3,2];h33:=H[3,3];
```

$$h11 := (a11\ b11 + a12\ b21 + a13\ b31)\ c11 + (a11\ b12 + a12\ b22 + a13\ b32)\ c21 + (a11\ b13 + a12\ b23 + a13\ b33)\ c31$$

$$h12 := (a11\ b11 + a12\ b21 + a13\ b31)\ c12 + (a11\ b12 + a12\ b22 + a13\ b32)\ c22 + (a11\ b13 + a12\ b23 + a13\ b33)\ c32$$

$$h13 := (a11\ b11 + a12\ b21 + a13\ b31)\ c13 + (a11\ b12 + a12\ b22 + a13\ b32)\ c23 + (a11\ b13 + a12\ b23 + a13\ b33)\ c33$$

$$h21 := (a21\ b11 + a22\ b21 + a23\ b31)\ c11 + (a21\ b12 + a22\ b22 + a23\ b32)\ c21 + (a21\ b13 + a22\ b23 + a23\ b33)\ c31$$

$$h22 := (a21\ b11 + a22\ b21 + a23\ b31)\ c12 + (a21\ b12 + a22\ b22 + a23\ b32)\ c22 + (a21\ b13 + a22\ b23 + a23\ b33)\ c32$$

$$h23 := (a21\ b11 + a22\ b21 + a23\ b31)\ c13 + (a21\ b12 + a22\ b22 + a23\ b32)\ c23 + (a21\ b13 + a22\ b23 + a23\ b33)\ c33$$

$$h31 := (a31\ b11 + a32\ b21 + a33\ b31)\ c11 + (a31\ b12 + a32\ b22 + a33\ b32)\ c21 + (a31\ b13 + a32\ b23 + a33\ b33)\ c31$$

$$h32 := (a31\ b11 + a32\ b21 + a33\ b31)\ c12 + (a31\ b12 + a32\ b22 + a33\ b32)\ c22 + (a31\ b13 + a32\ b23 + a33\ b33)\ c32$$

$$h33 := (a31\ b11 + a32\ b21 + a33\ b31)\ c13 + (a31\ b12 + a32\ b22 + a33\ b32)\ c23 + (a31\ b13 + a32\ b23 + a33\ b33)\ c33$$

```
> G:=B.C;
```

$$G := [b11\ c11 + b12\ c21 + b13\ c31,\ b11\ c12 + b12\ c22 + b13\ c32,\ b11\ c13 + b12\ c23 + b13\ c33]$$

$$[b21\ c11 + b22\ c21 + b23\ c31,\ b21\ c12 + b22\ c22 + b23\ c32,\ b21\ c13 + b22\ c23 + b23\ c33]$$

$$[b31\ c11 + b32\ c21 + b33\ c31,\ b31\ c12 + b32\ c22 + b33\ c32,\ b31\ c13 + b32\ c23 + b33\ c33]$$

```
> J:=A.G;
```

$$J := [a11\ (b11\ c11 + b12\ c21 + b13\ c31) + a12\ (b21\ c11 + b22\ c21 + b23\ c31) + a13\ (b31\ c11 + b32\ c21 + b33\ c31),\ a11\ (b11\ c12 + b12\ c22 + b13\ c32)$$

$$+ a12\ (b21\ c12 + b22\ c22 + b23\ c32) + a13\ (b31\ c12 + b32\ c22 + b33\ c32),\ a11\ (b11\ c13 + b12\ c23 + b13\ c33) + a12\ (b21\ c13 + b22\ c23 + b23\ c33) + a13\ (b31\ c13 + b32\ c23 + b33\ c33)]$$

$$[a21\ (b11\ c11 + b12\ c21 + b13\ c31) + a22\ (b21\ c11 + b22\ c21 + b23\ c31) + a23\ (b31\ c11 + b32\ c21 + b33\ c31),\ a21\ (b11\ c12 + b12\ c22 + b13\ c32) + a22\ (b21\ c12 + b22\ c22 + b23\ c32)$$

$$+ a23\,(b31\,c12 + b32\,c22 + b33\,c32),$$
$$a21\,(b11\,c13 + b12\,c23 + b13\,c33)$$
$$+ a22\,(b21\,c13 + b22\,c23 + b23\,c33)$$
$$+ a23\,(b31\,c13 + b32\,c23 + b33\,c33)]$$
$$[a31\,(b11\,c11 + b12\,c21 + b13\,c31)$$

$$+ a32\,(b21\,c11 + b22\,c21 + b23\,c31)$$
$$+ a33\,(b31\,c11 + b32\,c21 + b33\,c31),$$
$$a31\,(b11\,c12 + b12\,c22 + b13\,c32)$$
$$+ a32\,(b21\,c12 + b22\,c22 + b23\,c32)$$
$$+ a33\,(b31\,c12 + b32\,c22 + b33\,c32),$$

$$a31\,(b11\,c13 + b12\,c23 + b13\,c33)$$
$$+ a32\,(b21\,c13 + b22\,c23 + b23\,c33)$$
$$+ a33\,(b31\,c13 + b32\,c23 + b33\,c33)]$$

