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The low frequency spectrum of small Helmholtz resonators

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Abstract

We analyze the spectrum of the Laplace operator in a complex geometry, representing a small Helmholtz resonator. The domain is obtained from a bounded set $\Omega \subset \mathbb{R}^n$ by removing a small obstacle $\Sigma_\varepsilon \subset \Omega$ of size $\varepsilon > 0$. The set Σ_ε essentially separates an interior domain $\Omega_\varepsilon^{\mathrm{inn}}$ (the resonator volume) from an exterior domain $\Omega_\varepsilon^{\mathrm{out}}$, but the two domains are connected by a thin channel. For an appropriate choice of the geometry we identify the spectrum of the Laplace operator: It coincides with the spectrum of the Laplace operator on Ω, but contains an additional eigenvalue μ_ε^{-1}. We prove that this eigenvalue has the behavior $\mu_\varepsilon \approx V_\varepsilon L_\varepsilon / A_\varepsilon$, where V_ε is the volume of the resonator, L_ε is the length of the channel, and A_ε is the (area of the) cross-section of the channel. This justifies the well-known frequency formula $\omega_{\mathrm{HR}} = c_0 \sqrt{A/(LV)}$ for Helmholtz resonators, where c_0 is the speed of sound.

Keywords: Helmholtz resonator, spectral properties of the Laplace operator, complex geometry, sound attenuator

MSC: 35B34, 35P20, 47A40

1 Introduction

The Helmholtz resonator is an acoustic device which is important e.g. in the construction of sound attenuators. It consists of a resonator volume (the cavity), which is connected by a thin channel to the outer domain. The Helmholtz resonator has a resonance frequency ω_{HR} that depends only on elementary geometrical quantities of the resonator: the resonator volume V_ε, the channel length L_ε, and the channel cross-section A_ε. The well-known approximate formula for the resonance frequency states $\omega_{\mathrm{HR}} \approx c_0 \sqrt{A_\varepsilon/(L_\varepsilon V_\varepsilon)}$, where c_0 is the speed of sound. The formula indicates that it is possible to construct resonant devices that are much smaller than the wave length of the corresponding acoustic wave. Loosely speaking, a small volume V_ε corresponds to a large frequency, but this can be compensated by making the channel very thin.

In this contribution, we present a mathematical analysis of the spectral properties of small Helmholtz resonators, which are constructed as follows: Let $\Omega \subset \mathbb{R}^n$

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be an underlying bounded domain and let $\Sigma_\varepsilon \subset \Omega$ be a sequence of obstacles with diameter $\mathrm{diam}(\Sigma_\varepsilon) = O(\varepsilon)$ with $0 < \varepsilon \to 0$. We are interested in the spectral properties of the domain $\Omega_\varepsilon := \Omega \setminus \Sigma_\varepsilon$, which essentially consists of an outer domain $\Omega_\varepsilon^{\mathrm{out}}$ and an inner domain $\omega_\varepsilon = \Omega_\varepsilon^{\mathrm{inn}}$, separated by Σ_ε. Outer and inner domain are connected by a thin channel Γ_ε such that the domain of interest is $\Omega_\varepsilon = \Omega \setminus \Sigma_\varepsilon = \omega_\varepsilon \cup \Gamma_\varepsilon \cup \Omega_\varepsilon^{\mathrm{out}}$, where the latter is a disjoint union (compare Fig. 1).
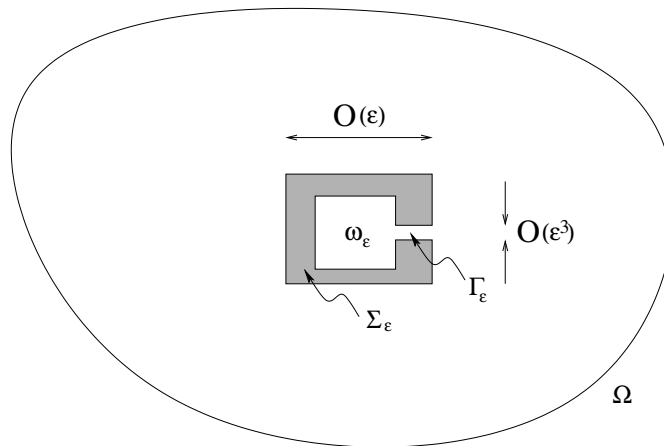

Figure 1: The geometry of the Helmholtz resonator (two-dimensional case). The resonator volume ω_ε has a diameter of order ε and a volume of order $V_\varepsilon \sim \varepsilon^n$. The channel Γ_ε has a length $L_\varepsilon \sim \varepsilon$ and a opening diameter of order ε^3 in dimension $n = 2$ and of order ε^2 in dimension $n = 3$.

Our main result is the characterization of the spectrum of the Laplace operator on the domain Ω_ε (with homogenous Neumann boundary conditions on $\partial \Sigma_\varepsilon$). Essentially, the spectrum of the Laplace operator on Ω_ε coincides with the spectrum on Ω, but there is an additional eigenvalue μ_ε^{-1}, which corresponds to the characteristic Helmholtz resonator frequency. We prove that this additional eigenvalue occurs, that no further (bounded) eigenvalues occur, and that the resonant frequency is characterized by the well-known formula $\mu_\varepsilon \approx L_\varepsilon V_\varepsilon / A_\varepsilon$, where L_ε is the length of Γ_ε, V_ε is the volume of $\Omega_\varepsilon^{\mathrm{inn}}$, and A_ε is the ($n-1$-dimensional) area of the cross-section of Γ_ε. A by-product of our analysis is a characterization of the corresponding eigenfunction: It is essentially constant in the cavity $\Omega_\varepsilon^{\mathrm{inn}}$ and it essentially vanishes in the outer domain $\Omega_\varepsilon^{\mathrm{out}}$.

To our knowledge, our result provides the first analytical derivation of the resonator frequency formula. We choose a setting of the problem that simplifies the analysis considerably — without being in conflict with physical experiments. We assume that the underlying domain $\Omega \subset \mathbb{R}^n$ is bounded; this implies that the spectrum of the Laplace operator consists of eigenvalues, we do neither have to deal with a continuous spectrum nor with the outgoing wave condition at $|x| = \infty$. Furthermore, we assume that the domain Ω is sufficiently small in order to have the smallest eigenvalue of $-\Delta$ on Ω larger than the resonator frequency. This second assumption again simplifies the analysis, since the two parts of the spectrum are separated. It is also physically legitimate: In an experiment with a Helmholtz resonator of diameter $1\,\mathrm{cm}$ and a resonant frequency of $300\,\mathrm{Hz}$ (such that the wave-length is of the order $1\,\mathrm{m}$), it is sufficient to restrict experiments to a domain Ω of less than $50\,\mathrm{cm}$ in diameter.

1.1 Literature

One of the first analytical approaches to the problem was presented in 1973 by Beale in the influential article [2]. The choice of the geometry is similar to ours: a cavity is (essentially) separated from an outer region, and there is only a thin channel that connects the two domains. Methods and results of [2] are very different from ours, mainly because [2] treats the more intricate problem of an unbounded exterior domain. Furthermore, the resonator keeps a finite size in [2], and only the thickness of the channel converges to zero, $A_\varepsilon \to 0$. For this reason, the results are quite different: The value $L_\varepsilon V_\varepsilon / A_\varepsilon$ tends to infinity and the Helmholtz resonator frequency cannot be identified in [2]. Instead, classical resonances of $\Omega_\varepsilon^{\mathrm{inn}}$ (eigenvalues of the Laplace operator in the cavity) are identified as contributions to the spectrum of the Laplace operator in the complex geometry.

In the spirit of Beale's contribution, much literature to the problem is available. Concerning the background in scattering theory we mention the older works by Lax and Phillips [22, 23] and refer to [9] for more modern approaches and further references. From different perspectives, the problem of [2] was also investigated in [11] and [7, 15]. It was furthermore treated with asymptotic expansions in [12, 13]. For the corresponding problem in an elasticity system see [10]. We mention [1] for a modern discussion of spectra in the presence of small obstacles; due to the geometrically simple perturbation, no additional eigenvalues appear.

Concerning a discussion of the Helmholtz resonator in more physical terms and a description of applications we mention [19, 24], connections with fluid mechanics are made in [25, 29]. Some more recent treatments concern the numerical analysis, see [8, 17, 18]. We are not aware of a discussion of the problem in our scaling, i.e. the case of small resonators that are in resonance with the underlying wave.

Periodic structures in the acoustic setting are analyzed in [27, 28], where the acoustic properties of a large number of small inclusions are investigated (but not in the sense that the single structure is in resonance with the wave). Periodic structures are also studied for related flow equations, see [26, 29], and they appear in the analysis of electromagnetic waves (which are, in many applications, also described by a Helmholtz equation). In this context, one is interested in waveguides and trapped modes (see e.g. [30]) and in meta-materials with surprising effective properties, see [3, 4, 5, 6, 20, 21].

Methods related to this contribution. We use tools that are borrowed from homogenization theory. In spirit, a strong relation can be seen to L. Tartar's energy method of homogenization (even though we do not exploit compensated compactness here). We use Sobolev and Poincaré estimates in complex geometries containing small structures. The only spectral theorem that we use is the characterization of the spectrum of a compact self-adjoint operator.

Abstract spectral convergence theorems are derived in [16]; once more, the applications come from homogenization theory. We refer, in particular, to Theorem 11.4 of [16], where the spectrum of an operator $\mathcal{B}_0$ on a Hilbert space $\mathcal{H}_0$ is related to the spectra of a family of operators $\mathcal{B}_\varepsilon$ on Hilbert spaces $\mathcal{H}_\varepsilon$. The topology in which the operators $\mathcal{B}_\varepsilon$ must be close to $\mathcal{B}_0$ is expressed in terms of restriction operators $\mathcal{R}_\varepsilon : \mathcal{H}_0 \to \mathcal{H}_\varepsilon$.

It seems that most of the assumptions of Theorem 11.4 of [16] could be verified for $\mathcal{B}_\varepsilon = T_\varepsilon$ and $\mathcal{B}_0 = T_0$ for our operators T_ε and T_0 below (using an appropriately

constructed restriction operator $\mathcal{R}_\varepsilon$ and choosing appropriate scalar products in the Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_\varepsilon$). Unfortunately, in this abstract description, the operators are no longer self-adjoint, hence Theorem 11.4 of [16] cannot be used. Here, we will therefore perform explicit constructions to characterize the spectrum of T_ε.

1.2   Geometry

We start our construction from a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ with $0 \in \Omega$. Our aim is to analyze the spectrum of the Laplace operator in a domain $\Omega_\varepsilon = \Omega \setminus \Sigma_\varepsilon$. We will later use homogeneous Neumann conditions on $\partial \Sigma_\varepsilon \subset \partial \Omega_\varepsilon$ and homogeneous Dirichlet conditions on $\partial \Omega \subset \partial \Omega_\varepsilon$.

