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On the possibility of teaching elements of Lobachevski Geometry at School

Today the secondary school in Russian Federation (grades 10 and 11) passes to the profile teaching. There are several profiles, including physical and mathematical ones, assigned to the profile teaching. The profile preparation assumes the intense study of the appropriate subjects. Therefore, it increases the possibility of intensive studies of mathematics, especially its parts connected with school program, but exceeding the limits of school curriculum. Special role in the organization of teaching in profile classes belongs to so-called elective courses, or courses that can be selected by pupils. They provide a possibility to perform lessons on sections of mathematics that are interesting to pupils, to expand their views of the subject and of methods of research of modern mathematical knowledge.

The interest in teaching elements of non-Euclidean hyperbolic (Lobachevski) geometry at school has heavily grown recently. Pupils of classes of mathematical profile show increased interest in studying qualities of figures on the Lobachevski plane. The main reason for this is the fact that studying the foundations of Lobachevski plane renders essential influence on understanding of the role and place of the parallelity axiom in the school course of elementary geometry that in turn results in more complete mastering of the school program.

The experience of teaching the elements of Lobachevski geometry to school pupils allows formulating the following purposes and problems of such teaching. The purpose of these lessons is to increase the general and mathematical cultural level of pupils and to stimulate and strengthen their interest to mathematics. The problem consists of mastering the material of the elementary concepts of Lobachevski plane geometry by the pupils and studying the importance of the parallelity axiom in the logical substantiation of the school course of elementary geometry. As a rule, the lessons should be carried out for those pupils who have finished studying plane geometry and have begun studies of space geometry.

The material of the course is designed for 8 lessons. As it is accepted at the Russian school, each lesson is 45 minutes long. The subject matter can be divided into four parts. The first part is introductory and deals with the repetition of the school material. It is necessary to indicate distinctly those statements of elementary geometry, proof of which does not depend of the parallelity axiom: these are statements of so-called absolute geometry. Furthermore, it is necessary to indicate the facts based on the properties of parallel straight lines. For the further study, it is necessary to prove three statements of absolute geometry: the first theorem on the external angle of a triangle (the external angle of a triangle is greater than any of internal angles that are not adjacent with it) and also the first and the second theorems of Legendre:

1) The sum of angles of a triangle is not greater than the sum of two right angles.

2) If the sum of angles of some triangle on a plane is equal to the sum of two right angles then the sum of angles of any other triangle is also equal to the straight angle.

Thus, it is very interesting to hear the answer of the pupils to the question "Why parallel straight lines do not cross on a plane? The experience shows, that in 90% of cases the pupils are at a loss to answer this question. The funniest answer is: "In the textbook, the axiom of parallelity is given, therefore, they exist".

The second part of the course consists of the facts from the history of attempts of the proof of the fifth postulate of Euclid. In the beginning of this part it is necessary to explain essence of a problem of the fifth postulate. It is necessary to tell about Euclid, his famous book "Elements of geometry". It is appropriate to explain to the pupils some concepts of the axiomatic method, to mention the basic definitions, axioms and postulates of Euclid and also formulation of the famous fifth postulate. Here it is necessary to explain the essence of the problem and reasons which have encouraged mathematicians since almost two thousand years to try to check the validity of this statement. It is well-known that all attempts to prove this postulate were logically wrong, as they implicitly used other statements actually equivalent to this postulate. We can mention the following statements: by Poseidonius: "On a plane, there exist at least three collinear points equidistant from a straight line"; by Farkas Bolyai: "On a plane, it is possible to draw a circle around any triangle"; by John Wallis: "On a plane there are two similar but not equal triangles"; and by Legendre: "A perpendicular line erected in any point of one of the sides of an acute angle crosses the second side of the angle". In conclusion of this part, it is necessary to tell about mathematicians who have discovered non-Euclidean geometry (Gauss, Lobachevski and Janos Bolyai).

