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Abstract

In this paper we propose an alternative model for electricity spot prices based on oscillating Ornstein-Uhlenbeck processes. This model captures the characteristics of empirical data, especially the oscillating shape of the autocorrelation function. Furthermore, we show that our model leads to explicit formulas for forwards and options on forwards.

1 Introduction

Ornstein-Uhlenbeck processes provide a popular class of stochastic models in different fields of applications. They were introduced as particular diffusion processes in the context of particle movements and also applied in the financial context (cf. [22]). Ornstein-Uhlenbeck processes became very popular since they provide tractable mean reverting models given by a stochastic differential equation $dX_t = -\lambda X_t dt + dB_t,\ X_0 = x_0$ with an explicit solution of the form $X_t = e^{-\lambda t} x_0 + \int_0^t e^{-\lambda(t-s)} dB_s$. The same nice properties still hold when the Brownian motion B is replaced by a Lévy process L leading to mean reverting processes with jumps and infinitely divisible marginals. These processes are also very popular in financial modelling, e.g. as volatility process in the Barndorff-Nielsen and Shephard model (cf.[1]) or models for electricity spot prices (cf. e.g. [7], [15]).

Choosing a suitable initial condition and a positive mean reverting parameter λ Ornstein-Uhlenbeck processes are stationary processes which might be represented as continuous time moving average processes of the form $\int_{-\infty}^t f(t-s) dL_s$ with kernel function $f(t-s) =$

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$\exp(-\lambda(t-s))$, $\lambda > 0$. In general continuous-time moving average processes offer a wide range for modelling, since they allow to combine a specific correlation structure given by the kernel function f with an infinitely divisible marginal distribution given by the driving Lévy process. In the following we will make use of these properties to obtain models for electricity prices, capturing the characteristic features like infinitely divisible distributed increments, mean reversion, oscillating behaviour of the autocorrelation function and the sample paths.

Classical models for electricity spot prices consist of a deterministic component capturing the seasonalities and a sum of Ornstein-Uhlenbeck processes with different speed of convergence (cf. [7]) or other stochastic processes like Lévy semistationary processes (cf. [2]) or CARMA processes (cf. [13], [5]) capturing the random and mean reverting effects. This type of modelling has the disadvantage from a statistical point of view that the resulting model is not stationary and in a first step of an estimating procedure the seasonalities have to be removed. We now propose a model based on a sum of different oscillating Ornstein-Uhlenbeck processes, i.e. processes of the form $\int_{-\infty}^{t} \sin(a(t-s)) \exp(-\lambda(t-s)) dL_s$ and $\int_{-\infty}^{t} \cos(b(t-s)) \exp(-\lambda(t-s)) dL_s$ with $\lambda, a > 0$, $b \geq 0$ or generalizations when the driving Lévy process is replaced by an additive process. These processes are still mean reverting, they possess an oscillating autocorrelation function and oscillating sample paths. The process with cos in the kernel function possess jumps, while the one with sin is continuous. Furthermore, if the driving process is a Lévy process, they are stationary. Hence our model may capture the characteristic features of empirical data with a stationary process. Furthermore, we are able to derive explicit formulas for forwards and options on the forwards in a similar way as in [7] and obtain their results as a special case.

The outline of the paper is the following: In section 2 we define generalized oscillating Ornstein-Uhlenbeck processes and derive some properties. In section 3 we introduce our model for the spot prices in electricity markets. In section 4 we derive the necessary formulas for moments and the autocorrelation function of this model. Section 5 deals with explicit formulas for forwards and options on forwards in our new model.

2 Definition of generalized oscillating Ornstein-Uhlenbeck process

In the following our aim is to introduce a class of processes which combines the mean-reverting property of Ornstein-Uhlenbeck processes with an oscillating behaviour of sample paths and autocorrelation function. As a starting-point we take the moving-average representation of an Ornstein-Uhlenbeck process and examine how we may change the involved quantities to reproduce the desired features.

As driving process we consider a one-dimensional additive process $(A_t)_{t\geq 0}$ that is an adapted stochastic process which is continuous in probability with cadlag sample path and independent increments. Consequently the characteristic function of $(A_t)_{t\geq 0}$ is given by

$$\varphi_A(t,y) := E(\exp(\mathrm{i}yA(t))) = \exp\big(\psi_A(t,y)\big)$$

with characteristic exponent

$$\psi_A(t,y) := \mathrm{i}y\gamma(t) - \frac{1}{2}y^2\sigma^2(t) + \int_0^t \int_{\mathbb{R}} \left(e^{\mathrm{i}yz} - 1 - \mathrm{i}yz\mathbb{1}_{|z|\leq 1}\right)\nu(dz,du),$$

where $\gamma : \mathbb{R}_+ \to \mathbb{R}$ denotes a continuous function, $\sigma^2 : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous increasing function and ν a σ-finite measure on the Borel-σ-algebra of $[0,\infty) \times \mathbb{R}$ which satisfies for $B \in \mathcal{B}(\mathbb{R}_+)$

$$\nu\left(B \times \{0\}\right) = 0, \quad \nu\left(\{t\} \times \mathbb{R}\right) = 0, \quad \textit{and} \quad \int_0^\infty \int_{\mathbb{R}} \min\left(1, y^2\right)\nu(dt,dy) < \infty.$$

Hence with the following notation $\gamma_t := \gamma(t)$, $\sigma_t^2 := \sigma^2(t)$ and the unique measure $\nu_t(A) = \nu([0,t] \times A)$ on $\mathcal{B}(\mathbb{R})$ for $t \geq 0$ and $A \in \mathcal{B}(\mathbb{R})$ the characteristic triplet of A_t is $(\gamma_t, \sigma_t^2, \nu_t)$ (for more details see [19]). Note that an additive process is a semimartingale if and only if γ is of finite variation. In the following we will assume this. If we consider the special case of also stationary increments, i.e. the quantities in the characteristic triplet only depend linearly on time, we are in the case of Lévy processes L. In this case the characteristic function reduces to $E(\exp(iuL_t)) = \exp(t\psi(u))$ with

$$\psi(u) = iu\gamma - \sigma^2\frac{u^2}{2} + \int_{-\infty}^{\infty} \Big(\exp(iux) - 1 - iux1_{|x|\leq 1}\Big)\nu(dx),$$

where the Lévy measure ν satisfies the integrability condition $\int_{-\infty}^{\infty} 1 \wedge x^2\,\nu(dx) < \infty$.

In the following we provide conditions on a kernel function $f(\cdot) : \mathbb{R} \to \mathbb{R}$ such that processes of the form

$$Z_t = \int_{-\infty}^t f(t-s)\,dA_s, \quad t \geq 0$$

exist. Here A denotes the two-sided version of the additive process which is defined in the canonical way by taking two independent copies $A^{(1)}$ and $A^{(2)}$ of an additive process and defining

$$A_t := \begin{cases} A_t^{(1)} & \text{if } t \geq 0 \\ -A_{-t-}^{(2)} & \text{if } t < 0. \end{cases}$$

Our integrals are defined in the sense of Rajput and Rosinski ([18]) where an independently scattered random measure Λ is associated with the two-sided additive process A. For details see [20] (in particular Theorem 3.2). The extension to $\mathbb{R}$ works in a canonical way. Λ is defined on the δ-ring of bounded Borel measurable sets in $\mathbb{R}$ and the integral $\int_{\mathbb{R}} g(s)\, d\Lambda_s$ is introduced in the usual way for deterministic step functions g. A function f is then called integrable if there exists a sequence $(g_n)_{n\in\mathbb{N}}$ of step functions such that

- $g_n \to f$ a.s with respect to the control measure of Λ (for details see [18])
- $\lim_{n\to\infty} \int_A g_n(s)\, d\Lambda_s$ exists for every $A \in \mathcal{B}(\mathbb{R})$.

