Optimal control of two variational inequalities arising in solid mechanics

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Contents

1. Introduction

We are concerned with the optimal control of elastoplastic contact problems. Our investigation is restricted to the static model of infinitesimal elastoplasticity with linear kinematic hardening and Signorini's problem, which are both represented by an elliptic variational inequality (VI) of the first kind. Thus, we have to deal with a system of two coupled VIs:

$$\begin{aligned} (A\boldsymbol{\Sigma}, \boldsymbol{T} - \boldsymbol{\Sigma}) - (\operatorname{div}^* \boldsymbol{u}, \boldsymbol{T} - \boldsymbol{\Sigma}) &\geq 0 & \forall\, \boldsymbol{T} \in \mathcal{K} \\ - (\operatorname{div} \boldsymbol{\Sigma}, \boldsymbol{v} - \boldsymbol{u}) &\geq \langle \ell, \boldsymbol{v} - \boldsymbol{u} \rangle & \forall\, \boldsymbol{v} \in \mathcal{C}. \end{aligned}$$

The solution operator of a VI is usually not Gâteaux differentiable, cf. [62] in case of the obstacle problem. The lack of differentiability substantially complicates the optimal control theory for only one VI - even in the finite-dimensional case, where control problems of this type are known as mathematical problems with equilibrium constraints, cf. e.g. [52, 65, 66, 69] and the references therein. Particularly, it is not possible to establish necessary optimality conditions in the form of Karush-Kuhn-Tucker conditions using the differentiability of the solution operator, which is the standard way. Instead several alternative stationarity concepts such as Clarke(C)-, Bouligand(B)- and strong stationarity have been introduced. There is a multitude of papers contributing to the field of optimal control of elliptic VIs. A common technique for the derivation of first-order necessary optimality conditions is to apply a Yosida-like regularization of the VI combined with a subsequent limit analysis w.r.t. the regularization parameter tending to zero. This approach was developed by Barbu [4] and adapted by many authors, see e.g. [15, 16, 30, 38, 45, 46, 50, 57]. In [5, 6, 7] Bergounioux used relaxation methods in order to obtain necessary conditions, which was modified in [47]. Moreover, there are various contributions employing different regularization and relaxation techniques, see e.g. [8, 9, 59, 67, 70]. Mignot and Puel proved the necessity of an optimality system for the optimal control of the obstacle problem solely based on the directional differentiability of the underlying control-to-state mapping, cf. [62, 63], and their findings were extended in [48]. Other direct approaches have been performed by Bermúdez and Saguez in [11, 12, 13], cf. also [10], by Jarušek et al. in [51] and by Wachsmuth in [76].
While sufficient conditions for optimal control problems governed by elliptic PDEs have been extensively investigated, see e.g. [20, 21, 22, 23, 24, 25, 26, 68], the literature for elliptic VIs is rather rare. In [62] it was proven that the obstacle control problem is convex if the desired state is behind the obstacle and thus not reachable. Kunisch and Wachsmuth presented second-order sufficient conditions for the

optimal control of a general obstacle problem, cf. [55].
In view of the difficulty mentioned above we will separately address the optimal control of static elastoplasticity and mechanical contact. The static model of infinitesimal elastoplasticity reads as follows: Given an inhomogeneity $\ell \in V'$ find $\boldsymbol{\Sigma} \in S^2$ and $\boldsymbol{u} \in V$ so that $\boldsymbol{\Sigma} \in \mathcal{K}$ and

$$\left.\begin{aligned} (A\boldsymbol{\Sigma}, \boldsymbol{T} - \boldsymbol{\Sigma}) + (\operatorname{div}^* \boldsymbol{u}, \boldsymbol{T} - \boldsymbol{\Sigma}) \geq 0 \quad & \text{for all } \boldsymbol{T} \in \mathcal{K} \\ \operatorname{div} \boldsymbol{\Sigma} = \ell \quad & \text{in } V' \end{aligned}\right\} \qquad (\mathbf{VI}_E)$$

is satisfied. Although this problem has only limited physical meaning, it is especially obtained via time discretization of a quasi-static counterpart modeling the plastic deformation of a body under the influence of external loads, cf. Figure 1.1. The VI in ($\mathbf{VI}_E$) represents the constitutive law of elastoplasticity and the equation is just the balance of momentum. Here $\boldsymbol{\Sigma} = (\boldsymbol{\sigma}, \boldsymbol{\chi})$ denotes the generalized stress, where $\boldsymbol{\sigma}$ is the Cauchy stress tensor resulting from the inhomogeneity ℓ and $\boldsymbol{\chi}$ is an internal force, which arises during hardening. Moreover the variable $\boldsymbol{u}$ denotes the displacement induced by $\boldsymbol{\Sigma}$.

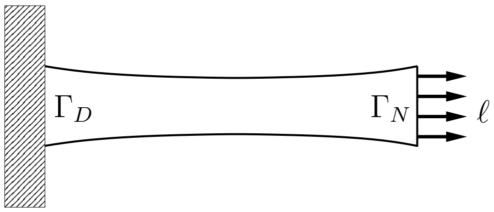

Figure 1.1.: Plastic deformation of a workpiece clamped at Γ_D, boundary loads acting on Γ_N

Optimal control problems governed by ($\mathbf{VI}_E$) have been well investigated. In particular, existence and regularity results as well as several stationarity conditions have been proven, cf. [40, 41, 42, 43, 44]. If one aims to establish sufficient optimality conditions one needs a certain differentiability result for the control-to-state map associated with ($\mathbf{VI}_E$). In [44] Herzog et al. showed that this operator is weakly directionally differentiable, which however is not satisfactory for the derivation of sufficient conditions. In Chapter 2 we will therefore enhance this result by proving Bouligand differentiability under additional regularity assumptions. As a consequence we will be enabled to deduce second-order sufficient conditions, which guarantee local optimality. The findings of Chapter 2 have already been published in large part in [14].
Signorini's problem describes the elastic deformation of a body which is pushed against a rigid obstacle, cf. Figure 1.2, and can be formulated in the following way: Given $\ell \in V'$ find $\boldsymbol{\sigma} \in S$ and $\boldsymbol{u} \in \mathcal{C}$ such that

$$\left.\begin{aligned} (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u})) \geq \langle \ell, \boldsymbol{v} - \boldsymbol{u} \rangle \quad \forall \boldsymbol{v} \in \mathcal{C} \\ \boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}). \end{aligned}\right\} \qquad (\mathbf{VI}_S)$$

The VI takes the contact conditions into account, while the equation is the law of linear elasticity. Again the variables $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ denote the Cauchy stress tensor and the corresponding displacement, respectively. We will specify all spaces, sets and operators involved in ($\mathbf{VI}_E$) and ($\mathbf{VI}_S$) later on. For a detailed introduction to elasticity, plasticity and Signorini's problem we refer to the books [35], [37], [53] and [60].

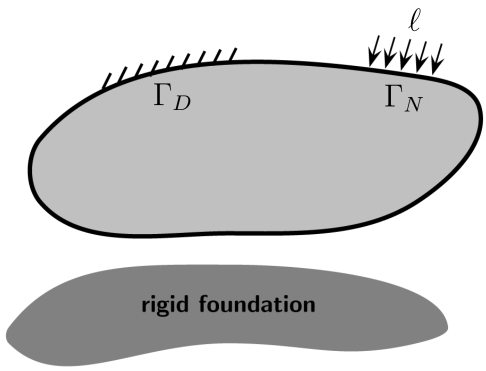

Figure 1.2.: Elastic workpiece clamped at Γ_D and pushed against a rigid foundation, boundary loads acting on Γ_N

The optimal control of Signorini contact problems has also been discussed w.r.t. the existence of solutions and necessary optimality conditions, cf. [12, 19, 29, 62]. The latter have been proven for either a simplified or a regularized Signorini problem. Furthermore, the solution operator of a simplified Signorini problem was shown to be directionally differentiable in [62]. To the best knowledge of the author there are no results concerning the differentiability of the solution operator associated with ($\mathbf{VI}_S$). By adapting the technique of [62] we will establish directional differentiability of this operator and derive first-order necessary conditions of strong stationary type, see Chapter 3.

Notation

In all what follows $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary Γ in dimension $d = 2, 3$. The boundary consists of three disjoint parts, the Dirichlet boundary Γ_D, the Neumann boundary Γ_N and the boundary of possible contact Γ_C. Vectors and tensors are represented by bold-face letters. We denote by $\mathbb{S} := \mathbb{R}^{d\times d}_{\text{sym}}$ the space of symmetric $d \times d$ matrices endowed with the Frobenius norm. For $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}$ the associated scalar product is denoted by $\boldsymbol{\sigma} : \boldsymbol{\tau} = \sum_{ij} \sigma_{ij}\tau_{ij}$. We write $\mathcal{L}(X, Y)$ for the space of linear and continuous operators from a normed space X into a normed space Y. If $X = Y$, then we abbreviate with $\mathcal{L}(X)$. The dual space of X and the adjoint operator of $T \in \mathcal{L}(X,Y)$ are denoted by X' and T^*, respectively. For a subset $M \subset X$ and $x \in M$, $\mathcal{K}_x(M)$ denotes the conical hull of $M - \{x\}$, i.e., $\mathcal{K}_x(M) = \{\alpha(w-x) \colon \alpha \geq 0,\, w \in M\}$. Furthermore, $M^+ = \{T \in X' \colon Tx \geq 0 \;\forall\, x \in M\}$ is the dual cone of M. We write M^0 for the negative of M^+, which is also called polar cone of M, and $[M]_H^0$ denotes the polar cone of M w.r.t. $H \in \mathcal{L}(X,X')$, i.e., $[M]_H^0 = \{x \in X \colon (Hx)y \leq 0 \;\forall\, y \in M\}$. Throughout, $c > 0$ represents a generic constant. Moreover, we unambiguously write $|\cdot|$ for the euclidean norm as well as for the Lebesgue measure. By $\mathcal{M}(K)$ and $\mathcal{M}^+(K)$ we denote the set of regular (signed) Borel measures on a compact set $K \subset \mathbb{R}^d$ and its subset containing all positive measures, respectively. The positive and negative parts of a real-valued function f are abbreviated by $f^+ = \max(f,0)$ and $f^- = -\min(f,0)$ such that $f = f^+ - f^-$. For frequently used function spaces we introduce the following abbreviations

$$\begin{aligned}
W_D^{1,p}\big(\Omega;\mathbb{R}^d\big) &:= \big\{\boldsymbol{u} \in W^{1,p}\big(\Omega;\mathbb{R}^d\big) : \tau\boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_D\big\}\\
V &:= W_D^{1,2}\big(\Omega;\mathbb{R}^d\big)\\
\tau_{\boldsymbol{\nu}}[V] &:= \big\{z \in H^{1/2}(\Gamma) \colon \exists\, \boldsymbol{v} \in V \text{ with } \tau_{\boldsymbol{\nu}}\boldsymbol{v} = z\big\}\\
W_D^{-1,p}\big(\Omega;\mathbb{R}^d\big) &:= \big(W_D^{1,p'}\big(\Omega;\mathbb{R}^d\big)\big)'\\
U &:= L^2\big(\Omega;\mathbb{R}^d\big) \times L^2\big(\Gamma_N;\mathbb{R}^d\big)\\
S &:= L^2(\Omega;\mathbb{S}),
\end{aligned}$$

where τ is the trace operator, $\tau_{\boldsymbol{\nu}}$ is the normal trace operator, cf. Section A.2, and p' is the integrability exponent conjugated to p, i.e., $1/p + 1/p' = 1$. The dual pairing between $W_D^{-1,p}\big(\Omega;\mathbb{R}^d\big)$ and $W_D^{1,p}\big(\Omega;\mathbb{R}^d\big)$ as well as between $(\tau_{\boldsymbol{\nu}}[V])'$ and $\tau_{\boldsymbol{\nu}}[V]$ and between $\big(H^1(\Omega)\big)'$ and $H^1(\Omega)$ is always denoted by $\langle\cdot,\,\cdot\rangle$. The scalar product in L^2-type spaces such as $L^2(\Omega)$, U, S, and S^2 is always denoted by $(\cdot,\,\cdot)$.

2. Optimal control of static elastoplasticity

Let $J\colon V \times U \to \mathbb{R}$ be a given objective functional. We consider the following optimal control problem:

$$\left.\begin{aligned} \text{Minimize} \quad & J(\boldsymbol{u}, \boldsymbol{f}) \\ \text{s.t.} \quad & \text{the elastoplasticity problem } (\mathbf{VI}_E) \text{ with } \ell \in V' \text{ defined by} \\ & \langle \ell, \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{f}_1 \cdot \boldsymbol{v} \,\mathrm{d}x + \int_{\Gamma_N} \boldsymbol{f}_2 \cdot \boldsymbol{v} \,\mathrm{d}s \quad \forall \boldsymbol{v} \in V. \end{aligned}\right\} \quad (\mathbf{P}_E)$$

The functions $\boldsymbol{f}_1 \in L^2(\Omega; \mathbb{R}^d)$ and $\boldsymbol{f}_2 \in L^2(\Gamma_N; \mathbb{R}^d)$ can be interpreted as volume and boundary forces, respectively, acting on the domain Ω. For $\boldsymbol{\Sigma} = (\boldsymbol{\sigma}, \boldsymbol{\chi})$, $\boldsymbol{T} = (\boldsymbol{\tau}, \boldsymbol{\mu}) \in S^2$ and $\boldsymbol{v} \in V$ the linear operators $A\colon S^2 \to S^2$ and $\operatorname{div}\colon S^2 \to V'$ appearing in $(\mathbf{VI}_E)$ are defined by

$$\begin{aligned} (A\boldsymbol{\Sigma}, \boldsymbol{T}) &= \int_\Omega \boldsymbol{\tau} : \mathbb{C}^{-1}\boldsymbol{\sigma} \,\mathrm{d}x + \int_\Omega \boldsymbol{\mu} : \mathbb{H}^{-1}\boldsymbol{\chi} \,\mathrm{d}x \\ \langle \operatorname{div} \boldsymbol{\Sigma}, \boldsymbol{v} \rangle &= -\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{d}x. \end{aligned}$$

Herein, $\mathbb{C}^{-1}(x)$ and $\mathbb{H}^{-1}(x)$ are linear maps from $\mathbb{S}$ to $\mathbb{S}$, which may depend on the spatial variable x, and

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = \frac{1}{2}\big(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^\top\big) \tag{2.1}$$

is the linearized strain tensor. The closed and convex set $\mathcal{K} \subset S^2$ of admissible stresses is determined by the von Mises yield condition, i.e.,

$$\mathcal{K} = \big\{\boldsymbol{\Sigma} \in S^2 : \phi(\boldsymbol{\Sigma}) \le 0 \text{ a.e. in } \Omega\big\} \tag{2.2}$$

with yield function ϕ. In case of linear kinematic hardening the yield function is given by

$$\phi(\boldsymbol{\Sigma}) = \frac{\|\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D\|_{\mathbb{S}}^2 - \sigma_0^2}{2}, \tag{2.3}$$

where $\sigma_0 > 0$ is the yield stress and

$$\boldsymbol{\sigma}^D = \boldsymbol{\sigma} - \frac{1}{d}(\operatorname{trace} \boldsymbol{\sigma})\boldsymbol{I}$$

with identity tensor $\boldsymbol{I} \in \mathbb{S}$ denotes the deviatoric part of $\boldsymbol{\sigma}$. In all what follows we abbreviate $\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D$ by

$$\mathcal{D}\boldsymbol{\Sigma} = \boldsymbol{\sigma}^D + \boldsymbol{\chi}^D.$$

Moreover it will be helpful for the subsequent analysis to note that the linear continuous operator $\mathcal{D}\colon S^2 \to S$ satisfies

$$\mathcal{D}^*\mathcal{D}\boldsymbol{\Sigma} = \begin{pmatrix} \mathcal{D}\boldsymbol{\Sigma} \\ \mathcal{D}\boldsymbol{\Sigma} \end{pmatrix}.$$

The following assumption is supposed to hold throughout this chapter:

Assumption 2.1.

(1) The domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, is a bounded domain with Lipschitz boundary Γ. The boundary consists of two disjoint measurable parts Γ_N and Γ_D such that $\Gamma = \Gamma_N \cup \Gamma_D$. While Γ_N is a relatively open subset, Γ_D is a relatively closed subset of Γ with positive measure.
In addition, the set $\Omega \cup \Gamma_N$ is regular in the sense of Gröger, cf. [34, Definition 2]. That is, for every point $x \in \Gamma$, there exists an open neighborhood $\mathcal{U}_x \subset \mathbb{R}^d$ of x and a bi-Lipschitz map (a Lipschitz continuous and bijective map with Lipschitz continuous inverse) $\Psi_x : \mathcal{U}_x \to \mathbb{R}^d$ such that $\Psi_x(x) = 0 \in \mathbb{R}^d$ and $\Psi_x\big(\mathcal{U}_x \cap (\Omega \cup \Gamma_N)\big)$ equals one of the following sets:

$$\begin{aligned} E_1 &:= \{y \in \mathbb{R}^d : |y| < 1,\ y_n < 0\} \\ E_2 &:= \{y \in \mathbb{R}^d : |y| < 1,\ y_n \le 0\} \\ E_3 &:= \{y \in E_2 : y_d < 0 \text{ or } y_1 > 0\}. \end{aligned}$$

(2) The fourth-order tensors $\mathbb{C}^{-1}$ and $\mathbb{H}^{-1}$ are elements of $L^\infty(\Omega; \mathcal{L}(\mathbb{S}))$. Moreover, both $\mathbb{C}^{-1}(x)$ and $\mathbb{H}^{-1}(x)$ are uniformly coercive on $\mathbb{S}$ and symmetric, i.e., $\boldsymbol{\sigma} : \mathbb{C}^{-1}(x)\boldsymbol{\sigma} \ge c\,\|\boldsymbol{\sigma}\|_{\mathbb{S}}^2$ with $c > 0$ independent of x, $\boldsymbol{\tau} : \mathbb{C}^{-1}(x)\boldsymbol{\sigma} = \boldsymbol{\sigma} : \mathbb{C}^{-1}(x)\boldsymbol{\tau}$ for all $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}$ and analogous relations hold for $\mathbb{H}^{-1}$.

Remark 2.2. *There is a broad class of non-smooth domains which satisfy Assumption 2.1(1). A characterization of such domains can be found in [36, Section 5]. Assumption 2.1(2) is for instance fulfilled by isotropic and homogeneous materials, where $\mathbb{C}^{-1}$ and $\mathbb{H}^{-1}$ are given by*

$$\mathbb{C}^{-1}\boldsymbol{\sigma} = \frac{1}{2\,\mu_L}\boldsymbol{\sigma} - \frac{\lambda_L}{2\,\mu_L\,(2\,\mu_L + d\,\lambda_L)}(\operatorname{trace}\boldsymbol{\sigma})\boldsymbol{I}, \qquad \mathbb{H}^{-1}\boldsymbol{\chi} = \frac{1}{k_1}\boldsymbol{\chi} \tag{2.4}$$

with Lamé constants μ_L, λ_L and hardening constant $k_1 > 0$. If $\mu_L > 0$ and $d\,\lambda_L + 2\,\mu_L > 0$, then $\mathbb{C}^{-1}$ is coercive:

$$\boldsymbol{\sigma} : \mathbb{C}^{-1}\boldsymbol{\sigma} = \frac{\|\boldsymbol{\sigma}\|_{\mathbb{S}}^2}{2\,\mu_L} - \underbrace{\frac{d\,\lambda_L}{2\,\mu_L(2\,\mu_L + d\,\lambda_L)}}_{<1/(2\,\mu_L)} \underbrace{\frac{(\operatorname{trace}\boldsymbol{\sigma})^2}{d}}_{=\frac{1}{d}\left(\sum_{i=1}^d \boldsymbol{\sigma}_{ii}\right)^2 \le \sum_{i=1}^d \boldsymbol{\sigma}_{ii}^2} \ge c\,\|\boldsymbol{\sigma}\|_{\mathbb{S}}^2.$$

2.1. Higher regularity

In this section we address the integrability of the solution to problem ($\mathbf{VI}_E$). First, we recall two known results concerning the existence and the uniqueness of a solution.

Proposition 2.3. *[40, Propositions 3.1, 3.2 and Lemma 3.3] For every $\ell \in V'$, problem ($\mathbf{VI}_E$) possesses a unique solution $(\boldsymbol{\Sigma}, \boldsymbol{u}) \in S^2 \times V$.*

In consequence of Proposition 2.3 we can introduce the control-to-state map associated with ($\mathbf{VI}_E$):

Definition 2.4. *The control-to-state map $V' \ni \ell \mapsto (\boldsymbol{\Sigma}, \boldsymbol{u}) \in S^2 \times V$ is denoted by G_E. We sometimes consider G_E with different domains and ranges. For the sake of convenience these operators are also denoted by G_E.*

By means of a unique slack variable, termed *plastic multiplier*, the variational inequality in ($\mathbf{VI}_E$) can equivalently be expressed as a complementarity system:

Theorem 2.5. *[43, Theorem 2.2] Let $\ell \in V'$ be given. The pair $(\boldsymbol{\Sigma}, \boldsymbol{u}) \in S^2 \times V$ is the unique solution of ($\mathbf{VI}_E$) if and only if there exists a multiplier $\lambda \in L^2(\Omega)$ such that*

$$A\boldsymbol{\Sigma} + \operatorname{div}^* \boldsymbol{u} + \lambda \mathcal{D}^*\mathcal{D}\boldsymbol{\Sigma} = \mathbf{0} \quad \text{in } S^2 \tag{2.5a}$$

$$\operatorname{div} \boldsymbol{\Sigma} = \ell \quad \text{in } V' \tag{2.5b}$$

$$0 \le \lambda(x) \perp \phi\big(\boldsymbol{\Sigma}(x)\big) \le 0 \quad \text{a.e. in } \Omega \tag{2.5c}$$

holds. Moreover, λ is unique.

The integrability of the solution to ($\mathbf{VI}_E$) improves, if the inhomogeneity ℓ is slightly more regular. The essential tool to prove this is the next theorem which relies on Assumption 2.1(1).

Theorem 2.6. *[41, Theorem 1.1] Let the nonlinear function $\boldsymbol{b}\colon \Omega \times \mathbb{S} \to \mathbb{S}$ satisfy*

$$\boldsymbol{b}(\cdot, \mathbf{0}) \in L^\infty(\Omega; \mathbb{S}) \tag{2.6a}$$

$$\boldsymbol{b}(\cdot, \boldsymbol{\sigma}) \text{ is measurable} \tag{2.6b}$$

$$\big(\boldsymbol{b}(x, \boldsymbol{\sigma}) - \boldsymbol{b}(x, \boldsymbol{\tau})\big) : (\boldsymbol{\sigma} - \boldsymbol{\tau}) \ge \underline{m}\, \|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_{\mathbb{S}}^2 \tag{2.6c}$$

$$\|\boldsymbol{b}(x, \boldsymbol{\sigma}) - \boldsymbol{b}(x, \boldsymbol{\tau})\|_{\mathbb{S}} \le \overline{m}\, \|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_{\mathbb{S}} \tag{2.6d}$$

f.a.a. $x \in \Omega$ and all $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}$ with constants $0 < \underline{m} \le \overline{m}$. Furthermore let $B_p\colon W^{1,p}_D\big(\Omega; \mathbb{R}^d\big) \to W^{-1,p}_D\big(\Omega; \mathbb{R}^d\big)$ be defined through

$$\langle B_p(\boldsymbol{u}), \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{b}(\cdot, \boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{d}x \quad \forall \boldsymbol{u} \in W^{1,p}_D\big(\Omega; \mathbb{R}^d\big),\ \boldsymbol{v} \in W^{1,p'}_D\big(\Omega; \mathbb{R}^d\big).$$

Then there exists $\hat{p} > 2$ such that the operator B_p is continuously invertible for all $p \in [2, \hat{p}]$. Moreover, the inverse is globally Lipschitz with a Lipschitz constant independent of $p \in [2, \bar{p}]$.

In addition to Theorem 2.6 we need the following auxiliary result.

Lemma 2.7 ([40, Lemma 4.1])**.** *Let H be a Hilbert space and $C \subset H$ be a nonempty closed convex set. Moreover let $P_C(x)$ denote the orthogonal projection of x onto C. Then the operator $F\colon H \to H$ defined by $F(x) = x - P_C(x)$ is monotone, i.e., it holds*

$$(F(x) - F(y),\, x - y) \geq 0 \quad \forall\, x, y \in H.$$

The elastoplasticity problem ($\mathbf{VI}_E$) is equivalent to a single nonlinear PDE in the displacement field $\boldsymbol{u}$ only whose underlying nonlinear function meets the conditions (2.6a)–(2.6d). This is why we can establish higher integrability for the solution of ($\mathbf{VI}_E$).

Theorem 2.8. *There exists $\hat{p} > 2$ such that for all $p \in [2, \hat{p}]$ and for any $\ell \in W_D^{-1,p}(\Omega; \mathbb{R}^d)$, the solution $(\boldsymbol{\Sigma}, \boldsymbol{u})$ of ($\mathbf{VI}_E$) belongs to $L^p(\Omega; \mathbb{S}^2) \times W_D^{1,p}(\Omega; \mathbb{R}^d)$. Moreover there exists a constant $L > 0$ such that*

$$\|\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2\|_{L^p(\Omega;\mathbb{S}^2)} + \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{W_D^{1,p}(\Omega;\mathbb{R}^d)} \leq L\, \|\ell_1 - \ell_2\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)},$$

i.e., the control-to-state map G_E is Lipschitz continuous from $W_D^{-1,p}(\Omega; \mathbb{R}^d)$ into $L^p(\Omega; \mathbb{S}^2) \times W_D^{1,p}(\Omega; \mathbb{R}^d)$.

Proof. The arguments are similar to [74, Theorem 4.4.4 and Proposition 4.4.5]. We reformulate ($\mathbf{VI}_E$) and (2.5), respectively, as a nonlinear PDE in $\boldsymbol{u}$ and apply Theorem 2.6.
Let $(\boldsymbol{\Sigma}, \boldsymbol{u})$ be given by the solution of (2.5). Testing (2.5a) with $(\boldsymbol{\tau}, \mathbf{0})$, $\boldsymbol{\tau} \in S$, we find

$$\mathbb{C}^{-1}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda \mathcal{D}\boldsymbol{\Sigma} = \mathbf{0} \quad \text{a.e. in } \Omega,$$

where the property $(\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D) : \boldsymbol{\tau}^D = (\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D) : \boldsymbol{\tau}$ was used. If we furthermore test with $(\mathbf{0}, \boldsymbol{\mu})$, $\boldsymbol{\mu} \in S$, we arrive at

$$\mathbb{H}^{-1}\boldsymbol{\chi} + \lambda \mathcal{D}\boldsymbol{\Sigma} = \mathbf{0} \quad \text{a.e. in } \Omega. \tag{2.7}$$

Combining both equations yields

$$\mathbb{C}^{-1}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathbb{H}^{-1}\boldsymbol{\chi} = \mathbf{0} \quad \text{a.e. in } \Omega. \tag{2.8}$$

Next we derive a pointwise version of ($\mathbf{VI}_E$). To this end let $x_0 \in \Omega$ be an arbitrary Lebesgue point of $\mathbb{C}^{-1}$, $\boldsymbol{\sigma}$, $\mathbb{H}^{-1}$, $\boldsymbol{\chi}$, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ and their products arising in the sequel. Moreover let $(\boldsymbol{\tau}, \boldsymbol{\mu}) \in K$ be given, where the closed and convex set K is defined by

$$K = \{\boldsymbol{T} \in \mathbb{S}^2 : \phi(\boldsymbol{T}) \leq 0\}.$$

For $\rho > 0$ such that $B_\rho(x_0) \subset \Omega$ we then define $(\tilde{\boldsymbol{\tau}}, \tilde{\boldsymbol{\mu}}) \in \mathcal{K}$ through

$$(\tilde{\boldsymbol{\tau}}, \tilde{\boldsymbol{\mu}}) = \begin{cases} (\boldsymbol{\tau}, \boldsymbol{\mu}), & x \in B_\rho(x_0) \\ (\boldsymbol{\sigma}, \boldsymbol{\chi}), & x \in \Omega \setminus B_\rho(x_0). \end{cases}$$

Testing $(\mathbf{VI}_E)$ with $(\tilde{\boldsymbol{\tau}}, \tilde{\boldsymbol{\mu}})$ results in

$$\begin{aligned} 0 &\leq \frac{1}{|B_\rho(x_0)|} \int_\Omega \mathbb{C}^{-1}\boldsymbol{\sigma} : (\tilde{\boldsymbol{\tau}} - \boldsymbol{\sigma}) + \mathbb{H}^{-1}\boldsymbol{\chi} : (\tilde{\boldsymbol{\mu}} - \boldsymbol{\chi}) - \boldsymbol{\varepsilon}(\boldsymbol{u}) : (\tilde{\boldsymbol{\tau}} - \boldsymbol{\sigma}) \, \mathrm{d}x \\ &= \frac{1}{|B_\rho(x_0)|} \int_{B(x_0,\rho)} \mathbb{C}^{-1}\boldsymbol{\sigma} : (\boldsymbol{\tau} - \boldsymbol{\sigma}) + \mathbb{H}^{-1}\boldsymbol{\chi} : (\boldsymbol{\mu} - \boldsymbol{\chi}) - \boldsymbol{\varepsilon}(\boldsymbol{u}) : (\boldsymbol{\tau} - \boldsymbol{\sigma}) \, \mathrm{d}x. \end{aligned}$$

We take the limit $\rho \searrow 0$ and obtain

$$\begin{aligned} \mathbb{C}^{-1}(x_0)\boldsymbol{\sigma}(x_0) : (\boldsymbol{\tau} - \boldsymbol{\sigma}(x_0)) + \mathbb{H}^{-1}(x_0)\boldsymbol{\chi}(x_0) : (\boldsymbol{\mu} - \boldsymbol{\chi}(x_0)) \\ -\boldsymbol{\varepsilon}(\boldsymbol{u}(x_0)) : (\boldsymbol{\tau} - \boldsymbol{\sigma}(x_0)) \geq 0. \end{aligned}$$

Since almost every point in Ω is a common Lebesgue point of $\mathbb{C}^{-1}$, $\boldsymbol{\sigma}$, $\mathbb{H}^{-1}$, $\boldsymbol{\chi}$, $\boldsymbol{\varepsilon}(\boldsymbol{u})$, and their respective products, we conclude

$$\begin{aligned} \mathbb{C}^{-1}(x)\boldsymbol{\sigma}(x) : (\boldsymbol{\tau} - \boldsymbol{\sigma}(x)) + \mathbb{H}^{-1}(x)\boldsymbol{\chi}(x) : (\boldsymbol{\mu} - \boldsymbol{\chi}(x)) \\ -\boldsymbol{\varepsilon}(\boldsymbol{u}(x)) : (\boldsymbol{\tau} - \boldsymbol{\sigma}(x)) \geq 0 \quad \forall\, (\boldsymbol{\tau}, \boldsymbol{\mu}) \in K. \end{aligned} \tag{2.9}$$

f.a.a. $x \in \Omega$. Plugging $(\boldsymbol{\sigma}(x), \boldsymbol{\mu})$ in (2.9) leads to

$$\mathbb{H}^{-1}(x)\boldsymbol{\chi}(x) : (\boldsymbol{\mu} - \boldsymbol{\chi}(x)) \geq 0 \quad \text{for all } \boldsymbol{\mu} \in \mathbb{S} \text{ such that } \boldsymbol{\mu} \in \bar{K} - \boldsymbol{\sigma}(x) \tag{2.10}$$

with convex and closed set $\bar{K}$ defined by

$$\bar{K} = \{\boldsymbol{\tau} \in \mathbb{S} : (\boldsymbol{\tau}, \mathbf{0}) \in K\}.$$

Note that $\boldsymbol{\mu} \in \bar{K} - \boldsymbol{\sigma}(x)$ is equivalent to $(\boldsymbol{\sigma}(x), \boldsymbol{\mu}) \in K$. The variational inequality (2.10) is the necessary and sufficient optimality condition for the convex optimization problem

$$\min_{\boldsymbol{\mu} \in \bar{K} - \boldsymbol{\sigma}(x)} \frac{1}{2} \|\boldsymbol{\mu}\|^2_{\mathbb{H}^{-1}(x)}.$$

Herein $\|\cdot\|_{\mathbb{H}^{-1}(x)}$ is the norm induced by the coercive operator $\mathbb{H}^{-1}(x)$, i.e.,

$$\|\boldsymbol{\mu}\|_{\mathbb{H}^{-1}(x)} = (\mathbb{H}^{-1}(x)\boldsymbol{\mu} : \boldsymbol{\mu})^{\frac{1}{2}}.$$

Therefore, f.a.a. $x \in \Omega$ the solution $\boldsymbol{\chi}(x)$ of (2.10) is given by

$$\boldsymbol{\chi}(x) = \mathrm{Proj}^{\mathbb{H}^{-1}(x)}_{\bar{K} - \boldsymbol{\sigma}(x)}(\mathbf{0}) = \mathrm{Proj}^{\mathbb{H}^{-1}(x)}_{\bar{K}}(\boldsymbol{\sigma}(x)) - \boldsymbol{\sigma}(x), \tag{2.11}$$

where, for a given closed and convex set $E \subset \mathbb{S}$, $\mathrm{Proj}^{\mathbb{H}^{-1}(x)}_E$ denotes the orthogonal projection on E w.r.t. the norm induced by $\mathbb{H}^{-1}(x)$. Inserting (2.11) in (2.8) yields

$$\mathbb{C}^{-1}(x)\boldsymbol{\sigma}(x) + \mathbb{H}^{-1}(x)(\boldsymbol{\sigma}(x) - \mathrm{Proj}^{\mathbb{H}^{-1}(x)}_{\bar{K}}(\boldsymbol{\sigma}(x))) = \boldsymbol{\varepsilon}(\boldsymbol{u})(x) \quad \text{a.e. in } \Omega. \tag{2.12}$$

The left-hand side can be expressed by means of the the nonlinear function $M_x\colon \mathbb{S} \to \mathbb{S}$ defined by

$$\boldsymbol{\tau} \mapsto \mathbb{C}^{-1}(x)\boldsymbol{\tau} + \mathbb{H}^{-1}(x)(\boldsymbol{\tau} - \mathrm{Proj}_{\bar{K}}^{\mathbb{H}^{-1}(x)}(\boldsymbol{\tau})).$$

In what follows we show that M_x is invertible f.a.a. $x \in \Omega$. On account of the monotonicity of $\mathbb{H}^{-1}(x)(\boldsymbol{\tau} - \mathrm{Proj}_{\bar{K}}^{\mathbb{H}^{-1}(x)}(\boldsymbol{\tau}))$, cf. Lemma 2.7, it follows for arbitrary $\boldsymbol{\tau}, \boldsymbol{\mu} \in \mathbb{S}$

$$\begin{aligned}
(M_x(\boldsymbol{\tau}) - M_x(\boldsymbol{\mu}),\, \boldsymbol{\tau} - \boldsymbol{\mu})_{\mathbb{S}} &= \left(\mathbb{C}^{-1}(x)(\boldsymbol{\tau} - \boldsymbol{\mu}),\, \boldsymbol{\tau} - \boldsymbol{\mu}\right)_{\mathbb{S}} \\
&\quad + \left(\mathbb{H}^{-1}(x)(\boldsymbol{\tau} - \mathrm{Proj}_{\bar{K}}^{\mathbb{H}^{-1}(x)}(\boldsymbol{\tau})),\, \boldsymbol{\tau} - \boldsymbol{\mu}\right)_{\mathbb{S}} \\
&\quad - \left(\mathbb{H}^{-1}(x)(\boldsymbol{\mu} - \mathrm{Proj}_{\bar{K}}^{\mathbb{H}^{-1}(x)}(\boldsymbol{\mu})),\, \boldsymbol{\tau} - \boldsymbol{\mu}\right)_{\mathbb{S}} \\
&\geq \left(\mathbb{C}^{-1}(x)(\boldsymbol{\tau} - \boldsymbol{\mu}),\, \boldsymbol{\tau} - \boldsymbol{\mu}\right)_{\mathbb{S}} \\
&\geq \underline{m}\, \|\boldsymbol{\tau} - \boldsymbol{\mu}\|_{\mathbb{S}}^2\,,
\end{aligned}$$

where $\underline{m} > 0$ is the coercivity constant of $\mathbb{C}^{-1}$. Hence the mapping $M_x(\cdot)$ is strongly monotone and coercive because of $M_x(\mathbf{0}) = \mathbf{0}$. Due to the boundedness of $\mathbb{C}^{-1}$ and $\mathbb{H}^{-1}$ and the non-expansiveness of the projection w.r.t. the norm induced by $\mathbb{H}^{-1}(x)$ we observe

$$\begin{aligned}
\|M_x(\boldsymbol{\tau}) - M_x(\boldsymbol{\mu})\|_{\mathbb{S}} &\leq \left(\|\mathbb{C}^{-1}\|_{L^\infty(\Omega;\mathcal{L}(\mathbb{S}))} + \|\mathbb{H}^{-1}\|_{L^\infty(\Omega;\mathcal{L}(\mathbb{S}))}\right) \|\boldsymbol{\tau} - \boldsymbol{\mu}\|_{\mathbb{S}} \\
&\quad + \frac{\|\mathbb{H}^{-1}\|_{L^\infty(\Omega;\mathcal{L}(\mathbb{S}))}}{\underline{h}^{\frac{1}{2}}} \left\|\mathrm{Proj}_{\bar{K}}^{\mathbb{H}^{-1}(x)}(\boldsymbol{\tau}) - \mathrm{Proj}_{\bar{K}}^{\mathbb{H}^{-1}(x)}(\boldsymbol{\mu})\right\|_{\mathbb{H}^{-1}(x)} \\
&\leq \left(\|\mathbb{C}^{-1}\|_{L^\infty(\Omega;\mathcal{L}(\mathbb{S}))} + \|\mathbb{H}^{-1}\|_{L^\infty(\Omega;\mathcal{L}(\mathbb{S}))}\right) \|\boldsymbol{\tau} - \boldsymbol{\mu}\|_{\mathbb{S}} \\
&\quad + \frac{\|\mathbb{H}^{-1}\|_{L^\infty(\Omega;\mathcal{L}(\mathbb{S}))}}{\underline{h}^{\frac{1}{2}}} \|\boldsymbol{\tau} - \boldsymbol{\mu}\|_{\mathbb{H}^{-1}(x)} \\
&\leq \left(\|\mathbb{C}^{-1}\|_{L^\infty(\Omega;\mathcal{L}(\mathbb{S}))} + \|\mathbb{H}^{-1}\|_{L^\infty(\Omega;\mathcal{L}(\mathbb{S}))}\right) \|\boldsymbol{\tau} - \boldsymbol{\mu}\|_{\mathbb{S}} \\
&\quad + \frac{\|\mathbb{H}^{-1}\|_{L^\infty(\Omega;\mathcal{L}(\mathbb{S}))}^{\frac{3}{2}}}{\underline{h}^{\frac{1}{2}}} \|\boldsymbol{\tau} - \boldsymbol{\mu}\|_{\mathbb{S}} \\
&\leq \overline{m}\, \|\boldsymbol{\tau} - \boldsymbol{\mu}\|_{\mathbb{S}}\,.
\end{aligned}$$

Here $\underline{h} > 0$ denotes the coercivity constant of $H^{-1}(x)$ and $\overline{m} > 0$ is given by

$$\overline{m} = \|\mathbb{C}^{-1}\|_{L^\infty(\Omega;\mathcal{L}(\mathbb{S}))} + \|\mathbb{H}^{-1}\|_{L^\infty(\Omega;\mathcal{L}(\mathbb{S}))} + \frac{\|\mathbb{H}^{-1}\|_{L^\infty(\Omega;\mathcal{L}(\mathbb{S}))}^{\frac{3}{2}}}{\underline{h}^{\frac{1}{2}}}.$$

According to the Browder-Minty theorem the inverse $M_x^{-1}(\cdot)$ w.r.t. $\boldsymbol{\tau}$ exists f.a.a. $x \in \Omega$. If we define f.a.a. $x \in \Omega$

$$M^{-1}(x, \boldsymbol{\tau}) := M_x^{-1}(\boldsymbol{\tau}),$$

then (2.12) is equivalent to $\boldsymbol{\sigma} = M^{-1}(\cdot, \boldsymbol{\varepsilon}(\boldsymbol{u}))$. In view of (2.5b) the function $\boldsymbol{u}$ is a solution of

$$\int_\Omega M^{-1}(\cdot, \boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{d}x = -\langle \ell, \boldsymbol{v}\rangle \quad \forall \boldsymbol{v} \in V, \tag{2.13}$$

which is the desired nonlinear PDE in $\boldsymbol{u}$. As we aim to make use of Theorem 2.6, we have to check that M^{-1} satisfies the conditions (2.6a)–(2.6d), i.e., $M^{-1}(\cdot, \mathbf{0}) \in L^\infty(\Omega; \mathbb{S})$, $M^{-1}(\cdot, \boldsymbol{\tau})$ is measurable, and M^{-1} is Lipschitz continuous and strongly monotone w.r.t. $\boldsymbol{\tau}$ f.a.a. $x \in \Omega$. The strong monotonicity of $M_x(\cdot)$ implies for every $\boldsymbol{\tau}, \boldsymbol{\mu} \in \mathbb{S}$

$$\begin{aligned}\left\|M_x^{-1}(M_x(\boldsymbol{\tau})) - M_x^{-1}(M_x(\boldsymbol{\mu}))\right\|_{\mathbb{S}}^2 = \|\boldsymbol{\tau} - \boldsymbol{\mu}\|_{\mathbb{S}}^2 &\le \frac{1}{\underline{m}}\left(M_x(\boldsymbol{\tau}) - M_x(\boldsymbol{\mu}),\, \boldsymbol{\tau} - \boldsymbol{\mu}\right)_{\mathbb{S}} \\ &\le \frac{1}{\underline{m}} \|M_x(\boldsymbol{\tau}) - M_x(\boldsymbol{\mu})\|_{\mathbb{S}} \|\boldsymbol{\tau} - \boldsymbol{\mu}\|_{\mathbb{S}}.\end{aligned}$$

Consequently, the inverse $M_x^{-1}(\cdot)$ is Lipschitz continuous with Lipschitz constant $1/\underline{m}$. The strong monotonicity and Lipschitz continuity of $M_x(\cdot)$ furthermore lead to the estimate

$$\begin{aligned}\left(M_x^{-1}(\boldsymbol{\tau}) - M_x^{-1}(\boldsymbol{\mu}),\, \boldsymbol{\tau} - \boldsymbol{\mu}\right)_{\mathbb{S}} &\ge \underline{m} \left\|M_x^{-1}(\boldsymbol{\tau}) - M_x^{-1}(\boldsymbol{\mu})\right\|_{\mathbb{S}}^2 \\ &\ge \frac{\underline{m}}{\overline{m}^2} \left\|M_x(M_x^{-1}(\boldsymbol{\tau})) - M_x(M_x^{-1}(\boldsymbol{\mu}))\right\|_{\mathbb{S}}^2 \\ &= \frac{\underline{m}}{\overline{m}^2} \|\boldsymbol{\tau} - \boldsymbol{\mu}\|_{\mathbb{S}}^2,\end{aligned}$$

which shows strong monotonicity of M_x^{-1}. Since moreover $M^{-1}(\cdot, \mathbf{0}) = \mathbf{0}$ belongs to $L^\infty(\Omega; \mathbb{S})$, it remains to be proven that $M^{-1}(x, \boldsymbol{\tau})$ is measurable w.r.t. x. Due to the measurability of $\mathbb{C}^{-1}$ and $\mathbb{H}^{-1}$ there exist simple functions $\mathbb{C}_n^{-1}, \mathbb{H}_n^{-1} \in L^\infty(\Omega; \mathcal{L}(\mathbb{S}))$, $n \in \mathbb{N}$, with

$$\left\|\mathbb{C}^{-1}(x) - \mathbb{C}_n^{-1}(x)\right\|_{\mathcal{L}(\mathbb{S})} \to 0 \quad \text{and} \quad \left\|\mathbb{H}^{-1}(x) - \mathbb{H}_n^{-1}(x)\right\|_{\mathcal{L}(\mathbb{S})} \to 0 \quad \text{a.e. in } \Omega.$$

Thus, there is $N_x^{\mathbb{C}} \in \mathbb{N}$, depending on x, such that

$$\begin{aligned}\boldsymbol{\tau} : \mathbb{C}_n^{-1}(x)\boldsymbol{\tau} &= \boldsymbol{\tau} : \mathbb{C}^{-1}(x)\boldsymbol{\tau} + \boldsymbol{\tau} : (\mathbb{C}_n^{-1}(x) - \mathbb{C}^{-1}(x))\boldsymbol{\tau} \\ &\ge \underline{m} \|\boldsymbol{\tau}\|_{\mathbb{S}}^2 - \left\|\mathbb{C}_n^{-1}(x) - \mathbb{C}^{-1}(x)\right\|_{\mathcal{L}(\mathbb{S})} \|\boldsymbol{\tau}\|_{\mathbb{S}}^2 \\ &\ge \underline{m}/2 \|\boldsymbol{\tau}\|_{\mathbb{S}}^2\end{aligned}$$

for all $n \ge N_x^{\mathbb{C}}$ and an analogous estimate holds true for $\mathbb{H}_n^{-1}$. We define

$$\tilde{\mathbb{C}}_n^{-1}(x) = \begin{cases}\mathbb{C}_n^{-1}(x), & n \ge N_x^{\mathbb{C}} \\ I_{\mathbb{S}}, & \text{else}\end{cases} \quad \text{and} \quad \tilde{\mathbb{H}}_n^{-1}(x) = \begin{cases}\mathbb{H}_n^{-1}(x), & n \ge N_x^{\mathbb{H}} \\ I_{\mathbb{S}}, & \text{else,}\end{cases}$$

where $I_{\mathbb{S}} \colon \mathbb{S} \to \mathbb{S}$ denotes the identity mapping. Thereby we obtain simple functions $\tilde{\mathbb{C}}_n^{-1} \colon \Omega \to \mathbb{S}$ and $\tilde{\mathbb{H}}_n^{-1} \colon \Omega \to \mathbb{S}$ with

$$\left\|\mathbb{C}^{-1}(x) - \tilde{\mathbb{C}}_n^{-1}(x)\right\|_{\mathcal{L}(\mathbb{S})} \to 0 \quad \text{and} \quad \left\|\mathbb{H}^{-1}(x) - \tilde{\mathbb{H}}_n^{-1}(x)\right\|_{\mathcal{L}(\mathbb{S})} \to 0 \quad \text{a.e. in } \Omega.$$

Both $\tilde{\mathbb{C}}_n^{-1}(x)$ and $\tilde{\mathbb{H}}_n^{-1}(x)$ are furthermore uniformly coercive with coercivity constant $\min(1,\underline{m}/2) > 0$ for all $n \in \mathbb{N}$. The same arguments as for M_x yield that $M_x^n \colon \mathbb{S} \to \mathbb{S}$ defined by

$$\boldsymbol{\tau} \mapsto \tilde{\mathbb{C}}_n^{-1}(x)\boldsymbol{\tau} + \tilde{\mathbb{H}}_n^{-1}(x)(\boldsymbol{\tau} - \mathrm{Proj}_{\bar{K}}^{\tilde{\mathbb{H}}_n^{-1}(x)}(\boldsymbol{\tau}))$$

is invertible f.a.a. $x \in \Omega$ and the inverse $(M_x^n)^{-1}(\cdot)$ is Lipschitz continuous with Lipschitz constant $L_x := 1/\min(1,\underline{m}/2) = \max(1, 2/\underline{m})$, independent of n. By construction

$$M^n(x,\boldsymbol{\tau}) := M_x^n(\boldsymbol{\tau}) \quad \text{and} \quad (M^n)^{-1}(x,\boldsymbol{\tau}) := (M_x^n)^{-1}(\boldsymbol{\tau})$$

are simple functions w.r.t. x. For $\boldsymbol{\mu} := \mathrm{Proj}_{\bar{K}}^{\mathbb{H}^{-1}(x)}(\boldsymbol{\tau})$ and $\boldsymbol{\mu}_n := \mathrm{Proj}_{\bar{K}}^{\tilde{\mathbb{H}}_n^{-1}(x)}(\boldsymbol{\tau})$ we find

$$\begin{aligned}
0 &\leq \mathbb{H}^{-1}(x)(\boldsymbol{\mu} - \boldsymbol{\tau}) : (\boldsymbol{\mu}_n - \boldsymbol{\mu}) + \tilde{\mathbb{H}}_n^{-1}(x)(\boldsymbol{\mu}_n - \boldsymbol{\tau}) : (\boldsymbol{\mu} - \boldsymbol{\mu}_n) \\
&= \mathbb{H}^{-1}(x)(\boldsymbol{\mu} - \boldsymbol{\tau}) : (\boldsymbol{\mu}_n - \boldsymbol{\mu}) - \tilde{\mathbb{H}}_n^{-1}(x)(\boldsymbol{\mu} - \boldsymbol{\tau}) : (\boldsymbol{\mu}_n - \boldsymbol{\mu}) \\
&\quad + \tilde{\mathbb{H}}_n^{-1}(x)(\boldsymbol{\mu} - \boldsymbol{\tau}) : (\boldsymbol{\mu}_n - \boldsymbol{\mu}) + \tilde{\mathbb{H}}_n^{-1}(x)(\boldsymbol{\mu}_n - \boldsymbol{\tau}) : (\boldsymbol{\mu} - \boldsymbol{\mu}_n) \\
&= (\mathbb{H}^{-1}(x) - \tilde{\mathbb{H}}_n^{-1}(x))(\boldsymbol{\mu} - \boldsymbol{\tau}) : (\boldsymbol{\mu}_n - \boldsymbol{\mu}) - \tilde{\mathbb{H}}_n^{-1}(x)(\boldsymbol{\mu}_n - \boldsymbol{\mu}) : (\boldsymbol{\mu}_n - \boldsymbol{\mu}).
\end{aligned}$$

In view of the uniform coercivity of $\tilde{\mathbb{H}}_n^{-1}(x)$ we infer

$$\left\|\mathbb{H}^{-1}(x) - \tilde{\mathbb{H}}_n^{-1}(x)\right\|_{\mathcal{L}(\mathbb{S})} \|\boldsymbol{\mu} - \boldsymbol{\tau}\|_{\mathbb{S}} \|\boldsymbol{\mu}_n - \boldsymbol{\mu}\|_{\mathbb{S}} \geq c \|\boldsymbol{\mu}_n - \boldsymbol{\mu}\|_{\mathbb{S}}^2$$

and therefore $\|\boldsymbol{\mu}_n - \boldsymbol{\mu}\|_{\mathbb{S}} \to 0$. Hence the sequence $\{\boldsymbol{\mu}_n(x)\}_{n\in\mathbb{N}} \subset \mathbb{S}$ is bounded, which implies

$$\begin{aligned}
&\left\|M_x^n(\boldsymbol{\tau}) - M_x(\boldsymbol{\tau})\right\|_{\mathbb{S}} = \\
&= \left\|(\tilde{\mathbb{C}}_n^{-1}(x) - \mathbb{C}^{-1}(x))\boldsymbol{\tau} + (\tilde{\mathbb{H}}_n^{-1}(x) - \mathbb{H}^{-1}(x))\boldsymbol{\tau} + \mathbb{H}^{-1}(x)\boldsymbol{\mu} - \tilde{\mathbb{H}}_n^{-1}(x)\boldsymbol{\mu}_n\right\|_{\mathbb{S}} \\
&\leq \left(\left\|\tilde{\mathbb{C}}_n^{-1}(x) - \mathbb{C}^{-1}(x)\right\|_{\mathcal{L}(\mathbb{S})} + \left\|\tilde{\mathbb{H}}_n^{-1}(x) - \mathbb{H}^{-1}(x)\right\|_{\mathcal{L}(\mathbb{S})}\right) \|\boldsymbol{\tau}\|_{\mathbb{S}} \\
&\quad + \left\|\mathbb{H}^{-1}(x)\right\|_{\mathcal{L}(\mathbb{S})} \|\boldsymbol{\mu} - \boldsymbol{\mu}_n\|_{\mathbb{S}} + \left\|\mathbb{H}^{-1}(x) - \tilde{\mathbb{H}}_n^{-1}(x)\right\|_{\mathcal{L}(\mathbb{S})} \|\boldsymbol{\mu}_n\|_{\mathbb{S}} \xrightarrow{n\to\infty} 0.
\end{aligned}$$

Thanks to the Lipschitz continuity of $(M_x^n)^{-1}(\cdot)$ we derive f.a.a. $x \in \Omega$

$$\begin{aligned}
&\left\|(M^n)^{-1}(x,\boldsymbol{\tau}) - M^{-1}(x,\boldsymbol{\tau})\right\|_{\mathbb{S}} = \\
&= \left\|(M_x^n)^{-1}\big(M_x^n((M_x^n)^{-1}(\boldsymbol{\tau}))\big) - (M_x^n)^{-1}\big(M_x^n(M_x^{-1}(\boldsymbol{\tau}))\big)\right\|_{\mathbb{S}} \\
&\leq L_x \left\|M_x^n((M_x^n)^{-1}(\boldsymbol{\tau})) - M_x^n(M_x^{-1}(\boldsymbol{\tau}))\right\|_{\mathbb{S}} \\
&= \max(1, 2/\underline{m}) \left\|M_x(M_x^{-1}(\boldsymbol{\tau})) - M_x^n(M_x^{-1}(\boldsymbol{\tau}))\right\|_{\mathbb{S}} \xrightarrow{n\to\infty} 0
\end{aligned}$$

so that $M^{-1}(\cdot,\boldsymbol{\tau})$ is indeed measurable. Altogether we have shown

- $M^{-1}(\cdot,\mathbf{0}) \in L^\infty(\Omega;\mathbb{S})$

- $M^{-1}(\cdot, \boldsymbol{\tau})$ is measurable
- $\big(M^{-1}(x,\boldsymbol{\tau}) - M^{-1}(x,\boldsymbol{\mu}),\, \boldsymbol{\tau} - \boldsymbol{\mu}\big)_{\mathbb{S}} \geq \frac{\underline{m}}{\overline{m}^2} \left\|\boldsymbol{\tau} - \boldsymbol{\mu}\right\|_{\mathbb{S}}^2$
- $\big\|M^{-1}(x,\boldsymbol{\tau}) - M^{-1}(x,\boldsymbol{\mu})\big\|_{\mathbb{S}} \leq \frac{1}{\underline{m}} \left\|\boldsymbol{\tau} - \boldsymbol{\mu}\right\|_{\mathbb{S}}$

f.a.a. $x \in \Omega$ and all $\boldsymbol{\tau}, \boldsymbol{\mu} \in \mathbb{S}$ with $\underline{m} \leq \overline{m}$. On account of Theorem 2.6 there exists $\hat{p} > 2$ such that (2.13) admits a unique solution $\boldsymbol{u} \in W_D^{1,p}\big(\Omega; \mathbb{R}^d\big)$ for all $p \in [2, \hat{p}]$ and every $\ell \in W_D^{-1,p}\big(\Omega; \mathbb{R}^d\big)$. In addition, the associated solution map $\ell \mapsto \boldsymbol{u}$ is Lipschitz continuous for all $p \in [2, \hat{p}]$. Because the inverse operator M_x^{-1} is Lipschitz continuous with Lipschitz constant $1/\underline{m}$, we conclude that $\boldsymbol{\sigma} = M^{-1}(\cdot, \boldsymbol{\varepsilon}(\boldsymbol{u}))$ is an element of $L^p(\Omega; \mathbb{S})$. It furthermore depends Lipschitz continuously on $\boldsymbol{u}$ and thus on ℓ. From equation (2.8) we then infer $\mathbb{H}^{-1}\boldsymbol{\chi} \in L^p(\Omega; \mathbb{S})$. Since $\mathbb{H}^{-1}$ is uniformly coercive by Assumption 2.1(2), the Lax-Milgram theorem yields a $\mathbb{H} \in L^\infty\big(\Omega; \mathcal{L}(\mathbb{S})\big)$ with

$$\mathbb{H}(x)\mathbb{H}^{-1}(x) = \boldsymbol{I}_{\mathbb{S}} \quad \text{a.e. in } \Omega,$$

which leads to $\boldsymbol{\chi} \in L^p(\Omega; \mathbb{S})$. Invoking equation (2.8) again, we deduce the Lipschitz continuous dependency of $\boldsymbol{\chi}$ on ℓ. □

We end this section with a short comment on the existence of globally optimal controls for $(\mathbf{P}_E)$.

Both the embedding $V \hookrightarrow L^2\big(\Omega; \mathbb{R}^d\big)$ and the trace $\tau_N \colon V \to L^2\big(\Gamma_N; \mathbb{R}^d\big)$ on Γ_N are compact operators, cf. [64, Section 2.6]. The linear continuous operator $R \colon U \to V'$ defined through

$$\langle R\boldsymbol{f},\, \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{f}_1 \cdot \boldsymbol{v} \,\mathrm{d}x + \int_{\Gamma_N} \boldsymbol{f}_2 \cdot \tau_N \boldsymbol{v} \,\mathrm{d}s \quad \forall\, \boldsymbol{v} \in V \tag{2.14}$$

is the adjoint of their product and according to Schauder's theorem also compact. Therefore we obtain the next proposition covering the existence of a global solution to $(\mathbf{P}_E)$.

Proposition 2.9. *Suppose the mapping $V \times U \ni (\boldsymbol{u}, \boldsymbol{f}) \mapsto J(\boldsymbol{u}, \boldsymbol{f}) \in \mathbb{R}$ is continuous w.r.t. $\boldsymbol{u}$ and weakly lower semicontinuous w.r.t. $\boldsymbol{f}$. If there exist $r > 0$ and $\hat{\boldsymbol{f}} \in U$ such that*

$$J\big(G_E(R\boldsymbol{f}), \boldsymbol{f}\big) \geq J\big(G_E(R\hat{\boldsymbol{f}}), \hat{\boldsymbol{f}}\big) \quad \forall\, \boldsymbol{f} \in U \text{ with } \big\|\boldsymbol{f} - \hat{\boldsymbol{f}}\big\|_U > r,$$

then Problem $(\mathbf{P}_E)$ admits a globally optimal solution.

Proof. Let $\{\boldsymbol{f}_n\}_{n\in\mathbb{N}} \subset U$ be a minimizing sequence so that

$$J\big(G_E(R\boldsymbol{f}_n), \boldsymbol{f}_n\big) \to \inf_{\boldsymbol{f}\in U} J\big(G_E(R\boldsymbol{f}), \boldsymbol{f}\big).$$

By assumption this sequence is contained in the weakly compact set

$$B(\hat{\boldsymbol{f}}, r) := \left\{\boldsymbol{f} \in U\colon \|\boldsymbol{f} - \hat{\boldsymbol{f}}\|_U \leq r\right\}.$$

Consequently, there is a subsequence, w.l.o.g. denoted by the same symbols, with $\boldsymbol{f}_n \rightharpoonup \bar{\boldsymbol{f}}$ in $B(\hat{\boldsymbol{f}}, r)$. Thanks to the Lipschitz continuity of $G_E\colon V' \to V$, cf. Theorem 2.8, and the compactness of the operator R, mentioned above, we moreover know

$$G_E(R\boldsymbol{f}_n) \to G_E(R\bar{\boldsymbol{f}}) \quad \text{in } V.$$

Hence the continuity of J in the first variable and the weak lower semicontinuity in the second variable imply

$$J\big(G_E(R\bar{\boldsymbol{f}}), \bar{\boldsymbol{f}}\big) \leq \liminf_{n\to\infty} J\big(G_E(R\boldsymbol{f}_n), \boldsymbol{f}_n\big) = \inf_{\boldsymbol{f}\in U} J\big(G_E(R\boldsymbol{f}), \boldsymbol{f}\big),$$

which shows global optimality of $\bar{\boldsymbol{f}} \in B(\hat{\boldsymbol{f}}, r)$ for $(\mathbf{P}_E)$. □

The solution cannot be expected to be unique due to the nonlinearity of G_E.

