technische universität
dortmund

Transmission conditions for the Helmholtz-equation in perforated domains

Christina Dörlemann, Martin Heida, Ben Schweizer

Preprint 2015-02 April 2015

Fakultät für Mathematik
Technische Universität Dortmund
Vogelpothsweg 87
44227 Dortmund

tu-dortmund.de/MathPreprints

Transmission conditions for the Helmholtz-equation in perforated domains

Christina Dörlemann, Martin Heida, Ben Schweizer*

April 7, 2015

Abstract

We study the Helmholtz equation in a perforated domain Ω_ε. The domain Ω_ε is obtained from an open set $\Omega \subset \mathbb{R}^3$ by removing small obstacles of typical size $\varepsilon > 0$, the obstacles are located along a 2-dimensional manifold $\Gamma_0 \subset \Omega$. We derive effective transmission conditions across Γ_0 that characterize solutions in the limit $\varepsilon \to 0$. We obtain that, to leading order $O(\varepsilon^0) = O(1)$, the perforation is invisible. On the other hand, at order $O(\varepsilon^1) = O(\varepsilon)$, inhomogeneous jump conditions for the pressure and the flux appear. The jumps can be characterized without cell problems by elementary expressions that contain the ε^0-order limiting pressure function and the volume of the obstacles.

Keywords: Helmholtz equation, perforated domain, transmission conditions, acoustic properties

MSC: 35B27, 74Q05

1 Introduction

Our aim is to study the acoustic properties of complex domains. Assuming that acoustic waves are described by the linear wave equation, the acoustic properties of a domain Ω_ε are determined by the Helmholtz equation

$$-\Delta p^\varepsilon = \omega^2 p^\varepsilon + f \qquad \text{in } \Omega_\varepsilon, \tag{1.1}$$

where ω is the frequency of waves and f is a right hand side that models sound sources in the domain $\Omega_\varepsilon \subset \mathbb{R}^3$. Equation (1.1) is accompanied by a boundary condition on $\partial\Omega_\varepsilon$.

We use a small parameter $\varepsilon > 0$ and write Ω_ε for the domain, since we assume that the domain contains structures of typical size ε. More specifically, we investigate a perforated domain: We investigate three-dimensional domains that contain many obstacles (the number of obstacles is of order ε^{-2}) with the small diameter $\varepsilon > 0$, we denote the

*Technische Universität Dortmund, Fakultät für Mathematik, Vogelpothsweg 87, D-44227 Dortmund, Germany. ben.schweizer@tu-dortmund.de

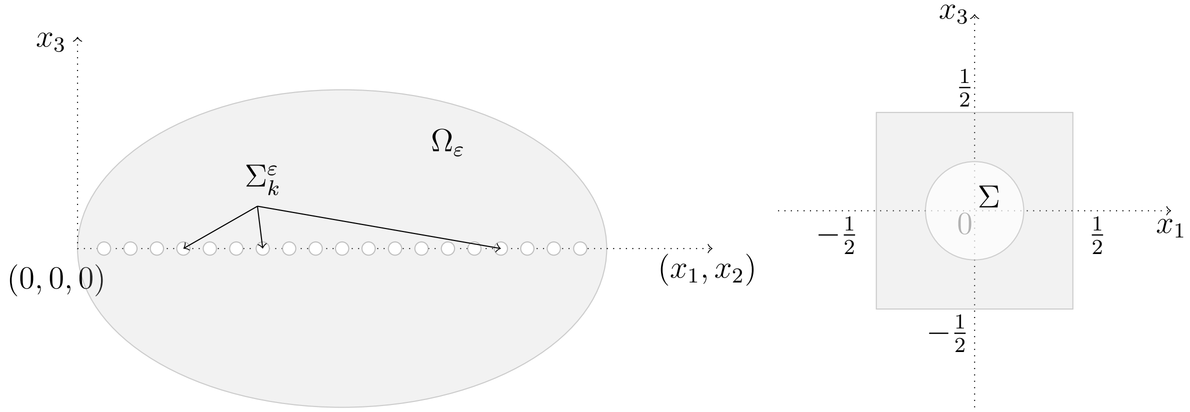

Figure 1: Left: The domain Ω_ε with many small obstacles Σ_k^ε. Right: Each obstacle is a scaled and shifted copy of a standard obstacle $\Sigma \subset \mathbb{R}^3$.

single obstacle by Σ_k^ε, where $k \in \mathbb{Z}^2$ is an index to number the obstacles. We assume that the obstacles are uniformly distributed along a 2-dimensional submanifold $\Gamma_0 \subset \mathbb{R}^3$. The domain Ω_ε is obtained from an ε-independent domain $\Omega \subset \mathbb{R}^3$ by removing the obstacles, $\Omega_\varepsilon = \Omega \setminus \bigcup_k \Sigma_k^\varepsilon$. Every point in the open set $\Omega \setminus \Gamma_0$ does not touch any obstacle for sufficiently small $\varepsilon > 0$ (compare Figure 1).

We ask for the effective influence of the perforations along Γ_0. A rigorous description can be obtained by the analysis of solution sequences p^ε to (1.1) in the sense of homogenization. Denoting a weak limit of the solution sequence p^ε by p, we ask for the system of equations that determines p. We will show rigorously that the limit p is characterized by the Helmholtz equation in the domain Ω, hence the effect of the perforation gets lost at leading order, see (1.6). This is in contrast to other claims in the literature as we will discuss below.

At first glance, our result seems to be counter-intuitive. One would expect some influence of the perforation, some jump conditions for the pressure function across Γ_0 and/or some jump conditions for the velocities $-\nabla p$ across Γ_0. But, typically, Dirichlet conditions survive under weak convergence in H^1, so one should not expect jumps of p across Γ_0, which we write briefly as $[p] = 0$. Moreover: If the flux into the obstacles vanishes on the ε-level ($\partial_n p^\varepsilon = 0$ on $\partial\Sigma_k^\varepsilon$), then no effective source can appear along Γ_0 and we expect $[\partial_\nu p] = 0$ along Γ_0, where ν denotes a normal vector on Γ_0. Both conditions are established rigorously in Theorem 1.1.

On the other hand: The intuition (and some rule-of-thumb equations of the more physical literature) can be confirmed if one considers first order effects in ε, i.e. the weighted difference $v^\varepsilon := (p^\varepsilon - p)/\varepsilon$. Our main result in Theorem 1.2 provides the characterization of the weak limit v of the sequence v^ε: The function v solves the Helmholtz equation on the domain $\Omega \setminus \Gamma_0$, and both v and ∇v satisfies jump conditions across Γ_0. These jump conditions contain the pressure function p and its derivatives, see (1.9): The jump $[v]$ of v is proportional to the slope of p in Γ_0, $\partial_\nu p$, and the jump $[\partial_\nu v]$ of the first-order velocity corrector is proportional to the curvature of p in Γ_0, $\partial_\nu^2 p$. The coefficients in the two laws do not depend on the shape of the obstacles, but only on the volume.