```
> 
j11:=J[1,1];j12:=J[1,2];j13:=J[1,3];j21:=J[2,1];j22:=J[2,2]
;j23:=J[2,3];j31:=J[3,1];j32:=J[3,2];j33:=J[3,3];
```

$$j11 := a11\,(b11\,c11 + b12\,c21 + b13\,c31) + a12\,(b21\,c11 + b22\,c21 + b23\,c31) + a13\,(b31\,c11 + b32\,c21 + b33\,c31)$$

$$j12 := a11\,(b11\,c12 + b12\,c22 + b13\,c32) + a12\,(b21\,c12 + b22\,c22 + b23\,c32) + a13\,(b31\,c12 + b32\,c22 + b33\,c32)$$

$$j13 := a11\,(b11\,c13 + b12\,c23 + b13\,c33) + a12\,(b21\,c13 + b22\,c23 + b23\,c33) + a13\,(b31\,c13 + b32\,c23 + b33\,c33)$$

$$j21 := a21\,(b11\,c11 + b12\,c21 + b13\,c31) + a22\,(b21\,c11 + b22\,c21 + b23\,c31) + a23\,(b31\,c11 + b32\,c21 + b33\,c31)$$

$$j22 := a21\,(b11\,c12 + b12\,c22 + b13\,c32) + a22\,(b21\,c12 + b22\,c22 + b23\,c32) + a23\,(b31\,c12 + b32\,c22 + b33\,c32)$$

$$j23 := a21\,(b11\,c13 + b12\,c23 + b13\,c33) + a22\,(b21\,c13 + b22\,c23 + b23\,c33) + a23\,(b31\,c13 + b32\,c23 + b33\,c33)$$

$$j31 := a31\,(b11\,c11 + b12\,c21 + b13\,c31) + a32\,(b21\,c11 + b22\,c21 + b23\,c31) + a33\,(b31\,c11 + b32\,c21 + b33\,c31)$$

$$j32 := a31\,(b11\,c12 + b12\,c22 + b13\,c32) + a32\,(b21\,c12 + b22\,c22 + b23\,c32) + a33\,(b31\,c12 + b32\,c22 + b33\,c32)$$

$$j33 := a31\,(b11\,c13 + b12\,c23 + b13\,c33) + a32\,(b21\,c13 + b22\,c23 + b23\,c33) + a33\,(b31\,c13 + b32\,c23 + b33\,c33)$$

The equation of the curve $h = (a.b).c$ is >