2.2. Bouligand differentiability

Based on the higher integrability of the solution to $(\mathbf{VI}_E)$, cf. Theorem 2.8, we will show Bouligand differentiability of the control-to-state map $G_E\colon \ell \to (\boldsymbol{\Sigma}, \boldsymbol{u})$ from $W_D^{-1,p}\big(\Omega;\mathbb{R}^d\big)$ to $S^2 \times V$ with $p > 2$.

Throughout this section let $\bar{\ell} \in W_D^{-1,p}\big(\Omega;\mathbb{R}^d\big)$ be fixed but arbitrary and $(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}}, \bar{\lambda})$ denote the solution of (2.5) with right hand side $\bar{\ell}$, i.e., $(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}}, \bar{\lambda}) \in S \times V \times L^2(\Omega)$ solves

$$A\bar{\boldsymbol{\Sigma}} + \operatorname{div}^* \bar{\boldsymbol{u}} + \bar{\lambda}\mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Sigma}} = \mathbf{0} \quad \text{in } S^2 \tag{2.15a}$$

$$\operatorname{div} \bar{\boldsymbol{\Sigma}} = \bar{\ell} \quad \text{in } V' \tag{2.15b}$$

$$0 \leq \bar{\lambda}(x) \perp \phi\big(\bar{\boldsymbol{\Sigma}}(x)\big) \leq 0 \quad \text{a.e. in } \Omega. \tag{2.15c}$$

In view of the complementarity (2.15c) we define the following subsets of Ω up to sets of zero measure

$$\bar{\mathcal{A}}_s = \big\{x \in \Omega : \bar{\lambda}(x) > 0\big\} \qquad \text{(strongly active set)} \tag{2.16a}$$

$$\bar{\mathcal{B}} = \big\{x \in \Omega : \phi\big(\bar{\boldsymbol{\Sigma}}(x)\big) = \bar{\lambda}(x) = 0\big\} \qquad \text{(biactive set)} \tag{2.16b}$$

$$\bar{\mathcal{I}} = \big\{x \in \Omega : \phi\big(\bar{\boldsymbol{\Sigma}}(x)\big) < 0\big\}. \qquad \text{(inactive set)} \tag{2.16c}$$

Note that $\Omega = \bar{\mathcal{A}}_s \cup \bar{\mathcal{B}} \cup \bar{\mathcal{I}}$.

The operator $G_E\colon V' \to S^2 \times V$ was already proven to be directionally differentiable in a weak sense:

Theorem 2.10. *[44, Theorem 3.2] For every $\bar{\ell} \in V'$ and every $\delta\ell \in V'$, the control-to-state map $G_E \colon V' \to S^2 \times V$ is weakly directionally differentiable at $\bar{\ell}$ in direction $\delta\ell$, i.e., there exists $\delta_w G_E(\bar{\ell}; \delta\ell) \in S^2 \times V$ such that*

$$\frac{G_E(\bar{\ell} + t\delta\ell) - G_E(\bar{\ell})}{t} \rightharpoonup \delta_w G_E(\bar{\ell}; \delta\ell) \quad \text{as } t \searrow 0.$$

The weak directional derivative $\delta_w G_E(\bar{\ell}; \delta\ell)$ is given by the unique solution $(\boldsymbol{\Sigma}', \boldsymbol{u}') \in \mathcal{S}_\ell \times V$ of the following variational inequality

$$\begin{aligned} \big(A\boldsymbol{\Sigma}', \boldsymbol{T} - \boldsymbol{\Sigma}'\big) + \big(\operatorname{div}^* \boldsymbol{u}', \boldsymbol{T} - \boldsymbol{\Sigma}'\big) \\ + \big(\bar{\lambda}, \mathcal{D}\boldsymbol{\Sigma}' : \mathcal{D}(\boldsymbol{T} - \boldsymbol{\Sigma}')\big) \geq 0 \quad \text{for all } \boldsymbol{T} \in \mathcal{S}_\ell \qquad (2.17a) \\ \operatorname{div} \boldsymbol{\Sigma}' = \delta\ell, \qquad (2.17b) \end{aligned}$$

where the convex cone $\mathcal{S}_\ell$ is defined by

$$\mathcal{S}_\ell := \Big\{ \boldsymbol{T} \in S^2 \colon \sqrt{\bar{\lambda}}\mathcal{D}\boldsymbol{T} \in S,\ \mathcal{D}\bar{\boldsymbol{\Sigma}}(x) : \mathcal{D}\boldsymbol{T}(x) \leq 0 \text{ a.e. in } \bar{\mathcal{B}}, \\ \mathcal{D}\bar{\boldsymbol{\Sigma}}(x) : \mathcal{D}\boldsymbol{T}(x) = 0 \text{ a.e. in } \bar{\mathcal{A}}_s \Big\}.$$

Again, by introducing a slack variable the variational inequality (2.17a) can be written as a complementarity system:

Theorem 2.11. *[44, Proposition 3.13] A pair $(\boldsymbol{\Sigma}', \boldsymbol{u}') \in S^2 \times V$ is the unique solution of (2.17) if and only if there exists a multiplier $\lambda' \in L^2(\Omega)$ such that*

$$\begin{aligned} A\boldsymbol{\Sigma}' + \operatorname{div}^* \boldsymbol{u}' + \bar{\lambda}\mathcal{D}^*\mathcal{D}\boldsymbol{\Sigma}' + \lambda'\mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Sigma}} &= \boldsymbol{0} && \text{in } S^2 && (2.18a) \\ \operatorname{div} \boldsymbol{\Sigma}' &= \delta\ell && \text{in } V' && (2.18b) \\ \mathbb{R} \ni \lambda'(x) \perp \mathcal{D}\bar{\boldsymbol{\Sigma}} : \mathcal{D}\boldsymbol{\Sigma}'(x) &= 0 && \text{a.e. in } \bar{\mathcal{A}}_s && (2.18c) \\ 0 \leq \lambda'(x) \perp \mathcal{D}\bar{\boldsymbol{\Sigma}} : \mathcal{D}\boldsymbol{\Sigma}'(x) &\leq 0 && \text{a.e. in } \bar{\mathcal{B}} && (2.18d) \\ 0 = \lambda'(x) \perp \mathcal{D}\bar{\boldsymbol{\Sigma}} : \mathcal{D}\boldsymbol{\Sigma}'(x) &\in \mathbb{R} && \text{a.e. in } \bar{\mathcal{I}} && (2.18e) \end{aligned}$$

holds. Moreover, λ' is unique.

Remark 2.12. *On account of (2.18d) the (weak) directional derivative is generally not linear w.r.t. to the direction and the control-to-state map G_E thus not (weakly) Gâteaux differentiable, if the biactive set $\bar{\mathcal{B}}$ has positive measure.*

If we want to improve the assertion of Theorem 2.10, we have to make additional assumptions.

Assumption 2.13.

(i) Let $\bar{\ell}, \delta\ell \in W_D^{-1,p}(\Omega; \mathbb{R}^d)$ with $p \in (2, \hat{p}]$ and $\hat{p}$ given by Theorem 2.8.

(ii) The solution of (2.15) *satisfies* $\bar{\boldsymbol{\chi}} \in L^s(\Omega; \mathbb{S})$ *with*

$$s > \frac{2p}{p-2}. \tag{2.19}$$

Moreover we set

$$q = \frac{sp}{s+p}. \tag{2.20}$$

Assumption 2.13 is supposed to hold for the rest of this section.

Remark 2.14. *(i) If the right hand sides ℓ and $\delta\ell$ in* (2.15) *and* (2.18)*, respectively, are defined as in* $(\mathbf{P}_E)$*, Assumption 2.13(i) is automatically fulfilled due to Sobolev's embedding theorems. To be more precise, the operator R defined by* (2.14) *continuously maps U into $W_D^{-1,4}(\Omega; \mathbb{R}^d)$ in case $d = 2$. If the spatial dimension is $d = 3$, then R continuously maps from U to $W_D^{-1,3}(\Omega; \mathbb{R}^d)$. Furthermore, R is compact as a mapping from U to $W_D^{-1,\beta}(\Omega; \mathbb{R}^d)$ with $\beta < 3$ in dimension $d = 3$ and with $\beta < 4$ in dimension $d = 2$, cf. [64, Section 2.3 and Section 2.6]*

(ii) We conclude from (2.19) *and* (2.20) *that $q > 2$ and $p' < q < p$:*

$$q = \frac{sp}{s+p} = \frac{p}{1+\frac{p}{s}} > \frac{p}{1+\frac{p}{\frac{2p}{p-2}}} = \frac{p}{1+\frac{p-2}{2}} = \frac{p}{\frac{p}{2}} = 2$$
$$> \frac{p}{1+\frac{p}{\frac{p}{p-2}}} = \frac{p}{1+p-2} = p',$$

where p' is the integrability exponent conjugated to p.

(iii) The solution operator G_E is not expected to map $W_D^{-1,p}(\Omega; \mathbb{R}^d)$ into $L^p(\Omega; \mathbb{S}) \times L^s(\Omega; \mathbb{S}) \times W_D^{1,p}(\Omega; \mathbb{R}^d)$. Assumption 2.13(ii) is only required for the particular solution $\bar{\boldsymbol{\chi}}$ of (2.15)*, where G_E is differentiated.*

In order to be able to show Bouligand differentiability of the control-to-state map, we collect some auxiliary results for the weak directional derivative $(\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda')$. First we consider the difference of the solution to (2.15) and the solution of the following perturbed problem

$$A\boldsymbol{\Sigma} + \operatorname{div}^* \boldsymbol{u} + \lambda \mathcal{D}^* \mathcal{D} \boldsymbol{\Sigma} = \mathbf{0} \quad \text{in } S^2 \tag{2.21a}$$
$$\operatorname{div} \boldsymbol{\Sigma} = \bar{\ell} + \delta\ell \quad \text{in } V' \tag{2.21b}$$
$$0 \le \lambda(x) \perp \phi\big(\boldsymbol{\Sigma}(x)\big) \le 0 \quad \text{a.e. in } \Omega. \tag{2.21c}$$

By Theorem 2.5 we know that (2.21) has a unique solution.

Lemma 2.15. *Let* $(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}}, \bar{\lambda})$ *and* $(\boldsymbol{\Sigma}, \boldsymbol{u}, \lambda)$ *be solution of* (2.15) *and* (2.21)*, respectively. Then it holds*

(i) $\|\boldsymbol{\Sigma} - \bar{\boldsymbol{\Sigma}}\|_{L^p(\Omega;\mathbb{S}^2)} + \|\boldsymbol{u} - \bar{\boldsymbol{u}}\|_{W^{1,p}_D(\Omega;\mathbb{R}^d)} \leq L\,\|\delta\ell\|_{W^{-1,p}(\Omega;\mathbb{R}^d)}$

(ii) $\mathcal{D}\boldsymbol{\Sigma} - \mathcal{D}\bar{\boldsymbol{\Sigma}} \to 0$ *in* $L^m(\Omega;\mathbb{S})\ \forall\, 1 \leq m < \infty$*, if* $\delta\ell \to 0$ *in* $W_D^{-1,p}(\Omega;\mathbb{R}^d)$

(iii) $\|\lambda - \bar{\lambda}\|_{L^q(\Omega)} \leq c\,\|\delta\ell\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)}$,

where $L > 0$ *is the Lipschitz constant given in Theorem 2.8.*

Proof. Let $(\delta\ell_n)_{n\in\mathbb{N}} \subset W_D^{-1,p}(\Omega;\mathbb{R}^d)$ be an arbitrary sequence with $\delta\ell_n \to 0$ for $n \to 0$ and let $(\boldsymbol{\Sigma}_n, \lambda_n)$ be given by the solution of (2.21) with right hand side $\bar{\ell} + \delta\ell_n$.
(i): Assertion (i) is a consequence of Theorem 2.8.
(ii): Due to (2.15c) and (2.21c) we observe

$$\|\mathcal{D}\boldsymbol{\Sigma}_n(x) - \mathcal{D}\bar{\boldsymbol{\Sigma}}(x)\|_{\mathbb{S}} \leq 2\sigma_0 \quad \text{a.e. in } \Omega,$$

so that $\mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}}$ is bounded in $L^m(\Omega;\mathbb{S})$ for every $m \geq 1$. In addition, we infer from (i) the existence of a subsequence, w.l.o.g. denoted by the same symbols, with

$$\mathcal{D}\boldsymbol{\Sigma}_n(x) - \mathcal{D}\bar{\boldsymbol{\Sigma}}(x) \to \boldsymbol{0} \quad \text{a.e. in } \Omega. \tag{2.22}$$

Lebesgue's theorem of dominated convergence implies (ii) for every subsequence satisfying (2.22) and hence for the whole sequence.
(iii): In view of (2.7) and (2.15c) we deduce

$$\sigma_0^2\bar{\lambda} = \bar{\lambda}\mathcal{D}\bar{\boldsymbol{\Sigma}} : \mathcal{D}\bar{\boldsymbol{\Sigma}} = -\mathbb{H}^{-1}\bar{\boldsymbol{\chi}} : \mathcal{D}\bar{\boldsymbol{\Sigma}} \quad \text{a.e. in } \Omega \tag{2.23}$$

and completely analogously we find

$$\sigma_0^2\lambda_n = \lambda_n\mathcal{D}\boldsymbol{\Sigma}_n : \mathcal{D}\boldsymbol{\Sigma}_n = -\mathbb{H}^{-1}\boldsymbol{\chi}_n : \mathcal{D}\boldsymbol{\Sigma}_n \quad \text{a.e. in } \Omega.$$

Subtraction of both equations consequently yields

$$\begin{aligned} \lambda_n - \bar{\lambda} &= \frac{1}{\sigma_0^2}\left(-\mathbb{H}_n^{-1}\bar{\boldsymbol{\chi}}_n : \mathcal{D}\boldsymbol{\Sigma}_n + \mathbb{H}^{-1}\bar{\boldsymbol{\chi}} : \mathcal{D}\boldsymbol{\Sigma}\right) \\ &= \frac{1}{\sigma_0^2}\left(-\mathbb{H}^{-1}\bar{\boldsymbol{\chi}} : (\mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}}) - \mathbb{H}^{-1}(\boldsymbol{\chi}_n - \bar{\boldsymbol{\chi}}) : \mathcal{D}\boldsymbol{\Sigma}_n\right). \end{aligned} \tag{2.24}$$

On account of (2.20), (2.21c), Remark 2.14(ii) and (i) this leads to

$$\begin{aligned} \|\lambda_n - \bar{\lambda}\|_{L^q(\Omega)} &\leq c\left(\|\bar{\boldsymbol{\chi}}\|_{L^s(\Omega;\mathbb{S})}\|\mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}}\|_{L^p(\Omega;\mathbb{S})} + \sigma_0\|\boldsymbol{\chi}_n - \bar{\boldsymbol{\chi}}\|_{L^q(\Omega;\mathbb{S})}\right) \\ &\leq c\,\|\delta\ell_n\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)}, \end{aligned}$$

which is the third assertion. □

Next, we establish higher regularity for the solution of (2.18). For that purpose we introduce another perturbed problem:

$$A\boldsymbol{\Sigma}_t + \operatorname{div}^* \boldsymbol{u}_t + \lambda_t \mathcal{D}^*\mathcal{D}\boldsymbol{\Sigma}_t = \mathbf{0} \quad \text{in } S^2 \tag{2.25a}$$
$$\operatorname{div} \boldsymbol{\Sigma}_t = \bar{\ell} + t\delta\ell \quad \text{in } V' \tag{2.25b}$$
$$0 \le \lambda_t(x) \perp \phi\big(\boldsymbol{\Sigma}_t(x)\big) \le 0 \quad \text{a.e. in } \Omega \tag{2.25c}$$

with $t > 0$ given.

Lemma 2.16. *Let* $(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}}, \bar{\lambda})$ *be the solution of* (2.15), $(\boldsymbol{\Sigma}_t, \boldsymbol{u}_t, \lambda_t)$ *the solution of* (2.25) *and* $(\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda')$ *the solution of* (2.18). *Then it holds*

(i) $\big(\frac{\boldsymbol{\Sigma}_t - \bar{\boldsymbol{\Sigma}}}{t}, \frac{\boldsymbol{u}_t - \bar{\boldsymbol{u}}}{t}\big) \rightharpoonup \big(\boldsymbol{\Sigma}', \boldsymbol{u}'\big)$ *in* $L^p\big(\Omega; \mathbb{S}^2\big) \times W_D^{1,p}\big(\Omega; \mathbb{R}^d\big)$ *as* $t \searrow 0$

(ii) $\frac{\lambda_t - \bar{\lambda}}{t} \rightharpoonup \lambda'$ *in* $L^q(\Omega)$ *as* $t \searrow 0$

(iii) $\|\boldsymbol{\Sigma}'\|_{L^p(\Omega;\mathbb{S}^2)} + \|\boldsymbol{u}'\|_{W_D^{1,p}(\Omega;\mathbb{R}^d)} \le L \, \|\delta\ell\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)}$

(iv) $\|\lambda'\|_{L^q(\Omega)} \le c \, \|\delta\ell\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)}$,

where $L > 0$ *is the Lipschitz constant given in Theorem 2.8.*

Proof. Let $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ be an arbitrary sequence of positive real numbers converging to zero and let $(\boldsymbol{\Sigma}_{t_n}, \boldsymbol{u}_{t_n}, \lambda_{t_n})$ be given by the solution of (2.25) with right hand side $\bar{\ell} + t_n\delta\ell$.
(i): According to Theorem 2.10 we know

$$\left(\frac{\boldsymbol{\Sigma}_{t_n} - \bar{\boldsymbol{\Sigma}}}{t_n}, \frac{\boldsymbol{u}_{t_n} - \bar{\boldsymbol{u}}}{t_n}\right) \rightharpoonup \big(\boldsymbol{\Sigma}', \boldsymbol{u}'\big) \quad \text{in } S^2 \times V.$$

Moreover it follows from Theorem 2.8 that

$$\left\|\frac{\boldsymbol{\Sigma}_{t_n} - \bar{\boldsymbol{\Sigma}}}{t_n}\right\|_{L^p(\Omega;\mathbb{S}^2)} + \left\|\frac{\boldsymbol{u}_{t_n} - \bar{\boldsymbol{u}}}{t_n}\right\|_{W_D^{1,p}(\Omega;\mathbb{R}^d)} \le L \, \|\delta\ell\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)} . \tag{2.26}$$

Thus, there exist a subsequence converging weakly in $L^p\big(\Omega, \mathbb{S}^2\big) \times W_D^{1,p}\big(\Omega; \mathbb{R}^d\big)$. Due to the uniqueness of the weak limit we conclude (i).
(ii): Similarly to (2.24) it can be shown that

$$\frac{\lambda_{t_n} - \bar{\lambda}}{t_n} = \frac{1}{\sigma_0^2}\left(-\mathbb{H}^{-1}\bar{\boldsymbol{\chi}} : \frac{\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \mathcal{D}\bar{\boldsymbol{\Sigma}}}{t_n} - \mathbb{H}^{-1}\frac{\boldsymbol{\chi}_{t_n} - \bar{\boldsymbol{\chi}}}{t_n} : \mathcal{D}\boldsymbol{\Sigma}_{t_n}\right). \tag{2.27}$$

From (i) and (2.20) we deduce

$$\mathbb{H}^{-1}\bar{\boldsymbol{\chi}} : \frac{\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \mathcal{D}\bar{\boldsymbol{\Sigma}}}{t_n} \rightharpoonup \mathbb{H}^{-1}\bar{\boldsymbol{\chi}} : \mathcal{D}\boldsymbol{\Sigma}' \quad \text{in } L^q(\Omega)$$

and Lemma 2.15(ii) with $m = s$ implies

$$\mathbb{H}^{-1}\frac{\boldsymbol{\chi}_{t_n} - \bar{\boldsymbol{\chi}}}{t_n} : \mathcal{D}\boldsymbol{\Sigma}_{t_n} \rightharpoonup \mathbb{H}^{-1}\boldsymbol{\chi}' : \mathcal{D}\bar{\boldsymbol{\Sigma}} \quad \text{in } L^q(\Omega).$$

Since $\mathcal{D}\boldsymbol{\Sigma}' : \mathcal{D}\bar{\boldsymbol{\Sigma}} = 0$ a.e. in $\bar{\mathcal{A}}_s$ and $\bar{\lambda} = 0$ a.e. in $\bar{\mathcal{B}} \cup \bar{\mathcal{I}}$, cf. (2.18c), (2.16) and (2.15c), equation (2.7) yields

$$\mathbb{H}^{-1}\bar{\boldsymbol{\chi}} : \mathcal{D}\boldsymbol{\Sigma}' = -\bar{\lambda}\mathcal{D}\boldsymbol{\Sigma}' : \mathcal{D}\bar{\boldsymbol{\Sigma}} = 0 \quad \text{a.e. in } \Omega. \tag{2.28}$$

Therefore, we derive

$$\frac{\lambda_{t_n} - \bar{\lambda}}{t_n} \rightharpoonup -\frac{1}{\sigma_0^2}\mathbb{H}^{-1}\boldsymbol{\chi}' : \mathcal{D}\bar{\boldsymbol{\Sigma}} \quad \text{in } L^q(\Omega). \tag{2.29}$$

By testing (2.18a) with $(\mathbf{0}, \boldsymbol{\mu})$, $\boldsymbol{\mu} \in S$, we obtain

$$\mathbb{H}^{-1}\boldsymbol{\chi}' + \bar{\lambda}\mathcal{D}\boldsymbol{\Sigma}' + \lambda'\mathcal{D}\bar{\boldsymbol{\Sigma}} = \mathbf{0} \quad \text{a.e. in } \Omega. \tag{2.30}$$

Furthermore, (2.16), (2.15c) and (2.18e) result in $\sigma_0^2\lambda' = \lambda'\mathcal{D}\bar{\boldsymbol{\Sigma}} : \mathcal{D}\bar{\boldsymbol{\Sigma}}$ a.e. in Ω so that

$$\sigma_0^2\lambda' = -\mathbb{H}^{-1}\boldsymbol{\chi}' : \mathcal{D}\bar{\boldsymbol{\Sigma}} - \bar{\lambda}\mathcal{D}\boldsymbol{\Sigma}' : \mathcal{D}\bar{\boldsymbol{\Sigma}} \quad \text{a.e. in } \Omega. \tag{2.31}$$

Taking (2.28) into account we arrive at

$$\lambda' = -\frac{1}{\sigma_0^2}\mathbb{H}^{-1}\boldsymbol{\chi}' : \mathcal{D}\bar{\boldsymbol{\Sigma}} \quad \text{a.e. in } \Omega,$$

which together with (2.29) leads to (ii).
(iii): Since closed and convex sets are weakly closed, Assertion (iii) is a consequence of (i) and (2.26).
(iv): From (2.20), (2.26), Remark 2.14(ii) and (2.27) we conclude

$$\begin{aligned}\left\|\frac{\lambda_{t_n} - \bar{\lambda}}{t_n}\right\|_{L^q(\Omega)} &\le c\left(\|\bar{\boldsymbol{\chi}}\|_{L^s(\Omega;\mathbb{S})}\left\|\frac{\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \mathcal{D}\bar{\boldsymbol{\Sigma}}}{t_n}\right\|_{L^p(\Omega;\mathbb{S})} + \sigma_0\left\|\frac{\boldsymbol{\chi}_{t_n} - \bar{\boldsymbol{\chi}}}{t_n}\right\|_{L^q(\Omega;\mathbb{S})}\right)\\ &\le c\,\|\delta\ell\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)}.\end{aligned}$$

By the same argument as in (iii) this shows $\|\lambda'\|_{L^q(\Omega)} \le c\,\|\delta\ell\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)}$. □

Before we are ready to prove Bouligand differentiability of G_E we need another two auxiliary lemmas.

Lemma 2.17. *Let X be a Hilbert space and $x_1, x_2 \in X$. If $\|x_1\|_X = \|x_2\|_X$, then it holds*

$$(x_1,\, x_1 - x_2)_X = \frac{1}{2}\,\|x_1 - x_2\|_X^2.$$

Proof. Because of the parallelogram law in Hilbert spaces it follows

$$\begin{aligned}(x_1,\, x_1 - x_2)_X &= (x_1,\, x_1)_X - (x_1,\, x_2)_X \\ &= \|x_1\|_X^2 - \frac{1}{4}\left(\|x_1 + x_2\|_X^2 - \|x_1 - x_2\|_X^2\right) \\ &= \|x_1\|_X^2 - \frac{1}{4}\left(2\,\|x_1\|_X^2 + 2\,\|x_2\|_X^2 - 2\,\|x_1 - x_2\|_X^2\right) \\ &= \frac{1}{2}\,\|x_1 - x_2\|_X^2\,,\end{aligned}$$

where $\|x_1\|_X = \|x_2\|_X$ was used for the last equation. □

The next lemma covers the convergence of bounded and a.e. convergent sequences in Lebesgue spaces.

Lemma 2.18. *Let $E \subset \mathbb{R}^d$ be measurable and bounded, $\nu \in (1, \infty)$ and $f, f_n \in L^\nu(E)$, $n \in \mathbb{N}$. If $\sup_{n\in\mathbb{N}} \|f_n\|_{L^\nu(E)} \leq c$ and $f_n \to f$ a.e. in E, then $f_n \to f$ in $L^\kappa(E)$ for $1 \leq \kappa < \nu$.*

Proof. Let $\kappa \in [1, \nu)$ and $g_n := |f_n - f|^\kappa$. We know $g_n(x) \to 0$ a.e. in E and g_n is bounded in the reflexive space $L^{\nu/\kappa}(E)$ with $\nu/\kappa > 1$. Hence there is a subsequence, w.l.o.g. denoted in the same way, converging weakly to $g \in L^{\frac{\nu}{\kappa}}(E)$. We assume that there exists $E_0 \subset E$ with $|E_0| > 0$ and $g > 0$ a.e. in E_0. Due to weak convergence we deduce

$$\int_{\tilde{E}} g_n \,\mathrm{d}x \to \int_{\tilde{E}} g \,\mathrm{d}x > 0 \quad \forall\, \tilde{E} \subset E_0 \text{ with } |\tilde{E}| > 0. \tag{2.32}$$

However, by Egorov's theorem there is $\hat{E} \subset E_0$ with $|\hat{E}| < |E_0|/2$ and

$$\|g_n\|_{L^\infty(E_0 \setminus \hat{E})} \to 0,$$

contradictory to (2.32). Therefore, the weak limit equals the pointwise limit, i.e., $g = 0$. Since the above arguments are independent of the chosen subsequence, the whole sequence $\{g_n\}_{n\in\mathbb{N}}$ converges weakly to zero so that

$$\int_E g_n \,\mathrm{d}x \to 0,$$

which implies the assertion. □

The main result of this section reads as follows:

Theorem 2.19. *Let $(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}})$ be given by the solution of (2.15), $(\boldsymbol{\Sigma}, \boldsymbol{u})$ by the solution of (2.21) and $(\boldsymbol{\Sigma}', \boldsymbol{u}')$ by the solution of (2.18). Then it holds*

$$\left\|\boldsymbol{\Sigma} - \bar{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}'\right\|_{S^2} + \left\|\boldsymbol{u} - \bar{\boldsymbol{u}} - \boldsymbol{u}'\right\|_V = o\left(\|\delta\ell\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)}\right), \tag{2.33}$$

i.e., the control-to-state map G_E is Bouligand differentiable from $W_D^{-1,p}(\Omega;\mathbb{R}^d)$ to $S^2 \times V$.

Remark 2.20. *We point out that a norm gap is needed in order to establish Bouligand differentiability of G_E. However, this is not surprising, since norm gaps are usually necessary for the differentiability of nonlinear operators, see e.g. [33], and the differentiability of solution operators associated with quasilinear PDEs, cf. [75, Theorem 3.3].*

Proof of Theorem 2.19. Let $(\delta\ell_n)_{n\in\mathbb{N}} \subset W_D^{-1,p}(\Omega;\mathbb{R}^d)$ be an arbitrary sequence with $\delta\ell_n \to 0$ for $n \to 0$. Furthermore let $(\boldsymbol{\Sigma}_n, \boldsymbol{u}_n, \lambda_n)$ denote the solution of (2.21) with right hand side $\bar{\ell} + \delta\ell_n$ and $(\boldsymbol{\Sigma}_n', \boldsymbol{u}_n', \lambda_n')$ the solution of (2.18) with right hand side $\delta\ell_n$.
Subtracting (2.15a) and (2.18a) from (2.21a) and testing with $\boldsymbol{\Sigma}_n - \bar{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}_n'$ leads to

$$\begin{aligned} &\big(A(\boldsymbol{\Sigma}_n - \bar{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}_n'), \boldsymbol{\Sigma}_n - \bar{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}_n'\big) + \underbrace{\big(\operatorname{div}^*(\boldsymbol{u}_n - \bar{\boldsymbol{u}} - \boldsymbol{u}_n'), \boldsymbol{\Sigma}_n - \bar{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}_n'\big)}_{=:I_1} \\ &\quad + \big(\lambda_n \mathcal{D}\boldsymbol{\Sigma}_n - \bar{\lambda}\mathcal{D}\bar{\boldsymbol{\Sigma}} - \bar{\lambda}\mathcal{D}\boldsymbol{\Sigma}_n' - \lambda_n'\mathcal{D}\bar{\boldsymbol{\Sigma}}, \mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}} - \mathcal{D}\boldsymbol{\Sigma}_n'\big) = 0. \end{aligned} \tag{2.34}$$

Thanks to (2.15b), (2.21b) and (2.18b) we observe

$$\operatorname{div}(\boldsymbol{\Sigma}_n - \bar{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}_n') = \bar{\ell} + \delta\ell_n - \bar{\ell} - \delta\ell_n = 0$$

and therefore $I_1 = 0$. As both $\mathbb{C}^{-1}$ and $\mathbb{H}^{-1}$ are uniformly coercive by Assumption 2.1(ii), the linear operator A induces an equivalent norm on S^2. In view of equation (2.34) we derive

$$\begin{aligned} c\,\big\|\boldsymbol{\Sigma}_n - \bar{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}_n'\big\|_{S^2}^2 &\le \big(A(\boldsymbol{\Sigma}_n - \bar{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}_n'), \boldsymbol{\Sigma}_n - \bar{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}_n'\big) \\ &= -\big(\bar{\lambda}\big(\mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}} - \mathcal{D}\boldsymbol{\Sigma}_n'\big), \mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}} - \mathcal{D}\boldsymbol{\Sigma}_n'\big) \\ &\quad - \big(\big(\lambda_n - \bar{\lambda} - \lambda_n'\big)\mathcal{D}\boldsymbol{\Sigma}_n, \mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}} - \mathcal{D}\boldsymbol{\Sigma}_n'\big) \\ &\quad - \big(\lambda_n'\big(\mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}}\big), \mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}} - \mathcal{D}\boldsymbol{\Sigma}_n'\big) \\ &= -\int_\Omega \underbrace{\bar{\lambda}}_{\ge 0}\, \underbrace{\big\|\mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}} - \mathcal{D}\boldsymbol{\Sigma}_n'\big\|_{\mathbb{S}}^2}_{\ge 0}\,\mathrm{d}x \\ &\quad - \int_\Omega \big(\lambda_n - \bar{\lambda} - \lambda_n'\big)\mathcal{D}\boldsymbol{\Sigma}_n : \big(\mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}} - \mathcal{D}\boldsymbol{\Sigma}_n'\big)\,\mathrm{d}x \\ &\quad \underbrace{- \int_\Omega \lambda_n'\big(\mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}}\big) : \big(\mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}} - \mathcal{D}\boldsymbol{\Sigma}_n'\big)\,\mathrm{d}x}_{=:I_\Omega} \\ &\le I_s + I_b + I_i + I_\Omega. \end{aligned} \tag{2.35}$$

Herein I_s is given by

$$I_s := -\int_{\bar{\mathcal{A}}_s} \big(\lambda_n - \bar{\lambda} - \lambda_n'\big)\mathcal{D}\boldsymbol{\Sigma}_n : \big(\mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}} - \mathcal{D}\boldsymbol{\Sigma}_n'\big)\,\mathrm{d}x.$$

Moreover, I_b and I_i are defined by the analogous integrals on the sets $\bar{\mathcal{B}}$ and $\bar{\mathcal{I}}$, respectively, cf. (2.16). By Lemma 2.15(i) and (iii) there exist subsequences $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ and $\{\mathbf{\Sigma}_{n_k}\}_{k\in\mathbb{N}}$ with

$$\lambda_{n_k}(x) \to \bar{\lambda}(x) \quad \text{a.e. on } \Omega \tag{2.36a}$$

$$\mathbf{\Sigma}_{n_k}(x) \to \bar{\mathbf{\Sigma}}(x) \quad \text{a.e. on } \Omega. \tag{2.36b}$$

For the sake of convenience we denote these subsequences again by λ_n and $\mathbf{\Sigma}_n$. Next we will estimate I_Ω, I_s, I_b and I_i separately.

Estimation of I_Ω:

According to Remark 2.14(ii) and Lemma 2.16(iv) there exists $m > 1$ with $1/q + 1/m + 1/2 = 1$ so that

$$\begin{aligned} I_\Omega &\leq \left\|\lambda_n'\right\|_{L^q(\Omega)} \left\|\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}}\right\|_{L^m(\Omega;\mathbb{S})} \left\|\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}} - \mathcal{D}\mathbf{\Sigma}_n'\right\|_S \\ &\leq c \left\|\delta\ell_n\right\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)} \left\|\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}}\right\|_{L^m(\Omega;\mathbb{S})} \left\|\mathbf{\Sigma}_n - \bar{\mathbf{\Sigma}} - \mathbf{\Sigma}_n'\right\|_{S^2}. \end{aligned} \tag{2.37}$$

Estimation of I_s:

If $x \in \bar{\mathcal{A}}_s$, then it holds $\bar{\lambda}(x) > 0$ and (2.15c) yields $\phi\big(\bar{\mathbf{\Sigma}}(x)\big) = 0$. Due to the pointwise convergence (2.36a) and the complementarity (2.21c) there exists $N_x \in \mathbb{N}$, depending on x, such that $\lambda_n(x) > 0$ and $\phi\big(\mathbf{\Sigma}_n(x)\big) = 0$, i.e., $\|\mathcal{D}\mathbf{\Sigma}_n(x)\|_{\mathbb{S}} = \big\|\mathcal{D}\bar{\mathbf{\Sigma}}(x)\big\|_{\mathbb{S}} = \sigma_0$, for all $n \geq N_x$ and f.a.a. $x \in \bar{\mathcal{A}}_s$. In view of Lemma 2.17 we conclude

$$z_n(x) := \frac{\mathcal{D}\bar{\mathbf{\Sigma}}(x) : (\mathcal{D}\bar{\mathbf{\Sigma}}(x) - \mathcal{D}\mathbf{\Sigma}_n(x))}{\big\|\mathcal{D}\bar{\mathbf{\Sigma}}(x)\big\|_{\mathbb{S}} \big\|\mathcal{D}\bar{\mathbf{\Sigma}}(x) - \mathcal{D}\mathbf{\Sigma}_n(x)\big\|_{\mathbb{S}}} \to 0 \quad \text{a.e. on } \bar{\mathcal{A}}_s. \tag{2.38}$$

Because $|z_n(x)| \leq 1$, it follows from Lebesgue's dominated convergence theorem that

$$z_n \to 0 \quad \text{in } L^\xi(\bar{\mathcal{A}}_s) \quad \forall\, 1 \leq \xi < \infty. \tag{2.39}$$

Since $\mathcal{D}\bar{\mathbf{\Sigma}} : \mathcal{D}\mathbf{\Sigma}_n' = 0$ a.e. in $\bar{\mathcal{A}}_s$, cf. (2.18c), we furthermore deduce

$$\begin{aligned} I_s &= -\int_{\bar{\mathcal{A}}_s} \big(\lambda_n - \bar{\lambda} - \lambda_n'\big)\mathcal{D}\mathbf{\Sigma}_n : \big(\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}} - \mathcal{D}\mathbf{\Sigma}_n'\big)\,\mathrm{d}x \\ &= -\int_{\bar{\mathcal{A}}_s} \big(\lambda_n - \bar{\lambda} - \lambda_n'\big)\big(\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}}\big) : \big(\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}} - \mathcal{D}\mathbf{\Sigma}_n'\big)\,\mathrm{d}x \\ &\quad - \int_{\bar{\mathcal{A}}_s} \big(\lambda_n - \bar{\lambda} - \lambda_n'\big)\mathcal{D}\bar{\mathbf{\Sigma}} : \big(\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}} - \mathcal{D}\mathbf{\Sigma}_n'\big)\,\mathrm{d}x \\ &= -\int_{\bar{\mathcal{A}}_s} \big(\lambda_n - \bar{\lambda} - \lambda_n'\big)\big(\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}}\big) : \big(\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}} - \mathcal{D}\mathbf{\Sigma}_n'\big)\,\mathrm{d}x \\ &\quad - \int_{\bar{\mathcal{A}}_s} \big(\lambda_n - \bar{\lambda} - \lambda_n'\big)\mathcal{D}\bar{\mathbf{\Sigma}} : \big(\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}}\big)\,\mathrm{d}x \\ &= -\int_{\bar{\mathcal{A}}_s} \big(\lambda_n - \bar{\lambda} - \lambda_n'\big)\big(\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}}\big) : \big(\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}} - \mathcal{D}\mathbf{\Sigma}_n'\big)\,\mathrm{d}x \\ &\quad + \int_{\bar{\mathcal{A}}_s} \big(\lambda_n - \bar{\lambda} - \lambda_n'\big) z_n \left\|\mathcal{D}\bar{\mathbf{\Sigma}}\right\|_{\mathbb{S}} \left\|\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}}\right\|_{\mathbb{S}}\,\mathrm{d}x. \end{aligned}$$

Again, on account of Remark 2.14(ii) there exist $m > 1$ and $\xi > 1$ with $\frac{1}{q}+\frac{1}{m}+\frac{1}{2} = 1$ and $\frac{1}{q} + \frac{1}{\xi} + \frac{1}{p} = 1$. In consequence of Lemma 2.15 and Lemma 2.16(iii)–(iv) we arrive at

$$\begin{aligned} I_s &\le \left\|\lambda_n - \bar\lambda - \lambda_n'\right\|_{L^q(\Omega)} \left\|\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}}\right\|_{L^m(\Omega;\mathbb{S})} \left\|\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}} - \mathcal{D}\mathbf{\Sigma}_n'\right\|_{S^2} \\ &\quad + \sigma_0 \left\|\lambda_n - \bar\lambda - \lambda_n'\right\|_{L^q(\Omega)} \|z_n\|_{L^\xi(\bar{\mathcal{A}}_s)} \left\|\mathbf{\Sigma}_n - \bar{\mathbf{\Sigma}}\right\|_{L^p(\Omega;\mathbb{S}^2)} \\ &\le c\,\|\delta\ell_n\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)} \left\|\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}}\right\|_{L^m(\Omega;\mathbb{S})} \left\|\mathbf{\Sigma}_n - \bar{\mathbf{\Sigma}} - \mathbf{\Sigma}_n'\right\|_{S^2} \\ &\quad + c\,\|\delta\ell_n\|^2_{W_D^{-1,p}(\Omega;\mathbb{R}^d)} \|z_n\|_{L^\xi(\bar{\mathcal{A}}_s)}\,. \end{aligned} \tag{2.40}$$

Estimation of I_b:
If $x \in \bar{\mathcal{B}}$, then we know $\phi\big(\bar{\mathbf{\Sigma}}(x)\big) = \bar\lambda(x) = 0$. Hence it either holds $\lambda_n(x) = 0$ and thus $\lambda_n(x) - \bar\lambda(x) = 0$ or $\lambda_n(x) > 0$ and $\phi\big(\mathbf{\Sigma}_n(x)\big) = 0$, i.e., $\|\mathcal{D}\mathbf{\Sigma}_n(x)\|_{\mathbb{S}} = \left\|\mathcal{D}\bar{\mathbf{\Sigma}}(x)\right\|_{\mathbb{S}} = \sigma_0$. Lemma 2.17 implies

$$(\lambda_n - \bar\lambda)(x)\big(\mathcal{D}\bar{\mathbf{\Sigma}} : (\mathcal{D}\bar{\mathbf{\Sigma}} - \mathcal{D}\mathbf{\Sigma}_n)\big)(x) = (\lambda_n - \bar\lambda)(x)\frac{1}{2}\left\|(\mathcal{D}\bar{\mathbf{\Sigma}} - \mathcal{D}\mathbf{\Sigma}_n)(x)\right\|^2_{\mathbb{S}} \tag{2.41}$$

in both cases. Moreover, due to (2.16) and the complementarity (2.18c)–(2.18e) we note that

$$\begin{aligned} \bar\lambda\mathcal{D}\bar{\mathbf{\Sigma}} : \mathcal{D}\mathbf{\Sigma}_n' = \lambda_n'\mathcal{D}\bar{\mathbf{\Sigma}} : \mathcal{D}\mathbf{\Sigma}_n' = 0 \quad &\text{a.e. in } \Omega \\ \mathcal{D}\bar{\mathbf{\Sigma}} : \mathcal{D}\mathbf{\Sigma}_n' \le 0 \quad \text{and} \quad 0 \le \lambda_n' \quad &\text{a.e. in } \bar{\mathcal{B}}. \end{aligned}$$

Therefore the following estimate is obtained

$$\begin{aligned} I_b &= -\int_{\bar{\mathcal{B}}} \big(\lambda_n - \bar\lambda - \lambda_n'\big)\mathcal{D}\mathbf{\Sigma}_n : \big(\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}} - \mathcal{D}\mathbf{\Sigma}_n'\big)\,\mathrm{d}x \\ &= -\int_{\bar{\mathcal{B}}} \big(\lambda_n - \bar\lambda - \lambda_n'\big)\big(\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}}\big) : \big(\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}} - \mathcal{D}\mathbf{\Sigma}_n'\big)\,\mathrm{d}x \\ &\quad - \int_{\bar{\mathcal{B}}} \big(\lambda_n - \bar\lambda\big)\mathcal{D}\bar{\mathbf{\Sigma}} : \big(\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}} - \mathcal{D}\mathbf{\Sigma}_n'\big)\,\mathrm{d}x \\ &\quad + \int_{\bar{\mathcal{B}}} \lambda_n'\mathcal{D}\bar{\mathbf{\Sigma}} : \big(\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}} - \mathcal{D}\mathbf{\Sigma}_n'\big)\,\mathrm{d}x \\ &= -\int_{\bar{\mathcal{B}}} \big(\lambda_n - \bar\lambda - \lambda_n'\big)\big(\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}}\big) : \big(\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}} - \mathcal{D}\mathbf{\Sigma}_n'\big)\,\mathrm{d}x \\ &\quad + \int_{\bar{\mathcal{B}}} \Big[\big(\lambda_n - \bar\lambda\big)\mathcal{D}\bar{\mathbf{\Sigma}} : \big(\mathcal{D}\bar{\mathbf{\Sigma}} - \mathcal{D}\mathbf{\Sigma}_n\big) + \underbrace{\lambda_n}_{\ge 0}\underbrace{\mathcal{D}\bar{\mathbf{\Sigma}} : \mathcal{D}\mathbf{\Sigma}_n'}_{\le 0} - \underbrace{\bar\lambda\mathcal{D}\bar{\mathbf{\Sigma}} : \mathcal{D}\mathbf{\Sigma}_n'}_{=0}\Big]\,\mathrm{d}x \\ &\quad + \int_{\bar{\mathcal{B}}} \Big[\lambda_n'\Big(\mathcal{D}\bar{\mathbf{\Sigma}} : \mathcal{D}\mathbf{\Sigma}_n - \underbrace{\left\|\mathcal{D}\bar{\mathbf{\Sigma}}\right\|^2_{\mathbb{S}}}_{=\sigma_0^2}\Big) - \underbrace{\lambda_n'\mathcal{D}\bar{\mathbf{\Sigma}} : \mathcal{D}\mathbf{\Sigma}_n'}_{=0}\Big]\,\mathrm{d}x \\ &\le -\int_{\bar{\mathcal{B}}} \big(\lambda_n - \bar\lambda - \lambda_n'\big)\big(\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}}\big) : \big(\mathcal{D}\mathbf{\Sigma}_n - \mathcal{D}\bar{\mathbf{\Sigma}} - \mathcal{D}\mathbf{\Sigma}_n'\big)\,\mathrm{d}x \\ &\quad + \int_{\bar{\mathcal{B}}} \big(\lambda_n - \bar\lambda\big)\mathcal{D}\bar{\mathbf{\Sigma}} : \big(\mathcal{D}\bar{\mathbf{\Sigma}} - \mathcal{D}\mathbf{\Sigma}_n\big)\,\mathrm{d}x + \int_{\bar{\mathcal{B}}} \underbrace{\lambda_n'}_{\ge 0}\big(\sigma_0\underbrace{\|\mathcal{D}\mathbf{\Sigma}_n\|_{\mathbb{S}}}_{\le \sigma_0} - \sigma_0^2\big)\,\mathrm{d}x. \end{aligned}$$

Taking (2.41) into account we derive

$$I_b \leq -\int_{\bar{\mathcal{B}}} \big(\lambda_n - \bar{\lambda} - \lambda_n'\big)\big(\mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}}\big) : \big(\mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}} - \mathcal{D}\boldsymbol{\Sigma}_n'\big) \,\mathrm{d}x$$
$$+ \int_{\bar{\mathcal{B}}} \big(\lambda_n - \bar{\lambda}\big)\frac{1}{2} \big\|\mathcal{D}\bar{\boldsymbol{\Sigma}} - \mathcal{D}\boldsymbol{\Sigma}_n\big\|_{\mathbb{S}}^2 \,\mathrm{d}x.$$

As in (2.40) there exist $m > 1$ and $\xi > 1$ with $\frac{1}{q} + \frac{1}{m} + \frac{1}{2} = 1$ and $\frac{1}{q} + \frac{1}{\xi} + \frac{1}{p} = 1$ so that

$$\begin{aligned} I_b &\leq \big\|\lambda_n - \bar{\lambda} - \lambda_n'\big\|_{L^q(\Omega)} \big\|\mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}}\big\|_{L^m(\Omega;\mathbb{S})} \big\|\mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}} - \mathcal{D}\boldsymbol{\Sigma}_n'\big\|_{S^2} \\ &\quad + \frac{1}{2} \big\|\lambda_n - \bar{\lambda}\big\|_{L^q(\Omega)} \big\|\mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}}\big\|_{L^\xi(\Omega;\mathbb{S})} \big\|\boldsymbol{\Sigma}_n - \bar{\boldsymbol{\Sigma}}\big\|_{L^p(\Omega;\mathbb{S}^2)} \\ &\leq c \left\|\delta\ell_n\right\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)} \big\|\mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}}\big\|_{L^m(\Omega;\mathbb{S})} \big\|\boldsymbol{\Sigma}_n - \bar{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}_n'\big\|_{S^2} \\ &\quad + c \left\|\delta\ell_n\right\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)}^2 \big\|\mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}}\big\|_{L^\xi(\Omega;\mathbb{S})}. \end{aligned} \tag{2.42}$$

Estimation of I_i:
For $x \in \bar{\mathcal{I}}$ it holds $\phi\big(\bar{\boldsymbol{\Sigma}}(x)\big) < 0$ and $\bar{\lambda}(x) = 0$, cf. (2.15c). Thanks to the continuity of ϕ, the pointwise convergence (2.36b) and the complementarity (2.21c) there exists $\tilde{N}_x \in \mathbb{N}$ with $\phi\big(\boldsymbol{\Sigma}_n(x)\big) < 0$ and $\lambda_n(x) = 0$ for all $n \geq \tilde{N}_x$ and f.a.a. $x \in \bar{\mathcal{I}}$. Hence we infer that

$$\frac{\lambda_n - \bar{\lambda}}{\left\|\delta\ell_n\right\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)}} \to 0 \quad \text{a.e. in } \bar{\mathcal{I}}.$$

Moreover, we know

$$\frac{\big\|\lambda_n - \bar{\lambda}\big\|_{L^q(\bar{\mathcal{I}})}}{\left\|\delta\ell_n\right\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)}} \leq c \quad \forall n \in \mathbb{N}$$

with $q > 2$ according to Lemma 2.15(iii) and Remark 2.14(ii). Therefore, from Lemma 2.18 it follows

$$\omega_n := \frac{\lambda_n - \bar{\lambda}}{\left\|\delta\ell_n\right\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)}} \to 0 \quad \text{in } L^2(\bar{\mathcal{I}}). \tag{2.43}$$

Because of (2.18e) we moreover deduce

$$\begin{aligned} I_i &= -\int_{\bar{\mathcal{I}}} \big(\lambda_n - \bar{\lambda} - \lambda_n'\big)\mathcal{D}\boldsymbol{\Sigma}_n : \big(\mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}} - \mathcal{D}\boldsymbol{\Sigma}_n'\big) \,\mathrm{d}x \\ &= -\int_{\bar{\mathcal{I}}} \big(\lambda_n - \bar{\lambda}\big)\mathcal{D}\boldsymbol{\Sigma}_n : \big(\mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}} - \mathcal{D}\boldsymbol{\Sigma}_n'\big) \,\mathrm{d}x \\ &= \int_{\bar{\mathcal{I}}} \omega_n \left\|\delta\ell_n\right\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)} \mathcal{D}\boldsymbol{\Sigma}_n : (\mathcal{D}\boldsymbol{\Sigma}_n - \mathcal{D}\bar{\boldsymbol{\Sigma}} - \mathcal{D}\boldsymbol{\Sigma}_n') \,\mathrm{d}x. \end{aligned}$$

Due to the complementarity (2.21c) this leads to

$$I_i \leq \sigma_0 \left\|\omega_n\right\|_{L^2(\bar{\mathcal{I}})} \left\|\delta\ell_n\right\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)} \big\|\boldsymbol{\Sigma}_n - \bar{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}_n'\big\|_{S^2}. \tag{2.44}$$

In summary, (2.37), (2.40), (2.42), and (2.44) together with (2.35) yield

$$\begin{aligned}\left\|\mathbf{\Sigma}_n-\bar{\mathbf{\Sigma}}-\mathbf{\Sigma}_n'\right\|_{S^2}^2 \leq{}& c\left\|\delta\ell_n\right\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)}\left\|\mathcal{D}\mathbf{\Sigma}_n-\mathcal{D}\bar{\mathbf{\Sigma}}\right\|_{L^m(\Omega;\mathbb{S})}\left\|\mathbf{\Sigma}_n-\bar{\mathbf{\Sigma}}-\mathbf{\Sigma}_n'\right\|_{S^2}\\ &+c\left\|\delta\ell_n\right\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)}^2\left(\left\|z_n\right\|_{L^\xi(\bar{\mathcal{A}}_s)}+\left\|\mathcal{D}\mathbf{\Sigma}_n-\mathcal{D}\bar{\mathbf{\Sigma}}\right\|_{L^\xi(\Omega;\mathbb{S})}\right)\\ &+c\left\|\delta\ell_n\right\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)}\left\|\omega_n\right\|_{L^2(\bar{\mathcal{I}})}\left\|\mathbf{\Sigma}_n-\bar{\mathbf{\Sigma}}-\mathbf{\Sigma}_n'\right\|_{S^2}.\end{aligned}$$

By applying Young's inequality we find

$$\begin{aligned}\frac{\left\|\mathbf{\Sigma}_n-\bar{\mathbf{\Sigma}}-\mathbf{\Sigma}_n'\right\|_{S^2}^2}{\left\|\delta\ell_n\right\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)}^2}\leq{}& c\left(\left\|\mathcal{D}\mathbf{\Sigma}_n-\mathcal{D}\bar{\mathbf{\Sigma}}\right\|_{L^m(\Omega;\mathbb{S})}+\left\|\omega_n\right\|_{L^2(\bar{\mathcal{I}})}\right)^2\\ &+c\left(\left\|z_n\right\|_{L^\xi(\bar{\mathcal{A}}_s)}+\left\|\mathcal{D}\mathbf{\Sigma}_n-\mathcal{D}\bar{\mathbf{\Sigma}}\right\|_{L^\xi(\Omega;\mathbb{S})}\right).\end{aligned}$$

Lemma 2.15 (ii), (2.39) and (2.43) then imply

$$\frac{\left\|\mathbf{\Sigma}_n-\bar{\mathbf{\Sigma}}-\mathbf{\Sigma}_n'\right\|_{S^2}^2}{\left\|\delta\ell_n\right\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)}^2}\xrightarrow{n\to\infty}0. \tag{2.45}$$

In order to prove the remainder term property for the displacement $\boldsymbol{u}$, we subtract (2.15a) and (2.18a) from (2.21a) and test the arising equation with

$$\tilde{\boldsymbol{T}}:=\big(\boldsymbol{\varepsilon}(\boldsymbol{u}_n)-\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})-\boldsymbol{\varepsilon}(\boldsymbol{u}_n'),-\boldsymbol{\varepsilon}(\boldsymbol{u}_n)+\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})+\boldsymbol{\varepsilon}(\boldsymbol{u}_n')\big)\in S^2.$$

Consequently, we obtain

$$\begin{aligned}&\big(A(\mathbf{\Sigma}_n-\bar{\mathbf{\Sigma}}-\mathbf{\Sigma}_n'),\,\tilde{\boldsymbol{T}}\big)+\big(\operatorname{div}^*(\boldsymbol{u}_n-\bar{\boldsymbol{u}}-\boldsymbol{u}_n'),\,\tilde{\boldsymbol{T}}\big)\\ &\quad+\underbrace{\big(\lambda_n\mathcal{D}\mathbf{\Sigma}_n-\bar{\lambda}\mathcal{D}\bar{\mathbf{\Sigma}}-\bar{\lambda}\mathcal{D}\mathbf{\Sigma}_n'-\lambda_n'\mathcal{D}\bar{\mathbf{\Sigma}},\,\mathcal{D}\tilde{\boldsymbol{T}}\big)}_{=:I_2}=0.\end{aligned} \tag{2.46}$$

As $\mathcal{D}\tilde{\boldsymbol{T}}=0$, we infer $I_2=0$ and therefore

$$\begin{aligned}\left\|\boldsymbol{u}_n-\bar{\boldsymbol{u}}-\boldsymbol{u}_n'\right\|_V^2&\leq c\int_\Omega\left\|\boldsymbol{\varepsilon}(\boldsymbol{u}_n-\bar{\boldsymbol{u}}-\boldsymbol{u}_n')\right\|_{\mathbb{S}}^2\,\mathrm{d}x\\ &=c\big(\operatorname{div}^*(\boldsymbol{u}_n-\bar{\boldsymbol{u}}-\boldsymbol{u}_n'),\,\tilde{\boldsymbol{T}}\big)\\ &\leq c\left|\big(A(\mathbf{\Sigma}_n-\bar{\mathbf{\Sigma}}-\mathbf{\Sigma}_n'),\,\tilde{\boldsymbol{T}}\big)\right|\\ &\leq c\left\|\boldsymbol{u}_n-\bar{\boldsymbol{u}}-\boldsymbol{u}_n'\right\|_V\left\|\mathbf{\Sigma}_n-\bar{\mathbf{\Sigma}}-\mathbf{\Sigma}_n'\right\|_{S^2}\end{aligned} \tag{2.47}$$

by Korn's inequality (Proposition A.25). Hence, (2.45) induces

$$\frac{\left\|\boldsymbol{u}_n-\bar{\boldsymbol{u}}-\boldsymbol{u}_n'\right\|_V}{\left\|\delta\ell_n\right\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)}}\xrightarrow{n\to\infty}0.$$

Since the above arguments hold for every subsequence of $(\mathbf{\Sigma}_n,\boldsymbol{u}_n,\lambda_n)$, we conclude (2.33) for the whole sequence. □

In view of Lemma 2.18 we can enhance the result of Theorem 2.19:

Corollary 2.21. *Let* $(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}})$ *be given by the solution of* (2.15), $(\boldsymbol{\Sigma}, \boldsymbol{u})$ *by the solution of* (2.21) *and* $(\boldsymbol{\Sigma}', \boldsymbol{u}')$ *by the solution of* (2.18). *Then it holds*

$$\left\|\boldsymbol{\Sigma} - \bar{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}'\right\|_{L^\beta(\Omega;\mathbb{S}^2)} + \left\|\boldsymbol{u} - \bar{\boldsymbol{u}} - \boldsymbol{u}'\right\|_{W^{1,\beta}_D(\Omega;\mathbb{R}^d)} = o\left(\|\delta\ell\|_{W^{-1,p}_D(\Omega;\mathbb{R}^d)}\right) \tag{2.48}$$

for all $1 \leq \beta < p$.

Proof. As in the proof of Theorem 2.19 let $(\delta\ell_n)_{n\in\mathbb{N}} \subset W^{-1,p}_D(\Omega;\mathbb{R}^d)$ be an arbitrary sequence with $\delta\ell_n \to 0$ for $n \to 0$. Furthermore, by $(\boldsymbol{\Sigma}_n, \boldsymbol{u}_n, \lambda_n)$ we denote the solution of (2.21) with right hand side $\bar{\ell} + \delta\ell_n$ and by $(\boldsymbol{\Sigma}'_n, \boldsymbol{u}'_n, \lambda'_n)$ the solution of (2.18) with right hand side $\delta\ell_n$. According to Theorem 2.19 there exist subsequences, w.l.o.g. denoted in the same way, such that

$$\frac{\boldsymbol{\Sigma}_n - \bar{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}'_n}{\|\delta\ell\|_{W^{-1,p}_D(\Omega;\mathbb{R}^d)}} \to \boldsymbol{0} \quad \text{and} \quad \frac{\boldsymbol{u}_n - \bar{\boldsymbol{u}} - \boldsymbol{u}'_n}{\|\delta\ell\|_{W^{-1,p}_D(\Omega;\mathbb{R}^d)}} \to \boldsymbol{0} \quad \text{a.e. in } \Omega.$$

Besides, these subsequences are bounded in $L^p\big(\Omega;\mathbb{S}^2\big)$ and $W^{1,p}_D\big(\Omega;\mathbb{R}^d\big)$, respectively, cf. Theorem 2.8 and Lemma 2.16. From Lemma 2.18 we infer (2.48) for arbitrary subsequences and thus for the whole sequence. ☐

In addition, Theorem 2.19 entails two consequences stated in the following corollaries:

Corollary 2.22. *Let the multipliers* $\bar{\lambda}$, λ *and* λ' *be given by the solutions of* (2.15), (2.21) *and* (2.18)*, respectively. Then it holds*

$$\left\|\lambda - \bar{\lambda} - \lambda'\right\|_{L^\gamma(\Omega)} = o\left(\|\delta\ell\|_{W^{-1,p}_D(\Omega;\mathbb{R}^d)}\right) \tag{2.49}$$

for all $1 \leq \gamma < q$.