1.1 Comparison with the literature

1.1.1 Effective description of the acoustic properties of a perforation

An effective description of the perforation that is used in the literature can be written as

$$\partial_\nu p^+ = \partial_\nu p^- = -i\frac{\omega\rho}{Z}(p^+ - p^-)\,. \tag{1.2}$$

In this formula, ρ denotes the density, ω the frequency, ν a normal vector on Γ_0, pointing into the domain Ω_+, and Z is a complex number, the *transmission impedance*, a parameter that characterizes the effective behavior of the obstacles (we cite from equation (2) of [9], where reference is given to [4]).

Let us compare the empirical formula (1.2) with our findings. As a first observation, we note that in both, in (1.2) and in our results, the normal component of the pressure gradient has no jump. The second equation in (1.2) seems to contradict our finding that also the effective pressure function p has no jump. But we may as well compare the pressure difference $p^+ - p^-$ with the jump of the first order corrector, scaled with ε, that is with $\varepsilon[v]$. If we do so, we may also say that (1.2) is consistent with $[v] = |\Sigma|\,\partial_\nu p$ from (1.9) if we set $Z = -i\omega\rho\varepsilon|\Sigma|$. In particular, the shape of the obstacles does not enter into the transmission impedance Z, only the relative volume $|\Sigma|$ of the obstacles is of relevance. We cannot confirm the frequency dependence that is suggested in (1.2).

1.1.2 The homogenization results of Rohan and Lukeš

The homogenization problem of this contribution has also been investigated in the more mathematical work [9]. In formula (29) of that work, an effective transmission condition for p is derived (which seems curious since we derive trivial transmission conditions for p in the work at hand). Their formula (29) contains coefficient matrices A, B, D, and F that are derived from cell-problems. Formula (29) can nevertheless be consistent with our results if most the coefficients A, B, D, and F are trivial.

1.1.3 Homogenization results and transmission conditions for perforated domains

Closely related to this work is [7], where a similar perforated geometry is studied. In [7], the problem with inhomogeneous boundary conditions at the small obstacles is studied. Using a flux condition that is scaled as ε^{-1}, the authors obtain a non-trivial effective problem (jump conditions appear also at lowest order, whereas a jump condition appear in our setting only in the first order term). Related works are [8], where the problem is further analyzed, and [6], where an oscillatory (on small scales) *boundary* instead of an *interface* is studied.

There are equations where order-1 effects are introduced by the perforation (even without an ε^{-1} boundary condition). An example is the Stokes flow in a perforated geometry, see [3, 10]. But even for the Helmholtz equation with a fixed frequency ω, order-1 effects are possible, namely in a Helmholtz resonator geometry. For a mathematical study of the Helmholtz resonator we refer to [11]. We emphasize that the lowest order effect of [11] is only possible by introducing *three* scales: The macroscopic scale (order 1, size of Ω), the microscopic scale ε (size of the resonator), and a sub-micro-scale which is small compared to ε (the diameter of a channel connecting the interior of the resonator to the exterior).

Effects of highest order by introducing small structures are also known from a related equation, namely the time homogeneous Maxwell equation (of which the Helmholtz equation is a special case): Using split-ring microscopic geometries, the effective behavior of solutions to Maxwell equations can be changed dramatically: Negative index materials with negative index of refraction can occur as homogenized materials, see [1, 5]. We note that in these works, again, three scales are used: Each microscopic element of size ε contains a substructure of a size that is small compared to ε (in this case: the diameter of the slit in the ring).

1.2 Mathematical description and results

Let $\Omega \subset \mathbb{R}^3$ be a domain with Lipschitz boundary, containing the origin. We use the unit cell $Y := \left[-\frac{1}{2}, \frac{1}{2}\right)^2 \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ and the obstacle shape $\Sigma \subset Y$. We assume that Σ is a domain with Lipschitz boundary, which is strictly contained in Y, i.e. $\overline{\Sigma} \subset \left(-\frac{1}{2}, \frac{1}{2}\right)^3$. To construct the obstacles in the complex geometry, we scale and shift the set Σ: We use $k \in \mathbb{Z}^2$ to label the different obstacles and set

$$Y_k^\varepsilon := \varepsilon\left(Y + (k_1, k_2, 0)\right)\,, \quad \Sigma_k^\varepsilon := \varepsilon\left(\Sigma + (k_1, k_2, 0)\right) \;\text{ for }\; k = (k_1, k_2) \in \mathbb{Z}^2\,. \tag{1.3}$$

The indices of cells inside Ω are $I_\varepsilon := \{k \in \mathbb{Z}^2 |\, Y_k^\varepsilon \subset \Omega\}$. The number of elements of I_ε is of order ε^{-2}. We denote by $\Sigma^\varepsilon := \bigcup_{k \in I_\varepsilon} \Sigma_k^\varepsilon$ the union of all obstacles in Ω and define the perforated domain by setting $\Omega_\varepsilon := \Omega \backslash \Sigma^\varepsilon$.

We denote by n the outer normal of Ω_ε on $\partial\Omega_\varepsilon$. The perforation Σ^ε is located along the submanifold $\Gamma_0 := (\mathbb{R}^2 \times \{0\}) \cap \Omega$. The submanifold Γ_0 separates the domain Ω into two subdomains:

$$\Omega_+ := \left[\mathbb{R}^2 \times (0, \infty)\right] \cap \Omega \;\text{ and }\; \Omega_- := \left[\mathbb{R}^2 \times (-\infty, 0)\right] \cap \Omega,$$

leading to the disjoint decomposition $\Omega = \Omega_+ \cup \Gamma_0 \cup \Omega_-$.

Our analysis concerns the following Helmholtz equation on Ω_ε:

$$\begin{aligned} -\Delta p^\varepsilon &= \omega^2 p^\varepsilon + f && \text{in } \Omega_\varepsilon, \\ \partial_n p^\varepsilon &= 0 && \text{on } \partial\Sigma^\varepsilon, \\ p^\varepsilon &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.4}$$

In this equation, $f \in L^2(\Omega)$ is a given source term and the frequency $\omega > 0$ is a fixed parameter. The natural space of solutions of (1.4) is

$$\mathcal{H}_\varepsilon := \left\{u \in H^1(\Omega_\varepsilon) \,|\, u|_{\partial\Omega} = 0\right\}\,.$$

The weak formulation of (1.4) is: find $p^\varepsilon \in \mathcal{H}_\varepsilon$ such that

$$\int_{\Omega_\varepsilon} \nabla p^\varepsilon \cdot \nabla\varphi = \int_{\Omega_\varepsilon} \omega^2 p^\varepsilon \varphi + \int_{\Omega_\varepsilon} f\varphi \qquad \forall \varphi \in \mathcal{H}_\varepsilon\,. \tag{1.5}$$

We assume that ω^2 is not an eigenvalue of the operator $(-\Delta)^{-1}$ to Dirichlet conditions on $\partial\Omega$, i.e. $\omega^2 \notin \sigma\left((-\Delta)^{-1}\right)$. In what follows, we denote by $\mathcal{P}_\varepsilon : L^2(\Omega_\varepsilon) \to L^2(\Omega)$ the extension operator that continues every function by 0 to all of Ω.

