

**A FAMILY OF NON-RESTRICTED  $D = 11$   
GEOMETRIC SUPERSYMMETRIES**

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# A FAMILY OF NON-RESTRICTED $D = 11$ GEOMETRIC SUPERSYMMETRIES

FRANK KLINKER

ABSTRACT. We construct a two parameter family of irreducible, eleven-dimensional, indecomposable, non-flat Cahen-Wallach spaces with non-restricted geometric supersymmetry of fraction  $\nu = 3/4$ . Its compactified moduli space can be parametrized by a compact interval with two points corresponding to two non-isometric, decomposable spaces. These singular spaces are associated to a restricted  $N = 4$  geometric supersymmetry with  $\nu = 1/2$  in dimension six and a non-restricted  $N = 2$  geometric supersymmetry with  $\nu = 3/4$  in dimension nine.

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## 1. INTRODUCTION

We will describe here in detail geometric supersymmetry of a family of eleven dimensional manifolds. Geometric supersymmetry is by definition an extension of the Lie algebra of the Killing vector fields to a super Lie algebra by purely geometric data. This roughly means that the odd part of the superalgebra is given by a linear subspace of the sections in a bundle over the manifold at hand compatible with the Killing vector fields – we will be more specific at the beginning of Section 5. Although the manifolds we will consider in this text are homogeneous spaces and, therefore, it

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would be sufficient to discuss the structure at one point, we will give all results in terms of local coordinates, too. On the one hand, we do this to emphasize the geometric nature and, on the other hand, because the local description may give an idea for similar constructions in the non homogeneous situation. However, the calculations are similar in both concepts.

First we will provide the general setup of the manifolds we will consider, the Cahen-Wallach spaces. We describe in detail the local structure of the metric and determine the Killing vector fields that yield the even part of our structure, see Sections 2 and 3. Then we turn to the odd part that will be spanned by sections in a spinor bundle that are parallel with respect to a given connection. Again we will give the local description and show how this depends on the elements at one point, see Section 4. These preparations lead to Section 5 where we first give a short introduction to geometric superalgebras and geometric supersymmetries. Then we describe in detail how the ingredients provided so far define a geometric superalgebra. In particular, we prove several compatibility conditions. In Section 6 we discuss whether there are situations in which the geometric superalgebra yields geometric supersymmetry. We formulate the obstruction and provide a full list. In the final Section we discuss the moduli space of geometric superalgebras and geometric supersymmetries. Furthermore, we associate the singularities to extended geometric supersymmetries in dimensions six and nine.

As a side note we emphasize that the spaces we discuss in this text are canonical candidates for supergravity backgrounds. For more details on this topic from the supergravity point of view we cordially refer the reader to the literature in the references.

## 2. THE GENERAL SETUP

The approach to classify solvable Lorentzian symmetric spaces by the construction presented below goes back to [3]. Let  $(V, \langle \cdot, \cdot \rangle)$  be an  $n$ -dimensional euclidean vector space and  $B$  be a symmetric endomorphism of  $V$ . We denote the symmetric bilinear form that is defined by  $B$  and  $\langle \cdot, \cdot \rangle$  by the same symbol  $B$  and we write  $*$  :  $V \rightarrow V^*$ ,  $v \mapsto v^*$  with  $v^*(w) = \langle v, w \rangle$  for the canonical identification of  $V$  and its dual. We define  $W := \mathbb{R}^{1,1} \oplus V$  and denote by  $\mathring{g}$  the the extension of  $\langle \cdot, \cdot \rangle$  to a block diagonal Lorentzian metric on  $W$ . Let  $\{e_+, e_-\}$  be a null basis of  $\mathbb{R}^{1,1}$  with respect to  $\mathring{g}|_{\mathbb{R}^{1,1}}$ . Then consider the following skew symmetric multiplication on  $\mathfrak{g} := V^* \oplus W$  that yields a Lie algebra structure on  $\mathfrak{g}$ :

$$[e_-, w] = w^*, \quad (1)$$

$$[v^*, e_-] = Bv, \quad (2)$$

$$[v^*, w] = -v^*(Bw) \cdot e_+ = -\langle Bv, w \rangle \cdot e_+, \quad (3)$$

for all  $w \in V$  and  $v^* \in V^*$ . The bilinear form  $\mathring{g}$  is extended to a bi-invariant metric on  $\mathfrak{g}$  be  $\mathring{g}(v^*, w^*) := \langle Bv, w \rangle$ .

Within  $\mathfrak{g}$  the factor  $V^*$  acts on  $W$ , the bracket of  $W$  with itself obeys  $[W, W] = V^*$  and  $\langle \cdot, \cdot \rangle$  is  $V^*$ -invariant. From (1)-(3) we see, that the embedding

$$V^* \longrightarrow \mathbb{R}_+ \otimes V \hookrightarrow \mathfrak{so}(W) = \mathfrak{so}(V) \oplus (\mathbb{R}_+ \otimes V) \oplus (\mathbb{R}_- \otimes V) \oplus (\mathbb{R}_+ \otimes \mathbb{R}_-) \quad (4)$$

is given by  $v^* \mapsto Bv \wedge e_+$  where  $x \wedge y(z) := \langle y, z \rangle x - \langle x, z \rangle y$ .

These data yield a  $(D = n + 2)$ -dimensional symmetric space  $M_B$  with Lorentz metric determined by  $\langle \cdot, \cdot \rangle$  and  $B$ . The resulting Lorentzian space  $M_B$  is indecomposable if and only if the symmetric map  $B$  is non-degenerate. This can best be seen from (1)-(3) if we recall that  $M_B$  is decomposable if there exists a  $V^*$ -invariant subspace  $\tilde{W} \subset W$  such that  $\hat{g}|_{\tilde{W} \times \tilde{W}}$  is non-degenerate, see [1, 18]. If  $B$  admits zero eigenvalues the space decomposes into a product of a Cahen-Wallach space and an euclidean space. This can also be deduced from the coordinate form of the metric, see (11).

### 3. THE METRIC AND THE KILLING VECTOR FIELDS

**3.1. The metric.** For the local description of  $M_B$  we may use the exponential map and write for  $x = x^+e_+ + x^-e_- + \vec{x} \in W$  with  $\vec{x} = \sum_i x^i e_i$

$$\mu(x) := \exp(x^+e_+) \exp(x^-e_-) \exp(\vec{x}).$$

This obeys

$$\partial_+ \mu = \exp(x^+e_+)e_+ \exp(x^-e_-) \exp(\vec{x}) = \mu(x)e_+ \quad (5)$$

$$\partial_i \mu = \exp(x^+e_+) \exp(x^-e_-) \exp(\vec{x})e_i = \mu(x)e_i \quad (6)$$

$$\begin{aligned} \partial_- \mu &= \exp(x^+e_+) \exp(x^-e_-)e_- \exp(\vec{x}) = \mu(x) \exp(-\vec{x})e_- \exp(\vec{x}) \\ &= \mu(x) \left( e_- + \sum_i x^i e_i^* - \frac{1}{2} \sum_{ij} B_{ij} x^i x^j e_+ \right) \end{aligned} \quad (7)$$

where we use  $\exp(\vec{x}) = \prod_i \exp(x^i e_i)$  and

$$\begin{aligned} e_j^* \exp(x^i e_i) &= \exp(x^i e_i) (e_j^* - B_{ij} x^i e_+) \\ e_- \exp(x^i e_i) &= \exp(x^i e_i) \left( e_- + x^i e_i^* - \frac{1}{2} B_{ii} (x^i)^2 e_+ \right) \end{aligned} \quad (8)$$

with the matrix  $(B_{ij})$  defined by  $B(e_i) = \sum_j B_{ij} e_j$ .

From this we read the two components of the Maurer-Cartan form  $\mu^{-1}d\mu = \omega + \theta \in \Omega^1(M_B) \otimes \mathfrak{g}$  with  $\omega \in \Omega^1(M_B) \otimes V^*$  and  $\theta \in \Omega^1(M_B) \otimes W$ :

$$\omega = \sum_i x^i dx^- \otimes e_i^*, \quad (9)$$

$$\theta = dx^- \otimes e_- + \sum_i dx^i \otimes e_i + \left( dx^+ - \frac{1}{2} \sum_{ij} B_{ij} x^i x^j dx^- \right) \otimes e_+. \quad (10)$$

With  $g_B = \hat{g}(\theta, \theta)$  we get the following local form of the metric on  $M_B$ :

$$g_B = 2dx^+ dx^- - \sum B_{ij} x^i x^j (dx^-)^2 + \sum_i (dx^i)^2. \quad (11)$$

In particular, the Levi-Civita connection of  $g_B$  is determined by the Christoffel symbols  $\Gamma_{i-;-} = -\Gamma_{--;i} = -\sum_j B_{ij} x^j$ . If we move from the coordinates to the adapted ON frame  $\{\partial_+, \partial_- + \frac{1}{2} \sum_{ij} B_{ij} x^i x^j \partial_+, \partial_i\}$  there is only one surviving component of the connection form, namely

$$\omega_{i-} = -\omega_{-i} = -\sum_j B_{ij} x^j dx^-. \quad (12)$$

The bi-invariant metric  $\hat{g}$  on  $\mathfrak{g}$  makes the decomposition  $\mathfrak{g} = V^* \oplus W$  an orthogonal splitting and the isometry algebra of  $M_B$  is given by

$$\mathbf{isom}(M_B) = \mathfrak{so}_B(V) \oplus V^* \oplus W \quad (13)$$

with

$$\begin{aligned} \mathfrak{so}_B(V) &= \{A \in \mathfrak{so}(V) \mid [A, B] = 0\} \\ &= \{A \in \mathfrak{so}(V) \mid [A, v^*] = (Av)^* \text{ for all } v^* \in V^*\}. \end{aligned} \quad (14)$$

**Remark 3.1.** To fix some notation, we like to mention that two Lorentzian spaces defined by symmetric maps  $B_1$  and  $B_2$  are isometric if and only if  $B_1$  and  $B_2$  are conformally equivalent, i.e. there exists a real scalar  $c > 0$  and an orthogonal transformation  $X$  such that  $B_2 = cX^t B_1 X$ .

Therefore, we may and will restrict to diagonal maps  $B$  such that the space  $M_B$  is defined by a sequence of real numbers  $\lambda_1^2, \dots, \lambda_n^2$ . We may also sort this sequence in the way  $\lambda_1^2 \leq \dots \leq \lambda_n^2$  and all non-vanishing if  $M_B$  is indecomposable.

**Example 3.2.** Consider  $D = 11$ , i.e.  $n = 9$ , and  $B = -4\beta^2 \begin{pmatrix} 4\mathbb{1}_3 & \\ & \mathbb{1}_6 \end{pmatrix}$ . Then  $M_B$  is indecomposable and the metric is given by

$$g_B = 2dx^+ dx^- + 4\beta^2 \left( 4 \sum_{i=1}^3 (x^i)^2 + \sum_{i=4}^9 (x^i)^2 \right) (dx^-)^2 + \sum_{i=1}^9 (dx^i)^2.$$

**3.2. The Killing vector fields.** A local basis of the isometry algebra of  $M_B$  is provided by the Killing vector fields, i.e. by those vector fields  $X$  that obey  $L_X g_B = 0$ . We will denote the Killing vector fields associated to the ON frame of  $\mathfrak{g}$  by  $K_{(+)}, K_{(-)}, K_{(i)}, K_{(i^*)}$  and those associated to the standard basis of  $\mathfrak{so}_B(V)$  by  $K_{(ij)}$ . In the following we consider  $B = \text{diag}(\lambda_1^2, \dots, \lambda_n^2)$ .