Proof. Due to (2.23), (2.31) and (2.28) we know

$$\begin{aligned}\sigma_0^2(\lambda - \bar{\lambda} - \lambda') &= -\,\mathbb{H}^{-1}\boldsymbol{\chi} : \mathcal{D}\boldsymbol{\Sigma} + \mathbb{H}^{-1}\bar{\boldsymbol{\chi}} : \mathcal{D}\bar{\boldsymbol{\Sigma}} + \mathbb{H}^{-1}\boldsymbol{\chi}' : \mathcal{D}\bar{\boldsymbol{\Sigma}} \\ &= -\,\mathbb{H}^{-1}(\boldsymbol{\chi} - \bar{\boldsymbol{\chi}}) : (\mathcal{D}\boldsymbol{\Sigma} - \mathcal{D}\bar{\boldsymbol{\Sigma}}) - \mathbb{H}^{-1}(\boldsymbol{\chi} - \bar{\boldsymbol{\chi}} - \boldsymbol{\chi}') : \mathcal{D}\bar{\boldsymbol{\Sigma}} \\ &\quad -\,\mathbb{H}^{-1}\bar{\boldsymbol{\chi}} : (\mathcal{D}\boldsymbol{\Sigma} - \mathcal{D}\bar{\boldsymbol{\Sigma}} - \mathcal{D}\boldsymbol{\Sigma}').\end{aligned}$$

Because of $q = 1/(1/s + 1/p)$, cf. (2.20), every $\gamma < q$ can be written as $\gamma = 1/(1/s + 1/\beta)$ for some $\beta < p$. The boundedness of $\mathcal{D}\bar{\boldsymbol{\Sigma}}(x)$ a.e. in Ω, cf. (2.15c), then implies

$$\begin{aligned}\left\|\lambda - \bar{\lambda} - \lambda'\right\|_{L^\gamma(\Omega)} \leq c\,\|\boldsymbol{\chi} - \bar{\boldsymbol{\chi}}\|_{L^\beta(\Omega;\mathbb{S})}\left\|\mathcal{D}\boldsymbol{\Sigma} - \mathcal{D}\bar{\boldsymbol{\Sigma}}\right\|_{L^s(\Omega;\mathbb{S})} \\ + c\left\|\boldsymbol{\chi} - \bar{\boldsymbol{\chi}} - \boldsymbol{\chi}'\right\|_{L^\beta(\Omega;\mathbb{S}^2)} \\ + c\,\|\bar{\boldsymbol{\chi}}\|_{L^s(\Omega;\mathbb{S})}\left\|\boldsymbol{\Sigma} - \bar{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}'\right\|_{L^\beta(\Omega;\mathbb{S}^2)}.\end{aligned}$$

Lemma 2.15 together with Corollary 2.21 yields the claim. □

As a result of Corollary 2.21 and Corollary 2.22 the operator G_E is directionally differentiable:

Corollary 2.23. *For all* $1 \leq \beta < p$ *the control-to-state map* $G_E \colon W_D^{-1,p}\big(\Omega;\mathbb{R}^d\big) \to L^\beta\big(\Omega;\mathbb{S}^2\big) \times W_D^{1,\beta}\big(\Omega;\mathbb{R}^d\big)$ *is directionally differentiable at every* $\bar{\ell} \in W_D^{-1,p}\big(\Omega;\mathbb{R}^d\big)$ *in all directions* $\delta\ell \in W_D^{-1,p}\big(\Omega;\mathbb{R}^d\big)$*. Furthermore, the mapping* $W_D^{-1,p}(\Omega;\mathbb{R}^d) \ni \bar{\ell} \mapsto \bar{\lambda} \in L^\gamma(\Omega)$ *is directionally differentiable for all* $1 \leq \gamma < q$*.*

Proof. Since the set $\mathcal{S}_\ell$ is a cone, the mapping $\delta\ell \mapsto \delta_w G_E(\bar{\ell}, \delta\ell)$ is positively homogeneous so that $(t\boldsymbol{\Sigma}', t\boldsymbol{u}', t\lambda')$ is the solution of (2.18) with right hand side $t\delta\ell$. Consequently, (2.48) leads to

$$\frac{\big\|\boldsymbol{\Sigma}_t - \bar{\boldsymbol{\Sigma}} - t\boldsymbol{\Sigma}'\big\|_{L^\beta(\Omega;\mathbb{S}^2)}}{t} + \frac{\|\boldsymbol{u}_t - \bar{\boldsymbol{u}} - t\boldsymbol{u}'\|_{W_D^{1,\beta}(\Omega;\mathbb{R}^d)}}{t} \xrightarrow{t\searrow 0} 0.$$

The second assertion follows analogously from (2.49). □

2.2.1. Directional differentiability

In order to derive the assertion of Corollary 2.23 we do not need Theorem 2.19. To be more precise, the integrability requirement (2.19) for the hardening variable can be relaxed such that $G_E \colon W_D^{-1,p}\big(\Omega;\mathbb{R}^d\big) \to L^\beta\big(\Omega;\mathbb{S}^2\big) \times W_D^{1,\beta}\big(\Omega;\mathbb{R}^d\big)$ is only directionally differentiable for all $1 \leq \beta < p$ without the remainder term property (2.48).

Assumption 2.24. *The solution of* (2.15) *satisfies the weaker condition* $\bar{\boldsymbol{\chi}} \in L^s(\Omega;\mathbb{S})$ *with*

$$s > \frac{p}{p-2}. \tag{2.50}$$

For the rest of this section Assumption 2.24 is supposed to hold instead of Assumption 2.13(ii).

Remark 2.25. *As in Remark 2.14(ii) we infer from* (2.50) *that* q *defined in* (2.20) *fulfills* $p' < q < p$*. Therefore, the statements of Lemma 2.15 and Lemma 2.16 remain unaffected by Assumption 2.24.*

By similar arguments as in the proof of Theorem 2.19 we can establish the following Theorem.

Theorem 2.26. *Let* $(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}}, \bar{\lambda})$ *be the solution of* (2.15), $(\boldsymbol{\Sigma}_t, \boldsymbol{u}_t, \lambda_t)$ *the solution of* (2.25) *and* $(\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda')$ *the solution of* (2.18). *Then it holds*

$$\left(\frac{\boldsymbol{\Sigma}_t - \bar{\boldsymbol{\Sigma}}}{t}, \frac{\boldsymbol{u}_t - \bar{\boldsymbol{u}}}{t}\right) \xrightarrow{t \searrow 0} (\boldsymbol{\Sigma}', \boldsymbol{u}') \quad \text{in } S^2 \times V, \tag{2.51}$$

i.e., the control-to-state map G_E *is directionally differentiable from* $W_D^{-1,p}(\Omega; \mathbb{R}^d)$ *to* $S^2 \times V$.

Proof. Let $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ be an arbitrary sequence of positive real numbers converging to zero and let $(\boldsymbol{\Sigma}_{t_n}, \boldsymbol{u}_{t_n}, \lambda_{t_n})$ be given by the solution of (2.25) with right hand side $\bar{\ell} + t_n \delta\ell$. Subtracting (2.15a) and (2.18a) from (2.25a) and testing with $(\boldsymbol{\Sigma}_{t_n} - \bar{\boldsymbol{\Sigma}})/t_n - \boldsymbol{\Sigma}'$ results in

$$\begin{aligned}
&\left(A\Big(\frac{\boldsymbol{\Sigma}_{t_n} - \bar{\boldsymbol{\Sigma}}}{t_n} - \boldsymbol{\Sigma}'\Big), \frac{\boldsymbol{\Sigma}_{t_n} - \bar{\boldsymbol{\Sigma}}}{t_n} - \boldsymbol{\Sigma}'\right) + \underbrace{\left(\operatorname{div}^*\Big(\frac{\boldsymbol{u}_{t_n} - \bar{\boldsymbol{u}}}{t_n} - \boldsymbol{u}'\Big), \frac{\boldsymbol{\Sigma}_{t_n} - \bar{\boldsymbol{\Sigma}}}{t_n} - \boldsymbol{\Sigma}'\right)}_{=:\mathfrak{I}_1^n} \\
&+ \left(\frac{\lambda_{t_n}\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \bar{\lambda}\mathcal{D}\bar{\boldsymbol{\Sigma}}}{t_n} - \bar{\lambda}\mathcal{D}\boldsymbol{\Sigma}' - \lambda'\mathcal{D}\bar{\boldsymbol{\Sigma}}, \frac{\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \mathcal{D}\bar{\boldsymbol{\Sigma}}}{t_n} - \mathcal{D}\boldsymbol{\Sigma}'\right) = 0.
\end{aligned}$$

Due to (2.15b), (2.25b) and (2.18b) we infer $\operatorname{div}\big((\boldsymbol{\Sigma}_{t_n} - \bar{\boldsymbol{\Sigma}})/t_n - \boldsymbol{\Sigma}'\big) = 0$ and thus $\mathfrak{I}_1^n = 0$ for all $n \in \mathbb{N}$. Analogously to (2.35) we obtain

$$\begin{aligned}
&c\left\|\frac{\boldsymbol{\Sigma}_{t_n} - \bar{\boldsymbol{\Sigma}}}{t_n} - \boldsymbol{\Sigma}'\right\|_{S^2}^2 \leq \\
&\leq -\left(\frac{\lambda_{t_n}\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \bar{\lambda}\mathcal{D}\bar{\boldsymbol{\Sigma}}}{t_n} - \bar{\lambda}\mathcal{D}\boldsymbol{\Sigma}' - \lambda'\mathcal{D}\bar{\boldsymbol{\Sigma}}, \frac{\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \mathcal{D}\bar{\boldsymbol{\Sigma}}}{t_n} - \mathcal{D}\boldsymbol{\Sigma}'\right) \\
&= \underbrace{\left(\frac{\lambda_{t_n}\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \bar{\lambda}\mathcal{D}\bar{\boldsymbol{\Sigma}}}{t_n} - \bar{\lambda}\mathcal{D}\boldsymbol{\Sigma}' - \lambda'\mathcal{D}\bar{\boldsymbol{\Sigma}}, \mathcal{D}\boldsymbol{\Sigma}'\right)}_{=:\mathfrak{I}_2^n} + \underbrace{\left(\bar{\lambda}\mathcal{D}\boldsymbol{\Sigma}', \frac{\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \mathcal{D}\bar{\boldsymbol{\Sigma}}}{t_n}\right)}_{=:\mathfrak{I}_3^n} \\
&\quad + \underbrace{\left(\lambda'\mathcal{D}\bar{\boldsymbol{\Sigma}}, \frac{\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \mathcal{D}\bar{\boldsymbol{\Sigma}}}{t_n}\right)}_{=:\mathfrak{I}_4^n} - \underbrace{\left(\frac{\lambda_{t_n} - \bar{\lambda}}{t_n}\mathcal{D}\boldsymbol{\Sigma}_{t_n}, \frac{\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \mathcal{D}\bar{\boldsymbol{\Sigma}}}{t_n}\right)}_{=:\mathfrak{I}_5^n} \\
&\quad - \underbrace{\left(\bar{\lambda}\frac{\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \mathcal{D}\bar{\boldsymbol{\Sigma}}}{t_n}, \frac{\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \mathcal{D}\bar{\boldsymbol{\Sigma}}}{t_n}\right)}_{=:\mathfrak{I}_6^n}.
\end{aligned} \tag{2.52}$$

Next the convergence of $\mathfrak{I}_2^n$, $\mathfrak{I}_3^n$, $\mathfrak{I}_4^n$, $\mathfrak{I}_5^n$ and $\mathfrak{I}_6^n$ as $n \to \infty$ is investigated. Thanks to the boundedness of $\mathcal{D}\bar{\boldsymbol{\Sigma}}$ and (2.23) the multiplier $\bar{\lambda}$ belongs to $L^s(\Omega)$. Moreover, in view of Remark 2.25 there exists $m > 0$ with $1/q + 1/m = 1/p'$ so that (2.20), Lemma 2.15(ii), Lemma 2.16(i) and (ii) yield

$$\frac{\lambda_{t_n} - \bar{\lambda}}{t_n}\mathcal{D}\boldsymbol{\Sigma}_{t_n} \rightharpoonup \lambda'\mathcal{D}\bar{\boldsymbol{\Sigma}}, \quad \bar{\lambda}\frac{\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \mathcal{D}\bar{\boldsymbol{\Sigma}}}{t_n} \rightharpoonup \bar{\lambda}\mathcal{D}\boldsymbol{\Sigma}' \quad \text{in } L^{p'}(\Omega; \mathbb{S})$$

and accordingly

$$\mathfrak{I}_2^n = \left(\frac{\lambda_{t_n} - \bar{\lambda}}{t_n}\mathcal{D}\boldsymbol{\Sigma}_{t_n} + \bar{\lambda}\frac{\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \mathcal{D}\bar{\boldsymbol{\Sigma}}}{t_n} - \lambda'\mathcal{D}\bar{\boldsymbol{\Sigma}} - \bar{\lambda}\mathcal{D}\boldsymbol{\Sigma}',\, \mathcal{D}\boldsymbol{\Sigma}'\right) \to 0. \tag{2.53}$$

From $\bar{\lambda}\mathcal{D}\boldsymbol{\Sigma}'$, $\lambda'\mathcal{D}\bar{\boldsymbol{\Sigma}} \in L^{p'}(\Omega;\mathbb{S})$, Lemma 2.16(i) and (2.18c)–(2.18e) we further conclude

$$\mathfrak{I}_3^n \to \left(\bar{\lambda}\mathcal{D}\boldsymbol{\Sigma}',\, \mathcal{D}\boldsymbol{\Sigma}'\right), \quad \mathfrak{I}_4^n \to \left(\lambda'\mathcal{D}\bar{\boldsymbol{\Sigma}},\, \mathcal{D}\boldsymbol{\Sigma}'\right) = 0. \tag{2.54}$$

The convergence of $\mathfrak{I}_5^n$ is separately discussed on the sets $\bar{\mathcal{A}}_s, \bar{\mathcal{B}}$ and $\bar{\mathcal{I}}$. By Lemma 2.15 there exist subsequences, w.l.o.g. denoted by the same symbols, with $\lambda_{t_n}(x) \to \bar{\lambda}(x)$ a.e. in Ω and $\boldsymbol{\Sigma}_{t_n}(x) \to \bar{\boldsymbol{\Sigma}}(x)$ a.e. in Ω. Similarly to (2.40) it can then be shown that

$$\begin{aligned}
&\int_{\bar{\mathcal{A}}_s} \frac{\lambda_{t_n} - \bar{\lambda}}{t_n}\mathcal{D}\boldsymbol{\Sigma}_{t_n} : \frac{\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \mathcal{D}\bar{\boldsymbol{\Sigma}}}{t_n}\,\mathrm{d}x = \\
&= \int_{\bar{\mathcal{A}}_s} \frac{\lambda_{t_n} - \bar{\lambda}}{t_n} z_{t_n} \left\|\mathcal{D}\boldsymbol{\Sigma}_{t_n}\right\|_{\mathbb{S}} \frac{\left\|\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \mathcal{D}\bar{\boldsymbol{\Sigma}}\right\|_{\mathbb{S}}}{t_n}\,\mathrm{d}x \\
&\le \sigma_0 \left\|\frac{\lambda_{t_n} - \bar{\lambda}}{t_n}\right\|_{L^q(\Omega)} \left\|z_{t_n}\right\|_{L^m(\bar{\mathcal{A}}_s)} \left\|\frac{\boldsymbol{\Sigma}_{t_n} - \bar{\boldsymbol{\Sigma}}}{t_n}\right\|_{L^p(\Omega;\mathbb{S})} \le c\left\|z_{t_n}\right\|_{L^m(\bar{\mathcal{A}}_s)},
\end{aligned} \tag{2.55}$$

where the function z_{t_n} is as defined in (2.38) with $\boldsymbol{\Sigma}_{t_n}$ instead of $\boldsymbol{\Sigma}_n$. Note here the boundedness of weakly convergent sequences, cf. Lemma 2.16. On account of (2.41) and Remark 2.25 it follows

$$\begin{aligned}
&\int_{\bar{\mathcal{B}}} \frac{\lambda_{t_n} - \bar{\lambda}}{t_n}\mathcal{D}\boldsymbol{\Sigma}_{t_n} : \frac{\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \mathcal{D}\bar{\boldsymbol{\Sigma}}}{t_n}\,\mathrm{d}x = \int_{\bar{\mathcal{B}}} \frac{\lambda_{t_n} - \bar{\lambda}}{t_n}\frac{1}{2}\frac{\left\|\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \mathcal{D}\bar{\boldsymbol{\Sigma}}\right\|_{\mathbb{S}}^2}{t_n}\,\mathrm{d}x \\
&\le \frac{1}{2}\left\|\frac{\lambda_{t_n} - \bar{\lambda}}{t_n}\right\|_{L^q(\Omega)} \left\|\frac{\boldsymbol{\Sigma}_{t_n} - \bar{\boldsymbol{\Sigma}}}{t_n}\right\|_{L^p(\Omega;\mathbb{S})} \left\|\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \mathcal{D}\bar{\boldsymbol{\Sigma}}\right\|_{L^m(\Omega;\mathbb{S})} \\
&\le c\left\|\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \mathcal{D}\bar{\boldsymbol{\Sigma}}\right\|_{L^m(\Omega;\mathbb{S})}
\end{aligned} \tag{2.56}$$

with $1/p + 1/q + 1/m = 1$. Finally, we observe

$$\begin{aligned}
\int_{\bar{\mathcal{I}}} \frac{\lambda_{t_n} - \bar{\lambda}}{t_n}\mathcal{D}\boldsymbol{\Sigma}_{t_n} : \frac{\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \mathcal{D}\bar{\boldsymbol{\Sigma}}}{t_n}\,\mathrm{d}x &\le \sigma_0 \left\|\frac{\lambda_{t_n} - \bar{\lambda}}{t_n}\right\|_{L^{p'}(\bar{\mathcal{I}})} \left\|\frac{\boldsymbol{\Sigma}_{t_n} - \bar{\boldsymbol{\Sigma}}}{t_n}\right\|_{L^p(\Omega;\mathbb{S})} \\
&\le c\left\|\frac{\lambda_{t_n} - \bar{\lambda}}{t_n}\right\|_{L^{p'}(\bar{\mathcal{I}})}
\end{aligned} \tag{2.57}$$

with $p' < q$ and analogously to (2.43) we deduce

$$\frac{\lambda_{t_n} - \bar{\lambda}}{t_n} \to 0 \quad \text{in } L^{p'}(\bar{\mathcal{I}}). \tag{2.58}$$

Lemma 2.15(ii), (2.39) and (2.55)–(2.58) lead to

$$\mathfrak{I}_5^n \to 0. \tag{2.59}$$

Let $f\colon L^p(\Omega,\mathbb{S}) \to \mathbb{R}$ be defined by

$$\boldsymbol{\tau} \mapsto \big(\bar{\lambda},\, \|\boldsymbol{\tau}\|_{\mathbb{S}}^2\big).$$

In consequence of (2.15c) the function $-f$ is concave. Moreover, from (2.20) and Remark 2.25 we infer $1/p + 1/s + 1/p < 1$ so that

$$\begin{aligned} f(\boldsymbol{\tau}_1) - f(\boldsymbol{\tau}_2) &= \int_\Omega \bar{\lambda}\big(\|\boldsymbol{\tau}_1\|_{\mathbb{S}}^2 - \|\boldsymbol{\tau}_2\|_{\mathbb{S}}^2 \big) \,\mathrm{d}x \\ &= \int_\Omega \bar{\lambda}\big(\|\boldsymbol{\tau}_1\|_{\mathbb{S}} - \|\boldsymbol{\tau}_2\|_{\mathbb{S}} \big)\big(\|\boldsymbol{\tau}_1\|_{\mathbb{S}} - \|\boldsymbol{\tau}_2\|_{\mathbb{S}} + 2\,\|\boldsymbol{\tau}_2\|_{\mathbb{S}} \big) \,\mathrm{d}x \\ &\le \big\|\bar{\lambda}\big\|_{L^s(\Omega)} \|\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2\|_{L^p(\Omega;\mathbb{S})}^2 \\ &\quad + \big\|\bar{\lambda}\big\|_{L^s(\Omega)} \|\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2\|_{L^p(\Omega;\mathbb{S})}\, 2\,\|\boldsymbol{\tau}_2\|_{L^p(\Omega;\mathbb{S})} \end{aligned} \tag{2.60}$$

for all $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in L^p(\Omega,\mathbb{S})$. This shows continuity of f and hence weak upper semi-continuity of $-f$, which implies

$$\begin{aligned} \limsup_{n\to\infty} \mathfrak{I}_6^n = \limsup_{n\to\infty} \left(-f\left(\frac{\mathcal{D}\boldsymbol{\Sigma}_{t_n} - \mathcal{D}\bar{\boldsymbol{\Sigma}}}{t_n} \right) \right) &\le -f\big(\mathcal{D}\boldsymbol{\Sigma}'\big) \\ &= -\big(\bar{\lambda}\mathcal{D}\boldsymbol{\Sigma}',\, \mathcal{D}\boldsymbol{\Sigma}'\big)\,. \end{aligned} \tag{2.61}$$

Together (2.52)–(2.54), (2.59) and (2.61) induce

$$\left\| \frac{\boldsymbol{\Sigma}_{t_n} - \bar{\boldsymbol{\Sigma}}}{t_n} - \boldsymbol{\Sigma}' \right\|_{S^2}^2 \xrightarrow{n\to\infty} 0. \tag{2.62}$$

The convergence result for the displacement is obtained similarly to (2.47). By subtracting (2.15a) and (2.18a) from (2.25a) and testing with

$$\hat{\boldsymbol{T}} := \left(\frac{\boldsymbol{\varepsilon}(\boldsymbol{u}_{t_n}) - \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})}{t_n} - \boldsymbol{\varepsilon}(\boldsymbol{u}'), -\frac{\boldsymbol{\varepsilon}(\boldsymbol{u}_{t_n}) - \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})}{t_n} + \boldsymbol{\varepsilon}(\boldsymbol{u}') \right) \in S^2$$

we arrive at

$$\int_\Omega \left\| \frac{\boldsymbol{\varepsilon}(\boldsymbol{u}_{t_n}) - \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})}{t_n} - \boldsymbol{\varepsilon}(\boldsymbol{u}') \right\|_{\mathbb{S}}^2 \mathrm{d}x \le c \left\| \frac{\boldsymbol{u}_{t_n} - \bar{\boldsymbol{u}}}{t_n} - \boldsymbol{u}' \right\|_V \left\| \frac{\boldsymbol{\Sigma}_{t_n} - \bar{\boldsymbol{\Sigma}}}{t_n} - \boldsymbol{\Sigma}' \right\|_{S^2}.$$

Korn's inequality (Proposition A.25) and (2.62) yield the claim for every subsequence $(\boldsymbol{\Sigma}_{t_n}, \boldsymbol{u}_{t_n}, \lambda_{t_n})$ and thus for the whole sequence. □

The next two corollaries can be derived analogously to Corollary 2.21 and Corollary 2.22.

Corollary 2.27. *Let* $(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}})$ *be given by the solution of* (2.15)*,* $(\boldsymbol{\Sigma}_t, \boldsymbol{u}_t)$ *by the solution of* (2.25) *and* $(\boldsymbol{\Sigma}', \boldsymbol{u}')$ *by the solution of* (2.18)*. Then it holds*

$$\left(\frac{\boldsymbol{\Sigma}_t - \bar{\boldsymbol{\Sigma}}}{t}, \frac{\boldsymbol{u}_t - \bar{\boldsymbol{u}}}{t} \right) \xrightarrow{t\searrow 0} (\boldsymbol{\Sigma}', \boldsymbol{u}') \quad \text{in } L^\beta\big(\Omega;\mathbb{S}^2\big) \times W^{1,\beta}\big(\Omega;\mathbb{R}^d\big)$$

for all $1 \le \beta < p$.

Corollary 2.28. *Let the multipliers $\bar{\lambda}$, λ_t and λ' be given by the solutions of* (2.15). (2.25) *and* (2.18), *respectively. Then for all $1 \leq \gamma < q$ it holds*

$$\frac{\lambda_t - \bar{\lambda}}{t} \xrightarrow{t \searrow 0} \lambda' \quad \text{in } L^\gamma(\Omega).$$

2.3. Second-order sufficient optimality conditions

Given the findings on the differentiability of the control-to state map G_E we can establish second-order sufficient optimality conditions for $(\mathbf{P}_E)$. We start by providing some preparatory auxiliary results. Then rather restrictive sufficient optimality conditions are presented, which however are applicable to a general smooth objective functional. Subsequently, we assume a specific structure of the objective functional such that these conditions can be relaxed.

Throughout this section we make the following assumption.

Assumption 2.29. *The objective functional $J\colon V \times U \to \mathbb{R}$ is twice continuously Fréchet-differentiable.*

According to Theorem 2.5 Problem $(\mathbf{P}_E)$ is equivalent to

$$\left.\begin{aligned} &\text{Minimize} \quad J(\boldsymbol{u}, \boldsymbol{f}) \\ &\text{s.t.} \quad \begin{cases} A\boldsymbol{\Sigma} + \operatorname{div}^* \boldsymbol{u} + \lambda \mathcal{D}^* \mathcal{D} \boldsymbol{\Sigma} = \mathbf{0} & \text{in } S^2 \\ \operatorname{div} \boldsymbol{\Sigma} = R\boldsymbol{f} & \text{in } V' \\ 0 \leq \lambda(x) \perp \phi(\boldsymbol{\Sigma}(x)) \leq 0 & \text{a.e. in } \Omega \end{cases} \end{aligned}\right\} \tag{2.63}$$

with $R\colon U \to V'$ given in (2.14). For the rest of this chapter we set

$$p = \min(\hat{p}, 3), \tag{2.64}$$

where $\hat{p}$ is as defined in Theorem 2.8. Hence, R continuously maps from U to $W_D^{-1,p}(\Omega; \mathbb{R}^d)$, cf. Remark 2.14(i). To simplify matters we denote R with range in $W_D^{-1,p}(\Omega; \mathbb{R}^d)$ by the same symbol.

Remark 2.30. *Let Assumption 2.1 hold. Furthermore let $(\boldsymbol{\Sigma}, \boldsymbol{u}, \lambda)$ be the state and multiplier associated with $\boldsymbol{f} \in U$ and $(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}}, \bar{\lambda})$ be the state and multiplier associated with $\bar{\boldsymbol{f}} \in U$. Due to the continuity of R the results of Section 2.2 imply*

(i) $\|\boldsymbol{\Sigma} - \bar{\boldsymbol{\Sigma}}\|_{L^p(\Omega;\mathbb{S}^2)} + \|\boldsymbol{u} - \bar{\boldsymbol{u}}\|_{W_D^{1,p}(\Omega;\mathbb{R}^d)} \leq L \, \|R\| \, \|\boldsymbol{f} - \bar{\boldsymbol{f}}\|_U$

(ii) $\|\lambda - \bar{\lambda}\|_{L^q(\Omega)} \leq c \, \|R\| \, \|\boldsymbol{f} - \bar{\boldsymbol{f}}\|_U$

(iii) $\|\boldsymbol{\Sigma}'\|_{L^p(\Omega;\mathbb{S}^2)} + \|\boldsymbol{u}'\|_{W^{1,p}_D(\Omega;\mathbb{R}^d)} \leq L\,\|R\|\,\|\boldsymbol{f}-\bar{\boldsymbol{f}}\|_U$

(iv) $\|\lambda'\|_{L^q(\Omega)} \leq c\,\|R\|\,\|\boldsymbol{f}-\bar{\boldsymbol{f}}\|_U$

(v) $\|\boldsymbol{\Sigma}-\bar{\boldsymbol{\Sigma}}-\boldsymbol{\Sigma}'\|_{L^\beta(\Omega;\mathbb{S}^2)} + \|\boldsymbol{u}-\bar{\boldsymbol{u}}-\boldsymbol{u}'\|_{W^{1,\beta}_D(\Omega;\mathbb{R}^d)} = o\big(\|\boldsymbol{f}-\bar{\boldsymbol{f}}\|_U\big)\ \ \forall\,\beta\in[1,p)$

(vi) $\|\lambda-\bar{\lambda}-\lambda'\|_{L^\gamma(\Omega)} = o\big(\|\boldsymbol{f}-\bar{\boldsymbol{f}}\|_U\big)\ \ \forall\,\gamma\in[1,q)$.

Here we used the abbreviation $\|R\| := \|R\|_{\mathcal{L}(U,W^{-1,p}_D(\Omega;\mathbb{R}^d))}$.

The yield function ϕ involved in the complementarity constraints in (2.63) is Lipschitz continuous on the admissible set $\mathcal{K}$:

Lemma 2.31. *Let* $\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2 \in \mathcal{K}$. *Then it holds*

$$\|\phi(\boldsymbol{\Sigma}_1)-\phi(\boldsymbol{\Sigma}_2)\|_{L^\nu(\Omega)} \leq \sigma_0\,\|\boldsymbol{\Sigma}_1-\boldsymbol{\Sigma}_2\|_{L^\nu(\Omega;\mathbb{S}^2)}$$

for every $\nu \geq 1$.

Proof. By definition of ϕ, cf. (2.3), we find

$$\begin{aligned}
\|\phi(\boldsymbol{\Sigma}_1)-\phi(\boldsymbol{\Sigma}_2)\|^\nu_{L^\nu(\Omega)} &= \int_\Omega \left|\frac{\|\mathcal{D}\boldsymbol{\Sigma}_1\|^2_{\mathbb{S}}-\sigma_0^2}{2} - \frac{\|\mathcal{D}\boldsymbol{\Sigma}_2\|^2_{\mathbb{S}}-\sigma_0^2}{2}\right|^\nu \mathrm{d}x \\
&= \int_\Omega \left|\frac{\|\mathcal{D}\boldsymbol{\Sigma}_1\|^2_{\mathbb{S}}}{2} - \frac{\|\mathcal{D}\boldsymbol{\Sigma}_2\|^2_{\mathbb{S}}}{2}\right|^\nu \mathrm{d}x \\
&= \int_\Omega \left|\frac{1}{2}(\mathcal{D}\boldsymbol{\Sigma}_2+\mathcal{D}\boldsymbol{\Sigma}_1) : (\mathcal{D}\boldsymbol{\Sigma}_1-\mathcal{D}\boldsymbol{\Sigma}_2)\right|^\nu \mathrm{d}x \\
&\leq \int_\Omega \frac{1}{2^\nu}\big(\,\|\mathcal{D}\boldsymbol{\Sigma}_2\|_{\mathbb{S}}+\|\mathcal{D}\boldsymbol{\Sigma}_1\|_{\mathbb{S}}\big)^\nu\,\|\mathcal{D}\boldsymbol{\Sigma}_1-\mathcal{D}\boldsymbol{\Sigma}_2\|^\nu_{\mathbb{S}}\ \mathrm{d}x \\
&\leq \int_\Omega \sigma_0^\nu\,\|\boldsymbol{\Sigma}_1-\boldsymbol{\Sigma}_2\|^\nu_{\mathbb{S}}\ \mathrm{d}x = \sigma_0^\nu\,\|\boldsymbol{\Sigma}_1-\boldsymbol{\Sigma}_2\|^\nu_{L^\nu(\Omega;\mathbb{S}^2)}\,,
\end{aligned}$$

where the last estimate follows from (2.2). ☐

For the derivation of the second-order sufficient conditions a particularly chosen Lagrange function is employed. To this end we introduce the space

$$S^2_\infty := \big\{\boldsymbol{T}\in S^2 : \mathcal{D}\boldsymbol{T}\in L^\infty(\Omega;\mathbb{S})\big\}.$$

Note that every solution of (2.15) and (2.21), respectively, belongs to S^2_∞ in consequence of (2.15c) and (2.21c). Equipped with an appropriate norm, S^2_∞ becomes a Banach space:

Proposition 2.32. *The space* S^2_∞ *endowed with the norm*

$$\|\boldsymbol{T}\|_{S^2_\infty} := \|\boldsymbol{T}\|_{S^2} + \|\mathcal{D}\boldsymbol{T}\|_{L^\infty(\Omega;\mathbb{S})}$$

is a Banach Space.

Proof. If $\{\boldsymbol{T}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in S^2_∞, then $\{\boldsymbol{T}_n\}_{n\in\mathbb{N}}$ and $\{\mathcal{D}\boldsymbol{T}_n\}_{n\in\mathbb{N}}$ are Cauchy sequences in the Banach spaces S^2 and $L^\infty(\Omega;\mathbb{S})$, respectively. Thus, there exist $\boldsymbol{T} \in S^2$ and $\boldsymbol{\mu} \in S$ with

$$\boldsymbol{T}_n \to \boldsymbol{T} \quad \text{in } S^2, \qquad \mathcal{D}\boldsymbol{T}_n \to \boldsymbol{\mu} \quad \text{in } L^\infty(\Omega;\mathbb{S}).$$

Moreover there are subsequences converging pointwisely so that $\boldsymbol{\mu} = \mathcal{D}\boldsymbol{T}$ a.e. in Ω. Therefore, the space S^2_∞ is complete. □

With S^2_∞ at hand we define the Lagrangian $\mathcal{L}\colon S^2_\infty \times V \times L^2(\Omega) \times U \times S^2 \times V \times L^2(\Omega) \times L^\infty(\Omega)' \to \mathbb{R}$ through

$$\begin{aligned}&\mathcal{L}(\boldsymbol{\Sigma}, \boldsymbol{u}, \lambda, \boldsymbol{f}, \boldsymbol{\Upsilon}, \boldsymbol{w}, \mu, \theta) = \\ &= J(\boldsymbol{u}, \boldsymbol{f}) + (A\boldsymbol{\Sigma} + \operatorname{div}^* \boldsymbol{u} + \lambda\mathcal{D}^*\mathcal{D}\boldsymbol{\Sigma},\, \boldsymbol{\Upsilon}) + \langle \operatorname{div} \boldsymbol{\Sigma} - R\boldsymbol{f},\, \boldsymbol{w}\rangle \\ &\quad - (\lambda,\, \mu) + \langle \phi(\boldsymbol{\Sigma}),\, \theta\rangle_{L^\infty(\Omega),L^\infty(\Omega)'} .\end{aligned} \tag{2.65}$$

On account of Assumption 2.29 and the definition of S^2_∞ we obtain the following result concerning the differentiability of $\mathcal{L}$.

Proposition 2.33. *The Lagrange function* $\mathcal{L}$ *defined in* (2.65) *is twice continuously Fréchet-differentiable.*

Proof. We show exemplarily that the nonlinear mapping $F_1\colon L^2(\Omega) \times S^2_\infty \to S$, $(\lambda, \boldsymbol{\Sigma}) \mapsto \lambda\mathcal{D}\boldsymbol{\Sigma}$, is twice continuously Fréchet differentiable. The definition of the norm on S^2_∞ implies

$$\begin{aligned} r_{F_1}\big((\delta\lambda, \delta\boldsymbol{\Sigma})\big) &:= \frac{\big\|F_1(\bar\lambda + \delta\lambda, \bar{\boldsymbol{\Sigma}} + \delta\boldsymbol{\Sigma}) - F_1(\bar\lambda, \bar{\boldsymbol{\Sigma}}) - (\delta\lambda\mathcal{D}\bar{\boldsymbol{\Sigma}} + \bar\lambda\mathcal{D}\delta\boldsymbol{\Sigma})\big\|_S}{\|(\delta\lambda, \delta\boldsymbol{\Sigma})\|_{L^2(\Omega)\times S^2_\infty}} \\ &= \frac{\big\|(\bar\lambda + \delta\lambda)(\mathcal{D}\bar{\boldsymbol{\Sigma}} + \mathcal{D}\delta\boldsymbol{\Sigma}) - \bar\lambda\mathcal{D}\bar{\boldsymbol{\Sigma}} - \delta\lambda\mathcal{D}\bar{\boldsymbol{\Sigma}} - \bar\lambda\mathcal{D}\delta\boldsymbol{\Sigma}\big\|_S}{\|(\delta\lambda, \delta\boldsymbol{\Sigma})\|_{L^2(\Omega)\times S^2_\infty}} \\ &= \frac{\|\delta\lambda\mathcal{D}\delta\boldsymbol{\Sigma}\|_S}{\|\delta\lambda\|_{L^2(\Omega)} + \|\delta\boldsymbol{\Sigma}\|_{S^2} + \|\mathcal{D}\delta\boldsymbol{\Sigma}\|_{L^\infty(\Omega;\mathbb{S})}} \\ &\le \|\mathcal{D}\delta\boldsymbol{\Sigma}\|_{L^\infty(\Omega;\mathbb{S})} \xrightarrow{(\delta\lambda,\delta\boldsymbol{\Sigma})\to 0} 0, \end{aligned}$$

which shows differentiability of F_1 with $F_1'(\bar\lambda, \bar{\boldsymbol{\Sigma}})(\delta\lambda, \delta\boldsymbol{\Sigma}) = \delta\lambda\mathcal{D}\bar{\boldsymbol{\Sigma}} + \bar\lambda\mathcal{D}\delta\boldsymbol{\Sigma}$. Since the first-order derivative is linear w.r.t. $(\bar\lambda, \bar{\boldsymbol{\Sigma}})$, we conclude that F_1 is twice continuously Fréchet differentiable. By the same arguments the nonlinear mapping

$F_2\colon S^2_\infty \to L^\infty(\Omega;\mathbb{S})$, $\boldsymbol{\Sigma} \mapsto \phi(\boldsymbol{\Sigma})$, is Fréchet differentiable with first-order derivative $F_2'(\bar{\boldsymbol{\Sigma}})\delta\boldsymbol{\Sigma} = \mathcal{D}\bar{\boldsymbol{\Sigma}} : \mathcal{D}\delta\boldsymbol{\Sigma}$. □

It will be crucial to estimate the last two expressions in (2.65) properly. More precisely, we need the next auxiliary lemma.

Lemma 2.34. *Let $(\boldsymbol{\Sigma}, \boldsymbol{u}, \lambda)$ be the state and multiplier associated with $\boldsymbol{f} \in U$ and $(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}}, \bar{\lambda})$ be the state and multiplier associated with $\bar{\boldsymbol{f}} \in U$. Furthermore let $\bar{\mu} \in L^\zeta(\Omega)$ and $\bar{\theta} \in L^r(\Omega)$ with*

$$\zeta, r > \frac{sp}{sp - p - 2s} \quad \text{and} \quad s > \frac{2p}{p-2}, \tag{2.66}$$

where p is as defined in (2.64). If $\bar{\mu} \geq 0$ in $\mathcal{A}_1 := \{x \in \Omega : -\kappa_1 \leq \phi(\bar{\boldsymbol{\Sigma}}) \leq 0\}$ for some $\kappa_1 > 0$ and $\bar{\theta} \geq 0$ in $\mathcal{A}_2 := \{x \in \Omega : 0 \leq \bar{\lambda} \leq \kappa_2\}$ for some $\kappa_2 > 0$, then there are $\nu_1, \nu_2 > 2$ such that

$$-\int_\Omega \lambda\bar{\mu} \,\mathrm{d}x \leq c \left\|\boldsymbol{f} - \bar{\boldsymbol{f}}\right\|_U^{\nu_1}, \qquad \int_\Omega \phi(\boldsymbol{\Sigma})\bar{\theta} \,\mathrm{d}x \leq c \left\|\boldsymbol{f} - \bar{\boldsymbol{f}}\right\|_U^{\nu_2}.$$

Remark 2.35. *The integrability condition (2.66) ensures that $1/\zeta < 1 - 1/q - 1/p$, or equivalently, $\zeta > qp/(qp - p - q)$ with q as defined in (2.20):*

$$\zeta > \frac{sp}{sp - p - 2s} = \frac{sp^2}{sp^2 - p(s+p) - sp} = \frac{\frac{sp}{s+p}p}{\frac{sp}{s+p}p - p - \frac{sp}{s+p}} = \frac{qp}{qp - p - q}.$$

Thanks to Remark 2.14(ii) we further know $1 - 1/q - 1/p > 1 - 1/p' - 1/p = 0$.

Proof of Lemma 2.34. By defining $\Omega_1 := \{x \in \Omega : \lambda(x) > 0\}$ and $E_1 := \Omega_1 \cap \mathcal{A}_1$, we observe

$$\begin{aligned} -\int_\Omega \lambda\bar{\mu} \,\mathrm{d}x &= -\int_{E_1} \underbrace{\lambda}_{>0} \underbrace{\bar{\mu}}_{\geq 0} \,\mathrm{d}x - \int_{\Omega_1 \setminus E_1} \lambda\bar{\mu} \,\mathrm{d}x \\ &\leq -\int_{\Omega_1 \setminus E_1} (\lambda - \bar{\lambda})\bar{\mu} \,\mathrm{d}x. \end{aligned}$$

Note here that $\bar{\lambda} = 0$ a.e. in $\Omega_1 \setminus E_1$ due to (2.15c). In view of Remark 2.35 there exists $\beta < p$ with $1/q + 1/\zeta + 1/\beta = 1$ so that we infer

$$-\int_\Omega \lambda\bar{\mu} \,\mathrm{d}x \leq \left\|\lambda - \bar{\lambda}\right\|_{L^q(\Omega)} \|\bar{\mu}\|_{L^\zeta(\Omega)} |\Omega_1 \setminus E_1|^{\frac{1}{\beta}}.$$

From the definition of $\mathcal{A}_1$ and (2.21c) it follows

$$\begin{aligned} |\Omega_1 \setminus E_1| &= \frac{1}{\kappa_1^p} \int_{\Omega_1 \setminus E_1} \kappa_1^p \,\mathrm{d}x \\ &< \frac{1}{\kappa_1^p} \int_{\Omega_1 \setminus E_1} \left|\phi(\bar{\boldsymbol{\Sigma}})\right|^p \,\mathrm{d}x = \frac{1}{\kappa_1^p} \int_{\Omega_1 \setminus E_1} \left|\phi(\boldsymbol{\Sigma}) - \phi(\bar{\boldsymbol{\Sigma}})\right|^p \,\mathrm{d}x. \end{aligned}$$

Because of Lemma 2.31, Remark 2.30(i) and (ii), we consequently arrive at

$$
\begin{aligned}
-\int_\Omega \lambda\bar{\mu}\,\mathrm{d}x &< c\left\|\lambda-\bar{\lambda}\right\|_{L^q(\Omega)}\|\bar{\mu}\|_{L^\zeta(\Omega)}\left\|\phi(\boldsymbol{\Sigma})-\phi(\bar{\boldsymbol{\Sigma}})\right\|^{\frac{p}{\beta}}_{L^p(\Omega)}\\
&\le c\left\|\lambda-\bar{\lambda}\right\|_{L^q(\Omega)}\|\bar{\mu}\|_{L^\zeta(\Omega)}\left\|\boldsymbol{\Sigma}-\bar{\boldsymbol{\Sigma}}\right\|^{\frac{p}{\beta}}_{L^p(\Omega;\mathbb{S}^2)}\\
&\le c\left\|\boldsymbol{f}-\bar{\boldsymbol{f}}\right\|_U^{1+\frac{p}{\beta}}
\end{aligned}
\tag{2.67}
$$

with $\nu_1 := 1+p/\beta > 2$. If we define $\Omega_2 := \{x\in\Omega : \phi(\boldsymbol{\Sigma}(x)) < 0\}$ and $E_2 := \Omega_2\cap\mathcal{A}_2$, then we obtain

$$
\begin{aligned}
\int_\Omega \phi(\boldsymbol{\Sigma})\bar{\theta}\,\mathrm{d}x &= \int_{E_2}\underbrace{\phi(\boldsymbol{\Sigma})}_{<0}\underbrace{\bar{\theta}}_{\ge 0}\,\mathrm{d}x + \int_{\Omega_2\setminus E_2}\phi(\boldsymbol{\Sigma})\bar{\theta}\,\mathrm{d}x\\
&\le \int_{\Omega_2\setminus E_2}\big(\phi(\boldsymbol{\Sigma})-\underbrace{\phi(\bar{\boldsymbol{\Sigma}})}_{=0}\big)\bar{\theta}\,\mathrm{d}x\\
&\le \left\|\phi(\boldsymbol{\Sigma})-\phi(\bar{\boldsymbol{\Sigma}})\right\|_{L^p(\Omega)}\left\|\bar{\theta}\right\|_{L^r(\Omega)}|\Omega_2\setminus E_2|^{\frac{1}{\gamma}},
\end{aligned}
$$

where the existence of $\gamma < q$ with $1/p+1/r+1/\gamma = 1$ was used. The definition of $\mathcal{A}_2$ and (2.21c) lead to

$$
\begin{aligned}
|\Omega_2\setminus E_2| &= \frac{1}{\kappa_2^q}\int_{\Omega_2\setminus E_2}\kappa_2^q\,\mathrm{d}x\\
&< \frac{1}{\kappa_2^q}\int_{\Omega_2\setminus E_2}\left|\bar{\lambda}\right|^q\,\mathrm{d}x = \frac{1}{\kappa_2^q}\int_{\Omega_2\setminus E_2}\left|\lambda-\bar{\lambda}\right|^q\,\mathrm{d}x.
\end{aligned}
$$

Thanks to Lemma 2.31, Remark 2.30(i) and (ii) it can be shown analogously to (2.67) that

$$
\int_\Omega \phi(\boldsymbol{\Sigma})\bar{\theta}\,\mathrm{d}x \le c\left\|\boldsymbol{f}-\bar{\boldsymbol{f}}\right\|_U^{1+\frac{q}{\gamma}}
$$

with $\nu_2 := 1+q/\gamma > 2$. □

Now we are ready to prove the first version of second-order sufficient optimality conditions for $(\mathbf{P}_E)$.

Theorem 2.36. *Let* $\bar{\boldsymbol{f}}\in U$ *and* $(\bar{\boldsymbol{\Sigma}},\bar{\boldsymbol{u}},\bar{\lambda})\in S^2\times V\times L^2(\Omega)$ *be the associated state and multiplier. Suppose further that there exist an adjoint state* $(\bar{\boldsymbol{\Upsilon}},\bar{\boldsymbol{w}})\in S^2\times V$ *and multipliers* $(\bar{\mu},\bar{\theta})\in L^2(\Omega)\times L^2(\Omega)$ *satisfying the following conditions:*

(1) $\bar{\boldsymbol{\chi}}\in L^s(\Omega;\mathbb{S})$, $\bar{\boldsymbol{\Upsilon}}\in L^\eta\big(\Omega;\mathbb{S}^2\big)$, $\bar{\mu}\in L^\zeta(\Omega)$ *and* $\bar{\theta}\in L^r(\Omega)$ *with*

$$
s > \frac{2p}{p-2},\quad \eta,\zeta,r > \frac{sp}{sp-p-2s}
\tag{2.68}
$$

and p *as defined in* (2.64).

(2) $(\bar{\boldsymbol{\Upsilon}}, \bar{\boldsymbol{w}})$ *and* $(\bar{\mu}, \bar{\theta})$ *solve*

$$
\begin{aligned}
A\bar{\boldsymbol{\Upsilon}} + \operatorname{div}^* \bar{\boldsymbol{w}} + \bar{\lambda}\mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Upsilon}} + \bar{\theta}\mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Sigma}} &= \mathbf{0} && \text{(2.69a)}\\
\operatorname{div} \bar{\boldsymbol{\Upsilon}} + \partial_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) &= \mathbf{0} && \text{(2.69b)}\\
R^*\bar{\boldsymbol{w}} - \partial_{\boldsymbol{f}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) &= \mathbf{0} && \text{(2.69c)}\\
\mathcal{D}\bar{\boldsymbol{\Sigma}} : \mathcal{D}\bar{\boldsymbol{\Upsilon}} &= \bar{\mu} && \text{(2.69d)}\\
\bar{\mu}\bar{\lambda} &= 0 \qquad \textit{a.e. in } \Omega && \text{(2.69e)}\\
\bar{\theta}\phi(\bar{\boldsymbol{\Sigma}}) &= 0 \qquad \textit{a.e. in } \Omega && \text{(2.69f)}\\
\bar{\mu} &\geq 0 \qquad \textit{a.e. in } \mathcal{A}_1 && \text{(2.69g)}\\
\bar{\theta} &\geq 0 \qquad \textit{a.e. in } \mathcal{A}_2, && \text{(2.69h)}
\end{aligned}
$$

where the sets $\mathcal{A}_1$ *and* $\mathcal{A}_1$ *are defined through*

$$
\begin{aligned}
\mathcal{A}_1 &= \{x \in \Omega : -\kappa_1 \leq \phi(\bar{\boldsymbol{\Sigma}}) \leq 0\}\\
\mathcal{A}_2 &= \{x \in \Omega : 0 \leq \bar{\lambda} \leq \kappa_2\}
\end{aligned}
$$

for some $\kappa_1, \kappa_2 > 0$*.*

(3) There is an $\alpha > 0$ *such that*

$$
\partial^2_{(\boldsymbol{\Sigma},\boldsymbol{u},\lambda,\boldsymbol{f})}\mathcal{L}\big(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}}, \bar{\lambda}, \bar{\boldsymbol{f}}, \bar{\boldsymbol{\Upsilon}}, \bar{\boldsymbol{w}}, \bar{\mu}, \bar{\theta}\big)\big(\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda', \boldsymbol{h}\big)^2 \geq \alpha \,\|\boldsymbol{h}\|_U^2 \qquad \text{(SSC)}
$$

for all $\boldsymbol{h} \in U$ *and* $(\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda')$ *solving* (2.18) *with* $\delta\ell = R\boldsymbol{h}$*.*

Then there exists an $\epsilon > 0$ *such that the quadratic growth condition*

$$
J(\boldsymbol{u}, \boldsymbol{f}) \geq J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) + \frac{\alpha}{4}\left\|\boldsymbol{f} - \bar{\boldsymbol{f}}\right\|_U^2 \tag{2.70}
$$

is fulfilled for all $\boldsymbol{f} \in U$ *with* $\left\|\boldsymbol{f} - \bar{\boldsymbol{f}}\right\|_U \leq \epsilon$*. Thus,* $\bar{\boldsymbol{f}}$ *is a strict local optimum of* $(\mathbf{P}_E)$*.*

Remark 2.37. *The Lagrangian* $\mathcal{L}$ *is twice continuously differentiable in its space of definition by Proposition 2.33. However, if the dual variables* $(\bar{\boldsymbol{\Upsilon}}, \bar{\theta})$ *fulfill the integrability conditions in* (2.68)*, the second derivative of* $\mathcal{L}$ *w.r.t. the primal variables* $(\boldsymbol{\Sigma}, \boldsymbol{u}, \lambda, \boldsymbol{g})$ *is given by*

$$
\begin{aligned}
&\partial^2_{(\boldsymbol{\Sigma},\boldsymbol{u},\lambda,\boldsymbol{f})}\mathcal{L}\big(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}}, \bar{\lambda}, \bar{\boldsymbol{f}}, \bar{\boldsymbol{\Upsilon}}, \bar{\boldsymbol{w}}, \bar{\mu}, \bar{\theta}\big)(\delta\boldsymbol{\Sigma}, \delta\boldsymbol{u}, \delta\lambda, \delta\boldsymbol{f})^2 = \\
&= \nabla^2_{(\boldsymbol{u},\boldsymbol{f})} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})(\delta\boldsymbol{u}, \delta\boldsymbol{f})^2 + 2\int_\Omega \delta\lambda \mathcal{D}\delta\boldsymbol{\Sigma} : \mathcal{D}\bar{\boldsymbol{\Upsilon}}\,dx + \int_\Omega \|\mathcal{D}\delta\boldsymbol{\Sigma}\|_{\mathbb{S}}^2\,\bar{\theta}\,dx
\end{aligned}
\tag{2.71}
$$

and defines a continuous bilinear form on the space $L^p\big(\Omega; \mathbb{S}^2\big) \times V \times L^q(\Omega) \times U$*, cf. Remark 2.35. Hence, the left-hand side in* (SSC) *is well-defined according to Remark 2.30(iii) and (iv). Hereafter we denote the integrals in* (2.71) *as* L^2*-scalar products by* $(\cdot, \cdot)$*.*

Remark 2.38. *Let us compare the sufficient optimality conditions in Theorem 2.36 with their finite-dimensional counterpart. In [69, Theorem 7] Scheel and Scholtes proved that*

- *strong stationarity ($\widehat{=}$ necessary conditions for local optimality)*
- *coercivity of the Hessian (w.r.t. the primal variables) of the Lagrangian on the cone of critical directions*

guarantee local optimality for a mathematical program with complementarity constraints (MPCC) in finite dimensions. In case of the infinite-dimensional MPCC $(\mathbf{P}_E)$ *Herzog et al. proved that C-stationarity conditions are necessary for local optimality, see [43, Theorem 3.16]. These conditions coincide with* (2.69)*, except that they claim a sign condition only for the product of the multipliers $\bar{\mu}$ and $\bar{\theta}$ in contrast to* (2.69g) *and* (2.69h)*. To the best knowledge of the author, necessary conditions of strongly stationary type can be established solely if an additional and physically meaningless control variable is introduced as inhomogeneity in* (2.5a)*, cf. [44, Theorem 4.5]. A strongly-stationary-like system for* $(\mathbf{P}_E)$ *would coincide with* (2.69)*, except that the individual sign conditions for the multipliers would hold on the biactive set $\bar{\mathcal{B}}$. The sets $\mathcal{A}_1$ and $\mathcal{A}_2$ involved in* (2.69g) *and* (2.69h)*, respectively, are even larger than $\bar{\mathcal{B}}$. Moreover, Theorem 2.36 requires higher integrability of the hardening variable $\bar{\boldsymbol{\chi}}$, the adjoint variable $\bar{\boldsymbol{\Upsilon}}$ and the multipliers $\bar{\mu}$ and $\bar{\theta}$, cf.* (2.68)*. Thus, we observe a significant gap between the necessary and the sufficient optimality conditions for* $(\mathbf{P}_E)$*. In addition, the set of directions, for which the coercivity* (SSC) *has to be fulfilled, is larger than the cone of critical directions. In the context of Problem* $(\mathbf{P}_E)$*, this cone consists of all directions $\big(\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda', \boldsymbol{h}\big)$, which solve* (2.18) *and satisfy* $J'(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})(\boldsymbol{u}', \boldsymbol{h}) = 0$.
To sum up, the sufficient conditions in Theorem 2.36 are quite restrictive in comparison with the sufficient conditions for finite dimensional MPCCs listed above. Later on we will improve the assertion of Theorem 2.36 and obtain conditions, which are more competitive with [69, Theorem 7], provided that the objective functional has a certain structure, cf. Theorem 2.41.

Proof of Theorem 2.36. At first we note that Assumption 2.13 is satisfied because of $s > 2p/(p-2)$ and (2.64). Therefore, the results of Section 2.2, enumerated in Remark 2.30, can be employed. Let $\boldsymbol{f} \in U$ be arbitrary with $\boldsymbol{f} \neq \bar{\boldsymbol{f}}$ and $(\boldsymbol{\Sigma}, \boldsymbol{u}, \lambda) \in S^2_\infty \times V \times L^2(\Omega)$ be the state and multiplier associated with $\boldsymbol{f}$.
We aim to deduce the quadratic growth condition (2.70) from a Taylor expansion of the Lagrangian. For this purpose we introduce the abbreviations

$$\boldsymbol{z} := (\boldsymbol{\Sigma}, \boldsymbol{u}, \lambda, \boldsymbol{f}), \qquad \bar{\boldsymbol{z}} := \big(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}}, \bar{\lambda}, \bar{\boldsymbol{f}}\big), \qquad \bar{\boldsymbol{\omega}} := \big(\bar{\boldsymbol{\Upsilon}}, \bar{\boldsymbol{w}}, \bar{\mu}, \bar{\theta}\big).$$

As $\mathcal{L}$ is twice continuously differentiable in its space of definition, cf. Proposition

2.33, there is a $t \in [0,1]$ such that

$$\mathcal{L}(\boldsymbol{z},\bar{\boldsymbol{\omega}}) = \mathcal{L}(\bar{\boldsymbol{z}},\bar{\boldsymbol{\omega}}) + \nabla_{\boldsymbol{z}}\mathcal{L}(\bar{\boldsymbol{z}},\bar{\boldsymbol{\omega}})(\boldsymbol{z}-\bar{\boldsymbol{z}}) + \frac{1}{2}\nabla^2_{\boldsymbol{z}}\mathcal{L}(\boldsymbol{z}_t,\bar{\boldsymbol{\omega}})(\boldsymbol{z}-\bar{\boldsymbol{z}})^2 \tag{2.72}$$

with $\boldsymbol{z}_t := \bar{\boldsymbol{z}} + t(\boldsymbol{z}-\bar{\boldsymbol{z}})$. In the following we discuss each expression in (2.72) separately.