Our first result characterizes limits p of solution sequences p^ε. We obtain that the perforation is invisible in the limit $\varepsilon \to 0$.

Theorem 1.1 (Limit behavior of solutions). *Let $f \in L^2(\Omega)$ be a source function and let $p^\varepsilon \in \mathcal{H}_\varepsilon$ be a sequence of weak solutions to* (1.4). *We assume that ω^2 is no eigenvalue of $-\Delta$ on Ω.*
Effective system: *$\|\mathcal{P}_\varepsilon p^\varepsilon\|_{L^2(\Omega)}$ and $\|\mathcal{P}_\varepsilon \nabla p^\varepsilon\|_{L^2(\Omega)}$ are bounded and there exists $p \in H^1_0(\Omega)$ such that $\mathcal{P}_\varepsilon p^\varepsilon \to p$ strongly in $L^2(\Omega)$ and $\mathcal{P}_\varepsilon \nabla p^\varepsilon \rightharpoonup \nabla p$ weakly in $L^2(\Omega)$. The limit p is the unique weak solution of*

$$-\Delta p = \omega^2 p + f \quad \text{in } \Omega\,. \tag{1.6}$$

Rate of convergence: *If f has the regularity $H^1 \cap C^0$ in an open neighborhood of Γ_0 and if $\partial\Omega$ is of class C^3 in a neighborhood of $\bar{\Gamma}_0 \cap \partial\Omega$, then there exists a constant $C = C(f) > 0$, independent of $\varepsilon > 0$, such that*

$$\|p - \mathcal{P}_\varepsilon p^\varepsilon\|_{L^2(\Omega)} + \|\nabla p - \mathcal{P}_\varepsilon \nabla p^\varepsilon\|_{L^2(\Omega)} + \|\Delta p - \mathcal{P}_\varepsilon \Delta p^\varepsilon\|_{L^2(\Omega)} \le C\varepsilon^{1/2}\,. \tag{1.7}$$

The rate of convergence in (1.7) is consistent with the following picture: The values of p^ε and p differ by the order of ε in the neighborhood of the perforation, the gradients ∇p^ε and ∇p differ by the order of 1 in the neighborhood of the perforation. The deviations of p^ε from p are present in an ε-neighborhood of Γ_0, which is consistent with $\|\nabla p - \mathcal{P}_\varepsilon \nabla p^\varepsilon\|_{L^2(\Omega)} \le C\varepsilon^{1/2}$. In particular, we expect that the rate $\varepsilon^{1/2}$ is the optimal rate of convergence.

If p^ε is the solution to the ε-problem (1.4) and $p \in H^1(\Omega)$ the homogenized limit according to Theorem 1.1, we define v^ε as the variation of order ε:

$$v^\varepsilon := \frac{p^\varepsilon - p}{\varepsilon} \quad \text{on } \Omega_\varepsilon\,. \tag{1.8}$$

For $v \in H^1(\Omega)$ with $\Delta v \in L^2(\Omega \backslash \Gamma_0)$, we denote by $[v]$ and $[\partial_\nu v]$ the jump of v and of its normal derivatives across Γ_0 (see Section 2; in our setting, we have $\nu = e_3$). We denote by $\mathcal{H}^2$ the two dimensional Hausdorff measure. We obtain the following corrector result:

Theorem 1.2 (First order behavior). *Let v^ε be defined through* (1.8), *where $p^\varepsilon \in \mathcal{H}_\varepsilon$ is a sequence of weak solutions to* (1.4) *and p solves* (1.6). *We assume that f is of class $H^1 \cap C^0$ in an open neighborhood of Γ_0, and that $\partial\Omega$ is of class C^3. On the sequence v^ε we assume that there is $v \in W^{1,1}(\Omega\backslash\Gamma_0)$ such that $\mathcal{P}_\varepsilon v^\varepsilon \rightharpoonup v$ in $L^1(\Omega)$ and $\mathcal{P}_\varepsilon \nabla v^\varepsilon \rightharpoonup \nabla v + [v]\,\nu\mathcal{H}^2\lfloor_{\Gamma_0}$ weakly as measures. Then, the function v is determined by the following system of equations*

$$\begin{aligned} -\Delta v &= \omega^2 v && \text{in } \Omega\backslash\Gamma_0\,,\\ [v] &= |\Sigma|\,\partial_\nu p && \text{on } \Gamma_0\,,\\ [\partial_\nu v] &= -\,|\Sigma|\,\partial_\nu^2 p && \text{on } \Gamma_0\,. \end{aligned} \tag{1.9}$$

Remark 1.3. Relation $(1.9)_1$ implies that $[\partial_\nu v]$ is well-defined on Γ_0 as a distribution. Since f is of class $H^1(\Omega)$, the solution p of the Helmholtz equation in Ω is of class $H^3(\Omega)$. For this reason, the right hand side of $(1.9)_{2,3}$ is well defined in the sense of traces.

Remark 1.4. If v^ε is bounded in $W^{1,1}(\Omega_\varepsilon)$ and bounded in $H^1(\tilde{\Omega})$ for every $\tilde{\Omega} \subset\subset \Omega_+$ and every $\tilde{\Omega} \subset\subset \Omega_-$, we obtain the existence of the function v by compactness in $BV(\Omega)$.

2 Preliminaries

For $Q \subset \mathbb{R}^3$, we write $L^2(Q)$ for the space of square integrable functions over Q and $H^k(Q) = W^{k,2}(Q)$ for the Bessel-potential spaces. We further denote $H^k_0(Q)$ the closure of $C_c^k(Q)$ in $H^k(Q)$. For a measurable domain $Q \subset \mathbb{R}^3$ of finite measure and $g \in L^1(Q)$, we write $⨍_Q g := |Q|^{-1} \int_Q g$ for the average of g over Q.