Because the metric coefficients only depend on the  $x^i$ , we immediately see that  $\partial_+$  and  $\partial_-$  are Killing vector fields and we write  $K_{(+)} = -\partial_+$  and  $K_{(-)} = -\partial_-$ . The ansatz

$$\begin{aligned} K_{(i)} &= \alpha_i(x^-) \partial_i + \beta_i(x^-) x^i \partial_+ \\ K_{(i^*)} &= \alpha_i^*(x^-) \partial_i + \beta_i^*(x^-) x^i \partial_+ \end{aligned} \quad (15)$$

inserted into  $\mathcal{L}_K g_B = 0$  yields

$$\frac{\partial \beta_i^{(*)}}{\partial x^-} = \lambda_i^2 \alpha_i^{(*)}, \quad \frac{\partial \alpha_i^{(*)}}{\partial x^-} = -\beta_i^{(*)},$$

or

$$\frac{\partial^2 \beta_i^{(*)}}{\partial (x^-)^2} = -\lambda_i^2 \beta_i^{(*)}, \quad \frac{\partial^2 \alpha_i^{(*)}}{\partial (x^-)^2} = -\lambda_i^2 \alpha_i^{(*)}.$$

This motivates the further specialization to

$$\alpha_i = a_i \cos(\lambda_i x^-), \quad \beta_i = b_i \sin(\lambda_i x^-), \quad \alpha_i^* = a_i^* \sin(\lambda_i x^-), \quad \beta_i^* = b_i^* \cos(\lambda_i x^-),$$

and the coefficients are related by  $\lambda_i a_i = b_i$ ,  $-\lambda_i a_i^* = b_i^*$ . By claiming the commutation relations (1)-(3) we fix the remaining free parameters:<sup>1</sup> (1) and (2) yield  $a_i^* = -\lambda_i a_i$  and (3) yields  $a_i^2 = 1$ . The resulting Killing fields that are adapted to  $e_+, e_-$  and the orthonormal eigenbasis  $\{e_i\}$  of  $B$  are  $K_{(+)} = -\partial_+$ ,  $K_{(-)} = -\partial_-$ , and

$$K_{(i)} = \cos(\lambda_i x^-) \partial_i + \lambda_i \sin(\lambda_i x^-) x^i \partial_+, \quad (16)$$

$$K_{(i^*)} = -\lambda_i \sin(\lambda_i x^-) \partial_i + \lambda_i^2 \cos(\lambda_i x^-) x^i \partial_+. \quad (17)$$

<sup>1</sup>We have to take into account that the vector fields obey the commutation relations only up to sign. This is due to the difference between right and left invariance when we turn from the group structure to the structure on the coset space, see [14] for more details on this fact.

The additional Killing vector fields that come from  $\mathfrak{so}_B(V)$  are given by the usual  $\mathfrak{so}(V)$  generators subject to condition (14), i.e.

$$K_{(ij)} = x^j \partial_i - x^i \partial_j \quad (18)$$

with  $i, j \in I_\alpha$  for some  $\alpha$  where  $\{1, \dots, n\} = \bigcup_{\alpha=1}^{\hat{n}} I_\alpha$  with  $I_\alpha = \{r_{\alpha-1} + 1, \dots, r_\alpha\}$  is the disjoint decomposition coming from  $\lambda_1^2 = \dots = \lambda_{r_1}^2 < \lambda_{r_1+1}^2 = \dots = \lambda_{r_2}^2 < \dots < \lambda_{r_{\hat{n}-1}+1}^2 = \dots = \lambda_n^2$  and  $r_0 = 0, r_{\hat{n}} = n$ .

#### 4. CONNECTIONS AND PARALLEL SPINORS

We will consider special connections on a the spinor bundle  $S(M_B)$  of  $M_B$  in the following, namely connections that are compatible with the homogeneous structure of  $M_B$ . If  $S(M_B)$  is associated to the irreducible Clifford module such connections are described by  $V^*$ -equivariant linear maps  $\mathfrak{g} = V^* \oplus W \rightarrow \text{Cl}(W)$  with the property that  $\rho : \mathfrak{so}(W) \supset V^* \rightarrow \text{Cl}(W)$  coincides with the spin representation, i.e.  $\rho(v^*) = \Gamma(v^*)$ . Here  $\text{Cl}(W)$  denotes the Clifford algebra of  $W$ . In [13] such connections are discussed in detail, and we will state the result in Propositions 4.1 and 4.3.

**4.1. Preliminaries.** Mainly to fix our notation, we recall some facts on the Clifford algebra in this special situation. We consider  $\text{Cl}(\mathbb{R}^{1,1}) = \mathfrak{gl}_2 \mathbb{C}$  with generators  $\gamma_+ = \gamma(e_+) = \frac{1}{\sqrt{2}}(i\sigma_2 + \sigma_1) = \sqrt{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $\gamma_- = \gamma(e_-) = \frac{1}{\sqrt{2}}(i\sigma_2 - \sigma_1) = \sqrt{2} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$  and we denote the two-dimensional volume element by  $\sigma := \frac{1}{2} [\gamma_+, \gamma_-] = -\sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ . If we denote the generators of  $\text{Cl}(V)$  by  $\{\gamma_i\}_{1 \leq i \leq n}$ , those of  $\text{Cl}(W) = \mathfrak{gl}_2 \mathbb{C} \hat{\otimes} \text{Cl}(V)$  are given by  $\{\Gamma_\mu\}_{\mu \in \{+, -, i\}} = \{\gamma_+ \otimes \mathbb{1}, \gamma_- \otimes \mathbb{1}, \sigma \otimes \gamma_i\}$ . In particular,

$$\mathfrak{gl}_2 \mathbb{C} \ni r \mapsto r \hat{\otimes} \mathbb{1} = r \otimes \mathbb{1} \in \text{Cl}(W) \quad (19)$$

$$\text{Cl}(V) \ni a \mapsto \mathbb{1} \hat{\otimes} a = \mathbb{1} \otimes a^0 + \sigma \otimes a^1 \in \text{Cl}(W) \quad (20)$$

where  $a = a^0 + a^1 \in \text{Cl}(V)$  is the decomposition into its even and odd part. In this regard, we consider the map  $\bar{\cdot} : \text{Cl}(V) \rightarrow \text{Cl}(V)$  with  $\overline{a^0 + a^1} = a^0 - a^1$ . Consider the irreducible Clifford modules  $S_2$  and  $S(V)$  of  $\text{Cl}(\mathbb{R}^{1,1})$  and  $\text{Cl}(V)$ . The first one decomposes into a sum of two one dimensional half spinor spaces  $S_2^\pm = \ker(\Gamma_\mp)$  given by the  $\pm 1$ -eigenspaces of  $\sigma$ . If we denote the two projections on the two eigenspaces by  $\sigma_\pm = \frac{1}{2}(\mathbb{1} \pm \sigma) = -\frac{1}{2}\gamma_\mp \gamma_\pm$  then (20) is rewritten as  $\mathbb{1} \hat{\otimes} a = \sigma_- \hat{\otimes} a + \sigma_+ \hat{\otimes} a = \sigma_- \otimes \bar{a} + \sigma_+ \otimes a$ . In our choice of  $\gamma$ -matrices the eigendirections are given by  $\vec{e}_1 = (1, 0)^t$  and  $\vec{e}_2 = (0, 1)^t$  such that a spinor in  $S(W) = S_2 \hat{\otimes} S(V) = S_2^- \otimes S(V) \oplus S_2^+ \otimes S(V) =:$

$S^-(W) \oplus S^+(W)$  can be written as  $\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \vec{e}_1 \otimes \eta_1 + \vec{e}_2 \otimes \eta_2$ . The action of

$\text{Cl}(W)$  on  $S(W)$  is now given by

$$(r \hat{\otimes} a) \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} r_{11} \bar{a} & r_{12} a \\ r_{21} \bar{a} & r_{22} a \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = r \begin{pmatrix} \bar{a} \eta_1 \\ a \eta_2 \end{pmatrix} \quad (21)$$

for  $r \in \mathfrak{gl}_2\mathbb{C}$  and  $a \in \text{Cl}(V)$ . In particular, the image of  $v^* \in V^*$  considered as an element of  $\mathfrak{so}(W) \subset \text{Cl}(W)$  under the spin representation, is given by

$$\begin{aligned} v^* = Bv \wedge e_+ &\mapsto \frac{1}{4}((\gamma_+ \otimes \mathbb{1})(\sigma \otimes Bv) - (\sigma \otimes Bv)(\gamma_+ \otimes \mathbb{1})) \\ &= \frac{1}{4}(\gamma_+ \sigma - \sigma \gamma_+) \otimes Bv = \frac{1}{2}\gamma_+ \otimes Bv = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & Bv \\ 0 & 0 \end{pmatrix}. \end{aligned} \quad (22)$$

**4.2. The algebraic description.** In terms of the notation introduced above, the relevant spinor connections of  $M_B$  are singled out as follows.

**Proposition 4.1.** *The  $V^*$  equivariant linear maps that define homogeneous connections on the spinor bundle are*

$$\begin{aligned} \rho(v^*) &= \frac{1}{2}\gamma_+ \hat{\otimes} Bv = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & Bv \\ 0 & 0 \end{pmatrix}, \\ \rho(e_+) &= \frac{1}{2}\gamma_+ \hat{\otimes} a = \begin{pmatrix} 0 & \sqrt{2}a \\ 0 & 0 \end{pmatrix}, \\ \rho(e_-) &= \sigma_- \hat{\otimes} c + \sigma_+ \hat{\otimes} d + \gamma_- \hat{\otimes} b + \gamma_+ \hat{\otimes} \epsilon = \begin{pmatrix} \bar{c} & \sqrt{2}\epsilon \\ \sqrt{2}b & d \end{pmatrix}, \\ \rho(w) &= -\sigma_- \hat{\otimes} wb - \sigma_+ \hat{\otimes} bw - \frac{1}{2}\gamma_+ \hat{\otimes} s_{\bar{c},d}(w) = \begin{pmatrix} wb & -\frac{1}{\sqrt{2}}s_{\bar{c},d}(w) \\ 0 & -bw \end{pmatrix}. \end{aligned}$$

with  $a, b, c, d, \epsilon \in \text{Cl}(V)$  and

$$s_{\bar{c},d} : \text{Cl}(V) \rightarrow \text{Cl}(V), \quad s_{\bar{c},d}(x) = \bar{c}x - xd.$$

The two parameters  $a, b$  are fixed to be pseudo-scalars  $a = \alpha + \beta\gamma^*$  and  $b = -\alpha + \beta\gamma^*$  if  $\dim(V)$  is even, and scalars  $a = -b = \alpha$  if  $\dim(V)$  is odd.

**Remark 4.2.** We consider  $\mathfrak{so}_B(V)$  acting in the usual way on  $W$ . Then it is compatible with the equivariant map  $\rho$  if it is extended by  $\rho(A) := \Gamma(A)$  for all  $A \in \mathfrak{so}_B(V)$  where  $\Gamma$  is the spin representation.

The curvature of a connection given by the equivariant map  $\rho$  is determined by its values in  $W$  and given by

$$\mathcal{R}^\rho(X, Y) = [\rho(X), \rho(Y)] - \rho([X, Y]_W) - \Gamma([X, Y]_{V^*}).$$

Therefore, an equivariant map  $\rho$  from Proposition 4.1 yields a flat connection if and only if  $\rho$  is a representation.

For example, if we assume scalar parameters  $a = b = \alpha$  the surviving components of the curvature are

$$R^\rho(e_-, e_+) = \begin{pmatrix} -\alpha^2 & \frac{\alpha}{\sqrt{2}}(\bar{c} - d) \\ 0 & \alpha^2 \end{pmatrix} \quad (23)$$

$$R^\rho(e_i, e_j) = \begin{pmatrix} 2\alpha^2\gamma_{ij} & -\sqrt{2}\alpha\{s_{\bar{c},d}(e_{[i]}, \gamma_{j]}\} \\ 0 & 2\alpha^2\gamma_{ij} \end{pmatrix} \quad (24)$$

$$R^\rho(e_-, e_i) = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & q_{\bar{c},d}(e_i) + B(e_i) \\ 0 & 0 \end{pmatrix} \quad (25)$$

with

$$q_{\bar{c},d} : \text{Cl}(V) \rightarrow \text{Cl}(V), \quad q_{\bar{c},d}(x) = s_{\bar{c},d}^2(x) = \bar{c}^2x + xd^2 - 2\bar{c}xd.$$

The flat connections from Proposition 4.1 are singled out by the following Proposition 4.3 together with Remark 4.6.

**Proposition 4.3.** *An equivariant map  $\rho$  with  $\rho(e_+) = 0$  defines a flat connection if and only if*

$$\begin{aligned} \rho(v^*) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & Bv \\ 0 & 0 \end{pmatrix}, \quad \rho(w) = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & s_{\bar{c},d}(w) \\ 0 & 0 \end{pmatrix}, \\ \rho(e_-) &= \begin{pmatrix} \bar{c} & \sqrt{2}\epsilon \\ 0 & d \end{pmatrix}. \end{aligned} \quad (26)$$

with  $(\bar{c}, d)$  subject to

$$q_{\bar{c},d}(v) = -B(v) \quad (27)$$

for all  $v \in V$ . In particular,  $\epsilon \in \text{Cl}(V)$  remains a free parameter of the flat connection.