```
h:=h11*x^2+h22*y^2+h33+(h12+h21)*x*y+(h13+h31)*x+(h23+h32)*
y;
```

$$h := ((a11\,b11 + a12\,b21 + a13\,b31)\,c11 + (a11\,b12 + a12\,b22 + a13\,b32)\,c21 + (a11\,b13 + a12\,b23 + a13\,b33)\,c31)\,x^2 + ((a21\,b11 + a22\,b21 + a23\,b31)\,c12 + (a21\,b12 + a22\,b22 + a23\,b32)\,c22$$

$$+ (a21\,b13 + a22\,b23 + a23\,b33)\,c32)\,y^2 + (a31\,b11 + a32\,b21 + a33\,b31)\,c13 + (a31\,b12 + a32\,b22 + a33\,b32)\,c23 + (a31\,b13 + a32\,b23 + a33\,b33)\,c33 + ((a11\,b11 + a12\,b21 + a13\,b31)\,c12$$

$$+ (a11\,b12 + a12\,b22 + a13\,b32)\,c22 + (a11\,b13 + a12\,b23 + a13\,b33)\,c32 + (a21\,b11 + a22\,b21 + a23\,b31)\,c11 + (a21\,b12 + a22\,b22 + a23\,b32)\,c21 + (a21\,b13 + a22\,b23 + a23\,b33)\,c31)\,x\,y + ($$

$(a11\, b11 + a12\, b21 + a13\, b31)\, c13$
$+ (a11\, b12 + a12\, b22 + a13\, b32)\, c23$
$+ (a11\, b13 + a12\, b23 + a13\, b33)\, c33$
$+ (a31\, b11 + a32\, b21 + a33\, b31)\, c11$
$+ (a31\, b12 + a32\, b22 + a33\, b32)\, c21$

$+ (a31\, b13 + a32\, b23 + a33\, b33)\, c31)\, x + ($
$(a21\, b11 + a22\, b21 + a23\, b31)\, c13$
$+ (a21\, b12 + a22\, b22 + a23\, b32)\, c23$
$+ (a21\, b13 + a22\, b23 + a23\, b33)\, c33$
$+ (a31\, b11 + a32\, b21 + a33\, b31)\, c12$

$+ (a31\, b12 + a32\, b22 + a33\, b32)\, c22$
$+ (a31\, b13 + a32\, b23 + a33\, b33)\, c32)\, y$

The equation of the curve j=a.(b.c) is

```
> j:=j11*x^2+j22*y^2+j33+(j12+j21)*x*y+(j13+j31)*x+(j23+j32)*y;
```

$j := (a11\, (b11\, c11 + b12\, c21 + b13\, c31)$
$+ a12\, (b21\, c11 + b22\, c21 + b23\, c31)$
$+ a13\, (b31\, c11 + b32\, c21 + b33\, c31))\, x^2 + ($
$a21\, (b11\, c12 + b12\, c22 + b13\, c32)$
$+ a22\, (b21\, c12 + b22\, c22 + b23\, c32)$

$+ a23\, (b31\, c12 + b32\, c22 + b33\, c32))\, y^2$
$+ a31\, (b11\, c13 + b12\, c23 + b13\, c33)$
$+ a32\, (b21\, c13 + b22\, c23 + b23\, c33)$
$+ a33\, (b31\, c13 + b32\, c23 + b33\, c33) + ($
$a11\, (b11\, c12 + b12\, c22 + b13\, c32)$

$+ a12\, (b21\, c12 + b22\, c22 + b23\, c32)$
$+ a13\, (b31\, c12 + b32\, c22 + b33\, c32)$
$+ a21\, (b11\, c11 + b12\, c21 + b13\, c31)$
$+ a22\, (b21\, c11 + b22\, c21 + b23\, c31)$
$+ a23\, (b31\, c11 + b32\, c21 + b33\, c31))\, x\, y + ($

$a11\,(b11\,c13 + b12\,c23 + b13\,c33)$
$+ a12\,(b21\,c13 + b22\,c23 + b23\,c33)$
$+ a13\,(b31\,c13 + b32\,c23 + b33\,c33)$
$+ a31\,(b11\,c11 + b12\,c21 + b13\,c31)$
$+ a32\,(b21\,c11 + b22\,c21 + b23\,c31)$

$+ a33\,(b31\,c11 + b32\,c21 + b33\,c31))\,x + ($
$a21\,(b11\,c13 + b12\,c23 + b13\,c33)$
$+ a22\,(b21\,c13 + b22\,c23 + b23\,c33)$
$+ a23\,(b31\,c13 + b32\,c23 + b33\,c33)$
$+ a31\,(b11\,c12 + b12\,c22 + b13\,c32)$

$+ a32\,(b21\,c12 + b22\,c22 + b23\,c32)$
$+ a33\,(b31\,c12 + b32\,c22 + b33\,c32))\,y$

```
> simplify(h-j);
```

$$0$$

Theorem3. The quadrate of any non degenerated curve is an imaginer ellipsis.sy
Proof. We have (taking **A** symmetric)

```
A:=Matrix([[a11,a12,a13],[a12,a22,a23],[a13,a23,a33]]);
```

$$A := \begin{bmatrix} a11 & a12 & a13 \\ a12 & a22 & a23 \\ a13 & a23 & a33 \end{bmatrix}$$

Let now

