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A DISTRIBUTIONAL LIMIT THEOREM FOR THE REALIZED POWER VARIATION OF LINEAR FRACTIONAL STABLE MOTIONS

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ABSTRACT. In this article we deduce a distributional theorem for the realized power variation of linear fractional stable motions. This theorem is proven by choosing the technique of subordination to reduce the proof to a Gaussian limit theorem based on Malliavin-calculus.

1. INTRODUCTION

Once the link between the mathematical concept of quadratic variation and integrated volatility was established, this was the starting point for the use of power variation. The realized power variation was introduced by Barndorff-Nielsen and Shephard [BNS02, BNS03, BNS04a, BNS04b] in the context of stochastic volatility models as an estimator of the integrated volatility.

In various articles the limit behaviour of the realized power variation is analysed in different models, e.g. for stochastic volatility models in [Woe05], for functionals of semi-martingales in [Jac08] and for Gaussian processes with non-stationary increments in [MN14]. There are also limit theorems for the bipower variation e.g. for semi-martingales in [BNGJ$^+$06]. Both concepts are investigated in [BNCP09, BNCPW09] for Gaussian processes with stationary increments and in [Pod14] for ambit fields.

In this article we derive a limit theorem for the power variation of linear fractional stable motions. These processes combine the distributional property of α-stable Lévy processes and the dependence structure of fractional Brownian motions. They possess a representation as fractional Lévy process and can be defined by

$$X_t^H := \int_{-\infty}^{\infty} (t-s)_+^{\gamma} - (-s)_+^{\gamma} \, dL_s^{\alpha}, \quad t \in \mathbb{R},$$

where L^α is a two-sided α-stable Lévy process, $\alpha \in (0,2)$ and $\gamma \in \left(-\frac{1}{\alpha}, 1-\frac{1}{\alpha}\right)$.

In Gaussian models central and non-central limit theorems are deduced with the help of very powerful results developed in the context of Wiener/Itô/Malliavin calculus (see e.g. [HN05]). We use the technique of subordination to find an elegant way to reduce the proof of a distributional limit theorem for the power variation of linear fractional stable motions (Theorem 4.1) to a Malliavin based limit theorem (Theorem 3.2). By subordination we get a conditionally Gaussian process and the deduction of a limit theorem for the power variation for this process is similar to the Gaussian limit theorem provided by [MN14, Theorem 1] for the power variation of non-stationary Gaussian processes. Because we use subordination we have to restrict ourselves to the case $\alpha \in (1,2)$.

This article is structured as follows: in the first section we define linear fractional stable motions as a special case of fractional Lévy processes and we state their marginal distributions. The second section contains the Malliavin based

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limit theorem that we apply in the third section to prove our main result. The proof is divided into three parts: a specific construction to be able to apply the technique of subordination such that the linear fractional stable motion X^H can be represented as a conditionally Gaussian process G, the deduction of a Gaussian limit theorem to the realized power variation of the process G and finally the proof of the main result.

2. Basics of Linear Fractional Stable Motions

Linear fractional stable motions were introduced by [ST00] as self-similar processes with non-Gaussian marginal distributions. We consider linear fractional stable motions from the view of fractional Lévy processes which can be introduced as processes of the form

$$X_t^\gamma := \int_{\mathbb{R}} f_\gamma(s,t)\, dL_s,$$

where L is a two-sided Lévy process and the integral is defined in the sense of [RR89, Definition 2.5]. Linear fractional stable motions are one particular case of these processes and for $\alpha \in (1,2)$ they can be defined as follows: let L^α be a two-sided symmetric α-stable Lévy process such that its characteristic function possess the following representation:

$$\mathbb{E}\left[e^{iuL_t^\alpha}\right] = \exp\left(t\int_{\mathbb{R}}\left(e^{iux} - 1 - iux\right)\, d\nu(x)\right),$$

where in this case the Lévy measure ν is absolutely continuous with respect to the Lebesgue measure with density $g(x) = \frac{c}{|x|^{1+\alpha}}$. Consider the so-called kernel function

$$f_\gamma(t,s) := (t-s)_+^\gamma - (-s)_+^\gamma,$$

where $\gamma \in \left(-\frac{1}{\alpha}, 1-\frac{1}{\alpha}\right)$ and we always exclude the case $\gamma = 0$. Then, since $\int_{\mathbb{R}} |f_\gamma(t,s)|^\alpha\, ds < \infty$ the integral of $f_\gamma(t,s)$ with respect to L^α exists for all $t \in \mathbb{R}$ in the sense of [RR89, Definition 2.5] and we call the process $X^H = (X_t^H)_{t\in\mathbb{R}}$ defined by

$$X_t^H := \int_{-\infty}^{\infty} f_\gamma^+(t,s)\, dL_s^\alpha, \quad t \in \mathbb{R},$$

linear fractional stable motion. The parameter H given by $H := \gamma + \frac{1}{\alpha}$ is the self-similarity index of the process X^H, which means that for all $a > 0$ the finite dimensional distributions of $\left(X_{at}^H\right)_{t\in\mathbb{R}}$ are the same as those of $\left(a^H X_t^H\right)_{t\in\mathbb{R}}$. Since our construction to prove our main result only works for the case $\alpha \in (1,2)$ we restrict ourselves to this case in the definition. In the other cases one has only to change the characteristic exponent ψ above as it is described e.g. in [EW13].

From [RR89, Proposition 2.6] we can deduce the characteristic function of the marginal distributions of general fractional Lévy processes and in particular for linear fractional stable motions as follows.

Proposition 2.1. *The process X^H as defined above has stationary increments. Moreover, for $m \in \mathbb{N}$, $t_1, \dots, t_m \in \mathbb{R}$ and $u_1, \dots, u_m \in \mathbb{R}$ its finite dimensional distributions exhibit the characteristic function given by*

$$\mathbb{E}\left[\exp\left\{i\sum_{j=1}^m u_j X_{t_j}^H\right\}\right] = \exp\left\{\int_{\mathbb{R}} \psi\left(\sum_{j=1}^m u_j f_\gamma^+(t_j,s)\right)\, ds\right\},$$

where

$$\psi(y) = \int_{\mathbb{R}} \left(e^{ixy} - 1 - ixy \right) \, d\nu(x), \quad y \in \mathbb{R}.$$

Additionally, the distribution of X_t^H is infinitely divisible for all $t \in \mathbb{R}$.

Proof. The statement is a consequence of [RR89, Proposition 2.6] and the proof is worked out in detail in [EW13, Proposition 4] for fractional Lévy processes. □

For a stochastic process $Z = (Z_t)_{t \geq 0}$ the realized power variation is defined by

$$V_n := V_p^n(Z)_t := \sum_{j=1}^{\lfloor nt \rfloor} \left| Z_{\frac{j}{n}} - Z_{\frac{j-1}{n}} \right|^p.$$

In this article we only consider the case $t = 1$. In [Gla14, Theorem 2] we have seen that the following limit theorem for the realized power variation holds: for any $0 < p < \alpha$ the following convergence is satisfied:

$$n^{-1+pH} \sum_{j=1}^{n} \left| X^H_{\frac{j}{n}} - X^H_{\frac{j-1}{n}} \right|^p \xrightarrow{\mathbb{P}} \mathbb{E}\left[\left| X_1^H \right|^p \right] \quad \text{as } n \to \infty.$$

Now, we want to go one step further and deduce a distributional limit theorem for the power variation of linear fractional stable motions. Therefore, we use the technique of subordination to reduce the proof of the distributional theorem to a well known limit theorem based on Malliavin-calculus for Gaussian processes which is introduced in the next section.

3. A Limit Theorem based on Malliavin-calculus

In order to deduce a distributional limit theorem for the power variation of linear fractional stable motions we need a limit theorem based on Malliavin calculus. Hence, we give a short introduction to Malliavin calculus in order to be able to formulate a central limit theorem for sequences of random variables that admit a Wiener chaos representation. For a more detailed insight to Malliavin calculus based on Wiener chaos decomposition we refer to [Nua95].

We start with the Wiener chaos decomposition and generalised multiple Wiener integrals. To this end we first define isonormal Gaussian processes on some Hilbert spaces.

Definition 3.1. Let H be a real, separable Hilbert space with inner product $\langle .,. \rangle_H$ and $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space. A family of random variables $W = \{W(h) | h \in H\}$ is called *isonormal Gaussian process* on H if W is a centred Gaussian family of random variables such that for all $g, h \in H$ it holds $\mathbb{E}[W(h)W(g)] = \langle h, g \rangle_H$.

Classically one would start with some given Hilbert space H and construct the Wiener chaos decomposition for square integrable random variables which are measurable with respect to the filtration given be an isonormal Gaussian process. Instead of this we start with a given Gaussian process G and construct a Hilbert space where an isonormal Gaussian process can be defined on. In this way we ensure that the power variation of the given process G satisfies the measurability condition of the Wiener chaos decomposition (c.f. [Nua95, Theorem 1.1.1]) and as a consequence it admits a series representation given by a Wiener chaos decomposition. The approach chosen here is based on the appendix of [MN14].

Let $T > 0$ and G be a centred, real valued Gaussian process on some complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $(\pi^n)_{n \in \mathbb{N}}$ be a sequence of partitions of $[0, T]$, this means

$$\pi^n := \{ t_j^n \mid 0 \leq t_0^n < t_1^n < \cdots < t_{m_n}^n \leq T \}.$$

We define $\Delta_j^n G := G(t_j^n) - G(t_{j-1}^n)$ and $w_{j,n} := \left(\mathbb{E}\left[\Delta_j^n G^2\right]\right)^{\frac{1}{2}}$. Then

$$W := \left\{ \left. \frac{\Delta_j^n G}{w_{j,n}} \right| j = 1, \ldots, n,\ n \in \mathbb{N} \right\}$$

is a collection of standard normal random variables. Let $\mathcal{H}$ be the closure of all finite linear combinations of elements of W with respect to the norm of $L^2 := L^2(\Omega, \mathcal{A}, \mathbb{P})$. Under this assumptions the space $\mathcal{H}$ is a Hilbert space with inner product being the covariance of its elements. As a consequence the identity map on $\mathcal{H}$ is an isonormal Gaussian process on $\mathcal{H}$.

Let H_m be the mth Hermite polynomial defined by $H_0(x) \equiv 1$ and

$$H_m(x) := (-1)^m e^{\frac{x^2}{2}} \frac{d^m}{dx^m} e^{-\frac{x^2}{2}}, \quad m \geq 1.$$

For each $m \geq 1$ we define $\mathcal{H}_m$ as the closed linear subspace of $L^2(\Omega, \mathcal{A}, \mathbb{P})$ generated by the set of random variables

$$\{H_m(h) \mid h \in \mathcal{H} :\ \|h\|_{\mathcal{H}} = 1\}.$$

For $m = 0$ we define $\mathcal{H}_0$ as the set of constants. For $m \geq 0$ the space $\mathcal{H}_m$ is called *mth Wiener chaos*.

Let $\mathcal{G}$ be the σ-algebra generated by the elements of $\mathcal{H}_1 = \mathcal{H}$. By [Nua95, Theorem 1.1.1] the space $L^2(\Omega, \mathcal{G}, \mathbb{P})$ has a decomposition into the infinite orthogonal sum of the subspaces $\mathcal{H}_m$, $m \geq 0$, this means

$$L^2(\Omega, \mathcal{G}, \mathbb{P}) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m.$$

We denote by J_m the projection of $L^2(\Omega, \mathcal{G}, \mathbb{P})$ onto the mth Wiener chaos $\mathcal{H}_m$.