The zero-order terms:

On account of (2.15a), (2.15b), (2.69e) and (2.69f) we observe

$$\begin{aligned}\mathcal{L}(\bar{\boldsymbol{z}},\bar{\boldsymbol{\omega}}) &= J(\bar{\boldsymbol{u}},\bar{\boldsymbol{f}}) + \underbrace{\left(A\bar{\boldsymbol{\Sigma}} + \operatorname{div}^*\bar{\boldsymbol{u}} + \bar{\lambda}\mathcal{D}^\star\mathcal{D}\bar{\boldsymbol{\Sigma}},\, \bar{\boldsymbol{\Upsilon}}\right)}_{=0} + \underbrace{\left\langle \operatorname{div}\bar{\boldsymbol{\Sigma}} - R\bar{\boldsymbol{f}},\, \bar{\boldsymbol{w}}\right\rangle}_{=0} \\ &\qquad - \underbrace{\left(\bar{\lambda},\, \bar{\mu}\right)_{L^2(\Omega)}}_{=0} + \underbrace{\left(\phi(\bar{\boldsymbol{\Sigma}}),\, \bar{\theta}\right)_{L^2(\Omega)}}_{=0} \\ &= J(\bar{\boldsymbol{u}},\bar{\boldsymbol{f}}).\end{aligned} \tag{2.73}$$

In addition, (2.21a) and (2.21b) show

$$\begin{aligned}\mathcal{L}(\boldsymbol{z},\bar{\boldsymbol{\omega}}) &= J(\boldsymbol{u},\boldsymbol{f}) + \underbrace{\left(A\boldsymbol{\Sigma} + \operatorname{div}^*\boldsymbol{u} + \lambda\mathcal{D}^\star\mathcal{D}\boldsymbol{\Sigma},\, \bar{\boldsymbol{\Upsilon}}\right)}_{=0} + \underbrace{\left\langle \operatorname{div}\boldsymbol{\Sigma} - R\boldsymbol{f},\, \bar{\boldsymbol{w}}\right\rangle}_{=0} \\ &\qquad - (\lambda,\, \bar{\mu})_{L^2(\Omega)} + \left(\phi(\boldsymbol{\Sigma}),\, \bar{\theta}\right)_{L^2(\Omega)} \\ &= J(\boldsymbol{u},\boldsymbol{f}) - (\lambda,\, \bar{\mu})_{L^2(\Omega)} + \left(\phi(\boldsymbol{\Sigma}),\, \bar{\theta}\right)_{L^2(\Omega)}\end{aligned} \tag{2.74}$$

so that Lemma 2.34 leads to

$$\mathcal{L}(\boldsymbol{z},\bar{\boldsymbol{\omega}}) \le J(\boldsymbol{u},\boldsymbol{f}) + o\left(\left\|\boldsymbol{f}-\bar{\boldsymbol{f}}\right\|^2_U\right). \tag{2.75}$$

The first-order term:

The first derivative of $\mathcal{L}$ at $\bar{\boldsymbol{z}}$ in direction $(\boldsymbol{z}-\bar{\boldsymbol{z}})$ is given by

$$\begin{aligned}&\nabla_{\boldsymbol{z}}\mathcal{L}(\bar{\boldsymbol{z}},\bar{\boldsymbol{z}}')(\boldsymbol{z}-\bar{\boldsymbol{z}}) = \\ &= \left\langle \partial_{\boldsymbol{u}}J(\bar{\boldsymbol{u}},\bar{\boldsymbol{f}}),\, \boldsymbol{u}-\bar{\boldsymbol{u}}\right\rangle + \left(\partial_{\boldsymbol{f}}J(\bar{\boldsymbol{u}},\bar{\boldsymbol{f}}),\, \boldsymbol{f}-\bar{\boldsymbol{f}}\right) + \left(A(\boldsymbol{\Sigma}-\bar{\boldsymbol{\Sigma}}),\, \bar{\boldsymbol{\Upsilon}}\right) \\ &\quad + \left(\operatorname{div}^*(\boldsymbol{u}-\bar{\boldsymbol{u}}),\, \bar{\boldsymbol{\Upsilon}}\right) + \left((\lambda-\bar{\lambda})\mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Sigma}},\, \bar{\boldsymbol{\Upsilon}}\right) + \left(\bar{\lambda}\mathcal{D}^*\mathcal{D}(\boldsymbol{\Sigma}-\bar{\boldsymbol{\Sigma}}),\, \bar{\boldsymbol{\Upsilon}}\right) \\ &\quad + \left\langle \operatorname{div}(\boldsymbol{\Sigma}-\bar{\boldsymbol{\Sigma}}),\, \bar{\boldsymbol{w}}\right\rangle - \left\langle R(\boldsymbol{f}-\bar{\boldsymbol{f}}),\, \bar{\boldsymbol{w}}\right\rangle - \left(\lambda-\bar{\lambda},\, \bar{\mu}\right) + \left(\mathcal{D}\bar{\boldsymbol{\Sigma}} : \mathcal{D}(\boldsymbol{\Sigma}-\bar{\boldsymbol{\Sigma}}),\, \bar{\theta}\right) \\ &= \left(A\bar{\boldsymbol{\Upsilon}} + \operatorname{div}^*\bar{\boldsymbol{w}} + \bar{\lambda}\mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Upsilon}} + \bar{\theta}\mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Sigma}},\, \boldsymbol{\Sigma}-\bar{\boldsymbol{\Sigma}}\right) + \left\langle \operatorname{div}\bar{\boldsymbol{\Upsilon}} + \partial_{\boldsymbol{u}}J(\bar{\boldsymbol{u}},\bar{\boldsymbol{f}}),\, \boldsymbol{u}-\bar{\boldsymbol{u}}\right\rangle \\ &\quad + \left(\partial_{\boldsymbol{f}}J(\bar{\boldsymbol{u}},\bar{\boldsymbol{f}}) - R^*\bar{\boldsymbol{w}},\, \boldsymbol{f}-\bar{\boldsymbol{f}}\right) - \left(\lambda-\bar{\lambda},\, \bar{\mu}\right) + \left(\lambda-\bar{\lambda},\, \mathcal{D}\bar{\boldsymbol{\Sigma}} : \mathcal{D}\bar{\boldsymbol{\Upsilon}}\right).\end{aligned}$$

From (2.69a)–(2.69d) we thus conclude

$$\nabla_{\boldsymbol{z}}\mathcal{L}(\bar{\boldsymbol{z}},\bar{\boldsymbol{z}}')(\boldsymbol{z}-\bar{\boldsymbol{z}}) = 0. \tag{2.76}$$

The second-order term:

In view of Remark 2.37 it is already known that

$$\begin{aligned}&\nabla^2_{\boldsymbol{z}}\mathcal{L}(\boldsymbol{z}_t,\bar{\boldsymbol{\omega}})(\boldsymbol{z}-\bar{\boldsymbol{z}})^2 = \\ &= \nabla^2_{(\boldsymbol{u},\boldsymbol{f})}J(\boldsymbol{u}_t,\boldsymbol{f}_t)(\boldsymbol{u}-\bar{\boldsymbol{u}},\boldsymbol{f}-\bar{\boldsymbol{f}})^2 + 2\left((\lambda-\bar{\lambda})\mathcal{D}^*\mathcal{D}(\boldsymbol{\Sigma}-\bar{\boldsymbol{\Sigma}}),\, \bar{\boldsymbol{\Upsilon}}\right) \\ &\quad + \left(\left\|\mathcal{D}\boldsymbol{\Sigma}-\mathcal{D}\bar{\boldsymbol{\Sigma}}\right\|^2_{\mathbb{S}},\, \bar{\theta}\right).\end{aligned} \tag{2.77}$$

Since $\nabla^2_{(\boldsymbol{u},\boldsymbol{f})} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})[\cdot,\cdot]$ defines a bilinear function on $V \times U$, we find

$$
\begin{aligned}
&\nabla^2_{(\boldsymbol{u},\boldsymbol{f})} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})(\boldsymbol{u}', \boldsymbol{f} - \bar{\boldsymbol{f}})^2 - \nabla^2_{(\boldsymbol{u},\boldsymbol{f})} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})(\boldsymbol{u} - \bar{\boldsymbol{u}}, \boldsymbol{f} - \bar{\boldsymbol{f}})^2 = \\
= &\partial^2_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})[\boldsymbol{u}', \boldsymbol{u}'] - \partial^2_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})[\boldsymbol{u} - \bar{\boldsymbol{u}}, \boldsymbol{u} - \bar{\boldsymbol{u}}] - 2\partial_{\boldsymbol{f}}\partial_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})[\boldsymbol{u} - \bar{\boldsymbol{u}} - \boldsymbol{u}', \boldsymbol{f} - \bar{\boldsymbol{f}}] \\
= & - \partial^2_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})[\boldsymbol{u} - \bar{\boldsymbol{u}} - \boldsymbol{u}', \boldsymbol{u}'] - \partial^2_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})[\boldsymbol{u} - \bar{\boldsymbol{u}}, \boldsymbol{u} - \bar{\boldsymbol{u}} - \boldsymbol{u}'] \\
&- 2\partial_{\boldsymbol{f}}\partial_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})[\boldsymbol{u} - \bar{\boldsymbol{u}} - \boldsymbol{u}', \boldsymbol{f} - \bar{\boldsymbol{f}}] \\
= & - \partial^2_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})[\boldsymbol{u} - \bar{\boldsymbol{u}} - \boldsymbol{u}', \boldsymbol{u} - \bar{\boldsymbol{u}} - \boldsymbol{u}'] - 2\partial^2_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})[\boldsymbol{u} - \bar{\boldsymbol{u}} - \boldsymbol{u}', \boldsymbol{u}'] \\
&- 2\partial_{\boldsymbol{f}}\partial_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})[\boldsymbol{u} - \bar{\boldsymbol{u}} - \boldsymbol{u}', \boldsymbol{f} - \bar{\boldsymbol{f}}].
\end{aligned}
$$

Consequently, equation (2.77) can be rewritten as

$$
\begin{aligned}
&\nabla^2_{\boldsymbol{z}} \mathcal{L}(\boldsymbol{z}_t, \bar{\boldsymbol{\omega}})(\boldsymbol{z} - \bar{\boldsymbol{z}})^2 = \\
= &\underbrace{\nabla^2_{(\boldsymbol{u},\boldsymbol{f})} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})(\boldsymbol{u}', \boldsymbol{f} - \bar{\boldsymbol{f}})^2}_{=:D_1} + \Big(\nabla^2_{(\boldsymbol{u},\boldsymbol{f})} J(\boldsymbol{u}_t, \boldsymbol{f}_t) - \nabla^2_{(\boldsymbol{u},\boldsymbol{f})} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})\Big) (\boldsymbol{u} - \bar{\boldsymbol{u}}, \boldsymbol{f} - \bar{\boldsymbol{f}})^2 \\
&+ 2\partial_{\boldsymbol{f}}\partial_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})[\boldsymbol{u} - \bar{\boldsymbol{u}} - \boldsymbol{u}', \boldsymbol{f} - \bar{\boldsymbol{f}}] + \partial^2_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})(\boldsymbol{u} - \bar{\boldsymbol{u}} - \boldsymbol{u}')^2 \\
&+ 2\partial^2_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})[\boldsymbol{u} - \bar{\boldsymbol{u}} - \boldsymbol{u}', \boldsymbol{u}'] + 2\left((\lambda - \bar{\lambda})\mathcal{D}^*\mathcal{D}(\boldsymbol{\Sigma} - \bar{\boldsymbol{\Sigma}}),\, \bar{\boldsymbol{\Upsilon}}\right) \\
&+ \Big(\left\|\mathcal{D}\boldsymbol{\Sigma} - \mathcal{D}\bar{\boldsymbol{\Sigma}}\right\|^2_{\mathbb{S}},\, \bar{\theta}\Big).
\end{aligned}
$$

Next we derive estimates for the last two expressions. Thanks to the regularity condition (2.68) there exist $q < \beta < p$ and $\gamma < q$ such that $1/q + 1/\beta + 1/\eta < 1$ and $1/\gamma + 1/p + 1/\eta < 1$, cf. Remark 2.35. In view of Remark 2.30(ii) and (iii) we deduce

$$
\begin{aligned}
&\left((\lambda - \bar{\lambda})(\mathcal{D}\boldsymbol{\Sigma} - \mathcal{D}\bar{\boldsymbol{\Sigma}}),\, \mathcal{D}\bar{\boldsymbol{\Upsilon}}\right) = \\
= &\underbrace{\left(\lambda'\mathcal{D}\boldsymbol{\Sigma}',\, \mathcal{D}\bar{\boldsymbol{\Upsilon}}\right)}_{=:D_2} + \left((\lambda - \bar{\lambda})(\mathcal{D}\boldsymbol{\Sigma} - \mathcal{D}\bar{\boldsymbol{\Sigma}} - \mathcal{D}\boldsymbol{\Sigma}'),\, \mathcal{D}\bar{\boldsymbol{\Upsilon}}\right) + \left((\lambda - \bar{\lambda} - \lambda')\mathcal{D}\boldsymbol{\Sigma}',\, \mathcal{D}\bar{\boldsymbol{\Upsilon}}\right) \\
\geq &\left(\lambda'\mathcal{D}\boldsymbol{\Sigma}',\, \mathcal{D}\bar{\boldsymbol{\Upsilon}}\right) - c\left\|\lambda - \bar{\lambda}\right\|_{L^q(\Omega)} \left\|\boldsymbol{\Sigma} - \bar{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}'\right\|_{L^\beta(\Omega;\mathbb{S}^2)} \left\|\bar{\boldsymbol{\Upsilon}}\right\|_{L^\eta(\Omega;\mathbb{S}^2)} \\
&- c\left\|\lambda - \bar{\lambda} - \lambda'\right\|_{L^\gamma(\Omega)} \left\|\boldsymbol{\Sigma}'\right\|_{L^p(\Omega;\mathbb{S}^2)} \left\|\bar{\boldsymbol{\Upsilon}}\right\|_{L^\eta(\Omega;\mathbb{S}^2)} \\
\geq &\left(\lambda'\mathcal{D}\boldsymbol{\Sigma}',\, \mathcal{D}\bar{\boldsymbol{\Upsilon}}\right) - c\left\|\boldsymbol{f} - \bar{\boldsymbol{f}}\right\|_U \Big(\left\|\boldsymbol{\Sigma} - \bar{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}'\right\|_{L^\beta(\Omega;\mathbb{S}^2)} + \left\|\lambda - \bar{\lambda} - \lambda'\right\|_{L^\gamma(\Omega)}\Big).
\end{aligned}
$$

Besides, (2.68) yields $1/p + 1/\beta + 1/r < 1$, which implies

$$
\begin{aligned}
&\Big(\left\|\mathcal{D}\boldsymbol{\Sigma} - \mathcal{D}\bar{\boldsymbol{\Sigma}}\right\|^2_{\mathbb{S}},\, \bar{\theta}\Big) = \\
= &\underbrace{\Big(\left\|\mathcal{D}\boldsymbol{\Sigma}'\right\|^2_{\mathbb{S}},\, \bar{\theta}\Big)}_{=:D_3} + \Big(\left\|\mathcal{D}\boldsymbol{\Sigma} - \mathcal{D}\bar{\boldsymbol{\Sigma}} - \mathcal{D}\boldsymbol{\Sigma}'\right\|^2_{\mathbb{S}},\, \bar{\theta}\Big) + 2\left((\mathcal{D}\boldsymbol{\Sigma} - \mathcal{D}\bar{\boldsymbol{\Sigma}} - \mathcal{D}\boldsymbol{\Sigma}') : \mathcal{D}\boldsymbol{\Sigma}',\, \bar{\theta}\right) \\
\geq &\Big(\left\|\mathcal{D}\boldsymbol{\Sigma}'\right\|^2_{\mathbb{S}},\, \bar{\theta}\Big) - c\left\|\boldsymbol{\Sigma} - \bar{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}'\right\|_{L^p(\Omega;\mathbb{S})} \left\|\boldsymbol{\Sigma} - \bar{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}'\right\|_{L^\beta(\Omega;\mathbb{S}^2)} \left\|\bar{\theta}\right\|_{L^r(\Omega)} \\
&- c\left\|\boldsymbol{\Sigma}'\right\|_{L^p(\Omega;\mathbb{S})} \left\|\boldsymbol{\Sigma} - \bar{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}'\right\|_{L^\beta(\Omega;\mathbb{S}^2)} \left\|\bar{\theta}\right\|_{L^r(\Omega)} \\
\geq &\Big(\left\|\mathcal{D}\boldsymbol{\Sigma}'\right\|^2_{\mathbb{S}},\, \bar{\theta}\Big) - c\left\|\boldsymbol{f} - \bar{\boldsymbol{f}}\right\|_U \left\|\boldsymbol{\Sigma} - \bar{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}'\right\|_{L^\beta(\Omega;\mathbb{S}^2)}
\end{aligned}
$$

according to Remark 2.30(i) and (iii). Recalling again Remark 2.37, we furthermore note

$$D_1 + 2D_2 + D_3 = \nabla_{\boldsymbol{z}}^2 \mathcal{L}(\bar{\boldsymbol{z}}, \bar{\boldsymbol{\omega}})\big(\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda', \boldsymbol{f} - \bar{\boldsymbol{f}}\big)^2.$$

Thus, because of Remark 2.30(iii)–(vi) and (SSC) we obtain

$$\begin{aligned}
&\nabla_{\boldsymbol{z}}^2 \mathcal{L}(\boldsymbol{z}_t, \bar{\boldsymbol{\omega}})(\boldsymbol{z} - \bar{\boldsymbol{z}})^2 = \\
&\geq \nabla_{\boldsymbol{z}}^2 \mathcal{L}(\bar{\boldsymbol{z}}, \bar{\boldsymbol{\omega}})\big(\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda', \boldsymbol{f} - \bar{\boldsymbol{f}}\big)^2 \\
&\quad - c \left\| \nabla_{(\boldsymbol{u},\boldsymbol{f})}^2 J(\boldsymbol{u}_t, \boldsymbol{f}_t) - \nabla_{(\boldsymbol{u},\boldsymbol{f})}^2 J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) \right\|_{\mathcal{L}(V\times U,(V\times U)')} \left\| \boldsymbol{f} - \bar{\boldsymbol{f}} \right\|_U^2 \\
&\quad - c \left\| \partial_{\boldsymbol{f}} \partial_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) \right\|_{\mathcal{L}(U,V')} \left\| \boldsymbol{u} - \bar{\boldsymbol{u}} - \boldsymbol{u}' \right\|_V \left\| \boldsymbol{f} - \bar{\boldsymbol{f}} \right\|_U \\
&\quad - c \left\| \partial_{\boldsymbol{u}}^2 J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) \right\|_{\mathcal{L}(V,V')} \left\| \boldsymbol{u} - \bar{\boldsymbol{u}} - \boldsymbol{u}' \right\|_V^2 \\
&\quad - c \left\| \partial_{\boldsymbol{u}}^2 J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) \right\|_{\mathcal{L}(V,V')} \left\| \boldsymbol{u} - \bar{\boldsymbol{u}} - \boldsymbol{u}' \right\|_V \left\| \boldsymbol{u}' \right\|_V \\
&\quad - c \left\| \boldsymbol{f} - \bar{\boldsymbol{f}} \right\|_U \Big(\left\| \boldsymbol{\Sigma} - \bar{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}' \right\|_{L^\beta(\Omega;\mathbb{S}^2)} + \left\| \lambda - \bar{\lambda} - \lambda' \right\|_{L^\gamma(\Omega)} \Big) \\
&\geq \alpha \left\| \boldsymbol{f} - \bar{\boldsymbol{f}} \right\|_U^2 - o\Big(\left\| \boldsymbol{f} - \bar{\boldsymbol{f}} \right\|_U^2 \Big),
\end{aligned} \tag{2.78}$$

where we used that J is twice continuously differentiable by Assumption 2.29. From (2.72), (2.73) and (2.75)–(2.78) it follows

$$\begin{aligned}
J(\boldsymbol{u}, \boldsymbol{f}) &\geq J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) + \frac{\alpha}{2} \left\| \boldsymbol{f} - \bar{\boldsymbol{f}} \right\|_U^2 - o\Big(\left\| \boldsymbol{f} - \bar{\boldsymbol{f}} \right\|_U^2 \Big) \\
&= J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) + \left(\frac{\alpha}{2} - \frac{o\Big(\left\| \boldsymbol{f} - \bar{\boldsymbol{f}} \right\|_U^2 \Big)}{\left\| \boldsymbol{f} - \bar{\boldsymbol{f}} \right\|_U^2} \right) \left\| \boldsymbol{f} - \bar{\boldsymbol{f}} \right\|_U^2 .
\end{aligned}$$

Hence there exists $\epsilon > 0$ such that

$$J(\boldsymbol{u}, \boldsymbol{f}) \geq J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) + \frac{\alpha}{4} \left\| \boldsymbol{f} - \bar{\boldsymbol{f}} \right\|_U^2 \quad \forall\, \boldsymbol{f} \in U \text{ with } \left\| \boldsymbol{f} - \bar{\boldsymbol{f}} \right\|_U \leq \epsilon,$$

which is the desired quadratic growth condition. □

Remark 2.39. *For every $p > 2$, there are numbers $s, \eta, \zeta, r \in [2, \infty[$ satisfying the regularity conditions* (2.68)*. However, if p tends to* 2*, then the bounds for s, η, ζ, and r tend to ∞, cf. also Remark 2.14(ii) and Remark 2.35. As $p > 2p/(p-2)$ for all $p > 4$, the assumption for the integrability exponent s in* (2.68) *is automatically fulfilled in case $p > 4$. In view of Sobolev's embedding theorem such high integrability cannot be expected for controls in $U = L^2\big(\Omega; \mathbb{R}^d\big) \times L^2\big(\Gamma_N; \mathbb{R}^d\big)$ but for controls in $L^{\nu_1}\big(\Omega; \mathbb{R}^d\big) \times L^{\nu_2}\big(\Gamma_N; \mathbb{R}^d\big)$ with ν_1, ν_2 sufficiently large, cf. Remark 2.14(i), provided that the problem data are smooth enough.*

If the objective functional provides a particular structure, we are able to relax the second-order condition (SSC). To be more precise, we can restrict the coercivity of the Hessian of $\mathcal{L}$ to the cone of critical directions so that (SSC) becomes a classical condition, cf. [69, Theorem 7]. Moreover we are allowed to weaken the integrability requirement for the hardening variable $\bar{\boldsymbol{\chi}}$ in (2.68).

As in Theorem 2.36 let $\bar{\boldsymbol{f}} \in U$ and $(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}}, \bar{\lambda})$ be the associated state and multiplier. For the rest of this section we assume that J fulfills the next assumption.

Assumption 2.40. *The objective functional $J\colon V \times U \to \mathbb{R}$ is given by $J(\boldsymbol{u}, \boldsymbol{f}) = i(\boldsymbol{u}) + j(\boldsymbol{f})$. Moreover it holds*

i) $i\colon V \to \mathbb{R}$ is twice continuously Fréchet-differentiable

ii) $j\colon U \to \mathbb{R}$ is twice continuously Fréchet-differentiable and there exists a $\nu > 0$ with

$$j''(\bar{\boldsymbol{f}})\boldsymbol{h}^2 \geq \nu \, \|\boldsymbol{h}\|_U^2 \quad \forall \boldsymbol{h} \in U.$$

The second version of sufficient optimality conditions reads as follows.

Theorem 2.41. *Let $\bar{\boldsymbol{f}} \in U$ and $(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}}, \bar{\lambda}) \in S^2 \times V \times L^2(\Omega)$ be the associated state and multiplier. Suppose further that there are an adjoint state $(\bar{\boldsymbol{\Upsilon}}, \bar{\boldsymbol{w}}) \in S^2 \times V$ and multipliers $(\bar{\mu}, \bar{\theta}) \in L^2(\Omega) \times L^2(\Omega)$ satisfying the following conditions:*

(1) $\bar{\boldsymbol{\chi}} \in L^s(\Omega; \mathbb{S})$, $\bar{\boldsymbol{\Upsilon}} \in L^\eta\big(\Omega; \mathbb{S}^2\big)$, $\bar{\mu} \in L^\zeta(\Omega)$ and $\bar{\theta} \in L^r(\Omega)$ with

$$s > \frac{p}{p-2}, \quad \eta, \zeta, r > \frac{sp}{sp - p - 2s}, \tag{2.79}$$

and p as defined in (2.64).

(2) $(\bar{\boldsymbol{\Upsilon}}, \bar{\boldsymbol{w}})$ and $(\bar{\mu}, \bar{\theta})$ solve

$$A\bar{\boldsymbol{\Upsilon}} + \operatorname{div}^* \bar{\boldsymbol{w}} + \bar{\lambda}\mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Upsilon}} + \bar{\theta}\mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Sigma}} = \boldsymbol{0} \tag{2.80a}$$
$$\operatorname{div} \bar{\boldsymbol{\Upsilon}} + i'(\bar{\boldsymbol{u}}) = \boldsymbol{0} \tag{2.80b}$$
$$R^*\bar{\boldsymbol{w}} - j'(\bar{\boldsymbol{f}}) = \boldsymbol{0} \tag{2.80c}$$
$$\mathcal{D}\bar{\boldsymbol{\Sigma}} : \mathcal{D}\bar{\boldsymbol{\Upsilon}} = \bar{\mu} \tag{2.80d}$$
$$\bar{\mu}\bar{\lambda} = 0 \qquad \text{a.e. in } \Omega \tag{2.80e}$$
$$\bar{\theta}\phi(\bar{\boldsymbol{\Sigma}}) = 0 \qquad \text{a.e. in } \Omega \tag{2.80f}$$
$$\bar{\mu} \geq 0 \qquad \text{a.e. in } \mathcal{A}_1 \tag{2.80g}$$
$$\bar{\theta} \geq 0 \qquad \text{a.e. in } \mathcal{A}_2 \tag{2.80h}$$

where the sets $\mathcal{A}_1$ and $\mathcal{A}_1$ are defined through

$$\mathcal{A}_1 = \{x \in \Omega : -\kappa_1 \leq \phi(\bar{\boldsymbol{\Sigma}}) \leq 0\}$$
$$\mathcal{A}_2 = \{x \in \Omega : 0 \leq \bar{\lambda} \leq \kappa_2\}$$

for some $\kappa_1, \kappa_2 > 0$.

(3) There is an $\alpha > 0$ such that

$$\partial^2_{(\boldsymbol{\Sigma},\boldsymbol{u},\lambda,\boldsymbol{f})}\mathcal{L}\big(\bar{\boldsymbol{\Sigma}},\bar{\boldsymbol{u}},\bar{\lambda},\bar{\boldsymbol{f}},\bar{\boldsymbol{\Upsilon}},\bar{\boldsymbol{w}},\bar{\mu},\bar{\theta}\big)\big(\boldsymbol{\Sigma}',\boldsymbol{u}',\lambda',\boldsymbol{h}\big)^2 \geq \alpha\,\|\boldsymbol{h}\|_U^2 \qquad (\widetilde{\text{SSC}})$$

for all $\boldsymbol{h} \in U$ and $(\boldsymbol{\Sigma}',\boldsymbol{u}',\lambda')$ solving (2.18) with $\delta\ell = R\boldsymbol{h}$ such that

$$i'(\bar{\boldsymbol{u}})\boldsymbol{u}' + j'(\bar{\boldsymbol{f}})\boldsymbol{h} = 0.$$

Then there exist $\epsilon > 0$ and $\delta > 0$ such that the quadratic growth condition

$$J(\boldsymbol{u},\boldsymbol{f}) \geq J(\bar{\boldsymbol{u}},\bar{\boldsymbol{f}}) + \delta\,\big\|\boldsymbol{f}-\bar{\boldsymbol{f}}\big\|_U^2 \tag{2.81}$$

is fulfilled for all $\boldsymbol{f} \in U$ with $\big\|\boldsymbol{f}-\bar{\boldsymbol{f}}\big\|_U \leq \epsilon$. Thus, $\bar{\boldsymbol{f}}$ is a strict local optimum of $(\mathbf{P}_E)$.

Proof. The proof is similar to [55, Theorem 2.12]. We argue by contradiction and assume the opposite of the quadratic growth condition (2.81). Then there exist a sequence $\{\boldsymbol{f}_n\}_{n\in\mathbb{N}} \subset U$ with $\bar{\boldsymbol{f}} \neq \boldsymbol{f}_n \to \bar{\boldsymbol{f}}$ and by Lemma 2.15 a sequence of associated states $\{(\boldsymbol{\Sigma}_n,\boldsymbol{u}_n,\lambda_n)\}_{n\in\mathbb{N}}$ with $(\boldsymbol{\Sigma}_n,\boldsymbol{u}_n,\lambda_n) \to (\bar{\boldsymbol{\Sigma}},\bar{\boldsymbol{u}},\bar{\lambda})$ in $L^p\big(\Omega;\mathbb{S}^2\big) \times W^{1,p}_D\big(\Omega;\mathbb{R}^d\big) \times L^q(\Omega)$ satisfying

$$J(\bar{\boldsymbol{u}},\bar{\boldsymbol{f}}) + \frac{1}{n}\,\big\|\boldsymbol{f}_n-\bar{\boldsymbol{f}}\big\|_U^2 > J(\boldsymbol{u}_n,\boldsymbol{f}_n). \tag{2.82}$$

For the sake of convenience we introduce the abbreviations

$$\rho_n := \big\|\boldsymbol{f}_n-\bar{\boldsymbol{f}}\big\|_U\,,\ \boldsymbol{h}_n := \frac{\boldsymbol{f}_n-\bar{\boldsymbol{f}}}{\rho_n},\ \boldsymbol{Z}_n := \frac{\boldsymbol{\Sigma}_n-\bar{\boldsymbol{\Sigma}}}{\rho_n},\ \boldsymbol{v}_n := \frac{\boldsymbol{u}_n-\bar{\boldsymbol{u}}}{\rho_n},\ \iota_n := \frac{\lambda_n-\bar{\lambda}}{\rho_n}.$$

Because of the boundedness of $\boldsymbol{h}_n$ and the compactness of the operator R, cf. Remark 2.14(i), there is a subsequence, w.l.o.g. denoted in the same way, with

$$\boldsymbol{h}_n \rightharpoonup \boldsymbol{h} \quad \text{in } U, \qquad R\boldsymbol{h}_n \to R\boldsymbol{h} \quad \text{in } W^{-1,\beta}_D\big(\Omega;\mathbb{R}^d\big) \quad \forall\, 1 \leq \beta < p. \tag{2.83}$$

Let $(\boldsymbol{\Sigma}_{\rho_n},\boldsymbol{u}_{\rho_n},\lambda_{\rho_n})$ denote the state and multiplier associated with $\bar{\boldsymbol{f}} + \rho_n\boldsymbol{h}$ and $(\boldsymbol{\Sigma}'_{\boldsymbol{h}},\boldsymbol{u}'_{\boldsymbol{h}},\lambda'_{\boldsymbol{h}})$ denote the solution of (2.18) with right-hand side $R\boldsymbol{h}$. Then Theorem 2.8 implies

$$\begin{aligned}
&\big\|\boldsymbol{Z}_n-\boldsymbol{\Sigma}'_{\boldsymbol{h}}\big\|_{L^\beta(\Omega;\mathbb{S}^2)} = \\
&\leq \left\|\frac{\boldsymbol{\Sigma}_{\rho_n}-\bar{\boldsymbol{\Sigma}}}{\rho_n}-\boldsymbol{\Sigma}'_{\boldsymbol{h}}\right\|_{L^\beta(\Omega;\mathbb{S}^2)} + \left\|\frac{\boldsymbol{\Sigma}_n-\boldsymbol{\Sigma}_{\rho_n}}{\rho_n}\right\|_{L^\beta(\Omega;\mathbb{S}^2)} \\
&\leq \left\|\frac{\boldsymbol{\Sigma}_{\rho_n}-\bar{\boldsymbol{\Sigma}}}{\rho_n}-\boldsymbol{\Sigma}'_{\boldsymbol{h}}\right\|_{L^\beta(\Omega;\mathbb{S}^2)} + L\left\|\frac{R(\boldsymbol{f}_n-\bar{\boldsymbol{f}}-\rho_n\boldsymbol{h})}{\rho_n}\right\|_{W^{-1,\beta}_D(\Omega;\mathbb{R}^d)} \\
&\leq \left\|\frac{\boldsymbol{\Sigma}_{\rho_n}-\bar{\boldsymbol{\Sigma}}}{\rho_n}-\boldsymbol{\Sigma}'_{\boldsymbol{h}}\right\|_{L^\beta(\Omega;\mathbb{S}^2)} + L\,\|R\boldsymbol{h}_n-R\boldsymbol{h}\|_{W^{-1,\beta}_D(\Omega;\mathbb{R}^d)}\,.
\end{aligned}$$

From Corollary 2.27 and (2.83) we thus infer

$$\left\| \boldsymbol{Z}_n - \boldsymbol{\Sigma}'_{\boldsymbol{h}} \right\|_{L^\beta(\Omega;\mathbb{S}^2)} \xrightarrow{n\to\infty} 0.$$

Analogous arguments hold true for $\boldsymbol{v}_n$ and ι_n, cf. Lemma 2.15(iii) and Corollary 2.28. Consequently, we observe

$$\begin{aligned} \boldsymbol{Z}_n &\to \boldsymbol{\Sigma}'_{\boldsymbol{h}} && \text{in } L^\beta(\Omega;\mathbb{S}^2) && \forall\, 1 \le \beta < p && (2.84\text{a})\\ \boldsymbol{v}_n &\to \boldsymbol{u}'_{\boldsymbol{h}} && \text{in } W^{1,\beta}_D(\Omega;\mathbb{R}^d) && \forall\, 1 \le \beta < p && (2.84\text{b})\\ \iota_n &\to \lambda'_{\boldsymbol{h}} && \text{in } L^\gamma(\Omega) && \forall\, 1 \le \gamma < q && (2.84\text{c}) \end{aligned}$$

with q as defined in (2.20). Because J is twice continuously differentiable, there is an element $(\tilde{\boldsymbol{u}}_n, \tilde{\boldsymbol{f}}_n)$ between $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})$ and $(\boldsymbol{u}_n, \boldsymbol{f}_n)$ such that

$$\begin{aligned} &J(\boldsymbol{u}_n, \boldsymbol{f}_n) - J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) = \\ &= i'(\bar{\boldsymbol{u}})(\boldsymbol{u}_n - \bar{\boldsymbol{u}}) + j'(\bar{\boldsymbol{f}})(\boldsymbol{f}_n - \bar{\boldsymbol{f}}) + \frac{1}{2} i''(\tilde{\boldsymbol{u}}_n)(\boldsymbol{u}_n - \bar{\boldsymbol{u}})^2 + \frac{1}{2} j''(\tilde{\boldsymbol{f}}_n)(\boldsymbol{f}_n - \bar{\boldsymbol{f}})^2 \\ &\quad + \underbrace{\big(A(\boldsymbol{\Sigma}_n - \bar{\boldsymbol{\Sigma}}) + \operatorname{div}^*(\boldsymbol{u}_n - \bar{\boldsymbol{u}}) + \lambda_n \mathcal{D}^*\mathcal{D}\boldsymbol{\Sigma}_n - \bar{\lambda}\mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Sigma}},\, \bar{\boldsymbol{\Upsilon}}\big)}_{=0} \\ &\quad + \underbrace{\big\langle \operatorname{div}(\boldsymbol{\Sigma}_n - \bar{\boldsymbol{\Sigma}}) - R\boldsymbol{f}_n + R\bar{\boldsymbol{f}},\, \bar{\boldsymbol{w}} \big\rangle}_{=0} + \big(\bar{\lambda}\mathcal{D}^*\mathcal{D}\boldsymbol{\Sigma}_n,\, \bar{\boldsymbol{\Upsilon}}\big) - \big(\bar{\lambda}\mathcal{D}^*\mathcal{D}\boldsymbol{\Sigma}_n,\, \bar{\boldsymbol{\Upsilon}}\big) \\ &= i'(\bar{\boldsymbol{u}})(\boldsymbol{u}_n - \bar{\boldsymbol{u}}) + j'(\bar{\boldsymbol{f}})(\boldsymbol{f}_n - \bar{\boldsymbol{f}}) + \frac{1}{2} i''(\tilde{\boldsymbol{u}}_n)(\boldsymbol{u}_n - \bar{\boldsymbol{u}})^2 + \frac{1}{2} j''(\tilde{\boldsymbol{f}}_n)(\boldsymbol{f}_n - \bar{\boldsymbol{f}})^2 \\ &\quad + \big(A\bar{\boldsymbol{\Upsilon}} + \operatorname{div}^* \bar{\boldsymbol{w}} + \bar{\lambda}\mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Upsilon}},\, \boldsymbol{\Sigma}_n - \bar{\boldsymbol{\Sigma}}\big) + \big\langle \operatorname{div} \bar{\boldsymbol{\Upsilon}},\, \boldsymbol{u}_n - \bar{\boldsymbol{u}} \big\rangle - \big(R^*\bar{\boldsymbol{w}},\, \boldsymbol{f}_n - \bar{\boldsymbol{f}}\big) \\ &\quad + \big((\lambda_n - \bar{\lambda})\mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Upsilon}},\, \boldsymbol{\Sigma}_n\big), \end{aligned}$$

where (2.15a), (2.15b), (2.21a) and (2.21b) was used. On account of (2.80a)–(2.80c) we obtain

$$\begin{aligned} J(\boldsymbol{u}_n, \boldsymbol{f}_n) - J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) &= \frac{1}{2} i''(\tilde{\boldsymbol{u}}_n)(\boldsymbol{u}_n - \bar{\boldsymbol{u}})^2 + \frac{1}{2} j''(\tilde{\boldsymbol{f}}_n)(\boldsymbol{f}_n - \bar{\boldsymbol{f}})^2 \\ &\quad - \big(\bar{\theta}\mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Sigma}},\, \boldsymbol{\Sigma}_n - \bar{\boldsymbol{\Sigma}}\big) + \big((\lambda_n - \bar{\lambda})\mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Upsilon}},\, \boldsymbol{\Sigma}_n\big). \end{aligned}$$

Hence, from (2.82) it follows

$$\begin{aligned} &\big((\lambda_n - \bar{\lambda})\mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Upsilon}},\, \boldsymbol{\Sigma}_n\big) - \big(\bar{\theta}\mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Sigma}},\, \boldsymbol{\Sigma}_n - \bar{\boldsymbol{\Sigma}}\big) \\ &\qquad < \frac{\rho_n^2}{n} - \frac{1}{2}\Big(i''(\tilde{\boldsymbol{u}}_n)(\boldsymbol{u}_n - \bar{\boldsymbol{u}})^2 + j''(\tilde{\boldsymbol{f}}_n)(\boldsymbol{f}_n - \bar{\boldsymbol{f}})^2 \Big), \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\big(\iota_n \mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Upsilon}},\, \boldsymbol{\Sigma}_n\big) - \big(\bar{\theta}\mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Sigma}},\, \boldsymbol{Z}_n\big) \\ &\qquad < \frac{\rho_n}{n} - \frac{1}{2}\Big(i''(\tilde{\boldsymbol{u}}_n)[\boldsymbol{u}_n - \bar{\boldsymbol{u}}, \boldsymbol{v}_n] + j''(\tilde{\boldsymbol{f}}_n)[\boldsymbol{f}_n - \bar{\boldsymbol{f}}, \boldsymbol{h}_n] \Big). \end{aligned} \tag{2.85}$$

Thanks to the integrability conditions (2.79) there are $\gamma < q$ and $\beta < p$ with $1/\gamma + 1/\eta + 1/p \leq 1$ and $1/p + 1/\beta + 1/r \leq 1$, cf. Remark 2.35, so that Lemma 2.15, (2.84a) and (2.84c) imply

$$\big(\iota_n \mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Upsilon}},\, \boldsymbol{\Sigma}_n\big) \to \big(\lambda'_{\boldsymbol{h}} \mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Upsilon}},\, \bar{\boldsymbol{\Sigma}}\big)\,, \quad \big(\bar{\theta}\mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Sigma}},\, \boldsymbol{Z}_n\big) \to \big(\bar{\theta}\mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Sigma}},\, \boldsymbol{\Sigma}'_{\boldsymbol{h}}\big)\,. \tag{2.86}$$

As the second derivatives of i and j are continuous bilinear forms by Assumption 2.40, we infer

$$\begin{aligned}\big|i''(\tilde{\boldsymbol{u}}_n)[\boldsymbol{u}_n - \bar{\boldsymbol{u}}, \boldsymbol{v}_n]\big| &\leq \big|i''(\tilde{\boldsymbol{u}}_n)[\boldsymbol{u}_n - \bar{\boldsymbol{u}}, \boldsymbol{v}_n] - i''(\bar{\boldsymbol{u}})[\boldsymbol{u}_n - \bar{\boldsymbol{u}}, \boldsymbol{v}_n]\big| \\ &\quad + \big|i''(\bar{\boldsymbol{u}})[\boldsymbol{u}_n - \bar{\boldsymbol{u}}, \boldsymbol{v}_n]\big| \\ &\leq \big\|i''(\tilde{\boldsymbol{u}}_n) - i''(\bar{\boldsymbol{u}})\big\|_{\mathcal{L}(V,V')} \|\boldsymbol{u}_n - \bar{\boldsymbol{u}}\|_V \|\boldsymbol{v}_n\|_V \\ &\quad + \big\|i''(\bar{\boldsymbol{u}})\big\|_{\mathcal{L}(V,V')} \|\boldsymbol{u}_n - \bar{\boldsymbol{u}}\|_V \|\boldsymbol{v}_n\|_V\end{aligned}$$

and a similar estimate for $j''(\tilde{\boldsymbol{f}}_n)[\boldsymbol{f}_n - \bar{\boldsymbol{f}}, \boldsymbol{h}_n]$. Thus, (2.83) and (2.84b) induce

$$i''(\tilde{\boldsymbol{u}}_n)[\boldsymbol{u}_n - \bar{\boldsymbol{u}}, \boldsymbol{v}_n] \to 0, \quad j''(\tilde{\boldsymbol{f}}_n)[\boldsymbol{f}_n - \bar{\boldsymbol{f}}, \boldsymbol{h}_n] \to 0. \tag{2.87}$$

Note that $\boldsymbol{v}_n$ as well as $\boldsymbol{h}_n$ is bounded in consequence of the (weak) convergence. From (2.85)–(2.87) we conclude

$$\big(\lambda'_{\boldsymbol{h}},\, \mathcal{D}\bar{\boldsymbol{\Upsilon}} : \mathcal{D}\bar{\boldsymbol{\Sigma}}\big) - \big(\bar{\theta},\, \mathcal{D}\bar{\boldsymbol{\Sigma}} : \mathcal{D}\boldsymbol{\Sigma}'_{\boldsymbol{h}}\big) \leq 0. \tag{2.88}$$

Moreover, (2.18c)–(2.18e) and (2.80d)–(2.80h) yield

$$\begin{aligned} \lambda'_{\boldsymbol{h}} = \bar{\theta} = 0 &\quad \text{in } \bar{\mathcal{I}}, & \mathcal{D}\bar{\boldsymbol{\Upsilon}} : \mathcal{D}\bar{\boldsymbol{\Sigma}} = \mathcal{D}\bar{\boldsymbol{\Sigma}} : \mathcal{D}\boldsymbol{\Sigma}'_{\boldsymbol{h}} = 0 &\quad \text{in } \bar{\mathcal{A}}_s \\ \lambda'_{\boldsymbol{h}} \geq 0 &\quad \text{in } \bar{\mathcal{B}}, & \mathcal{D}\bar{\boldsymbol{\Upsilon}} : \mathcal{D}\bar{\boldsymbol{\Sigma}} \geq 0 &\quad \text{in } \bar{\mathcal{B}} \\ \bar{\theta} \geq 0 &\quad \text{in } \bar{\mathcal{B}}, & \mathcal{D}\bar{\boldsymbol{\Sigma}} : \mathcal{D}\boldsymbol{\Sigma}'_{\boldsymbol{h}} \leq 0 &\quad \text{in } \bar{\mathcal{B}} \end{aligned}$$

and therefore $\big(\lambda'_{\boldsymbol{h}},\, \mathcal{D}\bar{\boldsymbol{\Upsilon}} : \mathcal{D}\bar{\boldsymbol{\Sigma}}\big) - \big(\bar{\theta},\, \mathcal{D}\bar{\boldsymbol{\Sigma}} : \mathcal{D}\boldsymbol{\Sigma}'_{\boldsymbol{h}}\big) \geq 0$, which together with (2.88) leads to

$$\big(\lambda'_{\boldsymbol{h}},\, \mathcal{D}\bar{\boldsymbol{\Upsilon}} : \mathcal{D}\bar{\boldsymbol{\Sigma}}\big) - \big(\bar{\theta},\, \mathcal{D}\bar{\boldsymbol{\Sigma}} : \mathcal{D}\boldsymbol{\Sigma}'_{\boldsymbol{h}}\big) = 0. \tag{2.89}$$

Because of (2.80a)–(2.80c), (2.18a) and (2.18b) we further find

$$\begin{aligned} &\big(\lambda'_{\boldsymbol{h}},\, \mathcal{D}\bar{\boldsymbol{\Upsilon}} : \mathcal{D}\bar{\boldsymbol{\Sigma}}\big) - \big(\bar{\theta},\, \mathcal{D}\bar{\boldsymbol{\Sigma}} : \mathcal{D}\boldsymbol{\Sigma}'_{\boldsymbol{h}}\big) = \big(\lambda'_{\boldsymbol{h}}\mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Sigma}},\, \bar{\boldsymbol{\Upsilon}}\big) - \big(\bar{\theta}\mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Sigma}},\, \boldsymbol{\Sigma}'_{\boldsymbol{h}}\big) = \\ &= -\big(A\boldsymbol{\Sigma}'_{\boldsymbol{h}} + \operatorname{div}^* \boldsymbol{u}'_{\boldsymbol{h}} + \bar{\lambda}\mathcal{D}^*\mathcal{D}\boldsymbol{\Sigma}'_{\boldsymbol{h}},\, \bar{\boldsymbol{\Upsilon}}\big) + \big(A\bar{\boldsymbol{\Upsilon}} + \operatorname{div}^* \bar{\boldsymbol{w}} + \bar{\lambda}\mathcal{D}^*\mathcal{D}\bar{\boldsymbol{\Upsilon}},\, \boldsymbol{\Sigma}'_{\boldsymbol{h}}\big) \\ &= -\big\langle \operatorname{div} \bar{\boldsymbol{\Upsilon}},\, \boldsymbol{u}'_{\boldsymbol{h}}\big\rangle + \big\langle \operatorname{div} \boldsymbol{\Sigma}'_{\boldsymbol{h}},\, \bar{\boldsymbol{w}}\big\rangle = -\big\langle \operatorname{div} \bar{\boldsymbol{\Upsilon}},\, \boldsymbol{u}'_{\boldsymbol{h}}\big\rangle + (R^*\bar{\boldsymbol{w}},\, \boldsymbol{h}) \\ &= i'(\bar{\boldsymbol{u}})\boldsymbol{u}'_{\boldsymbol{h}} + j'(\bar{\boldsymbol{f}})\boldsymbol{h} \end{aligned}$$

and in view of (2.89) we arrive at

$$i'(\bar{\boldsymbol{u}})\boldsymbol{u}'_{\boldsymbol{h}} + j'(\bar{\boldsymbol{f}})\boldsymbol{h} = 0. \tag{2.90}$$

As in the proof of Theorem 2.36 we introduce the abbreviations

$$\boldsymbol{z}_n := (\boldsymbol{u}_n, \boldsymbol{\Sigma}_n, \lambda_n, \boldsymbol{f}_n), \qquad \bar{\boldsymbol{z}} := \big(\bar{\boldsymbol{u}}, \bar{\boldsymbol{\Sigma}}, \bar{\lambda}, \bar{\boldsymbol{f}}\big), \qquad \bar{\boldsymbol{\omega}} := \big(\bar{\boldsymbol{\Upsilon}}, \bar{\boldsymbol{w}}, \bar{\mu}, \bar{\theta}\big).$$

Analogously to (2.73), (2.74) and (2.76) it can be shown that $\mathcal{L}(\bar{\boldsymbol{z}},\bar{\boldsymbol{\omega}}) = J(\bar{\boldsymbol{u}},\bar{\boldsymbol{f}})$, $\mathcal{L}(\boldsymbol{z}_n,\bar{\boldsymbol{\omega}}) = J(\boldsymbol{u}_n,\boldsymbol{f}_n) - (\lambda_n,\,\bar{\mu}) + \big(\phi(\boldsymbol{\Sigma}_n),\,\bar{\theta}\big)$ and $\nabla_{\boldsymbol{z}}\mathcal{L}(\bar{\boldsymbol{z}},\bar{\boldsymbol{\omega}})(\boldsymbol{z}_n-\bar{\boldsymbol{z}}) = 0$. Consequently, we derive

$$\begin{aligned} J(\boldsymbol{u}_n,\boldsymbol{f}_n) - J(\bar{\boldsymbol{u}},\bar{\boldsymbol{f}}) &= \mathcal{L}(\boldsymbol{z}_n,\bar{\boldsymbol{\omega}}) + (\lambda_n,\,\bar{\mu}) - \big(\phi(\boldsymbol{\Sigma}_n),\,\bar{\theta}\big) - \mathcal{L}(\bar{\boldsymbol{z}},\bar{\boldsymbol{\omega}}) \\ &= \frac{1}{2}\nabla^2_{\boldsymbol{z}}\mathcal{L}(\tilde{\boldsymbol{z}}_n,\bar{\boldsymbol{\omega}})(\boldsymbol{z}_n-\bar{\boldsymbol{z}})^2 + (\lambda_n,\,\bar{\mu}) - \big(\phi(\boldsymbol{\Sigma}_n),\,\bar{\theta}\big)\,, \end{aligned} \tag{2.91}$$

where $\tilde{\boldsymbol{z}}_n$ denotes an element between $\bar{\boldsymbol{z}}$ and $\boldsymbol{z}_n$. From (2.82) and (2.91) it then follows

$$\frac{1}{2}\nabla^2_{\boldsymbol{z}}\mathcal{L}(\tilde{\boldsymbol{z}}_n,\bar{\boldsymbol{\omega}})(\boldsymbol{z}_n-\bar{\boldsymbol{z}})^2 < \frac{\rho_n^2}{n} - (\lambda_n,\,\bar{\mu}) + \big(\phi(\boldsymbol{\Sigma}_n),\,\bar{\theta}\big) \tag{2.92}$$

so that Lemma 2.34 yields

$$\frac{1}{2}\nabla^2_{\boldsymbol{z}}\mathcal{L}(\tilde{\boldsymbol{z}}_n,\bar{\boldsymbol{\omega}})\left(\frac{\boldsymbol{z}_n-\bar{\boldsymbol{z}}}{\rho_n}\right)^2 < \frac{1}{n} + \frac{o(\rho_n^2)}{\rho_n^2}, \tag{2.93}$$

which is equivalent to

$$\begin{aligned} &\frac{1}{2}\nabla^2_{\boldsymbol{z}}\mathcal{L}(\bar{\boldsymbol{z}},\bar{\boldsymbol{\omega}})\left(\frac{\boldsymbol{z}_n-\bar{\boldsymbol{z}}}{\rho_n}\right)^2 = \\ &< \frac{1}{n} + \frac{o(\rho_n^2)}{\rho_n^2} - \frac{1}{2}\left(\nabla^2_{\boldsymbol{z}}\mathcal{L}(\tilde{\boldsymbol{z}}_n,\bar{\boldsymbol{\omega}})\left(\frac{\boldsymbol{z}_n-\bar{\boldsymbol{z}}}{\rho_n}\right)^2 - \nabla^2_{\boldsymbol{z}}\mathcal{L}(\bar{\boldsymbol{z}},\bar{\boldsymbol{\omega}})\left(\frac{\boldsymbol{z}_n-\bar{\boldsymbol{z}}}{\rho_n}\right)^2\right). \end{aligned} \tag{2.94}$$

Recalling (2.71), we know

$$\begin{aligned} &\nabla^2_{\boldsymbol{z}}\mathcal{L}(\tilde{\boldsymbol{z}}_n,\bar{\boldsymbol{\omega}})\left(\frac{\boldsymbol{z}_n-\bar{\boldsymbol{z}}}{\rho_n}\right)^2 - \nabla^2_{\boldsymbol{z}}\mathcal{L}(\bar{\boldsymbol{z}},\bar{\boldsymbol{\omega}})\left(\frac{\boldsymbol{z}_n-\bar{\boldsymbol{z}}}{\rho_n}\right)^2 = \\ &= \left[\nabla^2_{(\boldsymbol{u},\boldsymbol{f})}J(\tilde{\boldsymbol{u}}_n,\tilde{\boldsymbol{f}}_n) - \nabla^2_{(\boldsymbol{u},\boldsymbol{f})}J(\bar{\boldsymbol{u}},\bar{\boldsymbol{f}})\right]\left(\frac{\boldsymbol{u}_n-\bar{\boldsymbol{u}}}{\rho_n},\,\frac{\boldsymbol{f}_n-\bar{\boldsymbol{f}}}{\rho_n}\right)^2. \end{aligned} \tag{2.95}$$

Since $\nabla^2_{(\boldsymbol{u},\boldsymbol{g})}J$ is continuous by assumption and the sequence $(\boldsymbol{z}_n-\bar{\boldsymbol{z}})/\rho_n$ is bounded due to (weak) convergence, cf. (2.83) and (2.84), the right-hand side of (2.94) converges to zero. For the discussion of the left-hand side we use again (2.71) and observe

$$\begin{aligned} \nabla^2_{\boldsymbol{z}}\mathcal{L}(\bar{\boldsymbol{z}},\bar{\boldsymbol{\omega}})\left(\frac{\boldsymbol{z}_n-\bar{\boldsymbol{z}}}{\rho_n}\right)^2 &= i''(\bar{\boldsymbol{u}})(\boldsymbol{v}_n)^2 + j''(\bar{\boldsymbol{f}})(\boldsymbol{h}_n)^2 \\ &\quad + 2\left(\iota_n\mathcal{D}^*\mathcal{D}\boldsymbol{Z}_n,\,\bar{\boldsymbol{\Upsilon}}\right) + \left(\|\boldsymbol{Z}_n\|^2_{\mathbb{S}},\,\bar{\theta}\right). \end{aligned} \tag{2.96}$$

Thanks to the coercivity condition in Assumption 2.40(ii) the continuous mapping $U \ni \boldsymbol{f} \mapsto j''(\bar{\boldsymbol{f}})\boldsymbol{f}^2 \in \mathbb{R}$ is convex and hence weakly lower semi-continuous. On account of (2.83) we deduce

$$j''(\bar{\boldsymbol{f}})(\boldsymbol{h})^2 \le \liminf_{n\to\infty} j''(\bar{\boldsymbol{f}})(\boldsymbol{h}_n)^2. \tag{2.97}$$

Furthermore, according to (2.79) we can chose $1 \leq \beta < p$ and $1 \leq \gamma < q$ with $1/\beta + 1/\gamma + 1/\eta \leq 1$ and $2/\beta + 1/r \leq 1$ so that (2.84) implies

$$\left(\iota_n \mathcal{D}^* \mathcal{D} \boldsymbol{Z}_n,\, \bar{\boldsymbol{\Upsilon}}\right) \to \left(\lambda'_{\boldsymbol{h}} \mathcal{D}^* \mathcal{D} \boldsymbol{\Sigma}'_{\boldsymbol{h}},\, \bar{\boldsymbol{\Upsilon}}\right), \quad \left(\|\boldsymbol{Z}_n\|_{\mathbb{S}}^2\,,\, \bar{\theta}\right) \to \left(\|\boldsymbol{\Sigma}'_{\boldsymbol{h}}\|_{\mathbb{S}}^2\,,\, \bar{\theta}\right). \tag{2.98}$$

Together, (2.94)–(2.98) and (2.84b) lead to

$$\nabla_{\boldsymbol{z}}^2 \mathcal{L}(\bar{\boldsymbol{z}}, \bar{\boldsymbol{\omega}}) \left(\boldsymbol{\Sigma}'_{\boldsymbol{h}}, \boldsymbol{u}'_{\boldsymbol{h}}, \lambda'_{\boldsymbol{h}}, \boldsymbol{h}\right)^2 \leq \liminf_{n\to\infty} \nabla_{\boldsymbol{z}}^2 \mathcal{L}(\bar{\boldsymbol{z}}, \bar{\boldsymbol{\omega}}) \left(\frac{\boldsymbol{z}_n - \bar{\boldsymbol{z}}}{\rho_n}\right)^2 \leq 0.$$

Invoking ($\widetilde{\text{SSC}}$) and (2.90), we infer $\boldsymbol{h} = \boldsymbol{0}$ and therefore

$$\boldsymbol{\Sigma}'_{\boldsymbol{h}} = \boldsymbol{0}, \quad \boldsymbol{u}'_{\boldsymbol{h}} = \boldsymbol{0}, \quad \lambda'_{\boldsymbol{h}} = 0. \tag{2.99}$$

Note that (2.18) is uniquely solvable. Since $\|\boldsymbol{h}_n\|_U = 1$ by definition, Assumption 2.40(ii) and (2.94)–(2.96) yield

$$\begin{aligned}
0 < \frac{\nu}{2} &\leq \frac{1}{2} j''(\bar{\boldsymbol{f}})(\boldsymbol{h}_n)^2 \\
&< \frac{1}{n} + \frac{o(\rho_n^2)}{\rho_n^2} - \frac{1}{2}\left[\nabla^2_{(\boldsymbol{u},\boldsymbol{f})} J(\tilde{\boldsymbol{u}}_n, \tilde{\boldsymbol{f}}_n) - \nabla^2_{(\boldsymbol{u},\boldsymbol{f})} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})\right] \left(\frac{\boldsymbol{u}_n - \bar{\boldsymbol{u}}}{\rho_n}, \frac{\boldsymbol{f}_n - \bar{\boldsymbol{f}}}{\rho_n}\right)^2 \\
&\quad - \frac{1}{2} i''(\bar{\boldsymbol{u}})(\boldsymbol{v}_n)^2 - \left(\iota_n \mathcal{D}^* \mathcal{D} \boldsymbol{Z}_n,\, \bar{\boldsymbol{\Upsilon}}\right) - \frac{1}{2}\left(\|\boldsymbol{Z}_n\|_{\mathbb{S}}^2\,,\, \bar{\theta}\right).
\end{aligned}$$

On account of (2.84b), (2.98), (2.99) and the continuity of $\nabla^2_{(\boldsymbol{u},\boldsymbol{g})} J$ the right-hand side converges to zero, which is the desired contradiction. □

Remark 2.42. *Again, we want to compare the sufficient optimality conditions given in Theorem 2.41 with [69, Theorem 7]. There is no analogue of the integrability requirement* (2.79) *in finite dimensions. In contrast to Theorem 2.36 the coercivity of the Hessian (w.r.t. the primal variable) of the Lagrangian is restricted to the cone of critical directions, cf.* ($\widetilde{\text{SSC}}$)*, and can hence be seen as the infinite-dimensional counterpart of the coercivity condition in [69, Theorem 7], cf. also Remark 2.38. We still had to tighten the sign conditions of strong stationarity, though. In [69, Theorem 7] it is sufficient, if the multipliers are nonnegative on the biactive set $\bar{\mathcal{B}}$, which generally is a genuine subset of both $\mathcal{A}_1$ and $\mathcal{A}_2$. The larger sets are needed in order to employ Lemma 2.34, cf. the estimates* (2.75) *and* (2.93)*. However, in finite dimensions this lemma is not necessary, as the crucial expressions already provide a sign:*
Let us assume that $\bar{\lambda}$, $\phi(\bar{\boldsymbol{\Sigma}})$, $\bar{\mu}$ and $\bar{\theta}$ are elements of $\mathbb{R}^m$ satisfying the following complementarity system

$$\bar{\mu}_i \bar{\lambda}_i = 0 \quad \forall\, i \in \{1, \ldots, m\} \tag{2.100a}$$
$$\bar{\theta}_i \phi(\bar{\boldsymbol{\Sigma}})_i = 0 \quad \forall\, i \in \{1, \ldots, m\} \tag{2.100b}$$
$$\bar{\mu}_i, \bar{\theta}_i \geq 0 \quad \forall\, i \in \bar{\mathcal{B}} = \left\{i \in \{1, \ldots, m\} \colon \bar{\lambda}_i = \phi(\bar{\boldsymbol{\Sigma}})_i = 0\right\}. \tag{2.100c}$$

Due to (2.100a) *we know* $\bar{\mu}_i = 0$ *for all* $i \in \bar{\mathcal{A}}_s = \big\{i \in \{1,\dots,m\} \colon \bar{\lambda}_i > 0\big\}$. *From the continuity of* ϕ *and the convergence* $\boldsymbol{\Sigma}_n \to \bar{\boldsymbol{\Sigma}}$ *we moreover deduce* $\phi(\boldsymbol{\Sigma}_n)_i < 0$ *and hence* $(\lambda_n)_i = 0$ *for all* $i \in \bar{\mathcal{I}} = \big\{i \in \{1,\dots,m\} \colon \phi(\bar{\boldsymbol{\Sigma}})_i < 0\big\}$, *if* n *is large enough. The sign condition* (2.100c) *together with* $\lambda_n \geq 0$ *then shows*

$$\bar{\mu}^\top \lambda_n = \sum_{i=1}^{m} \bar{\mu}_i (\lambda_n)_i = \sum_{i \in \mathcal{B}} \bar{\mu}_i (\lambda_n)_i \geq 0. \tag{2.101}$$

In view of $\lambda_n \to \bar{\lambda}$ *we find* $(\lambda_n)_i < 0$ *and consequently* $\phi(\boldsymbol{\Sigma}_n)_i = 0$ *for all* $i \in \bar{\mathcal{A}}_s$, *if* n *is large enough. In addition, thanks to* (2.100b) *we derive* $\bar{\theta}_i = 0$ *for all* $i \in \mathcal{I}$ *so that* (2.100c) *and* $\phi(\boldsymbol{\Sigma}_n) \leq 0$ *imply*

$$\bar{\theta}^\top \phi(\boldsymbol{\Sigma}_n) = \sum_{i=1}^{m} \bar{\theta}_i \phi(\boldsymbol{\Sigma}_n)_i = \sum_{i \in \mathcal{B}} \bar{\theta}_i \phi(\boldsymbol{\Sigma}_n)_i \leq 0. \tag{2.102}$$

Thus, the right-hand sides in (2.74) *and* (2.92) *can be estimated without involving Lemma 2.34. Since the above arguments, which lead to* (2.101) *and* (2.102), *cannot be applied in function spaces, the assertion of Theorem 2.41 seems to be a natural generalization of [69, Theorem 7].*

In the remainder of this section we establish an equivalent formulation of $(\widetilde{\text{SSC}})$. Therefor the next auxiliary lemma is required.