If a function f is integrable, we write $\int_{\mathbb{R}} f\, dA_s = \lim_{n\to\infty} \int_{\mathbb{R}} g_n(s)\, d\Lambda_s$. In the case $\int_{\mathbb{R}} 1_{[0,t]} f(s)\, dA_s = \int_0^t f(s)\, dA_s$ for $f \in C_b$ this integral coincides with the classical Itô integral.

First we look at the setting of a general kernel $K(\cdot,\cdot)\colon \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$. By the criteria of [18] (1989) [Theorem 2.7] for the existence of the integral we obtain that the stochastic integral $\int_{-\infty}^t K(t,s)\, dA_s$ is well defined if for $t \geq 0$

(i) $\int_{-\infty}^t |K(t,s)|\ d\gamma(s) + \int_{-\infty}^t \int_{\mathbb{R}} |K(t,s)z \left(\mathbb{1}_{|zK(t,s)|\leq 1}(z) - \mathbb{1}_{|z|\leq 1}(z)\right) |\ \nu(ds,dz) < \infty$,

(ii) $\int_{-\infty}^t |K(t,s)|^2\ d\sigma^2(s) < \infty$,

(iii) $\int_{-\infty}^t \int_{\mathbb{R}} \min\left(|zK(t,s)|^2, 1\right)\ \nu(ds,dz) < \infty$.

(cf. [3] equations (2.1)-(2.3) in the case of Lévy process L). Then the characteristic function is given by

$$\varphi_{Z_t}(y) := \varphi_Z(t,y) = \exp\left(\psi_{Z_t}(y)\right)$$

with

$$\begin{aligned}\psi_{Z_t}(y) := \psi_A\left(t, K(t,\cdot)\right) := \psi_Z(t,y) =&\mathrm{i}y \int_{-\infty}^t K(t,s) d\gamma(s) - \frac{1}{2} y^2 \int_{-\infty}^t K(t,s)^2 d\sigma^2(s) \\ &+ \int_{-\infty}^t \int_{\mathbb{R}} (e^{\mathrm{i}yzK(t,s)} - 1 - \mathrm{i}yzK(t,s)\mathbb{1}_{|z|\leq 1})\ \nu(ds,dz).\end{aligned}$$

This implies that Z_t is infinitely divisible with characteristic triplet $(\gamma^f(t), \sigma_f^2(t), \nu_f(t))$ where

$$\begin{aligned}\gamma_K(t) &= \int_{-\infty}^t K(t,s) d\gamma(s) + \int_{-\infty}^t \int_{\mathbb{R}} zK(t,s)(\mathbb{1}_{|zK(t,s)|\leq 1}(z) - \mathbb{1}_{|z|\leq 1}(z))\ \nu(ds,dz), \\ \sigma_K^2(t) &= \int_{-\infty}^t K(t,s)^2 d\sigma^2(s), \\ \nu_K^t(A) &= \nu\left(\{(s,z) \in (\infty, t] \times \mathbb{R}) : zK(t,s) \in A \setminus \{0\}\right) \quad \textit{for } t \geq 0 \textit{ and } A \in \mathcal{B}(\mathbb{R}).\end{aligned}$$

Now we can focus on the special case of generalized oscillating Ornstein-Uhlenbeck processes. For $t \geq 0$ we look at the two processes

$$X_t := \int_{-\infty}^{t} \sin\left(a(t-s)\right) e^{-\lambda(t-s)} dA_s,$$
$$Y_t := \int_{-\infty}^{t} \cos\left(a(t-s)\right) e^{-\lambda(t-s)} dA_s,$$

where $a, \lambda > 0$. We call these processes generalized oscillating Ornstein-Uhlenbeck processes or oscillating Ornstein-Uhlenbeck processes if we restrict ourselves to Lévy processes. We will see later that this name makes sense, namely the $\sin -$ or cos-factor in the kernel leads to an oscillating behaviour of the sample paths and the autocorrelation function.

Lemma 2.1. *Assume that $a, \lambda > 0$, then the generalized oscillating Ornstein-Uhlenbeck processes are well-defined and infinitely divisible, if the following conditions are satisfied for $j = 1, 2$*

(a) $\int_{-\infty}^{t} |f_j(t-s)| \; d\gamma(s) \; < \infty,$

(b) $\int_{-\infty}^{t} f_j(t-s)^2 \; d\sigma^2(s) \; < \infty,$

(c) $\int_{-\infty}^{t} \int_{|z|>1} e^{-\lambda(t-s)} |z| \; \nu(ds, dz) \; < \infty,$

using the notation

$$f_1(x) := \sin\left(ax\right) e^{-\lambda x} \; \mathbb{1}_{[0,\infty[}(x),$$
$$f_2(x) := \cos\left(ax\right) e^{-\lambda x} \; \mathbb{1}_{[0,\infty[}(x).$$

Proof. We have to check the inequalities (i) to (iii).
ad (i): For X we obtain

$$\begin{aligned}
&\int_{-\infty}^{t} \int_{\mathbb{R}} |f_1(t-s) z \left(\mathbb{1}_{|zf_1(t-s)|\leq 1}(z) - \mathbb{1}_{|z|\leq 1}(z)\right) | \; \nu(ds, dz) \\
&\leq \int_{-\infty}^{t} \int_{\mathbb{R}} e^{-\lambda(t-s)} |z| \left(\mathbb{1}_{|z \sin(a(t-s)| \leq \exp(\lambda(t-s))}(z) - \mathbb{1}_{|z|\leq 1}(z)\right) \nu(ds, dz) \\
&\leq \int_{-\infty}^{t} \int_{|z|>1} e^{-\lambda(t-s)} |z| \; \nu(ds, dz) \; < \infty.
\end{aligned}$$

Hence together with (a) we obtain (i). Y works analogously.
ad (ii): May be deduced directly from (b).
ad (iii): We show the condition for $X_t + Y_t$, this implies the existence of the single integrals.

$$\int_{-\infty}^{t} \int_{\mathbb{R}} \min\left(|zf_1(t-s)|^2, 1\right) \; \nu(ds, dz) + \int_{-\infty}^{t} \int_{\mathbb{R}} \min\left(|zf_2(t-s)|^2, 1\right) \; \nu(ds, dz)$$

$$\leq \int_{-\infty}^{t} \int_{\mathbb{R}} \min \left(|z e^{-\lambda(t-s)}|^2 \left(\sin(a(t-s))^2 + \cos(a(t-s))^2 \right), 2 \right) \nu(ds, dz) \ < \infty.$$

$\square$

Remark. Obviously for Y we might allow $b = 0$ which leads to a classical Ornstein-Uhlenbeck process.