The abstract multiple Wiener integral is defined as follows: if $\{e_k \mid k \geq 1\}$ is a complete orthogonal system of $\mathcal{H}$, then $\{e_{j_1} \otimes \cdots \otimes e_{j_m} \mid j_1, \ldots, j_m \geq 1\}$ is an orthonormal basis of the mth tensor product of $\mathcal{H}$, denoted by $\mathcal{H}^{\otimes m}$. We define the symmetrisation of $e_{j_1} \otimes \cdots \otimes e_{j_m}$ by

$$symm(e_{j_1} \otimes \cdots \otimes e_{j_m}) := \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} e_{\sigma(j_1)} \otimes \cdots \otimes e_{\sigma(j_m)}.$$

Then the set

$$\{symm(e_{j_1} \otimes \cdots \otimes e_{j_m}) \mid j_1, \ldots, j_m \geq 1\}$$

is an orthonormal basis of $\mathcal{H}^{\odot m}$, which is the symmetric mth tensor product of $\mathcal{H}$. The inner product $\langle ., . \rangle_{\mathcal{H} \otimes \mathcal{H}}$ on the tensor product $\mathcal{H} \otimes \mathcal{H}$ is given by the relationship

$$\langle g_1 \otimes h_1, g_2 \otimes h_2 \rangle_{\mathcal{H} \otimes \mathcal{H}} = \langle g_1, g_2 \rangle_{\mathcal{H}} \langle h_1, h_2 \rangle_{\mathcal{H}}.$$

We equip $\mathcal{H}^{\odot m}$ with the norm $\sqrt{m!}\|.\|_{\mathcal{H}^{\otimes m}}$. To a multiindex $d = (d_j)_{j \geq 1} \in \mathbb{N}_0^{\mathbb{N}}$ such that all terms except a finite number of them vanish we define the generalised Hermite polynomial $H_d(x)$, $x \in \mathbb{R}^{\mathbb{N}}$, by

$$H_d(x) = \prod_{j=1}^{\infty} H_{d_j}(x_j).$$

By the above condition on d this is well defined. We also set $d! := \prod_{j=1}^{\infty} d_j$, $|d| = \sum_{j=1}^{\infty} d_j$ and $\Phi_d := \sqrt{d!} \prod_{j=1}^{\infty} H_{d_j}(e_j)$. Note that for the last definition it is involved that the identity is the isonormal Gaussian process used here. The set

$\{\Phi_d \mid |d| = m\}$ is a complete orthonormal system of $\mathcal{H}_m$ (c.f. [Nua95, Proposition 1.1.1]). As a consequence the mapping $I_m \colon \mathcal{H}^{\odot m} \to \mathcal{H}_m$ defined by

$$I_m\left(symm\left(\bigotimes_{j=1}^{\infty} e_j^{\otimes d_j}\right)\right) := \sqrt{d!}\Phi_d$$

is an isometry. Consequently, for $h \in \mathcal{H}$ such that $\|h\|_{\mathcal{H}} = 1$ it is

$$I_m(h^{\otimes m}) = H_m(h) \tag{1}$$

and it holds

$$\mathbb{E}\left[I_m(f)\right]^2 = m!\|f\|_{\mathcal{H}^{\otimes m}} \tag{2}$$

for all $f \in \mathcal{H}^{\odot m}$.

We also define contractions of elements taken from tensor products of Hilbert spaces. Let $m, n \geq 2$ and suppose that $g \in \mathcal{H}^{\otimes m}$ and $h \in \mathcal{H}^{\otimes n}$ have the representation

$$\begin{aligned} g &= \sum_{j_1,\dots,j_m=1}^{\infty} a(j_1,\dots,j_m) e_{j_1} \otimes \cdots \otimes e_{j_m} \quad \text{respectively} \\ h &= \sum_{k_1,\dots,k_n=1}^{\infty} b(k_1,\dots,k_n) e_{k_1} \otimes \cdots \otimes e_{k_n}, \end{aligned}$$

where $a(j_1,\dots,j_m)$ and $b(k_1,\dots,k_n)$ are real numbers depending on the indices $j_1,\dots,j_m$ respectively $k_1,\dots,k_n$. Then for any $1 \leq \kappa \leq m \wedge n$ we can define the contraction of order κ of g and h by

$$\begin{aligned} g \otimes_\kappa h := &\sum_{z_1,\dots,z_{m+n-2\kappa}=1}^{\infty} \sum_{l_1,\dots,l_\kappa=1}^{\infty} a(l_1,\dots,l_\kappa,z_1,\dots,z_{m-\kappa}) \\ &\cdot b(l_1,\dots,l_\kappa,z_{m-\kappa+1},\dots,z_{m+n-2\kappa}) e_{z_1} \otimes \cdots \otimes e_{z_{m+n-2\kappa}}. \end{aligned}$$

Note that $g \otimes_\kappa h \in \mathcal{H}^{\otimes m+n-2\kappa}$.

With these definitions we are able to state the central limit theorem for random variables admitting a Wiener chaos representation. It can be found in [MN14, Theorem A.1] which is based on [HN05, Theorem 3 and Remark 1].

Theorem 3.2. *Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of square integrable, centred random variables with Wiener chaos representations given by*

$$F_n = \sum_{m=0}^{\infty} I_m(f_{m,n})$$

with some symmetric functions $f_{m,n} \in \mathcal{H}^{\odot m}$. Under the assumptions

(1) for every $n \geq 1,\ m \geq 1$ it holds $m!\|f_{m,n}\|_{\mathcal{H}^{\otimes m}} \leq \delta_m$, where $\sum_{m=1}^{\infty} \delta_m < \infty$;
(2) for every $m \geq 1$ there exists $\lim_{n\to\infty} m!\|f_{m,n}\|_{\mathcal{H}^{\otimes m}} =: \sigma_m^2$;
(3) for every $m \geq 2$ and $\kappa = 1,\dots,m-1$ it is $\lim_{n\to\infty} \|f_{m,n} \otimes_\kappa f_{m,n}\|^2_{\mathcal{H}^{\otimes 2(m-\kappa)}} = 0$

the sequence $(F_n)_{n\in\mathbb{N}}$ converges in distribution to a centred Gaussian random variable with variance given by $\sigma^2 = \sum_{m=1}^{\infty} \sigma_m^2$.

4. THE LIMIT THEOREM

In Gaussian models distributional limit theorems for the power variation are proven with the help of Malliavin calculus (c.f. e.g. [CNW06, BNCP09, MN14]). The article [CNW06] develops limit theorems for the power variation of fractional Brownian motions B^H. We are only interested in the case $H < \frac{3}{4}$. Then the result is the following: The expression

$$n^{\frac{1}{2}} \left(n^{-1+pH} V_p^n(B^H)_t - t\mathbb{E}\left[|B_1^H|^p\right] \right)$$

converges in law to a Gaussian limit distribution and the variance can be calculated exactly.

The article [GI15] provides a limit theorem for the power variation of stable Lévy processes. Let L be a stable Lévy process with parameters $(\alpha, \beta, 0, c)$. We define

$$C_n(\alpha, p) := \begin{cases} n^{-\frac{p}{\alpha}} \mathbb{E}\left[|L_1|^p\right] & \frac{\alpha}{2} < p < \alpha, \\ \mathbb{E}\left[\sin\left(n^{-1}|L_1|^\alpha\right)\right] & p = \alpha, \\ 0 & p > \alpha. \end{cases}$$

Then for $p > \frac{\alpha}{2}$

$$V_p^n(L)_t - ntC_n(\alpha, p) \xrightarrow{\mathcal{D}} L'_t \quad \text{as } n \to \infty,$$

where L' is an $\frac{\alpha}{p}$-stable process which is independent of L and whose Lévy measure is concentrated on $(0, \infty)$. In the case $p < \frac{\alpha}{2}$ the result can be deduced from the standard central limit theorem since $|L_1|^p$ has finite second moment. Under this condition it holds:

$$n^{-\frac{1}{2}+\frac{p}{\alpha}} V_p^n(L)_t - t\sqrt{n}\mathbb{E}\left[|L_1|^p\right] \xrightarrow{\mathcal{D}} Var\left(|L_1|^p\right) B_t \quad \text{as } n \to \infty,$$

where B is a Brownian motion and independent of L.

In this chapter we combine the properties of both classes of processes and consider linear fractional stable motions which have α-stable marginal distributions and whose dependence structure is the same as the one of fractional Brownian motions. Our goal is to prove the following limit theorem:

Theorem 4.1. *Let $1 < \alpha < 2,\ 0 < p < \alpha$ and X^H be a linear fractional α-stable motion with $\gamma \in \left(-\frac{1}{\alpha}, 1 - \frac{1}{\alpha}\right)$. If*

$$H < \begin{cases} \frac{3}{4} & \text{for } \gamma > 0, \\ \frac{1}{2} & \text{for } \gamma < 0, \end{cases}$$

the following limit theorem holds:

$$\sqrt{n}\left(n^{-1+pH} V_p^n(X^H)_1 - \mathbb{E}\left[|X_1^H|^p\right]\right) \xrightarrow{\mathcal{D}} \Xi, \tag{3}$$

where Ξ is a non-trivial random variable whose law is obtained as a mixture of Gaussian distributions.

To achieve this goal we choose an elegant way to reduce the proof of the above mentioned theorem to a Malliavin based limit theorem (c.f. Theorem 3.2) by using the technique of subordination. To apply Theorem 3.2 we follow the strategy developed in [MN14]. This article is the first one which provides a distributional limit theorem for the power variation of Gaussian processes relaxing the assumption of stationary increments to processes with locally stationary increments which is defined later.

We proceed in the following steps: we first construct a specific probability space to identify a representation of a linear fractional stable motion X^H as a conditionally Gaussian process G. The limit theorem [MN14, Theorem 1] is provided in thereafter. Unfortunately, this limit theorem cannot be applied to the Gaussian process constructed in the first section but the statement of [MN14, Theorem 1] still holds true for our conditionally Gaussian process. This will be shown in the third subsection of this section. We finish this section by proving our main result applying the limit theorem for the power variation of the process G we deduced in the subsection before.

4.1. **Representation of Linear Fractional Stable Motions as Conditionally Gaussian Processes.** In the following we give an explicit construction of how a linear fractional stable motion can be represented as some conditionally Gaussian process G. Therefore, we proceed as follows: since it is well known that a Brownian motion subordinated by an $\frac{\alpha}{2}$-stable subordinator yields a symmetric α-stable Lévy process (c.f. chapter 1.3) we start with two-sided analogues of the mentioned processes and observe that also a two-sided Brownian motion subordinated by a two-sided $\frac{\alpha}{2}$-stable subordinator yields a two-sided symmetric α-stable Lévy process. After that we use this result to see that linear fractional stable motions are conditionally Gaussian processes.

Let $\widetilde{B}$ be a two-sided standard Brownian motion on a filtered probability space $\left(\Omega_1, \mathcal{A}_1, \mathcal{G}^1, \mathbb{P}_1\right)$, where the filtration $\mathcal{G}^1 = (\mathcal{G}^1_t)_{t\in\mathbb{R}}$ is generated by $\widetilde{B}$. We also assume that the filtration satisfies the usual hypotheses (i.e. it is complete and right continuous).

Let $1 < \alpha < 2$ and $\widetilde{C} = (\widetilde{C}^{(1)}, \widetilde{C}^{(2)})$ be a two-dimensional $\frac{2}{\alpha}$-stable, spectral negative Lévy process with independent components defined on a probability space $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$. In particular the processes $\widetilde{C}^{(1)}$ and $\widetilde{C}^{(2)}$ have no positive jumps. We define the two-sided process $\widetilde{M}$ by

$$\widetilde{M}_t := \begin{cases} \sup\limits_{0\le s\le t} \widetilde{C}^{(1)}_s & t \ge 0, \\ -\sup\limits_{0\le s\le -t^-} \widetilde{C}^{(2)}_s & t < 0. \end{cases}$$

Let the filtration $\mathcal{G}^2$ on $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$ be the filtration generated by $\widetilde{M}$. We also assume that it fulfils the usual hypotheses. We define the two-sided process $\widetilde{\theta}$ by

$$\widetilde{\theta}_u := \begin{cases} \inf\left\{t \ge 0 \middle|\, \widetilde{C}^{(1)}_t > u\right\} & u \ge 0, \\ -\inf\left\{t \ge 0 \middle|\, \widetilde{C}^{(2)}_t > -u^-\right\} & u < 0 \end{cases}$$

with the convention that the infimum of the empty set is ∞. Then by [Sat99, Theorem 46.3] the process $\widetilde{\theta}$ is a two-sided $\frac{\alpha}{2}$-stable subordinator.