**Example 4.4** (Example 3.2 continued). For the metric associated to the symmetric map  $B = -4\beta^2 \begin{pmatrix} 4\mathbb{1}_3 & \\ & \mathbb{1}_6 \end{pmatrix}$  in eleven dimensions, the pair  $(\bar{c}, d)$  with

$$\bar{c} = -3\beta\gamma_{123}, \quad d = \beta\gamma_{123}$$

obeys (27). This means, the spinor connection that is defined by these data is flat. In fact this pair together with  $\epsilon = 0$  has been considered in [4] and [10] as a connection that provides a maximal amount of parallel spinors.

**Remark 4.5.** If the spinor bundle is not associated to the irreducible module but to an extension of type  $S \otimes \mathbb{C}^N$  we call this an  $N$ -extension. In particular, Proposition 4.3, remains almost the same with  $(\bar{c}, d)$  taking its values in  $\text{Cl}(W) \otimes \mathfrak{gl}_N \mathbb{C}$ .

In [13] we discuss in detail a large class of pairs  $(\bar{c}, d)$  that solve condition (27), the so called quadratic Clifford pairs. Furthermore we give an additional condition that makes the list of solutions we present complete. This condition arises naturally in the discussion of supersymmetry.

Although we will not need it later but for the sake of completeness, we will state the result analog to Proposition 4.3 for  $\rho(e_+) \neq 0$ .

**Remark 4.6.** For an equivariant map  $\rho$  with  $\rho(e_+) = 0$  to define a flat connection we need  $n$  even and  $B = -2\lambda^2 \mathbb{1}$ . Moreover,  $a = \Pi^\pm$  is mandatory. We consider the upper sign for which the map  $\rho$  is given by

$$\begin{aligned} \rho(v^*) &= -\sqrt{2}\lambda^2 \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}, \quad \rho(e_+) = \sqrt{2}\alpha \begin{pmatrix} 0 & \Pi^+ \\ 0 & 0 \end{pmatrix}, \\ \rho(e_-) &= \sqrt{2} \begin{pmatrix} \rho_0 - \sqrt{\alpha\beta + \lambda^2} - \bar{c}_-^+ & \beta\Pi^- + (\epsilon_-^+ + \epsilon_-^- + \epsilon_+^+) \\ -\alpha\Pi^- & \rho_0 + \sqrt{\alpha\beta + \lambda^2} + d_+^- \end{pmatrix}, \\ \rho(v) &= \begin{pmatrix} -\alpha\Pi^+ v & \sqrt{\alpha\beta + \lambda^2} v + s_{\bar{c}_-^+, d_+^-}(v) \\ 0 & \alpha\Pi^- v \end{pmatrix}. \end{aligned} \quad (28)$$

The free parameters are the scalars  $\alpha, \beta, \rho_0$  and the Clifford element  $\epsilon_\pm^\pm$ . The further contributions are related by  $\sqrt{\alpha\beta + \lambda^2} s_{\bar{c}_-^+, d_+^-}(v) = \alpha s_{\epsilon_-^+, -\epsilon_+^-}(v)$  for all  $v \in V$ .



**4.3. The local description.** The discussion so far took place in one particular point of the manifold and, therefore, was a discussion on Lie algebras and representations. In the following, we will translate the discussion to the manifold  $M_B$ .

The choice of coordinates on  $M_B$  yields a splitting of the spinor bundle  $S := S(M_B)$  of  $M_B$  as  $S^- \oplus S^+$  with the first (resp. second) summand being the  $-1$ -eigenspaces (resp.  $+1$ -eigenspace) of  $\sigma := \frac{1}{2}[\Gamma_+, \Gamma_-]$ , i.e. the kernel of  $\Gamma_+$  (resp.  $\Gamma_-$ ). The projections on the two subbundles are given by  $\sigma_{\pm} = -\frac{1}{2}\Gamma_{\mp}\Gamma_{\pm}$ . We will use here the notation  $\vec{\xi} = \xi_1 + \xi_2$  for the sections in  $S$ , too. In particular, the action of  $\Gamma_{\pm}$  interchanges the two subbundles whereas they are preserved by the action of  $\Gamma_i$ . This is due to  $\sigma_{\pm}\Gamma_{\pm} = 0$ ,  $\sigma_{\pm}\Gamma_{\mp} = \Gamma_{\mp}$ , and  $\sigma_{\pm}\Gamma_i = \Gamma_i\sigma_{\pm}$ .

The Levi-Civita connection on  $M_B$  induces a connection on the spinor bundle  $S$  via the Spin-representation. It is given by  $\nabla\vec{\xi} = d\vec{\xi} - \frac{1}{4}\sum_{\mu\nu}\omega^{\mu\nu}\Gamma_{\mu\nu}\vec{\xi}$  which in our situation is

$$\begin{aligned}\nabla_+\vec{\xi} &= \partial_+\vec{\xi}, \\ \nabla_-\vec{\xi} &= \partial_-\vec{\xi} - \frac{1}{2}\omega^{+i}\Gamma_{+i}\vec{\xi} = \partial_-\vec{\xi} - \frac{1}{2}\sum_i x^i\Gamma_{+B}(e_i)\xi_2, \\ \nabla_i\vec{\xi} &= \partial_i\vec{\xi}.\end{aligned}\tag{29}$$

In the following we will consider a general connection on  $S(M_B)$  defined by the equivariant map  $\rho$  as given in Proposition 4.1. For the discussion of spinor connections we restrict to the situation of Proposition 4.1 with scalars  $a = -b = \alpha$ , because we will later on discuss odd dimensional manifolds, only. We know about the dimension of the space of parallel sections,  $\mathcal{K}_1 \subset \mathcal{S}$ : It coincides with the dimension of the kernel of the curvature  $R^\rho$ . In fact, for a connection according to Proposition 4.3 we get  $\dim \mathcal{K}_1 = \dim S(W) = \text{rank } S$ . More general, we see from (24) and (25) that  $\alpha = 0$  is mandatory if we assume the kernel of  $R^\rho$  to be non-trivial. Moreover, in this case the kernel is given by  $\ker(\Gamma_+) = S^-$ , generically.

We will now consider this situation with  $\epsilon = 0$  such that the connection is entirely determined by the pair  $(\bar{c}, d)$ . In our local coordinates the connection is given by  $D_\mu = \nabla_\mu + \rho(e_\mu)$  for  $\mu \in \{+, -, i\}$  and the parallel spinors satisfy  $D\vec{\xi} = 0$ . After applying the projecton operators  $-\frac{1}{2}\Gamma_{\pm}\Gamma_{\mp}$  this is

$$0 = \partial_+\xi_\alpha,\tag{30}$$

$$0 = \partial_i\xi_1 - \frac{1}{2}\Gamma_{+s_{\bar{c},d}}(e_i)\xi_2,\tag{31}$$

$$0 = \partial_i\xi_2,\tag{32}$$

$$0 = \partial_-\xi_1 - \frac{1}{2}\Gamma_+\sum_j x^j B(e_j)\xi_2 + \bar{c}\xi_1,\tag{33}$$

$$0 = \partial_-\xi_2 + d\xi_2.\tag{34}$$

From (30) we see that  $\vec{\xi}$  is independent of  $x^+$  and from (32) and  $\xi_2$  is independent from  $x^i$ , such that (34) yields

$$\xi_2 = \xi_2(x^-) = \exp(-x^-d)\xi_2^0\tag{35}$$

for a constant spinor  $\xi_2^0$ . Moreover, (31) yields  $\partial_i \partial_j \xi_1 = 0$  such that

$$\xi_1 = \xi_1(x^-, x^j) = \xi_1'(x^-) - \frac{1}{2} \sum_i x^i \Gamma_{+s_{\bar{c},d}(e_i)} \xi_2(x^-) \quad (36)$$

for a spinor  $\xi_1'$  depending only on  $x^-$ .

Inserting both in (33) yields

$$\begin{aligned} 0 &= \partial_- \xi_1' + \bar{c} \xi_1' + \frac{1}{2} \Gamma_+ \left( \sum_i x^i s_{\bar{c},d}(e_i) d\xi_2 - \sum_i x^i \bar{c} s_{\bar{c},d}(e_i) \xi_2 - \sum_i x^i B(e_i) \xi_2 \right) \\ &= (\partial_- \xi_1' + \bar{c} \xi_1') - \frac{1}{2} \Gamma_+ \sum_i x^i (q_{\bar{c},d}(e_i) + B(e_i)) \xi_2 \end{aligned}$$

The vanishing of this term means that both summands have to vanish separately such that we end up with

$$(q_{\bar{c},d}(v) + B(v)) \xi_2 = 0, \quad (37)$$

$$\xi_1'(x^-) = \exp(x^- \bar{c}) \xi_1^0, \quad (38)$$

with  $\xi_2 = \exp(-x^- d) \xi_2^0$ . We recall that  $\xi_1^0, \xi_2^0$  are constant spinors that obey  $\Gamma_+ \xi_1^0 = \Gamma_- \xi_2^0 = 0$ .

**Remark 4.7.** In terms of the local coordinates we again see, that the space of parallel spinors is of dimension  $\frac{1}{2} \dim S(W)$  and parametrized by  $\xi_1^0$ , generically. In case of maximal  $\mathcal{K}_1$  we need the full freedom in the choice of  $\xi_\alpha^0$  in (37) and (38). In this case the vanishing of the bracket in (37) is needed. This, of course, is the same as the vanishing of the sole remaining curvature term in (25), i.e. (27).

**4.4. A family of eleven dimensional spaces.** From now on we are interested in non-flat connections and turn to dimension eleven. More precisely, we consider a connection that is given according to the discussion above by

$$\bar{c} := (\alpha \Gamma_I + \beta \Gamma_J) \Gamma_K, \quad d := (\alpha' \Gamma_I + \beta' \Gamma_J) \Gamma_K \quad (39)$$

with  $I, J, K \subset \{1, \dots, 9\}$  and  $I \cap J \cap K = \emptyset$ . We use projections

$$X_{IJ}^\pm := \frac{1}{2} (\mathbb{1} \pm \iota_{IJ} \Gamma_{IJ})$$

with  $\iota_{IJ} \in \{1, i\}$  such that  $(\iota_{IJ} \Gamma_{IJ})^2 = \mathbb{1}$ . In terms of  $X_{IJ}^\pm$  we write

$$q_{\bar{c},d}(e_i) = \alpha_i^+ \Gamma_i X_{IJ}^+ + \alpha_i^- \Gamma_i X_{IJ}^- \quad (40)$$

for some linear combinations  $\alpha_i^\pm \in \{\pm \alpha \pm \alpha' \pm \beta \pm \beta'\}$  where the specific arrangement of signs depend on whether  $i \in I, J, K$ , or  $(I \cup J \cup K)^c$ .

We will further specify our connection and consider  $|I| = |J| = |K| + 1 = 2$  or – without loss of generality –  $I = (12)$ ,  $J = (34)$ , and  $K = (5)$ , i.e.

$$\bar{c} = (\alpha_+ X_{1234}^+ + \alpha_- X_{1234}^-) \Gamma_{125}, \quad d = (\alpha'_+ X_{1234}^+ + \alpha'_- X_{1234}^-) \Gamma_{125}, \quad (41)$$

with

$$\alpha_\pm = \alpha \mp \beta, \quad \alpha'_\pm = \alpha' \mp \beta'.$$

For  $q_{\bar{c},d}$  we get in this case

$$q_{\bar{c},d}(e_i) = \begin{cases} (\alpha_- - \alpha'_+)^2 \Gamma_i X_{1234}^+ + (\alpha_+ - \alpha'_-)^2 \Gamma_i X_{1234}^- & \text{for } i \in \{1, 2\} \\ (\alpha_- + \alpha'_+)^2 \Gamma_i X_{1234}^+ + (\alpha_+ + \alpha'_-)^2 \Gamma_i X_{1234}^- & \text{for } i \in \{3, 4\} \\ (\alpha_+ - \alpha'_+)^2 \Gamma_i X_{1234}^+ + (\alpha_- - \alpha'_-)^2 \Gamma_i X_{1234}^- & \text{for } i \in \{5\} \\ (\alpha_+ + \alpha'_+)^2 \Gamma_i X_{1234}^+ + (\alpha_- + \alpha'_-)^2 \Gamma_i X_{1234}^- & \text{for } i \in \{6, 7, 8, 9\} \end{cases} \quad (42)$$

**Remark 4.8.** In (37) we have  $X_{1234}^- \xi_2 = \exp(-x^- d) X_{1234}^- \xi_2^0$  such that  $X_{1234}^- \xi_2 = 0 \iff X_{1234}^- \xi_2^0 = 0$ . Therefore, condition (38) can be read as

$$(q_{\bar{c},d}(v) + B(v)) \xi_2^0 = 0.$$

Collecting the discussion yields the following proposition.

