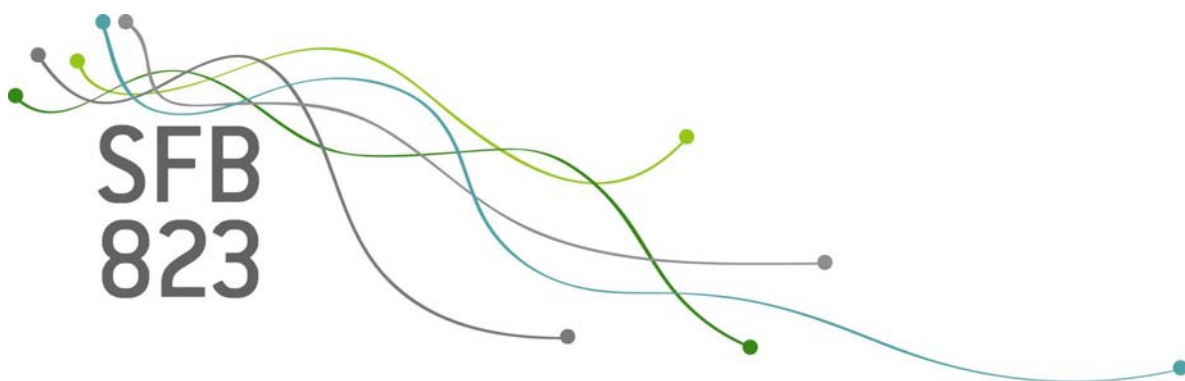# The sequential empirical process of a random walk in random scenery

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THE SEQUENTIAL EMPIRICAL PROCESS OF A RANDOM WALK IN RANDOM SCENERY

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ABSTRACT. A random walk in random scenery $(Y_n)_{n \in \mathbb{N}}$ is given by $Y_n = \xi_{S_n}$ for a random walk $(S_n)_{n \in \mathbb{N}}$ and iid random variables $(\xi(n))_{n \in \mathbb{N}}$. In this paper, we will show the weak convergence of the sequential empirical process, i.e. the centered and rescaled empirical distribution function. The limit process shows a new type of behavior, combining properties of the limit in form independent case (roughness of the paths) and of the long range dependent case (self-similarity).

1. INTRODUCTION

For a stationary, real valued sequence $(Y_n)_{n \in \mathbb{N}}$ of random variables with marginal distribution function F, the empirical distribution function F_n is defined by

$$(1) \qquad F_n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{Y_i \leq t\}}.$$

If the marginal distribution function F is continuous, we can without loss of generality assume that $F(t) = t$ (otherwise replacing Y_n by $F(Y_n)$). The sequential empirical process is a two parameter stochastic process $\left(W_n(s,t)\right)_{s,t \in [0,1]}$ defined by

$$(2) \qquad W_n(s,t) = \sum_{i=1}^{[ns]} \left(\mathbb{1}_{\{Y_i \leq t\}} - t \right),$$

where $[x]$ denotes the integer part of x. Note that we will have to rescale this process in order to obtain weak convergence, but as we need a different scaling for different kinds of stochastic processes, we have not included the scaling here. For iid random variables $(Y_n)_{n \in \mathbb{N}}$, Donsker [7] showed the weak convergence of the (non-sequential) empirical process $\left(\frac{1}{\sqrt{n}} W_n(1,t)\right)_{t \in [0,1]}$ to a Brownian bridge. This was extended by Müller [18] to the sequential empirical process $\left(\frac{1}{\sqrt{n}} W_n(s,t)\right)_{s,t \in [0,1]}$. The limit Gaussian process is the so called Kiefer-Müller process K, which is self-similar with exponent $b = \frac{1}{2}$, that means for any $a > 0$ the process $\left(K(as,t)\right)_{s,t \in [0,1]}$ has the same distribution as $\left(a^{\frac{1}{2}} K(s,t)\right)_{s,t \in [0,1]}$. For fixed $s \in [0,1]$, $\left(K(s,t)\right)_{t \in [0,1]}$ is a Brownian bridge, while for fixed $t \in [0,1]$ $\left(K(s,t)\right)_{s \in [0,1]}$ is a Brownian motion. This implies that there is an almost surely continuous modification of K, but the paths are not Hölder-continuous for any exponent $\gamma > \frac{1}{2}$.

This limit theorem has been extended to different kinds of short range dependent processes $(Y_n)_{n \in \mathbb{N}}$, where one still needs a $n^{-\frac{1}{2}}$ scaling and the limit process is still

self-similar with exponent $\frac{1}{2}$. For example, Berkes and Philipp [1] studied approximating functionals of strongly mixing sequences and Berkes, Hörmann, Schauer [2] so called S-mixing random variables. In the short range dependent case, the limit process is for fixed $t \in [0,1]$ a Brownian motion as in the independent case, so the paths are not smoother.

For long range dependent processes, the limit behavior is different in many aspects. For Gaussian sequences with slowly decaying covariances, Dehling and Taqqu [6] showed the convergence of sequential empirical process to a limit process that is self-similar with exponent $b > \frac{1}{2}$ and that is degenerate in the following sense: For fixed s, the process is not a Brownian bridge, but a deterministic function multiplied by a random variable. The paths for fixed s might be differentiable. For fixed t, the limit process is a fractional Brownian motion which is Hölder-continuous with exponent $\gamma > \frac{1}{2}$. For long range dependent linear processes, analog results were proved by Ho and Hsing [13].

