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Stochastic homogenization of plasticity equations

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Abstract: In the context of infinitesimal strain plasticity with hardening, we derive a stochastic homogenization result. We assume that the coefficients of the equation are random functions: elasticity tensor, hardening parameter and flow-rule function are given through a dynamical system on a probability space. A parameter $\varepsilon > 0$ denotes the typical length scale of oscillations. We derive effective equations that describe the behavior of solutions in the limit $\varepsilon \to 0$. Our results are based on the needle-problem approach: We verify that the stochastic coefficients "allow averaging": In average, a strain evolution $[0,T] \ni t \mapsto \xi(t) \in \mathbb{R}^{d\times d}_s$ induces a stress evolution $[0,T] \ni t \mapsto \Sigma(\xi)(t) \in \mathbb{R}^{d\times d}_s$. With the abstract result of [11] we conclude the stochastic homogenization result.

1 Introduction

In its history, mathematics has often been inspired by questions from continuum mechanics: Given a body of metal and given a force acting on it, what is the deformation that the body of metal is experiencing? Euler has been inspired by this question; much later, the development of linear and non-linear elasticity theory provided excellent models (and mathematical theories) for non-permanent deformations. In contrast, the description of permanent deformations with plasticity models is much less developed. The only well-established plasticity models are based on infinitesimal strain theories, ad-hoc decomposition rules of the strain tensor and flow rules for the plastic deformation tensor.

Homogenization theory is, in its origins, concerned with the following question: How does a heterogeneous material (composed of different materials) behave effectively? Can we characterize an effective material such that a heterogeneous medium (consisting of a very fine mixture) behaves like the effective material? This homogenization question has a positive answer in the context of linear elasticity: effective coefficients can be computed and bounds for these effective coefficients are available. The situation is quite different for plasticity models: Results have been obtained only in the last ten years and the effective model cannot be reduced to one effective macro-model (it remains a two-scale model).

With only two exceptions, so far, homogenization results in plasticity treat essentially the same system: Infinitesimal strains and an additive decomposition of the strain tensor are

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used, some hardening effect is included, and the homogenization is performed in a periodic setting. The two exceptions are [7] and [16]: In [7], no hardening effect is used and the limit system is much more involved. In [16], stochastic coefficients are permitted, but at the expence of a one-dimensional setting. The present article is based on [11] and provides the third exception: We treat a model with stochastic coefficients in dimensions 2 and 3.

We mention at this point the more abstract approach in the framework of energetic solutions, see [12, 13], and its application in gradient plasticity in [9].

Plasticity equations

We study a bounded domain $Q \subset \mathbb{R}^d$, $d \in \{2,3\}$, occupied by a heterogeneous material, and its evolution in some time interval $(0,T) \subset \mathbb{R}$. For a parameter $\varepsilon > 0$, we consider on $Q \times (0,T)$ the plasticity system

$$\begin{aligned} -\nabla \cdot \sigma^\varepsilon &= f\,, & \sigma^\varepsilon &= C_\varepsilon^{-1} e^\varepsilon\,, \\ \nabla^s u^\varepsilon &= e^\varepsilon + p^\varepsilon\,, & \partial_t p^\varepsilon &\in \partial\Psi_\varepsilon(\sigma^\varepsilon - B_\varepsilon p^\varepsilon)\,. \end{aligned} \tag{1.1}$$

The first relation is the quasi-static balance of forces in the body, f is a given load, σ the stress tensor. The second relation is Hooke's law which relates linearly the stress σ with the elastic strain e. The third relation is the additive decomposition of the infinitesimal strain $\nabla^s u = (\nabla u + (\nabla u)^T)/2$. The fourth relation is the flow rule for the plastic strain p, it uses the subdifferential $\partial\Psi$ of a convex function Ψ. We refer to [1, 8] for the modelling.

Our interest here is to study coefficients $B = B_\varepsilon$ (hardening), $C = C_\varepsilon$ (elasticity tensor), and $\Psi = \Psi_\varepsilon$ (convex flow rule function) that depend on the parameter $\varepsilon > 0$. We imagine ε to be the spatial length scale of the heterogeneities. Since the coefficients depend on ε, also the solution $(u,\sigma,e,p) = (u^\varepsilon,\sigma^\varepsilon,e^\varepsilon,p^\varepsilon)$ depends on ε.

We consider only positive and symmetric coefficient tensors, using the following setting: We denote by $\mathbb{R}^{d\times d}_s \subset \mathbb{R}^{d\times d}$ the space of symmetric matrices, $\mathcal{L}(\mathbb{R}^{d\times d}_s, \mathbb{R}^{d\times d}_s)$ is the space of linear mappings on $\mathbb{R}^{d\times d}_s$. For every $\varepsilon > 0$ and almost every $x \in Q$, the tensors $C_\varepsilon(x), B_\varepsilon(x) \in \mathcal{L}(\mathbb{R}^{d\times d}_s, \mathbb{R}^{d\times d}_s)$ are assumed to be symmetric with respect to the scalar product on $\mathbb{R}^{d\times d}_s$. Furthermore, for constants $\gamma, \beta > 0$, we assume the positivity and boundedness

$$\gamma\,|\xi|^2 \le \xi : (C_\varepsilon(x)\,\xi) \le \frac{1}{\gamma}\,|\xi|^2\,, \qquad \beta\,|\xi|^2 \le \xi : (B_\varepsilon(x)\,\xi) \le \frac{1}{\beta}\,|\xi|^2 \tag{1.2}$$

for every $\xi \in \mathbb{R}^{d\times d}_s$, a.e. $x \in Q$, and every $\varepsilon > 0$.

System (1.1) is accompanied by a Dirichlet boundary condition $u^\varepsilon = U$ on $\partial Q \times (0,T)$ and an initial condition for the plastic strain tensor (for simplicity, we assume here a vanishing initial plastic deformation). Finally, the load f must be imposed. We consider data

$$U \in H^1(0,T;H^1(Q;\mathbb{R}^d))\,, \quad f \in H^1(0,T;L^2(Q,\mathbb{R}^d))\,, \quad p^\varepsilon|_{t=0} \equiv 0\,. \tag{1.3}$$

The fundamental task of homogenization theory is the following: If $u^\varepsilon \rightharpoonup u$ converges in some topology as $\varepsilon \to 0$, what is the equation that characterizes u?

Known homogenization results and the needle-problem approach

The periodic homogenization of system (1.1) was performed in the last 10 years. The effective two-scale limit system was first stated in [2]. The rigorous derivation of the limit system (under different assumptions on the coefficients) was obtained by Visintin with two-scale convergence methods [20, 21, 22], by Alber and Nesenenko with phase-shift convergence [3, 14], and by Veneroni together with the second author with energy methods [17]. By the same authors, some progress was achieved regarding the monotone flow rule and a simplification of proofs in [19]. We refer to these publications also for a further discussion of the periodic homogenization of system (1.1).

The non-periodic homogenization of system (1.1) is much less treated. In particular, we are not aware of any stochastic homogenization result (with the exception of [16], but the analysis of the one-dimensional case is much simpler, since the stress variable can be obtained by a simple integration from the force f).

For the non-periodic case, a partial homogenization result has been obtained in [11]. That contribution is based on the needle-problem approach, which has its origin in [18]. The present article is based on [11] and we therefore describe in the next paragraph the needle-problem approach in more detail.

In the *needle-problem approach,* homogenization is seen as a two-step procedure. We describe the two steps here with the scalar model $-\nabla \cdot (a^\varepsilon \nabla u^\varepsilon) = f$ for a deformation $u^\varepsilon : Q \to \mathbb{R}$. Step 1 is concerned with cell-problems: One verifies that, on a representative elementary volume (REV, usually the unit square) and for a vanishing load, the material behaves in a well-defined way: An input (here: the averaged gradient ξ of the solution across the REV) results in a certain output (here: the averaged stress $\sigma(\xi) = a^*\xi$ for some matrix a^*). Step 2 is concerned with arbitrary domains Q and arbitrary loads f. The conclusion of Step 2 (justified with the needle-problem approach) is the following: If the REV-analysis provides the material law $\xi \mapsto \sigma(\xi)$, then the behavior of the material on the macroscopic scale is characterized by $-\nabla \cdot (\sigma(\nabla u)) = f$ in Q (in our example by $-\nabla \cdot (a^* \nabla u) = f$). In [18], methods are developed and the two-step scheme is illustrated in the linear model: The assumption of an averaging property on simplices implies the homogenization on the macroscopic scale with the corresponding law.

In [11], we performed Step 2 of the needle-problem approach in the context of plasticity. Our assumption was that the material parameters allow averaging: solutions on simplices with affine boundary data $x \mapsto \xi \cdot x$ and vanishing forces $f \equiv 0$ have convergent stress averages: in the limit $\varepsilon \to 0$, stress integrals converge to some deterministic quantity $\Sigma(\xi)$. Due to memory effects in plasticity problems, one has to find for every evolution of strains $\xi = \xi(t)$ an evolution of stresses $\Sigma(\xi)(t) = \Sigma(\xi(.))(t)$. In [11], we derived from this averaging assumption a homogenization result: For general domains Q, general boundary data U and general forces f, the effective problem for every limit $u = \lim_{\varepsilon \to 0} u^\varepsilon$ reads

$$-\nabla \cdot \Sigma(\nabla^s u) = f \quad \text{ in } Q \times (0,T)\,. \tag{1.4}$$

We recall the precise statement in Section 1.3, see Definition 1.9 and Theorem 1.10.

The stochastic homogenization result

In this contribution, we perform the stochastic homogenization of the plasticity system. Using the results of [11], we only have to verify the following: When the coefficient functions of system (1.1) are given by some ergodic stochastic process, then averaging occurs for the homogeneous plasticity system on simplices with affine boundary data.

The map $\Sigma : \xi \mapsto \sigma$ can be characterized in terms of a stochastic cell-problem. In the homogenized system (1.5), the first relation coincides with (1.4) for $\bar{z} = \Sigma(\nabla^s u)$. The other two relations in (1.5) characterize the map Σ, see Definition 2.2 of Σ.

Definition 1.1 (The limit problem)**.** Let the domain $Q \subset \mathbb{R}^d$ and the time horizon $T > 0$ be as above, let Ω be a probability space with ergodic dynamical system as in Section 1.1, let the stochastic coefficients C, B and Ψ be as in Assumption 1.5. We use the function spaces $L^2_{pot}(\Omega)$ and $L^2_{sol}(\Omega)$ of (1.11) and (1.12). For U and f as in (1.3) we consider the following problem: Find $u \in H^1(0,T;H^1(Q))$, $p \in H^1(0,T;L^2(Q;L^2(\Omega;\mathbb{R}^{d\times d})))$, $z \in H^1(0,T;L^2(Q;L^2_{sol}(\Omega)))$ and $\upsilon \in H^1(0,T;L^2(Q;L^2_{pot}(\Omega)))$ such that, with $\overline{z}(x,t) := \int_\Omega z(x,t,\omega)d\mathcal{P}(\omega)$, there holds

$$\begin{aligned} \int_0^T \int_Q \overline{z} : \nabla\varphi = \int_0^T \int_Q f \cdot \varphi \qquad \forall \varphi \in L^2(0,T;H^1_0(Q))\,, \\ \nabla^s_x u = Cz - \upsilon^s + p \quad \text{a.e. in } [0,T]\times Q \times \Omega\,, \\ \partial_t p \in \partial\Psi(z - Bp) \quad \text{a.e. in } [0,T]\times Q\times\Omega\,. \end{aligned} \tag{1.5}$$

Additionally, we demand that the boundary condition $u = U$ on $\partial Q \times (0,T)$ and the initial condition $p \equiv 0$ for $t = 0$ are satisfied in the sense of traces.

Our main result is the following homogenization statement.

Theorem 1.2 (Stochastic homogenization in plasticity)**.** *Let $Q \subset \mathbb{R}^d$ be a bounded domain, $d \in \{2,3\}$, $T > 0$. Let τ be an ergodic dynamical system on the probability space $(\Omega, \Sigma_\Omega, \mathcal{P})$ as in Section 1.1, let the stochastic coefficients B, C, Ψ and the data U and f be as in Assumption 1.5. Then, there exists a unique solution (u,p,z,υ) to the limit problem* (1.5) *of Definition 1.1. For $\omega \in \Omega$, let $(u^\varepsilon, \sigma^\varepsilon, e^\varepsilon, p^\varepsilon)$ be weak solutions to* (1.1)*. Then, for a.e. $\omega \in \Omega$, as $\varepsilon \to 0$,*

$$\begin{aligned} &u^\varepsilon \rightharpoonup u \qquad \textit{weakly in } H^1(0,T;H^1(Q)) \textit{ and} \\ &p^\varepsilon \rightharpoonup \int_\Omega p\, d\mathcal{P}\,,\ \ \sigma^\varepsilon \rightharpoonup \int_\Omega z\, d\mathcal{P} \qquad \textit{weakly in } H^1(0,T;L^2(Q))\,. \end{aligned}$$

Remarks. The weak solution concept for the ε-problem (1.1) is made precise in Definition 1.6. The unique existence of a solution for a.e. $\omega \in \Omega$ is guaranteed by Theorem 1.7.

The proof of Theorem 1.2 is concluded in Section 3.4. A sketch of the proof is presented at the end of Section 1.3.

1.1 Setting in stochastic homogenization

Let (Ω, d_Ω) be a compact metric space with the corresponding Borel-σ-algebra Σ_Ω and a Borel measure $\mathcal{P}$ such that $(\Omega, \Sigma_\Omega, \mathcal{P})$ is a probability space. Let $(\tau_x)_{x\in\mathbb{R}^d}$ be an ergodic dynamical system on $(\Omega, \Sigma_\Omega, \mathcal{P})$. We rely on the following definitions: A family $(\tau_x)_{x\in\mathbb{R}^d}$ of measurable bijective mappings $\tau_x : \Omega \mapsto \Omega$ is called a dynamical system on $(\Omega, \Sigma_\Omega, \mathcal{P})$ if it satisfies

(i) $\tau_x \circ \tau_y = \tau_{x+y}$, $\tau_0 = id$ (group property)

(ii) $\mathcal{P}(\tau_{-x}B) = \mathcal{P}(B)$ $\forall x \in \mathbb{R}^d$, $B \in \Sigma_\Omega$ (measure preservation)

(iii) $A : \mathbb{R}^d \times \Omega \to \Omega$ $(x, \omega) \mapsto \tau_x\omega$ is continuous (continuity property)

We say that the system $(\tau_x)_{x\in\mathbb{R}^d}$ is ergodic, if for every measurable function $f : \Omega \to \mathbb{R}$ holds

$$\left[f(\omega) = f(\tau_x\omega)\ \forall x \in \mathbb{R}^d,\ a.e.\ \omega \in \Omega\right] \Rightarrow [\exists c_0 \in \mathbb{R} :\ f(\omega) = c_0 \text{ for a.e. } \omega \in \Omega]\ . \tag{1.6}$$

An important property of ergodic dynamical systems is the fact that spatial averages can be related to expectations. For a quite general version of the ergodic theorem, we refer to [24].