Lemma 2.43. *Let* $\boldsymbol{h} \in U$ *and* $\{\boldsymbol{h}_n\}_{n\in\mathbb{N}} \subset U$ *be given. Moreover let* $(\boldsymbol{\Sigma}'_n, \boldsymbol{u}'_n, \lambda'_n)$ *be the solution of* (2.18) *with right-hand side* $R\boldsymbol{h}_n$. *If* $(\boldsymbol{\Sigma}'_n, \boldsymbol{u}'_n, \lambda'_n, \boldsymbol{h}_n) \rightharpoonup (\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda', \boldsymbol{h})$ *in* $S^2 \times V \times L^2(\Omega) \times U$, *then* $(\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda')$ *solves* (2.18) *with right-hand side* $R\boldsymbol{h}$.

Proof. According to Theorem 2.11 the system (2.18) with right-hand side $R\boldsymbol{h}_n$ is equivalent to

$$\begin{aligned}\big(A\boldsymbol{\Sigma}'_n,\, \boldsymbol{T}\big) - \big(A\boldsymbol{\Sigma}'_n,\, \boldsymbol{\Sigma}'_n\big) + \big(\operatorname{div}^* \boldsymbol{u}'_n,\, \boldsymbol{T}\big) - \big(\operatorname{div}^* \boldsymbol{u}'_n,\, \boldsymbol{\Sigma}'_n\big)&\\ + \big(\bar{\lambda},\, \mathcal{D}\boldsymbol{\Sigma}'_n : \mathcal{D}\boldsymbol{T}\big) - \Big(\bar{\lambda},\, \big\|\mathcal{D}\boldsymbol{\Sigma}'_n\big\|^2_{\mathbb{S}}\Big) \geq 0 \quad \forall\, \boldsymbol{T} \in \mathcal{S}_\ell& \quad (2.103a)\\ \operatorname{div} \boldsymbol{\Sigma}'_n = R\boldsymbol{h}_n.& \quad (2.103b)\end{aligned}$$

We will achieve the assertion by passing to the limit $n \to \infty$ in (2.103). To this end let $\boldsymbol{T} \in \mathcal{S}_\ell$ be fixed but arbitrary. On account of the weak continuity of div and R we conclude (2.17b) with ride-hand side $R\boldsymbol{h}$. Furthermore, the weak continuity of A yields

$$\big(A\boldsymbol{\Sigma}'_n,\, \boldsymbol{T}\big) \to \big(A\boldsymbol{\Sigma}',\, \boldsymbol{T}\big)\,, \quad \big(\operatorname{div}^* \boldsymbol{u}'_n,\, \boldsymbol{T}\big) \to \big(\operatorname{div}^* \boldsymbol{u}',\, \boldsymbol{T}\big)\,. \tag{2.104}$$

Because of (2.103b), the compactness of R and (2.17b) we obtain

$$\big(\operatorname{div}^* \boldsymbol{u}'_n,\, \boldsymbol{\Sigma}'_n\big) = \big\langle \boldsymbol{u}'_n,\, R\boldsymbol{h}_n\big\rangle \to \big\langle \boldsymbol{u}',\, R\boldsymbol{h}\big\rangle = \big(\operatorname{div}^* \boldsymbol{u}',\, \boldsymbol{\Sigma}'\big)\,. \tag{2.105}$$

By the same arguments as in the proof of Lemma 2.16 we infer

$$\bar{\lambda}\mathcal{D}\boldsymbol{\Sigma}'_n = -\mathbb{H}^{-1}\boldsymbol{\chi}'_n - \lambda'_n\mathcal{D}\bar{\boldsymbol{\Sigma}},$$

cf. (2.30). The left-hand side converges weakly in $L^1(\Omega;\mathbb{S})$. However, resulting from the boundedness of $\mathcal{D}\bar{\boldsymbol{\Sigma}}$ the right-hand side converges weakly in S so that

$$\bar{\lambda}\mathcal{D}\boldsymbol{\Sigma}'_n \rightharpoonup \bar{\lambda}\mathcal{D}\boldsymbol{\Sigma}' = -\mathbb{H}^{-1}\boldsymbol{\chi}' - \lambda'\mathcal{D}\bar{\boldsymbol{\Sigma}} \quad \text{in } S \tag{2.106}$$

and therefore

$$\big(\bar{\lambda},\, \mathcal{D}\boldsymbol{\Sigma}'_n : \mathcal{D}\boldsymbol{T}\big) \to \big(\bar{\lambda},\, \mathcal{D}\boldsymbol{\Sigma}' : \mathcal{D}\boldsymbol{T}\big)\,. \tag{2.107}$$

Due to Assumption 2.1(2) the operator A is continuous and coercive. This is why the mapping $S^2 \ni \boldsymbol{T} \mapsto -\,(A\boldsymbol{T},\, \boldsymbol{T}) \in \mathbb{R}$ is continuous and concave, which implies

$$\limsup_{n\to\infty} \big(-\,\big(A\boldsymbol{\Sigma}'_n,\, \boldsymbol{\Sigma}'_n\big)\big) \le -\,\big(A\boldsymbol{\Sigma}',\, \boldsymbol{\Sigma}'\big)\,. \tag{2.108}$$

For the discussion of the remaining term in (2.103a) we introduce the space

$$S^2_\lambda := \Big\{\boldsymbol{T} \in S^2 \colon \sqrt{\bar{\lambda}}\mathcal{D}\boldsymbol{T} \in S\Big\}$$

endowed with the scalar product

$$(\boldsymbol{\Sigma},\, \boldsymbol{T})_{S^2_\lambda} = (\boldsymbol{\Sigma},\, \boldsymbol{T}) + \big(\bar{\lambda},\, \mathcal{D}\boldsymbol{\Sigma} : \mathcal{D}\boldsymbol{T}\big)\,.$$

Similarly to the proof of Proposition 2.32 it can be derived that S^2_λ is complete and thus a Hilbert space. Next, we define the continuous function

$$f \colon S^2_\lambda \to \mathbb{R},\ \boldsymbol{T} \mapsto \big(\bar{\lambda},\, \mathcal{D}\boldsymbol{T} : \mathcal{D}\boldsymbol{T}\big)\,.$$

In view of (2.15c) the function $-f$ is concave and hence weakly upper semicontinuous. For S^2_λ is a Hilbert space, every linear functional on S^2_λ can be represented by a scalar product. Since (2.106) induces

$$\big(\bar{\lambda}\mathcal{D}\boldsymbol{\Sigma}'_n,\, \mathcal{D}\boldsymbol{T}\big) \to \big(\bar{\lambda}\mathcal{D}\boldsymbol{\Sigma}',\, \mathcal{D}\boldsymbol{T}\big) \quad \forall\, \boldsymbol{T} \in S^2_\lambda,$$

we find $\boldsymbol{\Sigma}'_n \rightharpoonup \boldsymbol{\Sigma}'$ in S^2_λ and consequently

$$\limsup_{n\to\infty} \big(-\,f(\boldsymbol{\Sigma}'_n)\big) \le -\Big(\bar{\lambda},\, \big\|\mathcal{D}\boldsymbol{\Sigma}'\big\|^2_{\mathbb{S}}\Big)\,. \tag{2.109}$$

Altogether, from (2.103a), (2.104), (2.105) and (2.107)–(2.109) we deduce (2.17a). Testing with $\mathcal{D}\boldsymbol{\Sigma}'$ in (2.106) and using the boundedness of $\mathcal{D}\bar{\boldsymbol{\Sigma}}$ yields $\sqrt{\bar{\lambda}}\mathcal{D}\boldsymbol{\Sigma}' \in S$. Since equality and inequality conditions remain valid for the weak limit, we observe $\boldsymbol{\Sigma}' \in \mathcal{S}_\ell$ so that $(\boldsymbol{\Sigma}', \boldsymbol{u}', R\boldsymbol{h})$ solves (2.17). By the same token it follows $\lambda' = 0$ a.e. in $\bar{\mathcal{I}}$. The argument which led from (2.30) to (2.31) and (2.106) then show that λ' coincides with the unique multiplier in (2.18). On account of Theorem 2.11 the weak limit $(\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda')$ is indeed the solution to (2.18) with ride-hand side $R\boldsymbol{h}$. □

Remark 2.44. *If the hardening variable $\bar{\boldsymbol{\chi}}$ satisfies the integrability condition in* (2.79)*, it is not necessary to introduce the Hilbert space $S^2_{\bar{\lambda}}$ in the proof of Lemma 2.43. According to Remark 2.30(iii) and the weak convergence $\boldsymbol{h}_n \rightharpoonup \boldsymbol{h}$ in U the sequence $\{\boldsymbol{\Sigma}'_n\}_{n\in\mathbb{N}}$ is bounded in $L^p(\Omega;\mathbb{S}^2)$. Hence there is a subsequence, w.l.o.g. denoted in the same way, converging weakly in $L^p(\Omega;\mathbb{S}^2)$. Moreover, analogously to* (2.60) *we obtain the continuity of f on $L^p(\Omega;\mathbb{S}^2)$ so that* (2.109) *is a consequence of $\boldsymbol{\Sigma}'_n \rightharpoonup \boldsymbol{\Sigma}'$ in $L^p(\Omega;\mathbb{S}^2)$.*

The weak convergence of the sequences $\{\boldsymbol{\Sigma}'_n\}_{n\in\mathbb{N}}$ and $\{\boldsymbol{u}'_n\}_{n\in\mathbb{N}}$ in Lemma 2.43 is in fact strong:

Corollary 2.45. *Let $\boldsymbol{h} \in U$, $\{\boldsymbol{h}_n\}_{n\in\mathbb{N}} \subset U$ be given and let $(\boldsymbol{\Sigma}'_n, \boldsymbol{u}'_n, \lambda'_n)$ be the solution of* (2.18) *with right-hand side $R\boldsymbol{h}_n$. If $(\boldsymbol{\Sigma}'_n, \boldsymbol{u}'_n, \lambda'_n, \boldsymbol{h}_n) \rightharpoonup (\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda', \boldsymbol{h})$ in $S^2 \times V \times L^2(\Omega) \times U$, then $(\boldsymbol{\Sigma}'_n, \boldsymbol{u}'_n) \to (\boldsymbol{\Sigma}', \boldsymbol{u}')$ in $S^2 \times V$.*

Proof. Due to Lemma 2.43 we know that $(\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda', R\boldsymbol{h})$ solves (2.18). By subtracting (2.18a) from the corresponding equation for $(\boldsymbol{\Sigma}'_n, \boldsymbol{u}'_n, \lambda'_n)$ and testing with $\boldsymbol{\Sigma}' - \boldsymbol{\Sigma}'_n$ we arrive at

$$\begin{aligned}&\big(A(\boldsymbol{\Sigma}' - \boldsymbol{\Sigma}'_n),\, \boldsymbol{\Sigma}' - \boldsymbol{\Sigma}'_n\big) + \big(\operatorname{div}^*(\boldsymbol{u}' - \boldsymbol{u}'_n),\, \boldsymbol{\Sigma}' - \boldsymbol{\Sigma}'_n\big)\\ &\quad + \big(\bar{\lambda}\mathcal{D}(\boldsymbol{\Sigma}' - \boldsymbol{\Sigma}'_n),\, \mathcal{D}(\boldsymbol{\Sigma}' - \boldsymbol{\Sigma}'_n)\big) + \big((\lambda' - \lambda'_n)\mathcal{D}\bar{\boldsymbol{\Sigma}},\, \mathcal{D}(\boldsymbol{\Sigma}' - \boldsymbol{\Sigma}'_n)\big) = 0.\end{aligned}$$

The coercivity of A implies

$$\begin{aligned} c\,\big\|\boldsymbol{\Sigma}' - \boldsymbol{\Sigma}'_n\big\|^2_{S^2} \le &- \underbrace{\big(\operatorname{div}^*(\boldsymbol{u}' - \boldsymbol{u}'_n),\, \boldsymbol{\Sigma}' - \boldsymbol{\Sigma}'_n\big)}_{=:I_n} - \underbrace{\Big(\bar{\lambda},\, \big\|\mathcal{D}(\boldsymbol{\Sigma}' - \boldsymbol{\Sigma}'_n)\big\|^2_{\mathbb{S}}\Big)}_{=:II_n}\\ &- \underbrace{\big((\lambda' - \lambda'_n)\mathcal{D}\bar{\boldsymbol{\Sigma}},\, \mathcal{D}(\boldsymbol{\Sigma}' - \boldsymbol{\Sigma}'_n)\big)}_{=:III_n}. \end{aligned} \tag{2.110}$$

By the same arguments as in the proof of Lemma 2.43, cf. (2.105) and (2.109), we infer

$$I_n \to 0 \quad \text{and} \quad \limsup_{n\to\infty}(-II_n) \le 0. \tag{2.111}$$

Thanks to (2.18c)–(2.18e) and the boundedness of $\mathcal{D}\bar{\boldsymbol{\Sigma}}$ the weak convergence leads to

$$\begin{aligned} III_n &= \underbrace{\big(\lambda'\mathcal{D}\bar{\boldsymbol{\Sigma}},\, \mathcal{D}\boldsymbol{\Sigma}'\big)}_{=0} - \big(\lambda'_n\mathcal{D}\bar{\boldsymbol{\Sigma}},\, \mathcal{D}\boldsymbol{\Sigma}'\big) - \big(\lambda'\mathcal{D}\bar{\boldsymbol{\Sigma}},\, \mathcal{D}\boldsymbol{\Sigma}'_n\big) + \underbrace{\big(\lambda'_n\mathcal{D}\bar{\boldsymbol{\Sigma}},\, \mathcal{D}\boldsymbol{\Sigma}'_n\big)}_{=0}\\ &= -\big(\lambda'_n\mathcal{D}\bar{\boldsymbol{\Sigma}},\, \mathcal{D}\boldsymbol{\Sigma}'\big) - \big(\lambda'\mathcal{D}\bar{\boldsymbol{\Sigma}},\, \mathcal{D}\boldsymbol{\Sigma}'_n\big) \xrightarrow{n\to\infty} -2\big(\lambda'\mathcal{D}\bar{\boldsymbol{\Sigma}},\, \mathcal{D}\boldsymbol{\Sigma}'\big) = 0, \end{aligned} \tag{2.112}$$

which together with (2.110) and (2.111) shows $\boldsymbol{\Sigma}'_n \to \boldsymbol{\Sigma}'$ in S^2. In order to prove strong convergence of $\boldsymbol{u}'_n$ we subtract again (2.18a) from the corresponding equation for $(\boldsymbol{\Sigma}'_n, \boldsymbol{u}'_n, \lambda'_n)$ and test with

$$\check{\boldsymbol{T}} := \big(\boldsymbol{\varepsilon}(\boldsymbol{u}') - \boldsymbol{\varepsilon}(\boldsymbol{u}'_n), -\boldsymbol{\varepsilon}(\boldsymbol{u}') + \boldsymbol{\varepsilon}(\boldsymbol{u}'_n)\big) \in S^2.$$

Since $\mathcal{D}\check{\boldsymbol{T}} = 0$, Korn's inequality (Proposition A.25) yields similarly to (2.46)–(2.47)

$$\begin{aligned}\left\|\boldsymbol{u}' - \boldsymbol{u}'_n\right\|_V^2 &\le c \int_\Omega \left\|\boldsymbol{\varepsilon}(\boldsymbol{u}' - \boldsymbol{u}'_n)\right\|_{\mathbb{S}}^2 \,\mathrm{d}x = c\left(\operatorname{div}^*(\boldsymbol{u}' - \boldsymbol{u}'_n),\, \check{\boldsymbol{T}}\right) \\ &\le c\left|\left(A(\boldsymbol{\Sigma}' - \boldsymbol{\Sigma}'_n),\, \check{\boldsymbol{T}}\right)\right| \le c\left\|\boldsymbol{u}' - \boldsymbol{u}'_n\right\|_V \left\|\boldsymbol{\Sigma}' - \boldsymbol{\Sigma}'_n\right\|_{S^2}\end{aligned}$$

so that $\boldsymbol{u}'_n \to \boldsymbol{u}'$ in V. □

By means of Lemma 2.18 we can extend the previous result:

Corollary 2.46. *Let $\boldsymbol{h} \in U$, $\{\boldsymbol{h}_n\}_{n\in\mathbb{N}} \subset U$ be given and let $(\boldsymbol{\Sigma}'_n, \boldsymbol{u}'_n, \lambda'_n)$ be the solution of* (2.18) *with right-hand side $R\boldsymbol{h}_n$. If $(\boldsymbol{\Sigma}'_n, \boldsymbol{u}'_n, \lambda'_n, \boldsymbol{h}_n) \rightharpoonup (\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda', \boldsymbol{h})$ in $L^p(\Omega;\mathbb{S}^2) \times V \times L^2(\Omega) \times U$ with $p \ge 2$, then $\boldsymbol{\Sigma}'_n \to \boldsymbol{\Sigma}'$ in $L^\beta(\Omega;\mathbb{S}^2)$ for all $1 \le \beta < p$.*

Proof. As a result of Lemma 2.45 there exists a subsequence, w.l.o.g. denoted in the same way, with $\boldsymbol{\Sigma}'_n \to \boldsymbol{\Sigma}'$ a.e. in Ω. Moreover the subsequence is bounded in $L^p(\Omega;\mathbb{S}^2)$ because of the weak convergence. Thus, Lemma 2.18 implies the claim for an arbitrary subsequence and hence for the whole sequence. □

Based on Corollary 2.45 and Corollary 2.46 we can now prove an equivalent reformulation of $(\widetilde{\mathrm{SSC}})$.

Theorem 2.47. *Let $(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}}, \bar{\lambda}) \in S^2 \times V \times L^2(\Omega)$ be the state associated to $\bar{\boldsymbol{f}} \in U$. If $\bar{\boldsymbol{\chi}} \in L^s(\Omega;\mathbb{S})$, $(\bar{\boldsymbol{\Upsilon}}, \bar{\boldsymbol{w}}) \in L^\eta(\Omega;\mathbb{S}^2) \times V$ and $(\bar{\mu}, \bar{\theta}) \in L^\zeta(\Omega) \times L^r(\Omega)$ with s, η, ζ and r satisfying* (2.79)*, then the following two statements are equivalent:*

a) There exists $\alpha > 0$ such that

$$\partial^2_{(\boldsymbol{\Sigma},\boldsymbol{u},\lambda,\boldsymbol{f})}\mathcal{L}\big(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}}, \bar{\lambda}, \bar{\boldsymbol{f}}, \bar{\boldsymbol{\Upsilon}}, \bar{\boldsymbol{w}}, \bar{\mu}, \bar{\theta}\big)\big(\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda', \boldsymbol{h}\big)^2 \ge \alpha \left\|\boldsymbol{h}\right\|_U^2$$

for all $\boldsymbol{h} \in U$ and $(\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda')$ solving (2.18) *with $\delta\ell = R\boldsymbol{h}$ such that $i'(\bar{\boldsymbol{u}})\boldsymbol{u}' + j'(\bar{\boldsymbol{f}})\boldsymbol{h} = 0$.*

b) For all $\boldsymbol{h} \in U \setminus \{\boldsymbol{0}\}$ and $(\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda')$ solving (2.18) *with $\delta\ell = R\boldsymbol{h}$ such that $i'(\bar{\boldsymbol{u}})\boldsymbol{u}' + j'(\bar{\boldsymbol{f}})\boldsymbol{h} = 0$ it holds*

$$\partial^2_{(\boldsymbol{\Sigma},\boldsymbol{u},\lambda,\boldsymbol{f})}\mathcal{L}\big(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}}, \bar{\lambda}, \bar{\boldsymbol{f}}, \bar{\boldsymbol{\Upsilon}}, \bar{\boldsymbol{w}}, \bar{\mu}, \bar{\theta}\big)\big(\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda', \boldsymbol{h}\big)^2 > 0.$$

Proof. It suffices to show that a) is a consequence of b). Similar to [23, Theorem 4.4] we argue by contradiction and suppose the opposite of a). Then for every $\alpha > 0$ there exist $\boldsymbol{h}_\alpha \in U$ and $(\boldsymbol{\Sigma}'_\alpha, \boldsymbol{u}'_\alpha, \lambda'_\alpha)$ solving (2.18) with $\delta\ell = R\boldsymbol{h}_\alpha$ such that $i'(\bar{\boldsymbol{u}})\boldsymbol{u}'_\alpha + j'(\bar{\boldsymbol{f}})\boldsymbol{h}_\alpha = 0$ and

$$\partial^2_{(\boldsymbol{\Sigma},\boldsymbol{u},\lambda,\boldsymbol{f})}\mathcal{L}\big(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}}, \bar{\lambda}, \bar{\boldsymbol{f}}, \bar{\boldsymbol{\Upsilon}}, \bar{\boldsymbol{w}}, \bar{\mu}, \bar{\theta}\big)\big(\boldsymbol{\Sigma}'_\alpha, \boldsymbol{u}'_\alpha, \lambda'_\alpha, \boldsymbol{h}_\alpha\big)^2 < \alpha \left\|\boldsymbol{h}_\alpha\right\|_U^2. \tag{2.113}$$

For S_ℓ is a cone, the solution map of (2.18) is positively homogeneous and we can assume that $\|\boldsymbol{h}_\alpha\|_U = 1$ for all $\alpha > 0$. From Remark 2.30(iii) and (iv) we derive boundedness of $(\boldsymbol{\Sigma}'_\alpha, \boldsymbol{u}'_\alpha, \lambda'_\alpha)$ in $L^p(\Omega;\mathbb{S}^2)\times W^{1,p}_D(\Omega;\mathbb{R})\times L^q(\Omega)$. Consequently, there exists a subsequence, w.l.o.g. denoted by the same symbols, with

$$(\boldsymbol{\Sigma}'_\alpha, \boldsymbol{u}'_\alpha, \lambda'_\alpha, \boldsymbol{h}_\alpha) \xrightarrow{\alpha\searrow 0} (\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda', \boldsymbol{h}) \quad \text{in } L^p(\Omega;\mathbb{S}^2) \times W^{1,p}_D(\Omega;\mathbb{R}^d) \times L^q(\Omega) \times U.$$

Thanks to Corollary 2.45 and Corollary 2.46 it thus follows

$$\boldsymbol{u}'_\alpha \to \boldsymbol{u}' \quad \text{in } V, \quad \boldsymbol{\Sigma}'_\alpha \to \boldsymbol{\Sigma}' \quad \text{in } L^\beta(\Omega;\mathbb{S}^2). \tag{2.114}$$

Moreover, the weak limit satisfies

$$i'(\bar{\boldsymbol{u}})\boldsymbol{u}' + j'(\bar{\boldsymbol{f}})\boldsymbol{h} = \lim_{\alpha\to 0} \big(\underbrace{i'(\bar{\boldsymbol{u}})\boldsymbol{u}'_\alpha + j'(\bar{\boldsymbol{g}})\boldsymbol{h}_\alpha}_{=0}\big) = 0. \tag{2.115}$$

In view of (2.113) and (2.71) we observe

$$\begin{aligned}
0 &\ge \limsup_{\alpha\to 0} \partial^2_{(\boldsymbol{\Sigma},\boldsymbol{u},\lambda,\boldsymbol{f})}\mathcal{L}\big(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}}, \bar{\lambda}, \bar{\boldsymbol{f}}, \bar{\boldsymbol{\Upsilon}}, \bar{\boldsymbol{w}}, \bar{\mu}, \bar{\theta}\big)\big(\boldsymbol{\Sigma}'_\alpha, \boldsymbol{u}'_\alpha, \lambda'_\alpha, \boldsymbol{h}_\alpha\big)^2 \\
&\ge \liminf_{\alpha\to 0} \partial^2_{(\boldsymbol{\Sigma},\boldsymbol{u},\lambda,\boldsymbol{f})}\mathcal{L}\big(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}}, \bar{\lambda}, \bar{\boldsymbol{f}}, \bar{\boldsymbol{\Upsilon}}, \bar{\boldsymbol{w}}, \bar{\mu}, \bar{\theta}\big)\big(\boldsymbol{\Sigma}'_\alpha, \boldsymbol{u}'_\alpha, \lambda'_\alpha, \boldsymbol{h}_\alpha\big)^2 \\
&= \liminf_{\alpha\to 0} \Big\{\partial^2 i(\bar{\boldsymbol{u}})(\boldsymbol{u}'_\alpha)^2 + \partial^2 j(\bar{\boldsymbol{f}})\boldsymbol{h}_\alpha^2 + 2\left(\lambda'_\alpha \mathcal{D}\boldsymbol{\Sigma}'_\alpha,\, \mathcal{D}\bar{\boldsymbol{\Upsilon}}\right) \\
&\quad + \Big(\big\|\mathcal{D}\boldsymbol{\Sigma}'_\alpha\big\|^2_{\mathbb{S}},\, \bar{\theta}\Big)\Big\}.
\end{aligned} \tag{2.116}$$

The integrability conditions for $\bar{\boldsymbol{\Upsilon}}$ and $\bar{\theta}$ yield the existence of $1 \le \beta < p$ with $1/q + 1/\beta + 1/\eta \le 1$ and $2/\beta + 1/r \le 1$, cf. Remark 2.35, so that the strong convergence (2.114) leads to

$$\left(\lambda'_\alpha \mathcal{D}\boldsymbol{\Sigma}'_\alpha,\, \mathcal{D}\bar{\boldsymbol{\Upsilon}}\right) \to \left(\lambda' \mathcal{D}\boldsymbol{\Sigma}',\, \mathcal{D}\bar{\boldsymbol{\Upsilon}}\right), \quad \Big(\big\|\mathcal{D}\boldsymbol{\Sigma}'_\alpha\big\|^2_{\mathbb{S}},\, \bar{\theta}\Big) \to \Big(\big\|\mathcal{D}\boldsymbol{\Sigma}'\big\|^2_{\mathbb{S}},\, \bar{\theta}\Big). \tag{2.117}$$

On account of Assumption 2.40(ii) the continuous mapping $U \ni \boldsymbol{f} \mapsto \partial^2 j(\bar{\boldsymbol{f}})\boldsymbol{f}^2 \in \mathbb{R}$ is convex and hence weakly lower semi-continuous. Due to (2.114), (2.116) and (2.117) we conclude

$$\begin{aligned}
0 &\ge \partial^2 i(\bar{\boldsymbol{u}})(\boldsymbol{u}')^2 + \partial^2 j(\bar{\boldsymbol{f}})\boldsymbol{h}^2 + 2\left(\lambda' \mathcal{D}\boldsymbol{\Sigma}',\, \mathcal{D}\bar{\boldsymbol{\Upsilon}}\right) + \Big(\big\|\mathcal{D}\boldsymbol{\Sigma}'\big\|^2_{\mathbb{S}},\, \bar{\theta}\Big) \\
&= \partial^2_{(\boldsymbol{u},\boldsymbol{\Sigma},\lambda,\boldsymbol{f})}\mathcal{L}(\bar{\boldsymbol{u}}, \bar{\boldsymbol{\Sigma}}, \bar{\lambda}, \bar{\boldsymbol{f}}, \bar{\boldsymbol{\Upsilon}}, \bar{\boldsymbol{w}}, \bar{\mu}, \bar{\theta})(\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda', \boldsymbol{h})^2.
\end{aligned} \tag{2.118}$$

As $(\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda')$ solves (2.18) with ride-hand side $R\boldsymbol{h}$ by Lemma 2.43, we deduce $\boldsymbol{h} = \boldsymbol{0}$ from (2.115) and b). Furthermore, (2.18) is uniquely solvable, cf. Theorem 2.11, which implies $(\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda') = (\boldsymbol{0}, \boldsymbol{0}, 0)$. Invoking $\|\boldsymbol{h}_\alpha\|_U = 1$, Assumption 2.40(ii) and (2.71), we infer

$$\begin{aligned}
\nu &= \nu \limsup_{\alpha\to 0} \|\boldsymbol{h}_\alpha\|^2_U \le \limsup_{\alpha\to 0} \partial^2 j(\bar{\boldsymbol{f}})\boldsymbol{h}_\alpha^2 \\
&\le \limsup_{\alpha\to 0} \Big\{\partial^2_{(\boldsymbol{\Sigma},\boldsymbol{u},\lambda,\boldsymbol{f})}\mathcal{L}\big(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}}, \bar{\lambda}, \bar{\boldsymbol{f}}, \bar{\boldsymbol{\Upsilon}}, \bar{\boldsymbol{w}}, \bar{\mu}, \bar{\theta}\big)\big(\boldsymbol{\Sigma}'_\alpha, \boldsymbol{u}'_\alpha, \lambda'_\alpha, \boldsymbol{h}_\alpha\big)^2 - \partial^2 i(\bar{\boldsymbol{u}})(\boldsymbol{u}'_\alpha)^2 \\
&\quad - 2\left(\lambda'_\alpha \mathcal{D}\boldsymbol{\Sigma}'_\alpha,\, \mathcal{D}\bar{\boldsymbol{\Upsilon}}\right) - \Big(\big\|\mathcal{D}\boldsymbol{\Sigma}'_\alpha\big\|^2_{\mathbb{S}},\, \bar{\theta}\Big)\Big\}.
\end{aligned}$$

Therefore, (2.114), (2.116) and (2.117) together with $(\boldsymbol{\Sigma}', \boldsymbol{u}', \lambda') = (\mathbf{0}, \mathbf{0}, 0)$ show

$$\begin{aligned}\nu &\leq \limsup_{\alpha\to 0} \partial^2_{(\boldsymbol{\Sigma},\boldsymbol{u},\lambda,\boldsymbol{f})}\mathcal{L}\big(\bar{\boldsymbol{\Sigma}}, \bar{\boldsymbol{u}}, \bar{\lambda}, \bar{\boldsymbol{f}}, \bar{\boldsymbol{\Upsilon}}, \bar{\boldsymbol{w}}, \bar{\mu}, \bar{\theta}\big)\big(\boldsymbol{\Sigma}'_\alpha, \boldsymbol{u}'_\alpha, \lambda'_\alpha, \boldsymbol{h}_\alpha\big)^2 \\ &\quad - \lim_{\alpha\to 0}\Big\{\partial^2 i(\bar{\boldsymbol{u}})(\boldsymbol{u}'_\alpha)^2 + 2\left(\lambda'_\alpha \mathcal{D}\boldsymbol{\Sigma}'_\alpha, \mathcal{D}\bar{\boldsymbol{\Upsilon}}\right) + \Big(\big\|\mathcal{D}\boldsymbol{\Sigma}'_\alpha\big\|^2_{\mathbb{S}}, \bar{\theta}\Big)\Big\} \\ &\leq 0,\end{aligned}$$

contradictory to $\nu > 0$. □

Remark 2.48. *The proof of Theorem 2.47 can analogously be applied to* (SSC) *provided that the objective functional J satisfies Assumption 2.40.*

2.4. An exact solution with non-vanishing biactive set

In Section 2.2 we have seen that the control-to-state map G_E associated with $(\mathbf{VI}_E)$ is Bouligand differentiable (Theorem 2.19) or at least directionally differentiable (Theorem 2.26), if additional integrability conditions are fulfilled by the solution to (2.15), where G_E is differentiated. However, if the biactive set of this solution has positive Lebesgue measure, then G_E is in general not Gâteaux differentiable, cf. Remark 2.12. Consequently, it is delicate to derive necessary optimality conditions from a reduced formulation, which would be the usual approach of optimal control theory, see e.g. [73, Section 4.6]. Furthermore, gradient-based optimization algorithms cannot carelessly be applied to $(\mathbf{P}_E)$, cf. [2, Section 1.3].
In the following we will show that the case of biactivity can indeed arise. More precisely, by means of the sufficient conditions stated in Theorem 2.36 we will find a locally optimal control for Problem $(\mathbf{P}_E)$ with non-vanishing biactive set.

The next assumption is supposed to hold throughout this section.

Assumption 2.49.

(i) *The spatial dimension is $d = 2$.*

(ii) *Both $\mathbb{C}^{-1}$ and $\mathbb{H}^{-1}$ are the identity mapping $\mathbb{S} \to \mathbb{S}$ f.a.a. $x \in \Omega$.*

(iii) *The domain Ω is the unit sphere in $\mathbb{R}^2$, i.e., $\Omega = \big\{x \in \mathbb{R}^2 : |x| < 1\big\}$.*

(iv) *The Dirichlet boundary Γ_D is the entire boundary of Ω, i.e., $\Gamma_D = \big\{x \in \mathbb{R}^2 \colon |x| = 1\big\}$.*

First, we aim to construct a biactive solution to $(\mathbf{VI}_E)$ with $\ell = R\boldsymbol{f}$ for some control $\boldsymbol{f} \in U = L^2\big(\Omega; \mathbb{R}^2\big)$. In view of Theorem 2.5 and Assumption 2.49 we look for $\big((\boldsymbol{\sigma},\boldsymbol{\chi}),\boldsymbol{u},\lambda,\boldsymbol{f}\big) \in S^2 \times H_0^1\big(\Omega;\mathbb{R}^2\big) \times L^2(\Omega) \times L^2\big(\Omega;\mathbb{R}^2\big)$ satisfying

$$\mathbb{C}^{-1}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda\big(\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D\big) = 0 \qquad \text{in } S \tag{2.119a}$$

$$\mathbb{H}^{-1}\boldsymbol{\chi} + \lambda\big(\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D\big) = 0 \qquad \text{in } S \tag{2.119b}$$

$$-\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x = 0 \qquad \forall\,\boldsymbol{v} \in H_0^1\big(\Omega;\mathbb{R}^2\big) \tag{2.119c}$$

$$0 \le \lambda \perp \phi(\boldsymbol{\sigma},\boldsymbol{\chi}) \le 0 \qquad \text{a.e. in } \Omega \tag{2.119d}$$

and $|\mathcal{B}| = \big|\{x \in \Omega : \lambda(x) = \phi\big(\boldsymbol{\sigma}(x),\boldsymbol{\chi}(x)\big) = 0\}\big| > 0$. To this end let $\bar{\boldsymbol{u}}\colon \overline{\Omega} \to \mathbb{R}^2$ be defined through

$$\bar{\boldsymbol{u}}(x) = \begin{pmatrix} U\big(|x|^2\big) \\ U\big(|x|^2\big) \end{pmatrix}, \tag{2.120}$$

where $U\colon \mathbb{R} \to \mathbb{R}$ is given by

$$U(t) = \begin{cases} -\sigma_0 t^2 + \frac{3}{2}\sigma_0 t - \frac{13}{16}\sigma_0, & t < \frac{1}{4} \\ \sigma_0\sqrt{t} - \sigma_0, & t \ge \frac{1}{4}. \end{cases} \tag{2.121}$$

The function U is twice continuously differentiable with first and second derivative

$$U'(t) = \begin{cases} -2\sigma_0 t + \frac{3}{2}\sigma_0, & t < \frac{1}{4} \\ \frac{\sigma_0}{2\sqrt{t}}, & t \ge \frac{1}{4}, \end{cases} \qquad U''(t) = \begin{cases} -2\sigma_0, & t < \frac{1}{4} \\ -\frac{\sigma_0}{4t^{3/2}}, & t \ge \frac{1}{4}. \end{cases} \tag{2.122}$$

Note that U' as well as U'' are continuous in $t = 1/4$. By the chain rule we know $\bar{\boldsymbol{u}} \in C^2\big(\overline{\Omega};\mathbb{R}^2\big)$ so that $\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}) \in C^1\big(\overline{\Omega},\mathbb{S}\big) \subset S$ and $\operatorname{div}\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}) \in C\big(\overline{\Omega};\mathbb{R}^2\big) \subset L^2\big(\Omega;\mathbb{R}^2\big)$. Since U vanishes in $t = 1$, it moreover follows $\bar{\boldsymbol{u}} \in H_0^1\big(\Omega;\mathbb{R}^2\big)$. Thereby we obtain a solution to (2.119), if we set

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}), \qquad \bar{\boldsymbol{\chi}} = \boldsymbol{0}, \qquad \bar{\lambda} = 0, \qquad \bar{\boldsymbol{f}} = \operatorname{div}\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}), \tag{2.123}$$

which is the system of linear elasticity, cf. [35, Section X.32]. In order to prove this the following auxiliary lemma is needed.

Lemma 2.50. *Let $U\colon \mathbb{R} \to \mathbb{R}$ be defined by* (2.121)*. Then it holds*

$$U'(t) < \frac{\sigma_0}{2\sqrt{t}} \quad \forall\, t \in (0, 1/4).$$

Proof. In consequence of (2.122) we have to check that $-2\sigma_0 t + 3/2\sigma_0 < \sigma_0/(2\sqrt{t})$ for all $t \in (0, 1/4)$, or equivalently,

$$-4t^{3/2} + 3\sqrt{t} < 1 \quad \forall\, t \in (0, 1/4). \tag{2.124}$$

We define $g\colon (0, 1/4] \to \mathbb{R}$, $t \mapsto -4t^{3/2} + 3\sqrt{t}$, and observe

$$g'(t) = -6\sqrt{t} + \frac{3}{2\sqrt{t}} = 0 \Leftrightarrow t = \frac{1}{4}. \tag{2.125}$$

Besides, the function g is concave, as its second derivative satisfies

$$g''(t) = -\frac{3}{\sqrt{t}} - \frac{3}{4t^{3/2}} \le 0 \quad \forall\, t \in (0, 1/4].$$

Hence the maximum value of g in $(0, 1/4]$ is $g(1/4) = 1$. From (2.125) we further deduce $g'(t) < 0$ for all $t \in (0, 1/4)$ so that g is strictly increasing in $(0, 1/4)$, which implies (2.124). □

Proposition 2.51. *Let $\bar{\boldsymbol{\sigma}}$, $\bar{\boldsymbol{\chi}}$, $\bar{\boldsymbol{u}}$, $\bar{\lambda}$ and $\bar{\boldsymbol{f}}$ be as defined in* (2.120) *and* (2.123). *Then $\big((\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}}), \bar{\boldsymbol{u}}, \bar{\lambda}, \bar{\boldsymbol{f}}\big) \in S^2 \times H^1_0\big(\Omega; \mathbb{R}^2\big) \times L^2(\Omega) \times L^2\big(\Omega; \mathbb{R}^2\big)$ solves* (2.119) *with associated biactive set $\bar{\mathcal{B}}$ and inactive set $\bar{\mathcal{I}}$ given by*

$$\bar{\mathcal{B}} = \big\{x \in \mathbb{R}^2 \colon 1/4 \le |x|^2 < 1\big\}, \qquad \bar{\mathcal{I}} = \big\{x \in \mathbb{R}^2 \colon 0 \le |x|^2 < 1/4\big\}. \tag{2.126}$$

Proof. Assumption 2.49(ii) together with $\bar{\boldsymbol{\sigma}} = \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})$, $\bar{\boldsymbol{\chi}} = \boldsymbol{0}$ and $\bar{\lambda} = 0$ yields (2.119a) and (2.119b). Furthermore, (2.119c) is the weak formulation of $\operatorname{div} \bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{f}}$, cf. Assumption 2.49(iv). For the verification of (2.119d) it suffices to show

$$\phi\big(\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}), \boldsymbol{0}\big) \le 0 \Leftrightarrow \big\|\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^D\big\|_{\mathbb{S}} \le \sigma_0 \quad \text{a.e. in } \Omega.$$

We recall that the linearized strain $\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})$ is defined through

$$\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}) = \begin{pmatrix} \frac{\partial \bar{\boldsymbol{u}}_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial \bar{\boldsymbol{u}}_1}{\partial x_2} + \frac{\partial \bar{\boldsymbol{u}}_2}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial \bar{\boldsymbol{u}}_1}{\partial x_2} + \frac{\partial \bar{\boldsymbol{u}}_2}{\partial x_1}\right) & \frac{\partial \bar{\boldsymbol{u}}_2}{\partial x_2} \end{pmatrix} \tag{2.127}$$

and its deviatoric part is

$$\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^D = \begin{pmatrix} \frac{1}{2}\left(\frac{\partial \bar{\boldsymbol{u}}_1}{\partial x_1} - \frac{\partial \bar{\boldsymbol{u}}_2}{\partial x_2}\right) & \frac{1}{2}\left(\frac{\partial \bar{\boldsymbol{u}}_1}{\partial x_2} + \frac{\partial \bar{\boldsymbol{u}}_2}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial \bar{\boldsymbol{u}}_1}{\partial x_2} + \frac{\partial \bar{\boldsymbol{u}}_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial \bar{\boldsymbol{u}}_2}{\partial x_2} - \frac{\partial \bar{\boldsymbol{u}}_1}{\partial x_1}\right) \end{pmatrix}. \tag{2.128}$$

Consequently, we find

$$\begin{aligned} \big\|\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^D\big\|_{\mathbb{S}}^2 &= 2\left(\frac{1}{2}\left(\frac{\partial \bar{\boldsymbol{u}}_1}{\partial x_1} - \frac{\partial \bar{\boldsymbol{u}}_2}{\partial x_2}\right)\right)^2 + 2\left(\frac{1}{2}\left(\frac{\partial \bar{\boldsymbol{u}}_1}{\partial x_2} + \frac{\partial \bar{\boldsymbol{u}}_2}{\partial x_1}\right)\right)^2 \\ &= \frac{1}{2}\left(\left(\frac{\partial \bar{\boldsymbol{u}}_1}{\partial x_1}\right)^2 - 2\frac{\partial \bar{\boldsymbol{u}}_1}{\partial x_1}\frac{\partial \bar{\boldsymbol{u}}_2}{\partial x_2} + \left(\frac{\partial \bar{\boldsymbol{u}}_2}{\partial x_2}\right)^2\right) \\ &\quad + \frac{1}{2}\left(\left(\frac{\partial \bar{\boldsymbol{u}}_1}{\partial x_2}\right)^2 + 2\frac{\partial \bar{\boldsymbol{u}}_1}{\partial x_2}\frac{\partial \bar{\boldsymbol{u}}_2}{\partial x_1} + \left(\frac{\partial \bar{\boldsymbol{u}}_2}{\partial x_1}\right)^2\right). \end{aligned}$$

In addition, the partial derivatives of $\bar{\boldsymbol{u}}$ satisfy

$$\frac{\partial \bar{\boldsymbol{u}}_1}{\partial x_1} = \frac{\partial \bar{\boldsymbol{u}}_2}{\partial x_1} = 2x_1 U'\big(|x|^2\big), \qquad \frac{\partial \bar{\boldsymbol{u}}_1}{\partial x_2} = \frac{\partial \bar{\boldsymbol{u}}_2}{\partial x_2} = 2x_2 U'\big(|x|^2\big) \tag{2.129}$$

which leads to

$$\left\|\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^D\right\|_{\mathbb{S}}^2 = \frac{1}{2}\left(8x_1^2 U'\big(|x|^2\big)^2 + 8x_2^2 U'\big(|x|^2\big)^2\right) = 4|x|^2 U'\big(|x|^2\big)^2.$$

If $x = 0$, then we know $\left\|\boldsymbol{\varepsilon}(\boldsymbol{u})^D\right\| = 0 \leq \sigma_0$. From (2.122) and Lemma 2.50 we moreover conclude

$$\left\|\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^D\right\|_{\mathbb{S}} = 2|x|U'\big(|x|^2\big) \leq 2|x|\frac{\sigma_0}{2|x|} = \sigma_0 \tag{2.130}$$

for all $x \neq 0$ and thus (2.119d). The estimate in (2.130) becomes an equation, if $x \in \big\{x \in \mathbb{R}^2 \colon 1/4 \leq |x|^2 < 1\big\}$, and a strict inequality otherwise. In view of (2.16) and $\bar{\lambda} = 0$ we obtain (2.126). □

Next, a particularly chosen objective functional J is considered such that $\bar{\boldsymbol{f}} \in L^2\big(\Omega;\mathbb{R}^2\big)$ with associated state and multiplier $\big((\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\chi}}),\bar{\boldsymbol{u}},\bar{\lambda}\big) \in S^2 \times H_0^1(\Omega;\mathbb{R}^2) \times L^2(\Omega)$ as defined in Proposition 2.51 is a locally optimal solution of Problem $(\mathbf{P}_E)$. We introduce the abbreviation

$$B_{1/2} = \big\{x \in \mathbb{R}^2 \colon 0 \leq |x| < 1/2\big\}$$

and define $J \colon V \times L^2\big(\Omega;\mathbb{R}^2\big) \to \mathbb{R}$ through

$$\begin{aligned} J(\boldsymbol{u},\boldsymbol{f}) = \frac{1}{2}\left\|\boldsymbol{u} - \boldsymbol{u}_{d1}\right\|_{L^2(\Omega;\mathbb{R}^2)}^2 &+ \frac{1}{2}\left\|\boldsymbol{u} - \boldsymbol{u}_{d2}\right\|_{L^2(\partial B_{1/2};\mathbb{R}^2)}^2 \\ &+ \frac{\nu}{2}\left\|\boldsymbol{f} - \boldsymbol{f}_d\right\|_{L^2(\Omega;\mathbb{R}^2)}^2. \end{aligned} \tag{2.131}$$

The functions $\boldsymbol{u}_{d1} \in L^2\big(\Omega;\mathbb{R}^2\big)$, $\boldsymbol{u}_{d2} \in L^2\big(\partial B_{1/2};\mathbb{R}^2\big)$ and $\boldsymbol{f}_d \in L^2\big(\Omega;\mathbb{R}^2\big)$ are given by

$$\boldsymbol{u}_{d1}(x) = \begin{cases} \bar{\boldsymbol{u}}(x) + 2\operatorname{div}\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}(x)), & x \in B_{1/2} \\ \bar{\boldsymbol{u}}(x) + \operatorname{div}\big(\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}(x)) + \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}(x))^S\big), & x \in \Omega \setminus B_{1/2} \end{cases} \tag{2.132a}$$

$$\boldsymbol{u}_{d2}(x) = \bar{\boldsymbol{u}}(x) - \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}(x))^D \frac{x}{|x|} \tag{2.132b}$$

$$\boldsymbol{f}_d = \bar{\boldsymbol{f}} - \frac{2}{\nu}\bar{\boldsymbol{u}}, \tag{2.132c}$$

where ν is a positive real number and

$$\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^S = \frac{1}{2}\big(\operatorname{trace}\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})\big)\boldsymbol{I} \tag{2.133}$$

with identity tensor $\boldsymbol{I} \in \mathbb{S}$ is the spherical part of $\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})$.

Proposition 2.52. *Let $J \colon V \times L^2\big(\Omega;\mathbb{R}^2\big) \to \mathbb{R}$ be as defined in* (2.131)*. Then $\bar{\boldsymbol{f}} \in L^2\big(\Omega;\mathbb{R}^2\big)$ with associated state and multiplier $\big((\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\chi}}),\bar{\boldsymbol{u}},\bar{\lambda}\big)$ defined in* (2.120) *and* (2.123) *is a locally optimal control for $(\mathbf{P}_E)$. In particular, the biactive set $\bar{\mathcal{B}}$ has positive Lebesgue measure.*

Proof. Because the objective functional J is twice continuously Fréchet differentiable, we can apply Theorem 2.36. In view of Assumption 2.49, (2.123) and Proposition 2.51 the function $\bar{\boldsymbol{f}}$ is a locally optimal control for $(\mathbf{P}_E)$, if there exist an adjoint state $\big((\bar{\boldsymbol{\zeta}},\bar{\boldsymbol{\psi}}),\bar{\boldsymbol{w}}\big) \in L^\eta(\Omega;\mathbb{S})^2 \times H_0^1\big(\Omega;\mathbb{R}^2\big)$ and multipliers $(\bar{\mu},\bar{\theta}) \in L^\zeta(\Omega) \times L^r(\Omega)$ with

$$\eta,\zeta,r > \frac{sp}{sp-p-2s}, \quad s > \frac{2p}{p-2} \tag{2.134}$$

and $p \in (2,3]$ as defined in (2.64), which fulfill the following optimality conditions:

(a) $\big((\bar{\boldsymbol{\zeta}},\bar{\boldsymbol{\psi}}),\bar{\boldsymbol{w}}\big)$ and $(\bar{\mu},\bar{\theta})$ solve

$$\bar{\boldsymbol{\zeta}} - \boldsymbol{\varepsilon}(\bar{\boldsymbol{w}}) + \bar{\theta}\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^D = 0 \tag{2.135a}$$

$$\bar{\boldsymbol{\psi}} + \bar{\theta}\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^D = 0 \tag{2.135b}$$

$$-\int_\Omega \bar{\boldsymbol{\zeta}} : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x + \big(\partial_{\boldsymbol{u}} J(\bar{\boldsymbol{u}},\bar{\boldsymbol{f}}),\,\boldsymbol{v}\big) = 0 \qquad \forall\,\boldsymbol{v} \in H_0^1\big(\Omega;\mathbb{R}^2\big) \tag{2.135c}$$

$$\bar{\boldsymbol{w}} - \partial_{\boldsymbol{f}} J(\bar{\boldsymbol{u}},\bar{\boldsymbol{f}}) = 0 \tag{2.135d}$$

$$\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^D : \big(\bar{\boldsymbol{\zeta}}^D + \bar{\boldsymbol{\psi}}^D\big) - \bar{\mu} = 0 \qquad \text{a.e. in } \Omega \tag{2.135e}$$

$$\bar{\theta}\phi\big((\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}),\mathbf{0})\big) = 0 \qquad \text{a.e. in } \Omega \tag{2.135f}$$

$$\bar{\mu} \ge 0 \qquad \text{a.e. in } \mathcal{A}_1 \tag{2.135g}$$

$$\bar{\theta} \ge 0 \qquad \text{a.e. in } \mathcal{A}_2 \tag{2.135h}$$

with $\mathcal{A}_1 := \{x \in \Omega : -\kappa_1 \le \phi(\bar{\boldsymbol{\Sigma}}) \le 0\}$ and $\mathcal{A}_2 := \{x \in \Omega : 0 \le \bar{\lambda} \le \kappa_2\}$ for some $\kappa_1,\kappa_2 > 0$.

