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On mathematical problems with elements of the game of chess

Introduction. Classes of mathematical circles must be built creatively to encourage the interest and imagination of students, to introduce them to the variety of problem-solving techniques, and to unleash children's imagination. The classes may focus on unusual, exciting problems, whose themes must not be limited by the curriculum. Compiling a selection of problems is the core question not only for teachers of mathematical circles, but also for mathematics teachers of compulsory education and specialists who organize mathematical Olympiads. Mathematical chess problems, as part of the field of recreational mathematics, stay in the centre of interests of mathematicians and puzzlers for many years. Colorful ideas from this field can be implemented by composition of various mathematical tasks.

Mathematical chess problems. The most popular topics here are placement of chess pieces, piece tours and permutations. The placement problems consider the independent dispositions of chess peaces or so-called domination problems. One of the famous examples is the eight Queens problem pronounced by Max Bezzel in 1848 (Gardner, 1968). Thorold Gosset proved the additional property of 12 basic eight Queens dispositions in 1914: it is not possible to combine all these dispositions for every square of board to be occupied by a Queen. The most popular problem is the Knight's tour puzzle. Only by computer calculations was it possible to find the number of all closed Knight's tours on the traditional chess board. The number of undirected tours is 13 267 364 410 532 (McKay, 1997). William Beverley found the semi-magic knight tournament in 1848, and only some years ago was the non-existence of the magic tour proved (Weisstein, 2003). Mathematicians are interested in relations of Knight's tour with regular graphs, prime numbers, Pick's theorem, calculations of areas, coloring problems. Form the huge collection of mathematical chess problems and open questions it is preferable to adapt ideas for composing "paper and pencil" problems according to the above-mentioned reasons.

Problem creation strategies. There are two eventual directions in the composition of mathematical problems: transformation and generation. To the first, more constructive direction relates simplification, reformulation and translation. Simplification is useful for formulating the introductory problem of some new topic, taking a special case of a more general problem. Reformulation allows to change the context of the problem and to consider the given conditions from different aspects. For demonstration of problem-solving strategies, some reasonable mathematical results could be "translated" in terms of elementary mathematics. Such problem posing techniques as “What if not...” are suitable for students to master the skills of analyzing, interpreting, reasoning, hypothesizing, questioning and experimenting:

„...The user lists each attribute of a situation, whether a theorem, a piece of equipment or a method of presentations, and negates one attribute at a time to generate a new mathematical situation to be explored.” (Small, 1993)

The problem-generation process is more creative. Only by research work and problem solution, experimentation, and generalization do a number of questions arise that specify additional fields of investigation. New mathematical problems can be re-formulated from the achieved results as a co-product of research activities.

Components of the problem. A combinatory problem with chess elements has three main components:

- Domain: game board or field and their properties;
- Objects: static or active chess pieces (or other objects) with their properties;
- Interactions: mutual interaction between objects or between an object and the board.

The disposition described poses a conflict situation or rouses a sequence of questions. On this ground a mathematical problem can be formulated.

The puzzlers are interested in many additional problem concepts that are derived from the rules of traditional chess. They use not only rectangular boards of arbitrary size but also domains of different type, e.g. orthogonal lattice, plane triangulation, hexagonal plane, three-dimensional or multi-dimensional action fields. For example, the existence problem of Knight’s tour on a rectangular board of arbitrary size is solved. Many complementary modified pieces stay in the range of interests, such as simple leapers Camel, Antelope, Zebra, or combined pieces, e.g. Amazon that is a combination of Knight and Queen. Different initial positions, given conditions, mutual interactions between objects and their properties initiate various mathematical tasks: calculation of the number of combinations, determination of the winning strategy, detection of extreme elements, proving problems of existence.

Construction of problems. Changing and combining the three above mentioned components offer unlimited opportunities for problem-posing. This possibility is useful for every teacher who wants to improve his or her classes. The teacher can find or construct problems of combinatory theory, graph theory, geometry, and vector algebra. It is possible to formulate problems by supplementing chess pieces with additional properties, for example:

Problem 1. All the squares of the chess board are colored green. Knight can erase the color from a rectangle sized 3x2 with his move. What is the minimum number of the Knight's moves to erase all squares on the board?

Different tasks can be constructed and general solutions can be obtained by placement of chess pieces on an unbounded board. Problems can be constructed using the Cartesian plane of integers:

Problem 2. Write the equation of a straight line that has the same direction as a particular Knight's move.

Problem 3. Create an algorithm for a symmetric Bishop's path with respect to the origin of plane.

Problem 4. Calculate the shortest Knight's path between any two given points and calculate the number of such paths.

Another way is the construction of such mathematical problems that use some well-known reasoning methods to find the solution. Such option is useful for students to master the method of invariants, the method of the extreme element, mathematical induction, the pigeonhole principle or other methods:

Problem 5. *Invariant method*. A hexagonal game board is tiled into 37 regular unit hexagons. The figure is a connection of two or more unit hexagons. Find such minimal figure that is not possible to completely cover the board (except the central hexagon) without overlapping these figures.

Problem 6. *Composition of circles.* All the Knight's moves on the hexagonal game board are defined. Construct a set of symmetrical circles so that they can be arranged in the Knight's path on the given game board.

Problem 7. *Pigeonhole principle*. Game board is an equilateral triangle that consists of 64 unit triangles. The pawn can move from one triangle to the next triangle by passing the common vertex in the move if non-adjacent edges of triangles are parallel. Calculate the maximum number of pawns that can be placed on the board so that no two of them attack each other.

Idea of solution. There are 4 independent sub-structures on the board according to the given conditions. If two pawns are placed on triangles of different structures they can't attack each other. There are 3 mutually symmetrical structures and one other:

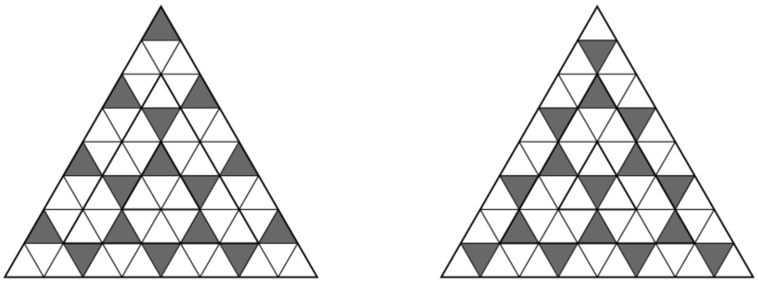

Figure 1. Different structures of pawns' moves.

The classification of triangles into separate structures and evaluation by the pigeonhole principle lead to the optimum number of pawns that can be placed on the board.

Significant tasks of education today are the representation of real-life problems and applications of mathematics. The interpretation of numerical, algebraic or other theoretical results are included in the lists of skills that students have to acquire. "Write a story about the given numerical expression!" is an initial challenge for students of younger grades. The challenge for older students as well as teachers could be the creation of a mathematical problem based on a story or a fairytale, for example:

"Once upon a time there lived one very suspicious Queen…"

Problem 8. The Queen is placed on a chess board. What is the minimal number of Knights so that every square could be attacked by a Knight in one or two moves, except the squares attacked by the Queen and except the squares occupied by pieces?

Conclusions. Books, journals, and internet resources offer a tremendous amount of manifold mathematical chess problems that can inspire every teacher or puzzler. Considering that there is no definite universal recipe for mathematical problem creation, the main way to construct mathematical problems is the research of various structures with various properties and various conditions using fantasy, imagination, inspiration, curiosity, experience, knowledge, courage, persistence, patience, and insight.

Literature

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