Our aim now is to describe the geometry in detail. Let $\omega_1 \subset \mathbb{R}^n$ be a bounded Lipschitz domain that represents the shape of the cavity. For notational convenience we make the assumption that ω_1 lies in the left hyperplane $\{x_1 < 0\}$ and that it has a flat part of the boundary: for some $\delta > 0$, we assume that the $(n-1)$-dimensional disc $\{0\} \times B_\delta^{n-1}(0) \subset \mathbb{R}^n$ is contained in the boundary, $\{0\} \times B_\delta^{n-1}(0) \subset \partial \omega_1$. The *cavity* of the small (ε-size) Helmholtz resonator is defined as $\Omega_\varepsilon^{\text{inn}} := \omega_\varepsilon := \varepsilon \omega_1$.

To describe the outer shape of the resonator, let $\Sigma_1^{\text{out}} \subset \mathbb{R}^n$ be the closure of a bounded Lipschitz domain with $\bar{\omega}_1 \subset (\Sigma_1^{\text{out}})^\circ$. We assume that, for some length parameter $L > 0$ and $\delta > 0$, the disc $\{L\} \times B_\delta^{n-1}(0)$ is contained in the boundary $\partial \Sigma_1^{\text{out}}$. The *outer shape* of the resonator is given by $\Sigma_\varepsilon^{\text{out}} := \varepsilon \Sigma_1^{\text{out}}$, and we write $\Omega_\varepsilon^{\text{out}} := \Omega \setminus \Sigma_\varepsilon^{\text{out}}$ for the outer domain.

We finally define the channel Γ_ε that connects the cavity $\Omega_\varepsilon^{\text{inn}} = \omega_\varepsilon$ with the outer domain $\Omega_\varepsilon^{\text{out}}$. For notational convenience we restrict ourselfs to straight channels with a uniform cross-section γ_ε, where $0 \in \gamma_\varepsilon \subset B_\delta^{n-1}(0) \subset \mathbb{R}^{n-1}$ is a family of bounded Lipschitz domains,

$$\Gamma_\varepsilon := [0, \varepsilon L] \times \gamma_\varepsilon , \qquad \text{diam}(\gamma_\varepsilon) = \begin{cases} O(\varepsilon^3) & \text{for } n = 2 , \\ O(\varepsilon^2) & \text{for } n = 3 , \end{cases} \tag{1.1}$$

where the typical diameters are given here only for illustration, they are not used as a mathematical assumption on Γ_ε. We assume that $\Gamma_\varepsilon \subset \Sigma_\varepsilon^{\text{out}}$ holds for every $\varepsilon > 0$ under consideration. Putting together the pieces, we define the (open) acoustic domain as $\Omega_\varepsilon := \omega_\varepsilon \cup \Gamma_\varepsilon \cup \Omega_\varepsilon^{\text{out}}$, and define accordingly the (closed) obstacle as $\Sigma_\varepsilon := \Sigma_\varepsilon^{\text{out}} \setminus (\omega_\varepsilon \cup \Gamma_\varepsilon)$. For later use, we denote the interfaces as $\gamma_\varepsilon^- := \bar{\omega}_\varepsilon \cap \bar{\Gamma}_\varepsilon = \{0\} \times \gamma_\varepsilon$ and $\gamma_\varepsilon^+ = \bar{\Omega}_\varepsilon^{\text{out}} \cap \bar{\Gamma}_\varepsilon = \{\varepsilon L\} \times \gamma_\varepsilon$.

The resonator frequency will not depend on the details of the geometry, but will only depend on three characteristic quantities: The (Lebesgue) measure V_ε of the resonator volume ω_ε, the channel length L_ε, and the area A_ε of the channel cross-section γ_ε. With numbers V, L, A we assume that

$$V_\varepsilon = |\omega_\varepsilon|_{\mathcal{L}^n} = V \varepsilon^n + o(\varepsilon^n) , \tag{1.2}$$

$$L_\varepsilon = \text{length}(\Gamma_\varepsilon) = L\varepsilon + o(\varepsilon) , \tag{1.3}$$

$$A_\varepsilon = |\gamma_\varepsilon|_{\mathcal{L}^{n-1}} = A \varepsilon^{n+1} + o(\varepsilon^{n+1}) = \begin{cases} A\varepsilon^3 + o(\varepsilon^3) & \text{for } n = 2 , \\ A\varepsilon^4 + o(\varepsilon^4) & \text{for } n = 3 . \end{cases} \tag{1.4}$$

We note that our choice of the geometry guarantees (1.2) with $V = |\omega_1|_{\mathcal{L}^n}$ and (1.3) with $L_\varepsilon = \varepsilon L$ (both without error terms). The only new assumption is (1.4),

which is in accordance with (1.1) (later on, we will only use (1.4)). The scaling of the channel cross-section is chosen such that

$$\frac{L_\varepsilon V_\varepsilon}{A_\varepsilon} \to \frac{L V}{A} =: \mu_0 \tag{1.5}$$

has a non-trivial limit as $\varepsilon \to 0$. Our main result shows that μ_0 occurs as a limit of eigenvalues in the ε-problems.

1.3 Main result

We analyze the spectral properties of the family of compact self-adjoint operators $T_\varepsilon : \mathcal{H}_\varepsilon \to \mathcal{H}_\varepsilon$,

$$\mathcal{H}_\varepsilon := L^2(\Omega_\varepsilon), \qquad T_\varepsilon := (-\Delta)^{-1} \text{ on } \Omega_\varepsilon, \tag{1.6}$$

where the Laplace operator is understood with a homogeneous Dirichlet boundary condition on $\partial\Omega$ and a homogeneous Neumann condition on $\partial\Sigma_\varepsilon$.

A related problem is given by the Laplace operator on Ω. We set

$$\mathcal{H}_* := L^2(\Omega), \qquad T_* := (-\Delta)^{-1} \text{ on } \Omega, \tag{1.7}$$

with homogeneous Dirichlet boundary condition on $\partial\Omega$. The operator T_* neglects the resonator and, as a consequence, misses the eigenvalues of T_ε that correspond to resonant frequencies. Our result is that, in the sense of spectral convergence, the limiting problem is better described by the Hilbert space $\mathcal{H}_0$ and the operator $T_0 : \mathcal{H}_0 \to \mathcal{H}_0$,

$$\mathcal{H}_0 := \mathbb{R} \times L^2(\Omega), \qquad T_0 := M_{\mu_0} \otimes T_* : (f_0, f_1) \mapsto (\mu_0 f_0, T_* f_1). \tag{1.8}$$

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $\lambda_1 \geq \lambda_2 \geq \dots$ be the ordered Dirichlet eigenvalues of $T_* = (-\Delta)^{-1}$ on Ω, repeated according to multiplicity. Let $\mu_0 > \lambda_1$ and $\delta \in (\lambda_{N+1}, \lambda_N)$ be two real numbers for some $N \in \mathbb{N}$. Let the shape $\Sigma_\varepsilon \subset \Omega$ of the small resonator be as described in Section 1.2 such that, in particular, (1.2)–(1.5) hold with $\mu_0 = LV/A$.*

Then there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon < \varepsilon_0$, the spectra of T_ε and T_0 are in a one-to-one correspondence on $\mathbb{C} \setminus B_\delta(0)$:

$$\Lambda_\varepsilon := \sigma(T_\varepsilon) \setminus B_\delta(0) = \{\mu_\varepsilon, \lambda_1^\varepsilon, \lambda_2^\varepsilon, \dots, \lambda_N^\varepsilon\}, \tag{1.9}$$

$$\Lambda_0 := \sigma(T_0) \setminus B_\delta(0) = \{\mu_0, \lambda_1, \lambda_2, \dots, \lambda_N\}, \tag{1.10}$$

and there holds $\mu_\varepsilon \to \mu_0$ and $\lambda_j^\varepsilon \to \lambda_j$ as $\varepsilon \to 0$ for every index $1 \leq j \leq N$.

Our proof provides additionally the convergence of eigenfunctions for the eigenvalues λ_j^ε. Moreover, we show that the sequence of eigenfunctions to μ_ε is concentrating in the cavity ω_ε, compare also Section 1.4 below.

Easy parts of the Theorem. The operator T_0 is compact and self-adjoint on $\mathcal{H}_0$, its spectrum consists of the eigenvalue μ_0 with eigenvector $(1, 0)$ and the eigenvalues λ_j of T_* with eigenvectors $(0, u_j)$, where u_j is the j-th eigenvector of T_*. By the choice of $\delta > 0$, this yields (1.10).

The compact and self-adjoint operator T_ε has a countable number of eigenvalues, and only a finite number of eigenvalues in $\mathbb{C} \setminus B_\delta(0)$. Labeling them in decreasing order and repeating eigenvalues according to their multiplicity we may write

$$\Lambda_\varepsilon = \{\mu_\varepsilon, \lambda_1^\varepsilon, \lambda_2^\varepsilon, ..., \lambda_{N_\varepsilon}^\varepsilon\}. \tag{1.11}$$

We have chosen here to denote the largest eigenvalue with a different letter. In order to prove the theorem, we have to show $N_\varepsilon = N$ for sufficiently small ε and the convergence of the eigenvalues.

We essentially proceed as follows.

1. There holds $\liminf_{\varepsilon \to 0} \mu_\varepsilon \geq \mu_0$ (Proposition 2.1).

2. The assumption $\mu_0 > \lambda_1$ implies that no eigenvalue λ_j is a cluster point of μ_ε. We conclude that μ_ε is a "sequence with concentration" (Lemma 3.2).

3. All sequences λ_j^ε are "sequences without concentration", hence every cluster point of a sequence λ_j^ε is an eigenvalue of T_* (Proposition 3.5).

4. The only cluster point of μ_ε is μ_0 (Proposition 3.7).

5. We count the eigenvalues with multiplicity (Proposition 4.2).

We treat item 1 in Section 2, items 2–3 in Section 3.1, item 4 in Section 3.2, and item 5 in Section 4.

More general geometries. 1.) Demanding that the boundaries of $\partial\omega_1$ and $\partial\Sigma_1^{\mathrm{out}}$ contain (adjacent) flat parts is not essential; this assumption just simplifies the description of the geometry. It guarantees that Ω_ε is a Lipschitz domain and that the interfaces $\gamma_\varepsilon^\pm$ are copies of γ_ε. 2.) The channel Γ_ε need not be straight; it may even have a variable cross-section. In this case a (harmonic) average of the cross-sectional areas must be calculated to obtain the effective area A. Similarly, multiple channels to the cavity could be considered. 3.) There could be multiple cavities. In this case, every cavity can generate its own eigenvalue.

1.4   Description of the localized resonant eigenfunction

The aim of this contribution is the spectral analysis of the Laplace operator in the domain Ω_ε. Before we turn to the rigorous analysis (i.e. the proof of Theorem 1.1), we present a loose description of the result, including an argument why $\mu_0 = LV/A$ is the resonant eigenvalue.

Since the inclusion Σ_ε is small, we can expect that most of the eigenvalues of $-\Delta$ on Ω_ε coincide (approximately) with eigenvalues of $-\Delta$ on Ω. This is indeed part of our main theorem.