(b) There is $\alpha > 0$ such that

$$\partial^2_{((\boldsymbol{\sigma},\boldsymbol{\chi}),\boldsymbol{u},\lambda,\boldsymbol{f})}\mathcal{L}\big((\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\chi}}),\bar{\boldsymbol{u}},\bar{\lambda},\bar{\boldsymbol{f}},(\bar{\boldsymbol{\zeta}},\bar{\boldsymbol{\psi}}),\bar{\boldsymbol{w}},\bar{\mu},\bar{\theta}\big)\big((\boldsymbol{\sigma}',\boldsymbol{\chi}'),\boldsymbol{u}',\lambda',\boldsymbol{h}\big)^2 \ge \alpha\,\|\boldsymbol{h}\|_U^2$$

for all $\boldsymbol{h} \in L^2\big(\Omega;\mathbb{R}^2\big)$ and $\big((\boldsymbol{\sigma}',\boldsymbol{\chi}'),\boldsymbol{u}',\lambda'\big)$ solving

$$\boldsymbol{\sigma}' - \boldsymbol{\varepsilon}(\boldsymbol{u}') + \lambda'\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^D = 0 \tag{2.136a}$$

$$\boldsymbol{\chi}' + \lambda'\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^D = 0 \tag{2.136b}$$

$$-\int_\Omega \boldsymbol{\sigma}' : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x - \int_\Omega \boldsymbol{h}\cdot\boldsymbol{v}\,\mathrm{d}x = 0 \qquad \forall\,\boldsymbol{v} \in H_0^1\big(\Omega;\mathbb{R}^2\big) \tag{2.136c}$$

$$0 \le \lambda' \perp \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^D : \big((\boldsymbol{\sigma}')^D + (\boldsymbol{\chi}')^D\big) \le 0 \qquad \text{a.e. in } \Omega \setminus B_{1/2} \tag{2.136d}$$

$$0 = \lambda' \perp \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^D : \big((\boldsymbol{\sigma}')^D + (\boldsymbol{\chi}')^D\big) \in \mathbb{R} \qquad \text{a.e. in } B_{1/2}. \tag{2.136e}$$

Herein, the Lagrangian $\mathcal{L}\colon S^2_\infty \times H_0^1\big(\Omega;\mathbb{R}^2\big) \times L^2(\Omega) \times L^2\big(\Omega;\mathbb{R}^2\big) \times S^2 \times H_0^1\big(\Omega;\mathbb{R}^2\big) \times L^2(\Omega) \times L^\infty(\Omega)'$ is given by

$$\begin{aligned}
&\mathcal{L}\big((\boldsymbol{\sigma},\boldsymbol{\chi}),\boldsymbol{u},\lambda,\boldsymbol{f},(\boldsymbol{\zeta},\boldsymbol{\psi}),\boldsymbol{w},\mu,\theta\big) = \\
&= J(\boldsymbol{u},\boldsymbol{f}) + \big(\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda\big(\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D\big),\,\boldsymbol{\zeta}\big) + \big(\boldsymbol{\chi} + \lambda\big(\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D\big),\,\boldsymbol{\psi}\big) \\
&\quad - (\boldsymbol{\sigma},\,\boldsymbol{\varepsilon}(\boldsymbol{w})) - (\boldsymbol{f},\,\boldsymbol{w}) - (\lambda,\,\mu) + \big\langle\phi\big((\boldsymbol{\sigma},\boldsymbol{\chi})\big),\,\theta\big\rangle_{L^\infty(\Omega),L^\infty(\Omega)'}\,.
\end{aligned}$$

We start with the construction of $\big((\bar{\boldsymbol{\zeta}}, \bar{\boldsymbol{\psi}}), \bar{\boldsymbol{w}}\big)$ and $(\bar{\mu}, \bar{\theta})$ such that (a) is satisfied. Subsequently we check the integrability condition (2.134) and (b). For that to happen, let $\bar{\theta}\colon \Omega \to \mathbb{R}$ be defined through

$$\bar{\theta} = \begin{cases} 0, & x \in B_{1/2} \\ 1, & x \in \Omega \setminus B_{1/2}. \end{cases} \tag{2.137}$$

On account of $\bar{\mathcal{B}} \cup \bar{\mathcal{I}} = \Omega$ and $\bar{\mathcal{I}} = B_{1/2}$, cf. (2.126), we obtain (2.135f) and (2.135h) with $\mathcal{A}_2 = \Omega$ due to $\bar{\lambda} = 0$. The choice

$$\bar{\boldsymbol{\psi}} = \begin{cases} \boldsymbol{0}, & x \in B_{1/2} \\ -\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^D, & x \in \Omega \setminus B_{1/2} \end{cases} \tag{2.138}$$

leads to (2.135b). Next, we set $\bar{\boldsymbol{w}} = 2\bar{\boldsymbol{u}}$ and observe

$$-\boldsymbol{\varepsilon}(\bar{\boldsymbol{w}}) + \bar{\theta}\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^D = -2\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}) + \bar{\theta}\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^D = \begin{cases} -2\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}), & x \in B_{1/2} \\ -\big(\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}) + \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^S\big), & x \in \Omega \setminus B_{1/2}. \end{cases}$$

If $\bar{\boldsymbol{\zeta}}\colon \Omega \to \mathbb{S}$ is defined by

$$\bar{\boldsymbol{\zeta}} = \begin{cases} 2\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}), & x \in B_{1/2} \\ \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}) + \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^S, & x \in \Omega \setminus B_{1/2}, \end{cases} \tag{2.139}$$

then it follows (2.135a). As $\bar{\boldsymbol{u}}$ is twice continuously differentiable, the function $\bar{\boldsymbol{\zeta}}$ is continuously differentiable in $B_{1/2}$ and $\Omega \setminus B_{1/2}$, respectively. Hence, integration by parts yields

$$\begin{aligned} -\int_\Omega \bar{\boldsymbol{\zeta}} : \boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{d}x &= -\int_{B_{1/2}} \bar{\boldsymbol{\zeta}} : \boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{d}x - \int_{\Omega \setminus B_{1/2}} \bar{\boldsymbol{\zeta}} : \boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{d}x \\ &= \int_{B_{1/2}} \operatorname{div} \bar{\boldsymbol{\zeta}} \cdot \boldsymbol{v} \,\mathrm{d}x - \int_{\partial B_{1/2}} \bar{\boldsymbol{\zeta}}|_{B_{1/2}} \frac{x}{|x|} \cdot \boldsymbol{v} \,\mathrm{d}s \\ &\quad + \int_{\Omega \setminus B_{1/2}} \operatorname{div} \bar{\boldsymbol{\zeta}} \cdot \boldsymbol{v} \,\mathrm{d}x - \int_{\partial B_{1/2}} \bar{\boldsymbol{\zeta}}|_{\Omega \setminus B_{1/2}} \left(-\frac{x}{|x|}\right) \cdot \boldsymbol{v} \,\mathrm{d}s \end{aligned}$$

for all $\boldsymbol{v} \in H_0^1\big(\Omega; \mathbb{R}^2\big)$, where we used that $x/\,|x|$ is the unit outward normal on the sphere $B_{1/2}$. Note here the density of $C_0^\infty\big(\Omega; \mathbb{R}^2\big)$ in $H_0^1\big(\Omega; \mathbb{R}^2\big)$. Since $\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})$ can be decomposed into its deviatoric and spherical parts, we arrive at

$$\begin{aligned} -\int_\Omega \bar{\boldsymbol{\zeta}} : \boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{d}x &= \int_\Omega \operatorname{div} \bar{\boldsymbol{\zeta}} \cdot \boldsymbol{v} \,\mathrm{d}x + \int_{\partial B_{1/2}} \big(\bar{\boldsymbol{\zeta}}|_{\Omega \setminus B_{1/2}} - \bar{\boldsymbol{\zeta}}|_{B_{1/2}}\big) \frac{x}{|x|} \cdot \boldsymbol{v} \,\mathrm{d}s \\ &= \int_\Omega \operatorname{div} \bar{\boldsymbol{\zeta}} \cdot \boldsymbol{v} \,\mathrm{d}x + \int_{\partial B_{1/2}} \big(-\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}) + \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^S\big) \frac{x}{|x|} \cdot \boldsymbol{v} \,\mathrm{d}s \\ &= \int_\Omega \operatorname{div} \bar{\boldsymbol{\zeta}} \cdot \boldsymbol{v} \,\mathrm{d}x + \int_{\partial B_{1/2}} \big(-\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^D - \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^S + \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^S\big) \frac{x}{|x|} \cdot \boldsymbol{v} \,\mathrm{d}s \\ &= \int_\Omega \operatorname{div} \bar{\boldsymbol{\zeta}} \cdot \boldsymbol{v} \,\mathrm{d}x + \int_{\partial B_{1/2}} -\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^D \frac{x}{|x|} \cdot \boldsymbol{v} \,\mathrm{d}s. \end{aligned}$$

Because of (2.132a) and (2.132b) the partial derivative $\partial_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})$ satisfies

$$\begin{aligned}\partial_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})\boldsymbol{v} &= \int_{\Omega} (\bar{\boldsymbol{u}} - \boldsymbol{u}_{d1}) \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\partial B_{1/2}} (\bar{\boldsymbol{u}} - \boldsymbol{u}_{d2}) \cdot \boldsymbol{v} \, \mathrm{d}s \\ &= \int_{B_{1/2}} -2 \operatorname{div} \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}) \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\Omega \setminus B_{1/2}} - \operatorname{div} \left(\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}) + \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^S\right) \cdot \boldsymbol{v} \, \mathrm{d}x \\ &\quad + \int_{\partial B_{1/2}} \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^D \frac{x}{|x|} \cdot \boldsymbol{v} \, \mathrm{d}s\end{aligned}$$

for all $\boldsymbol{v} \in H_0^1(\Omega; \mathbb{R}^2)$ so that (2.135c) is a result of (2.139). In addition, we infer $\partial_{\boldsymbol{f}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) = \nu(\bar{\boldsymbol{f}} - \boldsymbol{f}_d) = \bar{\boldsymbol{\omega}}$ from (2.132c), which shows (2.135d). Finally we set $\bar{\mu} = \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^D : \left(\bar{\boldsymbol{\zeta}}^D + \bar{\boldsymbol{\psi}}^D\right)$. Together, (2.138), (2.139) and $\left(\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^S\right)^D = \mathbf{0}$ imply

$$\bar{\boldsymbol{\zeta}}^D + \bar{\boldsymbol{\psi}}^D = \begin{cases} 2\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^D, & x \in B_{1/2} \\ \mathbf{0}, & x \in \Omega \setminus B_{1/2} \end{cases} \tag{2.140}$$

and hence $\bar{\mu} \geq 0$ in Ω, i.e., (2.135g) holds for arbitrary $\kappa_1 > 0$. For $(\bar{\boldsymbol{\zeta}}, \bar{\boldsymbol{\psi}}, \bar{\boldsymbol{w}}) \in L^\infty(\Omega; \mathbb{S}) \times L^\infty(\Omega; \mathbb{S}) \times H_0^1\left(\Omega; \mathbb{R}^2\right)$ and $(\bar{\mu}, \bar{\theta}) \in L^\infty(\Omega) \times L^\infty(\Omega)$ by the regularity of $\bar{\boldsymbol{u}}$, the integrability condition (2.134) is met. In order to verify (b) we recall that the second derivative of the Lagrangian w.r.t. the primal variables is given by

$$\begin{aligned}&\partial^2_{((\boldsymbol{\sigma},\boldsymbol{\chi}),\boldsymbol{u},\lambda,\boldsymbol{f})} \mathcal{L}\big((\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}}), \bar{\boldsymbol{u}}, \bar{\lambda}, \bar{\boldsymbol{f}}, (\bar{\boldsymbol{\zeta}}, \bar{\boldsymbol{\psi}}), \bar{\boldsymbol{w}}, \bar{\mu}, \bar{\theta}\big)\big((\boldsymbol{\sigma}', \boldsymbol{\chi}'), \boldsymbol{u}', \lambda', \boldsymbol{h}\big)^2 = \\ &= \left\|\boldsymbol{u}'\right\|^2_{L^2(\Omega;\mathbb{R}^2)} + \left\|\boldsymbol{u}'\right\|^2_{L^2(\partial B_{1/2};\mathbb{R}^2)} + \nu \left\|\boldsymbol{h}\right\|^2_{L^2(\Omega;\mathbb{R}^2)} \\ &\quad + 2\left(\lambda'\big((\boldsymbol{\sigma}')^D + (\boldsymbol{\chi}')^D\big), \big(\bar{\boldsymbol{\zeta}}^D + \bar{\boldsymbol{\psi}}^D\big)\right) + \left(\left\|(\boldsymbol{\sigma}')^D + (\boldsymbol{\chi}')^D\right\|^2_{\mathbb{S}}, \bar{\theta}\right),\end{aligned}$$

cf. (2.71). In view of (2.136e), (2.140) and (2.137) we deduce

$$\begin{aligned}&\partial^2_{((\boldsymbol{\sigma},\boldsymbol{\chi}),\boldsymbol{u},\lambda,\boldsymbol{f})} \mathcal{L}\big((\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}}), \bar{\boldsymbol{u}}, \bar{\lambda}, \bar{\boldsymbol{f}}, (\bar{\boldsymbol{\zeta}}, \bar{\boldsymbol{\psi}}), \bar{\boldsymbol{w}}, \bar{\mu}, \bar{\theta}\big)\big((\boldsymbol{\sigma}', \boldsymbol{\chi}'), \boldsymbol{u}', \lambda', \boldsymbol{h}\big)^2 = \\ &= \left\|\boldsymbol{u}'\right\|^2_{L^2(\Omega;\mathbb{R}^2)} + \left\|\boldsymbol{u}'\right\|^2_{L^2(\partial B_{1/2};\mathbb{R}^2)} + \nu \left\|\boldsymbol{h}\right\|^2_{L^2(\Omega;\mathbb{R}^2)} + \left(\left\|(\boldsymbol{\sigma}')^D + (\boldsymbol{\chi}')^D\right\|^2_{\mathbb{S}}, \bar{\theta}\right) \\ &\geq \nu \left\|\boldsymbol{h}\right\|^2_{L^2(\Omega;\mathbb{R}^2)}\end{aligned}$$

for all $\boldsymbol{h} \in L^2\left(\Omega; \mathbb{R}^2\right)$ and $\left((\boldsymbol{\sigma}', \boldsymbol{\chi}'), \boldsymbol{u}', \lambda'\right)$ solving (2.136). Thanks to Theorem 2.36 the control $\bar{\boldsymbol{f}}$ with associated state and multiplier $\left((\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}}), \bar{\boldsymbol{u}}, \bar{\lambda}\right)$ is thus locally optimal for $(\mathbf{P}_E)$. Moreover, according to Proposition 2.51 the biactive set $\bar{\mathcal{B}}$ fulfills $|\bar{\mathcal{B}}| > 0$. □

We end this section by providing a more detailed description of the local solution $\bar{\boldsymbol{f}}$ as well as the problem data $\boldsymbol{u}_{d1}$, $\boldsymbol{u}_{d2}$ and $\boldsymbol{f}_d$. From (2.123) and (2.127) we derive

$$\bar{\boldsymbol{f}} = \operatorname{div} \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}) = \begin{pmatrix} \frac{\partial^2 \bar{\boldsymbol{u}}_1}{\partial x_1^2} + \frac{1}{2}\left(\frac{\partial^2 \bar{\boldsymbol{u}}_1}{\partial x_2^2} + \frac{\partial^2 \bar{\boldsymbol{u}}_2}{\partial x_2 \partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial^2 \bar{\boldsymbol{u}}_1}{\partial x_1 \partial x_2} + \frac{\partial^2 \bar{\boldsymbol{u}}_2}{\partial x_1^2}\right) + \frac{\partial^2 \bar{\boldsymbol{u}}_2}{\partial x_2^2} \end{pmatrix}.$$

The second-order partial derivatives of $\bar{\boldsymbol{u}}$ are given by

$$\frac{\partial^2 \bar{\boldsymbol{u}}_1}{\partial x_1^2} = \frac{\partial^2 \bar{\boldsymbol{u}}_2}{\partial x_1^2} = 4x_1^2 U''\big(|x|^2\big) + 2U'\big(|x|^2\big) \tag{2.141a}$$

$$\frac{\partial^2 \bar{\boldsymbol{u}}_1}{\partial x_2^2} = \frac{\partial^2 \bar{\boldsymbol{u}}_2}{\partial x_2^2} = 4x_2^2 U''\big(|x|^2\big) + 2U'\big(|x|^2\big) \tag{2.141b}$$

$$\frac{\partial^2 \bar{\boldsymbol{u}}_1}{\partial x_1 \partial x_2} = \frac{\partial^2 \bar{\boldsymbol{u}}_2}{\partial x_1 \partial x_2} = \frac{\partial^2 \bar{\boldsymbol{u}}_1}{\partial x_2 \partial x_1} = \frac{\partial^2 \bar{\boldsymbol{u}}_2}{\partial x_2 \partial x_2} = 4x_1 x_2 U''\big(|x|^2\big), \tag{2.141c}$$

cf. also (2.129), so that

$$\bar{\boldsymbol{f}} = \begin{pmatrix} \left(4x_1^2 + 2x_2^2 + 2x_1x_2\right) U''\big(|x|^2\big) + 3U'\big(|x|^2\big) \\ \left(2x_1^2 + 4x_2^2 + 2x_1x_2\right) U''\big(|x|^2\big) + 3U'\big(|x|^2\big) \end{pmatrix}. \tag{2.142}$$

Consequently, (2.132c) leads to

$$\boldsymbol{f}_d = \begin{pmatrix} \left(4x_1^2 + 2x_2^2 + 2x_1x_2\right) U''\big(|x|^2\big) + 3U'\big(|x|^2\big) - \frac{2}{\nu} U\big(|x|^2\big) \\ \left(2x_1^2 + 4x_2^2 + 2x_1x_2\right) U''\big(|x|^2\big) + 3U'\big(|x|^2\big) - \frac{2}{\nu} U\big(|x|^2\big) \end{pmatrix}.$$

Moreover, (2.133) and (2.141) imply

$$\begin{aligned} \operatorname{div}\left(\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})^S\right) &= \operatorname{div} \begin{pmatrix} \frac{1}{2}\left(\frac{\partial \bar{\boldsymbol{u}}_1}{\partial x_1} + \frac{\partial \bar{\boldsymbol{u}}_2}{\partial x_2}\right) & 0 \\ 0 & \frac{1}{2}\left(\frac{\partial \bar{\boldsymbol{u}}_1}{\partial x_1} + \frac{\partial \bar{\boldsymbol{u}}_2}{\partial x_2}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(\frac{\partial^2 \bar{\boldsymbol{u}}_1}{\partial x_1^2} + \frac{\partial^2 \bar{\boldsymbol{u}}_2}{\partial x_1 \partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial^2 \bar{\boldsymbol{u}}_1}{\partial x_2 \partial x_1} + \frac{\partial^2 \bar{\boldsymbol{u}}_2}{\partial x_1^2}\right) \end{pmatrix} \\ &= \begin{pmatrix} 2(x_1^2 + x_1x_2) U''\big(|x|^2\big) + U'\big(|x|^2\big) \\ 2(x_2^2 + x_1x_2) U''\big(|x|^2\big) + U'\big(|x|^2\big) \end{pmatrix}. \end{aligned}$$

Due to (2.123), (2.132a) and (2.142) we therefore obtain

$$\boldsymbol{u}_{d1}(x) = \begin{cases} \begin{pmatrix} U\big(|x|^2\big) + \left(8x_1^2 + 4x_2^2 + 4x_1x_2\right) U''\big(|x|^2\big) + 6U'\big(|x|^2\big) \\ U\big(|x|^2\big) + \left(4x_1^2 + 8x_2^2 + 4x_1x_2\right) U''\big(|x|^2\big) + 6U'\big(|x|^2\big) \end{pmatrix}, & x \in B_{1/2} \\ \begin{pmatrix} U\big(|x|^2\big) + \left(6x_1^2 + 2x_2^2 + 4x_1x_2\right) U''\big(|x|^2\big) + 4U'\big(|x|^2\big) \\ U\big(|x|^2\big) + \left(2x_1^2 + 6x_2^2 + 4x_1x_2\right) U''\big(|x|^2\big) + 4U'\big(|x|^2\big) \end{pmatrix}, & \text{otherwise.} \end{cases}$$

Furthermore, (2.128), (2.129) and (2.132b) yield

$$\begin{aligned} \boldsymbol{u}_{d2}(x) &= \begin{pmatrix} U\big(|x|^2\big) \\ U\big(|x|^2\big) \end{pmatrix} - \frac{1}{|x|} \begin{pmatrix} \frac{1}{2}\left(\frac{\partial \bar{\boldsymbol{u}}_1}{\partial x_1} - \frac{\partial \bar{\boldsymbol{u}}_2}{\partial x_2}\right) x_1 + \frac{1}{2}\left(\frac{\partial \bar{\boldsymbol{u}}_1}{\partial x_2} + \frac{\partial \bar{\boldsymbol{u}}_2}{\partial x_1}\right) x_2 \\ \frac{1}{2}\left(\frac{\partial \bar{\boldsymbol{u}}_1}{\partial x_2} + \frac{\partial \bar{\boldsymbol{u}}_2}{\partial x_1}\right) x_1 + \frac{1}{2}\left(\frac{\partial \bar{\boldsymbol{u}}_2}{\partial x_2} - \frac{\partial \bar{\boldsymbol{u}}_1}{\partial x_1}\right) x_2 \end{pmatrix} \\ &= \begin{pmatrix} U\big(|x|^2\big) \\ U\big(|x|^2\big) \end{pmatrix} - \frac{1}{|x|} \begin{pmatrix} x_1^2 U'\big(|x|^2\big) + x_2^2 U'\big(|x|^2\big) \\ x_1^2 U'\big(|x|^2\big) + x_2^2 U'\big(|x|^2\big) \end{pmatrix} \\ &= \begin{pmatrix} U\big(|x|^2\big) - U'\big(|x|^2\big)|x| \\ U\big(|x|^2\big) - U'\big(|x|^2\big)|x| \end{pmatrix}. \end{aligned}$$

Because only the values on the boundary of $B_{1/2}$ matter, we conclude from (2.121) and (2.122) that

$$\boldsymbol{u}_{d2} = \begin{pmatrix} -\sigma_0 \\ -\sigma_0 \end{pmatrix} \quad \text{on } \partial B_{1/2} = \big\{x \in \mathbb{R}^2 \colon\ |x| = 1/2\big\}.$$

3. Optimal control of Signorini's problem

This chapter is dedicated to optimal control problems governed by ($\mathbf{VI}_S$). To be more precise, for a given objective functional $J\colon V \times U \to \mathbb{R}$ we now consider the optimization problem

$$\left.\begin{aligned} &\text{Minimize} && J(\boldsymbol{u}, \boldsymbol{f}) \\ &\text{s.t.} && \text{the Signorini problem } (\mathbf{VI}_S) \text{ with } \ell \in V' \text{ defined by} \\ & && \langle \ell, \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{f}_1 \cdot \boldsymbol{v} \,\mathrm{d}x + \int_{\Gamma_N} \boldsymbol{f}_2 \cdot \boldsymbol{v} \quad \forall\, \boldsymbol{v} \in V. \end{aligned}\right\} \tag{$\mathbf{P}_S$}$$

As in ($\mathbf{P}_E$), the functions $\boldsymbol{f}_1 \in L^2\big(\Omega; \mathbb{R}^d\big)$ and $\boldsymbol{f}_2 \in L^2\big(\Gamma_N; \mathbb{R}^d\big)$ represent volume and boundary loads, respectively, which are imposed on the domain Ω. By inserting the law of linear elasticity $\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u})$ in the variational inequality in ($\mathbf{VI}_S$), we see that the Signorini problem can be rewritten as follows: Given an inhomogeneity $\ell \in V'$ find a displacement $\boldsymbol{u} \in \mathcal{C}$ such that

$$\langle B\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u} \rangle \geq \langle \ell, \boldsymbol{v} - \boldsymbol{u} \rangle \quad \forall\, \boldsymbol{v} \in \mathcal{C}. \tag{3.1}$$

The linear operator $B\colon V \to V'$ is defined through

$$\langle B\boldsymbol{u}, \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{d}x \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V,$$

where $\mathbb{C}(x)$ is a linear mapping from $\mathbb{S}$ to $\mathbb{S}$, which may depend on the spatial variable x, and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the linearized strain tensor, cf. (2.1). The closed and convex set of admissible displacements is determined by

$$\mathcal{C} = \big\{\boldsymbol{v} \in V : \tau_{\boldsymbol{\nu}}\boldsymbol{v} \leq \psi \text{ a.e. on } \Gamma_C\big\}. \tag{3.2}$$

Herein, $\psi \in H^{1/2}(\Gamma)$ is given and $\tau_{\boldsymbol{\nu}}$ is the normal trace operator, cf. Section A.2. The function ψ represents the initial gap between Γ_C and the surface of the rigid obstacle, against which the domain Ω is pushed. From now on, when referring to ($\mathbf{VI}_S$), we will think of (3.1).

Throughout this chapter the next assumption is supposed to hold.

Assumption 3.1.

(1) *The domain* $\Omega \subset \mathbb{R}^d$*,* $d \in \{2,3\}$*, is bounded and of class* $C^{1,1}$*. Its boundary* Γ *consists of three disjoint measurable parts* Γ_D*,* Γ_N *and* Γ_C*, each of which has positive measure, such that* $\Gamma_D \cup \Gamma_N \cup \Gamma_C = \Gamma$*. While* Γ_N *is a relatively open subset,* Γ_D *and* Γ_C *are a relatively closed subsets of* Γ *with* $\operatorname{dist}(\Gamma_C, \Gamma_D) > 0$*.*

(2) *The forth-order tensor* $\mathbb{C}$ *is an element of* $L^\infty\big(\Omega; \mathcal{L}(\mathbb{S})\big)$*. Moreover,* $\mathbb{C}(x)$ *is uniformly coercive on* $\mathbb{S}$ *and symmetric, i.e.,* $\boldsymbol{\sigma} : \mathbb{C}(x)\boldsymbol{\sigma} \geq c\,\|\boldsymbol{\sigma}\|_{\mathbb{S}}^2$ *with* $c > 0$ *independent of* x *and* $\boldsymbol{\tau} : \mathbb{C}(x)\boldsymbol{\sigma} = \boldsymbol{\sigma} : \mathbb{C}(x)\boldsymbol{\tau}$ *hold for all* $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}$*.*

(3) *There exists* $\boldsymbol{v} \in V$ *such that* $\psi = \tau_{\boldsymbol{\nu}}\boldsymbol{v}$*.*

Remark 3.2. *Assumption 3.1(2) is for instance fulfilled, if* $\mathbb{C}$ *is the inverse of the tensor* $\mathbb{C}^{-1}$ *defined in* (2.4)*, i.e., if* $\mathbb{C}$ *is given by*

$$\mathbb{C}\boldsymbol{\sigma} = 2\,\mu_L\boldsymbol{\sigma} + \lambda_L(\operatorname{trace}\boldsymbol{\sigma})\boldsymbol{I}$$

with Lamé constants satisfying $\mu_L > 0$ *and* $d\,\lambda_L + 2\,\mu_L > 0$*:*

$$\boldsymbol{\sigma} : \mathbb{C}\boldsymbol{\sigma} = 2\,\mu_L\,\|\boldsymbol{\sigma}\|_{\mathbb{S}}^2 + \underbrace{d\,\lambda_L\frac{(\operatorname{trace}\boldsymbol{\sigma})^2}{d}}_{>-2\,\mu_L\|\boldsymbol{\sigma}\|_{\mathbb{S}}^2} \geq c\,\|\boldsymbol{\sigma}\|_{\mathbb{S}}^2,$$

cf. also Remark 2.2. This tensor is commonly known as elasticity tensor.

3.1. Existence of solutions

In the following we shortly comment on the existence of solutions to the Signorini problem ($\mathbf{VI}_S$) and the corresponding control problem ($\mathbf{P}_S$), respectively.

First, we recall a known result for variational inequalities.

Theorem 3.3 ([58, Theorem 2.1])**.** *Let* X *be a Hilbert space and* $C \subset X$ *a closed convex subset. Moreover let* $H \in \mathcal{L}(X, X')$ *be coercive and* $f \in X'$*. Then there exists a unique* $x \in C$ *such that*

$$\langle Hx,\, y - x\rangle_{X',X} \geq \langle f,\, y - x\rangle_{X',X} \quad \forall\, y \in C. \tag{3.3}$$

Furthermore, the mapping $f \mapsto x$ *is Lipschitz continuous from* X' *into* X*.*

The operator B and the set $\mathcal{C}$ involved in (3.1) meet the conditions of Theorem 3.3 so that we infer the existence and uniqueness of a solution to ($\mathbf{VI}_S$). Note that V is a Hilbert space in consequence of the continuity of the trace operator τ, cf. Theorem A.14.

Proposition 3.4. *For every $\ell \in V'$ problem* (3.1) *admits a unique solution $\boldsymbol{u} \in \mathcal{C}$. Furthermore, the mapping $\ell \mapsto \boldsymbol{u}$ is Lipschitz continuous from V' into V.*

Proof. Since $\mathbb{C}$ is uniformly coercive and Γ_D has positive measure by Assumption 2.1, Korn's inequality (Proposition A.25) implies that $B\colon V' \to V$ is coercive. In addition, it is continuous because of the boundedness of $\mathbb{C}$. For $\tau_{\boldsymbol{\nu}}$ is linear and continuous, cf. Theorem A.22, the set $\mathcal{C} \subset V$ is closed and convex. Hence, the assertion follows from Theorem 3.3. □

In view of Proposition 3.4 we are allowed to introduce the control-to-state map associated with ($\mathbf{VI}_S$).

Definition 3.5. *The control-to-state map $V' \ni \ell \mapsto \boldsymbol{u} \in V$ is denoted by G_S.*

Completely analogously to Proposition 2.9 the existence of a generally not unique solution to ($\mathbf{P}_S$) can be derived from the Lipschitz continuity of G_S and the compactness of the operator $R\colon U \to V'$ defined in (2.14).

Proposition 3.6. *Suppose the objective functional $J\colon V \times U \to \mathbb{R}$ is weakly lower semicontinuous. If there exist $r > 0$ and $\hat{\boldsymbol{f}} \in U$ such that*

$$J\big(G_S(R\boldsymbol{f}), \boldsymbol{f}\big) \geq J\big(G_S(R\hat{\boldsymbol{f}}), \hat{\boldsymbol{f}}\big) \quad \forall\, \boldsymbol{f} \in U \text{ with } \big\|\boldsymbol{f} - \hat{\boldsymbol{f}}\big\|_U > r,$$

then Problem ($\mathbf{P}_S$) admits a globally optimal solution.

In the remainder of this section three equivalent reformulations of ($\mathbf{VI}_S$) are presented, which will be useful for the subsequent analysis.

Thanks to its continuity and coercivity the operator B induces an equivalent norm on V. Moreover the Lax-Milgram theorem yields that B is invertible. Therefore, the function $\boldsymbol{u} \in \mathcal{C}$ solves (3.1) if and only if

$$\langle B(\boldsymbol{u} - \boldsymbol{w}),\, \boldsymbol{v} - \boldsymbol{u}\rangle \geq 0 \quad \forall\, \boldsymbol{v} \in \mathcal{C} \tag{3.4}$$

with $\boldsymbol{w} = B^{-1}\ell \in V$. The solution of (3.4) is the projection of $\boldsymbol{w}$ on the set $\mathcal{C}$ w.r.t. to the norm induced by B.

Definition 3.7. *The solution operator associated with* (3.4) *which maps $V \ni \boldsymbol{w} \mapsto \boldsymbol{u} \in \mathcal{C}$ is denoted by $P_{\mathcal{C}}^B$.*

Furthermore, due to the coercivity of B the mapping $\boldsymbol{u} \mapsto \langle B(\boldsymbol{u} - \boldsymbol{w}),\, \boldsymbol{u} - \boldsymbol{w}\rangle$ defines a strictly convex functional on V. As the operator B is symmetric resulting from the symmetry of $\mathbb{C}$, (3.4) is thus the necessary and sufficient optimality condition of the convex optimization problem

$$\left.\begin{array}{rl} \text{Minimize} & \dfrac{1}{2}\,\langle B(\boldsymbol{u}-\boldsymbol{w}),\, \boldsymbol{u}-\boldsymbol{w}\rangle \\ \text{s.t.} & \boldsymbol{u}\in V,\ F(\boldsymbol{u})\in K, \end{array}\right\} \tag{3.5}$$

where the convex cone K is given by

$$K = \big\{z \in \tau_{\boldsymbol{\nu}}[V] \colon z \le 0 \text{ a.e. on } \Gamma_C\big\}$$

and the affine mapping $F\colon V \to \tau_{\boldsymbol{\nu}}[V]$ is defined through

$$F(\boldsymbol{u}) = \tau_{\boldsymbol{\nu}}\boldsymbol{u} - \psi.$$

Note here that ψ is an element of $\tau_{\boldsymbol{\nu}}[V]$, cf. Assumption 3.1(3). The image $\tau_{\boldsymbol{\nu}}[V] \subset H^{1/2}(\Gamma)$ of $\tau_{\boldsymbol{\nu}}$ is a Hilbert space according to Corollary A.23. Since $\mathcal{K}_{\boldsymbol{u}}(V) = V$ and $F' = \tau_{\boldsymbol{\nu}}$ is surjective from V into $\tau_{\boldsymbol{\nu}}[V]$, every admissible function for (3.5) is regular in the sense of Zowe-Kurcyusz [77]:

Lemma 3.8. *Every $\boldsymbol{u} \in \mathcal{C}$ is regular in the sense of Zowe-Kurcyusz [77], i.e., for $\boldsymbol{u} \in \mathcal{C}$ it holds $F'(\boldsymbol{u})\mathcal{K}_{\boldsymbol{u}}(V) - \mathcal{K}_{F(\boldsymbol{u})}(K) = \tau_{\boldsymbol{\nu}}[V]$.*

On account of Lemma 3.8 the Signorini problem $(\mathbf{VI}_S)$ can be reformulated by means of a complementarity system:

Theorem 3.9. *A function $\boldsymbol{u} \in V$ is the solution of $(\mathbf{VI}_S)$ if and only if there exists a unique Lagrange multiplier $\lambda \in \big(\tau_{\boldsymbol{\nu}}[V]\big)'$ such that*

$$\langle B\boldsymbol{u},\, \boldsymbol{v}\rangle + \langle \lambda,\, \tau_{\boldsymbol{\nu}}\boldsymbol{v}\rangle = \langle \ell,\, \boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in V \tag{3.6a}$$

$$\lambda \in K^0, \quad \langle \lambda,\, F(\boldsymbol{u})\rangle = 0, \quad F(\boldsymbol{u}) \in K. \tag{3.6b}$$

Proof. Let $\boldsymbol{u} \in V$ be the solution of $(\mathbf{VI}_S)$, or equivalently, the solution of (3.5) with $\boldsymbol{w} = B^{-1}\ell$. Because $\boldsymbol{u}$ is regular in the sense of Zowe-Kurcyusz, the existence of a Lagrange multiplier is a corollary of [77, Theorem 3.1]. We suppose that there exist two multipliers $\lambda_1, \lambda_2 \in \big(\tau_{\boldsymbol{\nu}}[V]\big)'$ with $\lambda_1 \neq \lambda_2$ satisfying (3.6). Subtracting equation (3.6a) for λ_2 from the one for λ_1 leads to $\langle \lambda_1 - \lambda_2,\, \tau_{\boldsymbol{\nu}}\boldsymbol{v}\rangle = 0$ for all $\boldsymbol{v} \in V$. By definition this is equivalent to $\lambda_1 = \lambda_2$ in $(\tau_{\boldsymbol{\nu}}[V])'$. Consequently, the Lagrange multiplier is unique.
If $(\boldsymbol{u}, \lambda) \in V \times \big(\tau_{\boldsymbol{\nu}}[V]\big)'$ is a solution to (3.6), then for $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ and $\alpha \in [0,1]$ we find $F(\alpha\boldsymbol{v}_1 + (1-\alpha)\boldsymbol{v}_2) - \alpha F(\boldsymbol{v}_1) - (1-\alpha)F(\boldsymbol{v}_2) = 0 \in K$, which together with $\lambda \in K^0$ implies that $V \ni \boldsymbol{v} \mapsto \langle B(\boldsymbol{v}-\boldsymbol{w}),\, \boldsymbol{v}-\boldsymbol{w}\rangle + \langle \lambda,\, F(\boldsymbol{v})\rangle \in \mathbb{R}$ is a convex function. Hence, condition (3.6a) is necessary and sufficient for $\boldsymbol{u}$ to solve the problem

$$\begin{array}{rl} \text{Minimize} & \dfrac{1}{2}\,\langle B(\boldsymbol{v}-\boldsymbol{w}),\, \boldsymbol{v}-\boldsymbol{w}\rangle + \langle \lambda,\, F(\boldsymbol{v})\rangle \\ \text{s.t.} & \boldsymbol{v} \in V. \end{array}$$

Recall here the strict convexity of $\langle B(\cdot - \boldsymbol{w}),\, \cdot - \boldsymbol{w}\rangle$ and the symmetry of B. From (3.6b) we then infer

$$\begin{aligned}\frac{1}{2}\langle B(\boldsymbol{u}-\boldsymbol{w}),\, \boldsymbol{u}-\boldsymbol{w}\rangle &= \frac{1}{2}\langle B(\boldsymbol{u}-\boldsymbol{w}),\, \boldsymbol{u}-\boldsymbol{w}\rangle + \underbrace{\langle \lambda,\, F(\boldsymbol{u})\rangle}_{=0} \\ &\leq \frac{1}{2}\langle B(\boldsymbol{v}-\boldsymbol{w}),\, \boldsymbol{v}-\boldsymbol{w}\rangle + \underbrace{\langle \lambda,\, F(\boldsymbol{v})\rangle}_{\leq 0} \\ &\leq \frac{1}{2}\langle B(\boldsymbol{v}-\boldsymbol{w}),\, \boldsymbol{v}-\boldsymbol{w}\rangle\end{aligned}$$

for all $\boldsymbol{v} \in V$ with $F(\boldsymbol{v}) \in K$ so that $\boldsymbol{u}$ is the solution to (3.5). □

3.2. Directional differentiability

In this section we are concerned with the differentiability of the control-to-state map $G_S\colon V' \to V$. By adapting the technique of Mignot [62, Section 3] we will show that this operator is directionally differentiable.

The next Theorem states a criterion, which guarantees directional differentiability of a projection.

Theorem 3.10 ([62, Théorème 2.1])**.** *Let X be a Hilbert space, $C \subset X$ a closed convex subset and $H \in \mathcal{L}(X, X')$ be symmetric and coercive. Moreover let P_C^H denote the projection on the set C w.r.t. norm induced by H. Suppose further that*

$$\overline{\mathcal{K}_y(C) \cap [\operatorname{span}(x-y)]_H^0} = \overline{\mathcal{K}_y(C)} \cap [\operatorname{span}(x-y)]_H^0 := S^y \tag{3.7}$$

with $x \in X$ and $y = P_C^H(x)$. Then P_C^H is directionally differentiable at x and its directional derivative is given by the projection on the set S^y w.r.t. the norm induced by H. In particular it holds

$$P_C^H(x+th) = P_C^H(x) + tP_{S^y}^H(h) + o(t,h) \quad \forall\, h \in X,$$

where $o(t,h)/t \to 0$ as $t \to 0$.

In what follows we aim to verify (3.7) for the operator $P_{\mathcal{C}}^B$. To be more precise, we want to establish

$$\overline{\mathcal{K}_{\boldsymbol{u}}(\mathcal{C}) \cap [\operatorname{span}(\boldsymbol{w}-\boldsymbol{u})]_B^0} = \overline{\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})} \cap [\operatorname{span}(\boldsymbol{w}-\boldsymbol{u})]_B^0 \tag{3.8}$$

for all $\boldsymbol{w} \in V$ and $\boldsymbol{u} = P_{\mathcal{C}}^B(\boldsymbol{w})$. Because the control-to-state map G_S satisfies

$$G_S(\ell) = \left(P_{\mathcal{C}}^B \circ B^{-1}\right)(\ell), \tag{3.9}$$

cf. (3.1) and (3.4), Theorem 3.10 together with the linearity of B would imply the directional differentiability of G_S, if the density property (3.8) was fulfilled.

3.2.1. Extension of Riesz' representation theorem

In order to prove (3.8) we have to extend the Riesz representation theorem to positive functionals $f \in \big(H^1(\Omega)\big)'$. This will allow us to derive characterizations of the sets involved in the right-hand side of (3.8).

The following lemma is a first step towards this extension.

Lemma 3.11. *Let $f \in \big(H^1(\Omega)\big)'$ be positive, i.e., it holds $\langle f,\, v\rangle \geq 0$ for all $v \in H^1(\Omega)$ with $v \geq 0$ a.e. in Ω. Then there exists a unique measure $\mu \in \mathcal{M}^+(\overline{\Omega})$ such that*

$$\langle f,\, v\rangle = \int_{\overline{\Omega}} v \,\mathrm{d}\mu \tag{3.10}$$

for all $v \in H^1(\Omega) \cap C(\overline{\Omega})$.

Proof. Let $v \in H^1(\Omega) \cap C(\overline{\Omega})$ be arbitrary. For $\|v\|_{C(\overline{\Omega})} - v \geq 0$ in Ω, we infer from the positivity of f that

$$\langle f,\, v\rangle \leq \|v\|_{C(\overline{\Omega})} \langle f,\, 1\rangle\,. \tag{3.11}$$

Since the right-hand side in (3.11) defines a sublinear function $p\colon C(\overline{\Omega}) \to \mathbb{R}$, the Hahn-Banach theorem yields the existence of an extension $\ell \in \big(C(\overline{\Omega})\big)'$ with

$$\ell(v) = \langle f,\, v\rangle \quad \forall\, v \in H^1(\Omega) \cap C(\overline{\Omega}). \tag{3.12}$$

We assume there is another extension $\tilde{\ell} \in \big(C(\overline{\Omega})\big)'$, $\tilde{\ell} \neq \ell$, satisfying (3.12) and hence $\tilde{\ell}(v) = \ell(v)$ for all $v \in C^\infty(\overline{\Omega}) \subset H^1(\Omega) \cap C(\overline{\Omega})$. However, by the Stone-Weierstrass theorem we conclude $\tilde{\ell}(v) = \ell(v)$ for all $v \in C(\overline{\Omega})$ so that ℓ is unique. Moreover, due to the Riesz representation theorem there exists a unique $\mu \in \mathcal{M}(\overline{\Omega})$ with

$$\ell(v) = \int_{\overline{\Omega}} v \,\mathrm{d}\mu \quad \forall\, v \in C(\overline{\Omega}). \tag{3.13}$$

Together, (3.12) and (3.13) lead to

$$\langle f,\, v\rangle = \int_{\overline{\Omega}} v \,\mathrm{d}\mu$$

for all $v \in H^1(\Omega) \cap C(\overline{\Omega})$. It remains to be proven that $\mu \in \mathcal{M}^+(\overline{\Omega})$. For this purpose we consider $K \subset \mathcal{B}(\overline{\Omega})$ compact and the finite open covering

$$G_\epsilon := \bigcup_{i=1}^{n} B_\epsilon(x_i), \quad x_i \in K, \quad n \in \mathbb{N}, \quad \epsilon > 0.$$

Thanks to Lemma A.24 (partition of unity) there is $\varphi_K^\epsilon \in C_0^\infty(G_\epsilon)$ with $0 \leq \varphi_K^\epsilon \leq 1$ and $\varphi_K^\epsilon \equiv 1$ in K. As φ_K^ϵ is bounded by $1 \in H^1(\Omega) \cap C(\overline{\Omega}) \subset L^1(\overline{\Omega}; \mu)$ and

$$H^1(\Omega) \cap C(\overline{\Omega}) \ni \varphi_K^\epsilon \xrightarrow{\epsilon \to 0} \chi_K \quad \text{pointwisely in } \mathbb{R}^d,$$

we deduce from Lebesgue's dominated convergence theorem

$$\langle f, \varphi_K^\epsilon \rangle = \int_{\overline{\Omega}} \varphi_K^\epsilon \, \mathrm{d}\mu \xrightarrow{\epsilon \to 0} \int_{\overline{\Omega}} \chi_K \, \mathrm{d}\mu = \mu(K).$$

Note that Lebesgue's theorem can be applied for a general signed measure thanks to the Hahn-Jordan decomposition $\mu = \mu^+ - \mu^-$ with positive measures μ^+ and μ^-. Because of $\varphi_K^\epsilon \geq 0$ it follows $\langle f, \varphi_K^\epsilon \rangle \geq 0$ for all $\epsilon > 0$ by assumption and thus $\mu(K) \geq 0$. The regularity of μ implies for arbitrary $E \in \mathcal{B}(\overline{\Omega})$

$$\mu(E) = \sup\{\mu(K) \colon K \subset E, K \text{ compact}\} \geq 0$$

so that μ is indeed positive. □

Now we aim to get rid of the set $C(\overline{\Omega})$. More precisely, we want to show that every $v \in H^1(\Omega)$ has a particular representative satisfying (3.10). The proof will be performed in two steps:

1. We establish (3.10) for truncated $H^1(\Omega)$-functions by means of Lebesgue's dominated convergence theorem.

2. We conclude (3.10) for arbitrary functions $v \in H^1(\Omega)$ with the help of the monotone convergence theorem.

Therefor the convergence of truncated $H^1(\Omega)$-functions must be investigated. We start by providing two known auxiliary results.

Lemma 3.12 ([1, Lemma A1.17]). *Let $f \in L^1(\Omega)$ and $\{E_n\}_{n\in\mathbb{N}} \subset \Omega$ be a sequence of measurable sets with $|E_n| \to 0$. Then it holds*

$$\int_{E_n} f \, \mathrm{d}\mu \to 0.$$

While Lemma 3.12 states the convergence of the integral, if the measure of the domain converges, the next one claims the converse.

Lemma 3.13. *Let $M \subset \Omega$ be measurable, $f \in L^1(\Omega)$ and $\{f_n\}_{n\in\mathbb{N}} \subset L^1(\Omega)$ with $f_n \to f$ in $L^1(\Omega)$. If $f > 0$ a.e. in M, then it holds $|\{x \in M \colon f_n(x) \leq 0\}| \to 0$.*

Proof. In [44, Lemma A.2] Herzog et al. already proved under the same assumptions that $|\{x \in M \colon f_n(x) = 0\}| \to 0$. Their proof can readily be adapted to the sets $\{x \in M \colon f_n(x) \leq 0\}$. □

The space $H^1(\Omega)$ is closed under truncation by Stampaccia [54, Theorem II.A.1]:

Theorem 3.14. *Let $v \in H^1(\Omega)$. Then it holds $v^+ \in H^1(\Omega)$ with (weak) derivative given by*

$$\partial_{x_i}(v^+) = \begin{cases} \partial_{x_i} v & \text{in } \{x \in \Omega\colon v(x) > 0\} \\ 0 & \text{in } \{x \in \Omega\colon v(x) \leq 0\} \end{cases} \quad \forall\, i \in \{1, \dots, d\}. \tag{3.14}$$

From Theorem 3.14 we deduce the following convergence result.

Lemma 3.15. *Let $v \in H^1(\Omega)$ and $\{v_n\}_{n \in \mathbb{N}} \subset H^1(\Omega)$ be a sequence with $v_n \to v$. Then it holds*

(i) $v_n^+ \to v^+$ *in* $H^1(\Omega)$ *as* $n \to \infty$

(ii) $\min(v, m) \to v$ *in* $H^1(\Omega)$ *as* $m \to \infty$.

Proof. (i): The domain Ω can be decomposed as $\Omega = A_n \cup B_n \cup C_n \cup D_n$ with $A_n = \{x \in \Omega\colon v_n(x) > 0, v(x) \geq 0\}$, $B_n = \{x \in \Omega\colon v_n(x) \leq 0, v(x) > 0\}$, $C_n = \{x \in \Omega\colon v_n(x) > 0, v(x) < 0\}$ and $D_n = \{x \in \Omega\colon v_n(x) \leq 0, v(x) \leq 0\}$. In view of (3.14) we thus obtain

$$\begin{aligned}\left\|v_n^+ - v^+\right\|_{H^1(\Omega)}^2 = &\int_{A_n} \left(|v_n - v|^2 + |\nabla(v_n - v)|^2 \right) \mathrm{d}x + \int_{B_n} \left(|v|^2 + |\nabla v|^2 \right) \mathrm{d}x \\ &+ \int_{C_n} \left(|v_n|^2 + |\nabla v_n|^2 \right) \mathrm{d}x.\end{aligned}$$

Applying Minkowski's inequality leads to

$$\begin{aligned}
&\left\|v_n^+ - v^+\right\|_{H^1(\Omega)}^2 \leq \\
&\leq \|v_n - v\|_{H^1(\Omega)}^2 + \int_{B_n} \left(|v|^2 + |\nabla v|^2 \right) \mathrm{d}x \\
&\quad + \left(\left(\int_{C_n} |v_n - v|^2 \ \mathrm{d}x \right)^{\frac{1}{2}} + \left(\int_{C_n} |v|^2 \ \mathrm{d}x \right)^{\frac{1}{2}} \right)^2 \\
&\quad + \left(\left(\int_{C_n} |\nabla(v_n - v)|^2 \ \mathrm{d}x \right)^{\frac{1}{2}} + \left(\int_{C_n} |\nabla v|^2 \ \mathrm{d}x \right)^{\frac{1}{2}} \right)^2 \\
&\leq 2 \|v_n - v\|_{H^1(\Omega)}^2 + \int_{B_n} \left(|v|^2 + |\nabla v|^2 \right) \mathrm{d}x + \int_{C_n} \left(|v|^2 + |\nabla v|^2 \right) \mathrm{d}x \\
&\quad + 2 \left(\int_{C_n} |v_n - v|^2 \ \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{C_n} |v|^2 \ \mathrm{d}x \right)^{\frac{1}{2}} \\
&\quad + 2 \left(\int_{C_n} |\nabla(v_n - v)|^2 \ \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{C_n} |\nabla v|^2 \ \mathrm{d}x \right)^{\frac{1}{2}} \\
&\leq 2 \|v_n - v\|_{H^1(\Omega)}^2 + \int_{B_n} \left(|v|^2 + |\nabla v|^2 \right) \mathrm{d}x + \int_{C_n} \left(|v|^2 + |\nabla v|^2 \right) \mathrm{d}x \\
&\quad + 4 \|v_n - v\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} .
\end{aligned}$$

If we set $M = \{x \in \Omega \colon v > 0\}$ and $M = \{x \in \Omega \colon -v > 0\}$, respectively, Lemma 3.13 implies

$$|B_n| \to 0 \quad \text{and} \quad |C_n| \to 0.$$

Thus, the assertion follows due to Lemma 3.12.
(ii): We decompose the domain as $\Omega = A_m \cup B_m$ with $A_m = \{x \in \Omega \colon v(x) \geq m\}$ and $B_m = \{x \in \Omega \colon v(x) < m\}$. Theorem 3.14 then yields

$$\|v - \min(v, m)\|_{H^1(\Omega)}^2 = \int_{A_m} |v - m|^2 + |\nabla v|^2 \, \mathrm{d}x \leq \int_{A_m} |v|^2 + |\nabla v|^2 \, \mathrm{d}x,$$

where $\min(v, m) = m - (m - v)^+$ was used. As $A_{m+1} \subset A_m$, we moreover find

$$\lim_{m \to \infty} |A_m| = \left| \bigcap\nolimits_{m \in \mathbb{N}} A_m \right| = |\{x \in \Omega \colon v(x) = \infty\}| = 0$$

so that (ii) is a consequence of Lemma 3.12. □

On account of Lemma 3.15 we observe that $H_0^1(\Omega)$ is also closed under truncation:

Corollary 3.16. *Let $v \in H_0^1(\Omega)$ and $m \in \mathbb{N}$. Then $v^+, \min(v, m) \in H_0^1(\Omega)$.*

Proof. Due to the definition of $H_0^1(\Omega)$ there exists a sequence $\{\varphi_n\}_{n \in \mathbb{N}} \subset C_0^\infty(\Omega)$ with $\varphi_n \to v$ in $H^1(\Omega)$. From Lemma 3.15(i) we infer

$$C_0(\overline{\Omega}) \cap H^1(\Omega) \ni \varphi_n^+ \to v^+ \quad \text{in } H^1(\Omega).$$

Together with Theorem A.13 and Corollary A.16 this shows

$$\tau v^+ = \lim_{n \to \infty} \tau \varphi_n^+ = 0 \quad \text{a.e. on } \Gamma.$$

Because of $\min(v, m) = m - (m - v)^+$ we similarly obtain $\tau \min(v, m) = 0$. □

In order to derive (3.10) for $v \in H^1(\Omega)$ it will moreover be crucial that every set of zero capacity has μ-measure zero. To this end we need the next auxiliary result.

Lemma 3.17. *Let $E \subset \Omega$, $\mathcal{F} = \{v \in H_0^1(\Omega) \cap C(\Omega),\ v \geq 1 \text{ on } E\}$ and $\mathcal{F}_0 = \{v \in H_0^1(\Omega) \cap C(\Omega),\ v \geq 0 \text{ on } \Omega,\ v \geq 1 \text{ on } E\}$. Then it holds*

$$\inf_{\mathcal{F}} \left\{ \|v\|_{H_0^1(\Omega)}^2 \right\} = \inf_{\mathcal{F}_0} \left\{ \|v\|_{H_0^1(\Omega)}^2 \right\}.$$

Proof. Since $\mathcal{F}_0 \subset \mathcal{F}$, we know

$$\inf_{\mathcal{F}} \left\{ \|v\|_{H_0^1(\Omega)}^2 \right\} \leq \inf_{\mathcal{F}_0} \left\{ \|v\|_{H_0^1(\Omega)}^2 \right\}.$$

If $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ is a minimizing sequence, then Corollary 3.16 implies $\{v_n^+\}_{n\in\mathbb{N}} \subset \mathcal{F}_0$ with $\|v_n^+\|^2_{H_0^1(\Omega)} \leq \|v_n\|^2_{H_0^1(\Omega)}$. Therefore we conclude

$$\inf_{\mathcal{F}} \left\{ \|v\|^2_{H_0^1(\Omega)} \right\} \geq \inf_{\mathcal{F}_0} \left\{ \|v\|^2_{H_0^1(\Omega)} \right\},$$

which completes the proof. □

Lemma 3.17 enables us to prove that the measure μ in (3.10) vanishes on every set of zero capacity.

Lemma 3.18. *Let $\tilde{\Omega}$ be a bounded Lipschitz domain with $\overline{\Omega} \subsetneq \tilde{\Omega}$. Moreover let $f \in \big(H^1(\Omega)\big)'$ and $\mu \in \mathcal{M}^+(\overline{\Omega})$ such that*

$$\langle f, v\rangle = \int_{\overline{\Omega}} v \,\mathrm{d}\mu \quad \forall\, v \in H^1(\Omega) \cap C(\overline{\Omega}).$$

If $D \in \mathcal{B}(\tilde{\Omega})$ is a set of zero capacity, then $D \cap \overline{\Omega}$ belongs to $\mathcal{B}(\overline{\Omega})$ and satisfies $\mu(D \cap \overline{\Omega}) = 0$.

Proof. Let $K \subset \overline{\Omega}$ be compact. Furthermore let $\varphi_K \in H_0^1(\tilde{\Omega}) \cap C(\tilde{\Omega})$ be arbitrary with $\varphi_K \geq 0$ in $\tilde{\Omega}$ and $\varphi_K \geq 1$ in K. As φ_K is an element of $H^1(\Omega) \cap C(\overline{\Omega})$, we obtain

$$\begin{aligned}\mu(K) = \int_{\overline{\Omega}} \chi_K \,\mathrm{d}\mu \leq \int_{\overline{\Omega}} \varphi_K \,\mathrm{d}\mu = \langle f, \varphi_K\rangle \leq \|f\|_{(H^1(\Omega))'} \|\varphi_K\|_{H^1(\Omega)} \leq c\,\|\varphi_K\|_{H^1(\tilde{\Omega})} \\ \leq c\,\|\varphi_K\|_{H_0^1(\tilde{\Omega})},\end{aligned}$$

where Friedrich's inequality was used for the last estimate. Invoking Lemma 3.17 we arrive at

$$\mu(K) \leq c \inf \left\{ \|v\|_{H_0^1(\tilde{\Omega})} : v \in H_0^1(\tilde{\Omega}) \cap C(\tilde{\Omega}),\, v \geq 1 \text{ on } K \right\}.$$

Proposition A.3a) hence leads to

$$\mu(K) \leq c\,\mathrm{cap}(K; \tilde{\Omega})^{\frac{1}{2}}, \tag{3.15}$$

since we minimize on a smaller set. In view of Definition A.27 and Lemma A.28 we observe $D \cap \overline{\Omega} \in \mathcal{B}(\overline{\Omega})$. Consequently, due to the regularity of the measure μ and (3.15) it follows

$$\begin{aligned}\mu(D \cap \overline{\Omega}) &= \sup \big\{\mu(K) \colon K \subset D \cap \overline{\Omega},\, K \text{ compact}\big\} \\ &\leq c \sup \left\{ \mathrm{cap}(K; \tilde{\Omega})^{\frac{1}{2}} \colon K \subset D \cap \overline{\Omega},\, K \text{ compact}\right\}.\end{aligned}$$

From Theorem A.4i) we deduce $\mu(D \cap \overline{\Omega}) \leq c\,\mathrm{cap}(D \cap \overline{\Omega}; \tilde{\Omega})^{\frac{1}{2}} \leq c\,\mathrm{cap}(D; \tilde{\Omega})^{\frac{1}{2}}$ so that $\mu(D \cap \overline{\Omega}) = 0$. □

We are ready for the desired extension of the Riesz representation theorem to the space $\big(H^1(\Omega)\big)'$.

Theorem 3.19. *Let $f \in \big(H^1(\Omega)\big)'$ be positive, i.e., it holds $\langle f,\, v\rangle \geq 0$ for all $v \in H^1(\Omega)$ with $v \geq 0$ a.e. in Ω, and let $\tilde{\Omega} \supsetneq \overline{\Omega}$ be a bounded Lipschitz domain. Then there exists a unique measure $\mu \in \mathcal{M}^+(\overline{\Omega})$ such that*

$$\langle f,\, v\rangle = \int_{\overline{\Omega}} v \,\mathrm{d}\mu \tag{3.16}$$

for all $v \in H^1(\Omega)$, where (3.16) *is fulfilled by a representative of v, which is quasi-continuous in $\tilde{\Omega}$. Moreover, if $D \in \mathcal{B}(\tilde{\Omega})$ is a set of zero capacity, then $\mu(D \cap \overline{\Omega}) = 0$.*

Remark 3.20. *Resulting from Corollary A.10 and the second assertion of Theorem 3.19 the integral in* (3.16) *does not depend on the chosen quasi-continuous representative.*

Proof of Theorem 3.19. According to Lemma 3.11 and Lemma 3.18 there exists a unique $\mu \in \mathcal{M}^+(\overline{\Omega})$ vanishing on all sets $D \cap \overline{\Omega}$ with $\operatorname{cap}(D; \tilde{\Omega}) = 0$ such that (3.16) holds for every $v \in H^1(\Omega) \cap C(\overline{\Omega})$. By performing the two steps mentioned above, we show that $C(\overline{\Omega})$ can be omitted, if proper representatives are considered. Let $v \in H^1(\Omega)$ and $\tilde{v} \in H^1_0(\tilde{\Omega})$ with $\tilde{v} = v$ a.e. in Ω, see e.g. [31, Lemma 1.29]. We assume w.l.o.g. that $\tilde{v}$ is nonnegative a.e. in $\tilde{\Omega}$. If this is not the case, we apply the arguments to $\tilde{v}^+$ and $\tilde{v}^-$ separately.
Step 1: By definition of $H^1_0(\tilde{\Omega})$ there is a sequence $\{v_n\}_{n\in\mathbb{N}} \subset C^\infty_0(\tilde{\Omega})$ with $v_n \to \tilde{v}^m := \min(\tilde{v}, m)$ in $H^1_0(\tilde{\Omega})$ as $n \to \infty$ for every $m \in \mathbb{N}$, cf. also Corollary 3.16. Because of $\tilde{v}^m = m - (m - v)^+$ and Lemma 3.15(i) we thus infer

$$v^m_n := \min(v_n, m) \xrightarrow{n\to\infty} \tilde{v}^m \quad \text{in } H^1_0(\tilde{\Omega}).$$

Thanks to Lemma A.12 there exists a subsequence of quasi-continuous representatives, w.l.o.g. denoted by the same symbols, with

$$v^m_n \xrightarrow{n\to\infty} \tilde{v}^m \quad \text{q.e. in } \tilde{\Omega}.$$

Therefore, Lemma 3.18 yields

$$v^m_n \xrightarrow{n\to\infty} \tilde{v}^m \quad \mu\text{-a.e. in } \overline{\Omega}.$$

As $v^m_n \leq m$ a.e. in $\tilde{\Omega}$ for all $n \in \mathbb{N}$ by definition, we derive from Lemma A.9 and Lemma 3.18

$$v^m_n \leq m \quad \mu\text{-a.e. in } \overline{\Omega}.$$

Note that restrictions of Borel-measurable functions are measurable w.r.t. the respective Borel-sigma-algebra on the restricted domain by Corollary A.29. This property will frequently be used in the following without further reference. Since

$m \in H^1(\Omega) \cap C(\overline{\Omega})$ is μ-integrable, Lebesgue's dominated convergence theorem implies

$$\int_{\overline{\Omega}} v_n^m \, \mathrm{d}\mu \xrightarrow{n\to\infty} \int_{\overline{\Omega}} \tilde{v}^m \, \mathrm{d}\mu < \infty. \tag{3.17}$$

In addition, we find $H^1(\Omega) \cap C(\overline{\Omega}) \ni v_n^m \to \tilde{v}^m$ in $H^1(\Omega)$ as $n \to \infty$ so that

$$\int_{\overline{\Omega}} v_n^m \, \mathrm{d}\mu = \langle f,\, v_n^m \rangle \xrightarrow{n\to\infty} \langle f,\, \tilde{v}^m \rangle\,,$$

which combined with (3.17) leads to

$$\langle f,\, \tilde{v}^m \rangle = \int_{\overline{\Omega}} \tilde{v}^m \, \mathrm{d}\mu. \tag{3.18}$$

Step 2: Invoking Lemma A.9, we derive from $\tilde{v}^{m+1} \geq \tilde{v}^m$ a.e. in $\tilde{\Omega}$ that $\tilde{v}^{m+1} \geq \tilde{v}^m$ q.e. in $\tilde{\Omega}$. Let $E_m \in \mathcal{B}(\tilde{\Omega})$ denote the set of points $x \in \Omega$, where $\tilde{v}^{m+1} < \tilde{v}^m$. Due to Theorem A.4ii) the union of all sets E_m has zero capacity. Consequently, the sequence $\{\tilde{v}^m\}_{m\in\mathbb{N}}$ is monotonously increasing q.e. in $\tilde{\Omega}$ and thus μ-a.e. in $\overline{\Omega}$ by Lemma 3.18. Thanks to Lemma 3.15(ii) we moreover know

$$\tilde{v}^m \xrightarrow{m\to\infty} \tilde{v} \quad \text{in } H_0^1(\tilde{\Omega}). \tag{3.19}$$

Again, on account of Lemma A.12 there exists a subsequence of quasi-continuous representatives, w.l.o.g. denoted in the same way, with

$$\tilde{v}^m \xrightarrow{m\to\infty} \tilde{v} \quad \text{q.e. in } \tilde{\Omega}.$$

Hence, we conclude from Lemma 3.18

$$\tilde{v}^m \xrightarrow{m\to\infty} \tilde{v} \quad \mu\text{-a.e. in } \overline{\Omega}.$$

In view of (3.18) we have shown

- $0 \leq \tilde{v}^m \nearrow \tilde{v}$ μ-a.e. in $\overline{\Omega}$ as $m \to \infty$
- $\tilde{v}^m \in L^1(\overline{\Omega}; \mu)$.

Note that two different quasi-continuous representatives coincide μ-a.e. in $\overline{\Omega}$, cf. Corollary A.10 and Lemma 3.18. This is why (3.18) holds true for all quasi-continuous representatives of $\tilde{v}^m$. Because of the convergence (3.19) the sequence $\{\tilde{v}^m\}_{m\in\mathbb{N}}$ is bounded so that

$$\int_{\overline{\Omega}} \tilde{v}^m \, \mathrm{d}\mu = \langle f,\, \tilde{v}^m \rangle \leq \|f\|_{H^1(\Omega)^*} \, \|\tilde{v}^m\|_{H^1(\Omega)} \leq c.$$

From the monotone convergence theorem it then follows

$$\int_{\overline{\Omega}} \tilde{v}^m \, \mathrm{d}\mu \xrightarrow{m\to\infty} \int_{\overline{\Omega}} \tilde{v} \, \mathrm{d}\mu < \infty. \tag{3.20}$$

Since $\tilde{v}^m \to \tilde{v}$ in $H^1(\Omega)$ according to (3.19), we furthermore observe

$$\int_{\overline{\Omega}} \tilde{v}^m \,\mathrm{d}\mu = \langle f,\, \tilde{v}^m\rangle \xrightarrow{m\to\infty} \langle f,\, \tilde{v}\rangle\,. \tag{3.21}$$

Together, (3.20) and (3.21) imply

$$\langle f,\, \tilde{v}\rangle = \int_{\overline{\Omega}} \tilde{v}\,\mathrm{d}\mu.$$

For $\tilde{v} = v$ a.e. in Ω, we have found a quasi-continuous representative defined on $\overline{\Omega}$ such that (3.16) is fulfilled. □

In the subsequent section we will not have to deal with a positive $f \in \big(H^1(\Omega)\big)'$ but with $\lambda \in \big(\tau_{\boldsymbol{\nu}}[V]\big)'$ which is positive for all $z \in \tau_{\boldsymbol{\nu}}[V]$ with $z \geq 0$ a.e. on Γ_C. Nevertheless, as for every $z \in \tau_{\boldsymbol{\nu}}[V]$ there is a function $w \in H^1(\Omega)$ with $\tau w = z$ a.e. on Γ by the inverse trace theorem (Theorem A.19), such λ can fortunately be interpreted as a positive functional on $H^1(\Omega)$.

Lemma 3.21. *Let $\lambda \in \big(\tau_{\boldsymbol{\nu}}[V]\big)'$ such that $\langle\lambda,\, z\rangle \geq 0$ for all $z \in \tau_{\boldsymbol{\nu}}[V]$ with $z \geq 0$ a.e. on Γ_C and let $\tilde{\Omega} \supsetneq \overline{\Omega}$ be a bounded Lipschitz domain. Then there exists a unique measure $\mu \in \mathcal{M}^+(\overline{\Omega})$, concentrated on the set Γ_C, such that*

$$\langle\lambda,\, z\rangle = \int_{\Gamma_C} v\,\mathrm{d}\mu \tag{3.22}$$

for all $z \in \tau_{\boldsymbol{\nu}}[V]$, where (3.22) *is fulfilled by a representative of $v \in H^1(\Omega)$ with $\tau v = z$ a.e. on Γ_C, which is quasi-continuous in $\tilde{\Omega}$. Moreover it holds $\mu(D\cap\overline{\Omega}) = 0$ for all $D \in \mathcal{B}(\tilde{\Omega})$ with* $\operatorname{cap}(D;\tilde{\Omega}) = 0$ *and*

$$\int_{\Gamma_C} v_1\,\mathrm{d}\mu = \int_{\Gamma_C} v_2\,\mathrm{d}\mu \tag{3.23}$$

for all $v_1, v_2 \in H^1(\Omega)$ with $\tau v_1 = \tau v_2$ a.e. on Γ_C.