Theorem 1.3 (Ergodic theorem)**.** *Let $(\Omega, \Sigma_\Omega, \mathcal{P})$ be a probability space with an ergodic dynamical system $(\tau_x)_{x\in\mathbb{R}^d}$ on Ω. Let $f \in L^1(\Omega)$ be a function and $Q \subset \mathbb{R}^d$ be a bounded open set. Then, for $\mathcal{P}$-almost every $\omega \in \Omega$,*

$$\lim_{\varepsilon\to 0} \int_Q f(\tau_{\frac{x}{\varepsilon}}\omega)\, dx = |Q| \int_\Omega f\, d\mathcal{P}\, . \tag{1.7}$$

Furthermore, for every $f \in L^p(\Omega)$, $1 \le p \le \infty$, and a.e. $\omega \in \Omega$, the function $f_\omega(x) = f(\tau_x\omega)$ satisfies $f_\omega \in L^p_{loc}(\mathbb{R}^d)$. For $p < \infty$ holds $f_\omega(\cdot/\varepsilon) = f(\tau_{\cdot/\varepsilon}\omega) \rightharpoonup \int_\Omega f\, d\mathcal{P}$ weakly in $L^p_{loc}(\mathbb{R}^d)$ as $\varepsilon \to 0$.

For brevity of notation in calculations and proofs, we will often omit the symbol $d\mathcal{P}$ in Ω-integrals. We assume that the coefficients in (1.1) have the form

$$C_\varepsilon(x) = C(\tau_{\frac{x}{\varepsilon}}\omega)\, , \qquad B_\varepsilon(x) = B(\tau_{\frac{x}{\varepsilon}}\omega)\, , \qquad \Psi_\varepsilon(\sigma) = \Psi(\sigma; \tau_{\frac{x}{\varepsilon}}\omega) \tag{1.8}$$

for some functions B, C, and Ψ.

The stochastic homogenization result uses the function spaces $L^2_{pot}(\Omega)$ and $L^2_{sol}(\Omega)$ that we define next. We denote the set of continuous functions $\Omega \to \mathbb{R}$ by $C(\Omega)$. We note that Ω is a separable metric space and thus the Borel-σ-algebra is generated by countably many open sets. From [5] Theorem 4.13 we get that $L^p(\Omega)$ is separable for every $1 \le p < \infty$. From [4] Theorems 67.2 and 68.1 we get that $C(\Omega)$ is dense in $L^1(\Omega)$ and thus in $L^p(\Omega)$ for every $1 \le p < \infty$.

With the help of the dynamical system τ and the canonical basis $(e_i)_{1\le i\le d}$ of $\mathbb{R}^d$ we define derivatives of a function $f \in C(\Omega)$ by

$$\partial_{\omega,i}\, f(\omega) := \lim_{\mathbb{R}\ni h\to 0} \frac{f(\tau_{he_i}\omega) - f(\omega)}{h}\, , \tag{1.9}$$

if the limit exists. The space of continuously differentiable functions on Ω is

$$C^1(\Omega) := \{f \in C(\Omega) \mid \text{limit (1.9) exists for every } \omega \in \Omega,\, i \le d, \\ \text{and } \partial_{\omega,i} f \in C(\Omega) \text{ for every } 1 \le i \le d\}\,.$$

We use the gradient $\nabla_\omega f := (\partial_{\omega,1} f, \dots, \partial_{\omega,d} f)$ and define divergence $\nabla_\omega \cdot F$ and spaces $C^k(\Omega)$ in the usual way. The spaces $C^k(\Omega)$ are Banach spaces with respect to the norm

$$\|f\|_{C^k(\Omega)} := \max_{\omega\in\Omega} \sum_{|\alpha|\le k} |\partial_\omega^\alpha f(\omega)|\,.$$

From [24] (see note before Definition 2.2), it follows that $C^k(\Omega)$ is dense in $C^{k-1}(\Omega)$ and in $L^p(\Omega)$ for every $1 \le p < \infty$ and for every $k \in \mathbb{N}$.

For a set X, a function $f : \Omega \to X$ and $\omega \in \Omega$, the function $f_\omega : \mathbb{R}^d \to X$, $f_\omega(x) := f(\tau_x\omega)$ is called a realization (or the ω-realization) of f. For every continuous function f, the realizations $f_\omega : x \mapsto f(\tau_x\omega)$ are continuous by (iii) above. If $f \in C^1(\Omega)$, then f_ω is differentiable for every ω, since

$$\partial_i f_\omega(x) := \lim_{\mathbb{R}\ni h\to 0} \frac{f_\omega(x+he_i) - f_\omega(x)}{h} = \lim_{\mathbb{R}\ni h\to 0} \frac{f(\tau_{he_i}\tau_x\omega) - f(\tau_x\omega)}{h} = \partial_{\omega,i}\, f(\tau_x\omega)\,.$$

In particular, for $f \in C^1(\Omega)$, we find

$$\nabla_x f_\omega(x) = \nabla_\omega f(\tau_x\omega) \quad \forall \omega \in \Omega,\, x \in \mathbb{R}^d\,. \tag{1.10}$$

We usually drop the index x in x-derivatives and simply write ∇f_ω for the spatial gradient. With the derivatives of functions on Ω we can now define two useful function spaces as the closure and the orthogonal complement of subsets of $L^2(\Omega, \mathbb{R}^{d\times d})$,

$$L^2_{pot}(\Omega) := \mathrm{cl}_{L^2(\Omega)} \left\{\nabla_\omega u \,|\, u \in C^1(\Omega;\mathbb{R}^d)\right\}\,, \tag{1.11}$$

$$L^2_{sol}(\Omega) := \left(L^2_{pot}(\Omega)\right)^\perp\,. \tag{1.12}$$

Constant maps are in $L^2_{sol}(\Omega)$: for every vector $c \in \mathbb{R}^d$, every $u \in C^1(\Omega, \mathbb{R})$, and almost every $\omega \in \Omega$, we find by the ergodic theorem

$$\left|\int_\Omega \nabla_\omega u \cdot c\right| = \lim_{n\to\infty} \left|\frac{1}{(2n)^d} \int_{[-n,n]^d} \nabla_\omega u(\tau_x\omega)\cdot c\, dx\right| \le \lim_{n\to\infty} \frac{1}{(2n)^d} \int_{\partial[-n,n]^d} \|u\|_\infty |c| = 0\,.$$

We denote by $L^2_{loc}(\mathbb{R}^d;\mathbb{R}^{d\times d})$ the set of measurable functions f such that $f|_U \in L^2(U;\mathbb{R}^{d\times d})$ for every bounded set $U \subset \mathbb{R}^d$. We furthermore use the corresponding function spaces in the spatial variable x,

$$L^2_{pot,loc}(\mathbb{R}^d) := \left\{u \in L^2_{loc}(\mathbb{R}^d;\mathbb{R}^{d\times d}) \mid \forall U \text{ bounded domain, } \exists \varphi \in H^1(U;\mathbb{R}^d) \,:\, u = \nabla\varphi\right\}\,,$$

$$L^2_{sol,loc}(\mathbb{R}^d) := \left\{u \in L^2_{loc}(\mathbb{R}^d;\mathbb{R}^{d\times d}) \mid \int_{\mathbb{R}^d} u \cdot \nabla\varphi = 0 \;\forall \varphi \in C^1_c(\mathbb{R}^d)\right\}\,.$$

The following theorem is an important tool in stochastic homogenization, see e.g. [23] or [24], Lemma 2.3 and Theorem 2.1 for a generalized form.

Theorem 1.4 (Realizations of potentials and solenoidals)**.** *Let $v \in L^2_{pot}(\Omega)$. Then $\mathcal{P}$-almost all realizations $x \mapsto v(\tau_x\omega)$ belong to $L^2_{pot,loc}(\mathbb{R}^d)$. Let $w \in L^2_{sol}(\Omega)$. Then $\mathcal{P}$-almost all realizations $x \mapsto w(\tau_x\omega)$ belong to $L^2_{sol,loc}(\mathbb{R}^d)$.*

Remark. The above theory for $C^k(\Omega)$, $L^2_{pot}(\Omega)$ and $L^2_{sol}(\Omega)$ can also be developed for arbitrary stochastic geometries, see [10]. In particular, the compactness of Ω can be replaced by the assumption that $(\Omega, \sigma, \mathcal{P})$ is the probability space of a stochastic geometry.

Remark. The periodic homogenization setting is a special case of the stochastic setting. In the periodic case we recover known results. The problem on the periodicity cell is related to a problem on Ω, which is formulated in (1.5) with the help of the spaces $L^2_{pot}(\Omega)$ and $L^2_{sol}(\Omega)$.

1.2 Solution concepts and existence results

To formulate a stochastic setting, we consider $C, B \in L^\infty(\Omega; \mathcal{L}(\mathbb{R}^{d\times d}_s, \mathbb{R}^{d\times d}_s))$, pointwise symmetric, such that for $\gamma, \beta > 0$ holds

$$\gamma\,|\xi|^2 \le \xi : C(\omega)\xi \le \frac{1}{\gamma}\,|\xi|^2\,, \qquad \beta\,|\xi|^2 \le \xi : B(\omega)\xi \le \frac{1}{\beta}\,|\xi|^2\,, \tag{1.13}$$

for every $\xi \in \mathbb{R}^d$ and a.e. $\omega \in \Omega$. Let $\Psi : \mathbb{R}^{d\times d}_s \times \Omega \to (-\infty, +\infty]$ be measurable in $\mathbb{R}^{d\times d}_s \times \Omega$, lower semicontinuous and convex in $\mathbb{R}^{d\times d}_s$ for a.e. $\omega \in \Omega$, and with $\Psi(0,\omega) = 0$ for a.e. $\omega \in \Omega$. We furthermore assume that for a.e. $\omega \in \Omega$ there is $c(\omega) > 0$ such that the convex dual satisfies

$$|\Psi^*(\sigma; \tau_x\omega) - \Psi^*(\sigma; \tau_y\omega)| \le c(\omega)\; |x - y|\,|\sigma| \qquad \forall \sigma \in \mathbb{R}^{d\times d}_s\,,\ x, y \in \mathbb{R}^d\,. \tag{1.14}$$

Assumption 1.5 (Data)**.** *Let $C, B \in L^\infty(\Omega; \mathcal{L}(\mathbb{R}^{d\times d}_s, \mathbb{R}^{d\times d}_s))$ and $\Psi : \mathbb{R}^{d\times d}_s \times \Omega \to (-\infty, +\infty]$ be such that* (1.13)–(1.14) *are satisfied. We consider only parameters $\omega \in \Omega$ such that the ω-realizations $C_\omega(x) := C(\tau_x\omega)$, $B_\omega(x) := B(\tau_x\omega)$ are measurable and such that* (1.2) *and* (1.14) *hold. We furthermore assume that U and f satisfy the regularity* (1.3) *and the compatibility conditions $U|_{t=0} = 0$, $f|_{t=0} = 0$.*

Our aim is to study (1.1) with the coefficients defined in (1.8). Omitting the index ω whenever possible, we find that $C_\varepsilon := C_{\varepsilon,\omega}(x) := C(\tau_{\frac{x}{\varepsilon}}\omega)$ and $B_\varepsilon := B_{\varepsilon,\omega}(x) := B(\tau_{\frac{x}{\varepsilon}}\omega)$ satisfy (1.2) and Ψ_ε satisfies

$$\left|\Psi^*_{\varepsilon,\omega}(\sigma; x_1) - \Psi^*_{\varepsilon,\omega}(\sigma; x_2)\right| \le c(\varepsilon,\omega)\,|x_1 - x_2|\,|\sigma|\,. \tag{1.15}$$

The condition (1.15) is of technical nature and is used only in the proof of the existence result, Theorem 1.7. Using the methods of Section 2, one can prove Theorem 1.7 also without this Lipschitz condition.

Definition 1.6 (Weak formulation of the ε-problem)**.** We say that $(u^\varepsilon, \sigma^\varepsilon, e^\varepsilon, p^\varepsilon)$ is a weak solution to the ε-problem (1.1) on Q with boundary condition U if the following is satisfied: There holds $u^\varepsilon = v^\varepsilon + U$ with

$$v^\varepsilon \in H^1(0,T; H^1_0(Q))\,, \quad e^\varepsilon, p^\varepsilon, \sigma^\varepsilon \in H^1(0,T; L^2(Q; \mathbb{R}^{d\times d}_s))\,,$$

equation $(1.1)_1$ holds in the distributional sense and the other relations of (1.1) hold pointwise almost everywhere in $Q \times (0,T)$.

We note that, due to the regularity of σ^ε, every weak solution to (1.1) satisfies

$$\int_0^T \int_Q \sigma^\varepsilon : \nabla^s \varphi = \int_0^T \int_Q f \cdot \varphi \qquad \forall \varphi \in L^2(0,T; H^1_0(Q)) \,. \tag{1.16}$$

Theorem 1.2 of [11] provides the following:

Theorem 1.7 (Existence of solutions to the ε-problem)**.** *Let the coefficient functions C, B, Ψ, the parameter $\omega \in \Omega$, and the data U and f be as in Assumption 1.5. Then, for every $\varepsilon > 0$, there exists a unique weak solution $(u^\varepsilon, \sigma^\varepsilon, e^\varepsilon, p^\varepsilon)$ to the ε-problem* (1.1) *in the sense of Definition 1.6. The solutions satisfy the a priori estimate*

$$\|u^\varepsilon\|_{\mathcal{V}^1_1} + \|e^\varepsilon\|_{\mathcal{V}^1_0} + \|p^\varepsilon\|_{\mathcal{V}^1_0} + \|\sigma^\varepsilon\|_{\mathcal{V}^1_0} \le C \,, \tag{1.17}$$

in the spaces $\mathcal{V}^1_0 := H^1(0,T; L^2(Q; \mathbb{R}^{d\times d}_s))$ and $\mathcal{V}^1_1 := H^1(0,T; H^1_0(Q))$, the constant $C = C(U, f, \beta, \gamma)$ depends on β and γ from (1.2)*, but it does not depend on $\varepsilon > 0$ or $\omega \in \Omega$.*

1.3 The needle problem approach to plasticity

The main result of [11] is a homogenization theorem. Under the assumption that causal operators Σ and Π satisfy certain admissibility and averaging properties, we obtain the convergence of the ε-solutions u^ε to the solution u of the effective problem (1.4). We next recall the required properties. In the following, we use the space $H^1_*(0,T; \mathbb{R}^{d\times d}_s) := H^1(0,T; \mathbb{R}^{d\times d}_s) \cap \{\xi \mid \xi|_{t=0} = 0\}$ of evolutions with vanishing initial values.