In this paper, we will consider the random walk in random scenery, which is often considered to be another model for a long range dependent sequence of random variables. Let $(S_n)_{n \in \mathbb{N}}$ with $S_n = \sum_{i=1}^{n} X_i$ be a random walk in the normal domain of attraction of an α-stable Lévy process (with iid increments $(X_n)_{n \in \mathbb{N}}$) and $(\xi(n))_{n \in \mathbb{N}}$ a sequence of iid random variables (called scenery). Then the stationary process $(Y_n)_{n \in \mathbb{N}}$ with $Y_n = \xi_{S_n}$ is called random walk in random scenery and was first investigated by Kesten and Spitzer [14] and Borodin [4].

The behaviour of partial sum process Z_n with $Z_n(t) = \sum_{i=1}^{n} Y_i$ has been studied extensively. It converges weakly to a self-similar process with exponent $b > \frac{1}{2}$, which has smooth paths even if the random variables $(\xi(n))_{n \in \mathbb{N}}$ are in the domain of attraction of a Lévy process with jumps, see [14]. Other results include the law of the iterated logarithm (Khoshnevisan and Lewis [15]), large deviations (Gantert, König, and Shi [10]), extremes (Franke and Saigo [9]) and U-statistics (Guillotin-Plantard and Ladret [12], Franke, Pène, and Wendler [8]). As far as we know, there are no results on the empirical process of a random walk in random scenery.

2. Main Results

We will now give a functional non-central limit theorem for the sequential empirical process of a random walk in random scenery, that means the two-parameter process W_n with

$$(3) \qquad W_n(s,t) = \sum_{i=1}^{[ns]} \left(\mathbb{1}_{\{Y_i \leq t\}} - t \right), \quad \text{where} \quad Y_n = \xi_{S_n} \quad \text{and} \quad S_n := \sum_{i=1}^{n} X_i.$$

Let us first introduce the limit process W: Let $K = (K(s,t))_{s \in \mathbb{R}, t \in [0,1]}$ be a two sided Kiefer-Müller process, which is defined as follows: $K_1 = (K_1(s,t))_{s \in [0,\infty), t \in [0,1]}$ and K_{-1} be two independent, centered, continuous, two-parameter Gaussian process with covariance

$$(4) \qquad E\left[K_i(s,t)K_i(s',t')\right] = \min\{s,s'\}\left(\min\{t,t'\} - tt'\right) \quad \text{for} \quad i = 1, -1.$$

Set $K(s,t) = K_{\text{sgn}(s)}(|s|,t)$. Furthermore, let $(L_s(x))_{s \geq 0}$ be the local time of the limit process $(S_s^\star)_{s \geq 0}$ of the rescaled partial sum $(n^{-\frac{1}{\alpha}} \sum_{i=1}^{[ns]} X_i)_{s \geq 0}$, that means

$$(5) \qquad \int_0^t \mathbb{1}_{[a,b)}(S_s^\star)ds = \int_a^b L_s(x)dx.$$

For the existence of such a continuous time, see Getoor and Kesten [11]. Now the limit process W can be described by the following stochastic integral

$$(6) \qquad W(s,t) := \int_{\mathbb{R}} L_s(x) dK(x,t).$$

We will investigate the properties of this process after our main Theorem.

Theorem 1. *Let $(\xi_n)_{n \in \mathbb{N}}$ be an iid sequence of random variables uniformly distributed on $[0,1]$. If $(X_n)_{n \in \mathbb{N}}$ is another iid sequence and the law of X_n is in the normal domain of attraction of an α-stable law F_α with $1 < \alpha \le 2$, then we have the weak convergence*

$$(7) \qquad n^{-1+\frac{1}{2\alpha}} W_n \Rightarrow W$$

in the space $D\left([0,1]^2\right)$.

The space $D\left([0,1]^2\right)$ is the space of functions from $[0,1]^2$ to $\mathbb{R}$, for which the limit in each quadrant exists and which are continuous in each point coming from the upper right quadrant, equipped with the multidimensional Skorokhod distance (see Bickel and Wichura [3]). From the definition of W, we can see that for fixed t, the process $(W(s,t))_{s \in [0,1]}$ is the limit process of the random walk in random scenery as described by Kesten and Spitzer [14]. It is clear that the process W is self-similar with the same exponent $b = 1 - \frac{1}{2\alpha}$, that means $(W(as,t))_{s,t \in [0,1]}$ has the same distribution as $a^{1-\frac{1}{2\alpha}} W(s,t))_{s,t \in [0,1]}$.

On the other hand, for fixed s, the process $(W(s,t))_{t \in [0,1]}$ is a mixture of Brownian bridges (or a Brownian bridge with a random variance). So the process $(W(s,t))_{t \in [0,1]}$ has paths with the same properties as a Brownian bridge, and consequently they are Hölder-continuous for exponents $\gamma < \frac{1}{2}$, but not for exponents $\gamma > \frac{1}{2}$. In this sense, the limit process combines properties from the independent case (roughness of Kiefer-Müller process) and from the long range dependent case (self-similarity of the Dehling-Taqqu type limit process).

To give a deeper insight into the continuity properties of the process W, we need a generalization of the Kolmogorov-Chentsov theorem. There are several multidimensional versions of this theorem in the literature, see e.g. Mittmann and Steinwart [17] and the references therein, but they deal with uniform continuity, while our theorem allows for Hölder continuity with different exponents in different directions. The proof is nevertheless completely analogous and is hence omitted.

