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Three Modes of Description and Thinking of Linear Algebra Concepts at Upper Secondary Education

Many researchers point out that appropriate combinations of concept's representations lead to improved students' learning outcomes and translations between different representations support conceptional understanding (Ainsworth et al., 1997; Panasuk & Beyranevand, 2010). Multiple representations are important for acquiring deeper knowledge about a domain (van der Meij & de Jong, 2006). It is well known that quick and correct calculations or apparently fluent procedural skills are not necessarily proceeded by conceptual understanding. Previous research reports that one of the indicators of conceptual understanding is “the capability for recognizing structurally the same connections posed via multiple representations” (Panasuk & Beyranevand, 2010, p. 2). How can translations across more representations of linear algebra concepts be supported to maximize students' learning outcomes and effectiveness of multiple-representational learning environments? The phenomenon of dynamic multiple representations in computer based learning environments in comparison with single static representations, single dynamic representations and multiple static representations offers the most opportunities and challenges (van der Meij & de Jong, 2006). Let us first explain multiple modes of descriptions, representations and thinking in linear algebra more in details.

1. Three Modes of Description and Thinking

The theoretical framework on multiple modes incloses three modes of description: geometric, algebraic and abstract (Hillel, 2000) and three modes of thinking: synthetic-geometric, arithmetic and analytic-structural (Sierpinska, 2000) of linear algebra concepts. The three modes of thoughts in linear algebra (Dreyfus, Hillel & Sierpinska, 1998, p. 209) are as follows:

- The geometric language/ synthetic-geometric mode of thought refers to 2- and 3- space (points, lines, planes, directed line segments and geometric transformations).
- The arithmetic language/ analytic-arithmetic mode of thought refers to n-tuples, matrices, rank, solutions of systems of equations, etc.
- The algebraic language/ analytic-structural mode of thought refers to the general theory (vector spaces, subspaces, dimension, operators, kernels, etc.)

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2. Exemplification of Multiple Modes of Description and Thinking of Vectors, Dot Product of Vectors and Determinants

This section offers examples of three modes of description and thinking of the concepts: vectors, dot product of vectors and determinants.

Example 1. Vectors.

- *Geometric language/ synthetic-geometric mode of thought:* Vectors are classes of parallel, same directed and equal in length arrows.
- *Arithmetic language/ analytic-arithmetic mode of thought:* Vectors are n-tuples.
- *Algebraic language/ analytic-structural mode of thought:* Vectors are elements of vector spaces.

Example 2. Dot Product of Vectors.

- *Geometric language/ synthetic-geometric mode of thought:* Dot product of vectors is the product of vectors' magnitudes and the cosine of the angle between them.

  $\vec{u}\cdot\vec{v}=|\vec{u}||\vec{v}|cos\varphi$ or $\vec{u}\cdot\vec{v}=|\vec{u}|\cdot(\pm|\vec{v}_{\vec{u}}|)=(\pm|\vec{u}_{\vec{v}}|)\cdot|\vec{v}|$
- *Arithmetic language/ analytic-arithmetic mode of thought:* Dot product of vectors is the sum of the products of corresponding vectors' components.

  $$\vec{u}\cdot\vec{v}=u_1 v_1+u_2 v_2+\ldots+u_n v_n=\sum_{i=1}^{n} u_i v_i$$
- *Algebraic language/ analytic-structural mode of thought:* Dot product of vectors is defined by three axioms for: bilinearity (additive and homogeneity), symmetry and positivity.

Example 3. Determinants.

- *Geometric language/ synthetic-geometric mode of thought:* Determinants are oriented volumes (areas) of parallelepiped (parallelograms) spanned by vectors.
- *Arithmetic language/ analytic-arithmetic mode of thought:* Determinants are sums of permutations:

  $$A=(a_{ij})_{i,j}\in K^{n\times n}, det(A)=\sum_{\pi\in\sigma_n} sgn\pi\cdot\alpha_{\pi(1),1}\cdot\ldots\cdot\alpha_{\pi(n),n}$$

- *Algebraic language/ analytic-structural mode of thought:* Determinants are functions satisfying three axioms: multilinearity, norm and two equal rows in a matrix, give zero value to its determinant.

As a note, it is worth mentioning that another choice of the axioms in the last mode is possible, but is equivalent to the offered one.

The growth in cognition required for students' absorbing particular mode of description and thinking is in close relation to the appropriate level of education. This development is illustrated in the following Table 1.

Level of education	*Vectors*	*Dot product of vectors*	*Determinants*	*Modes of description/ thinking*
Lower secondary	Vector quantities	/	/	Geometric/ Synthetic-geometric
Upper secondary	Classes of arrows	•Vectors' magnitudes and angle's cosine •Vectors' projections	Oriented areas of parallelograms/ volumes of parallelepiped	Geometric/ Synthetic-geometric
	n-tuples (ordered pairs/ triples)	Vectors' components	Arithmetic calculations	Algebraic/ Arithmetic
University and further	Elements of vector spaces	Axioms	Axioms	Abstract/ Analytic-structural

Table 1. Three Modes of Description and Thinking of Linear Algebra Concepts through the Levels of Education

The above Table 1 shows that each stage in the growth in cognition does not replace previous modes of description and thinking, but aims to integrate the existing modes with the new, through establishing connections. This process is not trivial as it may seem on the first appearance. On the contrary, it deserves a lot of attention. The primary aim at upper secondary level of education is to support the recognition, translation and utilization of multiple modes of reasoning in linear algebra and analytic geometry.

3. Proposal for Supporting Multiple Modes of Description and Thinking in a Dynamic Geometry Environment (DGE)

The question that arises now is, whether the analytic-structural mode of description and though can be brought into context closer to upper high school students and how can the gap in transition be overcome. This article suggests that it can be done to a certain extent with the aid of DGE. Such proposals are offered in (Filler, Donevska-Todorova, 2012) for the concepts of vectors and in (Donevksa-Todorova, 2012a; 2012b) for the concepts of determinants. These proposals support connections between geometric and algebraic modes of description, and moreover give exemplary dynamic applets for teaching and learning of properties which construct concepts' axiomatic definitions. Students' performance (investigations, conjectures and proves) and competences for translating among all three modes of description and thinking in such designed DGE could be analyzed through the instrumental orchestration interpretative theoretical framework (Drijvers et al., 2010).

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