```
C:=Multiply(A,A);
```

$C :=$

$[a11^2 + a12^2 + a13^2\,,\, a12\,a11 + a12\,a22 + a13\,a23\,,$
$a11\,a13 + a12\,a23 + a13\,a33]$
$[a12\,a11 + a12\,a22 + a13\,a23\,,\, a12^2 + a22^2 + a23^2\,,$

$a13\,a12 + a23\,a22 + a23\,a33]$
$[a11\,a13 + a12\,a23 + a13\,a33\,,\, a13\,a12 + a23\,a22 + a23\,a33\,,$
$a13^2 + a23^2 + a33^2]$

We remark here: the quadrate of any symmtric matrix is a symmetric matrix. We have

```
c11:=a11^2+a12^2+a13^2;c12:=a11*a12+a12*a22+a13*a23;c13:=a1
1*a13+a12*a23+a13*a33;c22:=a12^2+a22^2+a23^2;c23:=a12*a13+a
22*a23+a23*a33;c33:=a13^2+a23^2+a33^2;
```

$$c11 := a11^2 + a12^2 + a13^2$$

$$c12 := a12\ a11 + a12\ a22 + a13\ a23$$

$$c13 := a11\ a13 + a12\ a23 + a13\ a33$$

$$c22 := a12^2 + a22^2 + a23^2$$

$$c23 := a13\ a12 + a23\ a22 + a23\ a33$$

$$c33 := a13^2 + a23^2 + a33^2$$

Evidently

(1) **c11 + c22 >0**

We calculate

```
C33:=c11*c22-c12^2=(a12*a23-a22*a13)^2+(a13*a12-a11*a23)^2+(a11*a22-a12^2)^2;
```

$$C33 := (a11^2 + a12^2 + a13^2)(a12^2 + a22^2 + a23^2) - (a12\ a11 + a12\ a22 + a13\ a23)^2 = (a12\ a23 - a22\ a13)^2 + (a13\ a12 - a11\ a23)^2 + (a11\ a22 - a12^2)^2$$

so that

(2) **C33>0**

Because of

```
simplify(Determinant(C)-Determinant(A)^2);
```

$$0$$

it follows

(3) . **Determinant(C)=Determinant(A)^2)**

From (1),(2) and (3) follows Theorem 3.

Theorem4. The unit imaginer circle ($x^2 + y^2 + 1 = 0$) plays the roll of a unite in the set of all curves of second order.

Let E is the unite matrix of order 3. It is the matrix of the imaginer unite circle.

The assertion follows from:

```
simplify(Multiply (A,E)-A);
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Inverse curve of a given curve.
If

```
a:=a11*x^2+a22*y^2+a33+(a12+a21)*x*y+(a13+a31)*x+(a23+a32)+y=0
```

$$a := a11\, x^2 + a22\, y^2 + a33 + (a12 + a21)\, x\, y + (a13 + a31)\, x + (a23 + a32)\, y = 0$$

is a curve of second order, which matrix $A = (a_{ij})$ is symmetric, using its inverse matrix $A^{-1} = (A_{ij})$,which is also symmetric, the curve

```
J(a):=A11*x^2+A22*y^2+A33+(A12+A21)*x*y+(A13+A31)*x+(A23+A32)*y=0;
```

$$\mathrm{J}(a) := A11\, x^2 + A22\, y^2 + A33 + (A12 + A21)\, x\, y + (A13 + A31)\, x + (A23 + A32)\, y = 0$$

is called **inverse curve** to the curve `a`. Evidently we have

Theorem 5. The product of any curve and its inverse curve is the unite imaginer circle.

As a consequence of the above theorems we have the following

Theorem 6. The set of all non degenerated curves of second order is a commutative and associative group.

Theorem 7. The operation "multiplication of curves" commutes with the rotation of the coordinate system.
Proof. Let us have a curve in respect to the orthogonal coordinate system Oxy

```
a:=a11*x^2+a22*y^2+2*a12*x*y+2*a13*x+2*a23*y+a33;
```

$$a := a11\, x^2 + a22\, y^2 + 2\, a12\, x\, y + 2\, a13\, x + 2\, a23\, y + a33$$

