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Indirect Addition: Theoretical, Methodological and Educational Considerations

The development of fundamentally important arithmetic principles related to the four basic operations, and of arithmetic strategies that are based on these principles, is an intriguing and important element of psychological, mathematical and math educational research. As far as addition and subtraction are concerned, we have, for instance, the following principles: (a) the commutativity principle, which says that the order of the addends is irrelevant to their sum (a + b = b + a); (b) the principle prescribing that if nothing is added to or removed from a collection its cardinal value remains unchanged (a + 0 = 0; a - 0 = a); (c) the principle that adding an amount to a collection can be undone by subtracting the same amount and vice versa (a + b - b = a or a - b + b = a); and (d) the principle that if a + b = c, then c-b = a or c - a = b. Previous theorizing and research shows that understanding these principles plays an important role in children's construction of the additive composition of number and in additive reasoning. Moreover, the implicit or explicit application of these principles can also considerably facilitate people's arithmetic performance by eliminating computational effort and increasing solution efficiency (Baroody, Torbeyns, & Verschaffel, 2009). For example, the first principle underlies the well-known computation shortcut for solving additions starting with the smaller given number (like 2 + 9 or 4 + 58), that consists of reversing the order of operands and adding the smaller addend to the larger one. The fourth principle underlies the computation shortcut for solving subtractions involving a small difference between the two integers (like 11 - 9 or 61 - 59), by determining how much has to be added to the smaller integer to make the larger one. Whereas the first three above-mentioned principles and their accompanying computational shortcut strategies have already received a great amount of research attention (Verschaffel, Greer, & De Corte, 2007), the fourth principle has not. In this contribution, we will present a series of closely related studies in the domain of elementary subtraction that we have done so far on this fourth principle and its accompanying computational shortcut, namely *indirect addition* (IA). We will use the term *direct subtraction* (DS) for the more common straightforward strategy for doing subtraction whereby the smaller number is directly taken away from the smaller one.

Use of IA in Young Adults

In our first study, 25 university students solved a series of three-digit subtractions (Torbeyns, Ghesquière, & Verschaffel, 2009). We made a distinction among three types of subtractions on the basis of the difference between the two given numbers, i.e., subtractions with a small (812 - 783), medium (821 - 475), and large difference (813 - 176). Adopting the choice/no-choice method (Siegler & Lemaire, 1997), all participants were instructed to solve these subtractions individually in one choice and two no-choice conditions. In the choice condition, participants could choose between IA *or* DS. In the first no-choice condition participants were instructed to solve all subtractions with IA; in the second no-choice condition they always had to apply DS. In all three conditions, they had to verbally report the strategy used *after* each trial. We registered the accuracy and speed of responding in each condition on a trial-by-trial basis.

When we analyzed the data from the choice condition, we found that participants solved about half of the subtractions with IA. The data from the two no-choice conditions revealed that, as expected, IA was significantly quicker than DS on the small-difference subtractions. However, we unexpectedly also found that this speed advantage of IA did also hold for medium- and even large-difference subtractions. In other words, IA was executed faster than DS not only when there was a small difference between the two integers, but also on the two other subtraction types where the computational advantage of using IA seems less clear.

Because we were surprised by the efficiency results of this first study, we replicated it in a follow-up study with a similar group of students and with a similar design (Torbeyns, Ghesquière, et al., 2009). The only important difference was that only subtractions with small to medium differences were presented, divided into four problem types on the basis of the size of the difference. The results of this follow-up study were similar to those from the first study, except that the results for the no-choice conditions revealed that IA was not only executed faster but also more accurately than DS. Furthermore, IA was (again) executed more efficiently than DS on the subtractions with a very small difference between the two integers as well as on the other problem types for which the computational advantage of solving the subtraction by IA seems less straightforward.

Still somewhat puzzled by these findings on the overall superior efficiency of IA compared to DS, we conducted a second replication study wherein we administered another set of subtractions in a similar group of young adults, again using the choice/no-choice method (Torbeyns, De Smedt, Peters, Ghesquière, & Verschaffel, 2009). Participants were offered two types of three-digit subtractions, namely subtractions with a very small difference (713 - 695) and subtractions with an extremely large difference, i.e., subtractions with a three-digit minuend and a *two*-digit subtrahend (756 - 78). We reasoned that the latter subtractions would favor DS par excellence and that it (thus) would be really striking if participants still solved these subtractions more efficiently with IA than with DS. Unexpectedly, we again observed that participants solved both subtraction types - even the subtractions with extremely large differences - more quickly and more accurately by means of IA than with DS. In other words, the results from our first three studies indicate that adults use IA frequently and highly efficiently on multi-digit subtractions, even on subtractions where the computational advantage of using IA is less clear.

