Ildar SAFUANOV, Moskau

Symmetry and elements of Galois Theory at school

In Soviet Union, since 30-es of 20-th century, schools with mathematical bias and also mathematical circles at leading universities have been established. The pupils of studying at these schools and circles usually solved a lot of difficult problems and participated in mathematical Olympiads. In addition, they attended lectures of university professors and working mathematicians on various topics of modern and classic advanced mathematics. For example, since 30-es such prominent mathematicians as academicians, professors of Moscow university A.Kolmogorov, P.Aleksandrov, S.Sobolev, S.Yanovskaya, B.Delaunay, I.Gelfand and others gave lectures on difficult themes of abstract algebra, number theory, topology, mechanics etc.

For example, here is the list of some of these lectures:

Aleksandrov P. Transfinite numbers.

Kolomogorov A. Modular arithmetic and its applications in technology.

Kolomogorov A. Fundamental theorem of algebra.

Gelfand I. Fundamental concepts of the set theory.

Hinchin A. Continuous fractions.

Gelfond A. Prime numbers.

Kurosh A. What is algebra?

Pontryagin L. What is topology?

Shafarevich. I. Solving algebraic equations by radicals (i.e. using roots only).

Shnirelman L. Group theory and its application in the solving of the equations of 3-d degree.

Boltyansky V. Continuous fractions and musical gamma.

Yefremovich V. Non-Euclidean geometry.

These lectures had a great impact on the audience, and some of pupils that attended the lectures later became prominent mathematicians themselves (e.g.V.Arnold and A.Kirillov).

Today, the process of the differentiation of schools and higher education takes place. Many of schools are converted into gymnasiums, lyceums, vocational schools etc. Various supplements to programs, special and optional courses are included in curricula of schools.

The profile preparation of the pupils at school assumes their profound specialization in the senior grades on the appropriate scientific direction, in particular in the field of physical and mathematical education

In the traditions of Soviet mathematics education, it is useful to acquaint school pupils, especially gifted ones, with interesting and important topics of modern mathematics.

As an example of such topic one can choose symmetry and elements of Galois Theory. However, Galois Theory is extremely difficult even for university students and school mathematics teachers. Therefore, it is necessary to thoroughly select, adapt and elaborate concepts and results to be taught to school pupils. We propose our system of teaching this topic at school.

We think that it is possible to explain the elementary facts on symmetry and Galois Theory within a short course consisting of several, usually 4, lectures (one hour long each).

We begin with the definition of a symmetry of a geometrical figure as a rigid motion (i.e. 1-1 transformation preserving distances) that transforms a figure onto itself. For example, an isosceles triangle not being equilateral has only two symmetries: the identity transformation (leaving every point unchanged) and an axial symmetry - reflection around the straight line bisecting the angle between equal sides of the triangle. An equilateral triangle has six symmetries: except for identical transformation e, there are rotations by 120^0 and 240^0 (we will designate them a and b), and also three axial symmetries c, d and f around bisectors of the angles. A rectangle (not square one) has, in turn, four symmetries (identity, two axial symmetries and the rotation by 180^0.

Similarly, it is possible to define a symmetry of a geometrical body in the space as a rigid motion of space translating body onto itself. It appears, for example, that a regular tetrahedron has 24 symmetries. Note that the more symmetric a figure or a body looks, the more symmetries it has. For example, a circle and a sphere have infinitely many symmetries.

Furthermore, we introduce the notion of the composition of symmetries as their consecutive performance. Moreover, a transformation that is inverse to a symmetry is also a symmetry.

Now it is possible to introduce the concept of a group. It is important to show that number systems w.r. to addition and sets of non-zero rational or real numbers w.r. to multiplication constitute groups. Further, rising to the higher abstraction level, we introduce permutations and groups of permutations, i.e. symmetric groups (e.g. S_3).

Thus, in the second lecture we acquaint pupils with elementary notions of theories of groups, rings and fields: Abelian groups, Cayley tables of finite groups, subgroups, normal subgroups, polycyclic groups, solvable groups (without introducing the notion of a quotient group).

In the third lecture, we begin the study of the solvability of algebraic equations of higher degrees, namely the problem of finding all roots of the polynomial

$$f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots a_{n-1} x + a_n .$$

First, we tell the pupils about attempts of finding the universal method of solving algebraic equations of any degree, acquaint them with formulas of Cardano and Ferrari for solving equations of 3-rd and 4-th degrees, with the theorem of K.F.Gauss on the existence of a complex root for a polynomial in one variable.

However, for equations of the 5-th degree it appeared to be impossible to find a formula for expressing solutions by radicals (i.e. using four arithmetic operations and computing roots of numbers, beginning with the coefficients of the polynomial).

It is important to include in the course elements of the history of the subject. In the research of the reason of the absence of a formula for solving `equations of 5-th degree such famous scientists as Josef-Louis Lagrange (1736-1813), Paolo Ruffini (1765-1822) and Niels Henrik Abel (who proved that general equations of the degree 5 and above are unsolvable by radicals) were engaged, and finally the problem was solved by Evariste Galois (1809-1830), who discovered and proved the necessary and sufficient conditions for solvability of the equations of any degree by radicals.

J.L.Lagrange was the first to notice that the solvability of the equation of 2-nd, the 3-rd 0r 4-th degree with rational coefficients is connected to the search of such polynomial with rational coefficients in several variables which preserves its value on the roots of a given equation accept under any permutation of these roots.

For example, polynomials in the left part of the Vieta formulas

$$x_1+x_2=-p,$$

$$x_1x_2=q$$

for the equation

$$x^2 + px + q$$

do not change their values under any permutation of two roots. However, not always such polynomials preserve their values under any permutation of roots. Consider, for example, the equation

$$x^3 - x = 0$$

with roots $x_1 = 0, x_2 = 1$ and $x_3 = -1$. The polynomial

$$\varphi(x_1, x_2, x_3) = x_1 - x_2 - x_3,$$

apparently, preserves its value 0 only under two permutations: identity permutation and transposition of x_1 and x_2.

It is possible to show, that for any equation of n-th degree (that, as we have noticed above, has n roots, some of which may be equal), the set of permutations preserving values of any polynomial that takes rational values on roots, is a subgroup in the group S_n of all permutations of n roots.

Thus, in the last lecture, we introduce the definition of a Galois group of the equation of n-th degree and formulate Galois criterion:

The equation of n-th degree is solvable by radicals if and only if its Galois group is solvable.

We provide the example of the equation with unsolvable Galois group (therefore, this equation is unsolvable by radicals):

$$x^5 - 13x + 13 = 0.$$

Of course, actually Galois theory is much deeper and more complicated. However, the approach described here allows pupils to get some certain views of the history and modern state of the problem of solving equations of any degree and of Galois' approach using symmetries and groups.

Literatur

Artin, E. (1998): Galois Theory. Dover Publications.

Postnikov, M. M. (2004): Foundations of Galois Theory. Dover Publications.