Proposition 2.1. *Let $(X_t)_{t \in [0,1]^d}$ be a stochastic process such that for some $m \ge 1$, $c_1, \ldots, c_d, \beta_1, \ldots, \beta_d$ and for all $t = (t_1, \ldots, t_d)$, $s = (s_1, \ldots, s_d)$ we have*

$$(8) \qquad E\left[|X_t - X_s|^m\right] \le \sum_{i=1}^{d} c_i \, |t_i - s_i|^{d+\beta_i} .$$

Then for all $\gamma_1, \ldots, \gamma_d$ with $\gamma_i < \frac{\beta_i}{m}$, where exists a modification $\tilde{X}$ of X and a almost surely finite random variable $C_{\gamma_1, \ldots, \gamma_d}$, such that for all $t = (t_1, \ldots, t_d)$, $s = (s_1, \ldots, s_d)$

$$(9) \qquad \left|\tilde{X}_t - \tilde{X}_s\right| \le C_{\gamma_1, \ldots, \gamma_d} \sum_{i=1}^{d} |t_i - s_i|^{\gamma_i} .$$

While for fixed s, the process $(W(s,t))_{t\in[0,1]}$ has the same modulus of continuity, no matter what the properties of the random walk S_n are, it will turn out for higher α, the limit process $(W(s,t))_{s\in[0,1]}$ for fixed t is Hölder continuous with a higher expoenten γ.

Proposition 2.2. *For any $\gamma < 1 - \frac{1}{2\alpha}$, $\gamma' < \frac{1}{2}$, there is a modification $\tilde{W}$ of W and a almost surely finite random variable $C_{\gamma,\gamma'}$, such that for all $s,t,s',t' \in [0,1]$*

$$(10) \qquad |W(s,t) - W(s',t')| \le C_{\gamma,\gamma'}\left(|s-s'|^{\gamma} + |t-t'|^{\gamma'}\right).$$

The exponent of Hölder continuity is linked to the exponent of self-similarity $b = 1 - \frac{1}{2\alpha}$. The same effect is known from fractional Brownian motion (see e.g. the book of Nourdin, [19], p. 8).

3. A Lemma on occupation times

The occupation time $N_n(x)$ is defined as the number of visits of the random walk $(S_i)_{i=1,\ldots,n}$ to x:

$$(11) \qquad N_n(x) := \sum_{i=1}^{n} \mathbb{1}_{\{S_i=x\}}.$$

The following Lemma gives a relation to the local time of the limiting process of the random walk, similar to Lemma 6 of Kesten and Spitzer [14]. In our proofs, C denotes a generic constant with might have different values in different inequalities, but does not dependent on n.

Lemma 3.1. *For any $k \in \mathbb{N}$, $s_1,\ldots,s_k \in [0,1]$, the random vector*

$$(12) \qquad \left(n^{-2+\frac{1}{\alpha}} \sum_{x\in\mathbb{Z}} N_{[ns_i]}(x)N_{[ns_j]}(x)\right)_{i,j\in\{1,\ldots,k\}}$$

converges in distribution to

$$(13) \qquad \left(\int L_{s_i}(x)L_{s_j}(x)dx\right)_{i,j\in\{1,\ldots,k\}}.$$

Proof. By the Cramér-Wold theorem, it suffices to show that for any $\theta_{ij} \in \mathbb{R}$, $i,j = 1,\ldots,k$, we have the weak convergence

$$(14) \qquad n^{-2+\frac{1}{\alpha}} \sum_{i,j=1}^{n} \theta_{ij} \sum_{x\in\mathbb{Z}} N_{[ns_i]}(x)N_{[ns_j]}(x) \Rightarrow \sum_{i,j=1}^{n} \theta_{ij} \int L_{s_i}(x)L_{s_j}(x)dx.$$

In order to show this, we will split the sum on the left side into several parts. Let $\tau > 0$, $M > 0$, $a(l,n) = \tau ln^{1/\alpha}$ and define

$$(15) \qquad Q(l,n) := n^{-2} \sum_{i,j=1}^{n} \theta_{ij} \sum_{a(l,n)\le x,y<a(l+1,n)} N_{[ns_i]}(x)N_{[ns_j]}(x)$$

$$(16) \qquad V(\tau,M,n) := \tau^{-1} \sum_{l=-M}^{M} Q(l,n)$$

$$(17) \qquad U(\tau,M,n) := n^{-2+\frac{1}{\alpha}} \sum_{|x|>M\tau n^{1/\alpha}} \sum_{i,j=1}^{n} \theta_{ij} N_{[ns_i]}(x)N_{[ns_j]}(x).$$

We now can decompose the sum into four parts:

$$(18) \quad n^{-2+\frac{1}{\alpha}} \sum_{i,j=1}^{n} \theta_{ij} \sum_{x \in \mathbb{Z}} N_{[ns_i]}(x) N_{[ns_j]}(x)$$

$$= V(\tau, M, n) + U(\tau, M, n)$$

$$+ \sum_{|l| \leq M} \sum_{a(l,n) \leq x < a(l+1,n)} n^{-2+\frac{1}{\alpha}} \left(\sum_{i,j=1}^{k} \theta_{ij} N_{[ns_i]}(x) N_{[ns_j]}(x) - \frac{n^2 Q(l,n)}{[a(l+1,n) - a(l,n)]} \right)$$

$$+ \sum_{|l| \leq M} \left(n^{\frac{1}{\alpha}} [a(l+1,n) - a(l,n)]^{-1} - 1 \right) Q(l,n)$$

$$= V(\tau, M, n) + U(\tau, M, n) + I(\tau, M, n) + I\!I(\tau, M, n).$$

We will treat the four summands separately. First note by Lemma 6 of Kesten and Spitzer [14] and the continuous mapping theorem, we have for $n \to \infty$ the convergence in distribution
$$(19)$$
$$V(\tau, M, n) \Rightarrow \tau^{-1} \sum_{i,j=1}^{n} \theta_{ij} \sum_{|l| \leq M} \int_{\tau l}^{\tau(l+1)} L_{s_i}(x) dx \int_{\tau l}^{\tau(l+1)} L_{s_j}(x) dx =: V(\tau, M).$$