Now we have to take a closer look under which assumptions on the characteristic quantities of A the conditions are satisfied.

Example 2.2.

1. *Let L be a Lévy process with characteristic triplet (γ, σ, ν), i.e. in the framework of additive processes*

$$\gamma(t) = t\gamma, \quad \sigma^2(t) = t\sigma^2, \quad \nu(dt, dz) = \nu(dz)dt.$$

If we assume that L possesses a finite first moment, namely $\int_{\mathbb{R}} |z| \mathbb{1}_{|z|\geq 1} \nu(dz) = C < \infty$, then the conditions of Lemma 2.1 are satisfied, for $j = 1, 2$

$$\int_{-\infty}^{t} |f_j(t-s)| ds \leq e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda s} ds = \frac{1}{\lambda} < \infty,$$
$$\int_{-\infty}^{t} |f_j(t-s)|^2 ds \leq e^{-2\lambda t} \int_{-\infty}^{t} e^{2\lambda s} ds = \frac{1}{2\lambda} < \infty,$$
$$\int_{-\infty}^{t} e^{-\lambda(t-s)} \int_{|z|\geq 1} |z| \nu(dz, ds) = C \int_{-\infty}^{t} e^{-\lambda(t-s)} ds < \infty.$$

Hence X and Y are well-defined.

2. *Let A be an additive process with characteristic triplet $(\gamma_t, \sigma_t^2, \nu_t)$. We assume that γ and σ^2 are differentiable in t with $\gamma' \in \mathcal{C}_b$ and $(\sigma^2)' \in \mathcal{C}_b$, then the conditions (i) and (ii) in Lemma 2.1 are satisfies, namely for X we obtain*

$$\int_{-\infty}^{t} |f_1(t-s)| d\gamma(s) \leq C_1 \int_{-\infty}^{t} |\sin\left(a(t-s)\right) e^{\lambda(t-s)}| ds < \infty,$$
$$\int_{-\infty}^{t} f_1(t-s)^2 d\sigma^2(s) \leq C_2 \int_{-\infty}^{t} |\sin\left(a(t-s)\right) e^{\lambda(t-s)}|^2 ds < \infty.$$

for convenient $C_1, C_2 > 0$. With Y we may proceed analogously.

3. *A popular example of an additive process is a time-inhomogeneous compound Poisson process*

$$A_t := \sum_{i=1}^{N(t)} X_i.$$

Here $(X_i)_{i\in\mathbb{N}}$ denotes a sequence of iid random variables with $E(X_1) < \infty$ and denotes $N(t)$ a Poisson process with bounded intensity function $\rho : \mathbb{R} \to \mathbb{R}$. Hence the characteristic triplet of A_t is $(0, 0, \nu)$ with

$$\nu(ds, dz) = \rho(s)F_{X_1}(dz)ds,$$

where F_{X_1} denotes the distribution function of X_1. In this framework X and Y exist, (a) and (b) are obviously satisfied and for (c) we obtain for $C > 0$

$$\begin{aligned}&\left|\int_{-\infty}^{t}\int_{|z|\geq 1} e^{-\lambda(t-s)}|z|\ \nu(ds, dz)\right|\\ \leq&\int_{-\infty}^{t}\int_{|z|\geq 1} e^{-\lambda(t-s)}|z|\ |\rho(s)|F_{X_1}(dz)ds\\ \leq& C\cdot\sup_{s\in\mathbb{R}}|\rho(s)|\int_{-\infty}^{t} e^{-\lambda(t-s)}\ ds < \infty\end{aligned}$$

Lemma 2.3. *Let $X_t = \int_{-\infty}^{t}\exp(-\lambda(t-s))\sin(a(t-s))\,dA_s$ and $Y_t = \int_{-\infty}^{t}\exp(-\lambda(t-s))\cos(a(t-s))\,dA_s$ with $a, \lambda > 0$, then we obtain that $X_t - X_{t-} = 0$ and $Y_t - Y_{t-} = A_t - A_{t-}$.*

Proof. Using the following formulas for trigonometric functions

$$\begin{aligned}\cos(a(t-s)) &= \cos(at)\cos(-as) - \sin(at)\sin(-as)\\ \sin(a(t-s)) &= \sin(at)\cos(-as) + \cos(at)\sin(-as)\end{aligned}$$

we can decompose X in the following way

$$\begin{aligned}X_t = &\sin(at)e^{-\lambda t}\int_{-\infty}^{0} e^{\lambda s}\cos(-as)dA_s + \sin(at)e^{-\lambda t}\int_{0}^{t} e^{\lambda s}\cos(-as)dA_s\\ &+\cos(at)e^{-\lambda t}\int_{-\infty}^{0} e^{\lambda s}\sin(-as)dA_s + \cos(at)e^{-\lambda t}\int_{0}^{t} e^{\lambda s}\sin(-as)dA_s.\end{aligned}$$

Hence all components are semimartingales which yields

$$\begin{aligned}X_t - X_{t-} &= (A_t - A_{t-})(\sin(at)\cos(-at) + \cos(at)\sin(-at))\\ &= (A_t - A_{t-})\sin(0) = 0.\end{aligned}$$

Applying similar calculations to Y we obtain

$$Y_t - Y_{t-} = (A_t - A_{t-})\cos(0) = A_t - A_{t-}.$$

□

We can see that while X possesses continuous paths, Y possesses the jumps of the driving additive process as in the case of an Ornstein-Uhlenbeck process. Note that this behaviour is induced by the behaviour of the kernel in zero.

Next we derive a system of coupled stochastic differential equations.

Lemma 2.4. *Let $X_t = \int_{-\infty}^t \exp(-\lambda(t-s))\sin(a(t-s))\,dA_s$ and $Y_t = \int_{-\infty}^t \exp(-\lambda(t-s))\cos(a(t-s))\,dA_s$ with $a, \lambda > 0$, then we obtain*

$$\begin{aligned} dX_t &= (aY_t - \lambda X_t)dt \\ dY_t &= (-\lambda Y_t - aX_t)dt + dA_t. \end{aligned}$$

Proof. We use the same decomposition of X and Y together with the following integration by parts formulas

$$\begin{aligned} \int_0^t e^{\lambda s}\sin(-as)dA_s &= [A_s e^{\lambda s}\sin(-as)]_0^t - \int_0^t (\lambda e^{\lambda s}\sin(-as) - ae^{\lambda s}\cos(-as))A_s ds \\ &= \sin(-at)A_t e^{\lambda t} - \lambda\int_0^t e^{\lambda s}\sin(-as)A_s ds + a\int_0^t e^{\lambda s}\cos(-as)A_s ds \\ \int_0^t e^{\lambda s}\cos(-as)dA_s &= \cos(-at)A_t e^{\lambda t} - \lambda\int_0^t e^{\lambda s}\cos(-as)A_s ds - a\int_0^t e^{\lambda s}\sin(-as)A_s ds \end{aligned}$$

and the notation $C = \int_{-\infty}^0 e^{\lambda s}\cos(-as)dA_s$ and $S = \int_{-\infty}^0 e^{\lambda s}\sin(-as)dA_s$ and obtain