Definition 1.8 (Averaging)**.** We say that a map $F : H^1_*(0,T; \mathbb{R}^{d\times d}_s) \to H^1(0,T; \mathbb{R}^{d\times d}_s)$ defines a *causal operator,* if, for almost every $t \in [0,T]$, the value $F(\xi, t) := F(\xi)(t)$ is independent of $\xi|_{(t,T]}$. We say that the coefficients C_ε, B_ε and Ψ_ε *allow averaging,* if there exist causal operators Σ and Π such that the following property holds: For every simplex $\mathcal{T} \subset Q$, every boundary condition $\xi \in H^1_*(0,T; \mathbb{R}^{d\times d}_s)$ and every additive constant $a \in H^1(0,T; \mathbb{R}^d)$, the corresponding solution $(u^\varepsilon, \sigma^\varepsilon, e^\varepsilon, p^\varepsilon)$ of the ε-problem (1.1) on $\mathcal{T}$ with $f = 0$ and $U(x,t) = \xi(t)x + a(t)$ satisfies the following: As $\varepsilon \to 0$, for a.e. $t \in (0,T)$, the averages of p^ε and σ^ε converge:

$$⨍_{\mathcal{T}} p^\varepsilon(t) \to \Pi(\xi)(t) \,, \qquad ⨍_{\mathcal{T}} \sigma^\varepsilon(t) \to \Sigma(\xi)(t) \,. \tag{1.18}$$

Here, $⨍_{\mathcal{T}} = |\mathcal{T}|^{-1} \int_{\mathcal{T}}$ denotes averages. In particular, we demand that limits of (averages of) stress and plastic strain depend only on the (time-dependent) boundary condition ξ, not on a and not on the simplex $\mathcal{T}$.

Definition 1.9 (Effective equation in the needle problem approach)**.** The effective plasticity problem in the needle problem approach is given by

$$-\nabla \cdot \Sigma(\nabla^s u) = f \quad \text{ in } Q \times (0,T) \,, \tag{1.19}$$

with boundary condition $u = U$ on $\partial Q \times (0,T)$. A function u is a solution to this limit problem if $u = U + v$ holds with $v \in H^1(0,T; H^1_0(Q; \mathbb{R}^d))$ and (1.19) is satisfied in the distributional sense. Regarding the expression $\Sigma(\nabla^s u)$ we note that, for a.e. $x \in Q$, the map $t \mapsto \nabla^s u(x,t)$ is in the space $H^1_*(0,T; \mathbb{R}^{d\times d}_s)$, hence $\Sigma(\nabla^s u)$ is well-defined for almost every point in $Q \times (0,T)$.

Result of the needle problem approach. Theorem 1.6 and Remark 5 of [11] (Proposition 4.6 in a revised version) provide the following result:

Theorem 1.10 (Needle-approach homogenization theorem in plasticity). *Let $Q \subset \mathbb{R}^d$ be open and bounded, let the data f and U be as in Assumption 1.5, let the coefficients C_ε, B_ε and Ψ_ε be as above, satisfying* (1.2). *Let the data allow averaging in the sense of Definition 1.8 with causal operators Σ and Π, and let Σ satisfy the admissibility condition of Definition 1.11. Let $(u^\varepsilon, \sigma^\varepsilon, e^\varepsilon, p^\varepsilon)$ be the weak solutions to the ε-problems* (1.1). *Then, as $\varepsilon \to 0$, there holds*

$$\begin{aligned} &u^\varepsilon \rightharpoonup u \quad \textit{weakly in } H^1(0,T;H^1_0(Q;\mathbb{R}^d))\,,\\ &p^\varepsilon \rightharpoonup \Pi(\nabla^s u), \quad \sigma^\varepsilon \rightharpoonup \Sigma(\nabla^s u) \quad \textit{weakly in } H^1(0,T;L^2(Q;\mathbb{R}^{d\times d}))\,, \end{aligned}$$

where u is the unique weak solution to the homogenized problem

$$-\nabla\cdot\Sigma(\nabla^s u) = f \qquad \textit{on } Q\times(0,T)$$

with boundary condition U in the sense of Definition 1.9.

An assumption that implies admissibility. For arbitrary $h > 0$, we use a polygonal domain $Q_h \subset Q$ and a triangulation $\mathbb{T}_h$ with the properties

$$\begin{aligned} &\mathbb{T}_h := \{\mathcal{T}_k\}_{k\in\Lambda_h} \quad \text{is a triangulation of } Q_h, \quad \operatorname{diam}(\mathcal{T}_k) < h \quad \forall\,\mathcal{T}_k \in \mathbb{T}_h,\\ &Q_h \text{ has the property that } x \in Q, \operatorname{dist}(x,\partial Q) \geq h \text{ implies } x \in Q_h\,, \end{aligned} \tag{1.20}$$

where $\mathcal{T}_k$ are disjoint open simplices and $\Lambda_h \subset \mathbb{N}$ is a finite set of indices. We always assume that the sequence of meshes is regular in the sense of [6], Section 3.1. As in [18], we consider the finite element space of continuous and piecewise linear functions with vanishing boundary values,

$$Y_h := \left\{\phi \in H^1_0(Q) \mid \phi|_{\mathcal{T}_k} \text{ is affine } \forall\,\mathcal{T}_k \in \mathbb{T}_h,\ \phi \equiv 0 \text{ on } Q\setminus Q_h\right\}\,. \tag{1.21}$$

Discretization of boundary conditions: We may extend the triangulation of Q_h by a finite amount of simplices with diameter not greater than h to obtain a grid $\tilde{\mathbb{T}}_h$ that covers Q in the sense $Q \subset \bigcup_{\mathcal{T}_k\in\tilde{\mathbb{T}}_h} \bar{\mathcal{T}}_k$ and introduce $\tilde{Y}_h := \left\{\phi \in H^1(Q) \mid \phi|_{\mathcal{T}_k\cap Q} \text{ is affine } \forall\,\mathcal{T}_k \in \tilde{\mathbb{T}}_h\right\}$. Denoting by $R_{Q,h}$ the H^1-orthogonal Riesz-projection $H^1(Q) \to \tilde{Y}_h$, we set $U_h := R_{Q,h}(U)$ and observe that $U_h \to U$ converges strongly in $H^1(0,T;H^1(Q))$ as $h \to 0$.

Definition 1.11 (Sufficient condition for admissibility of Σ). We consider a causal operator $\Sigma : H^1_*(0,T;\mathbb{R}^{d\times d}_s) \to H^1(0,T;\mathbb{R}^{d\times d}_s)$. We say that Σ *satisfies the sufficient condition for admissibility* if the following property holds: Let $h \to 0$ be a sequence of positive numbers, let $\mathbb{T}_h$ be a sequence of regular grids satisfying (1.20), and let $v_h \in L^2(0,T;Y_h)$ be a corresponding sequence of solutions to the discretized problems (the existence is guaranteed in [11])

$$\int_Q \Sigma\left(\nabla^s\left(v_h + U_h\right)\right) : \nabla\varphi_h = \int_Q f\varphi_h \qquad \forall\varphi_h \in L^2(0,T;Y_h)\,.$$

Assume furthermore that the solutions converge, $v_h \rightharpoonup v$ weakly in $H^1(0,T;H^1_0(Q))$ as $h \to 0$. Then v is a solution to

$$\int_Q \Sigma\left(\nabla^s\left(v + U\right)\right) : \nabla\varphi = \int_Q f\varphi \qquad \forall\varphi \in L^2(0,T;H^1_0(Q))\,.$$

Using Theorem 1.10, our stochastic homogenization result of Theorem 1.2 can be shown as follows: For stochastic parameters C_ε, B_ε and Ψ_ε we define causal operators Σ and Π with cell-problems on Ω. For these operators, we only have to check the averaging property of Definition 1.8 and the admissibility condition of Definition 1.11.

2 Stochastic cell problem and definition of Σ

We investigate an auxiliary problem, using the space $H^1_*(0,T;\mathbb{R}^{d\times d}_s) := H^1(0,T;\mathbb{R}^{d\times d}_s) \cap \{\xi | \xi|_{t=0} = 0\}$ of evolutions with vanishing initial values. For any function $\xi \in H^1_*(0,T;\mathbb{R}^{d\times d}_s)$ we consider the ordinary differential equation (inclusion) for $p(t,\,.\,) \in L^2(\Omega;\mathbb{R}^{d\times d}_s)$,

$$\partial_t p(t,\omega) \in \partial\Psi\left(z(t,\omega) - B(\omega)\, p(t,\omega)\,;\,\omega\right)\,, \tag{2.1}$$

with the initial condition $p(0,\omega) = 0$. In order to close the system, the function $z(t)$ must be determined through $\xi(t)$ and $p(t)$. We search for a map $z(t) \in L^2_{sol}(\Omega)$, symmetric in every point ω, i.e. $z(t,\omega) = z^s(t,\omega)$, such that the equality

$$Cz(t) = \xi(t) + \upsilon^s(t) - p(t) \tag{2.2}$$

holds in $L^2(\Omega)$ for a function $\upsilon \in L^2(0,T;L^2_{pot}(\Omega))$. We remark that $\upsilon \in L^2_{pot}(\Omega)$ does not imply $\upsilon^s \in L^2_{pot}(\Omega)$. Equation (2.2) is (up to the matrix factor C and the symmetrization) a Helmholz decomposition of $\xi(t) - p(t)$.

The next theorem provides the crucial existence result. Furthermore, the function spaces regarding the time dependence are made precise.

Theorem 2.1. *Let C, B and Ψ be as in Assumption 1.5. Then, for $\xi \in H^1_*(0,T;\mathbb{R}^{d\times d}_s)$, there exists a unique solution $(p,z,\upsilon) \in H^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_s)) \times H^1(0,T;L^2_{sol}(\Omega;\mathbb{R}^{d\times d})) \times H^1(0,T;L^2_{pot}(\Omega;\mathbb{R}^{d\times d}))$ with $z = z^s$ to (2.1)–(2.2) satisfying the a priori estimate*

$$\|p\|_{\mathcal{V}^1_0} + \|z\|_{\mathcal{V}^1_0} + \|\upsilon\|_{\mathcal{V}^1_0} \le C\,\|\xi\|_{H^1(0,T)}\,, \tag{2.3}$$

where $\mathcal{V}^1_0 := H^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_s))$. The solution $(p,z,\upsilon) \in (\mathcal{V}^1_0)^3$ depends continuously on $\xi \in H^1_*(0,T;\mathbb{R}^{d\times d}_s)$ with respect to the weak topologies in both spaces.

Theorem 2.1 permits us to define the operators Σ and Π.

Definition 2.2 (The effective plasticity operators)**.** For arbitrary $\xi \in H^1_*(0,T;\mathbb{R}^{d\times d}_s)$, let (p,z,υ) be the solution of (2.1)–(2.2) with $z = z^s$. We set

$$\Sigma(\xi)(t) := \int_\Omega z(t,\omega)\,d\mathcal{P}(\omega)\,, \qquad \Pi(\xi)(t) := \int_\Omega p(t,\omega)\,d\mathcal{P}(\omega)\,. \tag{2.4}$$

We note that the operators $\Sigma, \Pi : H^1_*(0,T;\mathbb{R}^{d\times d}_s) \to H^1(0,T;\mathbb{R}^{d\times d}_s)$ are well defined and continuous by Theorem 2.1.

The rest of this section is devoted to the proof of Theorem 2.1. We proceed as follows: In Section 2.1, we introduce a Galerkin approximation scheme for (2.1)–(2.2), using additionally a regularization of Ψ. In 2.2, we recall some results from the theory of convex functions, in 2.3 we provide a Korn's inequality on Ω. In Section 2.4 we prove existence and uniqueness of solutions to the approximate problems and show that these solutions satisfy uniform bounds. Finally, in Section 2.5, we show that the solutions of the approximate problems converge to the unique solution of the original system (2.1)–(2.2).

2.1 Galerkin method and regularization

Finite dimensional approximation. In what follows, let $\langle\varphi,\psi\rangle_\Omega := \int_\Omega \varphi : \psi \, d\mathcal{P}$ denote the scalar product in $L^2(\Omega) := L^2(\Omega;\mathbb{R}^{d\times d})$. We choose complete orthonormal systems $\{e_k\}_{k\in\mathbb{N}}$ of $L^2_{pot}(\Omega)$ and $\{\tilde{e}_k\}_{k\in\mathbb{N}}$ of $L^2_{sol}(\Omega)$ and consider the finite dimensional spaces

$$\tilde{L}^2_n(\Omega) := \operatorname{span}\{e_k\}_{k=1,\dots,n} \oplus \operatorname{span}\{\tilde{e}_k\}_{k=1,\dots,n}\,, \qquad L^2_n(\Omega) := \tilde{L}_n(\Omega) \oplus \left\{v^s \,|\, v\in \tilde{L}_n(\Omega)\right\}\,,$$
$$L^2_{pot,n}(\Omega) := L^2_{pot}(\Omega)\cap L^2_n(\Omega)\,, \qquad L^2_{sol,n}(\Omega) := L^2_{sol}(\Omega)\cap L^2_n(\Omega)\,.$$

We furthermore set $L^2_s(\Omega) := L^2(\Omega;\mathbb{R}^{d\times d}_s)$ and $L^2_{n,s}(\Omega) := \{v^s \,|\, v\in L^2_n(\Omega)\}$. Since constants are in $L^2_{sol}(\Omega)$, we can assume that they are in $L^2_n(\Omega)$ and thus in $L^2_{sol,n}(\Omega)$ for every $n\geq d^2$. We finally introduce the orthogonal projection $P_n : L^2(\Omega)\to L^2_n(\Omega)$ and note that $P_n\varphi\to\varphi$ strongly in $L^2(\Omega;\mathbb{R}^{d\times d})$ as $n\to\infty$ for every $\varphi\in L^2(\Omega;\mathbb{R}^{d\times d})$.