For the summand $I(\tau, M, n)$, we introduce the mean occupation time of an interval $[a(l,n), a(l+1,n)]$:

$$(20) \qquad \bar{N}_{s_i,l} := \frac{1}{[a(l+1,n) - a_l]} \sum_{a(l,n) \leq x < a(l+1,n)} N_{[ns_i]}(x).$$

Now we can rewrite $I(\tau, M, n)$ and apply the triangle inequality.

$$(21) \quad |I(\tau, M, n)|$$

$$= \sum_{|l| \leq M} \sum_{a(l,n) \leq x < a(l+1,n)} n^{-2+\frac{1}{\alpha}} \left(\sum_{i,j=1}^{k} \theta_{ij} \left(N_{[ns_i]}(x) N_{[ns_j]}(x) - \bar{N}_{s_i,l} \bar{N}_{s_j,l} \right) \right)$$

$$\leq \sum_{|l| \leq M} \sum_{a(l,n) \leq x < a(l+1,n)} n^{-2+\frac{1}{\alpha}} \left(\sum_{i,j=1}^{k} \theta_{ij} \left| N_{[ns_i]}(x) - \bar{N}_{s_i,l} \right| N_{[ns_j]}(x) \right)$$

$$+ \sum_{|l| \leq M} \sum_{a(l,n) \leq x < a(l+1,n)} n^{-2+\frac{1}{\alpha}} \left(\sum_{i,j=1}^{k} \theta_{ij} \bar{N}_{s_i,l} \left| N_{[ns_j]}(x) - \bar{N}_{s_j,l} \right| \right)$$

$$\leq \theta^{\star} \sum_{|l| \leq M} \sum_{a(l,n) \leq x < a(l+1,n)} n^{-2+\frac{1}{\alpha}} \sum_{i,j=1}^{k} \left| N_{[ns_i]}(x) - \bar{N}_{s_i,l} \right| N_{[ns_j]}(x)$$

$$+ \theta^{\star} \sum_{|l| \leq M} \sum_{a(l,n) \leq x < a(l+1,n)} n^{-2+\frac{1}{\alpha}} \sum_{i,j=1}^{k} \bar{N}_{s_i,l} \left| N_{[ns_j]}(x) - \bar{N}_{s_j,l} \right|$$

$$=: A_n + B_n$$

with $\theta^{\star} := \max\{\theta_1, \ldots, \theta_k\}$. By Lemma 1 of Kesten and Spitzer [14], we have that

$$(22) \qquad\qquad E\left(N_{[ns_i]}^2(x) \right) \leq C n^{2-\frac{2}{\alpha}},$$

and by Lemma 3 of [14] in combination with their line (2.26)

$$(23) \qquad E\left(N_{[ns_i]}(x) - N_{[ns_i]}(y)\right)^2 \le Cn^{1-\frac{1}{\alpha}}|x-y|^{\alpha-1}.$$

Keep in mind that $a(l+1,n) - a(l.n) \le C\tau n^{\frac{1}{\alpha}}$. By the Cauchy-Schwarz inequality and the definition of $\bar{N}_{s_i,l}$, we obtain

(24)

$$
\begin{aligned}
E\left|A_n\right| &\le \theta^\star \sum_{|l|\le M} \sum_{a(l,n)\le x<a(l+1,n)} n^{-2+\frac{1}{\alpha}} \sum_{i,j=1}^{k} \left\|N_{[ns_i]}(x) - \bar{N}_{s_i,l}\right\|_2 \left\|N_{[ns_j]}(x)\right\|_2 \\
&\le \theta^\star \sum_{|l|\le M} \sum_{i,j=1}^{k} \sum_{a(l,n)\le x,y<a(l+1,n)} \frac{n^{-2+\frac{1}{\alpha}} \left\|N_{[ns_j]}(x)\right\|_2}{[a(l+1,n) - a(l.n)]} \left\|N_{[ns_i]}(x) - N_{[ns_i]}(y)\right\|_2 \\
&\le C\theta^\star(2M+1)k^2 \sum_{a(l,n)\le x,y<a(l+1,n)} \frac{n^{-2+\frac{1}{\alpha}}\sqrt{n^{2-\frac{2}{\alpha}}}}{n^{\frac{1}{\alpha}}} \sqrt{n^{1-\frac{1}{\alpha}} n^{\frac{1}{\alpha}(\alpha-1)}} \\
&= C\theta^\star(2M+1)k^2 \sum_{a(l,n)\le x,y<a(l+1,n)} n^{-\frac{2}{\alpha}} \le C\theta^\star(2M+1)k^2 n^{-\frac{1}{\alpha}}.
\end{aligned}
$$

With the same arguments and using the fact that

$$(25) \qquad \left\|\bar{N}_{s_i,l}\right\|_2 \le \frac{1}{[a(l+1,n) - a_l]} \sum_{a(l,n)\le x<a(l+1,n)} \left\|N_{[ns_i]}(x)\right\|_2,$$

it follows that $E\left|B_n\right| \le C\theta^\star(2M+1)k^2 n^{-\frac{1}{\alpha}}$ and

$$(26) \qquad I(\tau,M,n) = A_n + B_n \xrightarrow{n\to\infty} 0$$

in probability. For the summand $I\!\!I(\tau,M,n)$, note that $Q(l.n)$ converges in distribution to $\sum_{i,j=1}^{k} \int_{\tau l}^{\tau(l+1)} Ls_i(x)dx \int_{\tau l}^{\tau(l+1)} Ls_j(x)dx$. Furthermore, $n^{\frac{1}{\alpha}}[a(l+1,n) - a(l,n)]^{-1} - 1 \to 0$ as $n \to \infty$ and consequently