$$\begin{aligned} X_t - X_0 &= (\sin(at)e^{-\lambda t})C + (\cos(at)e^{-\lambda t} - 1)S \\ &\quad + \sin((at)\cos(-at)A_t - \lambda\sin(at)e^{-\lambda t}\int_0^t e^{\lambda s}\cos(-as)A_s ds \\ &\quad - a\sin(at)e^{-\lambda t}\int_0^t e^{\lambda s}\sin(-as)A_s ds + \sin(-as)\cos(at)A_t \\ &\quad - \lambda\cos(at)e^{-\lambda t}\int_0^t e^{\lambda s}\sin(-as)A_s ds + a\cos(at)e^{-\lambda t}\int_0^t e^{\lambda s}\cos(-as)A_s ds \\ &= \int_0^t [(a\cos(ax)e^{-\lambda x} - \lambda\sin(ax)e^{-\lambda x})C + (-a\sin(ax)e^{-\lambda x} - \lambda\cos(ax)e^{-\lambda x})S \\ &\quad + (-a\lambda\cos(ax)e^{-\lambda x} + \lambda^2\sin(ax)e^{-\lambda x})\int_0^x e^{\lambda s}\cos(-as)A_s ds \\ &\quad - \lambda\sin(ax)\cos(-ax)A_x \\ &\quad + (-a^2\cos(ax)e^{-\lambda x} + a\lambda\sin(ax)e^{-\lambda x})\int_0^x e^{\lambda s}\sin(-as)A_s ds \\ &\quad - a\sin(ax)\sin(-ax)A_x + (a\lambda\sin(ax)e^{-\lambda x} + \lambda^2\cos(ax)e^{-\lambda x})\int_0^x e^{\lambda s}\sin(-as)A_s ds \end{aligned}$$

$$
\begin{aligned}
&-\lambda\cos(ax)\sin(-ax)A_x \\
&+(-a^2\sin(ax)e^{-\lambda x} - a\lambda\cos(ax)e^{-\lambda x})\int_0^x e^{\lambda s}\cos(-as)A_s ds \\
&+a\cos(ax)\cos(-ax)A_x]dx \\
= &\int_0^t (a\cos(ax)e^{-\lambda x} - \lambda\sin(ax)e^{-\lambda x})C + (-a\sin(ax)e^{-\lambda x} - \lambda\cos(ax)e^{-\lambda x})S \\
&-\lambda\sin(ax)e^{-\lambda x}\int_0^x e^{\lambda s}\cos(-as)dA_s - a\sin(ax)e^{-\lambda x}\int_0^x e^{\lambda s}\sin(-as)dA_s \\
&-\lambda\cos(ax)e^{-\lambda x}\int_0^x e^{\lambda s}\sin(-as)dA_s + a\sin(ax)e^{-\lambda x}\int_0^x e^{\lambda s}\cos(-as)dA_s dx \\
= &\int_0^t a\cos(ax)e^{-\lambda x}\int_{-\infty}^x e^{\lambda s}\cos(-as)dA_s - \lambda\sin(ax)e^{-\lambda x}\int_{-\infty}^x e^{\lambda s}\cos(-as)dA_s \\
&-a\sin(ax)e^{-\lambda x}\int_{-\infty}^x e^{\lambda s}\sin(-as)dA_s - \lambda\cos(ax)e^{-\lambda x}\int_{-\infty}^x e^{\lambda s}\sin(-as)dA_s dx \\
= &\int_0^t aY_x - \lambda X_x dx
\end{aligned}
$$

Analogously we obtain

$$
Y_t - Y_0 = A_t + \int_0^t (-\lambda Y_x - aX_x)dx
$$

□

Remark. Again we can see that for Y letting $a = 0$ leads to the Ornstein Uhlenbeck equation.

Using the methods outlined in [9] we can also simulate oscillating Ornstein-Uhlenbeck processes. We see in figure 1 that the sample paths of both oscillating Ornstein-Uhlenbeck processes show an oscillating and mean reverting behaviour. Furthermore, we see that X is continuous while Y possesses jumps.

3 Modelling of spot prices in electricity markets

Electricity demand and hence also electricity prices have a strong seasonal behaviour on different scales such as on a daily, a weekly and a yearly scale. To illustrate this behaviour we have plotted in figure 2 the price curves of hourly data from the EEX (European Energy Exchange) in Leipzig from the year 2010.

Empirical analysis of electricity data like e.g. in [10], [16] or [17] have provided three main stylized facts of electricity data, which is a seasonal effect, a mean reverting property and

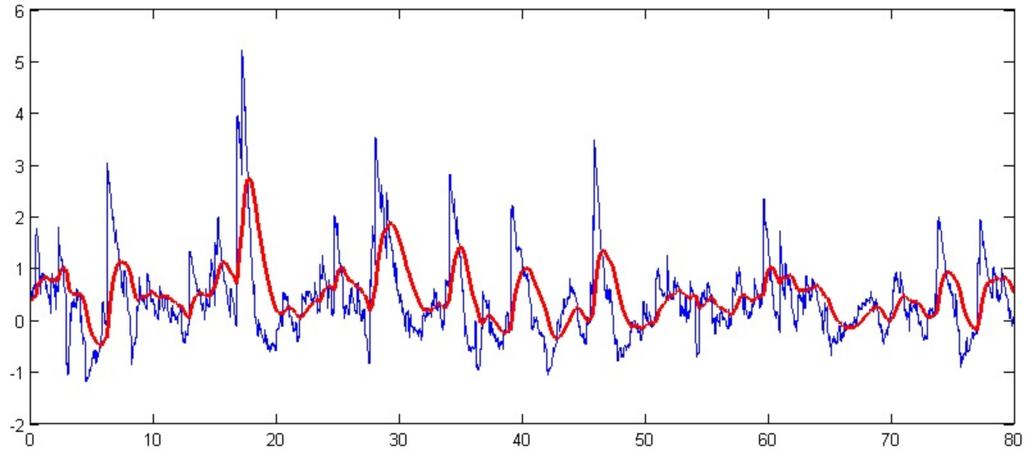

Figure 1: Simulated path of X_t (blue line) and Y_t (red line) with $a = 1$, $\lambda = 0.8$ and the same driving Lévy-Process L. Here L is a Normal inverse Gaussian process.

price spikes. Classically this features are modelled by a deterministic seasonal function and e.g. a superposition of Ornstein-Uhlenbeck processes with different speed of mean reversion and at least partly driven by Lévy processes, cf. [7], [15].

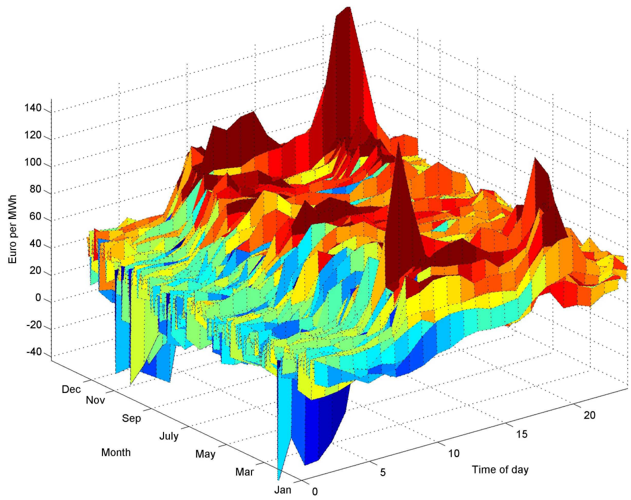

Figure 2: Price curves of intraday market of the EEX from the year 2010 (see [12]).