Additionally, there can appear one eigenvalue that corresponds to a concentrated eigenfunction. Let us assume that f_ε with $T_\varepsilon f_\varepsilon = \mu_\varepsilon f_\varepsilon$ is such a concentrated eigenfunction. We assume $f_\varepsilon \approx 0$ outside the resonator (i.e. in $\Omega_\varepsilon^{\mathrm{out}}$) and $f_\varepsilon \approx 1$ inside the resonator $\Omega_\varepsilon^{\mathrm{inn}} = \omega_\varepsilon$ (we note that we have not normalized f_ε in the space $\mathcal{H}_\varepsilon = L^2(\Omega_\varepsilon)$).

The geometrical constraints let us expect that f_ε has the typical gradient $\nabla f_\varepsilon(x) \approx -L_\varepsilon^{-1} e_1$ inside the channel, the direction of descent is the first unit

coordinate e_1. The slope is determined by the fact that the function f_ε decreases on the length L_ε from 1 to 0.

On the other hand, we may integrate the equation $-\Delta f_\varepsilon = \mu_\varepsilon^{-1} f_\varepsilon$ over the interior ω_ε. The normal derivative of f_ε vanishes on $\partial\omega_\varepsilon$, except for the interface γ_ε^- with the channel. Since γ_ε^- has the area A_ε and since the normal derivative of f_ε on γ_ε^- is approximately $e_1 \cdot \nabla f_\varepsilon \approx -L_\varepsilon^{-1}$, we can expect (with the outer normal ν on $\partial\omega_\varepsilon$)

$$\mu_\varepsilon^{-1} V_\varepsilon \approx \mu_\varepsilon^{-1} \int_{\omega_\varepsilon} f_\varepsilon = \int_{\omega_\varepsilon} (-\Delta f_\varepsilon) = -\int_{\partial\omega_\varepsilon} \nu \cdot (\nabla f_\varepsilon) \approx \int_{\gamma_\varepsilon^-} L_\varepsilon^{-1} = A_\varepsilon L_\varepsilon^{-1} .$$

This argument suggests $\mu_\varepsilon = L_\varepsilon V_\varepsilon / A_\varepsilon \to LV/A$.

Physical parameters. Denoting the speed of sound by c_0, sound waves are described by the wave equation $\partial_t^2 u = c_0^2 \Delta u$, where the field $u : \Omega_\varepsilon \to \mathbb{R}$ describes pressure variations. The time harmonic ansatz $u(x,t) = u(x)e^{i\omega_0 t}$ with a frequency $\omega_0 > 0$ leads to the Helmholtz equation $-\Delta u = (\omega_0/c_0)^2 u$. Our theorem identifies $\mu_0 = LV/A$ as a limiting eigenvalue of T_ε, related to a nontrivial solution of $-\Delta u \approx \mu_0^{-1} u$. Therefore, in physical terms, an approximation for the resonant frequency is

$$\omega_{\mathrm{HR}} = c_0 \sqrt{\mu_0^{-1}} = c_0 \sqrt{\frac{A}{LV}} . \tag{1.12}$$

This is the well-known formula for the frequency of a Helmholtz resonator.

2 The norm of the operator T_ε

We now start the proof of Theorem 1.1. Since the norm of T_ε coincides with its largest eigenvalue, a lower bound for norm of T_ε is also a lower bound for the largest eigenvalue. For this reason, the subsequent Proposition settles item 1 of the above list.

Proposition 2.1 (The norm of T_ε)**.** *We consider the operator* $T_\varepsilon = (-\Delta)^{-1}$ *on* $\mathcal{H}_\varepsilon = L^2(\Omega_\varepsilon)$ *and its norm* $\|T_\varepsilon\| = \|T_\varepsilon\|_{\mathcal{L}(\mathcal{H}_\varepsilon, \mathcal{H}_\varepsilon)}$. *Boundary conditions and geometry are as in Theorem 1.1. Then there holds, for arbitrary* $\varepsilon > 0$,

$$\|T_\varepsilon\| \geq \frac{L_\varepsilon V_\varepsilon}{A_\varepsilon} . \tag{2.1}$$

Proof. We consider the special right hand side $f_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ defined as

$$f_\varepsilon(x) := \begin{cases} 1 & \text{for } x \in \omega_\varepsilon \\ 0 & \text{else.} \end{cases} \tag{2.2}$$

Our aim is to analyze the solution $u_\varepsilon = T_\varepsilon f_\varepsilon$ of $-\Delta u_\varepsilon = f_\varepsilon$. The lemma will follow from the fact that u_ε has (at least) the averaged value $L_\varepsilon V_\varepsilon / A_\varepsilon$ in the cavity ω_ε.

Step 1: The averaged gradient in the channel. We construct a special test-function $\chi_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ which is constant in the cavity ω_ε, vanishes in the outer domain $\Omega_\varepsilon^{\mathrm{out}}$, and is affine in the channel Γ_ε. More precisely, we set

$$\chi_\varepsilon(x) := \begin{cases} 1 & \text{for } x \in \omega_\varepsilon , \\ 1 - x_1/L_\varepsilon & \text{for } x \in \Gamma_\varepsilon , \\ 0 & \text{for } x \in \Omega_\varepsilon^{\mathrm{out}} . \end{cases} \tag{2.3}$$

This function is of class $H^1(\Omega_\varepsilon)$ and satisfies a homogeneous Dirichlet condition on $\partial\Omega$. Its gradient in the channel Γ_ε is $\nabla\chi_\varepsilon = -L_\varepsilon^{-1}e_1$.

We use χ_ε as a test-function in the relation $-\Delta u_\varepsilon = f_\varepsilon$ and find

$$V_\varepsilon = \int_{\omega_\varepsilon} 1 \cdot 1 = \int_{\Omega_\varepsilon} f_\varepsilon \cdot \chi_\varepsilon = \int_{\Omega_\varepsilon} (-\Delta u_\varepsilon) \cdot \chi_\varepsilon = \int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla \chi_\varepsilon = -\frac{1}{L_\varepsilon} \int_{\Gamma_\varepsilon} \partial_{x_1} u_\varepsilon \,.$$
$$(2.4)$$

This provides an information about the averaged slope of u_ε in Γ_ε. Loosely speaking, it provides the desired information: Because of $|\Gamma_\varepsilon| = A_\varepsilon L_\varepsilon$, the average slope of u_ε in Γ_ε is $V_\varepsilon/A_\varepsilon$. Therefore, with a channel of length L_ε, the values of u_ε on ω_ε can be expected to be at least $L_\varepsilon V_\varepsilon/A_\varepsilon$.

In the sequel, we will obtain this result in a very convenient way, using u_ε as a test function.

Step 2: The values of u_ε in ω_ε. We use u_ε as a test function in the equation $-\Delta u_\varepsilon = f_\varepsilon$. With the Cauchy-Schwarz inequality we obtain

$$\int_{\omega_\varepsilon} u_\varepsilon = \int_{\Omega_\varepsilon} f_\varepsilon \cdot u_\varepsilon = \int_{\Omega_\varepsilon} (-\Delta u_\varepsilon) \cdot u_\varepsilon = \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \geq \int_{\Gamma_\varepsilon} |\partial_{x_1} u_\varepsilon|^2$$

$$\overset{(CS)}{\geq} |\Gamma_\varepsilon|^{-1} \left(\int_{\Gamma_\varepsilon} (-\partial_{x_1} u_\varepsilon) \right)^2 \overset{(2.4)}{=} (A_\varepsilon L_\varepsilon)^{-1} (L_\varepsilon V_\varepsilon)^2 = V_\varepsilon \left(L_\varepsilon V_\varepsilon/A_\varepsilon \right). \quad (2.5)$$

The inequality implies that the average of u_ε over ω_ε is at least $L_\varepsilon V_\varepsilon/A_\varepsilon$.

Step 3: The norm of T_ε. We use the Cauchy-Schwarz inequality to conclude

$$\frac{\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2}{\|f_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2} \geq \left(\int_{\omega_\varepsilon} |u_\varepsilon|^2 \right) \Big/ \int_{\omega_\varepsilon} 1 \overset{(CS)}{\geq} \left(\int_{\omega_\varepsilon} u_\varepsilon \right)^2 \Big/ |\omega_\varepsilon|^2 \overset{(2.5)}{\geq} \left(L_\varepsilon V_\varepsilon/A_\varepsilon \right)^2 .$$

Because of $\|T_\varepsilon\| \geq \|u_\varepsilon\|_{\mathcal{H}_\varepsilon}/\|f_\varepsilon\|_{\mathcal{H}_\varepsilon}$, this provides the desired lower bound for the norm as claimed in (2.1). $\qquad\square$

Proposition 2.1 yields a crucial information on the spectrum of T_ε. Since T_ε is a self-adjoint compact and non-negative operator, the norm of T_ε coincides with its largest eigenvalue. Denoting the largest eigenvalue of T_ε by μ_ε as in (1.11), we find

$$\mu_\varepsilon = \|T_\varepsilon\|, \quad \text{and hence} \quad \mu_\varepsilon = \|T_\varepsilon\| \geq \frac{L_\varepsilon V_\varepsilon}{A_\varepsilon} \to \frac{LV}{A} = \mu_0 . \quad (2.6)$$

We emphasize that we *do not* obtain (yet) the convergence $\mu_\varepsilon \to \mu_0$, but we *do* obtain the inequality $\liminf_{\varepsilon\to 0} \mu_\varepsilon \geq \mu_0$.

3   Sequences with and without concentration

We fix a sequence $\varepsilon \to 0$. Let $(\lambda_\varepsilon)_\varepsilon$ be any sequence of eigenvalues of T_ε. It is useful to distinguish two classes of such sequences. We call the sequence a *sequence with concentration* if and only if the following property is satisfied:

for every subsequence $(\varepsilon_j)_{j\in\mathbb{N}}$ and

every sequence of normed eigenfunctions $(f^j)_j$,

$T_{\varepsilon_j} f^j = \lambda_{\varepsilon_j} f^j, \quad \|f^j\|_{\mathcal{H}_{\varepsilon_j}} = 1, \quad$ and every radius $r > 0$ holds: $\qquad (3.1)$

$$\limsup_{j\to\infty} \int_{\Omega\setminus B_r(0)} |f^j|^2 = 0 \,.$$

Every other sequence of eigenvalues $(\lambda_\varepsilon)_\varepsilon$ is called a *sequence without concentration*. A sequence without concentration is characterized by the property

$$
\begin{aligned}
&\exists \text{ subsequence } (\varepsilon_j)_{j\in\mathbb{N}} \text{ and } (f^j)_j\,, \\
&T_{\varepsilon_j} f^j = \lambda_{\varepsilon_j} f^j\,, \|f^j\|_{\mathcal{H}_{\varepsilon_j}} = 1\,, \quad \text{and a radius } r > 0 : \\
&\liminf_{j\to\infty} \int_{\Omega\setminus B_r(0)} |f^j|^2 > 0\,.
\end{aligned}
\tag{3.2}
$$

Lemma 3.1. *Let $0 < \delta \le \lambda_\varepsilon \to \lambda \in \bar{\mathbb{R}}$ for $\varepsilon \to 0$ be a convergent sequence of eigenvalues of T_ε without concentration. Then λ is finite and an eigenvalue of T_*.*

Proof. Since λ_ε is a sequence without concentration, we can choose a subsequence $(\varepsilon_j)_{j\in\mathbb{N}}$, a sequence $(f^j)_j$ of eigenfunctions, and a radius $r > 0$ as in (3.2). We write λ_j instead of λ_{ε_j}. We extend each function f^j trivially to Ω to define a bounded sequence $\tilde{f}^j \in L^2(\Omega)$. By weak compactness we can choose a further subsequence (we do not relabel) and a limit function f such that $\tilde{f}^j \rightharpoonup f \in L^2(\Omega)$, weakly in $L^2(\Omega)$ as $j \to \infty$.