$$(27) \qquad I\!\!I(\tau,M,n) = \sum_{|l|\le M} \left(n^{\frac{1}{\alpha}}[a(l+1,n) - a(l,n)]^{-1} - 1\right) Q(l,n) \xrightarrow{n\to\infty} 0$$

in probability. For the last summand, we have

$$(28) \quad P\left(U(\tau,M,n)\ne 0\right) \le P\left(N_n(x) > 0 \text{ for an } x \text{ with} |x| > M\tau n^{\frac{1}{\alpha}}\right) \le \epsilon(M\tau),$$

where $\epsilon(z) \to 0$ as $z \to \infty$, see Lemma 1 of Kesten and Spitzer [14]. Note that the local time L has almost surely a compact support, since the paths of the process $(S_s^\star)_{s\in[0,1]}$ are almost surely bounded, so we have for $V(\tau,M)$ defined in (19) the following limes

$$(29) \quad V(\tau) = \lim_{M\to\infty} V(\tau,M) := \tau^{-1} \sum_{i,j=1}^{n} \theta_{ij} \sum_{l\in\mathbb{Z}} \int_{\tau l}^{\tau(l+1)} L_{s_i}(x)dx \int_{\tau l}^{\tau(l+1)} L_{s_j}(x)dx$$

almost surely. By the almost sure continuity of the local time L additionally

$$(30) \quad \lim_{\tau \to 0} V(\tau) = \lim_{\tau \to 0} \sum_{i,j=1}^{n} \theta_{ij} \sum_{l \in \mathbb{Z}} \int_{\tau l}^{\tau(l+1)} L_{s_i}(x) \left(\tau^{-1} \int_{\tau l}^{\tau(l+1)} L_{s_j}(y) dy \right) dx$$

$$= \sum_{i,j=1}^{n} \theta_{ij} \sum_{i,j=1}^{n} \theta_{ij} \int L_{s_i}(x) L_{s_j}(x) dx =: V.$$

Finally, we combine the convergence of the different parts. Let $d(X,Y)$ denote the Prokhorov distance of the distributions of X and Y (so convergence with respect to d is equivalent to weak convergence and $P(|X - Y| \geq \epsilon) \leq \epsilon$ implies $d(x,y) \leq \epsilon$). For any $\epsilon > 0$, use (19) to choose $\tau > 0$ small enough, such that $P(|V(\tau) - V| \geq \epsilon/6) \leq \epsilon/6$. Next, for $M > 0$ large enough, we have by (29) that $P(|V(\tau, M) - V(\tau)| \geq \epsilon/6) \leq \epsilon/6$ and $\epsilon(\tau M) \leq \epsilon/6$ by (28). Now we can choose $n_0 \in \mathbb{N}$ with the help of (19), (26) and (27), such that for all $n \geq n_0$ we have $d(V(\tau, M, n), V(\tau, M)) \leq \epsilon/6$, and for the last two terms $P(|I(\tau, M, n)| \geq \epsilon/6) \leq \epsilon/6$ and $P(|I\!\!I(\tau, M, n)| \geq \epsilon/6) \leq \epsilon/6$ and arrive with the help of the triangle inequality at

$$(31) \quad d(V(\tau, M, n) + U(\tau, M, n) + I(\tau, M, n) + I\!\!I(\tau, M, n), V)$$

$$\leq d\Big(n^{-2 + \frac{1}{\alpha}} \sum_{i,j=1}^{n} \theta_{ij} \sum_{x \in \mathbb{Z}} N_{[ns_i]}(x) N_{[ns_j]}(x), V(\tau, M, n) + U(\tau, M, n) + I(\tau, M, n)\Big)$$

$$+ d(V(\tau, M, n) + U(\tau, M, n) + I(\tau, M, n), V(\tau, M, n) + U(\tau, M, n))$$

$$+ d(V(\tau, M, n) + U(\tau, M, n), V(\tau, M, n)) + d(V(\tau, M, n), V(\tau, M))$$

$$+ d(V(\tau, M), V(\tau)) + d(V(\tau), V) \leq \epsilon.$$

$\square$

4. Proof of the Main Results

Proof of Theorem 1. We will first prove the convergence of the finite dimensional distributions, tightness will be established later. We will make use of the Cramér-Wold theorem and show that for $\theta_1, \ldots, \theta_k \in \mathbb{R}$, $s_1, \ldots, s_k \in [0,1]$, $t_1, \ldots, t_k \in [0,1]$, we have the weak convergence

$$(32) \quad n^{-1 + \frac{1}{2\alpha}} \sum_{j=1}^{k} \theta_j \sum_{i=1}^{[ns_j]} \left(\mathbb{1}_{\{Y_i \leq t_j\}} - t_j \right) = n^{-1 + \frac{1}{2\alpha}} \sum_{j=1}^{k} \theta_j \sum_{x \in \mathbb{Z}} N_{[ns_j]}(x) \xi_j(x)$$

$$\Rightarrow \sum_{j=1}^{k} \theta_j \int L_{s_j}(x) dK(x, t_j),$$

with $\xi_j(x) = \mathbb{1}_{\{\xi_x \leq t_j\}} - t_j$. For this, we will study the characteristic function and apply Lévy's continuity theorem:

$$(33)\quad \varphi_n(\lambda) := E\left(\exp\left(i\lambda n^{-1+\frac{1}{2\alpha}}\sum_{j=1}^{k}\theta_j\sum_{x\in\mathbb{Z}}N_{[ns_j]}(x)\xi_j(x)\right)\right)$$

$$= E\left(\prod_{x\in\mathbb{Z}}\exp\left(i\lambda n^{-1+\frac{1}{2\alpha}}\sum_{j=1}^{k}\theta_j N_{[ns_j]}(x)\xi_j(x)\right)\right)$$