**Proposition 4.9.** *For the four parameter family of connections (39) the space of parallel spinors is of dimension  $\frac{3}{4} \dim S(W)$  determined by  $S(V) \oplus X_{IJ}^+ S(V) \subset S(W)$  for a metric with at most four different eigenvalues determined by  $\alpha_i^+$  from (40).*

*In the particular situation of (41) this is true for the metric defined by<sup>2</sup>*

$$B = -\text{diag} \left( (\alpha_- - \alpha'_+)^2 \mathbb{1}_2, (\alpha_- + \alpha'_+)^2 \mathbb{1}_2, (\alpha_+ - \alpha'_-)^2 \mathbb{1}_1, (\alpha_+ + \alpha'_-)^2 \mathbb{1}_4 \right). \quad (43)$$

**Remark 4.10.** The Clifford elements  $c, d$  that define the connection are invariant with respect to  $\mathfrak{so}_B(V)$ , i.e.  $[c, A] = [d, A] = 0$  for all  $A \in \mathfrak{so}_B(V) \subset \text{Cl}(W)$ .

From the calculations in Section 3.2 the space of Killing vector fields  $\mathcal{K}_0$  of the metric (43) is spanned by

$$\begin{aligned} K_{(+)} &= -\partial_+, & K_{(-)} &= -\partial_-, \\ K_{(i)} &= \cos(\lambda_i x^-) \partial_i + \lambda_i \sin(\lambda_i x^-) x^i \partial_+ \\ K_{(i^*)} &= -\lambda_i \sin(\lambda_i x^-) \partial_i + \lambda_i^2 \cos(\lambda_i x^-) x^i \partial_+ \\ K_{(ij)} &= x^j \partial_i - x^i \partial_j. \end{aligned} \quad (44)$$

for  $1 \leq i \leq 9$  and  $(ij) \in \{1, 2\}^2 \cup \{3, 4\}^2 \cup \{6, 7, 8, 9\}^2$ .

Analogously, for the space of spinors that are parallel with respect to the connection defined by the pair  $(\bar{c}, d)$  according to (41) the results of Section 4.3, namely (35)-(38), yield

$$\mathcal{K}_1 = \left\{ \vec{\xi} \in \mathcal{S} \left| \begin{array}{l} \vec{\xi} = \vec{\xi}(\xi_1^0, \xi_2^0) = \exp(-x^- \bar{c}) \xi_1^0 \\ \quad + \left( 1 - \frac{1}{2} \sum_i \Gamma_+ x^i s_{\bar{c},d}(e_i) \right) \exp(-x^- d) \xi_2^0, \\ \xi_1^0, \xi_2^0 \text{ constant}, \Gamma_- \xi_2^0 = \Gamma_+ \xi_1^0 = 0, X_{1234}^- \xi_2^0 = 0 \end{array} \right. \right\} \quad (45)$$

## 5. GEOMETRIC SUPERALGEBRAS

**5.1. Introduction.** In this section we will not give the definition of geometric superalgebras and geometric supersymmetry in general, but consider the special situation from Proposition 4.9. Nevertheless, we will shortly recall the idea.

<sup>2</sup>From here on we use  $\lambda_1 = \lambda_2 = -i(\alpha_- - \alpha'_+)$ ,  $\lambda_3 = \lambda_4 = -i(\alpha_- + \alpha'_+)$ ,  $\lambda_5 = -i(\alpha_+ - \alpha'_-)$ , and  $\lambda_6 = \dots = \lambda_9 = -i(\alpha_+ + \alpha'_-)$ .

A manifold  $M$  is said to admit a geometric superalgebra if there exists an extension of a Lie (sub)algebra of Killing vector fields  $\mathcal{K}_0(M)$  on  $M$  to a graded skew-symmetric superalgebra where the even part acts on the odd part as derivatives. It is called to admit geometric supersymmetry if this extension is a Lie superalgebra. The odd part of this algebra is assumed to be purely geometric in the following sense: We consider a vector bundle  $E$  over  $M$  together with a connection  $D$  such that  $\mathcal{K}_0(M)$  acts on a subspace of the space of parallel sections

$$\mathcal{K}_1(M) \subseteq \{s \in \mathcal{E} \mid Ds = 0\}. \quad (46)$$

This action then defines the even/odd-bracket of the (Lie) superalgebra. That means, there exists a map  $\mathcal{L}$  with  $\mathcal{L}_X s \in \mathcal{K}_1$  and  $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X, Y]}$  for all  $X, Y \in \mathcal{K}_0(M)$ ,  $s \in \mathcal{K}_1(M)$ . If there exists an extension with equality in (46) the extension is called non-restricted.

One non-trivial, important ingredient of such extension is the pairing  $\mathcal{K}_1(M) \times \mathcal{K}_1(M) \rightarrow \mathcal{K}_0(M)$  that is compatible with  $\mathcal{L}$  and defines the algebra structure.

The bundle  $E$  usually is a spinor bundle over the base manifold  $M$  and a geometric superalgebra or geometric supersymmetry is called irreducible if the spinor bundle is modeled on an irreducible Clifford module, say  $S_0$ . If it is modeled on the reducible Clifford module  $S_0 \otimes \mathbb{C}^N$  then we call it  $N$ -extended, see also Remark 4.5.

In our special situation we consider the following data:

- The rank-32 bundle  $S = S(M_B)$  of spinors over the eleven dimensional Cahen-Wallach space  $M_B$  that is defined by (43).
- The connection  $D$  on  $S$  defined by  $(\bar{c}, d)$  according to (41).
- The Lie algebra of Killing vector fields  $\mathcal{K}_0$  and the space  $\mathcal{K}_1 \subset \mathcal{S}$  of dimension 24 that is given by the spinors parallel with respect to  $D$ , see (44) and (45).
- The charge conjugation  $C_W = C_V \otimes \sigma_2$  on the spinor space  $S(W) = S(V) \otimes S_2$  of  $W = V \oplus \mathbb{R}^{1,1}$  is skew-symmetric and can be described in terms of the symmetric charge conjugation  $C_V$  on the spinor space  $S(V)$  of  $V = \mathbb{R}^9$  and the skew-symmetric charge conjugation  $\sigma_2$  on  $\mathbb{R}^{1,1}$ . This yields a symmetric map by  $(\vec{\xi}, \vec{\eta}) \mapsto (C_W(\vec{\xi}, \Gamma_\mu \vec{\eta}))_{\mu \in \{+, -, i\}}$ , or, in terms of the two factors of  $C_W$ ,

$$\begin{aligned} C_W(\vec{\xi}, \Gamma_- \vec{\eta}) &= -\sqrt{2}i C_V(\xi_1, \eta_1), & C_W(\vec{\xi}, \Gamma_+ \vec{\eta}) &= \sqrt{2}i C_V(\xi_2, \eta_2), \\ C_W(\vec{\xi}, \Gamma_i \vec{\eta}) &= -i C_V(\xi_1, \gamma_i \eta_2) - i C_V(\eta_1, \gamma_i \xi_2). \end{aligned} \quad (47)$$

The skew-symmetry of  $C_W$  and the symmetry of the map  $S(W) \otimes S(W) \rightarrow W$  implies that for a fixed Clifford element  $a \in \text{Cl}(W)$  of degree  $\ell$  the symmetry of  $(\vec{\xi}, \vec{\eta}) \mapsto C_W(\vec{\xi}, a \vec{\eta}) = \Delta_\ell C_W(\vec{\eta}, a \vec{\xi})$  is given by

$$\Delta_\ell = -(-1)^{\frac{\ell(\ell+1)}{2}}. \quad (48)$$

- The charge conjugation on the fibers yields a spin-invariant bilinear form on the bundle  $S$  that we will denote by  $C$ , too. If we use the splitting of the bundle introduced at the beginning of Section 4.3, and consider  $\vec{\xi} = \xi_1 + \xi_2$  and the same for  $\vec{\eta}$  we have

$$C(\vec{\xi}, \vec{\eta}) = C(\xi_1, \eta_2) - C(\eta_1, \xi_2).$$

- The action of the Killing vector fields  $\mathcal{K}_0$  on the spinors is defined by the spinorial Lie derivative

$$\mathcal{L} : \mathcal{K}_0 \times \mathcal{S} \rightarrow \mathcal{S}, \quad (K, \vec{\xi}) \mapsto \mathcal{L}_K \vec{\xi} := \nabla_K \vec{\xi} - \Gamma(\nabla K) \vec{\xi}, \quad (49)$$

see [15]. We emphasize the fact that this definition works properly only for Killing vector fields, because in this case  $\nabla K$  is indeed skew symmetric.

Nevertheless, it has to be checked, whether parallel spinors from  $\mathcal{K}_1$  stay parallel after applying the Lie derivative, or, in other word, the connection is invariant under isometries.

## 5.2. Even-Odd and Even-Even-Odd.

**Proposition 5.1.** *Consider  $\mathcal{K}_0$  and  $\mathcal{K}_1$  as defined in (44) and (45). The Lie derivative when restricted to  $\mathcal{K}_0$  acts on  $\mathcal{K}_1$ .*

For the proof of Proposition 5.1, we need to know how the operators  $\mathcal{L}_K = \nabla_K - \Gamma(\nabla K)$  for  $K \in \mathcal{K}_0$  act on spinors from  $\mathcal{K}_1$ . We will do this for the basic elements  $\{K_{(\pm)}, K_{(i)}, K_{(i^*)}, K_{(ij)}\}$ .

We will make use of

$$\nabla_\mu (K_{(-)})_\nu = \partial_\mu (K_{(-)})_\nu + \sum_\kappa \Gamma_{\mu\kappa;\nu} (K_{(-)})^\kappa = \begin{cases} \lambda_\ell^2 x^\ell & \text{for } (\mu\nu) = (-\ell), \\ -\lambda_\ell^2 x^\ell & \text{for } (\mu\nu) = (\ell-), \\ 0 & \text{else,} \end{cases}$$

$$\nabla_\mu (K_{(+)})_\nu = \partial_\mu (K_{(+)})_\nu + \sum_\kappa \Gamma_{\mu\kappa;\nu} (K_{(+)})^\kappa = 0,$$

$$\begin{aligned} \nabla_\mu (K_{(i)})_\nu &= \partial_\mu (K_{(i)})_\nu + \Gamma_{\mu-;\nu} (K_{(i)})^- + \sum_j \Gamma_{\mu j;\nu} (K_{(i)})^j \\ &= \partial_\mu (K_{(i)})_\nu + \sum_j \Gamma_{\mu j;\nu} (K_{(i)})^j = \begin{cases} -\delta_{i\ell} \beta_i & \text{for } (\mu\nu) = (-\ell), \\ \delta_{i\ell} \beta_i & \text{for } (\mu\nu) = (\ell-), \\ 0 & \text{else,} \end{cases} \end{aligned}$$

$$\nabla_\mu (K_{(i^*)})_\nu = \partial_\mu (K_{(i^*)})_\nu = \begin{cases} -\delta_{i\ell} \beta_i^* & \text{for } (\mu\nu) = (-\ell), \\ \delta_{i\ell} \beta_i^* & \text{for } (\mu\nu) = (\ell-), \\ 0 & \text{else,} \end{cases}$$

and

$$\nabla_\mu (K_{(ij)})_\nu = \partial_\mu (K_{(ij)})_\nu + \sum_\ell \Gamma_{\mu\ell;\nu} (K_{(ij)})^\ell = \begin{cases} \delta_{kj} \delta_{i\ell} - \delta_{ik} \delta_{j\ell} & \text{for } (\mu\nu) = (k\ell), \\ 0 & \text{else.} \end{cases}$$