Our new class of generalized oscillating Ornstein-Uhlenbeck processes provides an alternative to model this features. If we restrict ourselves to Lévy processes as driving processes oscillating Ornstein-Uhlenbeck processes are stationary processes, as may be easily deduced from the characteristic function. Hence oscillating Ornstein-Uhlenbeck processes provide the possibility to include the seasonal behaviour directly into a stationary stochastic process. Furthermore, with the general class we are able to reproduce mean reverting behaviour, jumps and infinitely divisible marginals. In addition the processes allow for explicit formulas such as Ornstein-Uhlenbeck processes.

We consider an additive model for the spot prices such as in [7]. Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space satisfying the usual conditions. For $i = 1, \ldots, n$ define

$$\widetilde{S}_i(t) := \omega_1^i X_{i,a_i}(t) + \omega_2^i Y_{i,a_i}(t) + \omega_3^i Z_i(t) \tag{3.1}$$

with $\omega_j^i \geq 0$ and $\omega_3^i \geq \omega_1^i + \omega_2^i$ for $j = 1, 2, 3$. Furthermore, let

$$\begin{aligned} X_{i,a_i}(t) &= \int_{-\infty}^{t} e^{-\lambda_i(t-s)} \sin(a_i(t-s)) dA_i(s), \\ Y_{i,b_i}(t) &= \int_{-\infty}^{t} e^{-\lambda_i(t-s)} \cos(b_i(t-s)) dA_i(s) \quad \text{and} \\ Z_i(t) &= \int_{-\infty}^{t} e^{-\lambda_i(t-s)} dA_i(s) = Y_{i,0}(t) \end{aligned}$$

with $a_i, \lambda_i, b_i > 0$. A_i denote two-sided additive processes and are assumed to be pairwise independent.

Finally we define the spot-price process by

$$S(t) := S_t := \sum_{i=1}^{n} c_i \widetilde{S}_i(t) \tag{3.2}$$

with weights $c_i \geq 0$.
Note that the conditions on the weights c_i and ω_i ensure that the price process is non-negative if A is of pure jump-type with no negative jumps, cf. [1]. For modelling we will mainly assume this, though induced by the renewable energy act occasionally negative prices occur.

4 Correlation structure and moments of oscillating Ornstein-Uhlenbeck processes

In the following we will derive explicit formulas for derivatives based on our spot price model defined in the previous section. In the formulas moments and correlations between the different components of our price process play an important role, hence we first calculate this quantities we need later. In this section we restrict ourselves to driving Lévy processes, since in this case we may derive closed form expressions for the desired quantities.

Lemma 4.1. *Let* $X(t) = \int_{-\infty}^{t} e^{-\lambda(t-s)} \sin(a(t-s))dL(s)$*,* $Y(t) = \int_{-\infty}^{t} e^{-\lambda(t-s)} \cos(b(t-s))dL(s)$ *and* $Z(t) = \int_{-\infty}^{t} e^{-\lambda(t-s)}dL(s)$ *where* L *is a Lévy process with* $E(L(1)) = \mu$ *and* $Var(L(1)) = V < \infty$*, then we obtain:*

$$\begin{aligned} E(X_t) &= \mu\frac{a}{a^2+\lambda^2}, & E(Y_t) &= \mu\frac{\lambda}{b^2+\lambda^2}, & E(Z_t) &= \frac{\mu}{\lambda} \\ Var(X_t) &= V\frac{\lambda^2}{4\lambda(\lambda^2+a^2)}, & Var(Y_t) &= V\frac{2\lambda^2+b^2}{4\lambda(\lambda^2+b^2)}, & Var(Z_t) &= \frac{V}{2\lambda}. \end{aligned} \tag{4.3}$$

Proof. Follows directly from the characteristic functions. □

Lemma 4.2. *Let* $(X_t)_{t\geq 0}$*,* $(Y_t)_{t\geq 0}$ *and* $(Z_t)_{t\geq 0}$ *as in the previous Lemmas and* L *a Lévy process satisfying* $E(L(1)) = \mu$ *and* $Var(L(1)) = V < \infty$*, then we obtain for* $u > t$*:*

$$\begin{aligned} Cov(X_u, Z_t) &= Ve^{-\lambda(u-t)}\frac{2\lambda\sin(a(u-t)) + a\cos(a(u-t))}{4\lambda^2+a^2}, \\ Cov(Z_u, X_t) &= Ve^{-\lambda(u-t)}\frac{a}{4\lambda^2+a^2}, \\ Cov(Y_u, Z_t) &= Ve^{-\lambda(u-t)}\frac{2\lambda\cos(b(u-t)) - b\sin(b(u-t))}{4\lambda^2+b^2}, \\ Cov(Z_u, Y_t) &= Ve^{-\lambda(u-t)}\frac{2\lambda}{4\lambda^2+b^2}. \end{aligned}$$

Furthermore, we obtain

$$\begin{aligned} Cov(Y_u, X_t) &= Ve^{-\lambda(u-t)}\Bigg(\frac{(a+b)\cos(b(u-t)) + 2\lambda\sin(b(u-t)}{2(4\lambda^2+(a+b)^2)} \\ &\qquad + \frac{(a-b)\cos(b(u-t)) - 2\lambda\sin(b(u-t)}{2(4\lambda^2+(a-b)^2)}\Bigg), \\ Cov(X_u, Y_t) &= Ve^{-\lambda(u-t)}\Bigg(\frac{(a+b)\cos(a(u-t)) + 2\lambda\sin(a(u-t)}{2(4\lambda^2+(a+b)^2)} \\ &\qquad + \frac{(a-b)\cos(a(u-t)) + 2\lambda\sin(b(u-t)}{2(4\lambda^2+(a-b)^2)}\Bigg), \\ Cov(X_u, X_t) &= Ve^{-\lambda(u-t)}\frac{a^2\cos(a(u-t)) + a\lambda\sin(a(u-t))}{4\lambda(\lambda^2+a^2)}, \\ Cov(Y_u, Y_t) &= Ve^{-\lambda(u-t)}\frac{(2\lambda^2+b^2)\cos(b(u-t)) - a\lambda\sin(b(u-t))}{4\lambda(\lambda^2+b^2)}, \\ Cov(Z_u, Z_t) &= Ve^{-\lambda(u-t)}\frac{1}{2\lambda}. \end{aligned}$$

Proof. All terms may be derived in the same way and we show how to proceed with $Cov(Y_u, X_t)$. For the calculation of $Cov(Y_u, X_t)$ we apply the formula

$$Cov(Y_u, X_t) = \frac{1}{2}\left(E[Y_u^2] + E[X_t^2] - E[(Y_u - X_t)^2]\right) - E[Y_u]E[X_t]. \tag{4.4}$$

Hence we have to find the kernel for $Y_u - X_t$, namely

$$\begin{aligned}Y_u - X_t = &\int_{-\infty}^{t} e^{\lambda s}\big(e^{-\lambda u}\cos\left(b(u-s)\right) - e^{-\lambda t}\sin\left(a(t-s)\right)\big)dL(s)\\ &+\int_{t}^{u} e^{\lambda s}(e^{-\lambda u}\cos(b(u-s))dL(s).\end{aligned}$$