Multiplication of the eigenvalue equation $-\Delta f^j = \lambda_j^{-1} f^j$ with f^j and an integration over Ω_{ε_j} provides

$$
\int_{\Omega_{\varepsilon_j}} |\nabla f^j|^2 = \lambda_j^{-1} \int_{\Omega_{\varepsilon_j}} |f^j|^2 \le C\,,
\tag{3.3}
$$

with a constant C that does not depend on j. We find that the functions $g^j := \nabla f^j$ and their trivial extension $\tilde{g}^j \in L^2(\Omega)$ are uniformly bounded. Upon choice of a further subsequence we obtain a limit g such that $\tilde{g}^j \rightharpoonup g$ weakly in $L^2(\Omega)$. Test functions with support in $\Omega \setminus \{0\}$ provide $g = \nabla f$ almost everywhere in Ω.

The sequence f^j is uniformly bounded in $H^1(\Omega \setminus B_r(0))$. Compactness of the Rellich embedding ensures the strong convergence $f^j \to f$ in $L^2(\Omega \setminus B_r(0))$, the trace theorem ensures $f|_{\partial\Omega} = 0$. In particular, we obtain from (3.2) the property $f \ne 0$.

It remains to verify $-\Delta f = \lambda^{-1} f$ on all of Ω. With this aim, let $\varphi \in C_c^\infty(\Omega)$ be arbitrary. In the limit $j \to \infty$ there holds

$$
\lambda^{-1} \int_\Omega f\varphi \leftarrow \lambda_j^{-1} \int_\Omega \tilde{f}^j \varphi = \lambda_j^{-1} \int_{\Omega_{\varepsilon_j}} f^j \varphi
$$

$$
= \int_{\Omega_{\varepsilon_j}} \nabla f^j \cdot \nabla\varphi = \int_\Omega \tilde{g}^j \cdot \nabla\varphi \to \int_\Omega g \cdot \nabla\varphi = \int_\Omega \nabla f \cdot \nabla\varphi\,.
$$

This verifies $-\Delta f = \lambda^{-1} f$ in Ω (for unbounded sequences λ_j the relation $-\Delta f = 0$). Because of $f \ne 0$ and because of the homogeneous Dirichlet boundary values, we obtain that λ is finite and an eigenvalue of T_*. $\qquad\square$

Lemma 3.2. *The sequence $(\mu_\varepsilon)_\varepsilon$ is a sequence with concentration.*

Proof. We recall that Proposition 2.1 provides $\liminf_\varepsilon \mu_\varepsilon \ge \mu_0 > \lambda_1$.

For a contradiction argument, let us assume that $(\mu_\varepsilon)_\varepsilon$ is a sequence without concentration. We first choose a subsequence as in (3.2). We now choose a further subsequence and a limit $\mu \in \bar{\mathbb{R}}$ such the eigenvalues are convergent, $\mu_\varepsilon \to \mu \in \bar{\mathbb{R}}$. This subsequence is still a sequence without concentration. By Lemma 3.1, the limit μ is finite and an eigenvalue of T_*. This is the desired contradiction because of $\mu = \lim_{\varepsilon\to 0} \mu_\varepsilon > \lambda_1$. $\qquad\square$

3.1   Characterization of sequences with concentration

In the following we will exploit various Poincaré estimates that involve averages of functions. We will make use of the following fact, based on the Cauchy-Schwarz inequality: For an arbitrary domain $U \subset \mathbb{R}^n$, an $L^2(U)$-function $u : U \to \mathbb{R}$ and a number $m \in \mathbb{R}$ holds

$$
\left| \left(\int_U |u|^2 \right)^{1/2} - |U|^{1/2} m \right|^2 = \int_U |u|^2 - 2 \left(\int_U |u|^2 \right)^{1/2} |U|^{1/2} m + |U| \, m^2
$$

$$
\overset{(CS)}{\leq} \int_U |u|^2 - 2 \int_U m u + |U| \, m^2 = \int_U |u - m|^2 \,.
$$

$$(3.4)$$

The inequality implies that, having control of $\int_U |u - m|^2$, the $L^2(U)$-norm of u is comparable to the $L^2(U)$-norm of the constant function m.

Lemma 3.3 (Concentrating eigenfunctions). *Let $\varepsilon = \varepsilon_j \to 0$ be fixed and let f^ε be a sequence of normed eigenfunctions of T_ε. We assume that the corresponding eigenvalues satisfy $\lambda_\varepsilon \geq \delta$. We furthermore assume that f^ε is concentrating in the sense that*

$$
\forall r > 0 : \quad \limsup_{j \to \infty} \int_{\Omega \setminus B_r(0)} |f^\varepsilon|^2 = 0 \,.
$$

$$(3.5)$$

Then f^ε satisfies

$$
\int_{\omega_\varepsilon} \left| f^\varepsilon - \fint_{\omega_\varepsilon} f^\varepsilon \right|^2 \to 0 \,,
$$

$$(3.6)$$

$$
\int_{\Omega_\varepsilon^{\mathrm{out}} \cup \Gamma_\varepsilon} |f^\varepsilon|^2 \to 0 \,.
$$

$$(3.7)$$

We note that the H^1-boundedness (3.8) of f^ε is sufficient for (3.6). The H^1-boundedness together with the concentration property (3.5) implies (3.7).

Proof. The eigenvalue equation is $-\Delta f^\varepsilon = \lambda_\varepsilon^{-1} f^\varepsilon$. As in (3.3) we can use f^ε as a test function and obtain from boundedness of λ_ε^{-1} and of $\|f^\varepsilon\|_{L^2(\Omega_\varepsilon)} = 1$ an estimate

$$
\int_{\Omega_\varepsilon} \left\{ |f^\varepsilon|^2 + |\nabla f^\varepsilon|^2 \right\} \leq C \,.
$$

$$(3.8)$$

Step 1: Estimate in ω_ε. We apply a Poincaré inequality in ω_ε; smallness of the diameter of ω_ε provides the smallness of the corresponding constant. Technically, we define $\hat{f}^\varepsilon : \omega_1 \to \mathbb{R}$ by setting $\hat{f}^\varepsilon(y) := f^\varepsilon(\varepsilon y)$. We apply the Poincaré inequality in ω_1 with constant C_1 (using averages) and calculate

$$
\frac{1}{\varepsilon^n} \int_{\omega_\varepsilon} \left| f^\varepsilon - \fint_{\omega_\varepsilon} f^\varepsilon \right|^2 = \int_{\omega_1} \left| \hat{f}^\varepsilon - \fint_{\omega_1} \hat{f}^\varepsilon \right|^2 \leq C_1 \int_{\omega_1} \left| \nabla \hat{f}^\varepsilon \right|^2 = C_1 \frac{1}{\varepsilon^n} \varepsilon^2 \int_{\omega_\varepsilon} |\nabla f^\varepsilon|^2 \,.
$$

The boundedness of the integral on the right hand side implies the claim of (3.6).

Step 2: Estimate in $\Omega_\varepsilon^{\mathrm{out}}$. Since the outer shape Σ_1^{out} is bounded, we find a radius $R > 0$ with $\Sigma_1^{\mathrm{out}} \subset B_R(0)$. We can use a Poincaré inequality with constant C_1 in the domain $B_{2R}(0) \setminus \Sigma_1^{\mathrm{out}}$ and emphasize that we use here an average that

is taken only over a part of the domain, namely $B_{2R}(0) \setminus B_R(0)$. Rescaling the domain and using $\hat{f}^\varepsilon(.) := f^\varepsilon(\varepsilon\,.)$ as above, we obtain

$$\frac{1}{\varepsilon^n} \int_{\varepsilon(B_{2R}(0)\setminus\Sigma_1^{\mathrm{out}})} \left| f^\varepsilon - \fint_{\varepsilon(B_{2R}(0)\setminus B_R(0))} f^\varepsilon \right|^2 \le C_1 \frac{1}{\varepsilon^n} \varepsilon^2 \int_{\varepsilon(B_{2R}(0)\setminus\Sigma_1^{\mathrm{out}})} |\nabla f^\varepsilon|^2 \,. \quad (3.9)$$

We next use a Poincaré inequality with a constant $C_2(r)$ satisfying $C_2(r) \to 0$ as $r \to 0$. There holds

$$\int_{B_r(0)\setminus B_{\varepsilon R}(0)} |f^\varepsilon|^2 \le \int_{B_{2r}(0)\setminus B_r(0)} |f^\varepsilon|^2 + C_2(r) \int_{B_{2r}(0)\setminus B_{\varepsilon R}(0)} |\nabla f^\varepsilon|^2 \,. \quad (3.10)$$

This Poincaré inequality is obtained in the classical way by integrating the gradient of f^ε over rays $\{(x_1, \tilde{x}) | x_1 \in \mathbb{R}\}$ in the two regions $(B_{2r}(0) \setminus B_{\varepsilon R}(0)) \cap \{x_1 > 0\}$ and $(B_{2r}(0) \setminus B_{\varepsilon R}(0)) \cap \{x_1 < 0\}$. We exploit that each ray intersects each domain as a (connected) segment.

Combining the estimates, using $m := \fint_{\varepsilon(B_{2R}(0)\setminus B_R(0))} f^\varepsilon$ in the second line, we find

$$\int_{B_r(0)\setminus\Sigma_\varepsilon^{\mathrm{out}}} |f^\varepsilon|^2 = \int_{\varepsilon(B_R(0)\setminus\Sigma_1^{\mathrm{out}})} |f^\varepsilon|^2 + \int_{B_r(0)\setminus B_{\varepsilon R}(0)} |f^\varepsilon|^2$$

$$\overset{(3.4),(3.9)}{=} |\varepsilon(B_R(0) \setminus \Sigma_1^{\mathrm{out}})|\, m^2 + o(1) + \int_{B_r(0)\setminus B_{\varepsilon R}(0)} |f^\varepsilon|^2$$

$$\overset{(CS)}{\le} C \int_{B_r(0)\setminus B_{\varepsilon R}(0)} |f^\varepsilon|^2 + o(1)$$

$$\overset{(3.10)}{\le} C \int_{B_{2r}(0)\setminus B_r(0)} |f^\varepsilon|^2 + C C_2(r) \int_{B_{2r}(0)\setminus B_{\varepsilon R}(0)} |\nabla f^\varepsilon|^2 + o(1) \,. \quad (3.11)$$

Choosing first $r > 0$ small, we obtain that the second term is small (independent of $\varepsilon = \varepsilon_j$). Choosing then ε small, we obtain that the last term is small and, from the concentration property (3.5), that the first term is small. Using once more (3.5), we obtain the claim of (3.7) concerning $\Omega_\varepsilon^{\mathrm{out}}$.