With $\Omega \subset \mathbb{R}^3$ and Γ_0 as in the introduction, we note that Γ_0 cuts Ω into $\Omega_\pm := \{x \in \Omega \mid \pm x_3 > 0\}$. For $p \in H^1(\Omega \backslash \Gamma_0)$, we denote by $p^\pm$ the trace of $p|_{\Omega_\pm}$ on Γ_0, respectively. Further, if $\Delta p \in L^2(\Omega \backslash \Gamma_0)$, we denote

$$\partial_\nu^\pm p := \nabla p^\pm \cdot \nu \,,$$

where $\nu = e_3$ is the outer normal of Ω_- on Γ_0. The jumps of p and ∇p are introduced as

$$\begin{aligned} [p] &:= p^+ - p^- \,, \\ [\partial_\nu p] &:= \partial_\nu^+ p - \partial_\nu^- p \,. \end{aligned}$$

Note that $p \in H^1(\Omega \backslash \Gamma_0)$ together with $[p] = 0$ is equivalent to $p \in H^1(\Omega)$. This leads to the following observation:

Remark 2.1. Let $p \in H^1(\Omega \backslash \Gamma_0)$ and $f \in L^2(\Omega)$. Then, the partial differential equation

$$-\Delta p = \omega^2 p + f \text{ in } \Omega \tag{2.1}$$

is equivalent to the system

$$\begin{aligned} -\Delta p &= \omega^2 p + f && \text{in } \Omega \backslash \Gamma_0 \,, \\ [p] &= 0 && \text{auf } \Gamma_0 \,, \\ [\partial_\nu p] &= 0 && \text{auf } \Gamma_0 \,. \end{aligned} \tag{2.2}$$

Both equations (2.1) and $(2.2)_1$ are understood in the sense of distributions or, equivalently, in the weak sense. We emphasize that $(2.2)_1$ guarantees $\Delta p \in L^2(\Omega_\pm)$, hence $[\partial_\nu p]$ is well defined.

In the proofs of our main theorems, we are dealing with sequences $p^\varepsilon \in \mathcal{H}_\varepsilon$. Since these functions are defined on Ω_ε and not on Ω, we need suitable extension operators. The most elementary operator is the extension by 0, which we denote as $\mathcal{P}_\varepsilon : L^2(\Omega_\varepsilon) \to L^2(\Omega)$. Furthermore, it is well known, that there exists a family of extension operators $\tilde{\mathcal{P}}_\varepsilon : H^1(\Omega_\varepsilon) \to H^1(\Omega)$, such that

$$\left\| \tilde{\mathcal{P}}_\varepsilon p^\varepsilon \right\|_{H^1(\Omega)} \leq C \left\| p^\varepsilon \right\|_{H^1(\Omega_\varepsilon)} \tag{2.3}$$

for some $C > 0$ independent of ε ([2], Chapter 1). Essentially, $\tilde{\mathcal{P}}_\varepsilon$ is defined by using in each obstacle the harmonic extension of the boundary values.

Lemma 2.2. *Let $p^\varepsilon \in H^1(\Omega_\varepsilon)$ satisfy the a priori estimate $\|\mathcal{P}_\varepsilon p^\varepsilon\|_{L^2(\Omega)} + \|\mathcal{P}_\varepsilon \nabla p^\varepsilon\|_{L^2(\Omega)} \leq C$ for every $\varepsilon > 0$. Then, there exists $p \in H^1(\Omega)$ and a subsequence $\varepsilon \to 0$ such that $\mathcal{P}_\varepsilon p^\varepsilon \to p$ strongly in $L^2(\Omega)$, $\mathcal{P}_\varepsilon \nabla p^\varepsilon \rightharpoonup \nabla p$ weakly in $L^2(\Omega)$ and $\tilde{\mathcal{P}}_\varepsilon p^\varepsilon \rightharpoonup p$ weakly in $H^1(\Omega)$. Furthermore, if $p^\varepsilon|_{\partial\Omega} = 0$ holds for every $\varepsilon > 0$, then also $p|_{\partial\Omega} = 0$.*

Proof. In what follows, we successively pass to subsequences of p^ε, keeping the notation p^ε for each subsequence. Since $\|\mathcal{P}_\varepsilon p^\varepsilon\|_{L^2(\Omega)} + \|\mathcal{P}_\varepsilon \nabla p^\varepsilon\|_{L^2(\Omega)} \leq C$, upon changing the constant, there also holds $\left\| \tilde{\mathcal{P}}_\varepsilon p^\varepsilon \right\|_{H^1(\Omega)} \leq C$. Thus, there is $p \in H^1(\Omega)$ such that $\tilde{\mathcal{P}}_\varepsilon p^\varepsilon \rightharpoonup p$ weakly in $H^1(\Omega)$ and $\tilde{\mathcal{P}}_\varepsilon p^\varepsilon \to p$ strongly in $L^2(\Omega)$. By the trace theorem, the condition $p^\varepsilon|_{\partial\Omega} = 0$ for all $\varepsilon > 0$ implies $p|_{\partial\Omega} = 0$.

For $\delta > 0$ let $\phi_\delta \in L^\infty(\mathbb{R})$ be the indicator function $\phi_\delta(z) = 1$ for $|z| < \delta$ and $\phi_\delta(z) = 0$ for $|z| \geq \delta$. We set $\varphi_\delta : \Omega \to \mathbb{R}$, $\varphi_\delta(x) := \phi_\delta(x_3)$ and obtain for $\varepsilon < \delta$:

$$\begin{aligned}\limsup_{\varepsilon\to 0} \int_\Omega \left|\mathcal{P}_\varepsilon p^\varepsilon - \tilde{\mathcal{P}}_\varepsilon p^\varepsilon\right|^2 &= \limsup_{\varepsilon\to 0} \int_{\Sigma^\varepsilon} \left|\tilde{\mathcal{P}}_\varepsilon p^\varepsilon\right|^2 \leq \limsup_{\varepsilon\to 0} \int_\Omega \left|\tilde{\mathcal{P}}_\varepsilon p^\varepsilon\right|^2 \varphi_\delta^2 \\ &= \limsup_{\varepsilon\to 0} \left\|\varphi_\delta \tilde{\mathcal{P}}_\varepsilon p^\varepsilon\right\|_{L^2(\Omega)}^2 = \|\varphi_\delta p\|_{L^2(\Omega)}^2 \,.\end{aligned}$$

The last limit follows from the strong convergence $\varphi_\delta \tilde{\mathcal{P}}_\varepsilon p^\varepsilon \to \varphi_\delta p$ in $L^2(\Omega)$. Since $\delta > 0$ was arbitrary, the right hand side is arbitrarily small. We conclude that $\mathcal{P}_\varepsilon p^\varepsilon \to p$ converges strongly in $L^2(\Omega)$.