Let $(\Omega, \mathcal{A}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mathbb{P}_1 \otimes \mathbb{P}_2)$ be the product space equipped with the filtration $\mathcal{F}_t := \left(\bigcap\limits_{s>t} \mathcal{G}^1_s \otimes \mathcal{G}^2_s\right)^{\mathbb{P}}$. Hence, the filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbb{R}}$ is complete and right-continuous (i.e. it fulfils the usual hypotheses). For $\omega = (\omega_1, \omega_2)$ we define the processes B, θ and M by

$$B_t(\omega) := \widetilde{B}_t(\omega_1), \quad \theta_t(\omega) := \widetilde{\theta}_t(\omega_2) \quad \text{and} \quad M_t(\omega) := \widetilde{M}_t(\omega_2).$$

Then B is a two-sided standard $\mathcal{F}$-Brownian motion and θ is a two-sided $\frac{\alpha}{2}$-stable subordinator independent of B. By [Sat99, Example 30.6] the process L^α defined by

$$L^\alpha_t = B(\theta_t), \quad t \in \mathbb{R}$$

is a two-sided symmetric α-stable Lévy process.

We now apply this technique to linear fractional stable motions. Let X^H be the linear fractional stable motion driven by L^α. Hence, it can be represented as

$$X_t^H = \int_{-\infty}^{t} (t-s)_+^\gamma - (-s)_+^\gamma \, dL_s^\alpha = \int_{-\infty}^{t} (t-s)_+^\gamma - (-s)_+^\gamma \, dB(\theta_s).$$

This means that the linear fractional stable motion constructed above is a conditionally Gaussian process with covariance structure given by

$$\mathbb{E}\left[X_t^H X_s^H | \theta\right] = \int_{-\infty}^{s\wedge t} \left((t-r)_+^\gamma - (-r)_+^\gamma\right)\left((s-r)_+^\gamma - (-r)_+^\gamma\right) d\theta_r,$$

where the integral is a Lebesgue-Stieltjes integral which is well defined since θ is almost surely increasing. We consider the process X^H under the measure $\mathbb{P}_1$ so we introduce the following process G which for fixed $\omega_2 \in \Omega_2$ is defined by

$$G(t,\omega_2) := \int_{-\infty}^{t} (t-s)_+^\gamma - (-s)_+^\gamma \, d\widetilde{B}(\widetilde{\theta}_s(\omega_2)), \quad t \in \mathbb{R}. \tag{4}$$

This process G is defined on $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1)$ and $\mathbb{P}_2$-almost surely a Gaussian process with covariance structure

$$\mathbb{E}_1\left[G_t G_s\right] = \int_{-\infty}^{s\wedge t} \left((t-r)_+^\gamma - (-r)_+^\gamma\right)\left((s-r)_+^\gamma - (-r)_+^\gamma\right) d\theta_r,$$

where we suppress $\omega_2 \in \Omega_2$.

Observe that it is crucial for the construction made above that the driving Lévy process needs to have a representation as a subordinated Brownian motion in order to draw back the proof of our main result to a Gaussian limit theorem. Additionally, we give an example of a driving Lévy process L where this construction cannot be applied even if this process is closely related to L^α. Also the corresponding fractional Lévy process driven by L has the local self-similarity property. The process L arises from L^α by removing all jumps which are bigger than 1.

Example 4.2. *Let $1 < \alpha < 2$ and consider the Lévy process L defined by $L = L^\alpha - X$, where X is a stochastic process with $X_t := \sum_{\substack{0\le s\le t \\ \Delta L_s^\alpha > 1}} L_s^\alpha$. The fractional Lévy process driven by this process L is obviously a local self-similar process (since the Lévy measure of L is $d\nu(x) = \frac{1}{|x|^{1+\alpha}} \mathbb{1}_{|x|\le 1}\, dx$) but L cannot be obtained as a subordination of a Brownian motion by any subordinator Z. This is given because if the subordinator Z has jumps, the process $B(Z)$ has unbounded jumps. On the other hand if the subordinator is continuous the process $B(Z)$ is continuous as well.*

In the next subsection we state a limit theorem for the power variation of so-called locally stationary Gaussian processes. From the proof of this theorem the same result for the process G can be deduced. We will show this in the third subsection.

4.2. **Limit Theorem for the Power Variation of Gaussian Processes with Locally Stationary Increments.** The content of this subsection is taken from [MN14].

First we introduce the notation to state the limit theorem [MN14, Theorem 1]. Let $G = \{G(t) | t \in [0,1]\}$ be a zero mean Gaussian process defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The covariance function of G is the function

$\Gamma_G\colon [0,1]^2 \to \mathbb{R}$ defined by

$$\Gamma_G(s,t) := \mathbb{E}\left[G(s)G(t)\right], \quad s,t \in [0,1]$$

We denote the incremental variance function $\sigma_G^2\colon [0,1]^2 \to \mathbb{R}_+$ by

$$\sigma_G^2(s,t) := \mathbb{E}\left[(G(t)-G(s))^2\right], \quad s,t \in [0,1].$$

Let

$$\pi^n = \{0 \le t_0^n < t_1^n < \cdots < t_n^n \le 1\}$$

be a partition of $[0,1]$. Its mesh size is denoted by

$$\Delta_n := \sup\left\{t_j^n - t_{j-1}^n \mid j = 1,\ldots,n\right\}.$$

For a function $F\colon [0,1] \to \mathbb{R}$ we define by

$$\Delta_j^n F := F(t_j^n) - F(t_{j-1}^n)$$

its increment over the interval $[t_{j-1}^n, t_j^n]$. For a two-variable function $F\colon [0,1]^2 \to \mathbb{R}$ its double increment over the rectangle $[t_{j-1}^n, t_j^n] \times [t_{k-1}^n, t_k^n]$ is denoted by

$$\diamond_{j,k}^n F := F(t_j^n, t_k^n) - F(t_j^n, t_{k-1}^n) - F(t_{j-1}^n, t_k^n) + F(t_{j-1}^n, t_{k-1}^n).$$

In order to define Gaussian processes of locally stationary increments we need the following class of functions: let $R[0,1]$ be a set of functions $\rho\colon [0,1] \to \mathbb{R}_+$ such that ρ is continuous at zero, $\rho(0) = 0$ and for each $\delta \in (0,1)$, it holds

$$0 < \inf\{\rho(u) \mid u \in [\delta,1]\} \le \sup\{\rho(u) \mid u \in [\delta,1]\} < \infty.$$

Definition 4.3. Let $G = \{G(t) | t \in [0,1]\}$ be a zero-mean Gaussian stochastic process. We say G has *locally stationary increments* if there is a function $\rho \in R[0,1]$ such that the following holds:

(A1) there is a finite constant $c_1 > 0$ such that for all $s,t \in [0,1]$

$$\sigma_G(s,t) \le c_1 \rho\left(|t-s|\right);$$

(A2) for each $\varepsilon > 0$

$$\lim_{\delta \searrow 0} \sup\left\{ \left| \frac{\sigma_G(s,s+h)}{\rho(h)} - 1 \right| \;\middle|\; s \in [\varepsilon, 1),\ h \in (0, \delta \wedge (1-s)] \right\} = 0.$$

The interpretation of the limit theorem [MN14, Theorem 1] is the following: the function ρ approximates the local standard deviation. The process G is compared to a stationary, centred Gaussian process $\widetilde{G}$ whose incremental variance is given by

$$\sigma_{\widetilde{G}}^2(s,t) = \rho(|t-s|)^2.$$

If the process $\widetilde{G}$ fulfils a convergence condition (c.f. Condition (b) of Theorem 4.4 below), it satisfies a limit theorem for the power variation . If additionally the difference of the incremental variance of both processes G and $\widetilde{G}$ converges to zero as it is stated in Condition (c) of Theorem 4.4 below, then G satisfies a central limit theorem for the power variation.

We state [MN14, Theorem 1] after introducing the pth weighted power variation V_n of G, defined as

$$V_n := \Delta_n \sum_{j=1}^{n} \left(\frac{G(t_j^n) - G(t_{j-1}^n)}{\rho(\Delta_n)} \right)^p$$

and

$$\eta(k, \Delta_n) := \frac{\rho\left((k+1)\Delta_n\right)^2 + \rho\left((k-1)\Delta_n\right)^2 - 2\rho\left(k\Delta_n\right)^2}{2\rho\left(\Delta_n\right)^2} \tag{5}$$

Theorem 4.4. *Let $p > 0$ and let $G = \{G(t) | t \in [0,1]\}$ be a Gaussian process of locally stationary increments with $\rho \in R[0,1]$. Let $(\pi^n)_{n\in\mathbb{N}}$ be a sequence of partitions such that its mesh size Δ_n converges to zero as n tends to infinity. Suppose that*

(a) there is a constant $C_1 > 0$, such that $\sigma_G(s,t) \geq C_1 \rho(|t-s|)$ for all $s, t \in [0,1]$;

(b) for every integer $m \geq 2$, there is a real number Ψ_m such that

$$\lim_{n\to\infty} \sum_{k=1}^{y_n} \left(\eta(k, \Delta_n)\right)^m = \Psi_m \tag{6}$$

for every increasing and unbounded sequence of positive integers $(y_n)_{n\in\mathbb{N}}$ with values $y_n \leq n-1$ for each $n \geq 1$;

(c) for every integer $m \geq 2$,

$$\lim_{n\to\infty} \frac{\Delta_n}{\left[\rho(\Delta_n)\right]^2} \sum_{j,k=1}^{n} \left| \diamond_{j,k}^{n} \left[\Gamma_G - \tfrac{1}{2}\tilde{\rho} \right] \right|^m = 0,$$

where $\tilde{\rho}(s,t) := -\rho(|t-s|)^2$ for $s, t \in [0,1]$.

Then the central limit theorem

$$\Delta_n^{-1/2}\left(V_n - \mathbb{E}\left[V_n\right]\right) = \sqrt{\Delta_n} \sum_{j=1}^{n} \left[\left(\frac{|\Delta_j^n G|}{\rho(\Delta_n)} \right)^p - \mathbb{E}\left(\frac{|\Delta_j^n G|}{\rho(\Delta_n)} \right)^p \right] \xrightarrow{\mathcal{D}} \xi \text{ as } n \to \infty \tag{7}$$

holds, where ξ is a zero mean Gaussian random variable with variance

$$\mathbb{E}\xi^2 = \sum_{m=2}^{\infty} a_{p,m}^2 m! \left(1 + 2\Psi_m\right),$$

where Ψ_m is defined by (6), and the coefficients $a_{p,m}$ are given by

$$a_{p,m} := (m!)^{-1}\, \mathbb{E}\left[\left(|Z|^p - \mathbb{E}|Z|^p\right) H_m(Z)\right]$$

with H_m, $m \geq 2$, being the Hermite polynomials and Z being a standard normal random variable.

In order to prove our main result we would like to apply this theorem to the conditionally Gaussian process G we constructed in the last section. Unfortunately, it cannot be applied as stated above but in the next section we will see that under some slight modifications the statement of the last theorem still holds true for our process G constructed in the last section.

4.3. **Application of the Gaussian Limit Theorem.** In this subsection we use the notations and constructions we introduced in the first subsection in order to apply a modified version of Theorem 4.4 (c.f. Corollary 4.5) for fixed $\omega_2 \in \Omega_2$ to the process $G = (G_t)_{t\in\mathbb{R}}$ which is constructed in subsection 1. The process G is defined in Equation (4) by

$$G(t) = \int_{-\infty}^{t} (t-r)_+^{\gamma} - (-r)_+^{\gamma} \; d\widetilde{B}(\widetilde{\theta}_r).$$

The natural idea to apply Theorem 4.4 to the process G is using the function

$$\rho(u) := \mathbb{E}_1 \left[G(u)^2 \right]^{\frac{1}{2}}.$$

It turns out that under this assumption Condition (c) of Theorem 4.4 and Assumption (A2) are not satisfied in our model. But we found out that we can proceed analogously to the proof of Theorem 4.4 to deduce the same result (c.f. Corollary 4.5) for our process G. This is the goal of this subsection. Therefore, we introduce some notations.