After the rotation: `x:=cos(u)*x1-sin(u)*y1;y:=sin(u)*x1+cos(u)*y1;`

$$x := \cos(u)\, x1 - \sin(u)\, y1$$

$$y := \sin(u)\, x1 + \cos(u)\, y1$$

we get the curve

$$a11\, (\cos(u)\, x1 - \sin(u)\, y1)^2 + a22\, (\sin(u)\, x1 + \cos(u)\, y1)^2 + 2\, a12\, (\cos(u)\, x1 - \sin(u)\, y1)\, (\sin(u)\, x1 + \cos(u)\, y1) + 2\, a13\, (\cos(u)\, x1 - \sin(u)\, y1) + 2\, a23\, (\sin(u)\, x1 + \cos(u)\, y1) + a33$$

which coefficients are:

$$p11 := a11 \cos(u)^2 + a22 \sin(u)^2 + 2\, a12 \cos(u) \sin(u)$$

$$p22 := a11\sin(u)^2 + a22\cos(u)^2 - 2\,a12\cos(u)\sin(u)$$

$$p12 := -a11\cos(u)\sin(u) + a22\sin(u)\cos(u) + a12\cos(u)^2 - a12\sin(u)^2$$

$$p13 := a13\cos(u) + a23\sin(u)$$

$$p23 := -a13\sin(u) + a23\cos(u)$$

$$p33 := a33$$

$$p21 = p12\,,\, p31 = p13\,,\, p32 = p23$$

```
with(LinearAlgebra):
```

```
A:=Matrix([[a11,a12,a13],[a12,a22,a23],[a13,a23,a33]]);Dete
rminant(A);
```

$$A := \begin{bmatrix} a11 & a12 & a13 \\ a12 & a22 & a23 \\ a13 & a23 & a33 \end{bmatrix}$$

$$a11\,a22\,a33 - a11\,a23^2 + 2\,a12\,a23\,a13 - a12^2\,a33 - a22\,a13^2$$

```
P:=Matrix([[p11,p12,p13],[p12,p22,p23],[p13,p23,p33]]);
```

$$P := \begin{bmatrix} p11 & p12 & p13 \\ p12 & p22 & p23 \\ p13 & p23 & p33 \end{bmatrix}$$

```
 simplify(Determinant(P));simplify(Determinant(P)-
Determinant(A));
```

$$2\,a12\,a13\,a23 - a22\,a13^2 + a22\,a33\,a11 - a12^2\,a33 - a23^2\,a11$$

$$0$$

For the second curve correspondently we have:

```
b:=b11*x^2+b22*y^2+2*b12*x*y+2*b13*x+2*b23*y+b33;
```

```
B:=Matrix([[b11,b12,b13],[b12,b22,b23],[b13,b23,b33]]);
Determinant(B);
```

$$B := \begin{bmatrix} b11 & b12 & b13 \\ b12 & b22 & b23 \\ b13 & b23 & b33 \end{bmatrix}$$

$$b11\,b22\,b33 - b11\,b23^2 + 2\,b12\,b23\,b13 - b12^2\,b33 - b22\,b13^2$$

```
q11:=eval(p11,[a11=b11,a12=b12,a22=b22,a13=b13,a23=b23,a33=
b33]);q22:=eval(p22,[a11=b11,a12=b12,a22=b22,a13=b13,a23=b2
3,a33=b33]);q13:=eval(p13,[a11=b11,a12=b12,a22=b22,a13=b13,
a23=b23,a33=b33]);q23:=eval(p23,[a11=b11,a12=b12,a22=b22,a1
```

```
3=b13,a23=b23,a33=b33]);q12:=eval(p12,[a11=b11,a12=b12,a22=
b22,a13=b13,a23=b23,a33=b33]);q33:=eval(p33,[a11=b11,a12=b1
2,a22=b22,a13=b13,a23=b23,a33=b33]);
```

$$q11 := b11\cos(u)^2 + b22\sin(u)^2 + 2\,b12\cos(u)\sin(u)$$

$$q22 := b11\sin(u)^2 + b22\cos(u)^2 - 2\,b12\cos(u)\sin(u)$$

$$q13 := b13\cos(u) + b23\sin(u)$$

$$q23 := -b13\sin(u) + b23\cos(u)$$

$$q12 := -b11\cos(u)\sin(u) + b22\sin(u)\cos(u) + b12\cos(u)^2 - b12\sin(u)^2$$

$$q33 := b33$$

```
Q:=Matrix([[q11,q12,q13],[q12,q22,q23],[q13,q23,q33]]);simp
lify(Determinant(Q)-Determinant(B));
```

$Q :=$

$[b11\cos(u)^2 + b22\sin(u)^2 + 2\,b12\cos(u)\sin(u),$
$-b11\cos(u)\sin(u) + b22\sin(u)\cos(u) + b12\cos(u)^2 - b12\sin(u)^2$
$, b13\cos(u) + b23\sin(u)]$