We will also use

$$s_{\bar{c},d}(e_i) = \begin{cases} (\alpha_- - \alpha'_+) \Gamma_{125} \Gamma_i X_{1234}^+ + (\alpha_+ - \alpha'_-) \Gamma_{125} \Gamma_i X_{1234}^- & i = 1, 2 \\ (\alpha_- + \alpha'_+) \Gamma_{125} \Gamma_i X_{1234}^+ + (\alpha_+ + \alpha'_-) \Gamma_{125} \Gamma_i X_{1234}^- & i = 3, 4 \\ (\alpha_+ - \alpha'_+) \Gamma_{125} \Gamma_i X_{1234}^+ + (\alpha_- - \alpha'_-) \Gamma_{125} \Gamma_i X_{1234}^- & i = 5 \\ (\alpha_+ + \alpha'_+) \Gamma_{125} \Gamma_i X_{1234}^+ + (\alpha_- + \alpha'_-) \Gamma_{125} \Gamma_i X_{1234}^- & i = 6, \dots, 9 \end{cases}$$

which for  $\vec{\xi}(\xi_1^0, \xi_2^0)$  according to (45) becomes

$$s_{\bar{c},d}(e_i) \vec{\xi} = i \lambda_i \Gamma_{125} \Gamma_i X_{1234}^+ \vec{\xi}.$$

Furthermore we will need

$$\begin{aligned}\exp(-x^-d) &= (\cos(i\alpha'_+x^-) - i\sin(i\alpha'_+x^-)\Gamma_{125})X_{1234}^+ \\ &\quad + (\cos(i\alpha'_-x^-) - i\sin(i\alpha'_-x^-)\Gamma_{125})X_{1234}^-, \\ \exp(-x^-c) &= (\cos(i\alpha_+x^-) - i\sin(i\alpha_+x^-)\Gamma_{125})X_{1234}^+ \\ &\quad + (\cos(i\alpha_-x^-) - i\sin(i\alpha_-x^-)\Gamma_{125})X_{1234}^-.\end{aligned}$$

A careful examination yields

$$\begin{aligned}\mathcal{L}_{K_{(+)}}\vec{\xi} &= \nabla_+\vec{\xi} = 0, \\ \mathcal{L}_{K_{(-)}}\vec{\xi} &= \nabla_-\vec{\xi} + \frac{1}{2}\sum_j \lambda_j^2 x^j \Gamma_+ \Gamma_j \vec{\xi} \\ &= -\bar{c}\xi_1 - d\xi_2 + \frac{1}{2}\sum_j \lambda_j^2 x^j \Gamma_+ \vec{\xi} \\ &= -\exp(-x^-\bar{c})\bar{c}\xi_1^0 - \exp(-x^-d)d\xi_2^0 + \frac{1}{2}\Gamma_+ \sum_j x^j (\bar{c}s_{\bar{c},d}(e_j) + \lambda_j^2 \Gamma_j)\xi_2 \\ &= -\exp(-x^-\bar{c})\bar{c}\xi_1^0 - \exp(-x^-d)d\xi_2^0 + \frac{1}{2}\Gamma_+ \sum_j x^j s_{\bar{c},d}(e_j)d\xi_2 \\ &= -\vec{\xi}(\bar{c}\xi_1^0, d\xi_2^0),\end{aligned}$$

as well as

$$\begin{aligned}\mathcal{L}_{K_{(i)}}\vec{\xi} &= \alpha_i(x^-)\nabla_i\vec{\xi} + \beta_i(x^-)x^i\nabla_+\vec{\xi} + \frac{1}{2}\Gamma_+\beta_i\Gamma_i\vec{\xi} \\ &= \frac{1}{2}\Gamma_+(\alpha_i(x^-)s_{\bar{c},d}(e_i) + \beta_i(x^-)\Gamma_i)\xi_2 \\ &= \frac{i\lambda_i}{2}\Gamma_+(\cos(\lambda_i x^-)\Gamma_{125}\Gamma_i - i\sin(\lambda_i x^-)\Gamma_i)\xi_2,\end{aligned}$$

On the one hand – by recalling  $X_{1234}^+\xi_2 = \xi_2$  – we ave

$$\begin{aligned}\mathcal{L}_{K_{(i)}}\vec{\xi} &= \frac{i\lambda_i}{2}\Gamma_+(\cos(\lambda_i x^-)\Gamma_{125}\Gamma_i - i\sin(\lambda_i x^-)\Gamma_i)(\cos(i\alpha'_+x^-) - i\sin(i\alpha'_+x^-)\Gamma_{125})\xi_2^0 \\ &= \begin{cases} \frac{i\lambda_i}{2}\Gamma_+(\cos((\lambda_i + i\alpha'_+)x^-)\Gamma_{125}\Gamma_i - i\sin((\lambda_i + i\alpha'_+)x^-)\Gamma_i)\xi_2^0 & \text{for } i = 1, 2, 5 \\ \frac{i\lambda_i}{2}\Gamma_+(\cos((\lambda_i - i\alpha'_+)x^-)\Gamma_{125}\Gamma_i - i\sin((\lambda_i - i\alpha'_+)x^-)\Gamma_i)\xi_2^0 & \text{else} \end{cases} \\ &= \begin{cases} \frac{i\lambda_i}{2}\Gamma_+(\cos(i\alpha_-x^-)\Gamma_{125}\Gamma_i + i\sin(i\alpha_-x^-)\Gamma_i)\xi_2^0 & \text{for } i = 1, \dots, 4 \\ \frac{i\lambda_i}{2}\Gamma_+(\cos(i\alpha_+x^-)\Gamma_{125}\Gamma_i + i\sin(i\alpha_+x^-)\Gamma_i)\xi_2^0 & \text{for } i = 5, \dots, 9 \end{cases}\end{aligned}$$

and, on the other hand we have

$$\begin{aligned}\vec{\xi}(s_{\bar{c},d}(e_i)\xi_2^0, 0) &= \exp(-cx^-)s_{\bar{c},d}(e_i)\xi_2^0 \\ &= i\lambda_i(\cos(i\alpha_+x^-) - i\sin(i\alpha_+x^-)\Gamma_{125})X_{1234}^+\Gamma_i\Gamma_{125}\xi_2^0 \\ &\quad + i\lambda_i(\cos(i\alpha_-x^-) - i\sin(i\alpha_-x^-)\Gamma_{125})X_{1234}^-\Gamma_i\Gamma_{125}\xi_2^0 \\ &= \begin{cases} i\lambda_i(\cos(i\alpha_-x^-) - i\sin(i\alpha_-x^-)\Gamma_{125})\Gamma_{125}\Gamma_i\xi_2^0 & \text{for } i = 1, \dots, 4 \\ i\lambda_i(\cos(i\alpha_+x^-) - i\sin(i\alpha_+x^-)\Gamma_{125})\Gamma_{125}\Gamma_i\xi_2^0 & \text{for } i = 5, \dots, 6 \end{cases}\end{aligned}$$

such that

$$\mathcal{L}_{K_{(i)}} \vec{\xi} = \frac{1}{2} \vec{\xi}(\Gamma_{+s_{\bar{c},d}}(e_i) \xi_2^0, 0).$$

Doing analogous calculations for

$$\begin{aligned} \mathcal{L}_{K_{(i^*)}} \vec{\xi} &= \frac{1}{2} \Gamma_{+}(\alpha_i^*(x^-) s_{\bar{c},d}(e_i) + \beta_i^*(x^-) \Gamma_i) \xi_2 \\ &= \frac{\lambda_i^2}{2} \Gamma_{+}(\cos(\lambda_i x^-) \Gamma_i - i \sin(\lambda_i x^-) \Gamma_{125} \Gamma_i) \xi_2. \end{aligned}$$

we get

$$\mathcal{L}_{K_{(i^*)}} \vec{\xi} = -\frac{\lambda_i^2}{2} \Gamma_{+} \vec{\xi}(\Gamma_i \xi_2^0, 0).$$

In addition we obtain

$$\mathcal{L}_{K_{(ij)}} \vec{\xi} = \frac{1}{2} \vec{\xi}(\Gamma_{ij} \xi_1^0, \Gamma_{ij} \xi_2^0)$$

where we have to take into account  $(ij) \in \{1, 2\}^2 \cup \{3, 4\}^2 \cup \{6, \dots, 9\}^2$ . We collect the result as follows.

**Remark 5.2.** The Lie derivatives according to Proposition 5.1 are explicitly given by

$$\begin{aligned} \mathcal{L}_{K_{(+)}} \vec{\xi}(\xi_1^0, \xi_2^0) &= 0, \\ \mathcal{L}_{K_{(-)}} \vec{\xi}(\xi_1^0, \xi_2^0) &= -\vec{\xi}(\bar{c} \xi_1^0, d \xi_2^0), \\ \mathcal{L}_{K_{(i)}} \vec{\xi}(\xi_1^0, \xi_2^0) &= -\vec{\xi}(-\frac{1}{2} \Gamma_{+s_{\bar{c},d}}(e_i) \xi_2^0, 0), \\ \mathcal{L}_{K_{(i^*)}} \vec{\xi}(\xi_1^0, \xi_2^0) &= -\vec{\xi}(\frac{1}{2} \Gamma_{+} B(e_i) \xi_2^0, 0), \\ \mathcal{L}_{K_{(ij)}} \vec{\xi}(\xi_1^0, \xi_2^0) &= -\vec{\xi}(-\frac{1}{2} \Gamma_{ij} \xi_1^0, -\frac{1}{2} \Gamma_{ij} \xi_2^0). \end{aligned}$$

For  $\mu \in \{\pm, i, i^*, ij\}$  we can rewrite this as

$$\mathcal{L}_{K_{(\mu)}} \vec{\xi}(\xi_1^0, \xi_2^0) = -\vec{\xi}(\rho(e_\mu)(\xi_1^0, \xi_2^0)) \quad (50)$$

with  $\rho$  according to Proposition 4.1 and Remark 4.2 as well as  $e_{i^*} := e_i^*$  and  $\mathfrak{so}_B(V) = \text{span}\{e_{ij}\}$  with  $\rho(e_{ij}) = -\frac{1}{2} \Gamma_{ij}$ .

### 5.3. Odd-Odd and Even-Odd-Odd.

**Definition 5.3.** We use the charge conjugation  $C$  on  $S$  to define a symmetric map

$$\mathcal{K}_1 \otimes \mathcal{K}_1 \rightarrow \mathcal{K}_0.$$

Motivated by (47), we consider the projection

$$\mathcal{K}_1 \otimes \mathcal{K}_1 \rightarrow \text{span}\{K_{(+)}, K_{(-)}, K_{(i)}\} \subset \mathcal{K}_0.$$

given by

$$\{\vec{\xi}, \vec{\eta}\}^W = \{\vec{\xi}, \vec{\eta}\}^+ K_{(+)} + \{\vec{\xi}, \vec{\eta}\}^- K_{(-)} + \sum_i \{\vec{\xi}, \vec{\eta}\}^i K_{(i)} \quad (51)$$

with

$$\begin{aligned} \{\vec{\xi}, \vec{\eta}\}^- &= C(\xi_2^0, \Gamma_{+} \eta_2^0), \quad \{\vec{\xi}, \vec{\eta}\}^+ = C(\xi_1^0, \Gamma_{-} \eta_1^0), \\ \{\vec{\xi}, \vec{\eta}\}^i &= C(\xi_1^0, \Gamma_i \eta_2^0) + C(\eta_1^0, \Gamma_i \xi_2^0). \end{aligned} \quad (52)$$

We complete this projection to  $\{\cdot, \cdot\}$  by the two maps

$$\mathcal{K}_1 \otimes \mathcal{K}_1 \rightarrow \text{span}\{K_{(i^*)}\} \subset \mathcal{K}_0, \quad \{\vec{\xi}, \vec{\eta}\}^* = \sum_i \{\vec{\xi}, \vec{\eta}\}^{i^*} K_{(i^*)}, \quad (53)$$

and

$$\mathcal{K}_1 \otimes \mathcal{K}_1 \rightarrow \text{span}\{K_{(ij)}\} \subset \mathcal{K}_0, \quad \{\vec{\xi}, \vec{\eta}\}^{so} = \frac{1}{2} \sum_{ij} \{\vec{\xi}, \vec{\eta}\}^{ij} K_{(ij)}. \quad (54)$$

The coefficients therein are defined by

$$\begin{aligned} \{\vec{\xi}, \vec{\eta}\}^{i^*} &= C(\xi_1^0, s_{\bar{c}, d}(B^{-1}(e_i))\eta_2^0) - C(s_{\bar{c}, d}(B^{-1}(e_i))\xi_2^0, \eta_1^0) \\ &= \frac{i}{\lambda_i} C(\xi_1^0, \Gamma_{125}\Gamma_i\eta_2^0) + \frac{i}{\lambda_i} C(\eta_1^0, \Gamma_{125}\Gamma_i\xi_2^0) \end{aligned} \quad (55)$$

and

$$\begin{aligned} \{\vec{\xi}, \vec{\eta}\}^{ij} &= -\frac{1}{2} C(\xi_2^0, \Gamma_+(s_{d, \bar{c}}(e_j)\Gamma_i + \Gamma_i s_{\bar{c}, d}(e_j))\eta_2^0) \\ &= \begin{cases} i\lambda_1 C(\xi_2^0, \Gamma_+\Gamma_5\xi_2^0) & \text{for } (ij) = (12) \\ i\lambda_3 C(\xi_2^0, \Gamma_+\Gamma_{12345}\xi_2^0) & \text{for } (ij) = (34) \\ i\lambda_6 C(\xi_2^0, \Gamma_+\Gamma_{125ij}\xi_2^0) & \text{for } (ij) \in \{6, \dots, 9\}^2 \end{cases} \\ &=: i\epsilon_j \lambda_j C(\xi_2^0, \Gamma_+\Gamma_{125}\Gamma_{ij}\xi_2^0) \text{ for } (ij) \in \{1, 2\}^2 \cup \{3, 4\}^2 \cup \{6, \dots, 9\}^2 \end{aligned} \quad (56)$$

In (52), (55) and (56) the spinors  $\xi_1^0, \xi_2^0$  and  $\eta_1^0, \eta_2^0$  are the constant spinors that define the parallel spinors  $\vec{\xi}$  and  $\vec{\eta}$ , see Remark 4.7 and the calculations before.