Similarly, we obtain for every $\psi \in L^2(\Omega; \mathbb{R}^3)$:

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_\Omega \mathcal{P}_\varepsilon \nabla p^\varepsilon \cdot \psi &= \lim_{\varepsilon\to 0} \int_\Omega \mathcal{P}_\varepsilon \nabla p^\varepsilon \cdot \psi(1-\varphi_\delta) + \lim_{\varepsilon\to 0} \int_\Omega \mathcal{P}_\varepsilon \nabla p^\varepsilon \cdot \psi\varphi_\delta \\ &= \int_\Omega \nabla p \cdot \psi(1-\varphi_\delta) + \lim_{\varepsilon\to 0} \int_\Omega \mathcal{P}_\varepsilon \nabla p^\varepsilon \cdot \psi\varphi_\delta \,.\end{aligned}$$

Since $\limsup_{\varepsilon\to 0} \left|\int_\Omega \mathcal{P}_\varepsilon \nabla p^\varepsilon \cdot \psi\varphi_\delta\right| \leq \limsup_{\varepsilon\to 0} \|\mathcal{P}_\varepsilon \nabla p^\varepsilon\|_{L^2(\Omega)} \|\psi\varphi_\delta\|_{L^2(\Omega)} \to 0$ as $\delta \to 0$, we obtain $\mathcal{P}_\varepsilon \nabla p^\varepsilon \rightharpoonup \nabla p$ weakly in $L^2(\Omega)$. □

3 Limit of p^ε

Proof of Theorem 1.1. We will prove Theorem 1.1 in three steps: In Step 1, we prove the homogenization result under the assumption that $\|p^\varepsilon\|_{L^2(\Omega_\varepsilon)}$ is bounded. In Step 2, we use Step 1 to prove boundedness of $\|p^\varepsilon\|_{L^2(\Omega_\varepsilon)}$ by a contradiction argument. In Step 3, we prove (1.7) in case $f \in H^1(\Omega)$.

Step 1: Limit behavior of p^ε. We assume here that $\|p^\varepsilon\|_{L^2(\Omega_\varepsilon)}$ is bounded. We use p^ε as a test function in (1.5) and obtain

$$\|\nabla p^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq \|p^\varepsilon\|_{L^2(\Omega_\varepsilon)} \left(\omega^2 \|p^\varepsilon\|_{L^2(\Omega_\varepsilon)} + C\right) \,, \tag{3.1}$$

which implies boundedness of $\|\nabla p^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2$.

From the estimates for $\|p^\varepsilon\|_{L^2(\Omega_\varepsilon)}$ and $\|\nabla p^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2$ and Lemma 2.2, we conclude the existence of $p \in H_0^1(\Omega)$ such that $\mathcal{P}_\varepsilon p^\varepsilon \to p$ strongly in $L^2(\Omega)$ and $\mathcal{P}_\varepsilon \nabla p^\varepsilon \rightharpoonup \nabla p$ weakly in $L^2(\Omega)$ along a subsequence. We choose a test function $\varphi \in \mathcal{C}_c^\infty(\Omega)$, and obtain from (1.5) and Lemma 2.2

$$\int_\Omega \nabla p \cdot \nabla\varphi = \lim_{\varepsilon\to 0} \int_\Omega \mathcal{P}_\varepsilon \nabla p^\varepsilon \cdot \nabla\varphi = \lim_{\varepsilon\to 0} \int_\Omega \omega^2 \mathcal{P}_\varepsilon p^\varepsilon \varphi + \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} f\varphi = \int_\Omega \omega^2 p\,\varphi + \int_\Omega f\varphi \,. \tag{3.2}$$

This provides (1.6) and hence the homogenization result under the assumption of boundedness. We note that the above calculations also hold if in (1.5), f is replaced by a sequence $(f_\varepsilon)_{\varepsilon>0}$ with $f_\varepsilon \to f$ strongly in $L^2(\Omega)$ as $\varepsilon \to 0$.

Step 2: $L^2(\Omega)$-boundedness of p^ε. Let us assume for a contradiction argument that the sequence $\|p^\varepsilon\|_{L^2(\Omega_\varepsilon)}$ is not bounded. For every $\varepsilon > 0$, we define rescaled quantities by setting

$$\tilde{p}^\varepsilon := \frac{p^\varepsilon}{\|p^\varepsilon\|_{L^2(\Omega_\varepsilon)}} \text{ in } \Omega_\varepsilon \text{ and } \tilde{f}^\varepsilon := \frac{f}{\|p^\varepsilon\|_{L^2(\Omega_\varepsilon)}} \text{ in } \Omega\,. \tag{3.3}$$

We achieve $\|\tilde{p}^\varepsilon\|_{L^2(\Omega_\varepsilon)} = 1$ for every $\varepsilon > 0$ and $\left\|\tilde{f}^\varepsilon\right\|_{L^2(\Omega)} \to 0$ for $\varepsilon \to 0$. Since p^ε solves (1.4), we conclude that $\tilde{p}^\varepsilon$ solves

$$\begin{aligned} -\Delta\tilde{p}^\varepsilon &= \omega^2\tilde{p}^\varepsilon + \tilde{f}^\varepsilon && \text{in } \Omega_\varepsilon, \\ \partial_n\tilde{p}^\varepsilon &= 0 && \text{on } \partial\Sigma^\varepsilon\,. \end{aligned} \tag{3.4}$$

Since $\|\tilde{p}^\varepsilon\|_{L^2(\Omega_\varepsilon)}$ is bounded, we can apply Step 1 and obtain the existence of $\tilde{p} \in H_0^1(\Omega)$ such that $\mathcal{P}_\varepsilon\tilde{p}^\varepsilon \to \tilde{p}$ strongly in $L^2(\Omega)$ and $\mathcal{P}_\varepsilon\nabla\tilde{p}^\varepsilon \rightharpoonup \nabla\tilde{p}$ weakly in $L^2(\Omega)$, where $\tilde{p}$ solves

$$-\Delta\tilde{p} = \omega^2\tilde{p} \text{ in } \Omega. \tag{3.5}$$

Since $\tilde{p} \in H_0^1(\Omega)$ solves (3.5) and ω^2 is not an eigenvalue of $-\Delta$ on Ω, we conclude $\tilde{p} = 0$. We obtain the desired contradiction between the strong convergence $\mathcal{P}_\varepsilon\tilde{p}^\varepsilon \to 0$ in $L^2(\Omega)$ and $\|\mathcal{P}_\varepsilon\tilde{p}^\varepsilon\|_{L^2(\Omega)} = 1$ for every $\varepsilon > 0$.

Step 3: Rate of convergence. It remains to prove (1.7). For a contradiction argument, let us assume $\varepsilon^{-1/2}\|\mathcal{P}_\varepsilon p^\varepsilon - p\|_{L^2(\Omega)} \to \infty$, which also implies $G_\varepsilon := \varepsilon^{-1/2}\left\|\tilde{\mathcal{P}}_\varepsilon p^\varepsilon - p\right\|_{L^2(\Omega)} \to \infty$ by the uniform boundedness of p in Σ^ε. We study the sequence of functions $w^\varepsilon := G_\varepsilon^{-1}\varepsilon^{-1/2}(\tilde{\mathcal{P}}_\varepsilon p^\varepsilon - p)$ with $\|w^\varepsilon\|_{L^2(\Omega)} = 1$, satisfying

$$\begin{aligned} -\Delta w^\varepsilon &= \omega^2 w^\varepsilon && \text{in } \Omega_\varepsilon, \\ \partial_n w^\varepsilon &= -G_\varepsilon^{-1}\varepsilon^{-\frac{1}{2}}\partial_n p && \text{on } \partial\Sigma^\varepsilon, \\ w^\varepsilon &= 0 && \text{on } \partial\Omega, \end{aligned}$$

with the weak formulation

$$\int_{\Omega_\varepsilon}\nabla w^\varepsilon\cdot\nabla\varphi = -\int_{\partial\Omega_\varepsilon} G_\varepsilon^{-1}\varepsilon^{-\frac{1}{2}}\partial_n p\,\varphi\,d\mathcal{H}^2 + \int_{\Omega_\varepsilon}\omega^2 w^\varepsilon\varphi \qquad \forall\varphi \in H_0^1(\Omega)\,. \tag{3.6}$$