Definition of regularized convex functionals. In order to prove Theorem 2.1, we consider the family of Moreau-Yosida approximations

$$\Psi^\delta(\sigma,\omega) := \inf_{\xi\in\mathbb{R}^{d\times d}_s}\left\{\Psi(\xi,\omega) + \frac{|\xi-\sigma|^2}{2\delta}\right\}\,, \tag{2.5}$$

satisfying (see [15], Exercise 12.23; for the definition of the subdifferential $\partial\Psi^\delta$ see (2.10))

$$\begin{aligned} \Psi^\delta &: \mathbb{R}^{d\times d}_s \to \mathbb{R} \quad \text{is convex, coercive and continuously differentiable}\\ \partial\Psi^\delta &: \mathbb{R}^{d\times d}_s \to \mathbb{R}^{d\times d}_s \quad \text{is single valued and globally Lipschitz-continuous}\\ &\lim_{\delta\to 0}\Psi^\delta(\sigma;\omega) = \Psi(\sigma;\omega) \qquad \forall\sigma\in\mathbb{R}^{d\times d}_s\text{, and a.e. } \omega\in\Omega\,. \end{aligned} \tag{2.6}$$

Note that the last convergence is monotone, since $\Psi^{\delta_2}\geq\Psi^{\delta_1}$ for all $\delta_2<\delta_1$. Given Ψ and Ψ^δ, we consider the corresponding functionals

$$\Upsilon,\Upsilon^\delta :\; L^2_s(\Omega)\to\mathbb{R}\,,\quad \Upsilon(z) := \int_\Omega \Psi(z(\omega))\,d\mathcal{P}(\omega)\,,\; \Upsilon^\delta(z) := \int_\Omega \Psi^\delta(z(\omega))\,d\mathcal{P}(\omega)\,. \tag{2.7}$$

We denote by $\Upsilon_n : L^2_{n,s}(\Omega)\to\mathbb{R}$ the restriction of Υ to $L^2_{n,s}(\Omega)$. the subdifferential of Υ_n is $\partial\Upsilon_n$. Accordingly, we can define Υ^δ_n and $\partial\Upsilon^\delta_n$.

The approximate problem for (2.1)–(2.2)

We consider the following problem on discretized function spaces: Given $\xi\in H^1_*(0,T;\mathbb{R}^{d\times d}_s)$, we look for

$$p_{\delta,n}\in C^1(0,T;L^2_{n,s}(\Omega))\,,\qquad z_{\delta,n}\in H^1(0,T;L^2_{sol,n}(\Omega))\,,\qquad v_{\delta,n}\in H^1(0,T;L^2_{pot,n}(\Omega))\,,$$

with the symmetry $z_{\delta,n} = z^s_{\delta,n}$, satisfying

$$\partial_t p_{\delta,n} = \partial\Upsilon^\delta_n\left(z_{\delta,n} - B_n\, p_{\delta,n}\right) \tag{2.8}$$

and $C_n\, z_{\delta,n} = \xi + \upsilon^s_{\delta,n} - p_{\delta,n}$. The last equation can be written as

$$z_{\delta,n} = C_n^{-1} \left(\xi + \upsilon^s_{\delta,n} - p_{\delta,n} \right) \,. \tag{2.9}$$

Here, $B_n,\, C_n : L^2_{n,s}(\Omega) \to L^2_{n,s}(\Omega)$ are bounded positive (and thus invertible) operators defined through

$$\langle B_n \psi, \varphi \rangle_\Omega = \int_\Omega (B\psi) : \varphi \,, \quad \langle C_n \psi, \varphi \rangle_\Omega = \int_\Omega (C\psi) : \varphi \qquad \forall \varphi, \psi \in L^2_{n,s}(\Omega) \,.$$

We obtain the existence and uniqueness of solutions to (2.8)–(2.9) from the Picard-Lindelöf theorem: We show that the system can be understood as a single ordinary differential equation for $p_{\delta,n}$ with Lipschitz continuous right hand side, and that the solutions are uniformly bounded.

2.2 Convex functionals

Basic concepts of convex functions. We recall some well known results from convex analysis on a separable Hilbert space X with scalar product "$\cdot$". In the following, $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is a convex and lower-semicontinuous functional with $\varphi \not\equiv +\infty$. The domain of φ is $dom(\varphi) := \{\sigma \in X | \varphi(\sigma) < +\infty\}$, and the Legendre-Fenchel conjugate φ^* is defined by

$$\varphi^* : X \to \mathbb{R} \cup \{+\infty\}, \quad \varepsilon \mapsto \sup_{\sigma \in X} \{\varepsilon \cdot \sigma - \varphi(\sigma)\}.$$

The subdifferential $\partial\varphi : dom(\varphi) \to \mathcal{P}(X)$ is defined by

$$\partial\varphi(\sigma) = \{\varepsilon \in X \,|\, \varphi(\xi) \geq \varphi(\sigma) + \varepsilon \cdot (\xi - \sigma) \quad \forall\, \xi \in X\} \,. \tag{2.10}$$

A multivalued operator $f : dom(f) \subset X \to \mathcal{P}(X)$ is said to be *monotone* if

$$(\sigma_1 - \sigma_2) \cdot (\varepsilon_1 - \varepsilon_2) \geq 0, \quad \forall\, \varepsilon_i \in dom(f), \quad \sigma_i \in f(\varepsilon_i), \; (i = 1, 2).$$

In what follows, we frequently use the following properties of convex functionals [15].

Lemma 2.3. *For every convex and lower semicontinuous function φ on a Hilbert space X with $\varphi \not\equiv +\infty$ holds*

(i) φ^* *is convex, lower-semicontinuous, and* $dom(\varphi^*) \neq \emptyset$

(ii) $\partial\varphi,\ \partial\varphi^*$ *are monotone operators*

(iii) $\varphi(\sigma) + \varphi^*(\varepsilon) \geq \sigma \cdot \varepsilon \qquad \forall\, \sigma, \varepsilon \in X$

(iv) $\sigma \in dom(\varphi)$ *and* $\varepsilon \in \partial\varphi(\sigma) \;\Leftrightarrow\; \varepsilon \in dom(\varphi^*)$ *and* $\sigma \in \partial\varphi^*(\varepsilon)$

(v) $\varepsilon \in dom(\varphi^*)$ *and* $\sigma \in \partial\varphi^*(\varepsilon) \;\Leftrightarrow\; \varphi(\sigma) + \varphi^*(\varepsilon) = \sigma \cdot \varepsilon$

(vi) $\varphi^{**} = \varphi$.

We refer to (v) as Fenchel's equality and to (iii) as Fenchel's *in*equality.

Continuity properties of Υ and Υ^δ and subdifferentials

In order to obtain the subdifferential of the functional $\Upsilon : L^2_s(\Omega) \to \mathbb{R}$ we calculate

$$\begin{aligned} a \in \partial\Upsilon(z) \quad &\Leftrightarrow \quad \Upsilon(z+\psi) \geq \Upsilon(z) + \langle a, \psi\rangle_\Omega \quad \forall \psi \in L^2_s(\Omega) \\ &\Leftrightarrow \quad \int_\Omega \Psi(z+\psi) \geq \int_\Omega \Psi(z) + \langle a, \psi\rangle_\Omega \quad \forall \psi \in L^2_s(\Omega) \\ &\Leftrightarrow \quad a(\omega) \in \partial\Psi(z(\omega)) \text{ for a.e. } \omega \in \Omega\,. \end{aligned} \tag{2.11}$$

Similarly, $a \in \partial\Upsilon^\delta(z)$ if and only if $a \in \partial\Psi^\delta(z)$ almost everywhere. Both subdifferentials are therefore single-valued and we may identify $\partial\Upsilon^\delta(z) = \partial\Psi^\delta(z)$. We next determine the subdifferential of the restricted functional Υ^δ_n.

Lemma 2.4. *The functionals Υ^δ_n have a single valued subdifferential in every $z_0 \in L^2_{n,s}(\Omega)$, given through*

$$\partial\Upsilon^\delta_n(z_0) = P_n \partial\Psi^\delta(z_0)\,. \tag{2.12}$$

Proof. Let $a \in \partial\Upsilon^\delta_n(z_0) \subset L^2_{n,s}(\Omega)$ and let id be the identity on $L^2_s(\Omega)$. For arbitrary $\varphi \in L^2_s(\Omega)$ we set $\varphi_n := P_n\varphi$ and $\varphi_o := (id - P_n)\varphi$. We obtain

$$\begin{aligned} \int_\Omega \Psi^\delta(z_0 + t\varphi) = \Upsilon^\delta\left(z_0 + t\varphi_n + t\varphi_o\right) &\geq \Upsilon^\delta_n\left(z_0 + t\varphi_n\right) + t\left\langle \partial\Psi^\delta\left(z_0 + t\varphi_n\right), \varphi_o\right\rangle_\Omega \\ &\geq \Upsilon^\delta_n\left(z_0\right) + t\left\langle a, \varphi_n\right\rangle_\Omega + t\left\langle \partial\Psi^\delta\left(z_0 + t\varphi_n\right), \varphi_o\right\rangle_\Omega \end{aligned}$$

Since Ψ^δ is differentiable and $\partial\Psi^\delta$ is Lipschitz continuous, we obtain from the fact that the subdifferential coincides with the derivative and from the last inequality

$$\left\langle \partial\Psi^\delta(z_0), \varphi\right\rangle_\Omega = \lim_{t\to 0}\frac{1}{t}\left(\int_\Omega \Psi^\delta(z_0 + t\varphi) - \int_\Omega \Psi^\delta(z_0)\right) \geq \langle a, \varphi_n\rangle_\Omega + \left\langle \partial\Psi^\delta\left(z_0\right), \varphi_o\right\rangle_\Omega\,.$$

Replacing φ by $-\varphi$ in the above calculations, we obtain $\partial\Psi^\delta(z_0) = a + (id - P_n)\partial\Psi^\delta(z_0)$ or $P_n\partial\Psi^\delta(z_0) = a$. □

The Fenchel conjugate of Υ^δ_n in $L^2_{n,s}(\Omega)$ is

$$\Upsilon^{\delta*}_n(\sigma) := \sup\left\{\int_\Omega \sigma : e\, d\mathcal{P} - \Upsilon^\delta_n(e)\,|\, e \in L^2_{n,s}(\Omega)\right\}.$$

Since $-\Upsilon^\delta_n(\cdot)$ is coercive in a finite dimensional space, it has compact sublevels in $L^2_n(\Omega)$, and the supremum is indeed attained.

Lemma 2.5. *Let $\Upsilon^{\delta*}$ be the Fenchel conjugate of Υ^δ. For every $p \in L^2_s(\Omega)$ holds*

$$\Upsilon^*(p) = \int_\Omega \Psi^*(p)\, d\mathcal{P}\,, \qquad \Upsilon^{\delta*}(p) = \int_\Omega \Psi^{\delta*}(p)\, d\mathcal{P}\,, \tag{2.13}$$

and the functionals Υ, Υ^*, Υ^δ and $\Upsilon^{\delta*}$ are convex and weakly lower semicontinuous on $L^2_s(\Omega)$.

Proof. The functional Υ is convex with the conjugate

$$\Upsilon^*(p) := \sup\left\{\langle p, e\rangle_\Omega - \Upsilon(e) \,|\, e \in L^2_s(\Omega)\right\} \qquad \forall p \in L^2_s(\Omega)\,.$$

We first prove (2.13): Let $p \in dom\,\Upsilon^* = L^2_s(\Omega)$. Since Υ^* is convex, we know that $\partial\Upsilon^*(p) \neq \emptyset$. Lemma 2.3 (iv) yields for any $\sigma \in \partial\Upsilon^*(p)$ that $\sigma \in dom\,\Upsilon$ with $p \in \partial\Upsilon(\sigma)$ and Lemma 2.3 (v) then yields

$$\Upsilon^*(p) + \Upsilon(\sigma) = \langle p, \sigma\rangle_\Omega\,. \tag{2.14}$$

Since $p \in \partial\Upsilon(\sigma)$, (2.11) yields $p(\omega) \in \partial\Psi(\sigma(\omega);\omega)$ for a.e. $\omega \in \Omega$ and Lemma 2.3 (v) yields $\Psi^*(p)+\Psi(\sigma) = p : \sigma$ a.e.. Integrating the last equality over Ω and comparing with (2.14), we find $\Upsilon^*(p) = \int_\Omega \Psi^*(p)$ since $\Upsilon(\sigma) = \int_\Omega \Psi(\sigma)$. The proof for the second statement in (2.13) is similar.

We now prove the weak lower semicontinuity of Υ^*. Let $\sigma_i \in dom(\Psi)$, $i \in \mathbb{N}$, be dense in $dom(\Psi)$. We define Ψ^*_m as the maximum of finitely many functions

$$\Psi^*_m(p) := \max_{i=1,\dots,m} \{p : \sigma_i - \Psi(\sigma_i)\} \qquad \forall p \in \mathbb{R}^{d\times d}_s$$

and note that $\Psi^*_m(p) \leq \Psi^*(p)$ for every $p \in \mathbb{R}^{d\times d}_s$. For $z \in L^2_s(\Omega)$ and $i = 1,\dots,m$, we introduce the sets

$$\Omega_i := \{\omega \in \Omega \,|\, \Psi^*_m(z) = z : \sigma_i - \Psi(\sigma_i)\} \setminus \bigcup_{j<i} \Omega_j\,.$$

Let $(z_n)_n$ be a sequence such that $z_n \rightharpoonup z$ weakly in $L^2_s(\Omega)$. We find that

$$\begin{aligned}\liminf_{n\to\infty} \int_\Omega \Psi^*(z_n) &\geq \liminf_{n\to\infty} \sum_{i=1}^m \int_\Omega \Psi^*_m(z_n) = \liminf_{n\to\infty} \sum_i \int_{\Omega_i} \max_{j=1,\dots,m} (z_n : \sigma_j - \Psi(\sigma_j))\\ &\geq \liminf_{n\to\infty} \sum_i \int_{\Omega_i} (z_n : \sigma_i - \Psi(\sigma_i)) = \sum_i \int_{\Omega_i} (z : \sigma_i - \Psi(\sigma_i)) = \int_\Omega \Psi^*_m(z)\,.\end{aligned}$$

Since $\Psi^*(p) = \lim_{m\to\infty} \Psi^*_m(p)$ for every $p \in \mathbb{R}^{d\times d}_s$ by definition of Ψ^*_m, and since this convergence is monotone, we can apply the monotone convergence theorem and get $\int_\Omega \Psi^*_m(z) \to \int_\Omega \Psi^*(z) = \Upsilon^*(z)$. This yields the weak lower semicontinuity of Υ^*.