Step 3: Estimate in Γ_ε. The estimate in Γ_ε is again obtained with a Poincaré inequality. We define a continuation of the channel to the outside as $\tilde{\Gamma}_\varepsilon := (L\varepsilon, 2L\varepsilon) \times \gamma_\varepsilon$. A Poincaré inequality yields

$$\int_{\Gamma_\varepsilon} |f^\varepsilon|^2 \le \int_{\tilde{\Gamma}_\varepsilon} |f^\varepsilon|^2 + C_3(\varepsilon) \int_{\Gamma_\varepsilon \cup \tilde{\Gamma}_\varepsilon} |\nabla f^\varepsilon|^2 \,,$$

where $C_3(\varepsilon) \to 0$ as $\varepsilon \to 0$ since the diameter of the domain is of order ε. The extended channel lies in the outside, $\tilde{\Gamma}_\varepsilon \subset \Omega_\varepsilon^{\mathrm{out}}$, hence both terms on the right hand side vanish as $\varepsilon = \varepsilon_j \to 0$ by Step 2. We obtain the full estimate (3.7). □

Lemma 3.4 (There is only one sequence with concentration). *For a sequence $\varepsilon = \varepsilon_j \to 0$ let f_1^ε and f_2^ε be two sequences of eigenfunctions that are orthogonal and normalized, $\langle f_k^\varepsilon, f_l^\varepsilon \rangle_{L^2(\Omega_\varepsilon)} = \delta_{k,l}$ for $k, l \in \{1,2\}$, and let the corresponding eigenvalues be bounded from below by a number $\delta > 0$. Let f_1^ε be a sequence with concentration in the sense of (3.5). Then f_2^ε is not a sequence with concentration. Furthermore, there holds*

$$\lim_{0 < r \to 0} \liminf_{j \to \infty} \int_{\Omega_\varepsilon \setminus B_r(0)} |f_2^\varepsilon|^2 = 1 \,. \quad (3.12)$$

Proof. Let f_1^ε and f_2^ε be orthonormal sequences as in the lemma. We can use Lemma 3.3, which characterizes the concentrating sequence f_1^ε. We note that the result (3.6) is valid also for the (not necessarily concentrating) sequence f_2^ε. We choose a subsequence such that weighted ω_ε-averages converge (in the generalized sense in $\bar{\mathbb{R}}$) with limits in $\bar{\mathbb{R}}$,

$$|\omega_\varepsilon|^{1/2}\fint_{\omega_\varepsilon} f_k^\varepsilon \to A_k \in \bar{\mathbb{R}} \tag{3.13}$$

as $j \to \infty$. Orthogonality allows to calculate (we use (3.7) only for f_1^ε)

$$0 = \int_{\Omega_\varepsilon} f_1^\varepsilon \cdot f_2^\varepsilon \overset{(3.7)}{=} \int_{\omega_\varepsilon} f_1^\varepsilon \cdot f_2^\varepsilon + o(1)$$

$$\overset{(3.6)}{=} |\omega_\varepsilon| \left(\fint_{\omega_\varepsilon} f_1^\varepsilon\right)\left(\fint_{\omega_\varepsilon} f_2^\varepsilon\right) + o(1) \to A_1 \cdot A_2\,.$$

This provides that $A_1 = 0$ or $A_2 = 0$. The normalization provides for f_1^ε

$$1 = \int_{\Omega_\varepsilon} |f_1^\varepsilon|^2 \overset{(3.7)}{=} \int_{\omega_\varepsilon} |f_1^\varepsilon|^2 + o(1) \overset{(3.6)}{=} |\omega_\varepsilon|\left(\fint_{\omega_\varepsilon} f_1^\varepsilon\right)^2 + o(1) \to (A_1)^2$$

as $j \to \infty$ by (3.13). Hence $A_1 = 1$ and therefore $A_2 = 0$.

Let us assume, for a contradiction argument, that f_2^ε is also concentrating. Then the last calculation can be performed also for f_2^ε and provides $A_2 = 1$, a contradiction.

We now turn to the proof of (3.12). The orthonormality calculation above yields $A_1 = 1$ and $A_2 = 0$. For the sequence f_2^ε we use the Poincaré estimate (3.11) of the proof of Lemma 3.3, namely

$$\int_{B_r(0)\setminus\Sigma_\varepsilon^{\text{out}}} |f_2^\varepsilon|^2 \leq C \int_{B_{2r}(0)\setminus B_r(0)} |f_2^\varepsilon|^2 + CC_2(r)\int_{B_{2r}(0)\setminus B_{\varepsilon R}(0)} |\nabla f_2^\varepsilon|^2 + o(1)\,. \tag{3.14}$$

We consider a maximal radius $r_0 > 0$ with $B_{r_0}(0) \subset \Omega$ and a minimal radius $0 < \rho < r_0$, and the disjoint family of rings $B_{2\rho}(0)\setminus B_\rho(0)$, $B_{4\rho}(0)\setminus B_{2\rho}(0)$, $B_{8\rho}(0)\setminus B_{4\rho}(0)$, ..., $B_{2^{m+1}\rho}(0) \setminus B_{2^m\rho}(0)$ with $2^{m+1}\rho \leq r_0$. For small $\rho > 0$, the number $m = m(\rho) \in \mathbb{N}$ of such rings is large.

Let $\eta > 0$ be an arbitrary number. We choose first r_0 sufficiently small such that $C_2(r_0)$ is small compared to η. We can thus acchieve that the second integral on the right hand side of (3.14) (for $r = r_0$) is less than $\eta/3$ for every j. Restricting ourselfs to $j \geq J_0$ with $J_0 \in \mathbb{N}$ sufficiently large, also the third term is less than $\eta/3$.

We choose $\rho > 0$ sufficiently small so that $C/m(\rho) < \eta/3$ holds with C of the first integral on the right hand side of (3.14). For arbitrary $j \geq J_0$ we obtain that the left hand side of (3.14) is less than η (for some $r \geq \rho$). The $L^2(\omega_\varepsilon)$-norm of f_2^ε is small because of $A_2 = 0$. The $L^2(\Gamma_\varepsilon)$-norm of f_2^ε is small by the same argument as in Lemma 3.3 (Poincaré inequality using that the norm outside $\Sigma_\varepsilon^{\text{out}}$ is small). The fact that f_2^ε is normalized in $L^2(\Omega_\varepsilon)$ implies (3.12). □

Proposition 3.5. *Sequences of eigenvalues have the following properties.*

1. For some $\varepsilon_0 > 0$ holds: The eigenvalue μ_ε is simple for every $\varepsilon < \varepsilon_0$.

2. *For every sequence $\varepsilon_j \to 0$ and every sequence of indices $(k(j))_{j\in\mathbb{N}}$, $1 \leq k(j) \leq N_{\varepsilon_j}$ for every $j \in \mathbb{N}$, the sequence $\left(\lambda^{\varepsilon_j}_{k(j)}\right)_{j\in\mathbb{N}}$ is a sequence without concentration. Moreover, every corresponding sequence f^j of eigenfunctions is not concentrating (i.e. (3.5) does not hold).*

3. *Let $\lambda^{\varepsilon_j}_{k(j)} \to \lambda$ with $1 \leq k(j) \leq N_{\varepsilon_j}$ for every $j \in \mathbb{N}$ be a convergent sequence of eigenvalues. Then the limit point λ is of the form $\lambda = \lambda_K$ for some index $K \in \{1, 2, ..., N\}$.*

4. *Let $P \in \mathbb{N}$ be a multiplicity and let $\lambda^{\varepsilon_j}_{k_1(j)}$, ..., $\lambda^{\varepsilon_j}_{k_P(j)}$ be sequences of eigenvalues, distinct in the sense that $k_p(j) < k_{p+1}(j)$ for every $p < P$ (the eigenvalues may coincide). We assume that all sequences are convergent with the same limit, $\lambda^{\varepsilon_j}_{k_p(j)} \to \lambda$ as $j \to \infty$ for every $1 \leq p \leq P$. Then the multiplicity of λ (as an eigenvalue of T_*) is at least P.*

Proof. 1. For a contradiction argument we assume that, along a sequence $\varepsilon = \varepsilon_j \to 0$, every μ_ε is not simple. We can then choose two orthonormal sequences of eigenfunctions $(f_1^j)_j$ and $(f_2^j)_j$, $\langle f_k^j, f_l^j \rangle_{L^2(\Omega_{\varepsilon_j})} = \delta_{k,l}$ for every j. Both sequences must be concentrating by Lemma 3.2. This is in contradiction with Lemma 3.4.

2. By the simplicity result of item 1 there holds $\lambda^{\varepsilon_j}_{k(j)} \neq \mu_{\varepsilon_j}$. We note that the lower bound $\lambda^{\varepsilon_j}_{k(j)} \geq \delta$ is satisfied by construction of Λ_ε. The sequence $(\mu_\varepsilon)_\varepsilon$ is a sequence with concentration by Lemma 3.2. Accordingly, we can choose a concentrating sequence $(f_1^j)_j$ of eigenfunctions to μ_{ε_j}. Let $(f_2^j)_j$ be a normalized sequence of eigenfunctions to $\lambda^{\varepsilon_j}_{k(j)}$. Since the eigenvalues are different, $(f_1^j)_j$ and $(f_2^j)_j$ are orthogonal. Lemma 3.4 implies that $(f_2^j)_j$ is not concentrating. We therefore obtain that $\lambda^{\varepsilon_j}_{k(j)}$ is a sequence without concentration (and that arbitrary sequences of eigenfunctions are not concentrating).

3. By item 2, the sequence $\lambda^{\varepsilon_j}_{k(j)}$ is a sequence without concentration. By Lemma 3.1, the limit λ is an eigenvalue of T_*. Because of $\lambda \geq \liminf_j \lambda^{\varepsilon_j}_{k(j)} \geq \delta$, we find $\lambda = \lambda_K$ for some index $K \in \{1, 2, ..., N\}$.

4. We consider P corresponding families $f_p^j \in L^2(\Omega_{\varepsilon_j})$ of normalized eigenfunctions, $p = 1, ..., P$. Two eigenvalues $\lambda^{\varepsilon_j}_{k_p(j)}$ and $\lambda^{\varepsilon_j}_{k_q(j)}$ may coincide, but after an orthogonalization process we may assume the orthogonality $f_p^j \perp f_q^j$ for $p \neq q$.