**Remark 5.4.** The construction in Definition 5.3 can be made more general such that it provides a superalgebra for a wide class of connections according to Proposition 4.1. More precisely, it can be shown that this is the only possible algebra structure. This is used in [12] to start a systematic classification of supersymmetric extensions of Cahen-Wallach spaces that in particular covers the examples found in the literature, see [4–11, 16], for example. A first attempt of such systematic treatment has been started in [17] but with the restriction to  $\{\cdot, \cdot\} = \{\cdot, \cdot\}^W$ , which turns out to be too restrictive for allowing a superalgebra with non-trivial odd-odd bracket.

**Proposition 5.5.** *For any  $K \in \mathcal{K}_0$  and  $\xi \in \mathcal{K}_1$*

$$[K, \{\vec{\xi}, \vec{\xi}\}] = 2\{\mathcal{L}_K \vec{\xi}, \vec{\xi}\}.$$

The statement of Proposition 5.5 is clear for  $K = K_{(+)}$  such that we may restrict to  $K \in \{K_{(-)}, K_{(i)}, K_{(i^*)}, K_{(ij)}\}$ . The remaining proof needs the symmetry of the charge conjugation as stated in (48). For  $\vec{\xi} = \vec{\xi}(\xi_1^0, 0)$  we have

$$[K, \{\vec{\xi}, \vec{\xi}\}] = C(\xi_1^0, \Gamma_-\xi_1^0)[K, K_{(+)}] = 0 \text{ and } \{\mathcal{L}_K \vec{\xi}, \vec{\xi}\} = 0.$$

The last equation is only non obvious for  $K = K_{(-)}, K_{(ij)}$  and is then due to

$$\begin{aligned} C(\bar{c}\xi_1^0, \Gamma_-\xi_1^0) &= \Delta_0\Delta_3 C(\xi_1^0, \bar{c}\Gamma_-\xi_1^0) = C(\xi_1^0, \bar{c}\Gamma_-\xi_1^0) \\ &= \Delta_4 C(\xi_1^0, \bar{c}\Gamma_-\xi_1^0) = -C(\xi_1^0, \bar{c}\Gamma_-\xi_1^0), \\ C(\Gamma_{ij}\xi_1^0, \Gamma_-\xi_1^0) &= \Delta_0\Delta_2 C(\xi_1^0, \Gamma_{ij}\Gamma_-\xi_1^0) = -C(\xi_1^0, \Gamma_{ij}\Gamma_-\xi_1^0) \\ &= -\Delta_3 C(\xi_1^0, \Gamma_{ij}\Gamma_-\xi_1^0) = C(\xi_1^0, \Gamma_{ij}\Gamma_-\xi_1^0). \end{aligned}$$



If we consider  $\vec{\xi} = \vec{\xi}(0, \xi_2^0)$  we get

$$\begin{aligned} [K_{(-)}, \{\vec{\xi}, \vec{\xi}\}] &= \{\vec{\xi}, \vec{\xi}\}^- [K_{(-)}, K_{(-)}] + \frac{1}{2} \sum_{ij} \{\vec{\xi}, \vec{\xi}\}^{ij} [K_{(-)}, K_{(ij)}] = 0, \\ \{\mathcal{L}_{K_{(-)}}, \vec{\xi}, \vec{\xi}\} &= \{\vec{\xi}(0, d\xi_2^0), \vec{\xi}(0, \xi_2^0)\} \\ &= C(\xi_2^0, \Gamma + d\xi_2^0) K_{(-)} + \frac{i\lambda_6}{2} \sum_{ij=6}^9 C(\xi_2^0, \Gamma + \Gamma_{125ij} d\xi_2^0) K_{(ij)} \\ &\quad + i\lambda_1 C(\xi_2^0, \Gamma + \Gamma_5 d\xi_2^0) K_{(12)} + i\lambda_3 C(\xi_2^0, \Gamma + \Gamma_{12345} d\xi_2^0) K_{(34)} \\ &= 0, \end{aligned}$$

because for  $(ij) \in \{3, 4\}^2 \cup \{6, \dots, 9\}^2$  we have

$$\begin{aligned} C(\xi_2^0, \Gamma + \Gamma_{125ij} d\xi_2^0) &= \Delta_6 C(d\xi_2^0, \Gamma + \Gamma_{125ij} \xi_2^0) = \Delta_6 \Delta_0 \Delta_3 C(\xi_2^0, d\Gamma + \Gamma_{125ij} \xi_2^0) \\ &= -\Delta_6 \Delta_0 \Delta_3 C(\xi_2^0, \Gamma + \Gamma_{125ij} d\xi_2^0) = -C(\xi_2^0, \Gamma + \Gamma_{125ij} d\xi_2^0), \\ C(\xi_2^0, \Gamma + \Gamma_5 d\xi_2^0) &= \Delta_2 C(d\xi_2^0, \Gamma + \Gamma_5 \xi_2^0) = \Delta_2 \Delta_0 \Delta_3 C(\xi_2^0, d\Gamma + \Gamma_5 \xi_2^0) \\ &= -\Delta_2 \Delta_0 \Delta_3 C(\xi_2^0, \Gamma + \Gamma_5 d\xi_2^0) = -C(\xi_2^0, \Gamma + \Gamma_5 d\xi_2^0), \\ C(\xi_2^0, \Gamma + d\xi_2^0) &= \Delta_1 C(d\xi_2^0, \Gamma + \xi_2^0) = \Delta_1 \Delta_0 \Delta_3 C(\xi_2^0, d\Gamma + \xi_2^0) \\ &= -\Delta_1 \Delta_0 \Delta_3 C(\xi_2^0, \Gamma + d\xi_2^0) = -C(\xi_2^0, \Gamma + \Gamma_5 d\xi_2^0). \end{aligned}$$

For  $K = K_{(k\ell)}$  we get

$$\begin{aligned} [K_{(k\ell)}, \{\vec{\xi}, \vec{\xi}\}] &= \{\vec{\xi}, \vec{\xi}\}^- [K_{(k\ell)}, K_{(-)}] + \frac{1}{2} \sum_{ij} \{\vec{\xi}, \vec{\xi}\}^{ij} [K_{(k\ell)}, K_{(ij)}] \\ &= -i \sum_{ij} \epsilon_j \lambda_j C(\xi_2^0, \Gamma + \Gamma_{125} \Gamma_{ij} \xi_2^0) \delta_{i[k} K_{(\ell)j]} \\ &= \begin{cases} 0 & \text{for } (k\ell) = (12), (34) \\ i\lambda_6 \sum_{ij=6}^9 C(\xi_2^0, \Gamma + \Gamma_{125} \Gamma_{j[k} \xi_2^0) K_{(\ell)j]} & \text{for } (k\ell) \in \{6, \dots, 9\}^2 \end{cases} \\ 2\{\mathcal{L}_{K_{(k\ell)}}, \vec{\xi}, \vec{\xi}\} &= \{\vec{\xi}(0, \Gamma_{k\ell} \xi_2^0), \vec{\xi}(0, \xi_2^0)\} \\ &= \underbrace{C(\xi_2^0, \Gamma + \Gamma_{k\ell} \xi_2^0)}_{=0, \Delta_3=-1} K_{(-)} + \frac{i\lambda_6}{2} \sum_{ij=6}^9 C(\xi_2^0, \Gamma + \Gamma_{125ij} \Gamma_{k\ell} \xi_2^0) K_{(ij)} \\ &\quad + i\lambda_1 \underbrace{C(\xi_2^0, \Gamma + \Gamma_5 \Gamma_{k\ell} \xi_2^0)}_{=0, \Delta_4=-1} K_{(12)} + i\lambda_3 \underbrace{C(\xi_2^0, \Gamma + \Gamma_{12345} \Gamma_{k\ell} \xi_2^0)}_{=0, \Delta_4=\Delta_8=-1} K_{(34)} \\ &= \frac{i\lambda_6}{2} \sum_{ij=6}^9 C(\xi_2^0, \Gamma + \Gamma_{125} \Gamma_{ij} \Gamma_{k\ell} \xi_2^0) K_{(ij)} \\ &= \begin{cases} 0 & \text{for } (k\ell) = (12), (34) \\ i\lambda_6 \sum_{ij=6}^9 C(\xi_2^0, \Gamma + \Gamma_{125} \Gamma_{j[k} \xi_2^0) K_{(\ell)j]} & \text{for } (k\ell) \in \{6, \dots, 9\}^2 \end{cases} \end{aligned}$$

For  $K = K_{(k)}$  we get

$$[K_{(k)}, \{\vec{\xi}, \vec{\xi}\}] = \{\vec{\xi}, \vec{\xi}\}^- [K_{(k)}, K_{(-)}] + \frac{1}{2} \sum_{ij} \{\vec{\xi}, \vec{\xi}\}^{ij} [K_{(k)}, K_{(ij)}]$$

$$\begin{aligned}
&= C(\xi_2^0, \Gamma + \xi_2^+) K_{(k^*)} + \frac{1}{2} \sum_{ij} \{\vec{\xi}, \vec{\xi}\}_{ij} (\delta_{kj} K_{(i)} - \delta_{ki} K_{(j)}) \\
&= C(\xi_2^0, \Gamma + \xi_2^+) K_{(k^*)} + \{\vec{\xi}, \vec{\xi}\}_{12} (\delta_{k2} K_{(1)} - \delta_{k1} K_{(2)}) \\
&\quad + \{\vec{\xi}, \vec{\xi}\}_{34} (\delta_{k4} K_{(3)} - \delta_{k3} K_{(4)}) + \sum_{j=6}^9 \{\vec{\xi}, \vec{\xi}\}_{jk} K_{(j)}, \\
2\{\mathcal{L}_{K_{(k)}} \vec{\xi}, \vec{\xi}\} &= \{\vec{\xi}(\Gamma + s_{\bar{c}, d}(e_k) \xi_2^0, 0), \vec{\xi}(0, \xi_2^0)\} \\
&= \sum_j \frac{i}{\lambda_j} C(\Gamma + s_{\bar{c}, d}(e_k) \xi_2^0, \Gamma_{125} \Gamma_j \xi_2^0) K_{(j^*)} \\
&\quad + \sum_j C(\Gamma + s_{\bar{c}, d}(e_k) \xi_2^0, \Gamma_j \xi_2^0) K_{(j)} \\
&= - \sum_j \frac{\lambda_k}{\lambda_j} C(\xi_2^0, \Gamma + X_{1234}^+ \Gamma_k \Gamma_j X_{1234}^+ \xi_2^0) K_{(j^*)} \\
&\quad + i\lambda_k \sum_j C(\xi_2^0, \Gamma + X_{1234}^+ \Gamma_k \Gamma_{125} \Gamma_j \xi_2^0) K_{(j)} \\
&= C(\xi_2^0, \Gamma + \xi_2^0) K_{(j^*)} \\
&\quad + i\lambda_1 C(\xi_2^0, \Gamma + \Gamma_5 \xi_2^0) (\delta_{k2} K_{(1)} - \delta_{k1} K_{(2)}) \\
&\quad + i\lambda_3 C(\xi_2^0, \Gamma + \Gamma_{12345} \xi_2^0) (\delta_{k4} K_{(3)} - \delta_{k3} K_{(4)}) \\
&\quad + i\lambda_6 \sum_{j=6}^9 C(\xi_2^0, \Gamma + \Gamma_{125jk} \xi_2^0) K_{(j)}.
\end{aligned}$$