Since Ψ is convex and lower semicontinuous, we find $\Psi = \Psi^{**}$ and switching Ψ and Ψ^* in the above argumentation, the weak lower semicontinuity of Υ follows. The statements for Υ^δ and $\Upsilon^{\delta*}$ follow similarly. □

Convergence properties

We will later need additional lower semicontinuity properties: We have to analyze the behavior of, e.g., $\Upsilon^\delta(u_\delta)$.

Lemma 2.6 (Lower semicontinuity property of Ψ^δ and $\Psi^{\delta*}$)**.** *Let $U_s := \Omega \times (0,s)$ be the space-time cylinder and let $(u_\delta)_\delta$ be a weakly convergent sequence, $u_\delta \rightharpoonup u$ weakly in $L^2(U_s)$ as $\delta \to 0$. Then, for Ψ^δ, Ψ as above, we find*

$$\liminf_{\delta\to 0} \int_{U_s} \Psi^{\delta*}(u_\delta)\, d\mathcal{P}\, dt \geq \int_{U_s} \Psi^*(u)\, d\mathcal{P}\, dt\,. \tag{2.15}$$

For every sequence $(u_\delta)_\delta$ with $u_\delta \rightharpoonup u$ weakly in $L^2_s(\Omega)$ we find

$$\liminf_{\delta\to 0} \Upsilon^\delta(u_\delta) \geq \Upsilon(u)\,. \tag{2.16}$$

Proof. The proof of (2.15) is the same as in [17], Lemma 2.6.

Using the definition of Ψ^δ in (2.5), we choose, for every $\delta > 0$, a function $\pi_\delta \in L^2(\Omega;\mathbb{R}^{d\times d})$ such that

$$\int_\Omega \left(\frac{|\pi_\delta - u_\delta|^2}{\delta} + \Psi(\pi_\delta) \right) d\mathcal{P} \leq \int_\Omega \Psi^\delta(u_\delta)\, d\mathcal{P} + \delta\,.$$

Without loss of generality, we may assume $\liminf_{\delta\to 0} \int_\Omega \Psi^\delta(u_\delta)\, d\mathcal{P} < \infty$. Then we get for a subsequence $\int_\Omega |\pi_\delta - u_\delta|^2 \to 0$ as $\delta \to 0$ and hence $\pi_\delta \rightharpoonup u$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$ for this subsequence. Since $\int_\Omega |\pi_\delta - u_\delta|^2$ is positive and $\Upsilon(z) = \int_\Omega \Psi(z)$ is weakly lower semicontinuous, we find (2.16). □

The following lemma uses time-dependent functions and the discretization parameter $n \in \mathbb{N}$.

Lemma 2.7. *Let $s > 0$ and let $p \in L^2(0,s;L^2_s(\Omega))$ and $p_n \in L^2(0,s;L^2_n(\Omega))$ such that $p_n \rightharpoonup p$ weakly in $L^2(0,s;L^2(\Omega;\mathbb{R}^{d\times d}))$ as $n \to \infty$. Then, for $\Upsilon^{\delta*}_n$ and Υ^δ_n as above we find*

$$\liminf_{n\to\infty} \int_0^s \Upsilon^\delta_n(p_n)\, dt \geq \int_0^s \Upsilon^\delta(p)\, dt\,, \qquad \liminf_{n\to\infty} \int_0^s \Upsilon^{\delta*}_n(p_n)\, dt \geq \int_0^s \Upsilon^{\delta*}(p)\, dt\,. \tag{2.17}$$

Furthermore, if $z_n \to z$ strongly in $L^2(\Omega;\mathbb{R}^{d\times d})$ as $n \to \infty$, then

$$\lim_{n\to\infty} \Upsilon^\delta_n(z_n) = \Upsilon^\delta(z)\,. \tag{2.18}$$

Proof. Let $z_n \to z$ strongly in $L^2(\Omega;\mathbb{R}^{d\times d})$. Since Ψ^δ is Lipschitz continuous with $\Psi^\delta(0) = 0$, we find because of $\Upsilon^\delta_n(z_n) = \Upsilon^\delta(z_n)$

$$\lim_{n\to\infty} \Upsilon^\delta_n(z_n) = \lim_{n\to\infty} \int_\Omega \Psi^\delta(z_n) = \int_\Omega \Psi^\delta(z)$$

and thus (2.18). For $p_n \rightharpoonup p$ weakly in $L^2(U_s)$ with $p_n \in L^2(0,s;L^2_n(\Omega))$, the first inequality in (2.17) can be proved similarly to the weak lower semicontinuity results of Lemma 2.5, using $\Upsilon^\delta_n(p_n) = \Upsilon^\delta(p_n)$.

For the second inequality in (2.17), we choose finite sets $B_n = \{e^i_n \,|\, i = 1,\ldots,K_n\} \subset L^2_n(\Omega)$ with $K_n \geq n$ such that $B_n \subset B_{n+1}$ and $\bigcup_n B_n$ is dense in $L^2(\Omega;\mathbb{R}^{d\times d})$. For fixed $N \in \mathbb{N}$, the interval $[0,s]$ is split into subsets

$$\tilde{\mathbb{T}}^i_N := \left\{ t \in [0,s] \;|\; \max\left\{ \langle e, p(t)\rangle_\Omega - \Upsilon^\delta(e) \,|\, e \in B_N \right\} = \left\langle e^i_N, p(t)\right\rangle_\Omega - \Upsilon^\delta(e^i_N) \right\} \tag{2.19}$$

and we set $\mathbb{T}^1_N := \tilde{\mathbb{T}}^1_N$ and $\mathbb{T}^i_N := \tilde{\mathbb{T}}^i_N \setminus \bigcup_{j<i} \mathbb{T}^j_N$ for $i = 2,\ldots,K_N$. For $n \geq N$ we find, decomposing the time integral, taking the maximum, performing the weak limit, and using

the definition of $\mathbb{T}_N^i$:

$$\begin{aligned}
\liminf_{n\to\infty}\int_0^s \Upsilon_n^{\delta*}(p_n) &\geq \liminf_{n\to\infty}\sum_{i=1}^{K_N}\int_{\mathbb{T}_N^i}\max\left\{\langle e,p_n(t)\rangle_\Omega - \Upsilon_n^\delta(e)\,|\,e\in B_N\right\}dt\\
&\geq \liminf_{n\to\infty}\sum_{i=1}^{K_N}\int_{\mathbb{T}_N^i}\left(\left\langle e_N^i,p_n(t)\right\rangle_\Omega - \Upsilon_n^\delta(e_N^i)\right)dt\\
&= \sum_{i=1}^{K_N}\int_{\mathbb{T}_N^i}\left(\left\langle e_N^i,p(t)\right\rangle_\Omega - \Upsilon^\delta(e_N^i)\right)dt\\
&\overset{(2.19)}{=} \sum_{i=1}^{K_N}\int_{\mathbb{T}_N^i}\max\left\{\langle e,p(t)\rangle_\Omega - \Upsilon^\delta(e)\,|\,e\in B_N\right\}\,dt\\
&= \sup\left\{\int_0^s\left(\langle \tilde e,p(t)\rangle_\Omega - \Upsilon^\delta(\tilde e(t))\right)dt\,|\,\tilde e\in L^2(0,s;B_N)\right\}.
\end{aligned}$$

This inequality implies, due to density of $\bigcup_N B_N$ in $L^2(\Omega;\mathbb{R}^{d\times d})$,

$$\begin{aligned}
\liminf_{n\to\infty}\int_0^s \Upsilon_n^{\delta*}(p_n) &\geq \sup\left\{\int_0^s\int_\Omega\left(e:p-\Psi^\delta(e)\right)\,|\,e\in L^2(0,s;L^2(\Omega;\mathbb{R}^{d\times d}))\right\}\\
&= \int_0^s\int_\Omega \Psi^{\delta*}(p) = \int_0^s \Upsilon^{\delta*}(p)\,,
\end{aligned}$$

where we used (2.13) in the last equality. We have thus verified the second inequality of (2.17). □

2.3 A Korn's inequality on Ω

Lemma 2.8 (A Korn's inequality on Ω). *For all $\upsilon\in L^2_{pot}(\Omega)$ holds*

$$\|\upsilon\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \leq 2\,\|\upsilon^s\|_{L^2(\Omega;\mathbb{R}^{d\times d})}\,. \tag{2.20}$$

Proof. In what follows, we will often use the following: compactness of Ω implies boundednes of every $g\in C(\Omega)$. Denoting $Q_n := (-n,n)^d\subset\mathbb{R}^d$ the cube with volume $(2n)^d$ we find with the help of $|\partial Q_n| = 2d\,(2n)^{d-1}$

$$\lim_{n\to\infty}\left|\frac{1}{(2n)^d}\int_{\partial Q_n} g_\omega(x)\,dx\right| \leq \lim_{n\to\infty}\frac{d}{n}\sup_{\omega\in\Omega}|g(\omega)| = 0\,.$$

In what follows, we denote by ν the outer normal vector of the domain Q_n. For any $f\in C^2(\Omega)$, we get from the ergodic theorem 1.3 and relation (1.10) for $\mathcal{P}$-a.e. $\omega\in\Omega$

$$\begin{aligned}
\int_\Omega |\nabla_\omega^s f|^2\,d\mathcal{P} = \int_\Omega \nabla_\omega^s f:\nabla_\omega f\,d\mathcal{P} &\overset{1.3}{=} \lim_{n\to\infty}\frac{1}{2^d}\int_{(-1,1)^d}\nabla_\omega^s f(\tau_{nx}\omega):\nabla_\omega f(\tau_{nx}\omega)\,dx\\
= \lim_{n\to\infty}\frac{1}{(2n)^d}\int_{Q_n}\nabla_\omega^s f(\tau_x\omega):\nabla_\omega f(\tau_x\omega)\,dx &\overset{(1.10)}{=} \lim_{n\to\infty}\frac{1}{(2n)^d}\int_{Q_n}\nabla^s f_\omega:\nabla f_\omega\,dx\\
= \lim_{n\to\infty}\frac{1}{(2n)^d}\left(\int_{\partial Q_n}\nu\,(\nabla^s f_\omega)\,f_\omega - \int_{Q_n}\nabla\cdot(\nabla^s f_\omega)\,f_\omega\,dx\right)
\end{aligned}$$

$$= \lim_{n\to\infty} \frac{1}{(2n)^d} \left(- \int_{Q_n} \sum_i \left(\frac{1}{2}\partial_i \nabla \cdot f_\omega + \frac{1}{2}\Delta f_{\omega,i} \right) f_{\omega,i}\, dx \right)$$
$$= \lim_{n\to\infty} \frac{1}{(2n)^d} \frac{1}{2} \int_{Q_n} \left(|\nabla \cdot f_\omega|^2 + |\nabla f_\omega|^2 \right) dx$$
$$\geq \lim_{n\to\infty} \frac{1}{(2n)^d} \frac{1}{2} \int_{Q_n} |\nabla f_\omega|^2 = \lim_{n\to\infty} \frac{1}{(2n)^d} \frac{1}{2} \int_{Q_n} |(\nabla_\omega f)(\tau_x \omega)|^2\, dx = \frac{1}{2} \int_\Omega |\nabla_\omega f|^2\, d\mathcal{P}\,.$$

We have thus found (2.20) for $\upsilon = \nabla_\omega f$.

Let now $\upsilon \in L^2_{pot}(\Omega)$ be arbitrary. By definition of $L^2_{pot}(\Omega)$, there exists a sequence $(g_k)_{k\in\mathbb{N}}$ in $C^1(\Omega)$ such that $\nabla g_k \to \upsilon$ in $L^2(\Omega;\mathbb{R}^{d\times d})$, implying $\nabla^s g_k \to \upsilon^s$ in $L^2(\Omega;\mathbb{R}^{d\times d})$. On the other hand, for every $g_k \in C^1(\Omega)$ we find $f_k \in C^2(\Omega)$ with $\|f_k - g_k\|_{C^1(\Omega)} \leq \frac{1}{k}$ by density. Estimate (2.20) holds for $\upsilon_k = \nabla_\omega f_k$, hence, in the limit $k \to \infty$, the estimate holds for $\upsilon^s = \lim_{k\to\infty} \nabla^s_\omega f_k$ and $\upsilon = \lim_{k\to\infty} \nabla_\omega f_k$. □

2.4 Solutions to the approximate problem and a priori estimates

Lemma 2.9. *There exists a unique solution $p_{\delta,n}$, $z_{\delta,n}$, $\upsilon_{\delta,n}$ to problem* (2.8)–(2.9) *which satisfies the a priori estimate*

$$\|p_{\delta,n}\|_{\mathcal{V}^1_0} + \|z_{\delta,n}\|_{\mathcal{V}^1_0} + \|\upsilon_{\delta,n}\|_{\mathcal{V}^1_0} \leq c\left(\Upsilon^\delta_n(z_{\delta,n}(0)) + \|\xi\|_{H^1(0,T)} \right), \tag{2.21}$$

with $\mathcal{V}^1_0 := H^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_s))$ and c independent of δ and n.

Proof. In the following, all integrals over Ω are with respect to $\mathcal{P}$ and we omit $d\mathcal{P}$ for ease of notation. We will prove the lemma in two steps: we first show that the system (2.8)–(2.9) is equivalent to an ordinary differential equation for $p_{\delta,n}$ with Lipschitz continuous right hand side. Then, we show that the solution admits uniform a priori estimates.