Due to our assumptions on Ω and f, the functions Δp and ∇p are of class C^0 and bounded in an open neighborhood of Γ_0. This allows to estimate the boundary integral as

$$\begin{aligned} \left|\int_{\partial\Omega_\varepsilon}\varepsilon^{-\frac{1}{2}}\partial_n p\,\varphi\,d\mathcal{H}^2\right| &= \left|\sum_{k\in I_\varepsilon}\int_{\partial\Sigma_k^\varepsilon}\varepsilon^{-\frac{1}{2}}\partial_n p\,\varphi\,d\mathcal{H}^2\right| = \left|\sum_{k\in I_\varepsilon}\int_{\Sigma_k^\varepsilon}\varepsilon^{-\frac{1}{2}}\left(-\Delta p\varphi - \nabla p\cdot\nabla\varphi\right)\right| \\ &\leq \varepsilon^{-\frac{1}{2}}\left\||\Delta p| + |\nabla p|\right\|_{L^2(\Sigma^\varepsilon)}\cdot\left\||\varphi| + |\nabla\varphi|\right\|_{L^2(\Sigma^\varepsilon)} \leq C\,\|\varphi\|_{H^1(\Sigma^\varepsilon)}\,. \end{aligned} \tag{3.7}$$

Using $\varphi = w^\varepsilon$ as a test function in (3.6), exploiting $\|\nabla w^\varepsilon\|_{L^2(\Sigma^\varepsilon)} \leq C\,\|\nabla w^\varepsilon\|_{L^2(\Omega_\varepsilon)}$ from (2.3), we obtain

$$\int_{\Omega_\varepsilon}|\nabla w^\varepsilon|^2 \leq CG_\varepsilon^{-1}\,\|w^\varepsilon\|_{H^1(\Omega_\varepsilon)} + \omega^2\,\|w^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2\,, \tag{3.8}$$

and thus the boundedness of w^ε in $H^1(\Omega_\varepsilon)$. From the construction of w^ε and Lemma 2.2 we conclude that, for a limit function $w \in H_0^1(\Omega)$ and a subsequence, there holds $\mathcal{P}_\varepsilon\left(w^\varepsilon|_{\Omega_\varepsilon}\right) \to w$ strongly in $L^2(\Omega)$ and $\mathcal{P}_\varepsilon\left(\nabla w^\varepsilon|_{\Omega_\varepsilon}\right) \rightharpoonup \nabla w$ weakly in $L^2(\Omega)$ and $w^\varepsilon \to w$ strongly in $L^2(\Omega)$.

Since $G_\varepsilon^{-1} \to 0$ as $\varepsilon \to 0$, (3.6) yields the following limit equation for w:

$$\int_\Omega \nabla w \cdot \nabla\varphi = \int_\Omega \omega^2 w\varphi \qquad \forall\varphi \in H_0^1(\Omega)\,.$$

Since ω^2 is not an eigenvalue of $-\Delta$, we find $w = 0$. We obtain the desired contradiction, since the strong convergence of w^ε to 0 contradicts the normalization $\|w^\varepsilon\|_{L^2(\Omega)} = 1$.

With this contradiction to the assumption $G_\varepsilon \to \infty$, we have shown $\|p - \mathcal{P}_\varepsilon p^\varepsilon\|_{L^2(\Omega)} \le C\varepsilon^{\frac{1}{2}}$. Estimate (3.8) is valid in general and provides the estimate with improved regularity: boundedness of ∇w^ε in $L^2(\Omega_\varepsilon)$ and thus $\|\nabla p - \mathcal{P}_\varepsilon \nabla p^\varepsilon\|_{L^2(\Omega)} \le C\varepsilon^{\frac{1}{2}}$. The estimate $\|\Delta p - \mathcal{P}_\varepsilon \Delta p^\varepsilon\|_{L^2(\Omega)} \le C\varepsilon^{\frac{1}{2}}$ follows from the Helmholtz equations $(1.4)_1$ and (1.6). □

4 First order behavior

Proof of Theorem 1.2. We prove the theorem in three steps. In Step 1, we reduce the proof of the statement to the convergence behavior of a boundary integral. In Step 2, we prove the convergence of this boundary integral. In Step 3, we show that the weak limit problem is equivalent to the distributional formulation of (1.9).

Step 1: Reduction to one boundary integral. Our aim is to analyze the first order corrector function $v^\varepsilon := \varepsilon^{-1}(p^\varepsilon - p)$. The function v^ε solves the following Helmholtz equation:

$$\begin{aligned} -\Delta v^\varepsilon &= \omega^2 v^\varepsilon && \text{in } \Omega_\varepsilon\,,\\ \partial_n v^\varepsilon &= -\frac{1}{\varepsilon}\partial_n p && \text{on } \partial\Sigma^\varepsilon\,,\\ v^\varepsilon &= 0 && \text{on } \partial\Omega\,. \end{aligned} \tag{4.1}$$

System (4.1) has the following weak formulation: $v^\varepsilon \in H^1(\Omega_\varepsilon)$ satisfies $v^\varepsilon|_{\partial\Omega} = 0$ and

$$\int_{\Omega_\varepsilon} \nabla v^\varepsilon \cdot \nabla\varphi = -\int_{\partial\Omega_\varepsilon} \frac{1}{\varepsilon}\partial_n p\,\varphi\, d\mathcal{H}^2 + \int_{\Omega_\varepsilon} \omega^2 v^\varepsilon\varphi \qquad \forall\varphi \in H_0^1(\Omega)\,. \tag{4.2}$$

We use our assumption on the convergence behavior of v^ε and ∇v^ε and obtain from (4.2) in the limit $\varepsilon \to 0$

$$\int_\Omega \nabla v \cdot \nabla\varphi + \int_{\Gamma_0} [v]\nu \cdot \nabla\varphi\, d\mathcal{H}^2 = -\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} \frac{1}{\varepsilon}\partial_n p\,\varphi\, d\mathcal{H}^2 + \int_\Omega \omega^2 v\varphi \quad \forall\varphi \in C_c^\infty(\Omega)\,. \tag{4.3}$$

The main step of the proof is therefore to determine the limit of the boundary integral. We will derive in Step 2

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} \frac{1}{\varepsilon}\partial_n p\,\varphi\, d\mathcal{H}^2 = -|\Sigma| \int_{\Gamma_0} \left(\partial_\nu^2 p\,\varphi + \partial_\nu p\,\partial_\nu\varphi\right)\, d\mathcal{H}^2 \quad \forall\varphi \in C_c^\infty(\Omega)\,. \tag{4.4}$$

Inserting this characterization into (4.3) will provide the system of equations for v.