$$= E\left(E\left(\prod_{x\in\mathbb{Z}}\exp\left(i\lambda n^{-1+\frac{1}{2\alpha}}\sum_{j=1}^{k}\theta_j N_{[ns_j]}(x)\xi_j(x)\right)\bigg|(X_n)_{n\in\mathbb{N}}\right)\right)$$

$$= E\left(\prod_{x\in\mathbb{Z}}E\left(\exp\left(i\lambda n^{-1+\frac{1}{2\alpha}}\sum_{j=1}^{k}\theta_j N_{[ns_j]}(x)\xi_j(x)\right)\bigg|(X_n)_{n\in\mathbb{N}}\right)\right),$$

where we used the fact that the random variables $(\xi_x)_{x\in\mathbb{Z}}$ and thus also the random vectors $((\xi_1(x),\ldots,\xi_k(x)))_{x\in\mathbb{Z}}$ are independent and that inside conditional expectation, $(X_n)_{n\in\mathbb{N}}$ and thus $N_{[ns_1]}(x),\ldots,N_{[ns_k]}(x)$ are fixed. With $\varphi_{\xi_1(0),\ldots,\xi_k(0)}$, we denote the characteristic function of the random vector $(\xi_1(0),\ldots,\xi_k(0))$, so that

$$(34)\quad \varphi_n(\lambda) = E\left(\prod_{x\in\mathbb{Z}}\varphi_{\xi_1(0),\ldots,\xi_k(0)}\left(\lambda n^{-1+\frac{1}{2\alpha}}N_{[ns_1]}(x),\ldots,\lambda n^{-1+\frac{1}{2\alpha}}N_{[ns_k]}(x)\right)\right).$$

The next step will be a Taylor expansion, so we have to gather some statements about the conditional moments. Keep in mind that $E\xi_j(x) = 0$ and thus

$$(35)\qquad E\left(\sum_{j=1}^{k}n^{-1+\frac{1}{2\alpha}}\theta_j N_{[ns_j]}(x)\xi_j(x)\big|(X_n)_{n\in\mathbb{N}}\right) = 0.$$

Furthermore

$$(36)\quad E\left(\left(n^{-1+\frac{1}{2\alpha}}\sum_{j=1}^{k}\theta_j N_{[ns_j]}(x)\xi_j(x)\right)^2\bigg|(X_n)_{n\in\mathbb{N}}\right)$$

$$= \sum_{j,l=1}^{k}n^{-2+\frac{1}{\alpha}}\theta_j\theta_l N_{[ns_j]}(x)N_{[ns_l]}(x)\sigma_{jl}$$

with $\sigma_{jl} := \mathrm{Cov}(\xi_j(x),\xi_l(x))$. Finally, by Lemma 4 of Kesten and Spitzer the numbers $n^{-1+\frac{1}{2\alpha}}N_{[ns_j]}^3(x)$ converges uniformly to 0 in probability and by their Lemma 1

$$(37)\qquad E\left(\sum_{x\in\mathbb{Z}}N_{[ns_j]}^3(x)\right) \leq Cn^{3-\frac{3}{\alpha}}.$$

So we can conclude that

$$(38)\quad \varphi_n(\lambda)$$

$$= E\left(\prod_{x\in\mathbb{Z}}\left(1 - \frac{\lambda^2}{2}\sum_{j,l=1}^{k}n^{-2+\frac{1}{\alpha}}\theta_j\theta_l N_{[ns_j]}(x)N_{[ns_l]}(x)\sigma_{jl} + O\left(n^{-3+\frac{3}{2\alpha}}N_n^3(x)\right)\right)\right)$$

$$= E\Bigg(\exp\Big(\sum_{x\in\mathbb{Z}} \Big(-\frac{\lambda^2}{2} \sum_{j,l=1}^{k} n^{-2+\frac{1}{\alpha}} \theta_j \theta_l N_{[ns_j]}(x) N_{[ns_l]}(x) \sigma_{jl}$$

$$+ o\big(n^{-2+\frac{1}{\alpha}} N_n^2(x)\big) + O\big(n^{-3+\frac{3}{2\alpha}} N_n^3(x)\big)\Big)\Big)\Bigg)$$

$$\xrightarrow{n\to\infty} E\Big(\exp\Big(-\frac{\lambda^2}{2} \sum_{j,l=1}^{k} \theta_j \theta_l \sigma_{jl} \int L_{s_j} L_{s_l} dx \Big)\Big),$$

where we used Lemma 3.1 and the boundedness and continuity of the function $z \mapsto \exp(-z^2/2)$ to conclude that the expectation converges.

On the other hand, conditional on the Lévy-process $S^\star$, the linear combination $\sum_{j=1}^{k} \theta_j \int L_{s_j}(x) dK(x, t_j)$ is Gaussian with variance

$$(39) \qquad \sum_{j,l=1}^{k} \theta_j \theta_l \int L_{s_j}(x) L_{s_l}(x) \sigma_{jl} dx,$$

as $\sigma_{jl} = \mathrm{Cov}(\xi_j(x), \xi_l(x)) = \mathrm{Cov}(K(1, t_j), K(1, t_l))$ and the process K is centered. This implies that

$$(40) \quad E\left(\exp\Big(i\lambda \sum_{j=1}^{k} \theta_j \int L_{s_j}(x) dK(x, t_j) \Big) \right)$$

$$= E\left(E\left(\exp\Big(i\lambda \sum_{j=1}^{k} \theta_j \int L_{s_j}(x) dK(x, t_j) \Big) \Big| S^\star \right) \right)$$

$$= E\left(\exp\Big(-\frac{1}{2}\lambda^2 \sum_{j,l=1}^{k} \theta_j \theta_l \int L_{s_j}(x) L_{s_l}(x) \sigma_{jl} dx \Big) \right),$$

and by (38) and Lévy's continuity theorem the finite dimensional convergence follows. In order to prove tightness, we will establish a moment bound. First note that

$$(41) \qquad E\big(\mathbb{1}_{\{\xi_x \le t_1\}} - t_1 - \mathbb{1}_{\{\xi_x \le t_2\}} + t_2 \big)^2 \le |t_1 - t_2|,$$

$$(42) \qquad E\big(\mathbb{1}_{\{\xi_x \le t_1\}} - t_1 - \mathbb{1}_{\{\xi_x \le t_2\}} + t_2 \big)^4 \le |t_1 - t_2|.$$