In the last step all other summands vanish because of skew-symmetry, e.g.

$$\lambda_3 C(\xi_2^0, \Gamma + X_{1234}^+ \Gamma_4 \Gamma_{125} \Gamma_5 \xi_2^0) K_{(5)} = -\lambda_3 C(\xi_2^0, \Gamma + \Gamma_{124} \xi_2^0) K_{(5)} \stackrel{\Delta_4 = -1}{=} 0,$$

or because  $X_{1234}^- \xi_2^0 = 0$ , e.g.

$$\lambda_1 C(\xi_2^0, \Gamma + X_{1234}^+ \Gamma_5 \Gamma_{125} \Gamma_1 \xi_2^0) K_{(1)} = -\lambda_1 C(\xi_2^0, \Gamma + \Gamma_2 X_{1234}^- \xi_2^0) K_{(1)} = 0.$$

By similar arguments, we get that for  $K = K_{(k^*)}$  the following terms coincide:

$$\begin{aligned}
[K_{(k^*)}, \{\vec{\xi}, \vec{\xi}\}] &= \{\vec{\xi}, \vec{\xi}\}^- [K_{(k^*)}, K_{(-)}] + \frac{1}{2} \sum_{ij} \{\vec{\xi}, \vec{\xi}\}^{ij} [K_{(k^*)}, K_{(ij)}] \\
2\{\mathcal{L}_{K_{(k^*)}} \vec{\xi}, \vec{\xi}\} &= -\lambda_k^2 \{\vec{\xi}(\Gamma + \Gamma_k \xi_2^0, 0), \vec{\xi}(0, \xi_2^0)\}.
\end{aligned}$$

We collect the result of Section 5 in the following statement.

**Theorem 5.6.** *The indecomposable Cahen-Wallach space  $M_B$  and the connection  $D$  on its spinor bundle according to Proposition 4.9 define a non-restricted geometric superalgebra.*

- The even and odd part,  $\mathcal{K}_0$  and  $\mathcal{K}_1$ , of the underlying graded vector space are given by (44) and (45), respectively.
- The product structure is given by the usual commutator on  $\mathcal{K}_0$  and completed by the even-odd bracket defined by the Lie derivative, see Remark 5.2, and the odd-odd bracket according to (52), (55), and (56) from Definition 5.3.

We end this section with a short comment on the question if the odd part of the geometric superalgebra is minimal in a certain sense.

**Remark 5.7.** Because  $K_{(i^*)}$  acts by  $\mathcal{L}_{K_{(i^*)}}\vec{\xi}(0, \xi_2^0) = \lambda_i \vec{\xi}(\Gamma_+ \Gamma_i \xi_2^0, 0)$  we have

$$\mathcal{L}_{K_{(i^*)}}\vec{\xi}(0, \xi_2^0) \in \begin{cases} X_{1234}^+ S & \text{for } i = 5, \dots, 9, \\ X_{1234}^- S & \text{for } i = 1, \dots, 4. \end{cases}$$

Therefore, a reduction of  $\mathcal{K}_1$  in Theorem 5.6 is only possible if some of the eigenvalues of  $B$  vanish, i.e. if  $M_B$  is decomposable.<sup>3</sup>

There are special configurations of parameters in the metric (43) that yield decomposable spaces, namely  $\alpha_+ = \pm \alpha'_+$  and  $\alpha_- = \pm \alpha'_-$ . In fact a reduction is only possible if  $\alpha_- = \alpha'_+ = 0$  or  $\alpha_+ = \alpha'_- = 0$ .

## 6. GEOMETRIC SUPERSYMMETRY

**6.1. Odd-Odd-Odd.** In this section we will show that for a special set of parameters the geometric superalgebra from Theorem 5.6 in fact defines geometric supersymmetry.

We recall the fact that a superalgebra is a Lie superalgebra if the graded Jacobi identity is fulfilled, i.e. for all elements  $x, y, z$  we have

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||z|}[z, [x, y]] + (-1)^{|x||y|}[y, [z, x]] = 0$$

where  $|\cdot|$  denotes the  $\mathbb{Z}_2$ -degree.

If a Cahen-Wallach space  $M_B$  – or, more precisely,  $\mathcal{K}_0 \oplus \mathcal{K}_1$  – defines a geometric superalgebra, the only obstruction to geometric supersymmetry is the odd-odd-odd bracket. Furthermore, due to polarization it is enough to ask for the vanishing of

$$\begin{aligned} \mathcal{L}_{\{\vec{\xi}, \vec{\xi}\}}\vec{\xi} &= \{\vec{\xi}, \vec{\xi}\}^+ \mathcal{L}_{K_{(+)}}\vec{\xi} + \{\vec{\xi}, \vec{\xi}\}^- \mathcal{L}_{K_{(-)}}\vec{\xi} + \sum_i \{\vec{\xi}, \vec{\xi}\}^i \mathcal{L}_{K_{(i)}}\vec{\xi} \\ &+ \sum_i \{\vec{\xi}, \vec{\xi}\}^{i^*} \mathcal{L}_{K_{(i^*)}}\vec{\xi} + \frac{1}{2} \sum_{ij} \{\vec{\xi}, \vec{\xi}\}^{ij} \mathcal{L}_{K_{(ij)}}\vec{\xi} \end{aligned} \quad (57)$$

for all  $\vec{\xi} \in \mathcal{K}_1$ .

In our situation we use the notations from Section 5, in particular (50), (52), (55), and (56), such that the vanishing of  $\mathcal{L}_{\{\vec{\xi}, \vec{\xi}\}}\vec{\xi}$  for  $\vec{\xi} = \vec{\xi}(\xi_1^0, \xi_2^0)$  is

$$\begin{aligned} 0 &= -C(\xi_2^0, \Gamma_+ \xi_2^0) \vec{\xi}(\bar{\alpha} \xi_1^0, d\xi_2^0) + \frac{i\lambda_6}{4} \sum_{ij=6}^9 C(\xi_2^0, \Gamma_+ \Gamma_{125ij} \xi_2^0) \vec{\xi}(\Gamma_{ij} \xi_1^0, \Gamma_{ij} \xi_2^0) \\ &+ \frac{i\lambda_1}{2} C(\xi_2^0, \Gamma_+ \Gamma_5 \xi_2^0) \vec{\xi}(\Gamma_{12} \xi_1^0, \Gamma_{12} \xi_2^0) + \frac{i\lambda_3}{2} C(\xi_2^0, \Gamma_+ \Gamma_{12345} \xi_2^0) \vec{\xi}(\Gamma_{34} \xi_1^0, \Gamma_{34} \xi_2^0) \\ &+ i \sum_i \lambda_i C(\xi_1^0, \Gamma_i \xi_2^0) \vec{\xi}(\Gamma_+ \Gamma_{125} \Gamma_i \xi_2^0, 0) - i \sum_i \lambda_i C(\xi_1^0, \Gamma_{125} \Gamma_i \xi_2^0) \vec{\xi}(\Gamma_+ \Gamma_i \xi_2^0, 0). \end{aligned}$$

<sup>3</sup>As reduction we consider only those for which the resulting algebra is nontrivial, i.e.  $\Gamma_+ \mathcal{K}_1 \neq 0$ .

Together with  $\vec{\xi}(\xi_1^0, \xi_2^0) = 0 \Leftrightarrow \xi_1^0 = \xi_2^0 = 0$  this yields the following two equations for the two constant spinors

$$0 = -\alpha'_+ C(\xi_2^0, \Gamma_+ \xi_2^0) \Gamma_{125} \xi_2^0 + \frac{\alpha_+ + \alpha'_+}{4} \sum_{ij=6}^9 C(\xi_2^0, \Gamma_+ \Gamma_{125ij} \xi_2^0) \Gamma_{ij} \xi_2^0 \\ + \frac{\alpha_- - \alpha'_+}{2} C(\xi_2^0, \Gamma_+ \Gamma_5 \xi_2^0) \Gamma_{12} \xi_2^0 + \frac{\alpha_- + \alpha'_+}{2} C(\xi_2^0, \Gamma_+ \Gamma_{12345} \xi_2^0) \Gamma_{34} \xi_2^0$$

and

$$0 = -\frac{\alpha_+}{2} C(\xi_2^0, \Gamma_+ \xi_2^0) (\Gamma_{125} - \Gamma_{345}) \xi_1^0 - \frac{\alpha_-}{2} C(\xi_2^0, \Gamma_+ \xi_2^0) (\Gamma_{125} + \Gamma_{345}) \xi_1^0 \\ + \frac{\alpha_- - \alpha'_+}{2} C(\xi_2^0, \Gamma_+ \Gamma_5 \xi_2^0) \Gamma_{12} \xi_1^0 + \frac{\alpha_- + \alpha'_+}{2} C(\xi_2^0, \Gamma_+ \Gamma_{12345} \xi_2^0) \Gamma_{34} \xi_1^0 \\ + \frac{\alpha_+ + \alpha'_+}{4} \sum_{ij=6}^9 C(\xi_2^0, \Gamma_+ \Gamma_{125ij} \xi_2^0) \Gamma_{ij} \xi_1^0 \\ + (\alpha_- - \alpha'_+) \sum_{i=1}^2 (C(\xi_1^0, \Gamma_i \xi_2^0) \Gamma_{125} \Gamma_i \xi_2^0 - C(\xi_1^0, \Gamma_{125} \Gamma_i \xi_2^0) \Gamma_i \xi_2^0) \\ + (\alpha_- + \alpha'_+) \sum_{i=3}^4 (C(\xi_1^0, \Gamma_i \xi_2^0) \Gamma_{125} \Gamma_i \xi_2^0 - C(\xi_1^0, \Gamma_{125} \Gamma_i \xi_2^0) \Gamma_i \xi_2^0) \\ - (\alpha_+ - \alpha'_+) (C(\xi_1^0, \Gamma_5 \xi_2^0) \Gamma_{12} \xi_2^0 - C(\xi_1^0, \Gamma_{12} \xi_2^0) \Gamma_5 \xi_2^0) \\ + (\alpha_+ + \alpha'_+) \sum_{i=6}^9 (C(\xi_1^0, \Gamma_i \xi_2^0) \Gamma_{125} \Gamma_i \xi_2^0 - C(\xi_1^0, \Gamma_{125} \Gamma_i \xi_2^0) \Gamma_i \xi_2^0).$$

We may rewrite this in terms of  $V = \mathbb{R}^9$  by considering  $\xi_1^0, \xi_2^0 \in S(V)$  and using the conventions from Section A

$$0 = -\alpha'_+ C_V(\xi_2^0, \xi_2^0) \gamma_{125} \xi_2^0 + \frac{\alpha_+ + \alpha'_+}{4} \sum_{ij=6}^9 C_V(\xi_2^0, \gamma_{125ij} \xi_2^0) \gamma_{ij} \xi_2^0 \\ + \frac{\alpha_- - \alpha'_+}{2} C_V(\xi_2^0, \gamma_5 \xi_2^0) \gamma_{12} \xi_2^0 + \frac{\alpha_- + \alpha'_+}{2} C_V(\xi_2^0, \gamma_{12345} \xi_2^0) \gamma_{34} \xi_2^0 \quad (58)$$

and

$$0 = \frac{\alpha_+ + \alpha_-}{2} C_V(\xi_2^0, \xi_2^0) \gamma_{125} \xi_1^0 - \frac{\alpha_+ - \alpha_-}{2} C_V(\xi_2^0, \xi_2^0) \gamma_{345} \xi_1^0 \\ + \frac{\alpha_- - \alpha'_+}{2} C_V(\xi_2^0, \gamma_5 \xi_2^0) \gamma_{12} \xi_1^0 + \frac{\alpha_- + \alpha'_+}{2} C_V(\xi_2^0, \gamma_{12345} \xi_2^0) \gamma_{34} \xi_1^0 \\ + \frac{\alpha_+ + \alpha'_+}{4} \sum_{ij=6}^9 C_V(\xi_2^0, \gamma_{125ij} \xi_2^0) \gamma_{ij} \xi_1^0 \\ - 2(\alpha_- - \alpha'_+) \sum_{i=1}^2 (C_V(\xi_1^0, \gamma_i \xi_2^0) \delta_{i[1\gamma_2]5} \xi_2^0 - C_V(\xi_1^0, \delta_{i[1\gamma_2]5} \xi_2^0) \gamma_i \xi_2^0) \\ + (\alpha_- + \alpha'_+) \sum_{i=3}^4 (C_V(\xi_1^0, \gamma_i \xi_2^0) \gamma_{125i} \xi_2^0 - C_V(\xi_1^0, \gamma_{125i} \xi_2^0) \gamma_i \xi_2^0) \\ - (\alpha_+ - \alpha'_+) (C_V(\xi_1^0, \gamma_5 \xi_2^0) \gamma_{12} \xi_2^0 - C_V(\xi_1^0, \gamma_{12} \xi_2^0) \gamma_5 \xi_2^0) \quad (59)$$

$$+ (\alpha_+ + \alpha'_+) \sum_{i=6}^9 (C_V(\xi_1^0, \gamma_i \xi_2^0) \gamma_{125i} \xi_2^0 - C_V(\xi_1^0, \gamma_{125i} \xi_2^0) \gamma_i \xi_2^0). \quad (60)$$

A direct computation shows that (59) and (58) are obtained only for  $\alpha_+ = -3\alpha'_+$ .