The sequence of partitions $(\pi^n)_{n\in\mathbb{N}}$ is given by

$$\pi^n := \left\{ t_j^n = \tfrac{j}{n} \,\middle|\, j = 0, \dots, n \right\}.$$

For $1 \le j,k \le n$ we define $\Delta_j^n G := G(t_j^n) - G(t_{j-1}^n)$ and $r_n(j,k) := \frac{\mathbb{E}_1\left[\Delta_j^n G \Delta_k^n G\right]}{w_{j,n} w_{k,n}}$, where

$$w_{j,n} := \mathbb{E}_1 \left[\left(\Delta_j^n G \right)^2 \right]^{\frac{1}{2}}. \tag{8}$$

The choice of the function ρ in Theorem 4.4 is not unique. As it is described in [MN14, Remark 3] we can replace $\rho(\Delta_n)$ in Equation (7) by $w_{j,n}$. By doing this there is no need for introducing the function ρ, Assumptions (A1) and (A2) and Condition (a) of Theorem 4.4. The drawback is that we need to find alternatives to Hypotheses (b) and (c) of Theorem 4.4. Then the result of Theorem 4.4 reduces to

$$\sqrt{\frac{1}{n}} \sum_{j=1}^{n} \left[\left(\frac{|\Delta_j^n G|}{w_{j,n}} \right)^p - c_p \right] \xrightarrow{\mathcal{D}} \xi \text{ as } n \to \infty,$$

where $c_p = \mathbb{E}(|Z|^p)$ and Z is standard normal. This is the statement of [MN14, Remark 3].

The proof of Theorem 4.4 is reduced to exactly this case and is worked out in detail in [MN14]. It is based on Malliavin calculus, the corresponding limit theorem (Theorem 3.2) and on a decomposition of $r_n(j,k)$ into two parts. This is

$$r_n(j,k) = \frac{1}{v_{j,n} v_{k,n}} \left(\eta_n \left(|k-j| \right) + z_n(j,k) \right), \tag{9}$$

where $v_{j,n} := \frac{w_{j,n}}{\rho(\Delta_n)}$, η_n is given by (5) and the term $z_n(j,k)$ is defined by

$$z_n(j,k) := \frac{\diamond_{j,k}^n \left[\Gamma_G - \frac{1}{2}\tilde{\rho} \right]}{\rho(\Delta_n)^2},$$

where $\tilde{\rho}(s,t) := -\rho(|t-s|)^2$ (c.f. Condition (c) of Theorem 4.4). The interpretation is the following: assume that Condition (b) is true. Then, if it can be shown that the process G is '*almost stationary*' in the sense that the above mentioned decomposition holds and $z_n(j,k)$ satisfies the convergence condition (c) of Theorem 4.4, a central limit theorem holds for the power variation of G. In our case we do not have an analogous decomposition. Instead we show that $r_n(j,k)$ directly satisfies an equivalent condition to Condition (b) of Theorem 4.4.

We now state a corollary of the proof of Theorem 4.4 which provides the central limit theorem for the power variation of our process G. Note that the process G determines $1 < \alpha < 2$, $\gamma \in \left(-\frac{1}{\alpha}, 1 - \frac{1}{\alpha}\right)$ and $H = \gamma + \frac{1}{\alpha}$. In the remaining part of this subsection we prove this corollary. In the proof we focus on the changes compared to the proof of Theorem 4.4 presented by [MN14].

Corollary 4.5. *Let $1 < \alpha < 2$, $0 < p < \alpha$ and*

$$V_n := \frac{1}{n} \sum_{j=1}^{n} \left(\frac{\left| G(t_j^n) - G(t_{j-1}^n) \right|}{w_{j,n}} \right)^p .$$

Under the condition

$$H < \begin{cases} \frac{3}{4} & \text{for } \gamma > 0, \\ \frac{1}{2} & \text{for } \gamma < 0, \end{cases} \tag{10}$$

it holds

$$\lim_{n\to\infty} \frac{1}{n} \sum_{2 \le j < k \le n} (r_n(j,k))^m = 0 \tag{11}$$

for any integer $m \geq 2$. Additionally, the following convergence holds under the measure $\mathbb{P}_1$ and $\mathbb{P}_2$-almost surely:

$$\sqrt{n}\left(V_n - \mathbb{E}_1\left[V_n\right]\right) = \sqrt{\tfrac{1}{n}} \sum_{j=1}^{n} \left[\left(\frac{|\Delta_j^n G|}{w_{j,n}} \right)^p - c_p \right] \xrightarrow{\mathcal{D}_1} \xi \text{ as } n \to \infty, \tag{12}$$

where under the measure $\mathbb{P}_1$ the law of the random variable ξ is centred Gaussian with variance given by

$$\mathbb{E}_1 \xi^2 = \sum_{m=2}^{\infty} a_{p,m}^2 m!. \tag{13}$$

The coefficients $a_{p,m}$ are given by

$$a_{p,m} := (m!)^{-1} \, \mathbb{E}\left[\left(|Z|^p - \mathbb{E}|Z|^p \right) H_m(Z) \right]$$

with H_m, $m \geq 2$, being the Hermite polynomials and Z being a standard normal random variable.

Before we are able to prove this corollary we need two lemmas. The second one is a crucial detail for the proof of the above corollary. It uses a fact about stable random variables which is stated in the first lemma.

Lemma 4.6. *Let $\alpha < 1$ and consider the Lévy measure $d\nu(x) = \frac{1}{x^{1+\alpha}} \mathbb{1}_{x \geq 0} \, dx$. Let $c_X, c_Y > 0$ and X and Y be two α-stable random variables on $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$ with the same Lévy measure ν and characteristic functions given by*

$$\varphi_X(u) = e^{|u|^\alpha \int_{\mathbb{R}} e^{ix} - 1 \, d\nu(x) c_X},$$

respectively

$$\varphi_Y(u) = e^{|u|^\alpha \int_{\mathbb{R}} e^{ix} - 1 \, d\nu(x) c_Y}.$$

If $c_X > c_Y$, then for any $\delta > 0$ it holds

$$\mathbb{P}_2(X > \delta) \geq \mathbb{P}_2(Y > \delta).$$

Proof. Let $a = \left(\frac{c_X}{c_Y}\right)^{\frac{1}{\alpha}} > 1$. Then $X \overset{\mathcal{D}}{=} aY$ and

$$\mathbb{P}_2(X > \delta) = \mathbb{P}_2(aY > \delta) \geq \mathbb{P}_2(Y > \delta). \qquad \square$$

Lemma 4.7. *Let G be defined as in* (4) *and $H = \gamma + \frac{1}{\alpha}$. For $1 \leq j < k \leq n$ we define*

$$\tau_n(k-j) := n^{-2H} \begin{cases} \left(\frac{k-j+1}{2}\right)^{2H-2} & \text{for } \gamma > 0, \\ \left(\frac{k-j+1}{2}\right)^{H-1} & \text{for } \gamma < 0. \end{cases} \tag{14}$$

and consider

$$Y_n^{j,k} := \mathbb{E}_1\left[\left(G\left(\tfrac{j}{n}\right) - G\left(\tfrac{j-1}{n}\right)\right)\left(G\left(\tfrac{k}{n}\right) - G\left(\tfrac{k-1}{n}\right)\right)\right]$$

as random variable on $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$. Then for any $\varepsilon > 0$ it holds $\mathbb{P}_2$-almost surely

$$n^{-\varepsilon}\tau_n(k-j)^{-1}Y_n^{j,k} \to 0 \quad \text{as } n \to \infty. \tag{15}$$

Proof. From the definition of G we can conclude that

$$\begin{aligned} Y_n^{j,k} &:= \mathbb{E}_1\left[\left(G\left(\tfrac{j}{n}\right) - G\left(\tfrac{j-1}{n}\right)\right)\left(G\left(\tfrac{k}{n}\right) - G\left(\tfrac{k-1}{n}\right)\right)\right] \\ &= \int_{\mathbb{R}} \left(\left(\tfrac{j}{n} - s\right)_+^{\gamma} - \left(\tfrac{j-1}{n} - s\right)_+^{\gamma}\right)\left(\left(\tfrac{k}{n} - s\right)_+^{\gamma} - \left(\tfrac{k-1}{n} - s\right)_+^{\gamma}\right) d\widetilde{\theta}_s, \end{aligned}$$

where this integral is a Lebesgue-Stieltjes integral. Since $\gamma \in \left(-\frac{1}{\alpha}, 1 - \frac{1}{\alpha}\right)$ it also exists in the sense of [RR89, Definition 2.5] and both integrals coincide. Thus Proposition 2.1 can be applied to determine its characteristic function.

To prove the statement of the lemma we calculate the characteristic function of $Y_n^{j,k}$ and show that $\tau_n(k-j)^{-1}Y_n^{j,k}$ can be estimated in the sense of Lemma 4.6 by a (stable) random variable X on $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$ which is independent of j, k and n. The proof then finishes as follows: we define a family of sets $(A_n)_{n\in\mathbb{N}}$ by

$$A_n := \left\{\omega \in \Omega \middle| \; \left|n^{-\varepsilon}X\right| \geq \delta\right\}.$$

Then for all $n \in \mathbb{N}$ and any $\delta > 0$ it holds $A_{n+1} \subseteq A_n$. Additionally, for any $\delta > 0$ it holds $\lim\limits_{n\to\infty} \mathbb{P}_2(A_n) = 0$ which includes that for any $\delta > 0$

$$\mathbb{P}_2\left(\bigcap_{n_0\in\mathbb{N}} \bigcup_{n\geq n_0} n^{-\varepsilon}X \geq \delta\right) = \mathbb{P}_2\left(\bigcap_{n_0\in\mathbb{N}} A_{n_0}\right) = \lim_{n_0\to\infty} \mathbb{P}_2(A_{n_0}) = 0.$$

Then the following ensures the convergence in (15):

$$\begin{aligned} &\mathbb{P}_2\left(\lim_{n\to\infty} n^{-\varepsilon}\tau_n(k-j)^{-1}Y_n^{j,k} = 0\right) \\ =&\mathbb{P}_2\left(\forall\delta > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 :\ n^{-\varepsilon}\tau_n(k-j)^{-1}Y_n^{j,k} < \delta\right) \\ =&\mathbb{P}_2\left(\bigcap_{\delta>0} \bigcup_{n_0\in\mathbb{N}} \bigcap_{n\geq n_0} n^{-\varepsilon}\tau_n(k-j)^{-1}Y_n^{j,k} < \delta\right) \\ =&1 - \mathbb{P}_2\left(\bigcup_{\delta>0} \bigcap_{n_0\in\mathbb{N}} \bigcup_{n\geq n_0} n^{-\varepsilon}\tau_n(k-j)^{-1}Y_n^{j,k} \geq \delta\right) \end{aligned}$$

$$\geq 1-\mathbb{P}_2\left(\bigcup_{\delta\in\mathbb{Q}\cap(0,\infty)}\bigcap_{n_0\in\mathbb{N}}\bigcup_{n\geq n_0} n^{-\varepsilon}X\geq\delta\right)$$
$$=1.$$

To finish the proof we estimate the characteristic function of $Y_n^{j,k}$. Note that Proposition 2.1 can be applied to this random variable and its characteristic function is given by

$$\mathbb{E}_2\left[e^{iuY_n^{j,k}}\right]=\exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}} e^{ixu\left(\left(\frac{j}{n}-s\right)_+^\gamma-\left(\frac{j-1}{n}-s\right)_+^\gamma\right)\left(\left(\frac{k}{n}-s\right)_+^\gamma-\left(\frac{k-1}{n}-s\right)_+^\gamma\right)}-1\,d\nu(x)\,ds\right\}.$$

By the substitution

$$y=xu\left(\left(\tfrac{j}{n}-s\right)_+^\gamma-\left(\tfrac{j-1}{n}-s\right)_+^\gamma\right)\left(\left(\tfrac{k}{n}-s\right)_+^\gamma-\left(\tfrac{k-1}{n}-s\right)_+^\gamma\right)$$

and since $d\nu(x)=x^{-1-\frac{\alpha}{2}}\mathbb{1}_{x\geq 0}\,dx$ it follows that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} e^{ixu\left(\left(\frac{j}{n}-s\right)_+^\gamma-\left(\frac{j-1}{n}-s\right)_+^\gamma\right)\left(\left(\frac{k}{n}-s\right)_+^\gamma-\left(\frac{k-1}{n}-s\right)_+^\gamma\right)}-1\,d\nu(x)\,ds$$
$$=|u|^{\frac{\alpha}{2}}\int_{\mathbb{R}}\left|\left(\left(\tfrac{j}{n}-s\right)_+^\gamma-\left(\tfrac{j-1}{n}-s\right)_+^\gamma\right)\left(\left(\tfrac{k}{n}-s\right)_+^\gamma-\left(\tfrac{k-1}{n}-s\right)_+^\gamma\right)\right|^{\frac{\alpha}{2}}ds\int_{\mathbb{R}}e^{iy}-1\,d\nu(y).$$