By Lemma 2.1 of Guillotin-Plantard and Ladret [12]. we have that

$$(43) \qquad E\left(\sum_{x\in\mathbb{Z}} N_n^2(x) \right)^2 \le Cn^{4-\frac{2}{\alpha}},$$

$$(44) \qquad E\left(\sum_{x\in\mathbb{Z}} N_n^4(x) \right) \le Cn^{4-\frac{4}{\alpha}}.$$

Now we obtain the following moment bound for all $n_1 \leq n_2 \leq n$ and $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| \geq n^{-\frac{2}{\alpha}}$:

$$
(45) \quad E\left(\sum_{i=n_1+1}^{n_2} (\mathbb{1}_{\{\xi_x \leq t_1\}} - t_1) - \sum_{i=n_1+1}^{n_2} (\mathbb{1}_{\{\xi_x \leq t_2\}} - t_2) \right)^4
$$

$$
= E\left(E\left(\left(\sum_{i=n_1+1}^{n_2} (\mathbb{1}_{\{\xi_x \leq t_1\}} - t_1) - \sum_{i=n_1+1}^{n_2} (\mathbb{1}_{\{\xi_x \leq t_2\}} - t_2) \right)^4 \Big| (X_n)_{n\in\mathbb{N}} \right) \right)
$$

$$
\leq E\left(\sum_{x\in\mathbb{Z}} N_n^4(x)|t_1 - t_2| + \sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}} N_n^2(x)N_n^2(y)|t_1 - t_2|^2 \right)
$$

$$
\leq C\left(n^{4-\frac{4}{\alpha}}|t_1 - t_2| + Cn^{4-\frac{2}{\alpha}}|t_1 - t_2|^2 \right) \leq Cn^{4-\frac{2}{\alpha}}|t_1 - t_2|^2.
$$

In the case that $|t_1 - t_2| \leq n^{-\frac{2}{\alpha}}$, we have by monotonicity that for any $t \in (t_1, t_2)$

$$
(46) \quad \left| \sum_{i=n_1+1}^{n_2} (\mathbb{1}_{\{\xi_x \leq t\}} - t) - \sum_{i=n_1+1}^{n_2} (\mathbb{1}_{\{\xi_x \leq t_1\}} - t_1) \right|
$$

$$
\leq \left| \sum_{i=n_1+1}^{n_2} \mathbb{1}_{\{\xi_x \leq t\}} - \sum_{i=n_1+1}^{n_2} \mathbb{1}_{\{\xi_x \leq t_1\}} \right| + (n_2 - n_1)|t - t_1|
$$

$$
\leq \left| \sum_{i=n_1+1}^{n_2} \mathbb{1}_{\{\xi_x \leq t_2\}} - \sum_{i=n_1+1}^{n_2} \mathbb{1}_{\{\xi_x \leq t_1\}} \right| + (n_2 - n_1)|t_2 - t_1|
$$

$$
\leq \left| \sum_{i=n_1+1}^{n_2} (\mathbb{1}_{\{\xi_x \leq t_2\}} - t_2) - \sum_{i=n_1+1}^{n_2} (\mathbb{1}_{\{\xi_x \leq t_1\}} - t_1) \right| + 2(n_2 - n_1)|t_2 - t_1|
$$

$$
\leq \left| \sum_{i=n_1+1}^{n_2} (\mathbb{1}_{\{\xi_x \leq t_2\}} - t_2) - \sum_{i=n_1+1}^{n_2} (\mathbb{1}_{\{\xi_x \leq t_1\}} - t_1) \right| + 2n^{1-\frac{2}{\alpha}}.
$$

Following Bickel and Wichura [3], we introduce for a two-parameter stochastic process $(V(s,t))_{s,t\in[0,1]}$ the notation

$$
(47) \quad w_\delta''(V)
$$

$$
= \max\Bigg\{ \sup_{\substack{0 \leq t_1 \leq t \leq t_2 \leq 1 \\ t_2 - t_1 \leq \delta}} \min\left\{ \|V(\cdot, t_2) - V(\cdot, t)\|_\infty, \|V(\cdot, t) - V(\cdot, t_1)\|_\infty \right\},
$$

$$
\sup_{\substack{0 \leq s_1 \leq s \leq s_2 \leq 1 \\ s_2 - s_1 \leq \delta}} \min\left\{ \|V(s_2, \cdot) - V(s, \cdot)\|_\infty, \|V(s, \cdot) - V(s_1, \cdot)\|_\infty \right\} \Bigg\},
$$

where $\|\cdot\|_\infty$ denotes the supremum norm. Now define the index set $D_n := \left\{ 0, \frac{1}{n}, \frac{2}{n} \ldots, 1 \right\} \times \left\{ 0, [n^{\frac{2}{\alpha}}]^{-1}, 2[n^{\frac{2}{\alpha}}]^{-1}, \ldots, 1 \right\}$ and note that we have by (46)

$$
(48) \quad w_\delta''(n^{-1+\frac{1}{2\alpha}} W_n) \leq w_\delta''(n^{-1+\frac{1}{2\alpha}} W_{n|D_n}) + 2n^{-1+\frac{1}{2\alpha}} n^{1-\frac{2}{\alpha}},
$$

where $w_\delta''(n^{-1+\frac{1}{2\alpha}} W_{n|D_n})$ is calculated by restricting all suprema in (47) to the set D_n. Now by Theorem 3 (and the remarks following their theorem) of Bickel and