The characteristic function yields

$$\begin{aligned}E[(Y_u - X_t)^2] =&V\Bigg(e^{-2\lambda u}\int_{-\infty}^{t} e^{2\lambda s}\cos\left(b(u-s)\right)^2 ds + e^{-2\lambda u}\int_{t}^{u} e^{2\lambda s}\cos\left(b(u-s)\right)^2 ds\\ &+ e^{-2\lambda t}\int_{-\infty}^{t} e^{2\lambda s}\sin\left(a(t-s)\right)^2 ds\\ &- 2e^{-\lambda(t+u)}\int_{-\infty}^{t} e^{2\lambda s}\cos\left(b(u-s)\right)\sin\left(a(t-s)\right)ds\Bigg) + (E[Y_u - X_t])^2\\ =&Var(Y_u) + Var(X_t) + E(Y_u)^2 - 2E(Y_u)E(X_t) + E(X_t)^2\\ &- 2Ve^{-\lambda(t+u)}\int_{-\infty}^{t} e^{2\lambda s}\cos\left(b(u-s)\right)\sin\left(a(t-s)\right)\, ds\end{aligned}$$

and hence (4.4)

$$\begin{aligned}Cov(Y_u, X_t) &= Ve^{-\lambda(t+u)}\int_{-\infty}^{t} e^{2\lambda s}\cos\left(b(u-s)\right)\sin\left(a(t-s)\right)\, ds\\ &= Ve^{-\lambda(u-t)}\Bigg(\frac{(a+b)\cos(b(u-t)) + 2\lambda\sin(b(u-t)}{2(4\lambda^2 + (a+b)^2)}\\ &\qquad + \frac{(a-b)\cos(b(u-t)) - 2\lambda\sin(b(u-t)}{2(4\lambda^2 + (a-b)^2)}\Bigg),\end{aligned}$$

which is the desired formula. □

Note that these quantities may be used to draw inference on the involved quantities via a method of moments.

Now we can use the previous two lemmas to calculate the moments and correlation between different time-points of our building blocks $\widetilde{S} := \widetilde{S}_i$, when they are driven by Lévy processes. We obtain for $u > t$:

$$\begin{aligned}Cov(\widetilde{S}_u, \widetilde{S}_t) =&\omega_1^2 Cov(X_u, X_t) + \omega_2^2 Cov(Y_u, Y_t) + \omega_3^2 Cov(Z_u, Z_t)\\ &+ \omega_1\omega_2[Cov(X_u, Y_t) + Cov(Y_u, X_t)]\\ &+ \omega_1\omega_3[Cov(X_u, Z_t) + Cov(Z_u, X_t)]\\ &+ \omega_2\omega_3[Cov(Y_u, Z_t) + Cov(Z_u, Y_t)].\end{aligned} \tag{4.5}$$

and for the variance of $\widetilde{S}$ we obtain

$$\begin{aligned} Var(\widetilde{S}_u) =& \omega_1^2 Var(X_u) + \omega_2^2 Var(Y_u) + \omega_3^2 Var(Z_u) + 2\omega_1\omega_2 Cov(X_u, Y_u) \\ &+ 2\omega_1\omega_3 Cov(X_u, Z_u) + 2\omega_2\omega_3 Cov(Y_u, Z_u) = Var(\widetilde{S}_t). \end{aligned} \tag{4.6}$$

This results may be used to derive the correlation structure of the spot price process S.

Corollary 4.3. *For $i = 1, \dots, n$ let $\widetilde{S}_i(u)$ and S as defined in (3.1) and (3.2), then we obtain for $u > t$ the autocorrelation*

$$Corr(S_u, S_t) = \frac{\sum_{i=1}^n c_i^2 Cov(\widetilde{S}_i(u), \widetilde{S}_i(t))}{\sum_{i=1}^n c_i^2 Var(\widetilde{S}_i(u))},$$

where $Cov(\widetilde{S}_i(u), \widetilde{S}_i(t))$ and $Var(\widetilde{S}_i(u))$ are given in (4.5) and (4.6).

Hence we can see with the spot price models based on oscillating Ornstein Uhlenbeck processes we obtain an oscillating behaviour in the autocorrelation function as it occurs in empirical data. In figure 3 we fitted the autocorrelation function of S with $n = 4$ to the empirical autocorrelation function from intraday data of the EEX from the year 2012.

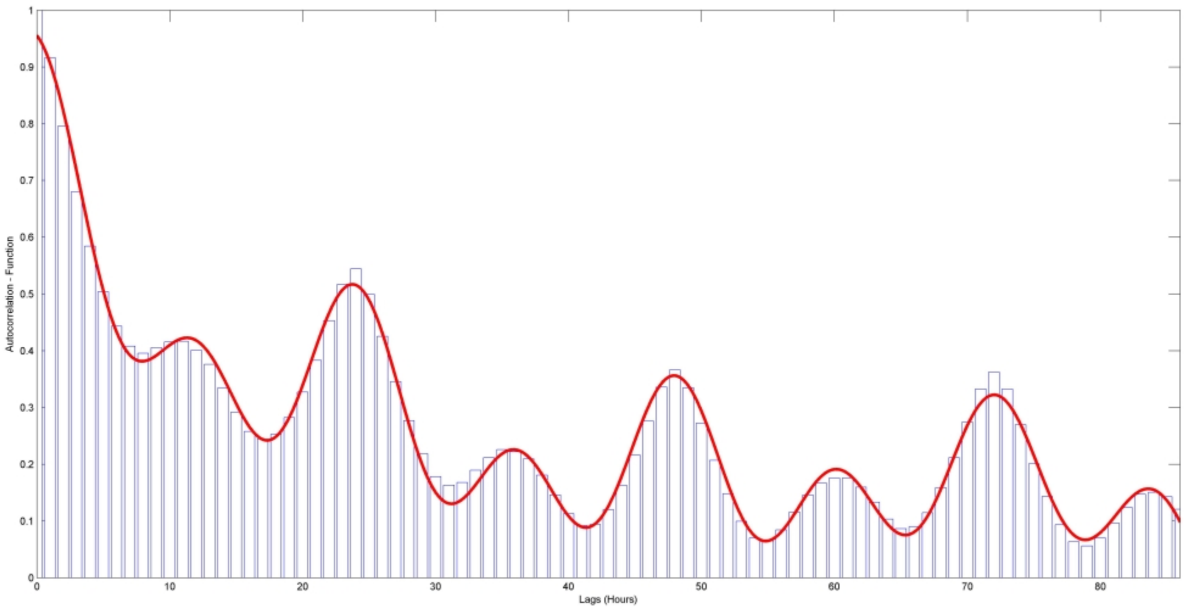

Figure 3: Empirical autocorrelation function of intraday market prices of the EEX from the year 2010 and fitted analytical autocorrelation function of our model S with $n = 4$.

5 Derivative pricing

In this section our aim is to deduce closed form pricing formulas for forwards based on generalized oscillating Ornstein-Uhlenbeck processes. And in a second step to derive also pricing formulas for options written on these forwards.