Note that only the last integral depends on j,k and n. To handle this term we proceed as follows:

$$\int_{\mathbb{R}}\left|\left(\left(\tfrac{j}{n}-s\right)_+^\gamma-\left(\tfrac{j-1}{n}-s\right)_+^\gamma\right)\left(\left(\tfrac{k}{n}-s\right)_+^\gamma-\left(\tfrac{k-1}{n}-s\right)_+^\gamma\right)\right|^{\frac{\alpha}{2}}ds$$
$$=\int_{\mathbb{R}}\left|\left(\left(\tfrac{k-j+1}{2n}-s\right)_+^\gamma-\left(\tfrac{k-j-1}{2n}-s\right)_+^\gamma\right)\left(\left(-\tfrac{k-j-1}{2n}-s\right)_+^\gamma-\left(-\tfrac{k-j+1}{2n}-s\right)_+^\gamma\right)\right|^{\frac{\alpha}{2}}ds$$
$$=\left(\tfrac{k-j+1}{2n}\right)^{\alpha\gamma+1}\int_{\mathbb{R}}\left|\left((1-r)_+^\gamma-\left(\tfrac{k-j-1}{k-j+1}-r\right)_+^\gamma\right)\left(\left(-\tfrac{k-j-1}{k-j+1}-r\right)_+^\gamma-(-1-r)_+^\gamma\right)\right|^{\frac{\alpha}{2}}dr,$$

where the last equation holds by substituting $r=\frac{2ns}{k-j+1}$. The statement of the lemma is already shown if $k-j=1$ since $\frac{k-j+1}{2}=1$. Then, the integral no longer depends on j,k and n, which means that $\tau_n(k-j)^{-1}Y_n^{j,k}=n^{2H}Y_n^{j,k}$ has the same distribution as the stable random variable

$$\int_{\mathbb{R}}\left((1-r)_+^\gamma-(-r)_+^\gamma\right)\left((-r)_+^\gamma-(-1-r)_+^\gamma\right)\,d\widetilde{\theta}_s.$$

To see this one easily observes that $\left(n^{2H}\right)^{\frac{\alpha}{2}}=n^{\alpha H}=n^{\alpha\gamma+1}$.

For the remaining part of the proof let $k-j\geq 2$ (this is only needed in Estimation (17)). To estimate the integral

$$\int_{\mathbb{R}}\left|\left((1-r)_+^\gamma-\left(\tfrac{k-j-1}{k-j+1}-r\right)_+^\gamma\right)\left(\left(-\tfrac{k-j-1}{k-j+1}-r\right)_+^\gamma-(-1-r)_+^\gamma\right)\right|^{\frac{\alpha}{2}}dr \tag{16}$$

the idea is to apply a Taylor expansion of the function $t \mapsto (t-r)_+^\gamma$ for both factors of the integrand. This can be done on the interval $(-\infty,-1)$. On the interval $\left[-1,-\frac{k-j-1}{k-j+1}\right]$ this can only be applied to the first factor and on the interval $\left(-\frac{k-j-1}{k-j+1},\infty\right)$ the integrand is zero. For the first factor we apply the Taylor expansion as follows: for any $r<0$ there exists $\xi \in \left(\frac{k-j-1}{k-j+1},1\right)$ such that

$$(1-r)_+^\gamma - \left(\tfrac{k-j-1}{k-j+1}-r\right)_+^\gamma = \gamma\left(1-\tfrac{k-j-1}{k-j+1}\right)(\xi-r)^{\gamma-1}.$$

Since $\gamma-1<0$ its absolute value can be estimated from above by $|\gamma|\left(\frac{2}{k-j+1}\right)(-r)^{\gamma-1}$. With the same arguments it holds for $r<-1$

$$\left|\left(-\tfrac{k-j-1}{k-j+1}-r\right)_+^\gamma - (-1-r)_+^\gamma\right| \le |\gamma|\left(\tfrac{2}{k-j+1}\right)\left|(-1-r)^{\gamma-1}\right|.$$

Note that the last term is not integrable at $r=-1$ iff $\gamma<0$. This means that for $\gamma<0$ the Taylor expansion can only be applied on the first factor in Equation (16). We split the integral in (16) into the integrals over the intervals

$$(-\infty,-1) \text{ and } \left(-1,-\tfrac{k-j-1}{k-j+1}\right).$$

On the interval $\left(-1,-\frac{k-j-1}{k-j+1}\right)$ it holds for any choice of γ:

$$\begin{aligned}
&\int_{-1}^{-\frac{k-j-1}{k-j+1}} \left|\left((1-r)_+^\gamma - \left(\tfrac{k-j-1}{k-j+1}-r\right)_+^\gamma\right)\left(-\tfrac{k-j-1}{k-j+1}-r\right)_+^\gamma\right|^{\frac{\alpha}{2}} dr\\
\le &\int_{-1}^{-\frac{k-j-1}{k-j+1}} \left|\left(\gamma\left(\tfrac{2}{k-j+1}\right)(-r)^{\gamma-1}\right)\left(-\tfrac{k-j-1}{k-j+1}-r\right)^\gamma\right|^{\frac{\alpha}{2}} dr\\
\le &|\gamma|\left(\tfrac{2}{k-j+1}\right)^{\frac{\alpha}{2}}\left(\tfrac{k-j-1}{k-j+1}\right)^{\gamma-1}\int_{-1}^{-\frac{k-j-1}{k-j+1}} \left|-\tfrac{k-j-1}{k-j+1}-r\right|^{\gamma\frac{\alpha}{2}} dr\\
\le &\left|\frac{\gamma}{\gamma\frac{\alpha}{2}+1}\right|\left(\tfrac{2}{k-j+1}\right)^{\frac{\alpha}{2}}\left(\tfrac{1}{3}\right)^{\gamma-1}\left(\tfrac{2}{k-j+1}\right)^{\gamma\frac{\alpha}{2}+1} =: \tilde{c}_1\cdot\left(\tfrac{2}{k-j+1}\right)^{\gamma\frac{\alpha}{2}+1+\frac{\alpha}{2}}.
\end{aligned} \tag{17}$$

Under the assumption $\gamma>0$ we estimate

$$\begin{aligned}
&\int_{-\infty}^{-1} \left|\left((1-r)_+^\gamma - \left(\tfrac{k-j-1}{k-j+1}-r\right)_+^\gamma\right)\left(\left(-\tfrac{k-j-1}{k-j+1}-r\right)_+^\gamma - (-1-r)_+^\gamma\right)\right|^{\frac{\alpha}{2}} dr\\
\le &\gamma^2\left(\tfrac{2}{k-j+1}\right)^\alpha \int_{-\infty}^{-1}\left|(-r)^{\gamma-1}(-1-r)^{\gamma-1}\right|^{\frac{\alpha}{2}} dr =: \tilde{c}_2\cdot\left(\tfrac{2}{k-j+1}\right)^\alpha,
\end{aligned}$$

where the last integral is finite. Since $\frac{2}{k-j+1}\le 1$ we have

$$\left(\tfrac{2}{k-j+1}\right)^\alpha \ge \left(\tfrac{2}{k-j+1}\right)^{\gamma\frac{\alpha}{2}+1+\frac{\alpha}{2}}$$

if and only if $\alpha \le \gamma\frac{\alpha}{2}+1+\frac{\alpha}{2}$ which is equivalent to $\gamma \ge 1-\frac{2}{\alpha}$. Since $\frac{2}{\alpha}>1$ and $\gamma>0$ the condition $\gamma\ge 1-\frac{2}{\alpha}$ is always satisfied for $\gamma>0$.

If $\gamma < 0$ the first factor can be estimated by

$$\left|\left((1-r)_+^\gamma - \left(\tfrac{k-j-1}{k-j+1}-r\right)_+^\gamma\right)\right|^{\frac{\alpha}{2}} \le |\gamma|^{\frac{\alpha}{2}} \left(\tfrac{2}{k-j+1}\right)^{\frac{\alpha}{2}} (-r)^{(\gamma-1)\frac{\alpha}{2}}.$$

Then we use the Cauchy-Schwarz inequality to calculate

$$\begin{aligned}
&\int_{-\infty}^{-1} \left|\left((1-r)_+^\gamma - \left(\tfrac{k-j-1}{k-j+1}-r\right)_+^\gamma\right)\left(\left(-\tfrac{k-j-1}{k-j+1}-r\right)_+^\gamma - (-1-r)_+^\gamma\right)\right|^{\frac{\alpha}{2}} dr \\
&\le |\gamma|^{\frac{\alpha}{2}} \left(\tfrac{2}{k-j+1}\right)^{\frac{\alpha}{2}} \int_{-\infty}^{-1} (-r)^{(\gamma-1)\frac{\alpha}{2}} \left|\left(-\tfrac{k-j-1}{k-j+1}-r\right)^\gamma - (-1-r)^\gamma\right|^{\frac{\alpha}{2}} dr \\
&\le |\gamma|^{\frac{\alpha}{2}} \left(\tfrac{2}{k-j+1}\right)^{\frac{\alpha}{2}} \left(\int_{-\infty}^{-1} (-r)^{(\gamma-1)\alpha}\, dr\right)^{\frac{1}{2}} \left(\int_{-\infty}^{-1} \left|\left(-\tfrac{k-j-1}{k-j+1}-r\right)^\gamma - (-1-r)^\gamma\right|^{\alpha} dr\right)^{\frac{1}{2}}.
\end{aligned}$$

By the observations made above the last integral is

$$\left(\int_{-\infty}^{-1} \left|\left(-\tfrac{k-j-1}{k-j+1}-r\right)^\gamma - (-1-r)^\gamma\right|^{\alpha} dr\right)^{\frac{1}{2}} = \left(\tfrac{2}{k-j+1}\right)^{\frac{\alpha\gamma+1}{2}} \left(\int_{-\infty}^{0} \left|(1-r)_+^\gamma - (-r)_+^\gamma\right|^{\alpha}\right)^{\frac{1}{2}}$$

and we can conclude

$$\begin{aligned}
&\int_{-\infty}^{-1} \left|\left((1-r)_+^\gamma - \left(\tfrac{k-j-1}{k-j+1}-r\right)_+^\gamma\right)\left(\left(-\tfrac{k-j-1}{k-j+1}-r\right)_+^\gamma - (-1-r)_+^\gamma\right)\right|^{\frac{\alpha}{2}} dr \\
&\le |\gamma|^{\frac{\alpha}{2}} \left(\tfrac{2}{k-j+1}\right)^{\frac{\alpha}{2}+\frac{\alpha\gamma+1}{2}} \left(\int_{-\infty}^{-1} (-r)^{(\gamma-1)\alpha}\, dr\right)^{\frac{1}{2}} \left(\int_{-\infty}^{0} \left|(1-r)_+^\gamma - (-r)_+^\gamma\right|^{\alpha}\right)^{\frac{1}{2}} \\
&=: \tilde{c}_3 \cdot \left(\tfrac{2}{k-j+1}\right)^{\frac{\alpha}{2}+\frac{\alpha\gamma+1}{2}},
\end{aligned}$$

where both integrals are finite. Since $\frac{\alpha}{2} + \frac{\alpha\gamma+1}{2} < \gamma\frac{\alpha}{2} + 1 + \frac{\alpha}{2}$ the term $\left(\frac{2}{k-j+1}\right)^{\frac{\alpha}{2}+\frac{\alpha\gamma+1}{2}}$ dominates the term $\left(\frac{2}{k-j+1}\right)^{\gamma\frac{\alpha}{2}+1+\frac{\alpha}{2}}$ given in Equation (17).