Proof. In view of Assumption 3.1(1) we know $\Gamma_C = \Gamma \cap E = \overline{\Omega} \cap \Omega^c \cap E$ with $E \subset \mathbb{R}^d$ closed. Hence, the set Γ_C is compact and the finite open covering

$$G_C := \bigcup_{i=1}^{n} B_{\delta/2}(x_i), \quad x_i \in \Gamma_C, \quad n \in \mathbb{N}$$

with $\delta := \operatorname{dist}(\Gamma_C, \Gamma_D) > 0$ intersects Γ_D trivially. Due to Lemma A.24 (partition of unity) there is a smooth function $\zeta_C \in C_0^\infty(G_C)$ with $0 \leq \zeta_C \leq 1$, $\zeta_C \equiv 1$ on Γ_C and $\zeta_C \equiv 0$ on Γ_D. From Theorem A.14 and Corollary A.18 we deduce

$$\tau(\zeta_C v) = 0 \quad \text{a.e. on } \Gamma_D \tag{3.24a}$$

$$\tau(\zeta_C v) = \tau v \quad \text{a.e. on } \Gamma_C \tag{3.24b}$$

for all $v \in H^1(\Omega)$. Note that ζ_C is locally Lipschitz continuous by the mean value theorem and thus globally Lipschitz continuous on the compact set $\overline{\Omega}$, i.e., $\zeta_C \in C^{0,1}(\overline{\Omega})$. Thanks to (3.24a) and Theorem A.22 there exists $\boldsymbol{v} \in V$ with $\tau_{\boldsymbol{\nu}}\boldsymbol{v} = \tau(\zeta_C v)$ and $\tau_T \boldsymbol{v} = \boldsymbol{0}$, where we used the identity $\tau\boldsymbol{v} = (\tau_{\boldsymbol{\nu}}\boldsymbol{v})\boldsymbol{\nu} + \tau_T\boldsymbol{v}$ leading to $\tau\boldsymbol{v} = \boldsymbol{0}$ a.e. on Γ_D. Therefore, the trace $\tau(\zeta_C v)$ is an element of $\tau_{\boldsymbol{\nu}}[V]$ for all $v \in H^1(\Omega)$. If $v \in H^1(\Omega)$ fulfills $v \geq 0$ a.e. in Ω, then Proposition A.21 implies $\tau(\zeta_C v) \geq 0$ a.e. on Γ_C. Accordingly we infer $\langle \lambda, \tau(v\zeta_C)\rangle \geq 0$ by assumption. In other words, the functional $f \in \big(H^1(\Omega)\big)'$ defined through $\langle f, v\rangle = \langle \lambda, \tau(\zeta_C v)\rangle$ is positive. Consequently, Theorem 3.19 yields

$$\langle \lambda, \tau(\zeta_C v)\rangle = \int_{\overline{\Omega}} v \,\mathrm{d}\mu \tag{3.25}$$

for a representative of v, which is quasi-continuous in $\tilde{\Omega}$, and with unique measure $\mu \in \mathcal{M}^+(\overline{\Omega})$ vanishing on $D \cap \overline{\Omega}$ if $\operatorname{cap}(D; \tilde{\Omega}) = 0$. Now let $z \in \tau_{\boldsymbol{\nu}}[V]$ and $v \in H^1(\Omega)$ with $\tau v - z = 0$ a.e. on Γ_C, cf. Theorem A.19. On account of (3.24b) we derive

$$\tau(\zeta_C v) - z = 0 \quad \text{a.e. on } \Gamma_C$$

so that the assumption on λ results in

$$\langle \lambda, \tau(\zeta_C v) - z\rangle \geq 0 \quad \text{and} \quad \langle \lambda, z - \tau(\zeta_C v)\rangle \geq 0, \tag{3.26}$$

i.e., $\langle \lambda, z\rangle = \langle \lambda, \tau(\zeta_C v)\rangle$. Together with (3.25) this shows (3.22) and by the same arguments we obtain

$$\int_{\overline{\Omega}} v_1 \,\mathrm{d}\mu = \int_{\overline{\Omega}} v_2 \,\mathrm{d}\mu \quad \forall\, v_1, v_2 \in H^1(\Omega) \text{ with } \tau v_1 = \tau v_2 \text{ a.e. on } \Gamma_C.$$

It remains to be proven that μ is concentrated on Γ_C. For $\epsilon > 0$ we consider the finite open covering of the compact set Γ_C

$$U_\epsilon := \bigcup_{i=1}^{k} B_\epsilon(x_i), \quad x_i \in \Gamma_C, \quad k \in \mathbb{N}.$$

Since $\overline{\Omega} \setminus U_\epsilon = \overline{\Omega} \cap U_\epsilon^c$ is also a compact set, there is a finite open covering

$$G_\epsilon := \bigcup_{i=1}^{l} B_{\epsilon/2}(x_i), \quad x_i \in \overline{\Omega} \setminus U_\epsilon, \quad l \in \mathbb{N}$$

with $G_\epsilon \cap \Gamma_C = \emptyset$. Analogously to above we can find $\zeta_\epsilon \in C_0^\infty(G_\epsilon)$ with $0 \leq \zeta_\epsilon \leq 1$ and $\zeta_\epsilon \equiv 1$ in $\overline{\Omega} \setminus U_\epsilon$. As ζ_ϵ is bounded by $1 \in H^1(\Omega)$ for every $\epsilon > 0$ and

$$\zeta_\epsilon \xrightarrow{\epsilon \to 0} \chi_{\overline{\Omega} \setminus \Gamma_C} \quad \text{pointwisely in } \mathbb{R}^d,$$

it follows from (3.25) and Lebesgue's dominated convergence theorem that

$$\langle \lambda, \tau(\zeta_C \zeta_\epsilon)\rangle = \int_{\overline{\Omega}} \zeta_\epsilon \,\mathrm{d}\mu \xrightarrow{\epsilon \to 0} \int_{\overline{\Omega}} \chi_{\overline{\Omega} \setminus \Gamma_C} \,\mathrm{d}\mu = \mu(\overline{\Omega} \setminus \Gamma_C).$$

Because of (3.24b) we further note $\tau(\zeta_C \zeta_\epsilon) = 0$ a.e. on Γ_C and hence $\langle \lambda, \tau(\zeta_C \zeta_\epsilon)\rangle = 0$ for every $\epsilon > 0$, cf. also (3.26). This is why we conclude $\mu(\overline{\Omega} \setminus \Gamma_C) = 0$. $\square$

Lemma 3.21 is independent of the chosen function $v \in H^1(\Omega)$ with $\tau v = z$ a.e. on Γ_C, cf. (3.23). In the remainder of this section we introduce a specifically chosen extension operator $E\colon H^{1/2}(\Gamma) \to H_0^1(\tilde{\Omega})$, customized for the purposes of the subsequent section, with $\tau E(z)|_\Omega = z$ a.e. on Γ_C, where $\tilde{\Omega} \supsetneq \overline{\Omega}$ is a bounded Lipschitz domain. Before we are in the position to define E, we need the following auxiliary lemma.

Lemma 3.22. *Let $\tilde{\Omega} \supsetneq \overline{\Omega}$ be a bounded Lipschitz domain such that $\tilde{\Omega} \setminus \overline{\Omega}$ is connected. If $f \in H^1(\Omega)$ and $g \in H^1(\tilde{\Omega} \setminus \overline{\Omega})$ satisfy $\tau f = \tau g$ a.e. on Γ, then the compound function h defined by*

$$h = \begin{cases} f, & \text{a.e. on } \Omega, \\ g, & \text{a.e. on } \tilde{\Omega} \setminus \overline{\Omega} \end{cases}$$

belongs to $H^1(\tilde{\Omega})$. Moreover, if $\tau g = 0$ a.e. on $\partial\tilde{\Omega}$, then $h \in H_0^1(\tilde{\Omega})$.

Remark 3.23. *The assumption on $\tilde{\Omega}$ in Lemma 3.22 is necessary in order to guarantee that the difference $\tilde{\Omega} \setminus \overline{\Omega}$ is also a Lipschitz domain. A set $\tilde{\Omega} \subset \mathbb{R}^d$ which satisfies this assumption can be constructed for instance by extending the exterior boundary of Ω.*

Proof of Lemma 3.22. Both $\mathcal{B}(\Omega)$ and $\mathcal{B}(\tilde{\Omega} \setminus \overline{\Omega})$ are subsets of $\mathcal{B}(\tilde{\Omega})$, which implies measurability of $h\colon (\tilde{\Omega}, \mathcal{B}(\tilde{\Omega})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with

$$\|h\|_{L^2(\tilde{\Omega})}^2 = \|f\|_{L^2(\Omega)}^2 + \|g\|_{L^2(\tilde{\Omega}\setminus\overline{\Omega})}^2 < \infty.$$

We have to verify if h has a weak derivative $\partial h \in L^2(\tilde{\Omega})$. To this end let $\varphi \in C_0^\infty(\tilde{\Omega})$ and $\boldsymbol{\nu}_i$ denote the ith component of the unit outward normal vector on Γ. Integration by parts yields

$$\begin{aligned} \int_{\tilde{\Omega}} h\, \partial_i\varphi \,\mathrm{d}x &= \int_\Omega f\, \partial_i\varphi \,\mathrm{d}x + \int_{\tilde{\Omega}\setminus\overline{\Omega}} g\, \partial_i\varphi \,\mathrm{d}x \\ &= \int_\Gamma \tau f\, \varphi\, \boldsymbol{\nu}_i \,\mathrm{d}s - \int_\Omega \partial_i f\, \varphi \,\mathrm{d}x + \int_\Gamma \tau g\, \varphi\, (-\boldsymbol{\nu}_i) \,\mathrm{d}s - \int_{\tilde{\Omega}\setminus\overline{\Omega}} \partial_i g\, \varphi \,\mathrm{d}x \\ &= -\int_\Omega \partial_i f\, \varphi \,\mathrm{d}x - \int_{\tilde{\Omega}\setminus\overline{\Omega}} \partial_i g\, \varphi \,\mathrm{d}x, \end{aligned}$$

where $\varphi \equiv 0$ on $\partial\tilde{\Omega}$ was used. Now we define $\partial h\colon \tilde{\Omega} \to \mathbb{R}$ through

$$\partial h = \begin{cases} \partial f, & \text{a.e. on } \Omega \\ \partial g, & \text{a.e. on } \tilde{\Omega} \setminus \overline{\Omega}. \end{cases}$$

The same arguments as above lead to $\partial h \in L^2(\tilde{\Omega})$ so that $h \in H^1(\tilde{\Omega})$ with weak derivative ∂h. According to Theorem A.15 and Theorem A.14 we derive $\tau h = \tau g$ a.e. on $\partial\tilde{\Omega}$. Therefore, the function h is an element of $H_0^1(\tilde{\Omega})$, if $\tau g = 0$ a.e. on $\partial\tilde{\Omega}$. $\square$

Due to Lemma 3.22 every $z \in H^{1/2}(\Gamma)$ can be extended to a function, which belongs to $H_0^1(\tilde{\Omega})$.

Corollary 3.24. *Let $\tilde{\Omega} \supsetneq \overline{\Omega}$ be a bounded Lipschitz domain such that $\tilde{\Omega} \setminus \overline{\Omega}$ is connected and let $z \in H^{1/2}(\Gamma)$. Then there exists $v \in H_0^1(\tilde{\Omega})$ with $\tau v|_\Omega = z$ a.e. on Γ, depending continuously on z. In particular it holds*

$$\|v\|_{H_0^1(\tilde{\Omega})} \le c\,\|z\|_{H^{1/2}(\Gamma)}\,.$$

Proof. With the help of the inverse trace operator we define

$$v = \begin{cases} \tau_{-1} z, & \text{a.e. on } \Omega \\ \tau_{-1}\tilde{z}, & \text{a.e. on } \tilde{\Omega} \setminus \overline{\Omega}, \end{cases} \tag{3.27}$$

where $\tilde{z} \in H^{1/2}(\partial(\tilde{\Omega} \setminus \overline{\Omega}))$ is given by

$$\tilde{z} = \begin{cases} z, & \text{a.e. on } \Gamma \\ 0, & \text{a.e. on } \partial\tilde{\Omega}. \end{cases} \tag{3.28}$$

Thanks to Lemma 3.22 we know $v \in H_0^1(\tilde{\Omega})$ and the estimate follows from Theorem A.19. □

For the calculation of the sets appearing in (3.8) we require an extension, which is not only continuous but also nonnegative whenever $z \ge 0$ a.e. on Γ_C.

Lemma 3.25. *Let $\tilde{\Omega} \supsetneq \overline{\Omega}$ be a bounded Lipschitz domain such that $\tilde{\Omega}\setminus\overline{\Omega}$ is connected and let $z \in H^{1/2}(\Gamma)$. Then there exists a unique solution $v \in H_0^1(\tilde{\Omega})$ of*

$$\begin{aligned} (\nabla v,\, \nabla \varphi)_{L^2(\tilde{\Omega};\mathbb{R}^d)} &= 0 \quad \forall\, \varphi \in K_0 \\ \tau v|_\Omega &= z \quad \textit{a.e. on } \Gamma_C \end{aligned} \tag{3.29}$$

with $K_0 := \{\varphi \in H_0^1(\tilde{\Omega})\colon \tau\varphi|_\Omega = 0$ a.e. on $\Gamma_C\}$. In addition, the solution operator associated with (3.29) *is linear continuous.*

Proof. Let us consider the minimization problem

$$\left.\begin{aligned} \text{Minimize} \quad & \frac{1}{2}\,\|v\|^2_{H_0^1(\tilde{\Omega})} \\ \text{s.t.} \quad & v \in K := \{\varphi \in H_0^1(\tilde{\Omega})\colon \tau\varphi|_\Omega = z \text{ a.e. on } \Gamma_C\}. \end{aligned}\right\} \tag{3.30}$$

In view of Corollary 3.24 the admissible set K is nonempty. It is convex and closed, as the trace operator is linear continuous, cf. Theorem A.14. The squared norm furthermore defines a continuous, radially unbounded and strictly convex functional on $H_0^1(\tilde{\Omega})$. Thus, Problem (3.30) admits a unique solution $v \in K$, whose necessary and sufficient optimality condition is given by

$$(\nabla v, \nabla(\varphi - v))_{L^2(\tilde{\Omega};\mathbb{R}^d)} \geq 0 \quad \forall \varphi \in K. \tag{3.31}$$

Since $K = \{v\} + K_0$, we derive (3.29) from (3.31). The solution of (3.29) depends linearly on z and Corollary 3.24 implies

$$\|v\|_{H_0^1(\tilde{\Omega})} \leq c\,\|z\|_{H^{1/2}(\Gamma)}\,,$$

where the optimality of v was used. □

The solution operator associated with Problem (3.29) is the extension operator we are looking for.

Definition 3.26. *The solution operator associated with* (3.29)*, which maps $z \in H^{1/2}(\Gamma) \mapsto v \in H_0^1(\tilde{\Omega})$, is denoted by E.*

In order to prove the nonnegativity of E mentioned above we have to improve the assertion of Corollary 3.16.

Lemma 3.27. *Let $v \in H^1(\Omega)$, $z \in H^{1/2}(\Gamma)$ and $\{z_n\}_{n\in\mathbb{N}} \subset H^{1/2}(\Gamma)$ a sequence with $z_n \to z$. Then it holds*

(i) $\tau v^+ = (\tau v)^+$ *a.e. on* Γ

(ii) $z_n^+ \to z^+$ *in* $H^{1/2}(\Gamma)$.

Proof. (i): On account of Theorem A.15 there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ with $\varphi_n \to v$ in $H^1(\Omega)$. Consequently, Theorem A.14 yields $\tau\varphi_n \to \tau v$ in $H^{1/2}(\Gamma)$ and therefore

$$\tau\varphi_n \to \tau v \quad \text{in } L^2(\Gamma).$$

We decompose the boundary as $\Gamma = A_n \cup B_n \cup C_n \cup D_n$ with $A_n = \{x \in \Gamma\colon \tau\varphi_n > 0, \tau v > 0\}$, $B_n = \{x \in \Gamma\colon \tau\varphi_n > 0, \tau v \leq 0\}$, $C_n = \{x \in \Gamma\colon \tau\varphi_n \leq 0, \tau v \leq 0\}$ and $D_n = \{x \in \Gamma\colon \tau\varphi_n \leq 0, \tau v > 0\}$. Then we obtain the estimate

$$\begin{aligned}
\left\|(\tau\varphi_n)^+ - (\tau v)^+\right\|_{L^2(\Gamma)}^2 &= \int_{A_n} |\tau\varphi_n - \tau v|^2 \,\mathrm{d}s + \int_{B_n} |\tau\varphi_n|^2 \,\mathrm{d}s + \int_{D_n} |\tau v|^2 \,\mathrm{d}s \\
&\leq \int_{A_n} |\tau\varphi_n - \tau v|^2 \,\mathrm{d}s + \int_{B_n} |\tau\varphi_n - \tau v|^2 \,\mathrm{d}s \\
&\quad + \int_{D_n} |\tau v - \tau\varphi_n|^2 \,\mathrm{d}s \\
&\leq 3\,\|\tau\varphi_n - \tau v\|_{L^2(\Gamma)}^2\,.
\end{aligned}$$

This shows $(\tau\varphi_n)^+ \to (\tau v)^+$ in $L^2(\Gamma)$ and hence

$$(\tau\varphi_n)^+ \to (\tau v)^+ \quad \text{a.e. on } \Gamma, \tag{3.32}$$

if a proper subsequence is considered. According to Theorem A.13, Corollary A.16 and Lemma 3.15(i) we moreover observe $(\tau\varphi_n)^+ = \tau\varphi_n^+ \to \tau v^+$ in $H^{1/2}(\Gamma)$. Similarly to (3.32), there exists a subsequence, w.l.o.g. denoted by the same symbols, such that

$$(\tau\varphi_n)^+ \to \tau v^+ \quad \text{a.e. on } \Gamma. \tag{3.33}$$

Together, (3.32) and (3.33) imply (i).
(ii): From the inverse trace theorem (Theorem A.19) we infer

$$\|\tau_{-1}z - \tau_{-1}z_n\|_{H^1(\Omega)} = \|\tau_{-1}(z - z_n)\|_{H^1(\Omega)} \le c\,\|z_n - z\|_{H^{1/2}(\Gamma)} \xrightarrow{n\to\infty} 0.$$

Thanks to Lemma 3.15(i) and Theorem A.14 we find

$$\tau(\tau_{-1}z_n)^+ \to \tau(\tau_{-1}z)^+ \quad \text{in } H^{1/2}(\Gamma)$$

so that (ii) is a result of (i). □

The next corollary covers the nonnegativity of $E(z)$ for every $z \in H^{1/2}(\Gamma)$ with $z \ge 0$ a.e. on Γ_C. This will be the crucial property of the operator E.

Corollary 3.28. *Let $z \in H^{1/2}(\Gamma)$ with $z \ge 0$ a.e. on Γ_C. Then the solution of (3.29) satisfies $v \ge 0$ a.e. in $\tilde{\Omega}$.*

Proof. Due to $\min(v,0) = -(-v)^+$ and Lemma 3.27(i) we can test with $\min(v,0) \in \{\varphi \in H_0^1(\tilde{\Omega})\colon \tau\varphi|_\Omega = 0 \text{ a.e. on } \Gamma_C\}$ in (3.29), which leads to

$$\|\min(v,0)\|_{H_0^1(\tilde{\Omega})} = 0.$$

Friedrich's inequality yields $\min(v,0) = 0$ a.e. in $\tilde{\Omega}$ and thus $v \ge 0$ a.e. in $\tilde{\Omega}$. □

3.2.2. The density property

With the results of the previous section at hand we can now establish (3.8). In [62, Section 3] Mignot proves the density (3.7) by exploiting that the closed and convex set $C \subset X$ satisfies

$$\{x\} + X^+ \subset C \quad \text{or} \quad \{x\} + X^- \subset C \quad \forall\, x \in C, \tag{3.34}$$

where X^+ and X^- consist of all positive and negative parts of functions in the Dirichlet space X, respectively. Since the convex set $\mathcal{C}$ involves the normal trace operator $\tau_{\boldsymbol{\nu}}$, cf. (3.2), condition (3.34) is generally not fulfilled in our setting. Therefore, we cannot directly transfer Mignot's technique of proof to $P^B_{\mathcal{C}}$. Instead a known result for infinite-dimensional optimization problems is employed.

First, we recall that $\boldsymbol{u} = P^B_{\mathcal{C}}(\boldsymbol{w})$ with $\boldsymbol{w} \in V$ is equivalent to $\boldsymbol{u}$ being a solution of the minimization problem (3.5), i.e.,

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{2}\langle B(\boldsymbol{u}-\boldsymbol{w}),\, \boldsymbol{u}-\boldsymbol{w}\rangle \\ \text{s.t.} \quad & \underbrace{\boldsymbol{u} \in V,\ F(\boldsymbol{u}) \in K}_{\Leftrightarrow\, \boldsymbol{u}\in\mathcal{C}}. \end{aligned}$$

Moreover we introduce the linearized cone of $\mathcal{C}$ at $\boldsymbol{u} \in \mathcal{C}$, which is given by

$$L_{\boldsymbol{u}}(\mathcal{C}) = \big\{\boldsymbol{v} \in V \colon F'(\boldsymbol{u})\boldsymbol{v} \in \mathcal{K}_{F(\boldsymbol{u})}(K)\big\}.$$

Note here that $\mathcal{K}_{\boldsymbol{u}}(V) = V$. As every $\boldsymbol{u} \in \mathcal{C}$ is regular in the sense of Zowe-Kurcyusz [77], cf. Lemma 3.8, the following proposition, which states the existence of multipliers for admissible but not necessarily optimal functions, can be deduced from the proof of [77, Theorem 3.1].

Proposition 3.29. *Let $\boldsymbol{u} \in \mathcal{C}$ and $\hat{\boldsymbol{u}} \in L_{\boldsymbol{u}}(\mathcal{C})^+$. Then there exists a multiplier $\lambda \in \big(\tau_{\boldsymbol{\nu}}[V]\big)'$ such that*

$$\begin{gathered} \hat{\boldsymbol{u}} + F'(\boldsymbol{u})^*\lambda = 0 \quad \textit{in } V' \\ \lambda \in K^0, \quad \langle \lambda,\, F(\boldsymbol{u})\rangle = 0, \quad F(\boldsymbol{u}) \in K. \end{gathered}$$

Besides, the linearized cone $L_{\boldsymbol{u}}(\mathcal{C})$ coincides with $\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})$:

Lemma 3.30. *For every $\boldsymbol{u} \in \mathcal{C}$ it holds $\mathcal{K}_{\boldsymbol{u}}(\mathcal{C}) = L_{\boldsymbol{u}}(\mathcal{C})$.*

Proof. Let $\boldsymbol{u} \in \mathcal{C}$ be arbitrary.
"$\subset$": If $\boldsymbol{v} \in \mathcal{K}_{\boldsymbol{u}}(\mathcal{C})$, then we observe

$$\begin{aligned} F'(\boldsymbol{u})\boldsymbol{v} = \tau_{\boldsymbol{\nu}}\boldsymbol{v} = \tau_{\boldsymbol{\nu}}(\alpha(\boldsymbol{s}-\boldsymbol{u})) &= \alpha(\tau_{\boldsymbol{\nu}}\boldsymbol{s} - \psi - \tau_{\boldsymbol{\nu}}\boldsymbol{u} + \psi) \\ &= \alpha(\tau_{\boldsymbol{\nu}}\boldsymbol{s} - \psi - F(\boldsymbol{u})) = \alpha(z - F(\boldsymbol{u})) \end{aligned}$$

with $\alpha \geq 0$, $\boldsymbol{s} \in \mathcal{C}$ and $z := \tau_{\boldsymbol{\nu}}\boldsymbol{s} - \psi \in K$. We infer $F'(\boldsymbol{u})\boldsymbol{v} \in \mathcal{K}_{F(\boldsymbol{u})}(K)$ and therefore $\boldsymbol{v} \in L_{\boldsymbol{u}}(\mathcal{C})$.
"$\supset$": Let $\boldsymbol{v} \in L_{\boldsymbol{u}}(\mathcal{C})$ so that

$$F'(\boldsymbol{u})\boldsymbol{v} = \tau_{\boldsymbol{\nu}}\boldsymbol{v} = \alpha(z - F(\boldsymbol{u})) = \alpha(z - \tau_{\boldsymbol{\nu}}\boldsymbol{u} + \psi)$$

with $\alpha \geq 0$ and $z \in K$. In case $\alpha = 0$ nothing is to show. Otherwise we set $\boldsymbol{s} = \boldsymbol{v}/\alpha + \boldsymbol{u} \in V$ and obtain

$$\tau_{\boldsymbol{\nu}}\boldsymbol{s} = \frac{1}{\alpha}\tau_{\boldsymbol{\nu}}\boldsymbol{v} + \tau_{\boldsymbol{\nu}}\boldsymbol{u} = z - \tau_{\boldsymbol{\nu}}\boldsymbol{u} + \psi + \tau_{\boldsymbol{\nu}}\boldsymbol{u} \leq \psi \quad \text{a.e. on } \Gamma_C,$$

where $z \in K$ was used for the estimate. Hence, the function $\boldsymbol{v}$ fulfills $\boldsymbol{v} = \alpha(\boldsymbol{s} - \boldsymbol{u})$ with $\boldsymbol{s} \in \mathcal{C}$, which implies $\boldsymbol{v} \in \mathcal{K}_{\boldsymbol{u}}(\mathcal{C})$. □

Based on Lemma 3.21 and Corollary 3.28 we next derive a characterization of the dual cone $[\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})]^0_B$. To this end we define for a quasi-continuous representative of $E(\tau_{\boldsymbol{\nu}}\boldsymbol{u} - \psi)$, w.l.o.g. denoted by the same symbol,

$$\mathcal{A}_{\boldsymbol{u}} = \big\{x \in \Gamma_C \colon E(\tau_{\boldsymbol{\nu}}\boldsymbol{u} - \psi) = 0\big\}, \qquad \text{(active set)}$$

cf. also Lemma A.11. Note that $\mathcal{A}_{\boldsymbol{u}}$ changes only on sets of zero capacity, if a different quasi-continuous representative is considered, cf. Corollary A.10.

Lemma 3.31. *For every $\boldsymbol{u} \in \mathcal{C}$ the dual cone $[\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})]^0_B$ is given by*

$$\begin{aligned}[\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})]^0_B = \Big\{&\boldsymbol{v} \in V \colon \exists\, \mu \in \mathcal{M}^+(\overline{\Omega}) \text{ concentrated on } \mathcal{A}_{\boldsymbol{u}} \text{ such that} \\ &\langle B\boldsymbol{v},\, \boldsymbol{s}\rangle = \int_{\Gamma_C} E\tau_{\boldsymbol{\nu}}\boldsymbol{s}\, \mathrm{d}\mu \ \forall\, \boldsymbol{s} \in V\Big\}.\end{aligned} \tag{3.35}$$

The integrals are defined in the sense of Lemma 3.21 and satisfy (3.23)*. Moreover, every measure $\mu \in \mathcal{M}^+(\overline{\Omega})$ in* (3.35) *vanishes on all sets $D \cap \overline{\Omega}$ with $D \in \mathcal{B}(\tilde{\Omega})$ and* $\operatorname{cap}(D; \tilde{\Omega}) = 0$.

Proof. Let $\boldsymbol{u} \in \mathcal{C}$ be arbitrary.
"$\subset$": If $\boldsymbol{v} \in [\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})]^0_B$, then we know $\langle B\boldsymbol{v},\, \boldsymbol{s}\rangle \le 0$ for all $\boldsymbol{s} \in \mathcal{K}_{\boldsymbol{u}}(\mathcal{C})$. As a result of Lemma 3.30 the functional $-B\boldsymbol{v} \in V'$ is an element of $L_{\boldsymbol{u}}(\mathcal{C})^+$. Thanks to Proposition 3.29 there is a $\lambda \in K^0$ with

$$-B\boldsymbol{v} + \tau_{\boldsymbol{\nu}}^*\lambda = 0 \quad \text{in } V', \quad \langle \lambda,\, \tau_{\boldsymbol{\nu}}\boldsymbol{u} - \psi\rangle = 0. \tag{3.36}$$

Since $\langle \lambda,\, z\rangle \ge 0$ for all $z \in \tau_{\boldsymbol{\nu}}[V]$ with $z \ge 0$ a.e. on Γ_C, Lemma 3.21 yields the existence of a unique measure $\mu \in \mathcal{M}^+(\overline{\Omega})$ such that

$$\mu(\overline{\Omega} \setminus \Gamma_C) = 0, \quad \langle \lambda,\, z\rangle = \int_{\Gamma_C} Ez\, \mathrm{d}\mu \quad \forall\, z \in \tau_{\boldsymbol{\nu}}[V] \tag{3.37}$$

and (3.23) is satisfied. In addition, it holds $\mu(D \cap \overline{\Omega}) = 0$ for all $D \in \mathcal{B}(\tilde{\Omega})$ with $\operatorname{cap}(D; \tilde{\Omega}) = 0$. Together, (3.36) and (3.37) show

$$\langle B\boldsymbol{v},\, \boldsymbol{s}\rangle = \langle \lambda,\, \tau_{\boldsymbol{\nu}}\boldsymbol{s}\rangle = \int_{\Gamma_C} E\tau_{\boldsymbol{\nu}}\boldsymbol{s}\, \mathrm{d}\mu, \quad \int_{\Gamma_C} E(\tau_{\boldsymbol{\nu}}\boldsymbol{u} - \psi)\, \mathrm{d}\mu = 0, \tag{3.38}$$

where Assumption 3.1(3) was used. From Corollary 3.28 and $\boldsymbol{u} \in \mathcal{C}$ it follows $E(\tau_{\boldsymbol{\nu}}\boldsymbol{u} - \psi) \le 0$ a.e. in $\tilde{\Omega}$ so that $E(\tau_{\boldsymbol{\nu}}\boldsymbol{u} - \psi) \le 0$ q.e. in $\tilde{\Omega}$ for every quasi-continuous representative by Lemma A.9 and thus

$$E(\tau_{\boldsymbol{\nu}}\boldsymbol{u} - \psi) \le 0 \quad \mu\text{-a.e. in } \overline{\Omega}. \tag{3.39}$$

Taking (3.38) into account we conclude

$$\mu\big(\big\{x \in \Gamma_C \colon\ |E(\tau_{\boldsymbol{\nu}} \boldsymbol{u} - \psi)| > 0\big\}\big) = 0.$$

The measure μ is therefore concentrated on $\mathcal{A}_{\boldsymbol{u}}$.
"$\supset$": Now let $\boldsymbol{v} \in V$ and $\mu \in \mathcal{M}^+(\overline{\Omega})$ be concentrated on $\mathcal{A}_{\boldsymbol{u}}$ with

$$\langle B\boldsymbol{v},\, \boldsymbol{s}\rangle = \int_{\Gamma_C} E\tau_{\boldsymbol{\nu}} \boldsymbol{s}\, \mathrm{d}\mu \quad \forall\, \boldsymbol{s} \in V.$$

Furthermore, if $\mu(D \cap \overline{\Omega}) = 0$ for all $D \in \mathcal{B}(\tilde{\Omega})$ with $\operatorname{cap}(D; \tilde{\Omega}) = 0$, we obtain $E(\tau_{\boldsymbol{\nu}} \boldsymbol{s} - \psi) \leq 0$ μ-a.e. in $\overline{\Omega}$ for every $\boldsymbol{s} \in \mathcal{C}$ by the same arguments leading to (3.39). Because the positive measure μ vanishes outside $\mathcal{A}_{\boldsymbol{u}} = \big\{x \in \Gamma_C \colon E(\tau_{\boldsymbol{\nu}} \boldsymbol{u} - \psi) = 0\big\}$, we find

$$\begin{aligned}
\langle B\boldsymbol{v},\, \boldsymbol{s} - \boldsymbol{u}\rangle &= \int_{\Gamma_C} E\tau_{\boldsymbol{\nu}}(\boldsymbol{s} - \boldsymbol{u})\, \mathrm{d}\mu \\
&= \underbrace{\int_{\Gamma_C} E(\tau_{\boldsymbol{\nu}} \boldsymbol{s} - \psi)\, \mathrm{d}\mu}_{\leq 0} + \underbrace{\int_{\Gamma_C} E(\psi - \tau_{\boldsymbol{\nu}} \boldsymbol{u})\, \mathrm{d}\mu}_{=0} \leq 0
\end{aligned}$$

for all $\boldsymbol{s} \in \mathcal{C}$, which is equivalent to $\boldsymbol{v} \in [\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})]^0_B$. $\square$

Lemma 3.31 combined with the bipolar theorem enables us to calculate the set $\overline{\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})}$:

Lemma 3.32. *For every $\boldsymbol{u} \in \mathcal{C}$ the closure of $\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})$ is given by*

$$\overline{\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})} = \big\{\boldsymbol{v} \in V \colon E\tau_{\boldsymbol{\nu}} \boldsymbol{v} \leq 0 \text{ q.e. on } \mathcal{A}_{\boldsymbol{u}}\big\},$$

where $E\tau_{\boldsymbol{\nu}} \boldsymbol{v}$ denotes a quasi-continuous representative.

Proof. Let $\boldsymbol{u} \in \mathcal{C}$ be arbitrary and $\boldsymbol{s} \in [\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})]^0_B$ so that

$$\langle B\boldsymbol{s},\, \boldsymbol{v}\rangle \leq 0 \quad \forall\, \boldsymbol{v} \in \mathcal{K}_{\boldsymbol{u}}(\mathcal{C})$$

is fulfilled. Then the functional $B\boldsymbol{s} \in V'$ is an element of the polar cone $(\mathcal{K}_{\boldsymbol{u}}(\mathcal{C}))^\circ$. Moreover, in view of the Lax-Milgram theorem there exists a unique $\boldsymbol{s} \in V$ for every $f \in (\mathcal{K}_{\boldsymbol{u}}(\mathcal{C}))^\circ$ with $B\boldsymbol{s} = f$ and $\langle B\boldsymbol{s},\, \boldsymbol{v}\rangle = \langle f,\, \boldsymbol{v}\rangle \leq 0$ for all $\boldsymbol{v} \in \mathcal{K}_{\boldsymbol{u}}(\mathcal{C})$. Consequently, we infer

$$(\mathcal{K}_{\boldsymbol{u}}(\mathcal{C}))^\circ = \big\{B\boldsymbol{s} \colon \boldsymbol{s} \in [\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})]^0_B\big\}.$$

The bipolar theorem and Lemma 3.31 yield

$$\begin{aligned}
\overline{\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})} = (\mathcal{K}_{\boldsymbol{u}}(\mathcal{C}))^{\circ\circ} &= \big\{\boldsymbol{v} \in V \colon \langle B\boldsymbol{s},\, \boldsymbol{v}\rangle \leq 0\ \forall\, \boldsymbol{s} \in [\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})]^0_B\big\} \\
&= \left\{\boldsymbol{v} \in V \colon \int_{\Gamma_C} E\tau_{\boldsymbol{\nu}} \boldsymbol{v}\, \mathrm{d}\mu \leq 0\ \forall\, \mu \in \mathfrak{M}_{\boldsymbol{u}}\right\}
\end{aligned}$$

with $\mathfrak{M}_{\boldsymbol{u}} \subset \mathcal{M}^+(\overline{\Omega})$ defined by

$$\mathfrak{M}_{\boldsymbol{u}} = \Big\{ \mu \in \mathcal{M}^+(\overline{\Omega}) \colon \mu \text{ is concentrated on } \mathcal{A}_{\boldsymbol{u}},\ \exists\, \boldsymbol{s} \in V \text{ such that}$$
$$\langle B\boldsymbol{s},\, \boldsymbol{v}\rangle = \int_{\Gamma_C} E\tau_{\boldsymbol{\nu}}\boldsymbol{v}\,\mathrm{d}\mu\ \forall\, \boldsymbol{v} \in V \Big\}.$$

Furthermore, for $\mu \in \mathcal{M}^+(\overline{\Omega})$ concentrated on $\mathcal{A}_{\boldsymbol{u}}$ and $\boldsymbol{v} \in V$ we observe

$$\int_{\Gamma_C} E\tau_{\boldsymbol{\nu}}\boldsymbol{v}\,\mathrm{d}\mu = \int_{\mathcal{A}_{\boldsymbol{u}}} E\tau_{\boldsymbol{\nu}}\boldsymbol{v}\,\mathrm{d}\mu.$$

According to the last assertion of Lemma 3.31 we therefore conclude

$$\overline{\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})} \supset \big\{\boldsymbol{v} \in V \colon E\tau_{\boldsymbol{\nu}}\boldsymbol{v} \le 0 \text{ q.e. on } \mathcal{A}_{\boldsymbol{u}}\big\}, \tag{3.40}$$

where the estimate is satisfied by a quasi-continuous representative of $E\tau_{\boldsymbol{\nu}}\boldsymbol{v}$.
In order to establish the reverse inclusion let $\boldsymbol{v} \in \overline{\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})}$ be arbitrary. Then there exist sequences $\{\boldsymbol{v}_n\}_{n\in\mathbb{N}} \subset \mathcal{C}$ and $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $t_n(\boldsymbol{v}_n - \boldsymbol{u}) \to \boldsymbol{v}$ in V. Invoking Theorem A.22, we deduce $t_n\tau_{\boldsymbol{\nu}}(\boldsymbol{v}_n - \boldsymbol{u}) \to \tau_{\boldsymbol{\nu}}\boldsymbol{v}$ in $H^{1/2}(\Gamma)$. Thanks to the continuity of E, cf. Lemma 3.25, it follows

$$E(t_n\tau_{\boldsymbol{\nu}}(\boldsymbol{v}_n - \boldsymbol{u})) \to E\tau_{\boldsymbol{\nu}}\boldsymbol{v} \quad \text{in } H^1_0(\tilde{\Omega})$$

so that Lemma A.12 leads to

$$E(t_n\tau_{\boldsymbol{\nu}}(\boldsymbol{v}_n - \boldsymbol{u})) \to E\tau_{\boldsymbol{\nu}}\boldsymbol{v} \quad \text{q.e. in } \tilde{\Omega}, \tag{3.41}$$

if quasi-continuous representatives of a proper subsequence are considered. From the definition of $\mathcal{A}_{\boldsymbol{u}}$ we derive $E\tau_{\boldsymbol{\nu}}(\boldsymbol{v}_n - \boldsymbol{u}) = E(\tau_{\boldsymbol{\nu}}\boldsymbol{v}_n - \psi)$ q.e. on $\mathcal{A}_{\boldsymbol{u}}$. Note here the linearity of E. In addition, the same arguments as in the proof of Lemma 3.31 show $E(\tau_{\boldsymbol{\nu}}\boldsymbol{v}_n - \psi) \le 0$ q.e. in $\tilde{\Omega}$ and hence

$$t_n E\tau_{\boldsymbol{\nu}}(\boldsymbol{v}_n - \boldsymbol{u}) \le 0 \quad \text{q.e. on } \mathcal{A}_{\boldsymbol{u}}.$$

Together with (3.41) this implies $E\tau_{\boldsymbol{\nu}}\boldsymbol{v} \le 0$ q.e. on $\mathcal{A}_{\boldsymbol{u}}$ and in particular

$$\overline{\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})} \subset \big\{\boldsymbol{v} \in V \colon E\tau_{\boldsymbol{\nu}}\boldsymbol{v} \le 0 \text{ q.e. on } \mathcal{A}_{\boldsymbol{u}}\big\},$$

which completes the proof. □

Remark 3.33. *For the proof of Lemma 3.32 it is crucial that the dual pairing* $\langle B\boldsymbol{s},\, \boldsymbol{v}\rangle$ *with* $\boldsymbol{s} \in [\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})]^0_B$ *and* $\boldsymbol{v} \in V$ *can be written as an integral w.r.t. a measure* $\mu \in \mathcal{M}^+(\overline{\Omega})$ *concentrated on* $\mathcal{A}_{\boldsymbol{u}}$. *Without this representation we are not able to exploit local information as performed in* (3.40)*. From the proof of Lemma 3.31 we know* $B\boldsymbol{s} = \lambda \circ \tau_{\boldsymbol{\nu}}$ *with* $\lambda \in K^0$, *cf.* (3.36)*. If one aimed to use a property of* $\boldsymbol{v} \in V$ *in a subset* $\Omega_0 \subset \Omega$ *or a property of* $\tau_{\boldsymbol{\nu}}\boldsymbol{v}$ *on a subset* $\Gamma_0 \subset \Gamma$ *solely based on* (3.36)*, it would be necessary to consider the dual pairings* $\langle \lambda,\, \tau_{\boldsymbol{\nu}}(\chi_{\Omega_0}\boldsymbol{v})\rangle$ *and* $\langle \lambda,\, \chi_{\Gamma_0}\tau_{\boldsymbol{\nu}}\boldsymbol{v}\rangle$*, respectively. However, in general it holds* $\chi_{\Omega_0}\boldsymbol{v} \notin V$ *and* $\chi_{\Gamma_0}\tau_{\boldsymbol{\nu}}\boldsymbol{v} \notin H^{1/2}(\Gamma)$*. The big advantage of the integral representation is thus the possibility to test with characteristic functions.*

With the help of Lemma 3.31 we can also calculate the set $[\operatorname{span}(\boldsymbol{w}-\boldsymbol{u})]_B^0$. To this end we state another auxiliary lemma.

Lemma 3.34. *Let $\boldsymbol{w} \in V$ and $\boldsymbol{u} = P_{\mathcal{C}}^B(\boldsymbol{w})$ with associated multiplier $\lambda \in \big(\tau_{\boldsymbol{\nu}}[V]\big)'$. Then it holds $B(\boldsymbol{w}-\boldsymbol{u}) = \lambda \circ \tau_{\boldsymbol{\nu}}$ in V' and $\boldsymbol{w}-\boldsymbol{u} \in [\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})]_B^0$. In particular, there exists a measure $\mu_\lambda \in \mathcal{M}^+(\overline{\Omega})$ concentrated on $\mathcal{A}_{\boldsymbol{u}}$ such that*

$$\langle \lambda,\, \tau_{\boldsymbol{\nu}}\boldsymbol{v}\rangle = \int_{\Gamma_C} E\tau_{\boldsymbol{\nu}}\boldsymbol{v}\,\mathrm{d}\mu_\lambda \quad \forall\, \boldsymbol{v} \in V. \tag{3.42}$$

The integral is defined in the sense of Lemma 3.21 and satisfies (3.23)*. Moreover, it holds $\mu_\lambda(D \cap \overline{\Omega}) = 0$ for all $D \in \mathcal{B}(\tilde{\Omega})$ with $\operatorname{cap}(D;\tilde{\Omega}) = 0$.*

Proof. The first assertion is a consequence of (3.6) with $\ell = B\boldsymbol{w}$. Let $\boldsymbol{0} \neq \boldsymbol{v} \in \mathcal{K}_{\boldsymbol{u}}(\mathcal{C})$ be arbitrary so that $\alpha\boldsymbol{v} + \boldsymbol{u} \in \mathcal{C}$ for some $\alpha > 0$. From (3.4) we infer $\langle B(\boldsymbol{w}-\boldsymbol{u}),\, (\alpha\boldsymbol{v}+\boldsymbol{u}) - \boldsymbol{u}\rangle \le 0$ and therefore $\langle B(\boldsymbol{w}-\boldsymbol{u}),\, \boldsymbol{v}\rangle \le 0$ for all $\boldsymbol{v} \in \mathcal{K}_{\boldsymbol{u}}(\mathcal{C})$, which yields $\boldsymbol{w}-\boldsymbol{u} \in [\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})]_B^0$. Lemma 3.31 implies (3.42) and the last claim. □

Since by definition $\boldsymbol{v} \in V$ belongs to $[\operatorname{span}(\boldsymbol{w}-\boldsymbol{u})]_B^0$ if and only if $\langle B(\boldsymbol{w}-\boldsymbol{u}),\, \boldsymbol{v}\rangle = 0$, we arrive at the following result.

Lemma 3.35. *Let $\boldsymbol{w} \in V$ and $\boldsymbol{u} = P_{\mathcal{C}}^B(\boldsymbol{w})$ with associated multiplier $\lambda \in \big(\tau_{\boldsymbol{\nu}}[V]\big)'$. Then there exists a measure $\mu_\lambda \in \mathcal{M}^+(\overline{\Omega})$ concentrated on $\mathcal{A}_{\boldsymbol{u}}$ such that*

$$[\operatorname{span}(\boldsymbol{w}-\boldsymbol{u})]_B^0 = \left\{\boldsymbol{v} \in V\colon\ \langle \lambda,\, \tau_{\boldsymbol{\nu}}\boldsymbol{v}\rangle = \int_{\Gamma_C} E\tau_{\boldsymbol{\nu}}\boldsymbol{v}\,\mathrm{d}\mu_\lambda = 0\right\}.$$

The integral is defined in the sense of Lemma 3.21 and satisfies (3.23)*. Moreover, it holds $\mu_\lambda(D \cap \overline{\Omega}) = 0$ for all $D \in \mathcal{B}(\tilde{\Omega})$ with $\operatorname{cap}(D;\tilde{\Omega}) = 0$.*

Having the characterizations of the sets $\overline{\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})}$ and $[\operatorname{span}(\boldsymbol{w}-\boldsymbol{u})]_B^0$ available, we are ready to prove the density property (3.8).

Theorem 3.36. *For every $\boldsymbol{w} \in V$ and $\boldsymbol{u} = P_{\mathcal{C}}^B(\boldsymbol{w})$ the set $\mathcal{K}_{\boldsymbol{u}}(\mathcal{C}) \cap [\operatorname{span}(\boldsymbol{w}-\boldsymbol{u})]_B^0$ is dense in $\overline{\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})} \cap [\operatorname{span}(\boldsymbol{w}-\boldsymbol{u})]_B^0$.*

Proof. Let $\boldsymbol{w} \in V$, $\boldsymbol{u} = P_{\mathcal{C}}^B(\boldsymbol{w})$, $\boldsymbol{0} \neq \boldsymbol{v} \in \overline{\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})} \cap [\operatorname{span}(\boldsymbol{w}-\boldsymbol{u})]_B^0$ and $\{\boldsymbol{v}_n\}_{n\in\mathbb{N}} \subset \mathcal{K}_{\boldsymbol{u}}(\mathcal{C})$ with $\boldsymbol{v}_n \to \boldsymbol{v}$ in V. We can assume w.l.o.g. that $\boldsymbol{v}_n \neq \boldsymbol{0}$ for all $n \in \mathbb{N}$. If this is not the case, we eliminate all zeros from the sequence. This is why there exists $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+ \setminus \{0\}$ such that $t_n\boldsymbol{v}_n + \boldsymbol{u} \in \mathcal{C}$, i.e., $t_n\tau_{\boldsymbol{\nu}}\boldsymbol{v}_n + \tau_{\boldsymbol{\nu}}\boldsymbol{u} \le \psi$ a.e. on Γ_C for all $n \in \mathbb{N}$. By manipulating $\tau_{\boldsymbol{\nu}}\boldsymbol{v}_n$ and applying Theorem A.22 we will obtain a sequence $\{\tilde{\boldsymbol{v}}_n\}_{n\in\mathbb{N}} \subset \mathcal{K}_{\boldsymbol{u}}(\mathcal{C}) \cap [\operatorname{span}(\boldsymbol{w}-\boldsymbol{u})]_B^0$ with $\tilde{\boldsymbol{v}}_n \to \boldsymbol{v}$ in V. For this purpose we define

$$z := \tau_{\boldsymbol{\nu}}\boldsymbol{v}, \qquad z_n := \tau_{\boldsymbol{\nu}}\boldsymbol{v}_n.$$

In view of Lemma 3.32 and Lemma 3.35 the quasi-continuous representatives of Ez satisfy

$$Ez \leq 0 \quad \text{q.e. on } \mathcal{A}_{\boldsymbol{u}} \tag{3.43a}$$

$$\int_{\Gamma_C} Ez \, \mathrm{d}\mu = 0 \text{ with } \mu \in \mathcal{M}^+(\overline{\Omega}) \text{ concentrated on } \mathcal{A}_{\boldsymbol{u}}. \tag{3.43b}$$

Due to (3.43a) we observe $(Ez)^+ = 0$ q.e. in $\mathcal{A}_{\boldsymbol{u}}$. Thus, the last assertion of Lemma 3.35 and (3.43b) result in

$$0 = \int_{\Gamma_C} Ez \, \mathrm{d}\mu = \int_{\Gamma_C} (Ez)^+ \, \mathrm{d}\mu - \int_{\Gamma_C} (Ez)^- \, \mathrm{d}\mu = -\int_{\Gamma_C} (Ez)^- \, \mathrm{d}\mu.$$

Moreover, from Lemma 3.27(i) and (3.29) we derive $\tau(Ez)^- = (\tau Ez)^- = z^- = \tau E z^-$ a.e. on Γ_C, which combined with (3.23) leads to

$$\int_{\Gamma_C} Ez^- \, \mathrm{d}\mu = 0. \tag{3.44}$$

Next, we introduce the abbreviation

$$\tilde{z}_n^- := \min(z_n^-, z^-).$$

Note that z^- and z_n^- as well as $\tilde{z}_n^-$ are elements of $H^{1/2}(\Gamma)$ according to Lemma 3.27(ii). Because of $z^- \geq \tilde{z}_n^-$ a.e. on Γ_C we deduce from Corollary 3.28, Lemma A.9 and Lemma 3.35 that

$$E(\tilde{z}_n^- - z^-) \leq 0 \quad \mu\text{-a.e. in } \overline{\Omega}.$$

Taking the linearity of E, (3.44) and the positivity of μ into account, we hence arrive at

$$\int_{\Gamma_C} E\tilde{z}_n^- \, \mathrm{d}\mu = \int_{\Gamma_C} E(\tilde{z}_n^- - z^-) \, \mathrm{d}\mu \leq 0.$$

Since $\tilde{z}_n^- \geq 0$ by definition, the same arguments yield

$$\int_{\Gamma_C} E\tilde{z}_n^- \, \mathrm{d}\mu = 0. \tag{3.45}$$

Considering $\boldsymbol{u} \in \mathcal{C}$ and $t_n z_n + \tau_{\boldsymbol{\nu}} \boldsymbol{u} \leq \psi$ a.e. on Γ_C we furthermore find

$$t_n z_n^+ + \tau_{\boldsymbol{\nu}} \boldsymbol{u} \leq \max\big(\tau_{\boldsymbol{\nu}} \boldsymbol{u}, t_n z_n + \tau_{\boldsymbol{\nu}} \boldsymbol{u}\big) \leq \psi \quad \text{a.e. on } \Gamma_C \tag{3.46}$$

and consequently

$$0 \leq z_n^+ \leq \frac{1}{t_n}(\psi - \tau_{\boldsymbol{\nu}} \boldsymbol{u}) \quad \text{a.e. on } \Gamma_C.$$

Again, thanks to Corollary 3.28, Lemma A.9 and Lemma 3.35 we conclude

$$0 \leq Ez_n^+ \leq \frac{1}{t_n} E(\psi - \tau_{\boldsymbol{\nu}} \boldsymbol{u}) \quad \mu\text{-a.e. in } \overline{\Omega}.$$

As the measure μ is concentrated on $\mathcal{A}_{\boldsymbol{u}}$, it follows

$$\int_{\Gamma_C} E z_n^+ \,\mathrm{d}\mu = 0. \tag{3.47}$$

Together, (3.45)–(3.47) show

$$\int_{\Gamma_C} E(z_n^+ - \tilde{z}_n^-) \,\mathrm{d}\mu = 0 \tag{3.48a}$$

$$t_n(z_n^+ - \tilde{z}_n^-) + \tau_{\boldsymbol{\nu}} \boldsymbol{u} \leq t_n z_n^+ + \tau_{\boldsymbol{\nu}} \boldsymbol{u} \leq \psi \quad \text{a.e. on } \Gamma_C, \tag{3.48b}$$

where we used $\tilde{z}_n^- \geq 0$ a.e. on Γ_C for the first estimate in (3.48b). By Theorem A.22 there exists a sequence $\{\boldsymbol{s}_n\}_{n\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^d)$ with $\tau_{\boldsymbol{\nu}} \boldsymbol{s}_n = (z_n^+ - \tilde{z}_n^-) - z$, $\tau_T \boldsymbol{s}_n = \boldsymbol{0}$ and

$$\|\boldsymbol{s}_n\|_{H^1(\Omega;\mathbb{R}^d)} \leq c \left\|(z_n^+ - \tilde{z}_n^-) - z\right\|_{H^{1/2}(\Gamma)}.$$

In view of Lemma 3.27(ii) we infer

$$\begin{aligned} z_n^+ - \tilde{z}_n^- &= z_n^+ - \min(z_n^-, z^-) \\ &= z_n^+ - (\min(z_n^- - z^-, 0) + z^-) \xrightarrow{n\to\infty} z^+ - z^- = z \quad \text{in } H^{1/2}(\Gamma) \end{aligned}$$

so that $\boldsymbol{s}_n \to \boldsymbol{0}$ in $H^1(\Omega;\mathbb{R}^d)$. In addition, we know

$$\tau \boldsymbol{s}_n = \tau_{\boldsymbol{\nu}} \boldsymbol{s}_n \boldsymbol{\nu} + \tau_T \boldsymbol{s}_n = ((z_n^+ - \tilde{z}_n^-) - z)\boldsymbol{\nu} = 0 \quad \text{a.e. on } \Gamma_D,$$

and therefore $\boldsymbol{s}_n \in V$. Recall here that $z = z_n = 0$ a.e. on Γ_D, as $\boldsymbol{v} \in V$ and $\boldsymbol{v}_n \in V$. This is why $\boldsymbol{v} + \boldsymbol{s}_n \to \boldsymbol{v}$ in V with

$$\tau_{\boldsymbol{\nu}}(\boldsymbol{v} + \boldsymbol{s}_n) = z + (z_n^+ - \tilde{z}_n^-) - z = z_n^+ - \tilde{z}_n^-.$$

From (3.48b) we derive $\boldsymbol{v} + \boldsymbol{s}_n \in \mathcal{K}_{\boldsymbol{u}}(\mathcal{C})$ and (3.48a) combined with Lemma 3.35 implies that $\boldsymbol{v} + \boldsymbol{s}_n$ belongs to $[\operatorname{span}(\boldsymbol{w} - \boldsymbol{u})]_B^0$. Finally, we set $\tilde{\boldsymbol{v}}_n = \boldsymbol{v} + \boldsymbol{s}_n$ to complete the proof. □

Theorem 3.10 and Theorem 3.36 lead to the directional differentiability of the control-to-state map G_S:

Corollary 3.37. *The control-to-state map $G_S \colon V' \to V$ is directionally differentiable. In particular, for $\ell \in V'$ and $\delta\ell \in V'$ it holds*

$$\frac{G_S(\ell + t\delta\ell) - G_S(\ell)}{t} \to \boldsymbol{\eta} \quad \text{as } t \searrow 0,$$

where $\boldsymbol{\eta} \in V$ is the solution of the following VI

$$\boldsymbol{\eta} \in S^{\boldsymbol{u}}, \quad \langle B\boldsymbol{\eta}, \boldsymbol{v} - \boldsymbol{\eta}\rangle \geq \langle \delta\ell, \boldsymbol{v} - \boldsymbol{\eta}\rangle \quad \forall \boldsymbol{v} \in S^{\boldsymbol{u}}. \tag{3.49}$$

The convex cone $S^{\boldsymbol{u}}$ is defined by

$$S^{\boldsymbol{u}} = \left\{\boldsymbol{v} \in V \colon E\tau_{\boldsymbol{\nu}} \boldsymbol{v} \leq 0 \text{ q.e. on } \mathcal{A}_{\boldsymbol{u}},\ \langle \lambda, \tau_{\boldsymbol{n}} \boldsymbol{v}\rangle = 0\right\} \tag{3.50}$$

with $\boldsymbol{u} = G_S(\ell)$ and associated multiplier $\lambda \in \left(\tau_{\boldsymbol{\nu}}[V]\right)'$.

Remark 3.38. *Due to the coercivity of B the variational inequality* (3.49) *admits a unique solution for every $\delta\ell \in V'$ and the mapping $\delta\ell \mapsto \eta$ is Lipschitz continuous, cf. Theorem 3.3.*

Remark 3.39. *Because of the inequality in* (3.50) *the directional derivative is generally not linear w.r.t. to the direction and the control-to-state map G_S thus not Gâteaux differentiable, if the active set $\mathcal{A}_{\boldsymbol{u}}$ has positive capacity. We point out that the capacity of a set can be positive, while its Lebesgue measure is zero, see e.g. [3, Section 5.8.2].*

Proof of Corollary 3.37. Since $G_S = P_{\mathcal{C}}^B \circ B^{-1}$, cf. (3.9), we deduce from Theorem 3.10 and Theorem 3.36

$$\frac{G_S(\ell + \delta\ell) - G_S(\ell)}{t} = \frac{P_{\mathcal{C}}^B(B^{-1}\ell + tB^{-1}\delta\ell) - P_{\mathcal{C}}^B(B^{-1}\ell)}{t} \xrightarrow{t \searrow 0} \boldsymbol{\eta},$$

where $\boldsymbol{\eta}$ denotes the projection of $B^{-1}\delta\ell$ on the set $S^{\boldsymbol{u}} = \overline{\mathcal{K}_{\boldsymbol{u}}(\mathcal{C})} \cap [\operatorname{span}(B^{-1}\ell - \boldsymbol{u})]_B^0$ w.r.t. the norm induced by B. To be more precise, the function $\boldsymbol{\eta}$ belongs to $S^{\boldsymbol{u}}$ and solves the variational inequality

$$\left\langle B(\boldsymbol{\eta} - B^{-1}\delta\ell),\, \boldsymbol{v} - \boldsymbol{\eta}\right\rangle = \langle B\boldsymbol{\eta},\, \boldsymbol{v} - \boldsymbol{\eta}\rangle - \langle \delta\ell,\, \boldsymbol{v} - \boldsymbol{\eta}\rangle \geq 0 \quad \forall\, \boldsymbol{v} \in S^{\boldsymbol{u}}.$$

Furthermore, Lemma 3.32 and Lemma 3.35 show that $S^{\boldsymbol{u}}$ satisfies (3.50). □

Despite Remark 3.39 we obtain Gâteaux differentiability of G_S on a dense subset of V' similarly to [62, Théorème 3.4]:

Theorem 3.40. *There exists a dense subset $\mathcal{V}' \subset V'$, where the control-to-state map G_S is Gâteaux differentiable. The Gâteaux derivative at $\ell \in \mathcal{V}'$ in direction $\delta\ell \in V'$ is the solution of the following variational equation*

$$\boldsymbol{\eta} \in \mathcal{S}^{\boldsymbol{u}}, \quad \langle B\boldsymbol{\eta},\, \boldsymbol{v}\rangle = \langle \delta\ell,\, \boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in \mathcal{S}^{\boldsymbol{u}} \tag{3.51}$$

with $\mathcal{S}^{\boldsymbol{u}} := \left\{\boldsymbol{v} \in V \colon E\tau_{\boldsymbol{\nu}}\boldsymbol{v} = 0 \text{ q.e. on } \mathcal{A}_{\boldsymbol{u}}\right\}$ and $\boldsymbol{u} = G_S(\ell)$.