Step 2: Proof of (4.4)**.** Let $\varphi \in C_c^\infty(\Omega)$ be a test function. For every $k \in I_\varepsilon$, we set

$$F^\varepsilon(k) := \varepsilon^{-2} \int_{\partial\Sigma_k^\varepsilon} \frac{1}{\varepsilon} \partial_n p \, \varphi \, d\mathcal{H}^2 .$$

An integration by parts can be used to rewrite $F^\varepsilon(k)$ as

$$F^\varepsilon(k) = \varepsilon^{-2} \int_{\Sigma_k^\varepsilon} \frac{-1}{\varepsilon} (\Delta p \, \varphi + \nabla p \cdot \nabla \varphi) = - |\Sigma| \fint_{\Sigma_k^\varepsilon} (\varphi \Delta p + \nabla p \cdot \nabla \varphi) \, ,$$

where we have used that n is the inner normal of Σ_k^ε and that the measure of obstacle k is $|\Sigma_k^\varepsilon| = \varepsilon^3 |\Sigma|$ for every $k \in I_\varepsilon$. For $y \in \Gamma_0$ and $\varepsilon > 0$ we choose the index $k(y, \varepsilon) \in \mathbb{Z}^2$ such that $y \in Y_{k(y,\varepsilon)}^\varepsilon$. The elliptic equation $-\Delta p = \omega^2 p + f$ and our regularity assumptions imply that the functions ∇p and Δp are of class C^0 in a neighborhood of Γ_0. This allows to calculate, for every point $y \in \Gamma_0$,

$$\begin{aligned} F(y) &:= \lim_{\varepsilon \to 0} F^\varepsilon(k(y, \varepsilon)) \\ &= - \lim_{\varepsilon \to 0} |\Sigma| \fint_{\Sigma_{k(y,\varepsilon)}^\varepsilon} (\Delta p \, \varphi + \nabla p \cdot \nabla \varphi) = - |\Sigma| \, (\Delta p \, \varphi + \nabla p \cdot \nabla \varphi) \, (y) \, . \end{aligned} \tag{4.5}$$

Our next step is to conclude from this point-wise convergence a convergence for integrals, more precisely, the convergence $\int_{\partial\Omega_\varepsilon} \frac{1}{\varepsilon} \partial_n p(y) \, \varphi(y) \; d\mathcal{H}^2(y) \to \int_{\Gamma_0} F(y) \, d\mathcal{H}^2(y)$ as $\varepsilon \to 0$. Since the interface area in the single cell is $|Y_k^\varepsilon \cap \Gamma_0|_{\mathcal{H}^2} = \varepsilon^2$ for every $k \in I_\varepsilon$, we obtain

$$\begin{aligned} \int_{\partial\Omega_\varepsilon} \frac{1}{\varepsilon} \partial_n p \, \varphi \, d\mathcal{H}^2 &= \sum_{k \in I_\varepsilon} \int_{\partial\Sigma_k^\varepsilon} \frac{1}{\varepsilon} \partial_n p \, \varphi \, d\mathcal{H}^2 \\ &= \sum_{k \in I_\varepsilon} F^\varepsilon(k) \, |Y_k^\varepsilon \cap \Gamma_0|_{\mathcal{H}^2} = \int_{\Gamma_0} F^\varepsilon(k(y, \varepsilon)) \, d\mathcal{H}^2(y) \, . \end{aligned} \tag{4.6}$$

By definition of F in $(4.5)_1$, we have the pointwise convergence $F^\varepsilon(k(y, \varepsilon)) \to F(y)$. Since ∇p and Δp are bounded in a neighborhood of Γ_0, the family $F^\varepsilon(k)$ is uniformly bounded. We can therefore apply Lebesgue's dominated convergence theorem and obtain, in the limit $\varepsilon \to 0$,

$$\int_{\partial\Omega_\varepsilon} \frac{1}{\varepsilon} \partial_n p \, \varphi \, d\mathcal{H}^2 \to \int_{\Gamma_0} F \, d\mathcal{H}^2 = - |\Sigma| \int_{\Gamma_0} (\Delta p + \nabla p \cdot \nabla \varphi) \; d\mathcal{H}^2 \, . \tag{4.7}$$

Since $\varphi|_{\Gamma_0} \in C_c^\infty(\Gamma_0)$, we may integrate by parts in the last expression with respect to the tangential coordinates x_1 and x_2, with vanishing boundary integrals. We obtain

$$\int_{\Gamma_0} F \, d\mathcal{H}^2 = - \int_{\Gamma_0} |\Sigma| \left(\partial_3^2 p \, \varphi + \partial_3 p \, \partial_3 \varphi \right) \; d\mathcal{H}^2 \, . \tag{4.8}$$

Because of $e_3 = \nu$, we have thus obtained (4.4).

Step 3: The limit equations. It remains to insert (4.4) into (4.3), which provides

$$\int_\Omega \nabla v \cdot \nabla \varphi + \int_{\Gamma_0} [v] \partial_\nu \varphi \, d\mathcal{H}^2 = \int_{\Gamma_0} |\Sigma| \left(\partial_\nu^2 p \, \varphi + \partial_\nu p \, \partial_\nu \varphi \right) \; d\mathcal{H}^2 + \int_\Omega \omega^2 v \varphi \quad \forall \varphi \in C_c^\infty(\Omega) \, .$$

This relation is the weak formulation of (1.9), since a formal integration by parts yields

$$-\int_\Omega \Delta v\,\varphi - \int_{\Gamma_0} [\partial_\nu v]\varphi\, d\mathcal{H}^2 + \int_{\Gamma_0} [v]\partial_\nu\varphi\, d\mathcal{H}^2 = \int_{\Gamma_0} |\Sigma| \left(\partial_\nu^2 p\,\varphi + \partial_\nu p\,\partial_\nu\varphi\right)\, d\mathcal{H}^2 + \int_\Omega \omega^2 v\varphi$$

for every smooth φ. Comparing the factors of φ in the bulk provides $-\Delta v = \omega^2 v$ (the equation thus holds rigorously in the sense of distributions in $\Omega\backslash\Gamma_0$). Comparing the factors of $\partial_\nu\varphi$ in boundary integrals provides $(1.9)_2$. Comparing the factors of φ in boundary integrals provides $(1.9)_3$. □

Acknowledgements: Support by the German Science foundation under grants DFG SCHW 639/5-1 and SCHW 639/6-1 are gratefully acknowledged.