Step 1: Existence. In order to study (2.8)–(2.9), we fix $\tilde{p} \in L^2_{n,s}(\Omega)$ and $\tilde{\xi} \in \mathbb{R}^{d\times d}_s$, and search for $\tilde{\upsilon} \in L^2(0,T;L^2_{pot,n}(\Omega))$ such that

$$\left\langle C_n^{-1}\tilde{\upsilon}^s, \zeta \right\rangle_\Omega = \left\langle C_n^{-1}\tilde{p}, \zeta \right\rangle_\Omega - \left\langle C_n^{-1}\tilde{\xi}, \zeta \right\rangle_\Omega \qquad \forall \zeta \in L^2_{pot,n}(\Omega)\,. \tag{2.22}$$

The Lax-Milgram theorem in combination with Korn's inequality (2.20) yields a unique solution $\tilde{\upsilon} \in H^1(0,T;L^2_{pot,n}(\Omega))$ of the last equality. We introduce the mapping $V_{\tilde{\xi}} : L^2_{n,s}(\Omega) \to L^2_{pot,n}(\Omega)$ with $V_{\tilde{\xi}}(\tilde{p}) = \tilde{\upsilon}$ and note that this operator is linear and bounded. We then look for a solution $p_{\delta,n} \in C^1(0,T;L^2_n(\Omega))$ to the following version of (2.8):

$$\partial_t p_{\delta,n} = \partial\Upsilon^\delta_n \left(C_n^{-1} \left(\xi + V_\xi(p_{\delta,n})^s - p_{\delta,n} \right) - B_n\, p_{\delta,n} \right).$$

Relation (2.12) yields the Lipschitz continuity of $\partial\Upsilon^\delta_n$. Therefore, since also $\partial\Upsilon^\delta_n$, C_n^{-1}, V^s_ξ and B_n are Lipschitz-continuous mappings $L^2_{n,s}(\Omega) \to L^2_{n,s}(\Omega)$, we find a unique solution $p_{\delta,n} \in C^1([0,T];L^2_{n,s}(\Omega))$ of the ordinary differential equation (a priori bounds are provided below). We furthermore set $\upsilon_{\delta,n} = V_\xi(p_{\delta,n}) \in C^1([0,T];L^2_{n,pot}(\Omega))$ and $z_{\delta,n} = C_n^{-1}\left(\xi + \upsilon^s_{\delta,n} - p_{\delta,n} \right) \in H^1(0,T;L^2_{n,s}(\Omega))$. From (2.22) and the definition of $\upsilon_{\delta,n}$, it follows that $z_{\delta,n} \in H^1(0,T;L^2_{n,sol}(\Omega))$. Note that $p_{\delta,n}$, $z_{\delta,n}$ and $\upsilon_{\delta,n}$ are constructed in such a way that (2.8)–(2.9) holds. The construction shows that the solution is uniquely determined.

Step 2: A priori estimates of order 0. We take the time derivative of (2.9), multiply by $z_{\delta,n}$ and integrate over $[0,t]\times\Omega$ for $t\in(0,T]$ to find

$$\begin{aligned}
\int_0^t\int_\Omega \partial_t\xi : z_{\delta,n} &\overset{(2.9)}{=} \int_0^t\int_\Omega \big((C_n\partial_t z_{\delta,n}) : z_{\delta,n} + \partial_t p_{\delta,n} : z_{\delta,n} - \partial_t \upsilon^s_{\delta,n} : z_{\delta,n}\big)\\
&= \frac{1}{2}\int_\Omega (p_{\delta,n} : B_n p_{\delta,n} + z_{\delta,n} : C_n z_{\delta,n})\Big|_0^t\\
&\quad + \int_0^t \langle \partial_t p_{\delta,n}, z_{\delta,n} - B_n p_{\delta,n}\rangle_\Omega - \int_0^t\int_\Omega z_{\delta,n} : \partial_t \upsilon_{\delta,n}\\
&\overset{(*)}{=} \frac{1}{2}\int_\Omega (p_{\delta,n} : (Bp_{\delta,n}) + z_{\delta,n} : (Cz_{\delta,n}))\Big|_0^t\\
&\quad + \int_0^t \big(\Upsilon^{\delta*}_n(\partial_t p_{\delta,n}) + \Upsilon^\delta_n(z_{\delta,n} - B_n p_{\delta,n})\big)\,. \qquad (2.23)
\end{aligned}$$

In $(*)$ we used the orthogonality of potentials and (symmetric) solenoidals, $\int_\Omega z_{\delta,n} : \partial_t\upsilon_{\delta,n} = 0$, and Lemma 2.3 (v), written as

$$\langle \partial_t p, z - Bp\rangle_\Omega = \Upsilon^\delta_n(z - Bp) + \Upsilon^{\delta*}_n(\partial_t p) \quad\Leftrightarrow\quad \partial_t p = \partial\Upsilon^\delta_n(z - Bp)\,.$$

A priori estimates of order 1. Taking the time derivative of (2.9), multiplying the result by $\partial_t z_{\delta,n}$ and integrating over Ω, we get

$$\begin{aligned}
\int_\Omega \partial_t\xi : \partial_t z_{\delta,n} &= \int_\Omega \partial_t z_{\delta,n} : \partial_t\left(p_{\delta,n} + C_n z_{\delta,n} - \upsilon_{\delta,n}\right) + \int_\Omega (B_n\partial_t p_{\delta,n} - B_n\partial_t p_{\delta,n}) : \partial_t p_{\delta,n}\\
&\overset{(2.8)}{=} \big\langle \partial_t z_{\delta,n} - B_n\partial_t p_{\delta,n}, \partial\Upsilon^\delta_n(z_{\delta,n} - B_n p_{\delta,n})\big\rangle_\Omega + \int_\Omega (B_n\partial_t p_{\delta,n}) : \partial_t p_{\delta,n}\\
&\quad + \int_\Omega (C_n\partial_t z_{\delta,n}) : \partial_t z_{\delta,n} - \int_\Omega \partial_t z_{\delta,n} : \partial_t\upsilon_{\delta,n}\\
&\overset{(*)}{=} \frac{d}{dt}\Upsilon^\delta_n(z_{\delta,n} - B_n p_{\delta,n}) + \int_\Omega \left((C\partial_t z_{\delta,n}) : \partial_t z_{\delta,n} + (B\partial_t p_{\delta,n}) : \partial_t p_{\delta,n}\right)\,,
\end{aligned}$$

where we used $\int_\Omega \partial_t z_{\delta,n} : \partial_t\upsilon_{\delta,n} = 0$ in $(*)$. We integrate the last equality over $(0,t)$ for $t\in(0,T]$ and obtain

$$\begin{aligned}
&\Upsilon^\delta_n(z_{\delta,n}(0)) + \int_0^t\int_\Omega \partial_t z_{\delta,n} : \partial_t\xi^s\\
&\qquad \geq \Upsilon^\delta_n(z_{\delta,n}(t) - B_n p_{\delta,n}(t)) + \int_0^t\int_\Omega \left((C\partial_t z_{\delta,n}) : \partial_t z_{\delta,n} + (B\partial_t p_{\delta,n}) : \partial_t p_{\delta,n}\right)\,. \qquad (2.24)
\end{aligned}$$

Since $\Upsilon^{\delta*}_n$ and Υ^δ_n are positive, we can neglect them in (2.23). Applying the Cauchy-Schwarz inequality to the right hand side of (2.23) and then Gronwall's inequality yields an estimate

$$\sup_{t\in[0,T]}\|z_{\delta,n}(t)\|_{L^2(\Omega;\mathbb{R}^{d\times d})} + \sup_{t\in[0,T]}\|p_{\delta,n}(t)\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \leq c\,\|\xi\|_{H^1}\,.$$

From positivity of Υ_n^δ on the right hand side of (2.24), it follows that

$$\int_0^t \int_\Omega ((C\partial_t z_{\delta,n}) : \partial_t z_{\delta,n} + (B\partial_t p_{\delta,n}) : \partial_t p_{\delta,n}) \leq \Upsilon_n^\delta(z_{\delta,n}(0)) + \|\xi\|_{H^1} \,.$$

The last two inequalities yield (2.21) for $z_{\delta,n}$ and $p_{\delta,n}$. The inequality for $\upsilon_{\delta,n}$ follows from equation (2.9). □

2.5 Proof of Theorem 2.1

Existence. Using the sequence $(p_{\delta,n}, z_{\delta,n}, \upsilon_{\delta,n})$ of solutions to (2.8)–(2.9), we can now prove Theorem 2.1. For $n \to \infty$, we find weakly convergent subsequences of $p_{\delta,n}$, $z_{\delta,n}$, $\upsilon_{\delta,n}$ in $\mathcal{V}_0^1$ with limits p_δ, z_δ, υ_δ. We note that $z_{\delta,n}(0)$ is the unique solution in $L^2_{n,sol}(\Omega)$ to

$$\int_\Omega (C_n z_{\delta,n}(0)) : \psi = \int_\Omega \xi(0) : \psi \qquad \forall \psi \in L^2_{n,sol}(\Omega)\,.$$

Hence, since we consider only ξ with $\xi(0) = 0$, the initial values $z_{\delta,n}(0)$ vanish identically. As a consequence, also $\Upsilon_n^\delta(z_{\delta,n}(0))$ in (2.21) vanishes. The estimate (2.21) therefore implies (2.3) for $(p_\delta, z_\delta, \upsilon_\delta)$.

Since $p_{\delta,n}$, $z_{\delta,n}$, $\upsilon_{\delta,n}$ satisfy (2.9), the limits p_δ, z_δ, υ_δ satisfy

$$Cz_\delta = \xi + \upsilon_\delta^s - p_\delta \,. \tag{2.25}$$

We take the limit $n \to \infty$ in (2.23), apply Lemma 2.7 and exploit the vanishing initial data to conclude that the functions p_δ, z_δ, υ_δ satisfy

$$\int_0^t \int_\Omega \left(\Psi^{\delta*}\left(\partial_t p_\delta\right) + \Psi^\delta\left(z_\delta - B\,p_\delta\right)\right) \leq \int_0^t \int_\Omega z_\delta : \partial_t \xi - \frac{1}{2} \int_\Omega \left(p_\delta : (B\,p_\delta) + z_\delta : (Cz_\delta)\right)\Big|_0^t \,. \tag{2.26}$$

In the limit $\delta \to 0$ we find weakly convergent subsequences of p_δ, z_δ, υ_δ with the respective weak limits p, z, υ satisfying the estimate (2.3). Passing to the limit $\delta \to 0$ in (2.25), we find that (p, z, υ) satisfies (2.2). Furthermore, passing to the limit in (2.26), using Lemma 2.6, we find that the functions p, z, υ satisfy

$$\int_0^t \int_\Omega \left(\Psi^*\left(\partial_t p\right) + \Psi\left(z - B\,p\right)\right) \leq \int_0^t \int_\Omega z : \partial_t \xi - \frac{1}{2}\left(\int_\Omega p : (Bp) + \int_\Omega z : (Cz)\right)\Big|_0^t \,.$$

We thus obtain

$$\begin{aligned}
\int_0^t \int_\Omega \left(\Psi^*\left(\partial_t p\right) + \Psi\left(z - B\,p\right)\right) &= \int_0^t \int_\Omega \left(z : \partial_t \xi - \partial_t p : Bp - \partial_t z : Cz\right) \\
&\overset{(2.2)}{=} \int_0^t \int_\Omega \left(z : C\partial_t z - z : \partial_t \upsilon^s + z : \partial_t p - \partial_t p : Bp - \partial_t z : Cz\right) \\
&= \int_0^t \int_\Omega \left(-z : \partial_t \upsilon^s + \partial_t p : (z - B\,p)\right) \;=\; \int_0^t \int_\Omega \partial_t p : (z - B\,p)
\end{aligned}$$

for every $t \in (0, T)$. On the other hand, since Lemma 2.3 (iii) yields $(\Psi^*(\partial_t p) + \Psi(z - Bp)) \geq \partial_t p : (z - Bp)$ pointwise a.e., we find

$$\left(\Psi^*\left(\partial_t p\right) + \Psi\left(z - Bp\right)\right) = \partial_t p : (z - Bp)$$

pointwise a.e. in $(0, T) \times \Omega$. The Fenchel equality of Lemma 2.3 (v) then yields (2.1).

Uniqueness and continuity. Let $\xi_1, \xi_2 \in H^1_*(0,T;\mathbb{R}^{d\times d}_s)$. Let $(p_i, z_i, \upsilon_i)_{i\in\{1,2\}}$ be two solutions to (2.1)-(2.2) for ξ_1, ξ_2 respectively with the difference $(\tilde{p}, \tilde{z}, \tilde{\upsilon}) := (p_1, z_1, \upsilon_1) - (p_2, z_2, \upsilon_2)$. We integrate $\tilde{z} : \partial_t(\xi_1 - \xi_2)$ over Ω and obtain from a calculation similar to (2.23)

$$\begin{aligned}
\int_\Omega \tilde{z} : (\xi_1 - \xi_2)\Big|_0^t - \int_0^t \int_\Omega \partial_t \tilde{z} : (\xi_1 - \xi_2) \\
&= \int_0^t \int_\Omega \tilde{z} : \partial_t(\xi_1 - \xi_2) = \int_\Omega \tilde{z} : \partial_t \left(C\tilde{z} - \tilde{p} + \tilde{\upsilon}\right) \\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega \left(\tilde{p} : (B\tilde{p}) + \tilde{z} : (C\tilde{z})\right) \\
&\quad + \int_\Omega \left[(z_1(t,\omega) - B(\omega)\, p_1(t,\omega)) - (z_2(t,\omega) - B(\omega)\, p_2(t,\omega)) \right] \left(\partial_t p_1 - \partial_t p_2\right) .
\end{aligned}$$

From the monotonicity of $\partial\Psi$ (Lemma 2.3 (ii)) and $(2.1)_{1,2}$, we find

$$\frac{1}{2}\int_\Omega \left(\tilde{p} : (B\tilde{p}) + \tilde{z} : (C\tilde{z})\right)\Big|_0^t \le \int_\Omega \tilde{z} : (\xi_1 - \xi_2)\Big|_0^t - \int_0^t \int_\Omega \partial_t \tilde{z} : (\xi_1 - \xi_2)$$

for every $t \in (0,T)$. Compactness of the embedding $H^1(0,T;\mathbb{R}^{d\times d}_s) \subset C([0,T];\mathbb{R}^{d\times d}_s)$ and boundedness of $\partial_t \tilde{z}$ provide the weak continuity of the mapping $\xi \mapsto (z, p, \upsilon)$. At the same time, it implies uniqueness of solutions, i.e. $(\tilde{p}, \tilde{z}, \tilde{\upsilon}) = (0,0,0)$ for $\xi_1 = \xi_2$. This completes the proof of Theorem 2.1.