Development and Use of IA in Children

Given both the results of our studies in adults and the fact that many math educators make a plea for giving IA a prominent place in elementary school children's mental arithmetic lessons, we next investigated children's use of this strategy. It is important to know that in Flanders children are, from the second grade on, intensively confronted with symbolically presented multi-digit subtractions of the form a - b = ?, which they are supposed to solve mentally (before they start learning the written algorithms). Mathematics instruction in mental subtraction typically focuses on the routine mastery of DS, with little or no systematic attention to IA. So, most Flemish teachers do not systematically teach IA and only allow children to apply alternative solution strategies such as IA as long as they can also demonstrate perfect mastery of the school-taught DS strategy and as long as they do not disturb the teacher's regular whole-class teaching with their alternative (self-discovered) strategies.

In line with this instructional tradition, 195 Flemish children who had not received systematic instruction in IA participated in our first study on children's use of IA (Torbeyns, De Smedt, Ghesquière, & Verschaffel, 2009a). Seventy-one second-, 71 third-, and 53 fourth-graders individually completed two tasks: (a) a *Spontaneous Strategy Use Task* (SST), consisting of different problem types designed to assess the use of diverse shortcut strategies, incl. two-digit small-difference subtractions (41 - 39) which can be efficiently solved with IA; children were instructed to solve each item as accurately and as fast as possible with their preferred strategy, and to verbally report both the answer and the strategy used immediately after solving each item; (b) a *Variability on Demand Task* (VDT), also consisting of various problem types, incl. small-difference subtractions; children had to solve each item with at least two different strategies and verbally report each strategy; the experimenter kept asking for another possible strategy until the child had either reported IA, stated that (s)he did not know any other strategy, or reported five other alternative solution methods.

This study had two main findings. First, the analysis of children's strategy repertoire in the SST revealed that less than 10% of the second- and third-graders and only 15% of the fourth-graders spontaneously applied IA at least once to answer the small-difference subtractions. Thus, children hardly used IA, even on items where this strategy can be considered to be extremely efficient. Second, all children reported various strategies for solving the small-difference items from the VDT, but only a minority of them reported IA as an alternative strategy, suggesting that IA was no part of the strategy repertoire of most children. In sum, these results indicate that elementary school children who did not receive systematic instruction in IA do not apply this strategy (even not small-difference subtractions) and are unable to generate IA as an alternative for their standard (DS) strategy.

To test the generalizability of these findings to children from other instructional backgrounds, we set up a new study wherein we compared the strategy performance of children from two Flemish schools that did not provide instruction in IA (= DS-oriented schools) with children from a third school in which IA did receive special instructional attention (= IA-oriented school) (Torbeyns, De Smedt, Ghesquière, & Verschaffel, 2009b). Children from the IA-oriented school were instructed to use IA when the difference between the two given numbers was small (i.e., a difference smaller than 10). The textbook also introduced a specific notation for the IA strategy: a little arrow or arc from the subtrahend to the minuend. In total, 54 second-, 54 third- and 49-fourth graders participated in this study. The number of children from the IA-oriented school and the two DS-oriented schools was, respectively, 53 and 104. All children completed a paper-and-pencil test with 16 two-digit subtractions. Half the items had a difference smaller than 10 (81 - 79), while the other half had a difference between 10 and 20 (72 - 58). Children were instructed to solve the subtractions in whatever way they wanted and to write down their solution strategy in the scrap paper area below each problem.

The major result of this study was surprising and, from an instructional perspective, quite disappointing. While children from the IA-favoring school used IA slightly more frequently than children from the two other schools, the frequency of IA was generally extremely low in all schools: 7.53% for the IA-oriented school and 0.19% for the DS-oriented schools. Because we could not exclude that the unexpectedly low number of IA strategies in this study was due to the technique being used to identify the children's solution strategies, namely a paper-and-pencil test, we set up a follow-up study with the same children and the same item set, but this time strategy performance was assessed during an individual interview. The overall frequency of reported IA increased only marginally - from 0.19% in the initial study to 2.43% in the follow-up study - implying that the type of data-gathering method used was clearly not a major cause of the remarkably low frequency of IA observed in the initial study.

In retrospect, the unexpectedly low frequency of IA strategies in the IA-oriented school was probably due to the weak instruction in IA as provided by the textbook and as implemented by the teachers, both from a quantitative and a qualitative perspective. Therefore, we conducted a third study with children, in which we tried to accelerate the emergence and further development of IA, using the microgenetic method (De Smedt, Torbeyns, Stassens, Ghesquière, & Verschaffel, 2010). The sample consisted of 35 third-graders who did not receive any previous instruction in IA and who did not apply IA on any subtraction during the initial test session. These 35 children were divided into the two groups on the basis of their general mathematical achievement level, resulting in two groups of equal mathematical ability: 20 children participated in the *strong instruction* (SI) group and 15 children in the *weak instruction* (WI) group. All children were individually administered three test sessions, four practice sessions, one transfer session, and one retention session. The test sessions, practice sessions, and retention session each consisted of a series of symbolic subtractions in the number domain 20-100. In the transfer session, children were offered two tasks: a symbolic subtraction task in the number domain up to 1000 and a subtractive word problem task in the number domain 20-100. In each session, three item types were included: items with a small, medium, or large difference between the minuend and the subtrahend.