For every $p \leq P$, and every subsequence $j \to \infty$, the family $(f_p^j)_{j\in\mathbb{N}}$ is a sequence without concentration by item 2. We can select a subsequence and P limit functions f_p with the weak convergence of the trivial extensions, $\tilde{f}_p^j \rightharpoonup f_p$ weakly in $L^2(\Omega)$. Since no subsequence is concentrating, the limit functions are all non-trivial, $f_p \neq 0$ for every $p \leq P$. As in the proof of Lemma 3.1 we furthermore obtain that every function f_p is an eigenfunction of T_*.

The families $(f_p^j)_{j\in\mathbb{N}}$ are orthogonal and they are all orthogonal to the family of eigenfunctions of the eigenvalue μ_{ε_j} (with concentration). All sequences $(f_p^j)_{j\in\mathbb{N}}$ therefore satisfy (3.12); loosely speaking, they do neither concentrate in ω_ε nor in any neighborhood of Σ_ε. This implies that also the weak limits f_p are orthogonal and the function space spanned by the weak limits f_p has dimension at least P. This shows the multiplicity claim. $\square$

At this point we know: μ_ε is a simple eigenvalue for every $\varepsilon < \varepsilon_0$. The other eigenvalues λ^ε can converge only to eigenvalues of T_*. There are at least as many

eigenvalues of T_* as for T_ε (minus one for the eigenvalue μ_ε). Hence (1.11) holds with $N_\varepsilon \leq N$ for every $\varepsilon < \varepsilon_0$.

3.2   The limit of μ_ε

The main result of this section is obtained in Proposition 3.7 below: The only cluster point of μ_ε is $\mu_0 = LV/A$. As a preparation, we show two estimates that are related to traces.

Lemma 3.6. *Let Ω_ε be as in Theorem 1.1 and let $f^\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ be a sequence of functions such that $\|\nabla f^\varepsilon\|_{L^2(\Omega_\varepsilon)}$ is bounded. Then, with the volume $V_\varepsilon = |\omega_\varepsilon|$ of the cavity ω_ε and the channel boundary portion $\gamma_\varepsilon^- \subset \partial \omega_\varepsilon$, averages satisfy*

$$\sqrt{V_\varepsilon} \left| \fint_{\gamma_\varepsilon^-} f^\varepsilon - \fint_{\omega_\varepsilon} f^\varepsilon \right| \to 0 \,. \tag{3.15}$$

Furthermore, if the concentration property $\int_{\Omega_\varepsilon^{\mathrm{out}}} |f^\varepsilon|^2 \to 0$ is satisfied, there holds

$$\sqrt{V_\varepsilon} \fint_{\gamma_\varepsilon^+} f^\varepsilon \to 0 \,. \tag{3.16}$$

Proof. In order to verify (3.15), we have to study the solution $\psi_\varepsilon : \omega_\varepsilon \to \mathbb{R}$ of the following auxiliary problem (ν denotes the outer normal to ω_ε):

$$-\Delta \psi_\varepsilon = A_\varepsilon V_\varepsilon^{-1} \text{ in } \omega_\varepsilon \,, \qquad \int_{\omega_\varepsilon} \psi_\varepsilon = 0 \,,$$

$$\partial_\nu \psi_\varepsilon = \begin{cases} 0 & \text{on } \partial \omega_\varepsilon \setminus \gamma_\varepsilon^- \,, \\ -1 & \text{on } \gamma_\varepsilon^- \,. \end{cases} \tag{3.17}$$

We note that the integrability condition is satisfied, hence there exists a unique solution $\psi_\varepsilon \in H^1(\omega_\varepsilon)$ to problem (3.17). We multiply the equation $-\Delta \psi_\varepsilon = A_\varepsilon V_\varepsilon^{-1}$ with f^ε and integrate over ω_ε,

$$A_\varepsilon \fint_{\omega_\varepsilon} f^\varepsilon = \int_{\omega_\varepsilon} A_\varepsilon V_\varepsilon^{-1} f^\varepsilon = \int_{\omega_\varepsilon} (-\Delta \psi_\varepsilon) f^\varepsilon = \int_{\omega_\varepsilon} \nabla \psi_\varepsilon \cdot \nabla f^\varepsilon - \int_{\partial \omega_\varepsilon} \partial_\nu \psi_\varepsilon \, f^\varepsilon \,.$$

Inserting the Neumann boundary condition of ψ_ε, we obtain

$$\sqrt{V_\varepsilon} \left| \fint_{\omega_\varepsilon} f^\varepsilon - \fint_{\gamma_\varepsilon^-} f^\varepsilon \right| = \frac{\sqrt{V_\varepsilon}}{A_\varepsilon} \left| \int_{\omega_\varepsilon} \nabla \psi_\varepsilon \cdot \nabla f^\varepsilon \right| \leq \frac{\sqrt{V_\varepsilon}}{A_\varepsilon} \|\nabla \psi_\varepsilon\|_{L^2(\omega_\varepsilon)} \|\nabla f^\varepsilon\|_{L^2(\Omega_\varepsilon)} \,. \tag{3.18}$$

Relation (3.15) is shown once that we verify, for arbitrary $\eta > 0$, that the right hand side of (3.18) is smaller than $C\eta$ for every $\varepsilon < \varepsilon_0 = \varepsilon_0(\eta)$, where C is independent of η and ε.

   In order to estimate $\|\nabla \psi_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2$, we multiply (3.17) with ψ_ε and integrate over ω_ε. We find

$$\int_{\omega_\varepsilon} |\nabla \psi_\varepsilon|^2 = \int_{\omega_\varepsilon} A_\varepsilon V_\varepsilon^{-1} \psi_\varepsilon + \int_{\gamma_\varepsilon^-} \partial_\nu \psi_\varepsilon \, \psi_\varepsilon = -\int_{\gamma_\varepsilon^-} \psi_\varepsilon \,. \tag{3.19}$$

   In order to exploit this equality, we need a trace estimate. On the ε-independent domain ω_1, let $\psi : \omega_1 \to \mathbb{R}$ be a function of class $H^1(\omega_1)$ with vanishing average. Then there exists an exponent $p > 2$ such that $\|\psi\|_{L^p(\partial \omega_1)} \leq C\|\nabla \psi\|_{L^2(\omega_1)}$.

Restricting ourselfs to a small portion $\gamma_1 \subset \partial\omega_1$ of the boundary provides $\|\psi\|_{L^2(\gamma_1)} \le C|\gamma_1|^{\alpha}\|\nabla\psi\|_{L^2(\omega_1)}$ for some exponent $\alpha > 0$.

Rescaling, we find for functions $\psi_{\varepsilon} : \omega_{\varepsilon} \to \mathbb{R}$ with vanishing average on ω_{ε} (we apply the above to $\hat{\psi}_{\varepsilon}(y) = \psi_{\varepsilon}(\varepsilon y)$ on ω_1 and with $\gamma_{\varepsilon} = \gamma_{\varepsilon}^{-} = \varepsilon\gamma_1 \subset \partial\omega_{\varepsilon}$)

$$\|\psi_{\varepsilon}|_{\gamma_{\varepsilon}}\|_{L^2(\gamma_{\varepsilon})} \le C \left(\frac{|\gamma_{\varepsilon}|}{|\partial\omega_{\varepsilon}|} \right)^{\alpha} \sqrt{\varepsilon}\|\nabla\psi_{\varepsilon}\|_{L^2(\omega_{\varepsilon})} . \tag{3.20}$$

The Cauchy-Schwarz inequality yields

$$\int_{\omega_{\varepsilon}} |\nabla\psi_{\varepsilon}|^2 \overset{(3.19)}{=} -\int_{\gamma_{\varepsilon}^{-}} \psi_{\varepsilon} \overset{(CS)}{\le} \left(\int_{\gamma_{\varepsilon}^{-}} 1 \right)^{1/2} \left(\int_{\gamma_{\varepsilon}^{-}} |\psi_{\varepsilon}|^2 \right)^{1/2} = |\gamma_{\varepsilon}|^{1/2}\|\psi_{\varepsilon}|_{\gamma_{\varepsilon}}\|_{L^2(\gamma_{\varepsilon})}$$

$$\overset{(3.20)}{\le} CA_{\varepsilon}^{1/2}\,\varepsilon^{1/2} \left(\frac{|\gamma_{\varepsilon}|}{|\partial\omega_{\varepsilon}|} \right)^{\alpha} \|\nabla\psi_{\varepsilon}\|_{L^2(\omega_{\varepsilon})} .$$

Dividing by the norm we obtain $\|\nabla\psi_{\varepsilon}\|_{L^2(\omega_{\varepsilon})} \le CA_{\varepsilon}^{1/2}\varepsilon^{1/2}(|\gamma_{\varepsilon}|/|\partial\omega_{\varepsilon}|)^{\alpha}$. For arbitrary $\eta > 0$ we can choose $\varepsilon_0 > 0$ such that the relative volume of the interface is small, $(|\gamma_{\varepsilon}|/|\partial\omega_{\varepsilon}|)^{\alpha} \le \eta \; \forall\varepsilon < \varepsilon_0$. In particular, we have verified for the right hand side of (3.18)

$$V_{\varepsilon}^{1/2}A_{\varepsilon}^{-1}\|\nabla\psi_{\varepsilon}\|_{L^2(\omega_{\varepsilon})}\|\nabla f^{\varepsilon}\|_{L^2(\Omega_{\varepsilon})} \le C\eta\, V_{\varepsilon}^{1/2}A_{\varepsilon}^{-1/2}\,\varepsilon^{1/2} \le C\eta\,\varepsilon^{n/2}\varepsilon^{-(n+1)/2}\varepsilon^{1/2} = C\eta\,,$$

as we had to show.

The proof of relation (3.16) is similar: We can define a family of auxiliary domains $\tilde{\omega}_{\varepsilon} \subset \Omega_{\varepsilon}^{\text{out}}$ that have γ_{ε}^{+} as a part of the boundary. The $L^2(\tilde{\omega}_{\varepsilon})$-norm of f^{ε} vanishes in the limit $\varepsilon \to 0$ by assumption. The relation (3.15) holds also on these domains and provides (3.16). $\qquad\square$

Proposition 3.7. *The only cluster point of μ_{ε} is $\mu_0 = LV/A$.*

Proof. We consider an $L^2(\Omega_{\varepsilon})$-normalized sequence of corresponding eigenfunctions f^{ε}. We recall that f^{ε} is a sequence with concentration by Lemma 3.2.