$[$
$-b11\cos(u)\sin(u) + b22\sin(u)\cos(u) + b12\cos(u)^2 - b12\sin(u)^2$
$, b11\sin(u)^2 + b22\cos(u)^2 - 2\,b12\cos(u)\sin(u),$
$-b13\sin(u) + b23\cos(u)]$

$[b13\cos(u) + b23\sin(u), -b13\sin(u) + b23\cos(u), b33]$

$$0$$

Let

```
C:=Multiply(A,B);
```

$C :=$

$[a11\,b11 + a12\,b12 + a13\,b13, a11\,b12 + a12\,b22 + a13\,b23,$
$a11\,b13 + a12\,b23 + a13\,b33]$
$[a12\,b11 + a22\,b12 + a23\,b13, a12\,b12 + a22\,b22 + a23\,b23,$
$a12\,b13 + a22\,b23 + a23\,b33]$

$[a13\,b11 + a23\,b12 + a33\,b13, a13\,b12 + a23\,b22 + a33\,b23,$
$a13\,b13 + a23\,b23 + a33\,b33]$

```
c11:=a11*b11+a12*b12+a13*b13;c12:=a11*b12+a12*b22+a13*b23;c
21:=a12*b11+a22*b12+a23*b13;c22:=a12*b12+a22*b22+a23*b23;c1
```

```
3:=a11*b13+a12*b23+a13*b33;c31:=a13*b11+a23*b12+a33*b13;c23
:=a12*b13+a22*b23+a23*b33;c32:=a13*b12+a23*b22+a33*b23;c33:
=a13*b13+a23*b23+a33*b33;
```

$$c11 := a11\, b11 + a12\, b12 + a13\, b13$$

$$c12 := a11\, b12 + a12\, b22 + a13\, b23$$

$$c21 := a12\, b11 + a22\, b12 + a23\, b13$$

$$c22 := a12\, b12 + a22\, b22 + a23\, b23$$

$$c13 := a11\, b13 + a12\, b23 + a13\, b33$$

$$c31 := a13\, b11 + a23\, b12 + a33\, b13$$

$$c23 := a12\, b13 + a22\, b23 + a23\, b33$$

$$c32 := a13\, b12 + a23\, b22 + a33\, b23$$

$$c33 := a13\, b13 + a23\, b23 + a33\, b33$$

```
R:=Multiply(P,Q);simplify(Determinant(R)-
Determinant(C));simplify(r11+r22-c11-
c22);f:=simplify(r11*r22-(r12+r21)^2/4-
c11*c22+(c12+c21)^2/4);
```

$R :=$

$[(a11 \cos(u)^2 + a22 \sin(u)^2 + 2\, a12 \cos(u) \sin(u))$
$(b11 \cos(u)^2 + b22 \sin(u)^2 + 2\, b12 \cos(u) \sin(u)) + ($
$- a11 \cos(u) \sin(u) + a22 \sin(u) \cos(u) + a12 \cos(u)^2 - a12 \sin(u)^2$

$) ($
$- b11 \cos(u) \sin(u) + b22 \sin(u) \cos(u) + b12 \cos(u)^2 - b12 \sin(u)^2$
$) + (a13 \cos(u) + a23 \sin(u)) (b13 \cos(u) + b23 \sin(u)),$
$(a11 \cos(u)^2 + a22 \sin(u)^2 + 2\, a12 \cos(u) \sin(u)) ($

$- b11 \cos(u) \sin(u) + b22 \sin(u) \cos(u) + b12 \cos(u)^2 - b12 \sin(u)^2$
$) + ($
$- a11 \cos(u) \sin(u) + a22 \sin(u) \cos(u) + a12 \cos(u)^2 - a12 \sin(u)^2$
$) (b11 \sin(u)^2 + b22 \cos(u)^2 - 2\, b12 \cos(u) \sin(u))$

$+ (a13 \cos(u) + a23 \sin(u)) (-b13 \sin(u) + b23 \cos(u))$,
$(a11 \cos(u)^2 + a22 \sin(u)^2 + 2\, a12 \cos(u) \sin(u))$
$(b13 \cos(u) + b23 \sin(u)) +$ (
$-a11 \cos(u) \sin(u) + a22 \sin(u) \cos(u) + a12 \cos(u)^2 - a12 \sin(u)^2$