Wichura [3] together with (45), we can conclude that for any $\epsilon > 0$

$$(49) \qquad P\left(\limsup_{n\to\infty} w''_\delta(n^{-1+\frac{1}{2\alpha}} W_{n|D_n}) > \epsilon\right) \xrightarrow{\delta\to 0} 0.$$

It follows by (48), that

$$(50) \qquad P\left(\lim_{n\to\infty} w''_\delta(n^{-1+\frac{1}{2\alpha}} W_n) > \epsilon\right) \xrightarrow{\delta\to 0} 0.$$

and thus the process is tight by corollary 1 of Bickel and Wichura [3]. $\qquad \square$

Proof of Proposition 2.2. We will use Proposition 2.1, so we have to establish a moment inequality. Let be $m \in \mathbb{N}$ even, $s, s', t \in [0, 1]$ with $s \le s'$. Note that conditional on the process $S^\star$, the process W is given by an Itō-Integral, and so it is Gaussian and we can apply the Itō-isometry. Furthermore, note that the difference of local times has the $L_{s'} - L_s$ has the same distribution as $L_{s'-s}$ shifted by $S^\star(s)$. We obtain

$$(51) \quad E\left(W(s',t) - W(s,t)\right)^m = E\left(\int (L_{s'}(x) - L_s(x))dK(x,t)\right)^m$$

$$= E\left(E\left(\left(\int (L_{s'}(x) - L_s(x))dK(x,t)\right)^m \Big| S^\star\right)\right)$$

$$= E\left(E\left(\left(\int L_{s'-s}(x)dK(x,t)\right)^m \Big| S^\star\right)\right)$$

$$= E\left(M(m,t)\left(\int L^2_{s'-s}(x)dx\right)^{\frac{m}{2}}\right),$$

where $M(m,t)$ is the m-th moment of $K(1,t)$ and thus $M(m,t) \le M_m$ for the m-th moment M_m of a standard normal random variable. Now we gather some facts about local time. Obviously

$$(52) \qquad \int L_s(x)dx = s.$$

By Theorem 1 of Davis [5], we have that for $L^\star_s := \sup_{x\in\mathbb{R}} L_s(x)$

$$(53) \qquad EL^{\star p}_s \le K_p s^{p\frac{\alpha-1}{\alpha}}$$

for a constant K_p (we use the form of the inequality as stated by Lacey [16], as there seems to be a misprint in [5]). Now we can proceed with the right side of (51):

$$(54) \quad E\left(M(m,t)\left(\int L^2_{s'-s}(x)dc\right)^{\frac{m}{2}}\right) \le M_m E\left(\left(\int L_{s'-s}(x)dx\right)^{\frac{m}{2}} \left(L^\star_{s'-s}\right)^{\frac{m}{2}}\right)$$

$$= M_m(s'-s)^{\frac{m}{2}} E\left(L^\star_{s'-s}\right)^{\frac{m}{2}} \le M_m K_{m/2}(s'-s)^{\frac{m}{2}}(s'-s)^{\frac{m}{2}\frac{\alpha-1}{\alpha}}$$

$$\le M_m K_{m/2}(s'-s)^{m(1-\frac{1}{2\alpha})}.$$

Now let be $t, t' \in [0, 1]$ with $t \le t'$. Note that the process $(K(s,t') - K(s,t))_{t\in\mathbb{R}}$ is a (two-sided) Brownian motion with variance $\mathrm{Var}(K(1,t') - K(1,t)) \le t' - t$. Now we proceed as above by conditioning on $S^\star$ and applying the Itō-isometry:

$$(55) \quad E\left(W(s,t') - W(s,t)\right)^m = E\left(\int L_s(x)d(K(x,t') - K(x,t))\right)^m$$

$$= E\left(E\left(\left(\int L_s(x)d(K(x,t') - K(x,t))\right)^m \middle| S^\star\right)\right)$$

$$\leq E\left(M_m\left((t'-t)\int L_s^2(x)dx\right)^{\frac{m}{2}}\right)$$

$$\leq M_m(t'-t)^{\frac{m}{2}}E\left(L^\star(s)\int L_s(x)dx\right)^{\frac{m}{2}}$$

$$\leq M_m(t'-t)^{\frac{m}{2}}EL^{\star\frac{m}{2}}(1) \leq M(m,t)K_{m/2}(t'-t)^{\frac{m}{2}}.$$

Combining (54) and (55), we arrive at

$$(56) \quad E\left(K(s',t') - K(s,t)\right)^m$$

$$\leq 2^{m-1}\left(E\left(K(s',t') - K(s,t')\right)^m + E\left(K(s,t') - K(s,t)\right)^m\right)$$

$$\leq 2^{m-1}M_mK_{m/2}(s'-s)^{m(1-\frac{1}{2\alpha})} + 2^{m-1}M_mK_{m/2}(t'-t)^{\frac{m}{2}}.$$

Now for any $\gamma < 1 - \frac{1}{2\alpha}$, $\gamma' < \frac{1}{2}$, we can choose m large enough such that

$$(57) \qquad\qquad \gamma < \frac{m(1-\frac{1}{2\alpha}) - 2}{m}, \quad \gamma' < \frac{\frac{m}{2} - 2}{m},$$

and the statement of this proposition follows from Proposition 2.1.

$\square$

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