3 Proof of the main theorem

3.1 Preliminaries

Lemma 3.1 (A time dependent ergodic theorem)**.** *Let $f \in L^p(0,T;L^p(\Omega))$, $1 \le p < \infty$ and $f_\omega(t,x) := f(t,\tau_x\omega)$. Then, for almost every $\omega \in \Omega$, there holds $f_\omega \in L^p(0,T;L^p_{loc}(\mathbb{R}^d))$. Furthermore, for almost every $\omega \in \Omega$, there holds*

$$\lim_{\varepsilon\to 0} \int_0^T \int_Q f(t, \tau_{\frac{x}{\varepsilon}}\omega)\, dx\, dt = |Q| \int_0^T \int_\Omega f(t,\omega)\, d\mathcal{P}(\omega)\, dt\,. \tag{3.1}$$

Proof. Since the mapping $(x,\omega) \mapsto \tau_x\omega$ is continuous, we find that $\tilde{f}(\omega,t,x) := f(t,\tau_x\omega)$ is $\mathcal{P}\otimes\mathcal{L}\otimes\mathcal{L}^d$-measurable. Since the mappings $\tau_x : \Omega \to \Omega$ are measure preserving, we find for every $x \in \mathbb{R}^d$

$$\int_0^T \int_\Omega |f(t,\omega)|^p \; d\mathcal{P}(\omega) dt = \int_0^T \int_\Omega |f(t,\tau_x\omega)|^p \; d\mathcal{P}(\omega) dt\,.$$

Integrating the last equation over $Q \subset \mathbb{R}^d$ and applying Fubini's theorem, we obtain

$$|Q| \int_0^T \int_\Omega |f(t,\omega)|^p \; d\mathcal{P}(\omega) dt = \int_\Omega \int_0^T \int_Q |f(t,\tau_x\omega)|^p \; dx\, dt\, d\mathcal{P}(\omega)\,.$$

Thus, $\tilde{f}$ has the integrability $\tilde{f} \in L^p(\Omega; L^p(0,T; L^p(Q)))$ and $f_\omega \in L^p(0,T; L^p(Q))$ for almost every $\omega \in \Omega$. In particular, $f_\omega \in L^1(0,T; L^1(Q))$. Setting $F(\omega) := \int_0^T f(t,\omega)\, dt$, we find as a consequence of Theorem 1.3:

$$\lim_{\varepsilon\to 0} \int_0^T \int_Q f(t, \tau_{\frac{x}{\varepsilon}}\omega)\, dx\, dt = \lim_{\varepsilon\to 0} \int_Q F(\tau_{\frac{x}{\varepsilon}}\omega)\, dx = |Q| \int_\Omega F\, d\mathcal{P} = |Q| \int_0^T \int_\Omega f\, d\mathcal{P}\, dt\,.$$

□

Lemma 3.2. *(Div-curl-lemma) Let $U \subset \mathbb{R}^d$ be an open and bounded set with Lipschitz-boundary ∂U. For a sequence $\varepsilon \to 0$ we consider sequences of functions u^ε and υ^ε as follows:*

$$u^\varepsilon \in L^2(0,T; L^2(U;\mathbb{R}^{d\times d})) \quad \textit{with} \quad \nabla \cdot u^\varepsilon(t) = 0 \quad \textit{in } \mathcal{D}'(U) \textit{ for a.e. } t \in [0,T]\,,$$
$$\upsilon^\varepsilon(t,x) := \upsilon(t, \tau_{\frac{x}{\varepsilon}}\omega) \quad \textit{with} \quad \upsilon \in L^2(0,T; L^2_{pot}(\Omega)) \quad \textit{and some } \omega \in \Omega\,,$$

with $\|u^\varepsilon\|_{L^2(0,T;L^2(U))} \le C_0$ bounded. Then, for almost every $\omega \in \Omega$, there holds

$$\lim_{\varepsilon\to 0} \int_0^T \int_U u^\varepsilon : \upsilon^\varepsilon = 0\,. \tag{3.2}$$

Proof. In this proof, we omit the time-dependence of u^ε and υ for simplicity of notation, i.e. we consider $u^\varepsilon \in L^2(U;\mathbb{R}^{d\times d})$ and $\upsilon \in L^2_{pot}(\Omega)$. In the time dependent case, one needs to apply Lemma 3.1 instead of the ergodic theorem 1.3.

We fix $\omega \in \Omega$ such that $\upsilon(\tau_x\omega) \in L^2_{pot,loc}(\mathbb{R}^d)$, which holds for a.e. $\omega \in \Omega$ due to Theorem 1.4. We consider a compact set $K \subset U$ and a cut-off function $\psi \in C^\infty(\mathbb{R}^d)$ with $\psi \equiv 1$ on K, $\psi \equiv 0$ on $\mathbb{R}^d \backslash U$ and $1 \ge \psi \ge 0$. For arbitrary $V \in C^1(\Omega;\mathbb{R}^d)$ we set $V^\varepsilon(x) := V(\tau_{\frac{x}{\varepsilon}}\omega)$ and $(\nabla_\omega V)^\varepsilon(x) := (\nabla_\omega V)(\tau_{\frac{x}{\varepsilon}}\omega) = \varepsilon \nabla_x V^\varepsilon(x)$. The triangle inequality and the Cauchy-Schwarz inequality imply

$$\left|\int_U u^\varepsilon : \upsilon^\varepsilon \psi\right| \le \|u^\varepsilon\|_{L^2(U)} \|\upsilon^\varepsilon - (\nabla_\omega V)^\varepsilon\|_{L^2(U)} + \left|\int_U u^\varepsilon : (\nabla_\omega V)^\varepsilon \psi\right|\,. \tag{3.3}$$

For the last integral in (3.3) we obtain

$$\int_U u^\varepsilon : (\nabla_\omega V)^\varepsilon \psi = \int_U u^\varepsilon : \nabla_x (\varepsilon V^\varepsilon \psi) - \int_U u^\varepsilon : (\varepsilon V^\varepsilon \otimes \nabla_x \psi) = -\int_U u^\varepsilon : (\varepsilon V^\varepsilon \otimes \nabla_x \psi)\,.$$

We take the limes superior as $\varepsilon \to \infty$. The last integral in (3.3) vanishes due to boundedness of V (remember that Ω is compact) and boundedness of $\nabla\psi$. The ergodic theorem 1.3 applied to the first term on the right hand side of (3.3) yields

$$\limsup_{\varepsilon\to 0} \left|\int_U u^\varepsilon : \upsilon^\varepsilon \psi\right| \le \limsup_{\varepsilon\to 0} \|u^\varepsilon\|_{L^2(U)} \|\upsilon^\varepsilon - (\nabla_\omega V)^\varepsilon\|_{L^2(U)} \le C_0 \sqrt{|U|}\, \|\upsilon - \nabla_\omega V\|_{L^2(\Omega)}\,. \tag{3.4}$$

Since $\{\nabla_\omega V \,|\, V \in C^1(\Omega)\}$ is dense in $L^2_{pot}(\Omega)$, the right hand side of (3.4) is arbitrarily small for an appropriate choice of $V \in C^1_b(\Omega)$.

Concerning the integral over $u^\varepsilon : \upsilon^\varepsilon(1-\psi)$, we find by the ergodic theorem 1.3

$$\left|\int_U u^\varepsilon : \upsilon^\varepsilon (1-\psi)\right| \le C_0 \|\upsilon^\varepsilon\|_{L^2(U\backslash K)} \to C_0 \|\upsilon\|_{L^2(\Omega;\mathbb{R}^{d\times d})} |U\backslash K|^{\frac{1}{2}} \tag{3.5}$$

as $\varepsilon \to 0$. Choosing first $K \subset U$ large and then $V \in C^1(\Omega)$ appropriately, we obtain that the integral in (3.2) is arbitrarily small. □

3.2 The averaging property of Σ and Π

Theorem 3.3 (Averaging property)**.** *Let the coefficients $B(\omega)$, $C(\omega)$, $\Psi(\,\cdot\,;\omega)$ be as in Assumption 1.5, let C_ε, B_ε, Ψ_ε be as in* (1.8)*. Then, for a.e. $\omega \in \Omega$, the coefficients allow averaging in sense of Definition 1.8 with the averaging operators Σ and Π given by* (2.4)*.*

Proof. We will prove a slightly stronger result: Given $\xi \in H^1_*(0,T;\mathbb{R}^{d\times d}_s)$, let (p, z, υ) be the unique solution of (2.1)–(2.2) (which exists by Theorem 2.1). Let $\omega \in \Omega$ be such that $p_\omega(t,x) := p(t,\tau_x\omega)$, $z_\omega(t,x) := z(t,\tau_x\omega)$ and $\upsilon_\omega(t,x) := \upsilon(t,\tau_x\omega)$ satisfy

$$p_\omega \in H^1(0,T;L^2_{loc}(\mathbb{R}^d;\mathbb{R}^{d\times d}_s))\,,\ z_\omega \in H^1(0,T;L^2_{sol,loc}(\mathbb{R}^d))\,,\ \upsilon_\omega \in H^1(0,T;L^2_{pot,loc}(\mathbb{R}^d))\,.$$

This regularity is valid for a.e. ω as can be seen applying Lemma 3.1 to time derivatives. Furthermore, we choose ω as in Assumption 1.5. For any $\varepsilon > 0$ let $\tilde p^\varepsilon(t,x) := p\left(t,\tau_{\frac{x}{\varepsilon}}\omega\right)$, $\tilde z^\varepsilon(t,x) := z\left(t,\tau_{\frac{x}{\varepsilon}}\omega\right)$, $\tilde \upsilon^\varepsilon(t,x) := \upsilon\left(t,\tau_{\frac{x}{\varepsilon}}\omega\right)$ be realizations. Let $\mathcal{T} \subset \mathbb{R}^d$ be a simplex and let u^ε, p^ε, σ^ε be the unique solution to

$$\begin{aligned} -\nabla\cdot\sigma^\varepsilon &= 0\,,\\ \nabla^s u^\varepsilon &= C_\varepsilon\sigma^\varepsilon + p^\varepsilon\\ \partial_t p^\varepsilon &\in \partial\Psi_\varepsilon(\sigma^\varepsilon - B_\varepsilon p^\varepsilon\,;\,.\,) \end{aligned} \tag{3.6}$$

on $\mathcal{T}$ (we recall $\partial\Psi_\varepsilon(\sigma\,;\,x) := \partial\Psi(\sigma\,;\,\tau_{\frac{x}{\varepsilon}}\omega)$) with boundary condition

$$u^\varepsilon(x) = \xi\cdot x \qquad \text{on } \partial\mathcal{T} \tag{3.7}$$

and initial condition $p^\varepsilon(0,\cdot) = 0$. We will prove that the realizations of the stochastic cell solutions and the plasticity solutions on $\mathcal{T}$ coincide in the limit $\varepsilon \to 0$; more precisely, we claim that

$$\lim_{\varepsilon\to 0}\Big(\|\sigma^\varepsilon - \tilde z^\varepsilon\|_{L^2(0,T;L^2(\mathcal{T}))} + \|p^\varepsilon - \tilde p^\varepsilon\|_{L^2(0,T;L^2(\mathcal{T}))}\Big) = 0\,. \tag{3.8}$$

Let us first show that (3.8) indeed implies Theorem 3.3: The ergodic theorem in the version of Lemma 3.1 and the definition of Σ and Π in (2.4) imply that $⨍_{\mathcal{T}} \tilde z^\varepsilon(.) \to \int_\Omega z(.) = \Sigma(\xi)(.)$ and $⨍_{\mathcal{T}} \tilde p^\varepsilon(.) \to \int_\Omega p(.) = \Pi(\xi)(.)$ holds in the space $L^2(0,T;\mathbb{R}^{d\times d}_s)$. Equation (3.8) therefore yields $⨍_{\mathcal{T}} \sigma^\varepsilon \to \Sigma(\xi)$ and $⨍_{\mathcal{T}} p^\varepsilon \to \Pi(\xi)$ in $L^2(0,T;\mathbb{R}^{d\times d}_s)$. This provides the averaging property (1.18) of Definition 1.8 (at first, for a subsequence $\varepsilon \to 0$ for almost every $t \in (0,T)$, then, since the limit is determined, along the original sequence $\varepsilon \to 0$).

Let us now prove (3.8). We will use a testing procedure and energy-type estimates. Due to (2.1)–(2.2), $\tilde z^\varepsilon$, $\tilde p^\varepsilon$ and $\tilde \upsilon^\varepsilon$ satisfy the following system of equations on $\mathcal{T}\times(0,T)$

$$\begin{aligned} -\nabla\cdot\tilde z^\varepsilon &= 0\,,\\ \xi &= C_\varepsilon\tilde z^\varepsilon + \tilde p^\varepsilon - (\tilde\upsilon^\varepsilon)^s\,,\\ \partial_t\tilde p^\varepsilon &\in \partial\Psi_\varepsilon(\tilde z^\varepsilon - B_\varepsilon\tilde p^\varepsilon\,;\,.\,)\,. \end{aligned} \tag{3.9}$$

In what follows we use the notation $|\zeta|^2_{B_\varepsilon} := \zeta : B_\varepsilon\zeta$ and $|\zeta|^2_{C_\varepsilon} := \zeta : C_\varepsilon\zeta$. We take the difference of $(3.6)_1$ and $(3.9)_1$, multiply the result by $(\partial_t u^\varepsilon - \partial_t(\xi\cdot x))$ and integrate over $\mathcal{T}$.