$) (-b13 \sin(u) + b23 \cos(u)) + (a13 \cos(u) + a23 \sin(u))\, b33$]
[(
$-a11 \cos(u) \sin(u) + a22 \sin(u) \cos(u) + a12 \cos(u)^2 - a12 \sin(u)^2$
$) (b11 \cos(u)^2 + b22 \sin(u)^2 + 2\, b12 \cos(u) \sin(u)) +$

$(a11 \sin(u)^2 + a22 \cos(u)^2 - 2\, a12 \cos(u) \sin(u))$ (
$-b11 \cos(u) \sin(u) + b22 \sin(u) \cos(u) + b12 \cos(u)^2 - b12 \sin(u)^2$
$) + (-a13 \sin(u) + a23 \cos(u)) (b13 \cos(u) + b23 \sin(u))$, (
$-a11 \cos(u) \sin(u) + a22 \sin(u) \cos(u) + a12 \cos(u)^2 - a12 \sin(u)^2$

) (
$-b11 \cos(u) \sin(u) + b22 \sin(u) \cos(u) + b12 \cos(u)^2 - b12 \sin(u)^2$
$) + (a11 \sin(u)^2 + a22 \cos(u)^2 - 2\, a12 \cos(u) \sin(u))$
$(b11 \sin(u)^2 + b22 \cos(u)^2 - 2\, b12 \cos(u) \sin(u))$

$+ (-a13 \sin(u) + a23 \cos(u)) (-b13 \sin(u) + b23 \cos(u))$, (
$-a11 \cos(u) \sin(u) + a22 \sin(u) \cos(u) + a12 \cos(u)^2 - a12 \sin(u)^2$
$) (b13 \cos(u) + b23 \sin(u)) +$
$(a11 \sin(u)^2 + a22 \cos(u)^2 - 2\, a12 \cos(u) \sin(u))$

$(-b13 \sin(u) + b23 \cos(u)) + (-a13 \sin(u) + a23 \cos(u))\, b33$]
$[(a13 \cos(u) + a23 \sin(u))$
$(b11 \cos(u)^2 + b22 \sin(u)^2 + 2\, b12 \cos(u) \sin(u)) +$
$(-a13 \sin(u) + a23 \cos(u))$ (

$-b11 \cos(u) \sin(u) + b22 \sin(u) \cos(u) + b12 \cos(u)^2 - b12 \sin(u)^2$
$) + a33 (b13 \cos(u) + b23 \sin(u)), (a13 \cos(u) + a23 \sin(u))($
$-b11 \cos(u) \sin(u) + b22 \sin(u) \cos(u) + b12 \cos(u)^2 - b12 \sin(u)^2$
$) + (-a13 \sin(u) + a23 \cos(u))$

$(b11 \sin(u)^2 + b22 \cos(u)^2 - 2\, b12 \cos(u) \sin(u))$
$+ a33 (-b13 \sin(u) + b23 \cos(u)),$
$(a13 \cos(u) + a23 \sin(u)) (b13 \cos(u) + b23 \sin(u))$
$+ (-a13 \sin(u) + a23 \cos(u)) (-b13 \sin(u) + b23 \cos(u)) + a33\, b33$
]

$$0$$

$$0$$

$$f := 0$$

where

```
>r11:=simplify(p11*q11+p12*q12+p13*q13);r12:=p11*q12+p12*q22
+p13*q23;r21:=p12*q11+p22*q12+p23*q13;r22:=p12*q12+p22*q22+
p23*q23;r13:=p11*q13+p12*q23+p13*q33;r31:=p13*q11+p23*q12+p
33*q13;r23:=p12*q13+p22*q23+p23*q33;r32:=p13*q12+p23*q22+p3
3*q23;r33:=p13*q13+p23*q23+p33*q33;
```

$r11 := \cos(u) \sin(u)\, a13\, b23 + a23 \sin(u)\, b13 \cos(u)$
$+ \sin(u)\, a12 \cos(u)\, b22 + \sin(u)\, a12\, b11 \cos(u)$
$+ \sin(u)\, a11 \cos(u)\, b12 + \sin(u)\, a22\, b12 \cos(u) + a23\, b23$
$+ a12\, b12 + \cos(u)^2\, a13\, b13 + a11 \cos(u)^2\, b11 - a22 \cos(u)^2\, b22$
$+ a22\, b22 - \cos(u)^2\, a23\, b23$