References

[1] G. Bouchitté and B. Schweizer. Homogenization of Maxwell's equations in a split ring geometry. *Multiscale Model. Simul.*, 8(3):717–750, 2010.

[2] D. Cioranescu and J. Saint Jean Paulin. *Homogenization of reticulated structures*, volume 136 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1999.

[3] C. Conca. Étude d'un fluide traversant une paroi perforée. I. Comportement limite près de la paroi. *J. Math. Pures Appl. (9)*, 66(1):1–43, 1987.

[4] R. Kirby and A. Cummings. The impedance of perforated plates subjected to grazing gas flow and backed by porous media. *Journal of Sound and Vibration*, 217(4):619–636, 1998.

[5] A. Lamacz and B. Schweizer. Effective Maxwell equations in a geometry with flat rings of arbitrary shape. *SIAM J. Math. Anal.*, 45(3):1460–1494, 2013.

[6] N. Neuss, M. Neuss-Radu, and A. Mikelić. Effective laws for the Poisson equation on domains with curved oscillating boundaries. *Appl. Anal.*, 85(5):479–502, 2006.

[7] M. Neuss-Radu and W. Jäger. Effective transmission conditions for reaction-diffusion processes in domains separated by an interface. *SIAM J. Math. Anal.*, 39(3):687–720 (electronic), 2007.

[8] M. Neuss-Radu, S. Ludwig, and W. Jäger. Multiscale analysis and simulation of a reaction-diffusion problem with transmission conditions. *Nonlinear Anal. Real World Appl.*, 11(6):4572–4585, 2010.

[9] E. Rohan and V. Lukeš. Homogenization of the acoustic transmission through a perforated layer. *J. Comput. Appl. Math.*, 234(6):1876–1885, 2010.

[10] J. Sanchez-Hubert and E. Sánchez-Palencia. Acoustic fluid flow through holes and permeability of perforated walls. *J. Math. Anal. Appl.*, 87(2):427–453, 1982.

[11] B. Schweizer. The low-frequency spectrum of small Helmholtz resonators. *Proc. R. Soc. A*, 2015.

Preprints ab 2012/12

2015-02 **Christina Dörlemann, Martin Heida, Ben Schweizer**
Transmission conditions for the Helmholtz-equation in perforated domains

2015-01 **Frank Klinker**
Program of the International Conference
Geometric and Algebraic Methods in Mathematical Physics
March 16-19, 2015, Dortmund

2014-10 **Frank Klinker**
An explicit description of SL $(2, \mathbb{C})$ in terms of $SO^{+}(3, 1)$ and vice versa

2014-09 **Margit Rösler and Michael Voit**
Integral representation and sharp asymptotic results for some Heckman-Opdam hypergeometric functions of type BC

2014-08 **Martin Heida and Ben Schweizer**
Stochastic homogenization of plasticity equations

2014-07 **Margit Rösler and Michael Voit**
A central limit theorem for random walks on the dual of a compact Grassmannian

2014-06 **Frank Klinker**
Eleven-dimensional symmetric supergravity backgrounds, their geometric superalgebras, and a common reduction

2014-05 **Tomáš Dohnal and Hannes Uecker**
Bifurcation of nonlinear Bloch waves from the spectrum in the Gross-Pitaevskii equation

2014-04 **Frank Klinker**
A family of non-restricted $D = 11$ geometric supersymmetries

2014-03 **Martin Heida and Ben Schweizer**
Non-periodic homogenization of infinitesimal strain plasticity equations

2014-02 **Ben Schweizer**
The low frequency spectrum of small Helmholtz resonators

2014-01 **Tomáš Dohnal, Agnes Lamacz, Ben Schweizer**
Dispersive homogenized models and coefficient formulas for waves in general periodic media

2013-16 **Karl Friedrich Siburg**
Almost opposite regression dependence in bivariate distributions

2013-15 **Christian Palmes and Jeannette H. C. Woerner**
The Gumbel test and jumps in the volatility process

2013-14 **Karl Friedrich Siburg, Katharina Stehling, Pavel A. Stoimenov, Jeannette H. C. Wörner**
An order for asymmetry in copulas, and implications for risk management

2013-13 **Michael Voit**
Product formulas for a two-parameter family of Heckman-Opdam hypergeometric functions of type BC

2013-12 **Ben Schweizer and Marco Veneroni**
Homogenization of plasticity equations with two-scale convergence methods

2013-11 **Sven Glaser**
A law of large numbers for the power variation of fractional Lévy processes

2013-10 **Christian Palmes and Jeannette H. C. Woerner**
The Gumbel test for jumps in stochastic volatility models

2013-09 **Agnes Lamacz, Stefan Neukamm and Felix Otto**
Moment bounds for the corrector in stochastic homogenization of a percolation model

2013-08 **Frank Klinker**
Connections on Cahen-Wallach spaces

2013-07 **Andreas Rätz and Matthias Röger**
Symmetry breaking in a bulk-surface reaction-diffusion model for signaling networks

2013-06 **Gilles Francfort and Ben Schweizer**
A doubly non-linear system in small-strain visco-plasticity

2013-05 **Tomáš Dohnal**
Traveling solitary waves in the periodic nonlinear Schrödinger equation with finite band potentials

2013-04 **Karl Friedrich Siburg, Pavel Stoimenov and Gregor N. F. Weiß**
Forecasting portfolio-value-at-risk with nonparametric lower tail dependence estimates

2013-03 **Martin Heida**
On thermodynamics of fluid interfaces

2013-02 **Martin Heida**
Existence of solutions for two types of generalized versions of the Cahn-Hilliard equation

2013-01 **Tomáš Dohnal, Agnes Lamacz, Ben Schweizer**
Dispersive effective equations for waves in heterogeneous media on large time scales

2012-19 **Martin Heida**
On gradient flows of nonconvex functional in Hilbert spaces with Riemannian metric and application to Cahn-Hilliard equations

2012-18 **Robert V. Kohn, Jianfeng Lu, Ben Schweizer and Michael I. Weinstein**
A variational perspective on cloaking by anomalous localized resonance

2012-17 **Margit Rösler and Michael Voit**
Olshanski spherical functions for infinite dimensional motion groups of fixed rank

2012-16 **Selim Esedoğlu, Andreas Rätz, Matthias Röger**
Colliding Interfaces in Old and New Diffuse-interface Approximations of Willmore-flow

2012-15 **Patrick Henning, Mario Ohlberger and Ben Schweizer**
An adaptive multiscale finite elment method

2012-14 **Andreas Knauf, Frank Schulz, Karl Friedrich Siburg**
Positive topological entropy for multi-bump magnetic fields

2012-13 **Margit Rösler, Tom Koornwinder and Michael Voit**
Limit transition between hypergeometric functions of type BC and Type A

2012-12 **Alexander Schnurr**
Generalization of the Blumenthal-Getoor index to the class of homogeneous diffusions with jumps and some applications