We integrate by parts yields and exploit that boundary integrals vanish by (3.7),

$$
\begin{aligned}
0 &= -\int_{\mathcal{T}} (\tilde{z}^\varepsilon - \sigma^\varepsilon) : (\partial_t \nabla^s u^\varepsilon - \partial_t \xi) \\
&= \int_{\mathcal{T}} (\tilde{z}^\varepsilon - \sigma^\varepsilon) : \partial_t \left(C_\varepsilon \tilde{z}^\varepsilon + \tilde{p}^\varepsilon - (\tilde{v}^\varepsilon)^s - C_\varepsilon \sigma^\varepsilon - p^\varepsilon \right) \\
&= \frac{1}{2}\frac{d}{dt} \int_{\mathcal{T}} \left[(\tilde{z}^\varepsilon - \sigma^\varepsilon) : (C_\varepsilon (\tilde{z}^\varepsilon - \sigma^\varepsilon)) + (\tilde{p}^\varepsilon - p^\varepsilon) : (B_\varepsilon (\tilde{p}^\varepsilon - p^\varepsilon)) \right] + \int_{\mathcal{T}} (\tilde{z}^\varepsilon - \sigma^\varepsilon) : \partial_t \tilde{v}^\varepsilon \\
&\quad + \int_{\mathcal{T}} (\partial_t \tilde{p}^\varepsilon - \partial_t p^\varepsilon) : ((\tilde{z}^\varepsilon - B_\varepsilon \tilde{p}^\varepsilon) - (\sigma^\varepsilon - B_\varepsilon p^\varepsilon)) \, . \\
&\in \frac{1}{2}\frac{d}{dt} \int_{\mathcal{T}} \left[|\tilde{z}^\varepsilon - \sigma^\varepsilon|^2_{C_\varepsilon} + |\tilde{p}^\varepsilon - p^\varepsilon|^2_{B_\varepsilon} \right] + \int_{\mathcal{T}} (\tilde{z}^\varepsilon - \sigma^\varepsilon) : \partial_t \tilde{v}^\varepsilon \\
&\quad + \int_{\mathcal{T}} \left(\partial \Psi_\varepsilon (\tilde{z}^\varepsilon - B_\varepsilon \tilde{p}^\varepsilon) - \partial \Psi_\varepsilon (\sigma^\varepsilon - B_\varepsilon p^\varepsilon) \right) : ((\tilde{z}^\varepsilon - B_\varepsilon \tilde{p}^\varepsilon) - (\sigma^\varepsilon - B_\varepsilon p^\varepsilon)) \, .
\end{aligned}
\tag{3.10}
$$

In the second line, we used $(3.6)_2$ and $(3.9)_2$. Additionally, we use symmetry of σ^ε and $\tilde{z}^\varepsilon$ to replace $(\tilde{v}^\varepsilon)^s$ by $\tilde{v}^\varepsilon$.

Concerning the second integral on the right hand side of (3.10), note that $\int_0^t \int_{\mathcal{T}} \tilde{z}^\varepsilon : \partial_t \tilde{v}^\varepsilon \to \int_0^t \int_{\mathcal{T}} \int_\Omega z : \partial_t v = 0$ by Lemma 3.1 and orthogonality of $L^2_{sol}(\Omega)$ and $L^2_{pot}(\Omega)$. Furthermore, $\int_0^t \int_{\mathcal{T}} \sigma^\varepsilon : \partial_t \tilde{v}^\varepsilon \to 0$ by Lemma 3.2. By monotonicity of $\partial\Psi_\varepsilon$, the last integral on the right hand side of (3.10) is positive. An integration over $(0,t)$ therefore provides

$$
\limsup_{\varepsilon \to 0} \int_{\mathcal{T}} \left[|\tilde{z}^\varepsilon - \sigma^\varepsilon|^2_{C_\varepsilon} + |\tilde{p}^\varepsilon - p^\varepsilon|^2_{B_\varepsilon} \right](t) \leq \limsup_{\varepsilon \to 0} \int_0^t \int_{\mathcal{T}} (\tilde{z}^\varepsilon - \sigma^\varepsilon) \partial_t \tilde{v}^\varepsilon = 0 \, . \tag{3.11}
$$

We have thus shown (3.8). This concludes the proof of Theorem 3.3. □

3.3 Admissibility of Σ

Theorem 3.4 (Admissibility)**.** *Let the coefficients $B(\omega)$, $C(\omega)$, $\Psi(\,\cdot\,;\omega)$ and data U, f be as in Assumption 1.5. Then the causal operator Σ of Definition 2.2 satisfies the sufficient condition for admissibility of Definition 1.11.*

Proof. We have to study solutions u_h of the discretized effective problem with the discretized boundary data $U_h \to U$ strongly in $H^1(0,T;H^1(Q))$ as $h \to 0$. With Σ given through (2.4), let $u_h \in H^1(0,T;H^1(Q))$ be a sequence with $u_h \in U_h + H^1(0,T;Y_h)$, satisfying the discrete system

$$
\int_0^T \int_Q \Sigma(\nabla^s u_h) : \nabla \varphi = \int_0^T \int_Q f \cdot \varphi \qquad \forall \varphi \in L^2(0,T;Y_h) \, . \tag{3.12}
$$

We can assume the weak convergence $u_h \rightharpoonup u \in H^1(0,T;H^1(Q;\mathbb{R}^d)$ as $h \to 0$ for some $u \in U + H^1(0,T;H^1_0(Q;\mathbb{R}^d))$. Our aim is to show that u solves the effective problem

$$
\int_0^T \int_Q \Sigma(\nabla^s u) : \nabla \varphi = \int_0^T \int_Q f \cdot \varphi \qquad \forall \varphi \in L^2(0,T;H^1_0(Q)) \, . \tag{3.13}
$$

Step 1. For every $x \in Q$, we denote by $p_h(t,x,\cdot)$, $z_h(t,x,\cdot)$, $\upsilon_h(t,x,\cdot)$ the solutions of (2.1)–(2.2) corresponding to $\xi(t) = \nabla^s u_h(t,x)$. By definition of Σ, there holds $\Sigma(\nabla^s u_h) = \int_\Omega z_h(\omega)\, d\mathcal{P}(\omega)$. The a priori estimate of Theorem 2.1 provides

$$\|p_h\|_{\mathcal{V}^1_{0,0}} + \|z_h\|_{\mathcal{V}^1_{0,0}} + \|\upsilon_h\|_{\mathcal{V}^1_{0,0}} \le C \, \|\nabla^s u\|_{H^1(0,T;L^2(Q))} \,,$$

where $\mathcal{V}^1_{0,0} := H^1(0,T;L^2(Q;L^2(\Omega;\mathbb{R}^{d\times d})))$. By this estimate, we obtain the weak convergence in $(\mathcal{V}^1_{0,0})^3$ of a subsequence, again denoted (p_h, z_h, υ_h), weakly converging to some limit (p, z, υ). The limit satisfies again the linear law (2.2),

$$Cz = \nabla^s u + \upsilon^s - p\,. \tag{3.14}$$

Equation (3.12) can be rewritten as

$$\int_0^T \int_Q \int_\Omega z_h(t,x,\omega)\, d\mathcal{P}(\omega) : \nabla\varphi(x)\, dx = \int_0^T \int_Q f\cdot\varphi \qquad \forall \varphi \in L^2(0,T;Y_h)\,,$$

and the limit $h \to 0$ provides

$$\int_Q \int_\Omega z : \nabla\varphi = \int_Q f\cdot\varphi \qquad \forall\varphi \in L^2(0,T;H^1_0(Q))\,. \tag{3.15}$$

Step 2. It remains to verify $\int_\Omega z = \Sigma(\nabla^s u)$. We use $\varphi = \partial_t (u_h - U_h)$ as a test function in (3.12) and exploit the orthogonality $0 = \int_Q \int_\Omega z_h : \partial_t \upsilon_h$. We follow the lines of the calculation in (2.23) to obtain

$$\begin{aligned}
&\int_Q f\cdot\partial_t (u_h - U_h) + \int_Q\int_\Omega z_h : \nabla\partial_t U_h \\
&\quad = \int_Q\int_\Omega z_h : \partial_t \nabla^s u_h = \int_Q\int_\Omega \left[z_h : C\partial_t z_h + z_h : \partial_t p_h - z_h : \partial_t \upsilon_h\right] \\
&\quad = \frac{1}{2}\frac{d}{dt}\left(\int_Q\int_\Omega p_h : Bp_h + \int_Q\int_\Omega z_h : Cz_h\right) + \int_Q\int_\Omega \left(\Psi^*\left(\partial_t p_h\right) + \Psi\left(z_h - Bp_h\right)\right)\,.
\end{aligned} \tag{3.16}$$

Taking weak limits in (3.16) yields

$$\begin{aligned}
&\int_0^T \int_Q\int_\Omega \left(\Psi^*\left(\partial_t p\right) + \Psi\left(z - Bp\right)\right) \\
&\le \int_0^T \int_Q f\cdot\partial_t (u - U) + \int_0^T\int_Q\int_\Omega z : \nabla\partial_t U - \frac{1}{2}\left(\int_Q\int_\Omega p : (Bp) + \int_Q\int_\Omega z : (Cz)\right)\Bigg|_0^T\,.
\end{aligned}$$

Relations (3.14) and (3.15) allow to perform the calculations of (3.16) also for the limit functions. We obtain from the last inequality

$$\int_0^T \int_Q\int_\Omega \left(\Psi^*\left(\partial_t p\right) + \Psi\left(z - Bp\right)\right) \le \int_0^T \int_Q\int_\Omega \partial_t p : (z - Bp)\,.$$

The Fenchel inequality of Lemma 2.3 (iii) yields $\partial_t p : (z - Bp) \leq \Psi^* (\partial_t p) + \Psi (z - Bp)$ pointwise. We can therefore conclude from the Fenchel equality

$$\partial_t p \in \partial\Psi(\sigma - Bp)\,. \tag{3.17}$$

Relations (3.14) and (3.17) imply that z is defined as in the definition of Σ, hence $\int_\Omega z(t,x,\,.\,) = \Sigma(\nabla^s u)(t,x,\,.\,)$ for every $t \in [0,T]$ and a.e. $x \in Q$. Therefore, (3.15) is equivalent with (3.13). □

3.4 Proof of the main theorem

Theorem 3.4 yields that Σ of (2.4) is admissible. Theorem 3.3 yields that, for almost every $\omega \in \Omega$, the coefficients $C_{\varepsilon,\omega}(x)$, $B_{\varepsilon,\omega}(x)$, $\Psi_{\varepsilon,\omega}(\sigma; x)$ allow averaging with the limit operator Σ.

We can therefore apply Theorem 1.10 and obtain

$$\begin{aligned}
&u^\varepsilon \rightharpoonup u \quad \text{weakly in } H^1(0,T; H^1_0(Q;\mathbb{R}^d))\\
&p^\varepsilon \rightharpoonup \Pi(\nabla^s u), \quad \sigma^\varepsilon \rightharpoonup \Sigma(\nabla^s u) \quad \text{weakly in } H^1(0,T; L^2(Q;\mathbb{R}^{d\times d}))\,,
\end{aligned}$$

where u is the unique weak solution to the homogenized problem

$$-\nabla \cdot \Sigma(\nabla^s u) = f$$

with boundary condition U in the sense of Definition 1.9. Definition 2.2 implies that

$$\Sigma(\nabla^s u) = \int_\Omega z\, d\mathcal{P}\,,$$

where the functions $z \in H^1(0,T; L^2(Q; L^2_{sol}(\Omega)))$, $v \in H^1(0,T; L^2(Q; L^2_{pot}(\Omega)))$, and $p \in H^1(0,T; L^2(Q; L^2(\Omega; \mathbb{R}^{d\times d}_s)))$ solve

$$\begin{aligned}
\nabla^s_x u = Cz - v^s + p \quad &\text{a.e. in } [0,T]\times Q\times\Omega\,,\\
\partial_t p \in \partial\Psi(z - Bp) \quad &\text{a.e. in } [0,T]\times Q\times\Omega\,.
\end{aligned}$$

In particular, (z, p, v) is a solution to (1.5). This concludes the proof.

A An example for the stochastic setting

Our aim here is to describe briefly a non-trivial example for a stochastic setting: the checker board construction of i.i.d. random variables. Our main goal is to show that the compactness assumption on Ω is not too restrictive and still permits the analysis of interesting problems.

We use $Y := [0,1[^d$ with the topology of the torus and the partition of $\mathbb{R}^d$ with unit cubes $\mathcal{C}_z := z + Y$ for $z \in \mathbb{Z}^d$. We consider the sets

$$\begin{aligned}
&\tilde\Omega := \left\{ u \in L^\infty(\mathbb{R}^d) \,|\, u|_{\mathcal{C}_z} \equiv c_z\,, \text{ for some } \ c : \mathbb{Z}^d \to [0,1], z \mapsto c_z \right\}\\
&\Omega := \left\{ u \in L^\infty(\mathbb{R}^d) \,|\, \exists \xi \in Y \text{ s.t. } u(.-\xi) \in \tilde\Omega \right\}.
\end{aligned}$$

For $u \in \Omega$ we denote a shift ξ from the above definition as $\xi(u)$. Since $L^1(\mathbb{R}^d)$ is separable, we infer from [5], Theorem III.28, that $L^\infty(\mathbb{R}^d)$ with the weak-$*$-topology is metrizable: With a countable and dense subset $(\phi_i)_{i\in\mathbb{N}}$ of $L^1(\mathbb{R}^d)$, a metric d on B_∞ is given by

$$d(u,v) := \sum_{i=1}^{\infty} \frac{1}{2^i} \left|\langle u-v, \phi_i\rangle\right| \,.$$

We infer that $B_\infty := B_1(0) \subset L^\infty(\mathbb{R}^d)$ with the weak-$*$-topology is a compact metric space. The sets $\tilde{\Omega}$ and Ω are closed subsets of (B_∞, d) and thus compact metric spaces.

The probability measure on Ω corresponding to i.i.d. random variables can be defined with the help of elementary subsets. For an open set $U \subseteq Y$, a number $k \in \mathbb{N}$, and relatively open intervals $I_z := ((a_z, b_z) \cap [0,1]) \subset [0,1]$, $z \in \mathbb{Z}^d$ and $a_z < b_z$, the sets

$$A\left(U, (I_z)_{z\in\mathbb{Z}^d}, k\right) = \left\{u \in \Omega \,\middle|\, \xi(u) \in U\,,\ u(.-\xi(u))|_{\mathcal{C}_z} \in I_z \,\forall z\,,\ |z| \le k\right\} \qquad \text{(A.1)}$$

are open and form a basis of the topology in Ω. For any such set $A(\,.\,)$ we define

$$\mathcal{P}\left(A\left(U, (I_z)_{z\in\mathbb{Z}^d}, k\right)\right) := |U| \prod_{|z|\le k} |b_z - a_z| \,.$$

We finally introduce $\tau_x : \Omega \to \Omega$ for every $x \in \mathbb{R}^d$ through $\tau_x u(\,.\,) = u(x + .)$. It is easy to check that the family $(\tau_x)_{x\in\mathbb{R}^d}$ is a dynamical system. Since $\mathcal{P}(A) = \mathcal{P}(\tau_x A)$ for A as in (A.1) and $x \in \mathbb{R}^d$, the dynamical system is measure preserving.

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