The children from the SI group solved the same series of items as the children from the WI group. All sessions, except the practice sessions, were exactly the same for both experimental groups. In the three test sessions, the transfer session, and the retention sessions, all children were asked to mentally solve all items with their preferred strategy. In the four practice sessions, the SI group was explicitly instructed to mentally solve each item once with DS and once with IA, while children of the WI group mentally solved each item twice with their preferred strategy without any further instruction. In the SI group, the IA strategy was also briefly demonstrated at the beginning of each practice session, and, if necessary, support with the execution of the IA strategy was provided during the practice session. In the WI group, the only extra instruction during practice sessions was an unusually large number of subtractions with a very small difference between the integers, compared to children's regular instructional practice at school, which typically contains little or no such problems.

In each session, children were asked to mentally solve each item as good and as fast as possible. Accuracy and speed of responding were registered per child and per item; children had to verbally report their strategy during (practice session) or immediately after (test, transfer, and retention sessions) solving each item. The exact sequence of the different sessions was: test 1, practice 1, practice 2, test 2, practice 3, practice 4, test 3, transfer, retention. Test, practice and transfer sessions were separated at least two days in time for each child. One month after the transfer session, children were offered the materials from the retention session.

The major results of the microgenetic study can be summarized as follows. First, as far as strategy frequency is concerned, IA was, quite surprisingly, not used on a single trial by any child from the WI group during any session. But also in the SI group, IA was used rather infrequently during the second and the third test session. Second, as far as the efficiency of IA in the SI group is concerned, we compared the accuracy and speed of this newly learnt and quite rarely used IA strategy with the accuracy of the familiar DS strategy. It turned out that as soon as children from the SI group started to apply the new IA strategy, they immediately did so more accurately and more quickly than the DS strategy, although only the greater accuracy in favor of IA reached significance.

Conclusion and discussion

Our research program on IA strategy use in children and adults has yielded quite an interesting contrast, which demands further research and reflection. Whereas young adults use IA frequently, efficiently, and adaptively to solve symbolically presented multi-digit subtractions, IA is almost completely absent in the strategy repertoire of 6- to 9-year-olds. Even when children were confronted with problems for which the computational advantage seems overwhelming or with an explicit invitation to demonstrate strategy variety, even when they reportedly got math education using a book that pays systematic attention to IA, even when they participated in an experiment wherein they actually got instruction and practice in IA, the number of IA strategies remained remarkably low. At the same time, children who (begin to) use IA immediately seem to demonstrate relatively high levels of accuracy and speed, compared to the efficiency of the systematically taught and intensively practiced DS strategy. Therefore, more research is needed to unravel why so many elementary school children stick so strongly and stubbornly to the DS strategy and move so slowly and reluctantly in the direction of IA strategy use. In our view, this is a result of a mixture of factors, educational as well as cognitive-psychological ones.

First of all, there are the math *educational* factors. One could argue that IA will only show up in children when this strategy has received intensive and high-quality instructional attention. Although the children from the strong instruction group in the last (microgenetic) study did receive intensive instruction in IA, it presumably was not of a high quality, given that it was completely individual, purely procedurally oriented, and not building on children's prior knowledge (their physical experiences and social interactions that lie at the roots of inversion; their knowledge of addition-based strategies for solving subtraction word problems of the missing addend type, etc.). Second, at a more general level, the children who participated in our studies all had received math education in a broader math education culture and practice that can be characterized as aiming at *routine* rather than at *adaptive expertise* (Baroody & Dowker, 2003). More particularly, they all had been immersed in a classroom practice and culture that values routine mastery of one single (taught) strategy rather than flexible use of various (self-invented) strategies. Aiming for such adaptive expertise would require a classroom climate and culture that systematically, from a very young age on, teaches for strategy variety and flexibility.

Besides these educational explanations for why elementary school children move so slowly and reluctantly in the direction of IA strategy use, there are also some explanatory factors that are of a more cognitive-psychological nature. First, there is the conceptual knowledge of the mathematical principle that underlies the meaningful use of IA, i.e., the *inverse principle*, which may be particularly difficult for children of that age, who are still in the transition from the pre-operational to the concrete-operational stage of their cognitive development. However, a proper test of this hypothesis would require a test of children's understanding of the underlying inverse principle, independent of their procedural knowledge of IA. A second possible cognitive factor relates to children's limited metacognitive or self-regulatory capacities, which may make it very difficult for them to suppress or inhibit certain tendencies, such as the tendency to execute the (direct) *subtraction* operation when confronted with a problem that contains the *minus* sign. These two intrinsically (meta)cognitive factors may explain why IA apparently originates and develops so slowly and laboriously in the vast majority of children of that age group, whereas other shortcut strategies for doing addition and subtraction, such as disregarding addend order when doing addition, seem to develop much earlier and easier.

From the above list of explanations, it becomes clear that there is probably no single explanation for the absence of IA in many children's repertoire of strategies for doing symbolic subtraction. Most probably, the phenomenon is the result of the complex interaction of various factors, psychological and educational.

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