Let $f(t,T)$ be the price of a forward at time $t > 0$ for the delivery of a certain quantity of electricity (e.g. 1 MWh) at some future time-point $T > t$. Furthermore, we denote by $F(t,T_1,T_2)$ the price of a forward at time t of a certain quantity electricity (e.g. 1 MWh) to be delivered in the time-period $[T_1,T_2]$ with $T_2 \geq T_1 > t \geq 0$. As in [7] we calculate the forward price based on our spot price model in the following way

$$F(t,T_1,T_2) = \frac{1}{T_2 - T_1} E_Q \left[\int_{T_1}^{T_2} S(u)\ du \middle| \mathcal{F}_t \right]. \tag{5.7}$$

The obvious question is now how to choose Q. In classical financial mathematics we have to choose a risk neutral measure which is equivalent to the objective measure P, hence we look for an equivalent martingale measure. In electricity markets we have much more flexibility, namely as electricity is non-storable, we need not have a martingale measure, but may choose any equivalent measure. However, a convenient one is often the one based on the Esscher transform. In the following as in [2] or [8] we consider a measure change based on the Esscher transform.

Theorem 5.1. *Let A be an additive process given via the triplet $(\gamma_t, \sigma_t^2, \nu_t)$. Let $c > 0$, such that*

$$\int_0^T \int_{|z|\geq 1} (e^{cz} - 1)\ \nu(dz, du) \quad < \infty.$$

for all $T > 0$. Furthermore, let

$$Z_\theta(t) := \exp\left(\int_0^t \theta(u)\ dA_u - \Theta\left(t, \theta(\cdot)\right) \right)$$

a density process where $\theta : [0,\infty[\to \mathbb{R}$ is continuous and bounded with

$$\sup_{t\leq T} |\theta(t)| \ < c,$$

and $\Theta\left(t, \theta\left(\cdot\right)\right) := \psi_A(t, -i\theta(\cdot))$ with ψ_A the characteristic exponent of A. We define the change of measure by

$$\left.\frac{dP}{dQ}\right|_{\mathcal{F}_t} = Z_\theta(t), \tag{5.8}$$

then under Q A is an additive process with triplet $(\widetilde{\gamma}_t, \sigma_t^2, \widetilde{\nu}_t)$, where

$$\begin{aligned} \widetilde{\gamma}_t = &\ \gamma_t + \int_0^t \int_{|z|<1} z(e^{\theta(u)z} - 1)\nu(du, dz) + \int_0^t \theta(u) d\sigma^2(u) \quad \textit{and} \\ \widetilde{\nu}_t = &\ e^{\theta(t)z}\nu_t. \end{aligned}$$

Proof. The proof is similar to [6], Proposition 4.4. □

Note that this change of measure applied to a two-sided additive process only changes the measure for $t \geq 0$. Hence also a risk premium may only be calculated for $t \geq 0$ which is no restriction in practice.

Example 5.1. *Note that our model together with the risk neutral measure based on the Esscher transform is able to reproduce the change of sign in the risk premium which was empirically observed e.g. in [11]. The risk premium is defined as*

$$RP(t,T) = F(t, T_1, T_2) - \frac{1}{T_2 - T_1} E_P \left[\int_{T_1}^{T_2} S(u)\ du \middle| \mathcal{F}_t \right].$$

For simplicity taking our model with $n = 1$ and the measure based on the previous theorem we obtain

$$\begin{aligned} RP(t,T) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \Bigg(\int_t^u \int_{\mathbb{R}} & e^{\lambda(u-s)} (\omega_1 \sin(a(u-s)) + \omega_2 \cos(a(u-s)) + \omega_3) \\ & z(e^{\theta(s)z} - 1)\ \nu(ds, dz) \Bigg) du. \end{aligned}$$

Now we can consider the following to cases: For $\theta(s) \equiv \theta > 0$ we see that $z(e^{\theta(s)z} - 1)$ is positive, whereas for $\theta(s) \equiv \theta < 0$ the term $z(e^{\theta(s)z} - 1)$ is negative. Hence we are able to reproduce the desired sign change.

In the following we will consider as driving processes additive processes with the following representation. Let A_i for $i = 1, \ldots n$ denote a two-sided additive process given by

$$A_i(t) = \int_{-\infty}^t \int_0^\infty z\ J_i(ds, dz) \tag{5.9}$$

with jumps measure J_i, whose support is concentrated on the positive real line, hence accounts only for positive jumps. Furthermore, we denote by

$$\widetilde{J}_i(ds, dz) := J_i(ds, dz) - \widetilde{\nu}_i(ds, dz)$$

the jump measure which is compensated under Q defined in 5.1 and by

$$\widetilde{A}_i(t) := \int_{-\infty}^{t} \int_{0}^{\infty} z \ \widetilde{J}_i(ds, dz)$$

the associated compensated process. Now we can derive a formula for the forward price $F(t, T_1, T_2)$ under our spot price model based on generalized oscillating Ornstein-Uhlenbeck processes.

Theorem 5.2. *Let S be the spot price process defined in (3.2) based on the building blocks $\widetilde{S}_i$ for $i = 1, \ldots, n$ as in (3.1) where we set $a_i = b_i$. Assume that the driving process A_i is of the form (5.9). Let Q denote a risk neutral measure, then we obtain for the forward price at time $t > 0$ for the delivery of one MWh electricity in the time interval $[T_1, T_2]$ with $T_2 > T_1 > t \geq 0$:*

$$F(t, T_1, T_2) = \sum_{i=1}^{n} c_i F_i(t, T_1, T_2)$$

with

$$\begin{aligned} F_i(t, T_1, T_2) = \frac{1}{T_2 - T_1} \Bigg[& X_{i,a_i}(t) \quad \Big(\omega_1^i D_i(t, T_1, T_2) - \omega_2^i E_i(t, T_1, T_2) \Big) \\ & + Y_{i,a_i}(t) \Big(\omega_1^i E_i(t, T_1, T_2) + \omega_2^i D_i(t, T_1, T_2) \Big) \\ & + Z_i(t) \quad \Big(\omega_3^i G_i(t, T_1, T_2) \Big) \\ & + \qquad \omega_1^i U_1^i(t, T_1, T_2) + \omega_2^i U_2^i(t, T_1, T_2)) + \omega_3^i U_3^i(t, T_1, T_2) \Bigg], \end{aligned}$$

where

$$\begin{aligned} D_i(t, T_1, T_2) :=& \frac{1}{\lambda_i^2 + a_i^2} \Big(e^{-\lambda_i (T_2 - t)} \big(a_i \sin(a_i(T_2 - t)) - \lambda_i \cos(a_i(T_2 - t)) \big) \\ & - e^{-\lambda_i (T_1 - t)} \big(a_i \sin(a_i(T_1 - t)) - \lambda_i \cos(a_i(T_1 - t)) \big) \Big), \\ E_i(t, T_1, T_2) :=& \frac{1}{\lambda_i^2 + a_i^2} \Big(e^{-\lambda_i (T_1 - t)} \big(a_i \cos(a_i(T_1 - t)) + \lambda_i \sin(a_i(T_1 - t)) \big) \\ & - e^{-\lambda_i (T_2 - t)} \big(a_i \cos(a_i(T_2 - t)) + \lambda_i \sin(a_i(T_2 - t)) \big) \Big) \end{aligned}$$

and

$$G_i(t, T_1, T_2) := \frac{1}{\lambda_i} \Big(e^{-\lambda_i (T_1 - t)} - e^{-\lambda_i (T_2 - t)} \Big),$$

$$U_1^i(t,T_1,T_2) := \int_{T_1}^{T_2}\left(\int_t^u\int_{\mathbb{R}} ze^{-\lambda_i(u-s)}\sin(a_i(u-s))\widetilde{\nu}_i(dz,ds)\right)\, du,$$
$$U_2^i(t,T_1,T_2) := \int_{T_1}^{T_2}\left(\int_t^u\int_{\mathbb{R}} ze^{-\lambda_i(u-s)}\cos(a_i(u-s))\widetilde{\nu}_i(dz,ds)\right)\, du,$$
$$U_3^i(t,T_1,T_2) := \int_{T_1}^{T_2}\left(\int_t^u\int_{\mathbb{R}} ze^{-\lambda_i(u-s)}\widetilde{\nu}_i(dz,ds)\right)\, du.$$

Here $\widetilde{\nu}$ denotes the compensator under Q.