The history of the discovery of Lobachevski geometry has exclusive methodological importance. Generally, the history of the discoveries and development of scientific theories, the practical needs that have brought to their formation, and the historical period of development of a civilization, during which the discoveries were made, are extremely important for the understanding of scientific ideas and ways of their development. In connection with this, the history of the discovery of non-Euclidean geometry has some peculiarities. While the discovery and study of properties of geometrical figures, of concepts and statements of algebra and trigonometry followed the needs of practical activity of people, the research of the problem of the fifth Euclid’s postulate had no practical purpose. It has arisen only as a logical problem; its research has resulted in the discovery of non-Euclidean geometry, in the substantiation of the axiomatic method in mathematical research.

In the third part the proofs of the statements formulated above should be carried out. First, the equivalence of the Playfair's axiom (that is present in all Russian Geometry textbooks as the parallelity axiom) to the fifth postulate of Euclid is proved. Then, we prove the equivalence of the axiom of parallelity to the following statement: “the sum of angles of any triangle on a plane is equal to the sum of two right angles”. It follows from the second theorem of Legendre that for the fulfillment of the axiom of parallelity, it is enough to prove that on a plane there exists, at least, one triangle with the sum of angles equal to the sum of two right angles. This statement is used for the proof of the equivalence between the statements of Poseidonius, Wallis and Legendre, on the one hand, and the axiom of parallelity, on the other hand. It is necessary to draw special attention to the statement of F.Bolyai. The experience shows that many teachers of Russian schools do not understand where in the proof of the statement “It is possible to describe a circle around any triangle” the axiom of parallelity is used. Therefore, the study of the proof of this statement is extremely useful for pupils, because it allows understanding the essence of the application of this axiom to the substantiation of the statements of elementary geometry.

The fourth part of the course is devoted to the facts of Lobachevski plane geometry and their interpretation on the Cayley - Klein model. We do not demonstrate strict proofs of properties of figures on the Lobachevski plane. The understanding of the main ideas of the proof suffices. The axiom of parallelity of straight lines on the Lobachevski plane is formulated as the logical negation of Playfair’s axiom. The Lobachevski’s concept of parallel straight lines with a common point is introduced. As the statements of Farkas Bolyai, Poseidonius and John Wallis are equivalent to Playfair’s axiom, on the Lobachevski plane there exist triangles, around which it is impossible to describe a circle. Furthermore, points, equidistant from the given straight line, do not belong to one straight line, and there is no similarity, i.e. one more sign of the equality of triangles is true: “The triangles are equal, if their corresponding angles are equal". From the first and second theorems of Legendre, and also from the equivalence of Playfair's axiom to the statement "The sum of angles of a triangle is equal to the sum of two right angles" follows, that on the Lobachevski plane the sum of angles of any triangle is strictly less than the sum of two right angles. Using the statement of Legendre, it is not difficult to explain major property of acute angles and segments on the Lobachevski plane, namely 1-1 correspondence between segments and their parallelity angles. From this fact easily follows that for each angle there is a straight line entirely laying inside the angle and parallel to its sides.

All these statements are extremely interesting to the pupils, cause many questions and stimulate them to the active understanding of the subject matter. In the last part of the course, it is useful to consider the Cayley - Klein model on the Euclidean plane. Usually, this model is constructed on a projective plane as the set of internal points of the real oval quadric. It can be also constructed on a Euclidean plane as a set of internal points of a circle. Under a straight line its part inside the circle is understood. Using such model, it is easy to interpret properties of straight lines on the Lobachevski plane: the Lobachevski's axiom of parallelity, the existence of parallel straight lines and of the straight line parallel to the sides of the angle. Properties of perpendicular straight lines are more difficultly interpreted. One can tell the pupils (without the proof) the following property of perpendicular straight lines on the Cayley - Klein model: Two straight lines are perpendicular, if each straight line on a Euclidean plane, containing on of them, passes through the pole of other straight line. Using this fact, it is easy to explain the statement of Legendre for the perpendiculars to the sides of an acute angle. Also it is possible to construct a common perpendicular to hyperparallels and to explain why it is unique.

The experience of the implementation of the course on elements of Lobachevski geometry with the pupils of mathematical classes has shown its efficiency. The pupils show the increased interest and, most important, the mathematics is no more a boring discipline for them, but rather a subject rich by ideas, beautiful theories and interesting reasonings.