The characteristic function of $\tau_n(k-j)^{-1}Y_n^{j,k}$ can be calculated by

$$\begin{aligned}
&\varphi_{\tau_n(k-j)^{-1}Y_n^{j,k}}(u) \\
=&\varphi_{Y_n^{j,k}}(\tau_n(k-j)^{-1}u) \\
=&\exp\Bigg\{ |\tau_n(k-j)^{-1}u|^{\frac{\alpha}{2}} \int_{\mathbb{R}} e^{iy} - 1\, d\nu(y) \left(\tfrac{k-j+1}{2n}\right)^{\alpha\gamma+1} \\
&\quad \cdot \int_{\mathbb{R}} \left|\left((1-r)_+^\gamma - \left(\tfrac{k-j-1}{k-j+1}-r\right)_+^\gamma\right)\left(\left(-\tfrac{k-j-1}{k-j+1}-r\right)_+^\gamma - (-1-r)_+^\gamma\right)\right|^{\frac{\alpha}{2}} dr \Bigg\}.
\end{aligned}$$

We first consider the case $\gamma > 0$. Then $\tau_n(k-j)^{-\frac{\alpha}{2}} = \left(\frac{k-j+1}{2n}\right)^{-\alpha H}\left(\frac{k-j+1}{2}\right)^{\alpha}$ and since $H = \gamma + \frac{1}{\alpha}$ it holds $|\tau_n(k-j)^{-1}|^{\frac{\alpha}{2}}\left(\frac{k-j+1}{2n}\right)^{\alpha\gamma+1} = \left(\frac{k-j+1}{2}\right)^{\alpha}$. By the calculations made above we have the following representation:

$$\varphi_{\tau_n(k-j)^{-1}Y_n^{j,k}}(u) = \exp\left\{|u|^{\frac{\alpha}{2}}\int_{\mathbb{R}} e^{iy} - 1\, d\nu(y) c_1\right\},$$

where

$$c_1 := \left(\frac{k-j+1}{2}\right)^{\alpha}\int_{\mathbb{R}}\left|\left((1-r)_+^{\gamma} - \left(\frac{k-j-1}{k-j+1} - r\right)_+^{\gamma}\right)\left(\left(-\frac{k-j-1}{k-j+1} - r\right)_+^{\gamma} - (-1-r)_+^{\gamma}\right)\right|^{\frac{\alpha}{2}} dr.$$

In the case $\gamma < 0$ it holds $|\tau_n(k-j)|^{-\frac{\alpha}{2}}\left(\frac{k-j+1}{2n}\right)^{\alpha\gamma+1} = \left(\frac{k-j+1}{2}\right)^{\frac{\alpha\gamma+1+\alpha}{2}}$ and the exponent of the term $\frac{k-j+1}{2}$ in the above representation for c_1 changes as follows: instead of α we have $\frac{\alpha\gamma+1+\alpha}{2}$. We have seen that the following estimation holds for the above integral:

$$\begin{aligned}
&\int_{\mathbb{R}}\left|\left((1-r)_+^{\gamma} - \left(\frac{k-j-1}{k-j+1} - r\right)_+^{\gamma}\right)\left(\left(-\frac{k-j-1}{k-j+1} - r\right)_+^{\gamma} - (-1-r)_+^{\gamma}\right)\right|^{\frac{\alpha}{2}} dr \\
&\leq \begin{cases} \tilde{c}_1\left(\frac{k-j+1}{2}\right)^{-\frac{\alpha\gamma+2+\alpha}{2}} + \tilde{c}_2\left(\frac{k-j+1}{2}\right)^{-\alpha} & \gamma > 0, \\ \tilde{c}_1\left(\frac{k-j+1}{2}\right)^{-\frac{\alpha\gamma+2+\alpha}{2}} + \tilde{c}_3\left(\frac{k-j+1}{2}\right)^{-\frac{\alpha\gamma+1+\alpha}{2}} & \gamma < 0, \end{cases} \\
&\leq \begin{cases} (\tilde{c}_1 + \tilde{c}_2)\left(\frac{k-j+1}{2}\right)^{-\alpha} & \gamma > 0, \\ (\tilde{c}_1 + \tilde{c}_3)\left(\frac{k-j+1}{2}\right)^{-\frac{\alpha\gamma+1+\alpha}{2}} & \gamma < 0. \end{cases}
\end{aligned}$$

If we define

$$c_2 := \begin{cases} \tilde{c}_1 + \tilde{c}_2 & \gamma > 0, \\ \tilde{c}_1 + \tilde{c}_3 & \gamma < 0, \end{cases}$$

it immediately follows $c_2 > c_1$. Now, we define a random variable X on $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$ by its characteristic function

$$\mathbb{E}_2\left[e^{iuX}\right] = \exp\left\{|u|^{\frac{\alpha}{2}}\int_{\mathbb{R}} e^{iy} - 1\, d\nu(y) c_2\right\}.$$

Then by Lemma 4.6 it holds

$$\mathbb{P}_2(X \geq \delta) \geq \mathbb{P}_2(\tau_n(k-j)^{-1}Y_n^{j,k} \geq \delta)$$

which finishes the proof. □

Now, we are able to prove Corollary 4.5.

Proof of Corollary 4.5. We proceed analogously to the proof of [MN14, Theorem 1]. We first determine the Wiener chaos representation of $\sqrt{n}\left(V_n - \mathbb{E}\left[V_n\right]\right)$. Then we show that Theorem 3.2 can be applied in our model.

Let $c_p := \mathbb{E}_1\left[|Z|^p\right]$, where Z is a standard normal random variable and

$$V_n := \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\left|G(t_j^n)-G(t_{j-1}^n)\right|}{w_{j,n}}\right)^p .$$

Then $\mathbb{E}_1[V_n] = c_p$. Now, we consider the term of interest. This is $\sqrt{n}\left(V_n - \mathbb{E}_1[V_n]\right)$. By a separation of the first summand we obtain

$$\sqrt{n}\left(V_n - \mathbb{E}_1[V_n]\right) = \sqrt{\tfrac{1}{n}}\left(\frac{\left|\Delta_1^n G\right|^p}{w_{1,n}^p} - c_p\right) + \sqrt{\tfrac{1}{n}}\sum_{j=2}^{n}\left(\frac{\left|\Delta_j^n G\right|^p}{w_{j,n}^p} - c_p\right) =: R_n + Y_n.$$

Analogous to the proof of [MN14, Theorem 1] we show that $R_n \overset{\mathbb{P}_1}{\to} 0$ as follows: by applying Tschebyscheff's inequality it holds for any $\delta > 0$

$$\begin{aligned}\mathbb{P}_1\left(|R_n| > \delta\right) &\le \delta^{-2}\frac{1}{n}\left(\mathbb{E}_1\left[\frac{\left|\Delta_1^n G\right|^{2p}}{w_{1,n}^{2p}}\right] - 2c_p\mathbb{E}_1\left[\frac{\left|\Delta_1^n G\right|^p}{w_{1,n}^p}\right] + c_p^2\right)\\ &= \delta^{-2}\frac{1}{n}\left(c_{2p} - c_p^2\right) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Hence, by Slutsky's lemma it is sufficient to show that for almost any $\omega_2 \in \Omega_2$

$$Y_n \overset{\mathcal{D}_1}{\to} \xi \quad \text{as } n \to \infty, \tag{18}$$

where ξ is a centred normally distributed random variable with variance given by (13). Therefore, we use Malliavin based technique.

Let $\mu = N(0,1)$, then the Hermite-polynomials H_m introduced in Chapter 1.5 are an orthogonal basis of the Hilbert-space $L^2(\mathbb{R},\mu)$. We define a function $H\colon \mathbb{R} \to \mathbb{R}$ by $H(x) := |x|^p - c_p$. Then $H \in L^2(\mathbb{R},\mu)$ which means that H can be expressed by the expansion

$$H = \sum_{m=0}^{\infty} a_m H_m.$$

Then the corresponding expansion of Y_n is given by

$$Y_n = \sum_{m=0}^{\infty}\left(a_m\sqrt{\tfrac{1}{n}}\sum_{j=2}^{n} H_m\left(\frac{\Delta_j^n G}{w_{j,n}}\right)\right) = \sum_{m=2}^{\infty}\left(a_m\sqrt{\tfrac{1}{n}}\sum_{j=2}^{n} H_m\left(\frac{\Delta_j^n G}{w_{j,n}}\right)\right),$$

where the second equality holds since for a standard normal random variable Z under the measure $\mathbb{P}_1$ it holds

$$\begin{aligned} a_0 &= \mathbb{E}_1\left[H_0(Z)H(Z)\right] = \mathbb{E}_1\left[|Z|^p - c_p\right] = 0,\\ a_1 &= \mathbb{E}_1\left[H_1(Z)H(Z)\right] = \mathbb{E}_1\left[Z\left(|Z|^p - c_p\right)\right] = \mathbb{E}_1\left[|Z|^p Z\right] = 0.\end{aligned}$$

Let I_m be the abstract multiple Wiener integral (c.f. Chapter 1.5). By the linearity of I_m and since the L^2-norm of $\frac{\Delta_i^n G}{w_{i,n}}$ equals one we have the following Wiener-chaos representation of Y_n:

$$Y_n = \sum_{m=2}^{\infty} I_m(f_{m,n}),$$

where

$$f_{m,n} := a_m \sqrt{\tfrac{1}{n}} \sum_{j=2}^{n} \left(\frac{\Delta_j^n G}{w_{j,n}} \right)^{\otimes m}.$$

Let J_m be the projection of $\mathcal{H}$ on the mth Wiener chaos $\mathcal{H}_m$. Then

$$J_m Y_n = a_m \sqrt{\tfrac{1}{n}} \sum_{j=2}^{n} H_m \left(\frac{\Delta_j^n G}{w_{j,n}} \right)$$

and by (2) it holds for any $n \geq 1$ and $m \geq 2$

$$m! \| f_{m,n} \|_{\mathcal{H}^{\otimes m}} = \mathbb{E}_1 \left[(J_m Y_n)^2 \right].$$

According to Theorem 3.2 the following conditions imply the convergence of Y_n as it is stated in (18):

(1) for every $n \geq 1,\ m \geq 1$ it holds $\mathbb{E}_1 \left[(J_m Y_n)^2 \right] \leq \delta_m$, where $\sum_{m=1}^{\infty} \delta_m < \infty$;
(2) for every $m \geq 1$, there exists $\lim_{n\to\infty} \mathbb{E}_1 \left[(J_m Y_n)^2 \right] =: \sigma_m^2$;
(3) for every $m \geq 2$ and $\kappa = 1, \dots, m-1$ it holds $\lim_{n\to\infty} \| f_{m,n} \otimes_\kappa f_{m,n} \|^2_{\mathcal{H}^{\otimes 2(m-\kappa)}} = 0$.

The variance of ξ is then given by $\mathbb{E}_1 \left[\xi^2 \right] = \sum_{m=2}^{\infty} \sigma_m^2$. By the orthogonality of the Hermite-polynomials and the resulting orthogonality of the Wiener chaoses it holds $J_1 Y_n = 0$ for each $n \geq 1$, so it suffices to prove Conditions (1) and (2) for $m \geq 2$.

For $n \geq 1$ and $2 \leq j,k \leq n$ we define

$$r_n(j,k) := \mathbb{E}_1 \left[\frac{\Delta_j^n G \Delta_k^n G}{w_{j,n} w_{k,n}} \right].$$

By the Cauchy-Schwarz inequality it is $r_n(j,k) \leq 1$ for all $n \geq 1$ and $2 \leq j,k \leq n$. Then for all $m \geq 2$ it holds

$$\left| \sum_{2 \leq j,k \leq n} r_n(j,k)^m \right| \leq \sum_{2 \leq j,k \leq n} r_n(j,k)^2.$$

By [Nua95, Lemma 1.1.1] it is

$$\mathbb{E}_1 \left[(J_m Y_n)^2 \right] = a_m^2 m! \frac{1}{n} \sum_{2 \leq j,k \leq n} r_n(j,k)^m$$

$$= a_m^2 m! \left(1 + 2\frac{1}{n} \sum_{2 \leq j < k \leq n} r_n(j,k)^m \right) \tag{19}$$

$$\leq a_m^2 m! \left(1 + 2\frac{1}{n} \sum_{2 \leq j < k \leq n} r_n(j,k)^2 \right). \tag{20}$$

In the proof of [MN14, Theorem 1] Conditions (1) - (3) are shown by using a decomposition of $r_n(j,k)$ into two terms $r_n(j,k) = \frac{1}{v_{j,n} v_{k,n}} \left(\eta_n(|k-j|) + z_n(j,k) \right)$ (c.f. (9)) and the fact that $r_n(j,k)$ behaves essentially as $\eta_n(|k-j|)$. We claim that Conditions (1) and (2) are satisfied if for each $m \geq 2$ the following limits are zero:

$$\lim_{n\to\infty} \tfrac{1}{n} \sum_{2 \leq j < k \leq n} r_n(j,k)^m = 0. \tag{21}$$

This equation is referred as [MN14, Equation (3.15)] and we later show that it is satisfied in our model. Combining this with (19) it follows that σ_m^2 in Condition (2) above is given by

$$a_m^2 m!$$

and additionally by (20) it holds that for all $m \geq 2$ the term δ_m can be bounded by $a_m^2 m!$. Under the hypothesis that (21) holds true Conditions (1) and (2) are proven since $\sum\limits_{m=2}^{\infty} a_m^2 m! = \mathbb{E}\left[(H(Z))^2\right] < \infty$, where Z is a standard normal random variable.