Multiplication of the eigenvalue relation $-\Delta f^{\varepsilon} = \mu_{\varepsilon}^{-1}f^{\varepsilon}$ with χ_{ε} of (2.3) yields

$$\mu_{\varepsilon}^{-1}\int_{\omega_{\varepsilon}} f^{\varepsilon} + O(|\Gamma_{\varepsilon}|^{1/2}) = \int_{\Omega_{\varepsilon}} \mu_{\varepsilon}^{-1}f^{\varepsilon}\chi_{\varepsilon} = \int_{\Omega_{\varepsilon}} \nabla f^{\varepsilon} \cdot \nabla\chi_{\varepsilon} = -\frac{1}{L_{\varepsilon}}\int_{\Gamma_{\varepsilon}} \partial_{x_1} f^{\varepsilon} . \tag{3.21}$$

This relates the ω_{ε}-average of f^{ε} to the Γ_{ε}-average of the slope of f^{ε}. We can transform this information on the slope of f^{ε} into an information on the values of f^{ε} at the interfaces. We calculate, using (3.21) in the last equality,

$$\fint_{\gamma_{\varepsilon}^{+}} f^{\varepsilon} - \fint_{\gamma_{\varepsilon}^{-}} f^{\varepsilon} = \fint_{\Gamma_{\varepsilon}} \partial_{x_1} f^{\varepsilon} = -L_{\varepsilon}\mu_{\varepsilon}^{-1}\fint_{\omega_{\varepsilon}} f^{\varepsilon} + O(L_{\varepsilon}|\Gamma_{\varepsilon}|^{1/2}) . \tag{3.22}$$

We multiply (3.22) with $\sqrt{V_{\varepsilon}}A_{\varepsilon}^{-1}$ and re-order terms to find

$$\sqrt{V_{\varepsilon}} \left\{ \fint_{\gamma_{\varepsilon}^{-}} f^{\varepsilon} - L_{\varepsilon}V_{\varepsilon}A_{\varepsilon}^{-1}\mu_{\varepsilon}^{-1}\fint_{\omega_{\varepsilon}} f^{\varepsilon} \right\} = \sqrt{V_{\varepsilon}}A_{\varepsilon}^{-1}O(L_{\varepsilon}|\Gamma_{\varepsilon}|^{1/2}) + \sqrt{V_{\varepsilon}}\fint_{\gamma_{\varepsilon}^{+}} f^{\varepsilon} . \tag{3.23}$$

The first term on the right hand side is of the order $\sqrt{V_{\varepsilon}}A_{\varepsilon}^{-1}O(L_{\varepsilon}|\Gamma_{\varepsilon}|^{1/2}) = O(V_{\varepsilon}^{1/2}A_{\varepsilon}^{-1/2}L_{\varepsilon}^{3/2}) = O(\varepsilon^{n/2}\varepsilon^{-(n+1)/2}\varepsilon^{3/2}) = O(\varepsilon)$, independent of the dimension n. We conclude that this error term is small for $\varepsilon \to 0$.

In order to proceed, we will use (3.15)–(3.16) of Lemma 3.6. These estimates can be used since, by normalization, f^ε satisfies $\|\nabla f^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 = \langle -\Delta f^\varepsilon, f^\varepsilon \rangle_{L^2} = \mu_\varepsilon^{-1} \|f^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 = \mu_\varepsilon^{-1}$, which is bounded by Proposition 2.1. The concentration property (3.7) assures that (3.16) holds.

Relation (3.16) implies that also the second term on the right hand side of (3.23) is small, hence the right hand side of (3.23) is $o(1)$ for $\varepsilon \to 0$.

On the left hand side of (3.23) we replace, using (3.15), the mean value over γ_ε^- by the mean value over ω_ε. We obtain

$$\left\{ 1 - L_\varepsilon V_\varepsilon A_\varepsilon^{-1} \mu_\varepsilon^{-1} \right\} \sqrt{V_\varepsilon} \fint_{\omega_\varepsilon} f^\varepsilon = o(1) \,. \tag{3.24}$$

We claim that the average of f^ε satisfies

$$\sqrt{V_\varepsilon} \fint_{\omega_\varepsilon} f^\varepsilon \to 1 \,. \tag{3.25}$$

Once this is shown, we conclude $\left\{ 1 - L_\varepsilon V_\varepsilon A_\varepsilon^{-1} \mu_\varepsilon^{-1} \right\} \to 0$ for $\varepsilon \to 0$. This implies the result, $\lim_\varepsilon \mu_\varepsilon = \lim_\varepsilon L_\varepsilon V_\varepsilon A_\varepsilon^{-1} = \mu_0$.

Verification of (3.25). The sequence μ_ε is a sequence with concentration by Lemma 3.2. We can therefore apply Lemma 3.3, which characterizes sequences with concentration. Normalization of f^ε and (3.7) yield

$$\int_{\omega_\varepsilon} |f^\varepsilon|^2 = 1 - \int_{\Omega_\varepsilon^{\mathrm{out}} \cup \Gamma_\varepsilon} |f^\varepsilon|^2 \to 1 \,. \tag{3.26}$$

Since relation (3.6) provides

$$\int_{\omega_\varepsilon} |f^\varepsilon|^2 - V_\varepsilon \left(\fint_{\omega_\varepsilon} f^\varepsilon \right)^2 = \int_{\omega_\varepsilon} \left| f^\varepsilon - \fint_{\omega_\varepsilon} f^\varepsilon \right|^2 \to 0 \,,$$

we find the claim of (3.25). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

4   Counting eigenvalues and proof of Theorem 1.1

For compact self-adjoint operators, the existence of approximate eigenfunctions implies the existence of nearby eigenvalues. This result is used e.g. as Lemma 11.2 in [16] for multiplicity $P = 1$. The case of general multiplicity is shown e.g. in [14], Theorem 9(bis). For convenience of the reader, we sketch the elementary proof below.

Lemma 4.1 (Approximate eigenfunctions imply the existence of eigenvalues)**.** *Let H be a Hilbert space and let T be a compact self-adjoint linear operator $T : H \to H$, let $P \in \mathbb{N}$ be a number. Then, for some constant $\sigma_0(P) > 0$, the following holds.*

Let $f_1, ..., f_P \in H$ be approximately orthonormal in the sense that $|\langle f_p, f_q \rangle_H - \delta_{p,q}| \leq \sigma_0(P)$ for all $1 \leq p, q \leq P$. Let them furthermore be approximate eigenvectors of T for the same eigenvalue $0 \neq \lambda \in \mathbb{R}$ in the sense that

$$\|T f_p - \lambda f_p\|_H \leq \alpha \, \|f_p\|_H \quad \forall 1 \leq p \leq P \,, \tag{4.1}$$

for some number $\alpha > 0$. Then T posesses P eigenvalues $\lambda_1, ..., \lambda_P$ (repeated according to multiplicity) with

$$|\lambda_p - \lambda| \leq \alpha \quad \forall 1 \leq p \leq P \,. \tag{4.2}$$

Proposition 4.2 ($N_\varepsilon \geq N$ for every $\varepsilon < \varepsilon_0$). *Let the situation be as in Theorem 1.1, let $0 \neq \lambda \in \mathbb{R}$ be an eigenvalue of $T_* = (-\Delta)^{-1}$ with multiplicity $P \in \mathbb{N}$ and let $\alpha > 0$ be an arbitrary radius. Then there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \leq \varepsilon_0$, the operator T_ε has P eigenvalues (counted with multiplicity) in $B_\alpha(\lambda) \subset \mathbb{C}$.*

Proof. We want to use Lemma 4.1 with $T = T_\varepsilon$, $H = \mathcal{H}_\varepsilon$, and with λ as in this proposition. We will construct P approximate eigenfunctions of T_ε. The natural choice is to use P orthonormal eigenfunctions $f_1, ..., f_P$ of T_*, corresponding to the eigenvalue λ. We cannot use these functions directly, since $f_p \in \mathcal{H}_* = L^2(\Omega)$ is not an element of $\mathcal{H}_\varepsilon = L^2(\Omega_\varepsilon)$, but it suffices to restrict f_p to the domain Ω_ε and to use $f_p^\varepsilon := f_p|_{\Omega_\varepsilon} \in \mathcal{H}_\varepsilon$.

We have to show for the family $(f_p^\varepsilon)_{p \leq P}$ two properties: The family is approximately orthonormal and it consists of approximate eigenfunctions.

The first property is immediate. We calculate

$$\langle f_p^\varepsilon, f_q^\varepsilon \rangle_{\mathcal{H}_\varepsilon} = \int_{\Omega_\varepsilon} f_p^\varepsilon f_q^\varepsilon = \int_{\Omega_\varepsilon} f_p f_q = \delta_{p,q} - \int_{\Sigma_\varepsilon} f_p f_q = \delta_{p,q} + O(\varepsilon^n) \,,$$

since normed eigenfunctions of the Laplace operator in Ω are bounded.

In order to estimate $\|T_\varepsilon f_p^\varepsilon - \lambda f_p^\varepsilon\|_{\mathcal{H}_\varepsilon}$ we consider $u_\varepsilon := T_\varepsilon f_p^\varepsilon - \lambda f_p^\varepsilon$. We note that $\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C_0$ holds with C_0 independent of ε by boundedness of T_ε. By definition of the solution operator T_ε we can deduce from $T_\varepsilon f_p^\varepsilon = u_\varepsilon + \lambda f_p^\varepsilon$ and $-\lambda \Delta f_p^\varepsilon = f_p^\varepsilon$ on Ω_ε the equation

$$-\Delta u_\varepsilon = -\Delta(u_\varepsilon + \lambda f_p^\varepsilon) + \lambda \Delta f_p^\varepsilon = f_p^\varepsilon - f_p^\varepsilon = 0 \,,$$

hence $u_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ is harmonic. The boundary condition for u_ε can be deduced from the fact that the solution $u_\varepsilon + \lambda f_p^\varepsilon$ satisfies a homogeneous Neumann condition,

$$\nu \cdot \nabla u_\varepsilon|_{\partial \Sigma_\varepsilon} = \nu \cdot \nabla(u_\varepsilon + \lambda f_p^\varepsilon)|_{\partial \Sigma_\varepsilon} - \lambda \nu \cdot \nabla f_p^\varepsilon|_{\partial \Sigma_\varepsilon} = -\lambda \nu \cdot \nabla f_p|_{\partial \Sigma_\varepsilon} \,.$$

An a priori estimate for u_ε is obtained by testing the equation $-\Delta u_\varepsilon = 0$ with u_ε,

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 = \int_{\partial \Sigma_\varepsilon} (-\lambda \nu \cdot \nabla f_p) u_\varepsilon \leq \|u_\varepsilon|_{\partial \Sigma_\varepsilon}\|_{L^2(\partial \Sigma_\varepsilon)} \|\lambda \nu \cdot \nabla f_p\|_{L^2(\partial \Sigma_\varepsilon)}$$
$$\leq C\lambda \left(\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \right) \sqrt{|\partial \Sigma_\varepsilon|} \,,$$

where we used a trace estimate and the fact that f_p is a smooth function. We can now exploit the boundedness $\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C_0$, the elementary inequality $\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq 1 + \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2$, and the smallness of $\sqrt{|\partial \Sigma_\varepsilon|}$ to conclude $\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \to 0$. The Poincaré inequality is valid on the domain $\Omega_\varepsilon^{\mathrm{out}} \cup \Gamma_\varepsilon = \Omega_\varepsilon \setminus \omega_\varepsilon$ (outer domain plus channel) and we find $\|u_\varepsilon\|_{L^2(\Omega_\varepsilon^{\mathrm{out}} \cup \Gamma_\varepsilon)} \to 0$. We note already here that this fact allows to apply (3.16) of Lemma 3.6, which provides an estimate for the average of u_ε on γ_ε^+.