**Theorem 6.1.** *The geometric superalgebra of the indecomposable Cahen-Wallach space  $M_B$  according to Theorem 5.6 yields non restricted geometric supersymmetry if and only if*

$$B = -\text{diag}((\alpha_- - \alpha'_+)^2 \mathbb{1}_2, (\alpha_- + \alpha'_+)^2 \mathbb{1}_2, 16\alpha'_+{}^2 \mathbb{1}_1, 4\alpha'_+{}^2 \mathbb{1}_4)$$

and  $(\bar{c}, d)$  given by<sup>4</sup>

$$\bar{c} = (-3\alpha'_+ X_{1234}^+ + \alpha_- X_{1234}^-) \Gamma_{125}, \quad d = \alpha'_+ X_{1234}^+ \Gamma_{125}.$$

## 7. THE MODULI SPACE OF GEOMETRIC SUPERSYMMETRY

### 7.1. The moduli space of geometric superalgebras and supersymmetries.

The moduli space of geometric superalgebras according to Theorem 5.6 is naturally parameterized by

$$(\alpha_-, \alpha'_+, \alpha_+) \in \mathbb{R}^3 \setminus \{(\pm\alpha'_+, \alpha'_+, \alpha_+), (\alpha_-, \alpha'_+, \pm\alpha'_+) \mid \alpha_+, \alpha_-, \alpha'_+ \in \mathbb{R}\}.$$

If we omit the euclidean configuration and divide out the isometries defined by the action of positive(!) scalars as well as the isometries defined by  $(\alpha_-, \alpha'_+, \alpha_+) \sim (-\alpha_-, \alpha'_+, \alpha_+)$  we obtain a subset of the closed disc

$$(\alpha'_+, \alpha_+) \in \mathcal{C}' = D^2 \setminus \{(\alpha'_+, \alpha_+) \mid \alpha_+ = \pm\alpha'_+ \text{ or } \alpha_+^2 + 2\alpha'_+{}^2 = 1\}, \quad (61)$$

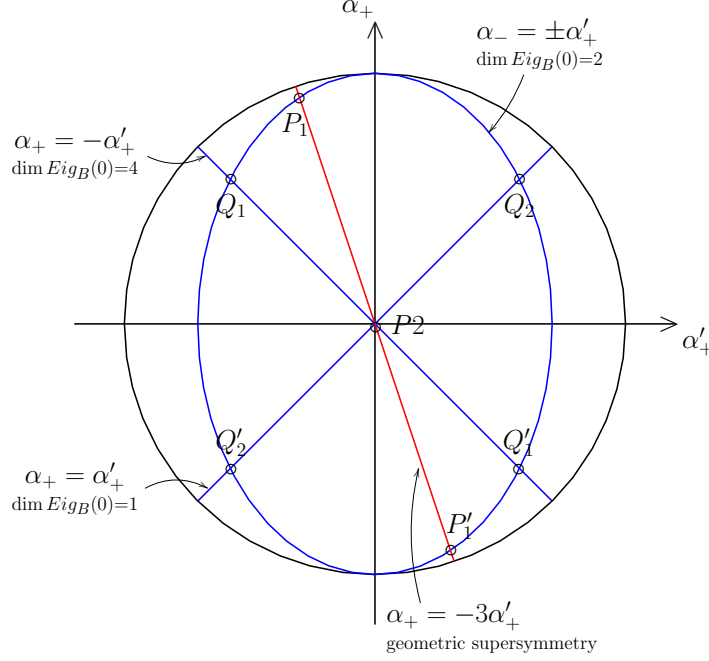
see Figure 1.

**Proposition 7.1.** *The compactified moduli space of geometric superalgebras according to Theorem 5.6 is  $\hat{\mathcal{C}}' = D^2$  and (61). Its obtained by adding the decomposable, non-euclidean configurations indicated by the diagonals and the ellipse in Figure 1.*

*If we divide out the remaining isometries defined by the antipodal map the resulting space is  $\hat{\mathcal{C}} = D^2$ , too, because  $S^1 = \mathbb{R}P^1$ . More precisely, the result is an  $S^1$ -cone over the base point  $P_2$  and the decomposable spaces are associated to conic sections: Two different line segments as well as one ellipse.*

---

<sup>4</sup>Strictly speaking, we may add an arbitrary summand  $\beta' X_{1234}^- \Gamma_{125}$  to  $d$  without changing the result, because  $X_{1234}^- \xi_2^0 = 0$ .

FIGURE 1. The moduli space  $\hat{\mathcal{C}}'$  of geometric superalgebras


The geometric supersymmetries according to Theorem 6.1 are parameterized by

$$(\alpha_-, \alpha'_+) \in \mathbb{R}^2 \setminus \{(\alpha_-, \pm\alpha_-), (\alpha_-, 0) \mid \alpha_- \in \mathbb{R}\}.$$

If we divide out the isometries given by the conformal equivalence as mentioned in Remark 3.1 as well as the isometries given by  $(\alpha'_+, \alpha_-) \mapsto (\alpha_+, \alpha_-)$  we get the following result.

**Proposition 7.2.** *The moduli space of geometric supersymmetries according to Theorem 6.1 is given by*

$$\mathcal{C}_0 = (0, \frac{\pi}{4}) \cup (\frac{\pi}{4}, \frac{\pi}{2}].$$

*The compactification of the moduli space is done by adding the decomposable, non-euclidean spaces. The result is the compact interval*

$$\hat{\mathcal{C}}_0 = [0, \frac{\pi}{2}].$$

**Remark 7.3.** The moduli of geometric supersymmetries can be identified with the line  $\alpha_+ = -3\alpha'_+$  in Figure 1, at least after identifying via antipodal map. The correspondence  $\hat{\mathcal{C}}_0 \rightarrow \hat{\mathcal{C}}'$  is

$$\phi \mapsto \frac{\pm 1}{\sqrt{1 + 9 \sin^2 \phi}} (\sin \phi, -3 \sin \phi).$$

**7.2. The singular points as  $N$ -extended supersymmetries.** As we saw above, there are two configurations of parameters in the compactified moduli  $\hat{\mathcal{C}}_0$  of geometric supersymmetries that yield decomposable spaces. They are associated to the points  $P_1 = P'_1$  and  $P_2$  in Figure 1, respectively

i.  $P_2$  ( $\alpha'_+ = 0$ ) with

$$B_0 = -\alpha_-^2 \text{diag}(\mathbb{1}_4, \mathbb{O}_5) .$$

In this case we have  $c = \alpha_- \Gamma_{125} X_{1234}^-$  and  $d = 0$ . Therefore, the action on  $\sigma_- X^+ S$  is trivial, and we may further reduce  $\mathcal{K}_1$  to

$$X_{1234}^- S_{11}^+ \oplus X_{1234}^+ S_{11}^- \subset S_{11}^+ \oplus S_{11}^- = S_{11} .$$

ii.  $P_1$  ( $\alpha'_+ = \pm \alpha_-$ ) with

$$B_{\pm} = -4\alpha_-^2 \text{diag}(4\mathbb{1}_1, \mathbb{1}_6, \mathbb{O}_2) .$$

In this case a further reduction is not possible.

We emphasize the fact, that in both cases i. and ii. conditions (58) and (59) don't see the Killing vector fields associated to the zero eigenvalues of  $B$ .

The geometric supersymmetries on the decomposable eleven dimensional spaces that are associated to the singular points of the moduli space can be interpreted as  $N$ -extended geometric supersymmetries in lower dimensions  $D = 11 - d$ , at least if we restrict the even part  $\mathcal{K}_0$  in a suitable way, i.e. to  $i, i^* \in \{1, \dots, D - 2\}$ .

i. The singular point  $P_2$  can be associated to restricted  $\nu = 1/2$ , 4-extended geometric supersymmetry in six dimensions. The ingredients are as follows

- The  $D = 6$  Cahen-Wallach space  $M_6$  associated to  $B = -\beta^2 \mathbb{1}_4$ .
- The spinor bundle  $S = S(M_6) \otimes \mathbb{C}^4$ ,
- The bilinear form  $C = C_6 \otimes C_5$  with  $C_5$  being the charge conjugation on  $S_5 = \mathbb{C}^4$ .
- The non-flat connection according to Proposition 4.1 that is defined by

$$\bar{c} = \beta X_{1234}^- \Gamma_{12}^{(6)} \otimes T, \quad d = 0$$

with  $T$  being some vector in  $\mathcal{C}\ell_1(\mathbb{R}^5)$ .

- The even part is defined by the Killing vector fields of  $M_6$ .
- The odd part  $\mathcal{K}_1$  is defined by

$$(X_{1234}^- S_6^- \oplus X_{1234}^+ S_6^+) \otimes \mathbb{C}^4 = \Pi^+ S_6 \otimes \mathbb{C}^4 .$$

The space we just described is exactly the  $D = 6, N = 4$  supergravity background discussed in [16].

**Remark 7.4.** We may take a closer look at Examples 3.2 and 4.4. A straight forward generalization yields non-restricted  $\nu = 1$  geometric superalgebras on Cahen-Wallach spaces of dimension eleven. The associated symmetric map is given by  $B = -\text{diag}((\alpha - \alpha')^2 \mathbb{1}_3, (\alpha + \alpha')^2 \mathbb{1}_6)$ . The connection according to Proposition 4.3 is defined by the pair  $(\bar{c}, d) = (\alpha \Gamma_{125}^{(11)}, \alpha' \Gamma_{125}^{(11)})$  and is flat.

In particular, there is a unique pair  $(\alpha, \alpha') = (-3\beta, \beta)$  for which geometric supersymmetry is achieved, see [4, 10].

Nevertheless, if we restrict the odd part of the geometric superalgebra to  $X_{1234}^- S_{11}^- \oplus X_{1234}^+ S_{11}^+$  and consider  $\alpha' = 0$  we see the following feature:

Although the analog to (58) and (59) is not obtained for the full summation  $1, \dots, 9$ , it is obtained for a summation  $1, \dots, 4$ .





The charge conjugation matrix for  $V$  obeys  $\gamma_i^t C_V = C_V \gamma_i$  and is given by

$$C_V = \sigma_3 \otimes \mathbb{1}.$$

From this we get the  $\gamma$ -matrices  $\{\Gamma_\mu\}_{\mu \in \{\pm, i\}}$  on  $W = \mathbb{R}^{1,10}$  by the procedure described in Section 4.1. In particular, the charge conjugation obeys  $\Gamma_\mu^t C_W = C_W \Gamma_\mu$  and is given by

$$C_W = \sigma_2 \otimes C_V.$$

Here as well as in the main text the matrices  $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ , and  $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  that obey  $\sigma_i \sigma_j = i \sum_{k=1}^3 \epsilon_{ijk} \sigma_k$  denote the Pauli-matrices.

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