Proof. The space V and hence its dual V' are Hilbert spaces and the operator G_S is Lipschitz continuous, cf. Proposition 3.4. Therefore, the existence of the set $\mathcal{V}'$ is a consequence of [62, Théorème 1.2]. If G_S is Gâteaux differentiable at $\ell \in \mathcal{V}'$, then we know $\boldsymbol{\eta} := \partial G_S(\ell; \delta\ell) = -\partial G_S(\ell; -\delta\ell)$ for all $\delta\ell \in V'$. From Corollary 3.37 we infer

$$\boldsymbol{\eta} \in S^{\boldsymbol{u}}, \quad \langle B\boldsymbol{\eta},\, \boldsymbol{v} - \boldsymbol{\eta}\rangle \geq \langle \delta\ell,\, \boldsymbol{v} - \boldsymbol{\eta}\rangle \quad \forall\, \boldsymbol{v} \in S^{\boldsymbol{u}} \tag{3.52}$$

and in addition

$$\begin{aligned} &-\boldsymbol{\eta} \in S^{\boldsymbol{u}}, \quad \langle B(-\boldsymbol{\eta}),\, \boldsymbol{v} - (-\boldsymbol{\eta})\rangle \geq \langle -\delta\ell,\, \boldsymbol{v} - (-\boldsymbol{\eta})\rangle \quad \forall\, \boldsymbol{v} \in S^{\boldsymbol{u}} \\ \iff\ &\boldsymbol{\eta} \in -S^{\boldsymbol{u}}, \quad \langle B\boldsymbol{\eta},\, \boldsymbol{v} - \boldsymbol{\eta}\rangle \geq \langle \delta\ell,\, \boldsymbol{v} - \boldsymbol{\eta}\rangle \quad \forall\, \boldsymbol{v} \in -S^{\boldsymbol{u}} \end{aligned} \tag{3.53}$$

with $S^{\boldsymbol{u}} = \{\boldsymbol{v} \in V : E\tau_{\boldsymbol{\nu}}\boldsymbol{v} \le 0 \text{ q.e. on } \mathcal{A}_{\boldsymbol{u}},\ \langle \lambda, \tau_{\boldsymbol{n}}\boldsymbol{v}\rangle = 0\}$. Together, (3.52) and (3.53) yield

$$\boldsymbol{\eta} \in S^{\boldsymbol{u}} \cap -S^{\boldsymbol{u}}, \quad \langle B\boldsymbol{\eta}, \boldsymbol{v} - \boldsymbol{\eta}\rangle \ge \langle \delta\ell, \boldsymbol{v} - \boldsymbol{\eta}\rangle \quad \forall \boldsymbol{v} \in S^{\boldsymbol{u}} \cap -S^{\boldsymbol{u}}.$$

Since $\mathcal{S}^{\boldsymbol{u}} := S^{\boldsymbol{u}} \cap -S^{\boldsymbol{u}} = \{\boldsymbol{v} \in V : E\tau_{\boldsymbol{\nu}}\boldsymbol{v} = 0 \text{ q.e. on } \mathcal{A}_{\boldsymbol{u}}\}$ is a subspace, we arrive at (3.51). Here we used that $\lambda \circ \tau_{\boldsymbol{\nu}}$ can be represented by an integral w.r.t. a measure $\mu_\lambda \in \mathcal{M}^+(\overline{\Omega})$ concentrated on the set $\mathcal{A}_{\boldsymbol{u}}$ and vanishing on all sets $D \cap \overline{\Omega}$ with $\operatorname{cap}(D; \tilde{\Omega})=0$, cf. Lemma 3.34. Finally, we note that the solution of (3.51) indeed depends linearly on $\delta\ell$. □

3.3. First-order necessary optimality conditions

Based on the directional differentiability of G_S it is possible to derive necessary optimality conditions for the optimal control problem ($\mathbf{P}_S$). In particular, we will establish first-order conditions, which are comparable with the strong stationarity system for mathematical programs with complementarity constraints (MPCCs) in finite dimensions given in [69, Theorem 2].

We make the following assumption, which is supposed to hold throughout this section.

Assumption 3.41. *The objective functional $J : V \times U \to \mathbb{R}$ is Fréchet differentiable.*

According to Theorem 3.9 Problem ($\mathbf{P}_S$) is equivalent to a MPCC in function space:

$$\text{Minimize} \quad J(\boldsymbol{u}, \boldsymbol{f})$$
$$\text{s.t.} \quad \begin{cases} B\boldsymbol{u} + \tau_{\boldsymbol{\nu}}^*\lambda = R\boldsymbol{f} & \text{in } V' \\ \langle \lambda, z\rangle \ge 0 & \forall z \in \tau_{\boldsymbol{\nu}}[V] \text{ with } z \ge 0 \text{ a.e. on } \Gamma_C \\ \tau_{\boldsymbol{\nu}}\boldsymbol{u}(x) - \psi(x) \le 0 & \text{a.e. on } \Gamma_C \\ \langle \lambda, \tau_{\boldsymbol{\nu}}\boldsymbol{u} - \psi\rangle = 0, & \end{cases}$$

with $R : U \to V'$ as defined in (2.14). Our first-order analysis requires directional differentiability of the reduced objective functional $J(G_S(R\cdot), \cdot) : U \to \mathbb{R}$ guaranteed by the next auxiliary lemma.

Lemma 3.42 ([44, Lemma 3.9]). *Let X and Y be normed vector spaces. Moreover let $\mathcal{G} : X \to Y$ be directionally differentiable at $x \in X$ and $I : Y \times X \to \mathbb{R}$ be Fréchet differentiable. Then the functional $j : X \to \mathbb{R}$ defined through $j(x) = I(\mathcal{G}(x), x)$ is directionally differentiable at x and its directional derivative in direction $\delta x \in X$ is given by*

$$\partial j(x; \delta x) = I'(\mathcal{G}(x), x)(\partial\mathcal{G}(x; \delta x), \delta x).$$

Now we can deduce a necessary optimality condition involving solely primal variables:

Lemma 3.43. *Let $\bar{\boldsymbol{f}} \in U$ be a local optimal solution of $(\mathbf{P}_S)$ with associated state $\bar{\boldsymbol{u}} \in V$. Then it holds*

$$\partial_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})\boldsymbol{\eta} + \partial_{\boldsymbol{f}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})\delta\boldsymbol{f} \geq 0 \quad \forall\, \delta\boldsymbol{f} \in U, \tag{3.54}$$

where $\boldsymbol{\eta}$ solves (3.49) *with $S^{\boldsymbol{u}} = S^{\bar{\boldsymbol{u}}}$ and $\delta\ell = R\delta\boldsymbol{f}$.*

Proof. In view of Corollary 3.37 and the linearity of the operator R the composition $G_S \circ R \colon U \to V$ is directionally differentiable with

$$\begin{aligned} &\frac{(G_S \circ R)(\boldsymbol{f} + t\delta\boldsymbol{f}) - (G_S \circ R)(\boldsymbol{f})}{t} = \\ &= \frac{G_S(R\boldsymbol{f} + tR\delta\boldsymbol{f}) - G_S(R\boldsymbol{f})}{t} \xrightarrow{t \searrow 0} \partial G_S(R\boldsymbol{f}; R\delta\boldsymbol{f}) \quad \forall\, \boldsymbol{f} \in U,\ \forall\, \delta\boldsymbol{f} \in U. \end{aligned}$$

As the objective functional $J \colon V \times U \to \mathbb{R}$ is Fréchet differentiable by assumption, Lemma 3.42 thus yields directional differentiability of the mapping $U \ni \boldsymbol{f} \mapsto j(\boldsymbol{f}) := J(G_S(R\boldsymbol{f}), \boldsymbol{f}) \in \mathbb{R}$ with

$$\partial j(\bar{\boldsymbol{f}}; \delta\boldsymbol{f}) = \partial_{\boldsymbol{u}} J(G_S(R\bar{\boldsymbol{f}}), \bar{\boldsymbol{f}})\partial G_S(R\bar{\boldsymbol{f}}; R\delta\boldsymbol{f}) + \partial_{\boldsymbol{f}} J(G_S(R\bar{\boldsymbol{f}}), \bar{\boldsymbol{f}})\delta\boldsymbol{f}.$$

Due to local optimality of $\bar{\boldsymbol{f}}$ we conclude the assertion. □

Because the space U includes distributed controls $\boldsymbol{f}_1 \in L^2(\Omega; \mathbb{R}^d)$, strong stationarity for $(\mathbf{P}_S)$ can be proven analogously to [27, Section 5]. For this purpose we need another auxiliary result.

Lemma 3.44. *Let $\bar{\boldsymbol{f}} \in U$ be a local optimal solution of $(\mathbf{P}_S)$ with associated state $\bar{\boldsymbol{u}} \in V$. Then there exists a $\bar{\boldsymbol{\omega}} \in V$ such that*

$$\partial_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})\boldsymbol{\eta} - \langle \delta\ell, \bar{\boldsymbol{\omega}} \rangle \geq 0 \quad \forall\, \delta\ell \in V',$$

where $\boldsymbol{\eta}$ solves (3.49) *with $S^{\boldsymbol{u}} = S^{\bar{\boldsymbol{u}}}$.*

Proof. Thanks to Lemma 3.43 it is already known that

$$\partial_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})\partial G_S(R\bar{\boldsymbol{f}}; R\delta\boldsymbol{f}) + \partial_{\boldsymbol{f}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})\delta\boldsymbol{f} \geq 0 \quad \forall\, \delta\boldsymbol{f} \in U. \tag{3.55}$$

The variational inequality (3.49) is uniquely solvable and the associated solution operator is Lipschitz continuous by Remark 3.38. Since $\partial G_S(R\bar{\boldsymbol{f}}; R\mathbf{0}) = \mathbf{0}$, we therefore arrive at the estimate

$$-\partial_{\boldsymbol{f}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})\delta\boldsymbol{f} \leq c \left\| \partial_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) \right\|_{V'} \left\| R\delta\boldsymbol{f} \right\|_{V'}. \tag{3.56}$$

Next, we show that the linear operator R is injective. If $\boldsymbol{f}, \boldsymbol{g} \in U$ satisfy $R\boldsymbol{f} = R\boldsymbol{g}$, then the density of $C_0^\infty(\Omega;\mathbb{R}^d)$ in $L^2(\Omega;\mathbb{R}^d)$, cf. [1, Satz 2.14], implies the existence of a sequence $\{\boldsymbol{h}_1^n\}_{n\in\mathbb{N}} \subset V$ with

$$0 = \langle R(\boldsymbol{f}-\boldsymbol{g}),\, \boldsymbol{h}_1^n\rangle = \int_\Omega (\boldsymbol{f}_1 - \boldsymbol{g}_1)\cdot \boldsymbol{h}_1^n \,\mathrm{d}x \xrightarrow{n\to\infty} \|\boldsymbol{f}_1 - \boldsymbol{g}_1\|^2_{L^2(\Omega;\mathbb{R}^d)} . \tag{3.57}$$

Now let $\tilde{\boldsymbol{f}}_2 \in L^2(\Gamma;\mathbb{R}^d)$ and $\tilde{\boldsymbol{g}}_2 \in L^2(\Gamma;\mathbb{R}^d)$ denote the extensions by zero of $\boldsymbol{f}_2$ and $\boldsymbol{g}_2$, respectively. According to Theorem A.20 there is a sequence $\{\boldsymbol{h}_2^n\}_{n\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^d)$ with

$$\tau\boldsymbol{h}_2^n \to \tilde{\boldsymbol{f}}_2 - \tilde{\boldsymbol{g}}_2 \quad \text{in } L^2(\Gamma;\mathbb{R}^d). \tag{3.58}$$

Similarly to the proof of Lemma 3.21 we furthermore find $\{\zeta_n\}_{n\in\mathbb{N}} \subset C^{0,1}(\overline{\Omega})$ with $0 \le \zeta_n \le 1$, $\zeta_n \equiv 0$ on Γ_D and

$$\zeta_n \to \chi_{\overline{\Omega}\setminus\Gamma_D} \quad \text{pointwisely in } \overline{\Omega}.$$

Corollary A.18, Theorem A.14 and (3.58) consequently lead to $\zeta_n\boldsymbol{h}_2^n \in V$ and

$$\tau(\zeta_n\boldsymbol{h}_2^n) = \zeta_n|_\Gamma \tau(\boldsymbol{h}_2^n) \to \chi_{\Gamma\setminus\Gamma_D}(\tilde{\boldsymbol{f}}_2 - \tilde{\boldsymbol{g}}_2) \quad \text{a.e. on } \Gamma,$$

in case a proper subsequence is considered. As a result of (3.58) and $\zeta_n \in [0,1]$ for all $n \in \mathbb{N}$ the sequence $\{\tau(\zeta_n\boldsymbol{h}_2^n)\}_{n\in\mathbb{N}}$ is bounded in $L^2(\Gamma;\mathbb{R}^d)$. Hence, there is a weakly convergent subsequence, w.l.o.g. denoted in the same way, whose weak limit coincides with the pointwise limit by Egorov's theorem, cf. the proof of Lemma 2.18, so that

$$0 = \langle R(\boldsymbol{f}-\boldsymbol{g}),\, \zeta_n\boldsymbol{h}_2^n\rangle = \int_{\Gamma_N} (\boldsymbol{f}_2 - \boldsymbol{g}_2)\cdot\tau_N(\zeta_n\boldsymbol{h}_2^n)\,\mathrm{d}s \to \|\boldsymbol{f}_2 - \boldsymbol{g}_2\|^2_{L^2(\Gamma_N;\mathbb{R}^d)} . \tag{3.59}$$

Together, (3.57) and (3.59) yield $\boldsymbol{f} = \boldsymbol{g}$ in U. The operator R is therefore bijective from U to its image $R[U]$ with linear inverse R_{-1} and the left-hand side in (3.56) can be seen as a linear functional on $R[U]$. This functional is dominated by the sublinear function $p\colon V' \to \mathbb{R}$ defined through

$$p(\ell) = c\,\big\|\partial_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})\big\|_{V'} \|\ell\|_{V'} .$$

In view of the Hahn-Banach theorem it follows that $-\partial_{\boldsymbol{f}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})R_{-1}$ can be extended to a linear and continuous functional on V'. Because the space V is a Hilbert space, we identify the extension with a function $\bar{\boldsymbol{\omega}} \in V$ and obtain

$$\langle \bar{\boldsymbol{\omega}},\, R\delta\boldsymbol{f}\rangle = -\partial_{\boldsymbol{f}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})\delta\boldsymbol{f} \quad \forall\, \delta\boldsymbol{f} \in U. \tag{3.60}$$

Moreover, on account of [1, Satz 2.14] the embedding $V \hookrightarrow L^2(\Omega;\mathbb{R}^d)$ is dense. Since the operator $\tilde{R}\colon L^2(\Omega;\mathbb{R}^d) \to \big(L^2(\Omega;\mathbb{R}^d)\big)'$ defined by

$$\langle \tilde{R}\boldsymbol{f},\, \boldsymbol{g}\rangle_{(L^2(\Omega;\mathbb{R}^d))', L^2(\Omega;\mathbb{R}^d)} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{g}\,\mathrm{d}x$$

is isometrically isomorphic, its image $\tilde{R}\big[L^2\big(\Omega;\mathbb{R}^d\big)\big]$ and thus $R[U]$ is densely embedded in V' due to Proposition A.26. Note here that $R\delta\boldsymbol{f} = \tilde{R}\delta\boldsymbol{f}_1$ holds for all $\delta\boldsymbol{f} = (\delta\boldsymbol{f}_1, \mathbf{0}) \in U$. From the Lipschitz continuity of $\partial G_S(R\bar{\boldsymbol{f}};\cdot)\colon V' \to V$, (3.55) and (3.60) we infer the assertion. □

The first-order necessary optimality conditions of strongly stationary type for $(\mathbf{P}_S)$ read as follows:

Theorem 3.45. *Let $\bar{\boldsymbol{f}} \in U$ be a locally optimal solution of $(\mathbf{P}_S)$ with associated state $\bar{\boldsymbol{u}} \in V$ and multiplier $\bar{\lambda} \in \big(\tau_{\boldsymbol{\nu}}[V]\big)'$. Then there exist an adjoint state $\bar{\boldsymbol{\omega}} \in V$ and a multiplier $\bar{\theta} \in \big(\tau_{\boldsymbol{\nu}}[V]\big)'$ such that*

$$
\begin{aligned}
B\bar{\boldsymbol{u}} + \tau_{\boldsymbol{\nu}}^*\bar{\lambda} &= R\bar{\boldsymbol{f}} \quad \text{in } V' && \text{(3.61a)}\\
\big\langle \bar{\lambda},\, z\big\rangle &\geq 0 \quad \forall\, z \in \tau_{\boldsymbol{\nu}}[V] \text{ with } z \geq 0 \text{ a.e. on } \Gamma_C && \text{(3.61b)}\\
\tau_{\boldsymbol{\nu}}\bar{\boldsymbol{u}}(x) - \psi(x) &\leq 0 \quad \text{a.e. on } \Gamma_C && \text{(3.61c)}\\
\big\langle \bar{\lambda},\, \tau_{\boldsymbol{\nu}}\bar{\boldsymbol{u}} - \psi\big\rangle &= 0 && \text{(3.61d)}\\
B\bar{\boldsymbol{\omega}} + \tau_{\boldsymbol{\nu}}^*\bar{\theta} &= \partial_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) \quad \text{in } V' && \text{(3.61e)}\\
\bar{\boldsymbol{\omega}} \in S^{\bar{\boldsymbol{u}}}, \quad \big\langle \bar{\theta},\, \tau_{\boldsymbol{\nu}}\boldsymbol{v}\big\rangle &\geq 0 \quad \forall\, \boldsymbol{v} \in S^{\bar{\boldsymbol{u}}} && \text{(3.61f)}\\
\bar{\boldsymbol{\omega}} + \partial_{\boldsymbol{f}_1} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) &= 0 \quad \text{a.e. in } \Omega && \text{(3.61g)}\\
\tau\bar{\boldsymbol{\omega}} + \partial_{\boldsymbol{f}_2} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) &= 0 \quad \text{a.e. on } \Gamma_N, && \text{(3.61h)}
\end{aligned}
$$

where the convex cone $S^{\bar{\boldsymbol{u}}}$ is as defined in (3.50).

Proof. We define $\boldsymbol{q} \in V$ as the unique solution of

$$
\langle B\boldsymbol{q},\, \boldsymbol{v}\rangle = \big\langle \partial_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}),\, \boldsymbol{v}\big\rangle \quad \forall\, \boldsymbol{v} \in V. \tag{3.62}
$$

Let moreover $\Pi\colon V \to V$ be defined by

$$
\Pi(\boldsymbol{v}) = \big(\partial G_S(R\bar{\boldsymbol{f}};\cdot) \circ B\big)(\boldsymbol{v}).
$$

In consequence of the symmetry of $\mathbb{C}$, cf. Assumption 3.1(2), the operator B is self-adjoint. From Lemma 3.44, (3.62) and $\partial G_S(R\bar{\boldsymbol{f}};\cdot) = \Pi \circ B^{-1}$ we therefore infer the estimate

$$
\begin{aligned}
0 &\leq \partial_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) \partial G_S(R\bar{\boldsymbol{f}}; \delta\ell) - \big\langle BB^{-1}\delta\ell,\, \bar{\boldsymbol{\omega}}\big\rangle\\
&= \big\langle B\boldsymbol{q},\, \partial G_S(R\bar{\boldsymbol{f}}; \delta\ell)\big\rangle - \big\langle B^{-1}\delta\ell,\, B\bar{\boldsymbol{\omega}}\big\rangle\\
&= \big\langle \Pi(B^{-1}\delta\ell),\, B(\boldsymbol{q} - \bar{\boldsymbol{\omega}})\big\rangle - \big\langle (I_V - \Pi)(B^{-1}\delta\ell),\, B\bar{\boldsymbol{\omega}}\big\rangle\\
&= \big\langle \Pi(B^{-1}\delta\ell),\, B(\boldsymbol{q} - \bar{\boldsymbol{\omega}})\big\rangle - \big\langle (I_V - \Pi)(B^{-1}\delta\ell),\, B\Pi(\bar{\boldsymbol{\omega}})\big\rangle\\
&\quad - \big\langle (I_V - \Pi)(B^{-1}\delta\ell),\, B(I_V - \Pi)(\bar{\boldsymbol{\omega}})\big\rangle
\end{aligned} \tag{3.63}
$$

for a $\bar{\boldsymbol{\omega}} \in V$ satisfying (3.60) and for all $\delta\ell \in V'$. Herein, the mapping $I_V\colon V \to V$ denotes the identity. The operator Π is the projection on the set $S^{\bar{\boldsymbol{u}}}$ w.r.t. to the norm induced by B, cf. the proof of Corollary 3.37. Hence, for every $\boldsymbol{\xi} \in V$ we know

$$\Pi(\boldsymbol{\xi}) \in S^{\bar{\boldsymbol{u}}}, \quad \langle B(\Pi(\boldsymbol{\xi}) - \boldsymbol{\xi}), \boldsymbol{v} - \Pi(\boldsymbol{\xi})\rangle \geq 0 \quad \forall \boldsymbol{v} \in S^{\bar{\boldsymbol{u}}}. \tag{3.64}$$

Because $S^{\bar{\boldsymbol{u}}}$ is a convex cone, we can test the variational inequality in (3.64) with $\boldsymbol{v} = 2\Pi(\boldsymbol{\xi})$ and $\boldsymbol{v} = \boldsymbol{0}$, which leads to

$$\langle B(I_V - \Pi)\boldsymbol{\xi}, \Pi(\boldsymbol{\xi})\rangle = 0 \quad \forall \boldsymbol{\xi} \in V. \tag{3.65}$$

Besides, Π is idempotent so that

$$\Pi \circ \Pi = \Pi, \; (I_V - \Pi) \circ (I_V - \Pi) = I_V - \Pi \tag{3.66a}$$
$$\Pi \circ (I_V - \Pi) = (I_V - \Pi) \circ \Pi = 0. \tag{3.66b}$$

We insert $\delta\ell = B(I_V - \Pi)(\bar{\boldsymbol{\omega}}) \in V'$ in (3.63) and obtain

$$\begin{aligned} 0 &\leq \big\langle \Pi(B^{-1}B(I_V - \Pi)(\bar{\boldsymbol{\omega}})), B(\boldsymbol{q} - \bar{\boldsymbol{\omega}})\big\rangle \\ &\quad - \big\langle (I_V - \Pi)(B^{-1}B(I_V - \Pi)(\bar{\boldsymbol{\omega}})), B\Pi(\bar{\boldsymbol{\omega}})\big\rangle \\ &\quad - \big\langle (I_V - \Pi)(B^{-1}B(I_V - \Pi)(\bar{\boldsymbol{\omega}})), B(I_V - \Pi)(\bar{\boldsymbol{\omega}})\big\rangle \\ &= \langle (I_V - \Pi)(\bar{\boldsymbol{\omega}}), B\Pi(\bar{\boldsymbol{\omega}})\rangle - \langle (I_V - \Pi)(\bar{\boldsymbol{\omega}}), B(I_V - \Pi)(\bar{\boldsymbol{\omega}})\rangle \\ &= -\langle B(I_V - \Pi)(\bar{\boldsymbol{\omega}}), (I_V - \Pi)(\bar{\boldsymbol{\omega}})\rangle, \end{aligned}$$

where (3.65), (3.66) and $B = B^*$ was used. Together with the coercivity of B this implies

$$\|\bar{\boldsymbol{\omega}} - \Pi(\bar{\boldsymbol{\omega}})\|_V^2 \leq c\,\langle B(\bar{\boldsymbol{\omega}} - \Pi(\bar{\boldsymbol{\omega}})), \bar{\boldsymbol{\omega}} - \Pi(\bar{\boldsymbol{\omega}})\rangle \leq 0,$$

i.e., $\bar{\boldsymbol{\omega}} = \Pi(\bar{\boldsymbol{\omega}}) \in S^{\bar{\boldsymbol{u}}}$. Now let $\boldsymbol{s} \in V$ be the solution of

$$\langle B\boldsymbol{s}, \boldsymbol{v}\rangle = \langle B(\bar{\boldsymbol{\omega}} - \boldsymbol{q}), \boldsymbol{v}\rangle \quad \forall \boldsymbol{v} \in V. \tag{3.67}$$

Plugging $\delta\ell = B\Pi(\boldsymbol{s}) \in V'$ in (3.63) and taking account of (3.65)–(3.67) yields

$$\begin{aligned} 0 &\leq \big\langle \Pi(B^{-1}B\Pi(\boldsymbol{s})), B(\boldsymbol{q} - \bar{\boldsymbol{\omega}})\big\rangle - \big\langle (I_V - \Pi)(B^{-1}B\Pi(\boldsymbol{s})), B\Pi(\bar{\boldsymbol{\omega}})\big\rangle \\ &\quad - \big\langle (I_V - \Pi)(B^{-1}B\Pi(\boldsymbol{s})), B(I_V - \Pi)(\bar{\boldsymbol{\omega}})\big\rangle \\ &= \langle \Pi(\boldsymbol{s}), B(\boldsymbol{q} - \bar{\boldsymbol{\omega}})\rangle = -\langle B\boldsymbol{s}, \Pi(\boldsymbol{s})\rangle = -\langle B\Pi(\boldsymbol{s}), \Pi(\boldsymbol{s})\rangle. \end{aligned}$$

Thanks to the coercivity of B we conclude $\Pi(\boldsymbol{s}) = \boldsymbol{0}$. In view of (3.64) it therefore follows

$$\langle B\boldsymbol{s}, \boldsymbol{v}\rangle \leq 0 \quad \forall \boldsymbol{v} \in S^{\bar{\boldsymbol{u}}},$$

which by (3.67) and (3.62) is equivalent to

$$\langle B\bar{\boldsymbol{\omega}}, \boldsymbol{v}\rangle \leq \langle B\boldsymbol{q}, \boldsymbol{v}\rangle = \big\langle \partial_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}), \boldsymbol{v}\big\rangle \quad \forall \boldsymbol{v} \in S^{\bar{\boldsymbol{u}}}.$$

We define $\bar{\zeta} = \partial_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) - B\bar{\boldsymbol{\omega}} \in V'$ and find

$$\begin{aligned} B\bar{\boldsymbol{\omega}} + \bar{\zeta} &= \partial_{\boldsymbol{u}} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) \\ \big\langle \bar{\zeta}, \boldsymbol{v}\big\rangle &\geq 0 \quad \forall \boldsymbol{v} \in S^{\bar{\boldsymbol{u}}}. \end{aligned} \tag{3.68}$$

Due to the linearity of the extension operator E we further deduce from Corollary A.10 and (3.68) that

$$\langle \bar{\zeta}, \boldsymbol{v} \rangle = 0 \quad \forall \boldsymbol{v} \in V \text{ with } \tau_{\boldsymbol{\nu}} \boldsymbol{v} = 0.$$

This is why the functional $\bar{\zeta} \in V'$ belongs to $\ker(\tau_{\boldsymbol{\nu}})^{\perp}$. As the mapping $\tau_{\boldsymbol{\nu}} \colon V \to \tau_{\boldsymbol{\nu}}[V]$ is surjective, the set $\ker(\tau_{\boldsymbol{\nu}})^{\perp}$ coincides with the range of $\tau_{\boldsymbol{\nu}}^*$, cf. [61, Theorem 6.6.2], and there exists $\bar{\theta} \in \big(\tau_{\boldsymbol{\nu}}[V]\big)'$ with $\bar{\zeta} = \tau_{\boldsymbol{\nu}}^* \bar{\theta}$. Finally, by testing with $(\boldsymbol{f}_1, \boldsymbol{0})$, $\boldsymbol{f}_1 \in L^2\big(\Omega; \mathbb{R}^d\big)$, and $(\boldsymbol{0}, \boldsymbol{f}_2)$, $\boldsymbol{f}_2 \in L^2\big(\Gamma_N; \mathbb{R}^d\big)$, in (3.60), we derive $\bar{\boldsymbol{\omega}} = \partial_{\boldsymbol{f}_1} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})$ a.e. in Ω and $\tau \bar{\boldsymbol{\omega}} = \partial_{\boldsymbol{f}_2} J(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}})$ a.e. on Γ_N, respectively. □

Remark 3.46. *We want to compare the strong stationarity system* (3.61) *with the one stated in [69, Theorem 2]. To this end we investigate a finite-dimensional version of* $(\mathbf{P}_S)$*. More precisely, we consider*

$$\left.\begin{aligned} &\text{Minimize} \quad J_h(\boldsymbol{u}, \boldsymbol{f}) \\ &\text{s.t.} \quad \begin{cases} M\boldsymbol{u} + N^{\top}\boldsymbol{\lambda} = Q\boldsymbol{f} \\ \boldsymbol{\lambda} \geq 0, \quad \boldsymbol{\lambda}^{\top}(N\boldsymbol{u} - \boldsymbol{\psi}) = 0, \quad N\boldsymbol{u} \leq \boldsymbol{\psi} \end{cases} \end{aligned}\right\} \tag{3.69}$$

with objective functional $J_h \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$*, vectors* $\boldsymbol{u}, \boldsymbol{f} \in \mathbb{R}^n$*,* $\boldsymbol{\lambda}, \boldsymbol{\psi} \in \mathbb{R}^m$*, and matrices* $M, Q \in \mathbb{R}^{n \times n}$ *and* $N \in \mathbb{R}^{m \times n}$*. The complementarity system in* (3.69) *is obtained e.g. via a finite element discretization of* (3.6)*. According to [69] a solution* $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{f}}) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$ *of* (3.69) *is called strongly stationary, if there exist multipliers* $\tilde{\boldsymbol{\omega}} \in \mathbb{R}^n$ *and* $\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\zeta}} \in \mathbb{R}^m$ *such that*

$$M^{\top}\tilde{\boldsymbol{\omega}} + N^{\top}\tilde{\boldsymbol{\theta}} = -\partial_{\boldsymbol{u}} J_h(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) \tag{3.70a}$$

$$N\tilde{\boldsymbol{\omega}} = \tilde{\boldsymbol{\zeta}} \tag{3.70b}$$

$$Q^{\top}\tilde{\boldsymbol{\omega}} = \partial_{\boldsymbol{f}} J_h(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) \tag{3.70c}$$

$$(N\boldsymbol{u} - \boldsymbol{\psi})_i \tilde{\boldsymbol{\theta}}_i = 0 \quad \forall i \in \{1, \ldots, m\} \tag{3.70d}$$

$$\bar{\boldsymbol{\lambda}}_i \tilde{\boldsymbol{\zeta}}_i = 0 \quad \forall i \in \{1, \ldots, m\} \tag{3.70e}$$

$$\tilde{\boldsymbol{\theta}}_i, \tilde{\boldsymbol{\zeta}}_i \geq 0 \quad \forall i \in \{1, \ldots, m\} \text{ with } \bar{\boldsymbol{\lambda}}_i = (N\bar{\boldsymbol{u}} - \boldsymbol{\psi})_i = 0. \tag{3.70f}$$

From (3.70b)*,* (3.70e) *and* (3.70f) *we deduce* $(N\tilde{\boldsymbol{\omega}})_i \geq 0$ *for all* $i \in \{1, \ldots, m\}$ *with* $\bar{\boldsymbol{\lambda}}_i = (N\bar{\boldsymbol{u}} - \boldsymbol{\psi})_i = 0$*,* $(N\tilde{\boldsymbol{\omega}})_i = 0$ *for all* $i \in \{1, \ldots, m\}$ *with* $\bar{\boldsymbol{\lambda}}_i > 0$ *and thus* $\bar{\boldsymbol{\lambda}}^{\top} N\tilde{\boldsymbol{\omega}} = 0$*. Moreover,* (3.70d) *and* (3.70f) *lead to* $\tilde{\boldsymbol{\theta}}_i = 0$ *for all* $i \in \{1, \ldots, m\}$ *with* $(N\bar{\boldsymbol{u}} - \boldsymbol{\psi})_i \neq 0$ *and* $\tilde{\boldsymbol{\theta}}_i \geq 0$ *for all* $i \in \{1, \ldots, m\}$ *with* $\bar{\boldsymbol{\lambda}}_i = (N\bar{\boldsymbol{u}} - \boldsymbol{\psi})_i = 0$*. If* $\boldsymbol{v} \in \mathbb{R}^n$ *satisfies* $(N\boldsymbol{v})_i \leq 0$ *for all* $i \in \{1, \ldots, m\}$ *with* $(N\bar{\boldsymbol{u}} - \boldsymbol{\psi})_i = 0$ *and* $\bar{\boldsymbol{\lambda}}^{\top} N\boldsymbol{v} = 0$*, then we observe* $\lambda_i (N\boldsymbol{v})_i = 0$ *for all* $i \in \{1, \ldots, m\}$ *and* $(N\boldsymbol{v})_i = 0$ *for all* $i \in \{1, \ldots, m\}$ *with* $\bar{\boldsymbol{\lambda}}_i > 0$ *so that* $\tilde{\boldsymbol{\theta}}^{\top} N\boldsymbol{v} \leq 0$*. Consequently, the strong stationarity system* (3.70) *implies*

$$M^{\top}\bar{\boldsymbol{\omega}} + N^{\top}\bar{\boldsymbol{\theta}} = \partial_{\boldsymbol{u}} J_h(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) \tag{3.71a}$$

$$\bar{\boldsymbol{\omega}} \in S_h^{\bar{\boldsymbol{u}}}, \quad \bar{\boldsymbol{\theta}}^{\top} N\boldsymbol{v} \geq 0 \quad \forall \boldsymbol{v} \in S_h^{\bar{\boldsymbol{u}}} \tag{3.71b}$$

$$Q^{\top}\bar{\boldsymbol{\omega}} + \partial_{\boldsymbol{f}} J_h(\bar{\boldsymbol{u}}, \bar{\boldsymbol{f}}) = 0, \tag{3.71c}$$

where $\bar{\boldsymbol{\omega}} := -\tilde{\boldsymbol{\omega}}$, $\bar{\boldsymbol{\theta}} := -\tilde{\boldsymbol{\theta}}$ *and* $S_h^{\bar{\mathbf{u}}}$ *is given by*

$$S_h^{\bar{\mathbf{u}}} = \left\{ \boldsymbol{v} \in \mathbb{R}^n \colon (N\boldsymbol{v})_i \leq 0 \ \forall\, i \in \{1, \ldots, n\} \text{ with } (N\bar{\mathbf{u}} - \boldsymbol{\psi})_i = 0,\ \bar{\boldsymbol{\lambda}}^\top N\boldsymbol{v} = 0 \right\}.$$

Since (3.61g) *together with* (3.61h) *is equivalent to* $R^*\bar{\boldsymbol{\omega}} + \partial_{\boldsymbol{f}} J(\bar{\mathbf{u}}, \bar{\boldsymbol{f}}) = 0$, *cf.* (3.60), *we can interpret* (3.71) *as the finite-dimensional analogue of* (3.61e)–(3.61h). *Having the surjectivity of the mapping* $\tau_{\boldsymbol{\nu}} \colon V \to \tau_{\boldsymbol{\nu}}[V]$ *in mind, we assume now that its counterpart* $N \in \mathbb{R}^{m \times n}$ *is also surjective. From* (3.71b) *it follows* $(N\bar{\boldsymbol{\omega}})_i \leq 0$ *for all* $i \in \{1, \ldots, m\}$ *with* $(N\bar{\mathbf{u}} - \boldsymbol{\psi})_i = 0$, $\bar{\boldsymbol{\lambda}}^\top N\bar{\boldsymbol{\omega}} = 0$ *and hence* $\bar{\boldsymbol{\lambda}}_i (N\bar{\boldsymbol{\omega}})_i = 0$ *for all* $i \in \{1, \ldots, m\}$. *Furthermore, the surjectivity of* N *yields* $\bar{\boldsymbol{\theta}}_i \leq 0$ *for all* $i \in \{1, \ldots, m\}$ *with* $(N\bar{\mathbf{u}} - \boldsymbol{\psi})_i = \bar{\boldsymbol{\lambda}}_i = 0$ *and* $\bar{\boldsymbol{\theta}}_i = 0$ *for all* $i \in \{1, \ldots, m\}$ *with* $(N\bar{\mathbf{u}} - \boldsymbol{\psi})_i \neq 0$ *so that* $\bar{\boldsymbol{\theta}}_i (N\bar{\mathbf{u}} - \boldsymbol{\psi})_i = 0$ *for all* $i \in \{1, \ldots, m\}$. *Altogether, we infer from* (3.71) *the strong stationarity system* (3.70) *with* $\tilde{\boldsymbol{\omega}} := -\bar{\boldsymbol{\omega}}$, $\tilde{\boldsymbol{\theta}} := -\bar{\boldsymbol{\theta}}$ *and* $\tilde{\boldsymbol{\zeta}} := -N\bar{\boldsymbol{\omega}}$, *which justifies that we used the same terminology in Theorem 3.45.*

4. Conclusion and outlook

In the previous chapters we investigated the optimal control of two variational inequalities arising in mechanical contact problems - the static model of infinitesimal elastoplasticity with linear kinematic hardening ($\mathbf{VI}_E$) and Signorini's problem ($\mathbf{VI}_S$). Solution operators associated with variational inequalities are commonly not Gâteaux differentiable and the same applied for ($\mathbf{VI}_E$) and ($\mathbf{VI}_S$). As a result the standard optimal control theory could not be employed. Nevertheless, in case of ($\mathbf{VI}_E$) we were able to prove Bouligand differentiability of the solution operator under additional regularity assumptions. By slightly weakening these assumptions it was still possible to show directional differentiability. With these results at hand we established two different second-order sufficient optimality conditions for the optimal control of ($\mathbf{VI}_E$). The first condition was based on the Bouligand differentiability of the solution operator and rather restrictive but applicable to a general smooth objective functional. For an objective functional with a particular structure we then stated a sufficient condition, which was comparable with its finite-dimensional counterpart in [69], using the directional differentiability of the solution operator. With the help of an extension of Riesz' representation theorem it was also possible to show directional differentiability of the solution operator associated with ($\mathbf{VI}_S$) by following the ideas of [62]. This allowed us to derive first-order necessary optimality conditions of strongly stationary type for optimal control problems subject to ($\mathbf{VI}_S$).

We have seen that the solution operator of ($\mathbf{VI}_E$) is Gâteaux differentiable if certain regularity requirements are satisfied and the biactive set of the solution, where the operator is differentiated, is empty. The same holds true for ($\mathbf{VI}_S$), provided that the active set vanishes. The topic of future research could be to exploit this information in order to design efficient optimization algorithms for optimal control problems governed by ($\mathbf{VI}_E$) and ($\mathbf{VI}_S$), respectively. Bundle methods for non-smooth problems or path-following approaches for regularized problems as in [56, 71] could be combined with gradient-based optimization methods. By observing the critical sets such algorithms would determine gradients via an adjoint calculus and perform Newton-like steps, whenever it seems suitable, and switch back to a bundle or path-following method otherwise. Maybe it is also possible to find descent directions under additional assumptions, although the gradient does not exist, so that a classical descent method can be performed, cf. [49]. Another alternative could be to use information about the gradient within a trust region algorithm, cf. [27]. In addition, the goal of future research should be of course to establish a substantial optimal control theory for the coupling of $(\mathbf{VI}_E)$ and $(\mathbf{VI}_S)$. In particular the existence of optimal solutions and subsequently the differentiability of the solution operator associated with the coupled system must be discussed. Furthermore, the underlying model must be extended. For instance the assumption of linear kinematic hardening is not realistic and should be replaced by a nonlinear hardening law.

A. Auxiliary results

A.1. Capacity theory

We provide basics of capacity theory obtained from the books [3], [17] and [39]. Let $\Omega \subset \mathbb{R}^d$ be open and bounded. The following definition can be found in [3, Section 5.8.2].

Definition A.1 (Capacity)**.**

a) For an open set $U \subset \Omega$ the capacity of U w.r.t. Ω is defined by

$$\operatorname{cap}(U;\Omega) := \inf\left\{\|v\|^2_{H^1_0(\Omega)} : v \in H^1_0(\Omega),\, v \geq 1 \text{ a.e. in } U\right\}.$$

b) This definition is extended to any subset E of Ω by

$$\operatorname{cap}(E;\Omega) := \inf\left\{\operatorname{cap}(U;\Omega)\colon E \subset U \subset \Omega,\, U \text{ open}\right\}.$$

Proposition A.2 ([3, Proposition 5.8.3])**.** *Let $E \subset$ be an arbitrary subset of Ω. Then it holds*

$$\operatorname{cap}(E;\Omega) = \inf\left\{\|v\|^2_{H^1_0(\Omega)} : v \in H^1_0(\Omega),\, v \geq 1 \text{ a.e. in a neighborhood of } E\right\}.$$

Proposition A.3 ([3, Proposition 5.8.4])**.** *Let $K \subset \Omega$ be compact and $U \subset \Omega$ be open. Then it holds*

a) $\operatorname{cap}(K;\Omega) = \inf\left\{\|v\|^2_{H^1_0(\Omega)} : v \in C^\infty_0(\Omega),\, v \geq 1 \text{ in } K\right\}$

b) $\operatorname{cap}(U;\Omega) = \sup\left\{\operatorname{cap}(K;\Omega)\colon K \subset U,\, K \text{ compact}\right\}.$

Heinonen et al. define capacity in [39, Section 2] for compact and open sets as in Proposition A.3. The definition is then extended to arbitrary sets as in Definition A.1 b).

Theorem A.4 ([39, Theorem 2.2])**.** *Let $A, B \subset \Omega$ with $A \subset B$ and $E = \bigcup_{n\in\mathbb{N}} E_n$ with $\{E_n\}_{n\in\mathbb{N}} \subset \mathfrak{P}(\Omega)$. Then it holds* *i)* $\mathrm{cap}(A;\Omega) \leq \mathrm{cap}(B;\Omega)$

ii) $\mathrm{cap}(E;\Omega) \leq \sum_{n\in\mathbb{N}} \mathrm{cap}(E_n;\Omega)$.

In [17, Section 6.4.3] capacity is defined only for Borel sets:

Definition A.5. *Let $E \in \mathcal{B}(\Omega)$ and $\alpha \in \mathbb{R}$.*

a) We say that $v \in H_0^1(\Omega)$ satisfies $v \geq \alpha$ in E in the sense of $H_0^1(\Omega)$, if there exists a sequence $v_n \to v$ in $H_0^1(\Omega)$ such that $v_n \geq \alpha$ a.e. in a neighborhood of E.

b) The capacity of E, in the sense of $H_0^1(\Omega)$, is defined as

$$\mathrm{c}(E;\Omega) = \inf\left\{\|v\|^2_{H_0^1(\Omega)} : v \geq 1 \text{ in } E \text{ in the sense of } H_0^1(\Omega)\right\}.$$

In fact, Definition A.1 and Definition A.5 are equivalent.

Lemma A.6. *Let $E \in \mathcal{B}(\Omega)$. Then it holds* $\mathrm{c}(E;\Omega) = \mathrm{cap}(E;\Omega)$.

Proof. By the choice $v_n = v$ for all $n \in \mathbb{N}$ we see that every $v \in H_0^1(\Omega)$ with $v \geq 1$ a.e. in a neighborhood of E also satisfies $v \geq 1$ in E in the sense of $H_0^1(\Omega)$. Thanks to Proposition A.2 it follows $\mathrm{cap}(E;\Omega) \geq \mathrm{c}(E;\Omega)$, since we minimize on a smaller set. In order to prove the inverse estimate let

$$\{v_n\}_{n\in\mathbb{N}} \subset \left\{v \in H_0^1(\Omega) \colon v \geq 1 \text{ in } E \text{ in the sense of } H_0^1(\Omega)\right\}$$

be a minimizing sequence so that

$$\|v_n\|^2_{H_0^1(\Omega)} \to \mathrm{c}(E;\Omega).$$

Consequently, for every $n \in \mathbb{N}$ there exists a sequence $\{v_k^n\}_{k\in\mathbb{N}} \subset H_0^1(\Omega)$ with $v_k^n \geq 1$ a.e. in a neighborhood of E and $v_k^n \to v_n$ as $k \to \infty$. If we extract $v_k \in \{v_k^n\}_{k\in\mathbb{N}}$ with

$$\left|\|v_n\|^2_{H_0^1(\Omega)} - \|v_k\|^2_{H_0^1(\Omega)}\right| < \frac{1}{k},$$

then $\{v_k\}_{k\in\mathbb{N}} \subset \left\{v \in H_0^1(\Omega) \colon v \geq 1 \text{ a.e. in a neighborhood of } E\right\}$ is also a minimizing sequence, which implies $\mathrm{cap}(E;\Omega) \leq \mathrm{c}(E;\Omega)$. □

Definition A.7 (Quasi-everywhere)**.** *We say that a property P is true quasi-everywhere (q.e.) on Ω, if P is true except on a set of zero capacity.*

Definition A.8 (Quasi-continuous)**.** *A function $f \colon \Omega \to \mathbb{R}$ is said to be quasi-continuous in Ω, if there exists a sequence $U_1 \supset U_2 \supset \ldots \supset U_n \supset U_{n+1} \supset \ldots$ of open sets in Ω such that f is continuous on $\Omega \setminus U_n$ and $\mathrm{cap}(U_n;\Omega) \to 0$ as $n \to \infty$.*

The following lemmas are taken from [17, Section 6.4.3]. Note that $\lim_{n\to\infty} f_n$ is a measurable function and $\{x \in \Omega \colon f \succ g\}$ with $\succ \in \{\leq, \geq, <, >, =\}$ is a measurable set, if the functions f_n, $n \in \mathbb{N}$, f and g are measurable. Therefore we will not have to distinguish between $\mathrm{c}(\cdot;\Omega)$ and $\mathrm{cap}(\cdot;\Omega)$.

Lemma A.9 ([17, Lemma 6.49])**.** *Let $f \colon \Omega \to \mathbb{R}$ be quasi-continuous and measurable. If $f \geq 0$ a.e. in Ω, then $\mathrm{cap}(\{x \in \Omega \colon f < 0\};\Omega) = 0$.*

Note that Ω must be an open set. Applied to the difference of two quasi-continuous and measurable functions, which coincide a.e. in Ω, Lemma A.9 yields the next corollary.

Corollary A.10. *Let $f, g \colon \Omega \to \mathbb{R}$ be quasi-continuous and measurable. If $f = g$ a.e. in Ω, then $f = g$ q.e. in Ω.*

Lemma A.11 ([17, Lemma 6.50])**.** *Let $f \in H_0^1(\Omega)$. Then f has a quasi-continuous representative.*

Lemma A.12 ([17, Lemma 6.52])**.** *Let $f_n \to f$ in $H_0^1(\Omega)$. Then there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that its quasi-continuous representatives, w.l.o.g. denoted by the same symbols, satisfy $f_{n_k} \to f$ q.e. in Ω.*

A.2. The trace operator

In this section we collect several results on the trace operator mainly obtained from Nečas [64]. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary Γ.

Theorem A.13 ([1, Theorem A6.6])**.** *There exists a unique linear continuous map $\gamma \colon H^1(\Omega) \to L^2(\Gamma)$ such that*

$$\gamma v = v|_\Gamma \quad \forall\, v \in H^1(\Omega) \cap C(\overline{\Omega}). \tag{A.1}$$

Theorem A.14 ([64, Theorems 2.4.2, 2.4.6 and 2.5.5])**.** *There exists a unique linear continuous map $\tau \colon H^1(\Omega) \to H^{1/2}(\Gamma)$ such that*

$$\tau v = v|_\Gamma \quad \forall\, v \in C^\infty(\overline{\Omega}). \tag{A.2}$$

The space $H^{1/2}(\Gamma) \subset L^2(\Gamma)$ is a Hilbert Space. For a detailed definition we refer to [32, Section 1.3] or [64, Section 2.5.2].

Theorem A.15 ([64, Theorem 2.3.1])**.** *The space $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$.*

Corollary A.16. *The operators γ and τ defined by* (A.1) *and* (A.2)*, respectively, are identical. In particular, it holds $\gamma v = \tau v$ a.e. on Γ for all $v \in H^1(\Omega)$.*

Proof. Let $v \in H^1(\Omega) \cap C(\overline{\Omega})$ be arbitrary. Due to Theorem A.15 there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ with $\varphi_n \to v$ in $H^1(\Omega)$. In view of Theorem A.13 and Theorem A.14 we deduce

$$\varphi_n|_\Gamma \to v|_\Gamma \quad \text{in } L^2(\Gamma) \quad \text{and} \quad \varphi_n|_\Gamma \to \tau v \quad \text{in } L^2(\Gamma).$$

Thus, the operator τ satisfies

$$\tau v = v|_\Gamma \quad \text{a.e. on } \Gamma \quad \forall\, v \in H^1(\Omega) \cap C(\overline{\Omega}),$$

which together with the uniqueness of γ implies the claim. □

The operator τ is well-known as *trace operator*. For a vector-valued Sobolev function $\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d)$ the componentwise application of the trace operator is also denoted by $\tau\boldsymbol{v}$.

Lemma A.17 ([64, Lemma 2.5.5])**.** *Let $v \in H^1(\Omega)$ and $h \in C^{0,1}(\overline{\Omega})$. Then it holds $vh \in H^1(\Omega)$ with*

$$\|vh\|_{H^1(\Omega)} \le c\,\|v\|_{H^1(\Omega)}\,\|h\|_{C^{0,1}(\overline{\Omega})}\,.$$

Corollary A.18. *Let $v \in H^1(\Omega)$ and $h \in C^{0,1}(\overline{\Omega})$. Then it holds*

$$\tau(vh) = \tau v \tau h \quad \textit{a.e. on } \Gamma. \tag{A.3}$$

Proof. Thanks to Theorem A.15 there is a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ with $\varphi_n \to v$ in $H^1(\Omega)$. On account of Lemma A.17 we arrive at the estimate

$$\|\varphi_n h - vh\|_{H^1(\Omega)} = \|(\varphi_n - v)h\|_{H^1(\Omega)} \le c\,\|\varphi_n - v\|_{H^1(\Omega)}\,\|h\|_{C^{0,1}(\overline{\Omega})}$$

so that the product $\varphi_n h$ converges to vh in $H^1(\Omega)$. Theorem A.14 consequently yields that

$$\tau\varphi_n \to \tau v \quad \text{in } L^2(\Gamma) \quad \text{and} \quad \tau(\varphi_n h) \to \tau(vh) \quad \text{in } L^2(\Gamma). \tag{A.4}$$

Because the property (A.3) is fulfilled by continuous functions, cf. Theorem A.13 and Corollary A.16, we moreover infer

$$\tau(\varphi_n h) = \tau\varphi_n \tau h \to \tau v \tau h \quad \text{in } L^1(\Gamma). \tag{A.5}$$

Together, (A.4) and (A.5) lead to the assertion. □

Theorem A.19 (Inverse trace theorem, [64, Theorem 2.5.7])**.** *The trace operator τ has a linear continuous right inverse $\tau_{-1}\colon H^{1/2}(\Gamma) \to H^1(\Omega)$. In particular, it holds*

$$\tau\tau_{-1}z = z \quad \forall\, z \in H^{1/2}(\Gamma)$$

and there exists a constant $c > 0$ such that

$$\|\tau_{-1}z\|_{H^1(\Omega)} \le c\,\|z\|_{H^{1/2}(\Gamma)}\,.$$

Theorem A.20 ([64, Lemma 2.4.9])**.** *The image of the trace operator τ is dense in $L^2(\Gamma)$, i.e., it holds*
$$\overline{\tau[H^1(\Omega)]} = L^2(\Gamma).$$

The next proposition is a result of the embedding $W^{1,\infty}(\Omega) \hookrightarrow C(\overline{\Omega})$, cf. [64, Theorem 2.3.8], and [54, Proposition II.5.2].

Proposition A.21. *The trace operator τ is positive, i.e., it holds $\tau v \geq 0$ a.e. on Γ for all $v \in H^1(\Omega)$ with $v \geq 0$ a.e. in Ω.*

Let $\boldsymbol{\nu}\colon \Gamma \to \mathbb{R}^d$ denote the unit outward normal vector field on Γ. The tangent space of $H^{1/2}(\Gamma; \mathbb{R}^d)$ is defined through
$$H_T^{1/2}\big(\Gamma; \mathbb{R}^d\big) = \big\{\boldsymbol{v} \in H^{1/2}\big(\Gamma; \mathbb{R}^d\big) \colon \boldsymbol{v} \cdot \boldsymbol{\nu} = 0\big\}.$$

Theorem A.22 ([53, Theorem 5.5])**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $C^{1,1}$-boundary Γ. The maps $\tau_{\boldsymbol{\nu}}\colon H^1\big(\Omega; \mathbb{R}^d\big) \to H^{1/2}(\Gamma)$ and $\tau_T\colon H^1\big(\Omega; \mathbb{R}^d\big) \to H_T^{1/2}\big(\Gamma; \mathbb{R}^d\big)$ defined by*
$$\tau_{\boldsymbol{\nu}} \boldsymbol{v} = \tau \boldsymbol{v} \cdot \boldsymbol{\nu}, \quad \tau_T \boldsymbol{v} = \tau \boldsymbol{v} - (\tau_{\boldsymbol{\nu}} \boldsymbol{v}) \boldsymbol{\nu}$$
are linear continuous. Moreover, for a given $(h, \boldsymbol{g}) \in H^{1/2}(\Gamma) \times H_T^{1/2}\big(\Gamma; \mathbb{R}^d\big)$, there exist $\boldsymbol{v} \in H^1\big(\Omega; \mathbb{R}^d\big)$ and a constant $c > 0$ such that
$$\begin{aligned} &\tau_{\boldsymbol{\nu}} \boldsymbol{v} = h \quad \text{and} \quad \tau_T \boldsymbol{v} = \boldsymbol{g}, \\ &\|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^d)} \leq c \left(\|h\|_{H^{1/2}(\Gamma)} + \|\boldsymbol{g}\|_{H_T^{1/2}(\Gamma;\mathbb{R}^d)} \right). \end{aligned}$$
Therefore, both $\tau_{\boldsymbol{\nu}}$ and τ_T are surjective.

The operator $\tau_{\boldsymbol{\nu}}$ is known as *normal trace operator*.

Corollary A.23. *Let $\Gamma_D \subset \Gamma$ be measurable. The image $\tau_{\boldsymbol{\nu}}[V]$ of the set $V = \big\{\boldsymbol{v} \in H^1\big(\Omega; \mathbb{R}^d\big) \colon \tau \boldsymbol{v} = 0$ a.e. on $\Gamma_D\big\}$ under the normal trace operator $\tau_{\boldsymbol{\nu}}$ is a Hilbert space.*

Proof. We aim to show that $\tau_{\boldsymbol{\nu}}[V]$ is closed in the Hilbert space $H^{1/2}(\Gamma)$ and hence complete. For this purpose let $\{z_n\}_{n\in\mathbb{N}} \subset \tau_{\boldsymbol{\nu}}[V]$ be a sequence with $z_n \to z$ in $H^{1/2}(\Gamma)$. In view of Theorem A.22 there exists $\{\boldsymbol{v}_n\}_{n\in\mathbb{N}} \subset H^1\big(\Omega; \mathbb{R}^d\big)$ with $\tau_{\boldsymbol{\nu}} \boldsymbol{v}_n = z_n$, $\tau_T \boldsymbol{v}_n = \boldsymbol{0}$ and
$$\|\boldsymbol{v}_n\|_{H^1(\Omega;\mathbb{R}^d)} \leq c \, \|z_n\|_{H^{1/2}(\Gamma)}. \tag{A.6}$$
By definition of $\tau_{\boldsymbol{\nu}}$ we know $\tau_{\boldsymbol{\nu}} \boldsymbol{v}_n = 0$ a.e. on Γ_D and accordingly
$$\tau \boldsymbol{v}_n = (\tau_{\boldsymbol{\nu}} \boldsymbol{v}_n) \boldsymbol{\nu} + \tau_T \boldsymbol{v} = \boldsymbol{0} \quad \text{a.e. on } \Gamma_D$$

so that $\{\boldsymbol{v}_n\}_{n\in\mathbb{N}} \subset V$. Furthermore, the convergence of z_n combined with (A.6) yields the existence of a subsequence, w.l.o.g. denoted in the same way, converging weakly to $\boldsymbol{v} \in V$. Since $\tau_{\boldsymbol{\nu}}$ is linear continuous and thus weakly continuous, we find $\tau_{\boldsymbol{\nu}}\boldsymbol{v}_n \rightharpoonup \tau_{\boldsymbol{\nu}}\boldsymbol{v}$ in $H^{1/2}(\Gamma)$. From the uniqueness of the weak limit we conclude $z = \tau_{\boldsymbol{\nu}}\boldsymbol{v}$ with $\boldsymbol{v} \in V$, which implies $z \in \tau_{\boldsymbol{\nu}}[V]$. Consequently, the space $\tau_{\boldsymbol{\nu}}[V]$ is a closed subset of $H^{1/2}(\Gamma)$. □

A.3. Miscellaneous

Lemma A.24 (Partition of unity, [18, Lemma 9.3]). *Let $K \subset \mathbb{R}^d$ be compact and let $U_1, U_2, \ldots, U_n$ be an open covering of K. Then there exist functions $\varphi_i \in C_0^\infty(U_i)$, $i = 1, \ldots, n$, with*

i) $0 \le \varphi_i \le 1 \;\forall\, i \in \{1, \ldots, n\}$

ii) $\sum_{i=1}^n \varphi_i(x) = 1 \;\forall\, x \in K$ *and* $\sum_{i=1}^n \varphi_i(x) \le 1 \;\forall\, x \in \mathbb{R}^d \setminus K$.

Proposition A.25 (Korn's inequality, [72, Proposition 1.1 and Remark 1.3]). *Let $\Gamma_D \subset \Gamma$ be measurable with positive measure. Then there exists a constant $c_K > 0$ such that*

$$\|\boldsymbol{u}\|^2_{H^1(\Omega;\mathbb{R}^d)} \le c_K \left(\|\boldsymbol{u}\|^2_{L^2(\Gamma_D;\mathbb{R}^d)} + \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{L^2(\Omega;\mathbb{S})} \right) \quad \forall\, \boldsymbol{u} \in H^1\big(\Omega; \mathbb{R}^d\big),$$

where $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the linearized strain tensor, cf. (2.1).

Proposition A.26 ([31, Bemerkung I.5.14]). *Let X and Y be Banach spaces. Moreover let X be reflexive and dense in Y such that*

$$\|x\|_Y \le c\, \|x\|_X \quad \forall\, x \in X.$$

Then the dual space Y' can be identified with a dense subspace of the dual space X' such that

$$\|f\|_{X'} \le c\, \|f\|_{Y'} \quad \forall\, f \in Y'.$$

Let $X \subset \mathbb{R}^d$ and $M \subset X$ be arbitrary. Moreover let $\mathfrak{A}$ denote a sigma-algebra on the set X.

Definition A.27. *The trace sigma-algebra of M in $\mathfrak{A}$ is defined as*

$$\mathfrak{A}|_M := \{A \cap M \colon A \in \mathfrak{A}\}.$$

Lemma A.28 ([28, Korollar I.4.5]). *Let the sigma-algebra $\mathfrak{A}$ be generated by the family $\mathfrak{E}$ of subsets of X. Then the trace sigma-algebra $\mathfrak{A}|_M$ is generated by the family*

$$\mathfrak{E}|_M := \{E \cap M \colon E \in \mathfrak{E}\}.$$

Corollary A.29. *If a mapping* $f\colon (X, \mathcal{B}(X)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *is measurable, then its restriction to* M, $f|_M\colon (M, \mathcal{B}(M)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, *is measurable.*

Proof. The inclusion mapping $j\colon (M, \mathcal{B}(X)|_M) \to (X, \mathcal{B}(X))$ is measurable. Hence, the composition $f|_M = f \circ j\colon (M, \mathcal{B}(X)|_M) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable. From Lemma A.28 we infer the assertion. □

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