It remains to derive the L^2-smallness of u_ε on ω_ε. Since the smallness is clear for the smooth function $-\lambda f_p^\varepsilon$, we concentrate on the part $v_\varepsilon := u_\varepsilon + \lambda f_p^\varepsilon = T_\varepsilon f_p^\varepsilon$. The corresponding estimate for u_ε carries over and we find, as in (3.16),

$$\sqrt{V_\varepsilon} \fint_{\gamma_\varepsilon^+} v_\varepsilon \to 0 \,. \tag{4.3}$$

The comparison of inside and outside traces is obtained as in previous proofs by testing the equation for v_ε with χ_ε of (2.3),

$$\left| \int_{\gamma_\varepsilon^+} v_\varepsilon - \int_{\gamma_\varepsilon^-} v_\varepsilon \right| = \left| \int_{\Gamma_\varepsilon} \partial_{x_1} v_\varepsilon \right| = \left| L_\varepsilon \int_{\Omega_\varepsilon} \nabla v_\varepsilon \nabla \chi_\varepsilon \right| = \left| L_\varepsilon \int_{\Omega_\varepsilon} f_p^\varepsilon \chi_\varepsilon \right| \le C\varepsilon^{n+1} \,.$$

This calculation implies that the estimate (4.3) for averages of v_ε remains valid also on γ_ε^-. Finally, we use (3.15) of Lemma 3.6 to conclude

$$\sqrt{V_\varepsilon} \fint_{\omega_\varepsilon} v_\varepsilon \to 0 \,. \tag{4.4}$$

With the estimate (3.6) (which uses only the boundedness of the gradient and no further properties of the integrand) we find

$$\int_{\omega_\varepsilon} |v_\varepsilon|^2 = V_\varepsilon \left(\fint_{\omega_\varepsilon} v_\varepsilon \right)^2 + \int_{\omega_\varepsilon} \left| v_\varepsilon - \fint_{\omega_\varepsilon} v_\varepsilon \right|^2 \to 0 \,.$$

In particular, we find $\|T_\varepsilon f_p^\varepsilon - \lambda f_p^\varepsilon\|_{\mathcal{H}_\varepsilon} = \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le \alpha$ for $\varepsilon > 0$ sufficiently small. Lemma 4.1 implies $|\lambda_p^\varepsilon - \lambda| \le \alpha$ for P eigenvalues λ_p^ε of T_ε. $\qquad\square$

Proof of Theorem 1.1. Proposition 3.5 provides $N_\varepsilon \le N$ and Proposition 4.2 provides $N_\varepsilon \ge N$ for every $\varepsilon < \varepsilon_0$ sufficiently small. We therefore obtain $N_\varepsilon = N$ for every $\varepsilon < \varepsilon_0$ and thus (1.9).

The sequence μ_ε converges to μ_0 by Proposition 3.7. By Proposition 3.5, item 3, the other eigenvalues $\lambda_{k(j)}^j$ can have limits only in the set $\Lambda_0 \setminus \{\mu_0\}$ in the spectrum. Since all eigenvalues are ordered, the one-to-one correspondence as claimed in Theorem 1.1 holds. $\qquad\square$

Approximate eigenvectors imply the existence of eigenvalues.

Proof of Lemma 4.1. We consider the linear subspace $F := \operatorname{span}\{f_1, ..., f_P\} \subset H$. We choose $\sigma_0(P) > 0$ sufficiently small in order to assure that the almost orthonormality of f_p implies $\dim(F) = P$. Since every basis vector of F has this property, there holds

$$\|Tv - \lambda v\|_H^2 \le \alpha^2 \|v\|^2 \qquad \forall v \in F \,. \tag{4.5}$$

We now consider the set of eigenvalues $\{\lambda_1, ..., \lambda_Q\}$ with the property $|\lambda_q - \lambda| \le \alpha$ for every $1 \le q \le Q$. We can assume that the set is finite since otherwise the statement of the lemma is shown. Let φ_q be corresponding orthonormal eigenfunctions and set $G := \operatorname{span}\{\varphi_1, ..., \varphi_Q\} \subset H$. Expanding the basis of G, we can assume that $(\varphi_j)_j$ is a basis of H, consisting of normalized and orthonormal eigenfunctions of T. An arbitrary vector $v \in G^\perp \subset H$ in the orthogonal complement of G can be expanded as $v = \sum_{j=Q+1}^\infty c_j \varphi_j$ and we can calculate

$$\|Tv - \lambda v\|_H^2 = \sum_{j=Q+1}^\infty |c_j|^2 |\lambda_j - \lambda|^2 \|\varphi_j\|_H^2 > \sum_{j=Q+1}^\infty |c_j|^2 \alpha^2 = \alpha^2 \|v\|^2 \,. \tag{4.6}$$

For a contradiction argument, we assume $Q < P$, i.e. $\dim(G) < \dim(F)$. In this case, there exists a vector $0 \ne v \in F \cap G^\perp$. For v, both properties (4.5) and (4.6) must hold, which results in a contradiction. $\qquad\square$

References

[1] H. Ammari and S. Moskow. Asymptotic expansions for eigenvalues in the presence of small inhomogeneities. *Math. Methods Appl. Sci.*, 26(1):67–75, 2003.

[2] J. T. Beale. Scattering frequencies of reasonators. *Comm. Pure Appl. Math.*, 26:549–563, 1973.

[3] G. Bouchitté, C. Bourel, and D. Felbacq. Homogenization of the 3D Maxwell system near resonances and artificial magnetism. *C. R. Math. Acad. Sci. Paris*, 347(9-10):571–576, 2009.

[4] G. Bouchitté and D. Felbacq. Homogenization of a wire photonic crystal: the case of small volume fraction. *SIAM J. Appl. Math.*, 66(6):2061–2084, 2006.

[5] G. Bouchitté and B. Schweizer. Homogenization of Maxwell's equations in a split ring geometry. *Multiscale Model. Simul.*, 8(3):717–750, 2010.

[6] G. Bouchitté and B. Schweizer. Plasmonic waves allow perfect transmission through sub-wavelength metallic gratings. *Networks and Heterogeneous Media*, 2014.

[7] R. Brown, P. D. Hislop, and A. Martinez. Eigenvalues and resonances for domains with tubes: Neumann boundary conditions. *J. Differential Equations*, 115(2):458–476, 1995.

[8] M. Clausel, M. Duruflé, P. Joly, and S. Tordeux. A mathematical analysis of the resonance of the finite thin slots. *Appl. Numer. Math.*, 56(10-11):1432–1449, 2006.

[9] D. Colton and R. Kress. *Inverse acoustic and electromagnetic scattering theory*, volume 93 of *Applied Mathematical Sciences*. Springer, New York, third edition, 2013.

[10] C. Fernández and G. P. Menzala. Resonances of an elastic resonator. *Appl. Anal.*, 76(1-2):41–49, 2000.

[11] C. A. Fernández. Resonances in scattering by a resonator. *Indiana Univ. Math. J.*, 34(1):115–125, 1985.

[12] R. R. Gadyl′shin. On analogues of the Helmholtz resonator in averaging theory. *Mat. Sb.*, 193(11):43–70, 2002.

[13] R. R. Gadyl′shin and A. M. Il′in. On the spectrum of the Neumann problem for Laplace equation in a domain with a narrow slit. *Asymptot. Anal.*, 67(3-4):167–189, 2010.

[14] I. M. Glazman. *Direct methods of qualitative spectral analysis of singular differential operators*. Translated from the Russian by the IPST staff. Israel Program for Scientific Translations, Jerusalem, 1965; Daniel Davey & Co., Inc., New York, 1966.

[15] P. D. Hislop and A. Martinez. Scattering resonances of a Helmholtz resonator. *Indiana Univ. Math. J.*, 40(2):767–788, 1991.

[16] V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik. *Homogenization of differential operators and integral functionals.* Springer-Verlag, Berlin, 1994. Translated from the Russian by G. A. Yosifian [G. A. Iosif′yan].

[17] P. Joly and S. Tordeux. Asymptotic analysis of an approximate model for time harmonic waves in media with thin slots. *M2AN Math. Model. Numer. Anal.*, 40(1):63–97, 2006.

[18] P. Joly and S. Tordeux. Matching of asymptotic expansions for waves propagation in media with thin slots. II. The error estimates. *M2AN Math. Model. Numer. Anal.*, 42(2):193–221, 2008.

[19] J. J. Keller and E. Zauner. On the use of Helmholtz resonators as sound attenuators. *Z. Angew. Math. Phys.*, 46(3):297–327, 1995.

[20] R. Kohn and S. Shipman. Magnetism and homogenization of micro-resonators. *Multiscale Modeling & Simulation*, 7(1):62–92, 2007.

[21] A. Lamacz and B. Schweizer. Effective Maxwell equations in a geometry with flat rings of arbitrary shape. *SIAM J. Math. Anal.*, 45(3):1460–1494, 2013.

[22] P. D. Lax and R. S. Phillips. Decaying modes for the wave equation in the exterior of an obstacle. *Comm. Pure Appl. Math.*, 22:737–787, 1969.

[23] P. D. Lax and R. S. Phillips. On the scattering frequencies of the Laplace operator for exterior domains. *Comm. Pure Appl. Math.*, 25:85–101, 1972.

[24] B. Lombard and J.-F. Mercier. Numerical modeling of nonlinear acoustic waves in a tube connected with Helmholtz resonators. *J. Comput. Phys.*, 259:421–443, 2014.

[25] R. Ma, P. E. Slaboch, and S. C. Morris. Fluid mechanics of the flow-excited Helmholtz resonator. *J. Fluid Mech.*, 623:1–26, 2009.

[26] A. Mielke and E. Rohan. Homogenization of elastic waves in fluid-saturated porous media using the Biot model. *Math. Models Methods Appl. Sci.*, 23(5):873–916, 2013.

[27] E. Rohan. Homogenization of acoustic waves in strongly heterogeneous porous structures. *Wave Motion*, 50(7):1073–1089, 2013.

[28] E. Rohan and V. Lukeš. Homogenization of the acoustic transmission through a perforated layer. *J. Comput. Appl. Math.*, 234(6):1876–1885, 2010.

[29] J. Sanchez-Hubert and E. Sánchez-Palencia. Acoustic fluid flow through holes and permeability of perforated walls. *J. Math. Anal. Appl.*, 87(2):427–453, 1982.

[30] S. P. Shipman and S. Venakides. Resonance and bound states in photonic crystal slabs. *SIAM J. Appl. Math.*, 64(1):322–342 (electronic), 2003.