On the other hand Condition (3) reduces to the following weaker condition:
for $m \geq 2,\ 1 \leq \kappa \leq m-1$ it holds

$$\frac{1}{n^2} \sum_{i,j,k,l=2}^{n} |r_n(i,j)|^{\kappa} |r_n(k,l)|^{\kappa} |r_n(i,k)|^{m-\kappa} |r_n(j,l)|^{m-\kappa} \to 0 \quad \text{as } n \to \infty. \tag{22}$$

To see this we observe that by the calculation made in the proof of [MN14, Theorem 1] $f_{m,n} \otimes_{\kappa} f_{m,n}$ is given by

$$f_{m,n} \otimes_{\kappa} f_{m,n} = \frac{1}{n}\, a_m^2 \sum_{j,k=2}^{n} r_n(j,k)^{\kappa} \left(h_{j,n}^{\otimes(m-\kappa)} \otimes h_{k,n}^{\otimes(m-\kappa)}\right),$$

where $h_{j,n} := \frac{\Delta_j^n G}{w_{j,n}}$. Then the square of the $\mathcal{H}^{\otimes 2(m-\kappa)}$-norm of this term is (again by the calculations done in [MN14]) given by

$$\|f_{m,n} \otimes_{\kappa} f_{m,n}\|^2_{\mathcal{H}^{\otimes 2(m-\kappa)}} = n^{-2} a_m^4 \sum_{i,j,k,l=2}^{n} r_n(i,j)^{\kappa} r_n(k,l)^{\kappa} r_n(i,k)^{m-\kappa} r_n(j,l)^{m-\kappa}.$$

Then, (22) implies Condition (3) above.

We later show that there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ there is a function $\tilde{\eta}_n \colon \mathbb{R} \to \mathbb{R}$ such that

$$|r_n(j,k)| \leq \tilde{\eta}_n(k-j)$$

for any $1 \leq j < k \leq n$ and $\tilde{\eta}_n$ satisfies the following condition: let H be as in (10), then for any $n \geq n_0$ and $m \geq 2$ it holds

$$\lim_{n\to\infty} \sum_{k=1}^{n} \tilde{\eta}_n(k)^m < \infty. \tag{23}$$

This condition can be identified with Condition (b) of [MN14, Theorem 1] and it is sufficient to show that the convergence in (22) is satisfied. For this step we proceed exactly as in [MN14, Pages 335-337] and it is worked out in the Appendix. Thus Equation (23) implies (22) and then Condition (3) is satisfied in our model.

It remains to prove the convergence in (21). To this end we show that the series $\frac{1}{n} \sum\limits_{2\leq j<k\leq n} r_n(j,k)^m$ is absolutely convergent for any $m \geq 2$. The denominator of $r_n(j,k)$ can be estimated from below as follows: it is

$$w_{j,n}^2 = \int\limits_{\mathbb{R}} \left(\left(\tfrac{j}{n} - s\right)_+^{\gamma} - \left(\tfrac{j-1}{n} - s\right)_+^{\gamma}\right)^2 d\widetilde{\theta}_s,$$

where the integral is defined as Lebesgue-Stieltjes integral. Since θ is $\mathbb{P}_2$-almost surely non-decreasing and since the integrand is non-negative it can simply be estimated from below by

$$w_{j,n}^2 \geq \int\limits_{\frac{j-2}{n}}^{\frac{j-1}{n}} \left(\left(\frac{j}{n} - s \right)^{\gamma} - \left(\frac{j-1}{n} - s \right)^{\gamma} \right)^2 d\widetilde{\theta}_s.$$

The simplest estimation from below is to replace the integrand by its minimum over the interval $\left[\frac{j-2}{n}, \frac{j-1}{n}\right]$. Since the integrand is convex, non-negative and increasing on this interval it has its minimal value at $\frac{j-2}{n}$ so we have the estimate

$$w_{j,n}^2 \geq \frac{(2^{\gamma}-1)^2}{n^{2\gamma}} \Delta_{j-1}^n \widetilde{\theta}.$$

By [Sat99, Proposition 47.13] as $n \to \infty$, we have the following estimate from below for $\Delta_{j-1}^n \widetilde{\theta}$:

$$\Delta_{j-1}^n \theta \gtrsim n^{-\frac{2}{\alpha}} \quad \mathbb{P}_2 - a.s.,$$

which means that for large n it holds:

$$w_{j,n}^2 \gtrsim n^{-2(\gamma+\frac{1}{\alpha})} = n^{-2H} \quad \mathbb{P}_2 - a.s,$$

which is obviously also satisfied by $w_{k,n}^2$. After all the denominator of $r_n(j,k)$ can be estimated as follows:

$$w_{j,n} w_{k,n} \gtrsim n^{-2H} \quad \mathbb{P}_2 - a.s. \tag{24}$$

For the numerator we apply Lemma 4.7 and conclude that there is some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $\varepsilon > 0$ it holds

$$\mathbb{E}_1 \left[\Delta_j^n G \Delta_k^n G \right] \leq n^{\varepsilon} \tau_n(k-j) \quad \mathbb{P}_2 - a.s.,$$

where $\tau_n(k-j)$ is defined in (14). This implies

$$r_n(j,k) \leq const \cdot n^{\varepsilon} \begin{Bmatrix} (k-j+1)^{2H-2} & \gamma > 0, \\ (k-j+1)^{H-1} & \gamma < 0 \end{Bmatrix} =: \tilde{\eta}_n(k-j) \quad \mathbb{P}_2 - a.s. \tag{25}$$

for all $n \geq n_0$ and for any $\varepsilon > 0$. If $\gamma > 0$ then for any $n \geq n_0$, $H < \frac{3}{4}$ and $0 < \varepsilon < \frac{3-4H}{2}$ it holds

$$\begin{aligned} &\frac{1}{n} \left| \sum_{2 \leq j < k \leq n} r_n(j,k)^m \right| \\ \leq &\frac{1}{n} \sum_{2 \leq j < k \leq n} r_n(j,k)^2 \\ \lesssim &\frac{1}{n} n^{2\varepsilon} \sum_{2 \leq j < k \leq n} (k-j+1)^{4H-4} \\ \sim &n^{2\varepsilon+4H-3} \to 0 \end{aligned}$$

$\mathbb{P}_2$-almost surely as $n \to \infty$ which implies the convergence in Equation (21).

In the case $\gamma < 0$ it holds

$$r_n(j,k) \leq const \cdot n^{\varepsilon}(k-j+1)^{H-1} = const \cdot n^{\varepsilon}(k-j+1)^{H-1} \quad \mathbb{P}_2 - a.s.$$

for all $n \geq n_0$ and for any $\varepsilon > 0$. Under this condition the convergence in Equation (21) holds by the same arguments with the restriction $H < \frac{1}{2}$ and $0 < \varepsilon < \frac{2H-1}{2}$. Note that this is no contradiction to the Gaussian limit theorems developed in [CNW06, MN14]. This is given because in the Gaussian model it is $\alpha = 2$ which means that $H = \gamma + \frac{1}{2}$. In this case $\gamma < 0$ implies $H < \frac{1}{2}$. Hence, Conditions (1) and (2) above (c.f. Theorem 3.2) are satisfied in our model.

By those calculations we also conclude that in the case $\gamma > 0$ for any $H < \frac{3}{4}$ for all $m \geq 2$ and $0 < \varepsilon < \frac{3-4H}{m}$ it holds

$$\lim_{n\to\infty} \sum_{k=1}^{n} \tilde{\eta}_n(k)^m < \infty$$

which implies the convergence in Equation (23). If $\gamma < 0$ the same result holds for $H < \frac{1}{2}$ and any $0 < \varepsilon < \frac{1-2H}{2}$. Hence, Condition (3) above (c.f. Theorem 3.2) is satisfied in our model which finishes the proof. □

In the next subsection we apply Corollary 4.5 in our model in order to prove Theorem 4.1.

4.4. **Proof of the Main Result.** Now, we are able to prove Theorem 4.1.

Proof of Theorem 4.1. Let $V^n := V_p^n(X^H)_1$. It is

$$V^n = \sum_{j=1}^{n} \left| X^H_{\frac{j}{n}} - X^H_{\frac{j-1}{n}} \right|^p = \sum_{j=1}^{n} \left| \Delta_j^n X^H \right|^p$$

and we define in analogy to Theorem 4.4

$$V_n := \frac{1}{n} \sum_{j=1}^{n} \left(\frac{\left|\Delta_j^n X^H\right|}{w_{j,n}} \right)^p,$$

where $w_{j,n}$ is defined by (8).

Now, we have the following for the left hand side of (3):

$$\begin{aligned}
&\sqrt{n}\left(n^{-1+pH}V^n - \mathbb{E}\left[\left|X_1^H\right|^p\right]\right) \\
=&\sqrt{n}\left(n^{-1+pH}\sum_{j=1}^{n}\left|X^H_{\frac{j}{n}} - X^H_{\frac{j-1}{n}}\right|^p - \mathbb{E}\left[\left|X_1^H\right|^p\right]\right) \\
=&\frac{1}{\sqrt{n}}\sum_{j=1}^{n} n^{pH}\left(\left|\Delta_j^n X^H\right|^p - \mathbb{E}\left[\left|\Delta_j^n X^H\right|^p\right]\right) \\
=&\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(n^{2H}w_{j,n}^2\right)^{\frac{p}{2}}\left(\left(\frac{\left|\Delta_j^n X^H\right|}{w_{j,n}}\right)^p - \mathbb{E}\left[\left(\frac{\left|\Delta_j^n X^H\right|}{w_{j,n}}\right)^p\right]\right)
\end{aligned}$$

By the observations made in the proof of Lemma 4.7 for any $n \in \mathbb{N}$ and any $1 \leq j \leq n$ it holds $n^{2H}w_{j,n}^2 \overset{\mathcal{D}}{=} w_{1,1}^2$ under $\mathbb{P}_2$. Since for any $n \in \mathbb{N}$ and $1 \leq j \leq n$ the random variables $w_{j,n}$ are independent of $\omega_1 \in \Omega_1$ this also holds under the measure $\mathbb{P}$. Additionally, it is

$$\mathbb{E}\left[\left(\frac{\left|\Delta_j^n X^H\right|}{w_{j,n}}\right)^p\right] = \mathbb{E}_2\mathbb{E}_1\left[\left(\frac{\left|\Delta_j^n X^H\right|}{w_{j,n}}\right)^p\right] = \mathbb{E}_2\mathbb{E}_1\left[|Z|^p\right] =: c_p,$$

where Z is standard normal under $\mathbb{P}_1$. Then the convergence of $\sqrt{n}\left(n^{-1+pH}V^n - \mathbb{E}\left[\left|X_1^H\right|^p\right]\right)$ to a mixture of Gaussian random variables is shown as follows: by Corollary 4.5 the following holds under $\mathbb{P}_1$ and $\mathbb{P}_2$ almost surely (note that G and X^H are the same processes under the measure $\mathbb{P}$):

$$\sqrt{n}\left(V_n - \mathbb{E}_1\left[V_n\right]\right) = \frac{1}{\sqrt{n}}\left(\sum_{j=1}^n\left[\left(\frac{\Delta_j^n X^H}{w_{j,n}}\right)^p - c_p\right]\right) \xrightarrow{\mathcal{D}_1} \xi \text{ as } n \to \infty. \tag{26}$$