$r12 := (a11 \cos(u)^2 + a22 \sin(u)^2 + 2\, a12 \cos(u) \sin(u))($
$-b11 \cos(u) \sin(u) + b22 \sin(u) \cos(u) + b12 \cos(u)^2 - b12 \sin(u)^2$
$) + ($
$-a11 \cos(u) \sin(u) + a22 \sin(u) \cos(u) + a12 \cos(u)^2 - a12 \sin(u)^2$
$) (b11 \sin(u)^2 + b22 \cos(u)^2 - 2\, b12 \cos(u) \sin(u))$
$+ (a13 \cos(u) + a23 \sin(u)) (-b13 \sin(u) + b23 \cos(u))$

$$r21 := (-a11\cos(u)\sin(u) + a22\sin(u)\cos(u) + a12\cos(u)^2 - a12\sin(u)^2)(b11\cos(u)^2 + b22\sin(u)^2 + 2\,b12\cos(u)\sin(u)) + (a11\sin(u)^2 + a22\cos(u)^2 - 2\,a12\cos(u)\sin(u))(-b11\cos(u)\sin(u) + b22\sin(u)\cos(u) + b12\cos(u)^2 - b12\sin(u)^2) + (-a13\sin(u) + a23\cos(u))(b13\cos(u) + b23\sin(u))$$

$$r22 := (-a11\cos(u)\sin(u) + a22\sin(u)\cos(u) + a12\cos(u)^2 - a12\sin(u)^2)(-b11\cos(u)\sin(u) + b22\sin(u)\cos(u) + b12\cos(u)^2 - b12\sin(u)^2) + (a11\sin(u)^2 + a22\cos(u)^2 - 2\,a12\cos(u)\sin(u))(b11\sin(u)^2 + b22\cos(u)^2 - 2\,b12\cos(u)\sin(u)) + (-a13\sin(u) + a23\cos(u))(-b13\sin(u) + b23\cos(u))$$

$$r13 := (a11\cos(u)^2 + a22\sin(u)^2 + 2\,a12\cos(u)\sin(u))(b13\cos(u) + b23\sin(u)) + (-a11\cos(u)\sin(u) + a22\sin(u)\cos(u) + a12\cos(u)^2 - a12\sin(u)^2)(-b13\sin(u) + b23\cos(u)) + (a13\cos(u) + a23\sin(u))\,b33$$

$$r31 := (a13\cos(u) + a23\sin(u))(b11\cos(u)^2 + b22\sin(u)^2 + 2\,b12\cos(u)\sin(u)) + (-a13\sin(u) + a23\cos(u))(-b11\cos(u)\sin(u) + b22\sin(u)\cos(u) + b12\cos(u)^2 - b12\sin(u)^2) + a33\,(b13\cos(u) + b23\sin(u))$$

$$r23 := (-a11\cos(u)\sin(u) + a22\sin(u)\cos(u) + a12\cos(u)^2 - a12\sin(u)^2)(b13\cos(u) + b23\sin(u)) + (a11\sin(u)^2 + a22\cos(u)^2 - 2\,a12\cos(u)\sin(u))(-b13\sin(u) + b23\cos(u)) + (-a13\sin(u) + a23\cos(u))\,b33$$

$$r32 := (a13\cos(u) + a23\sin(u))(-b11\cos(u)\sin(u) + b22\sin(u)\cos(u) + b12\cos(u)^2 - b12\sin(u)^2) + (-a13\sin(u) + a23\cos(u))(b11\sin(u)^2 + b22\cos(u)^2 - 2\,b12\cos(u)\sin(u)) + a33\,(-b13\sin(u) + b23\cos(u))$$

$$r33 := (a13\cos(u) + a23\sin(u))(b13\cos(u) + b23\sin(u)) + (-a13\sin(u) + a23\cos(u))(-b13\sin(u) + b23\cos(u)) + a33\,b33$$

So we have proved :

LEMA . Under the orthogonal transformation of the coordinate system

C1:= c11+c22=r11+r22 =:R1

C33:= c11*c22-(c12+c21)^2/4=r11*r22-(r12+r21)^2/2 =:R33

C3:= Det(C) = Det (R) =:R3

Remark. The theorem 7 is not true (for arbitrary curves) under the translation of the coordinate system. This fact we shall investigate later.

stanilov@fmi.uni-sofia.bg g-panayotova@mail.bg slavova@dobrich.net