Then for any continuous, bounded, real valued function $f \in \mathcal{C}_b^0(\mathbb{R})$ it holds

$$\begin{aligned}
&\mathbb{E}\left[f\left(\sqrt{n}\left(n^{-1+pH}V^n - \mathbb{E}\left[\left|X_1^H\right|^p\right]\right)\right)\right] \\
=&\mathbb{E}\left[f\left(w_{1,1}^p\frac{1}{\sqrt{n}}\sum_{j=1}^n\left(\left(\frac{\left|\Delta_j^n X^H\right|}{w_{j,n}}\right)^p - c_p\right)\right)\right] \\
\overset{Fubini}{=}&\mathbb{E}_2\mathbb{E}_1\left[f\left(w_{1,1}^p\frac{1}{\sqrt{n}}\sum_{j=1}^n\left(\left(\frac{\left|\Delta_j^n X^H\right|}{w_{j,n}}\right)^p - c_p\right)\right)\right].
\end{aligned}$$

Since Equation (26) holds $\mathbb{P}_2$-almost surely we can apply Lebesgue's theorem to the term above. Then we have the following convergence as $n \to \infty$:

$$\mathbb{E}_2\mathbb{E}_1\left[f\left(w_{1,1}^p\frac{1}{\sqrt{n}}\sum_{j=1}^n\left(\left(\frac{\left|\Delta_j^n X^H\right|}{w_{j,n}}\right)^p - c_p\right)\right)\right] \to \mathbb{E}_2\mathbb{E}_1\left[f\left(w_{1,1}^p\xi\right)\right] = \mathbb{E}\left[f\left(w_{1,1}^p\xi\right)\right].$$

Note that under $\mathbb{P}_1$ the random variable $w_{1,1}^p\xi$ is a Gaussian random variable, which means that under $\mathbb{P}$ it is a mixture of Gaussian random variables and in particular it is non-trivial. This finishes the proof of Theorem 4.1. □

Note that the distribution is not determined yet. For this purpose we did a simulation study which will be presented in the next section.

Appendix

For the proof of Corollary 4.5 it remains to show

$$\frac{1}{n^2}\sum_{i,j,k,l=2}^n |r_n(i,j)|^\kappa |r_n(k,l)|^\kappa |r_n(i,k)|^{m-\kappa} |r_n(j,l)|^{m-\kappa} \to 0 \quad \text{as } n \to \infty.$$

To this end we proceed exactly as it is worked out in [MN14, Pages 335-337]. The term above can be identified with the term B_n of the proof of [MN14, Theorem 1]. We have seen that there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ there is some function $\tilde{\eta}_n$ such that $r_n(j,k) \leq \tilde{\eta}_n(|k-j|)$ for almost every $\omega_2 \in \Omega_2$ and the limit of the series

$$\lim_{n\to\infty}\sum_{k=1}^n \tilde{\eta}_n(k)^m =: \lambda_m$$

exists for any $m \geq 2$. Then we have to show

$$\frac{1}{n^2}\sum_{i,j,k,l=2}^{n}\tilde{\eta}_n(|i-j|)^{\kappa}\tilde{\eta}_n(|k-l|)^{\kappa}\tilde{\eta}_n(|i-k|)^{m-\kappa}\tilde{\eta}_n(|j-l|)^{m-\kappa} \to 0 \quad \text{as } n \to \infty,$$

which is equivalent to

$$E_n := \frac{1}{n}\sum_{i,j,k=2}^{n}\tilde{\eta}_n(|i-j|)^{\kappa}\tilde{\eta}_n(k)^{\kappa}\tilde{\eta}_n(|i-k|)^{m-\kappa}\tilde{\eta}_n(k)^{m-\kappa} \to 0 \quad \text{as } n \to \infty.$$

By Hölder's inequality it holds

$$\begin{aligned}E_n \leq &\left(\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{k=1}^{n}\tilde{\eta}_n(|i-k|)^{m-\kappa}\tilde{\eta}_n(k)^{\kappa}\right)^2\right)^{\frac{1}{2}}\\ &\left(\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\tilde{\eta}_n(|i-j|)^{\kappa}\tilde{\eta}_n(j)^{m-\kappa}\right)^2\right)^{\frac{1}{2}} =: U_nW_n.\end{aligned}$$

Both factors can be treated similarly. Let $\varepsilon > 0$ and let $a, b \geq 1$ be two integers. By using again Hölder's inequality we have the following three bounds for W_n:

$$\begin{aligned}W_{1,n}(a,b) :=&\frac{1}{n}\sum_{i=1}^{\lceil n\varepsilon\rceil}\left(\sum_{j=1}^{n}\tilde{\eta}_n(|i-j|)^a\tilde{\eta}_n(j)^b\right)^2\\ \leq&\frac{1}{n}\sum_{i=1}^{\lceil n\varepsilon\rceil}\sum_{j=1}^{n}\tilde{\eta}_n(|i-j|)^{2a}\sum_{j=1}^{n}\tilde{\eta}_n(j)^{2b}\\ \leq&2\frac{1}{n}(\lceil n\varepsilon\rceil+1)\sum_{j=1}^{n}\tilde{\eta}_n(j)^{2a}\sum_{j=1}^{n}\tilde{\eta}_n(j)^{2b} \to 2\varepsilon\lambda_{2a}\lambda_{2b},\end{aligned}$$

as $n \to \infty$,

$$\begin{aligned}W_{2,n}(a,b) :=&\frac{1}{n}\sum_{i=\lceil n\varepsilon\rceil+1}^{n}\left(\sum_{j=1}^{\lceil\frac{n\varepsilon}{2}\rceil}\tilde{\eta}_n(|i-j|)^a\tilde{\eta}_n(j)^b\right)^2\\ \leq&\frac{1}{n}\sum_{i=\lceil n\varepsilon\rceil+1}^{n}\sum_{j=1}^{\lceil\frac{n\varepsilon}{2}\rceil}\tilde{\eta}_n(|i-j|)^{2a}\sum_{j=1}^{\lceil\frac{n\varepsilon}{2}\rceil}\tilde{\eta}_n(j)^{2b}\\ \leq&2\frac{1}{n}(n-\lceil n\varepsilon\rceil)\sum_{k=\lceil\frac{n\varepsilon}{2}\rceil}^{n}\tilde{\eta}_n(k)^{2a}\sum_{j=1}^{\lceil\frac{n\varepsilon}{2}\rceil}\tilde{\eta}_n(j)^{2b} \to 0,\end{aligned}$$

as $n \to \infty$, and

$$\begin{aligned}W_{3,n}(a,b) :=&\frac{1}{n}\sum_{i=\lceil n\varepsilon\rceil+1}^{n}\left(\sum_{j=\lceil\frac{n\varepsilon}{2}\rceil+1}^{n}\tilde{\eta}_n(|i-j|)^a\tilde{\eta}_n(j)^b\right)^2\\ \leq&\frac{1}{n}\sum_{i=\lceil n\varepsilon\rceil+1}^{n}\sum_{j=\lceil\frac{n\varepsilon}{2}\rceil+1}^{n}\tilde{\eta}_n(|i-j|)^{2a}\sum_{j=\lceil\frac{n\varepsilon}{2}\rceil+1}^{n}\tilde{\eta}_n(j)^{2b}\end{aligned}$$

$$\leq 2\frac{1}{n}(n-\lceil n\varepsilon\rceil)\sum_{k=1}^{n-\lceil n\varepsilon\rceil-1}\tilde{\eta}_n(k)^{2a}\sum_{j=1}^{\lceil\frac{n\varepsilon}{2}\rceil}\tilde{\eta}_n(j)^{2b}\to 0,$$

as $n\to\infty$. Hence

$$\begin{aligned}&\limsup_{n\to\infty}W_n^2\\ \leq&\limsup_{n\to\infty}(W_{1,n}(\kappa,m-\kappa)+2W_{2,n}(\kappa,m-\kappa)+2W_{3,n}(\kappa,m-\kappa))\\ \leq&2\varepsilon\lambda_{2\kappa}\lambda_{2(m-\kappa)}\end{aligned}$$

and, since $\varepsilon>0$ is arbitrary it holds $W_n\to 0$ as $n\to\infty$. Similarly, $U_n\to 0$ as $n\to\infty$ which implies the convergence in Equation (22). This finishes the proof of Corollary 4.5.

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REFERENCES

[BNCP09] O. E. Barndorff-Nielsen, J. M. Corcuera, and M. Podolskij. Power variation for Gaussian processes with stationary increments. *Stochastic Processes and Their Applications*, 119:1845–1865, 2009.

[BNCPW09] O.E. Barndorff-Nielsen, J.M. Corcuera, M. Podolskij, and J.H.C. Woerner. Bipower variation for gaussian processes with stationary increments. *Journal of Applied Probability*, 46(1):132–150, 2009.

[BNGJ$^+$06] O. E. Barndorff-Nielsen, S. E. Graversen, J. Jacod, M. Podolskij, and N. Shephard. A central limit theorem for realised power and bipower variations of continuous semimartingales. In *From Stochastic Calculus to Mathematical Finance*, pages 33–68, 2006.

[BNS02] O. E. Barndorff-Nielsen and N. Shephard. Econometric analysis of realized volatility and its use in estimating stochastic volatility models. *J. Roy. Statist. Soc. Ser. B*, 64(2):253–280, 2002.

[BNS03] O. E. Barndorff-Nielsen and N. Shephard. Realized power variation and stochastic volatility models. *Bernoulli*, 9(2):243–265, 2003.

[BNS04a] O. E. Barndorff-Nielsen and N. Shephard. Econometric analysis of realized covariation: high frequency based covariance, regression, and correlation in financial economics. *Econometrica*, 72(3):885–925, 2004.

[BNS04b] O. E. Barndorff-Nielsen and N. Shephard. Power and Bipower Variation with Stochastic Volatility and Jumps. *J. Fin. Econometrics*, 2(1):1–48, 2004.

[CNW06] J.M. Corcuera, D. Nualart, and J.H.C. Woerner. Power variation of some integral fractional processes. *Bernoulli*, 12(4):713–735, 2006.

[EW13] S. Engelke and J.H.C. Woerner. A unifying approach to fractional Lévy processes. *Stochastics and Dynamics*, 13(4), 2013.

[GI15] J.M. Gairing and P. Imkeller. Stable CLTs and Rates for Power Variation of α-Stable Lévy Processes. *Methodology and Computing in Applied Probability*, 17(1):73–90, 2015.

[Gla14] S. Glaser. A law of large numbers for the power variation of fractional Lévy processes. *Stochastic Analysis and Applications*, 2014.

[HN05] Y. Hu and D. Nualart. Renormalized self-intersection local time for Brownian Motion. *Annals of Probability*, 33:948–983, 2005.

[Jac08] J. Jacod. Asymptotic properties of realized power variations and related functionals of semimartingales. *Stochastic Processes and their Applications*, 118(4):517 – 559, 2008.

[MN14] R. Malukas and R. Norvaiša. A central limit theorem for a weighted power variation of a Gaussian process. *Lithuanian Mathematical Journal*, 54(3):323–344, 2014.

[Nua95] D. Nualart. *The Malliavin Calculus and Related Topics*. Probability and its applications : a series of the applied probability. Springer-Verlag, 1995.

[Pod14] M. Podolskij. *Ambit fields: survey and new challenges*. Available at http://www.podolskij.uni-hd.de/files/p2014.pdf, 2014.

[RR89] B.S. Rajput and J. Rosinski. Spectral representations of infinitely divisible processes. *Probability Theory and Related Fields*, 82:451–487, 1989.

[Sat99] K.-I. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1999.

[ST00] G. Samorodnitsky and S. Taqqu. *Stable non-Gaussian random processes*. Stochastic Modeling. Chapman and Hall/CRC, 2000.

[Woe05] J.H.C. Woerner. Estimation of integrated volatility in stochastic volatility models. *Applied Stochastic Models in Business and Industry*, 21:27–44, 2005.

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