Proof. We outline the proof for $n = 1$. For general n the calculations are similar. Let $X := X_{1,a_1}$, $Y := Y_{1,a_1}$, $Z := Z_1$, $A := A_1$ further $a := a_1$ and $\lambda := \lambda_1$.
In a first step we evaluate

$$\frac{1}{T_2-T_1}\int_{T_1}^{T_2} E_Q\left[X(u)|\mathcal{F}_t\right]\, du. \tag{5.10}$$

We consider the following representation for $0 \le t < u$:

$$\begin{aligned}
X(u) &= \int_{-\infty}^{t}\sin(a(u-s))e^{\lambda(u-s)}dA_s + \int_t^u \sin(a(u-s))e^{\lambda(u-s)}dA_s \\
&= \int_{-\infty}^{t}\sin\left(a\left(t-s+u-t\right)\right)e^{\lambda(t-s+u-t)}dA_s + \int_t^u \sin\left(a\left(u-s\right)\right)e^{\lambda(u-s)}dA_s \\
&= e^{-\lambda(u-t)}\Big[\cos\left(a\left(u-t\right)\right)\cdot X(t) + \sin\left(a\left(u-t\right)\right)\cdot Y(t)\Big] \qquad (*) \\
&\quad + \int_t^u \sin\left(a\left(u-s\right)\right)e^{\lambda(u-s)}dA_s. \qquad (**)
\end{aligned}$$

The representation $X(u) = (*) + (**)$ yields

$$\frac{1}{T_2-T_1}\int_{T_1}^{T_2} E_Q\left[(*)|\mathcal{F}_t\right]\, du + \frac{1}{T_2-T_1}\int_{T_1}^{T_2} E_Q\left[(**)|\mathcal{F}_t\right]\, du.$$

Now we look separately at the two terms.
By the$\mathcal{F}_t$-measuarability of $(*)$ we obtain

$$\begin{aligned}
&\int_{T_1}^{T_2} E_Q\left[(*)|\mathcal{F}_t\right] du \\
&\qquad = X(t)\int_{T_1}^{T_2} e^{\lambda(u-t)}\cos\left(a\left(u-t\right)\right)\, du + Y(t)\int_{T_1}^{T_2} e^{\lambda(u-t)}\sin\left(a\left(u-t\right)\right)\, du \\
&\qquad = X(t)\cdot D_1(t,T_1,T_2) \;+\; Y(t)\cdot E_1(t,T_1,T_2),
\end{aligned}$$

and similarly for the second term

$$
\begin{aligned}
&\int_{T_1}^{T_2} E_Q\left[(**)|\mathcal{F}_t\right]\,du \\
&\qquad = \int_{T_1}^{T_2} E_Q\left[\int_t^u \sin\left(a\left(u-s\right)\right)e^{\lambda(u-s)}dA_s|\mathcal{F}_t\right] \\
&\qquad = \int_{T_1}^{T_2}\left(\int_t^u\int_0^\infty ze^{\lambda(u-s)}\sin\left(a((u-s)\right)\ \widetilde{\nu}(dz,ds)\right)\,du \\
&\qquad\quad + \int_{T_1}^{T_2}\left(E_Q\Big[\int_t^u\ e^{\lambda(u-s)}\sin\left(a((u-s)\right)d\widetilde{A}(s)\big|\mathcal{F}_t\Big]\right)\,du \qquad (5.11) \\
&\qquad = U_1^1(t,T_1,T_2),
\end{aligned}
$$

where (5.11) vanishes by the martingale property of $\widetilde{A}$ under Q. Hence together we obtain

$$
\begin{aligned}
&\frac{1}{T_2-T_1}\int_{T_1}^{T_2} E_Q\left[X(u)|\mathcal{F}_t\right]\,du \\
&\qquad = \frac{1}{T_2-T_1}\Big(X(t)\cdot D_1(t,T_1,T_2)\ +\ Y(t)\cdot E_1(t,T_1,T_2)\ +\ U_1^1(t,T_1,T_2)\Big).
\end{aligned}
$$

In a second step we evaluate

$$
\frac{1}{T_2-T_1}\int_{T_1}^{T_2} E_Q\left[Y(u)|\mathcal{F}_t\right]\,du. \tag{5.12}
$$

Similarly to the first step we consider the representation

$$
\begin{aligned}
Y(u) &= \int_{-\infty}^t \sin(a(u-s))e^{\lambda(u-s)}dA_s + \int_t^u \sin(a(u-s))e^{\lambda(u-s)}dA_s \\
&= e^{-\lambda(u-t)}\Big[\cos\left(a\left(u-t\right)\right)\cdot Y(t) - \sin\left(a\left(u-t\right)\right)\cdot X(t)\Big] \qquad (*) \\
&\quad + \int_t^u \cos\left(a\left(u-s\right)\right)e^{\lambda(u-s)}dA_s. \qquad (**)
\end{aligned}
$$

and obtain for the two terms

$$
\begin{aligned}
&\int_{T_1}^{T_2} E_Q\left[(*)|\mathcal{F}_t\right]\,du \\
&\qquad = Y(t)\cdot D_1(t,T_1,T_2)\ -\ X(t)\cdot E_1(t,T_1,T_2).
\end{aligned}
$$

and

$$
\int_{T_1}^{T_2} E_Q\left[(**)|\mathcal{F}_t\right]\,du
$$

$$
\begin{aligned}
&= \int_{T_1}^{T_2} \left(\int_t^u \int_0^\infty z e^{\lambda(u-s)} \cos\left(a((u-s)\right) \ \widetilde{\nu}(dz, ds) \right) \, du \\
&= U_2^1(t, T_1, T_2),
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} E_Q \left[Y(u) | \mathcal{F}_t \right] \ du \\
&\qquad = \frac{1}{T_2 - T_1} \left(Y(t) \cdot D_1(t, T_1, T_2) \ - \ X(t) \cdot E_1(t, T_1, T_2) \ + \ U_2^1(t, T_1, T_2) \right).
\end{aligned} \tag{5.13}
$$

Finally note that

$$
\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} E_Q \left[Z(u) | \mathcal{F}_t \right] \ du,
$$

is a special case of (5.12) with $a = 0$. Hence we obtain

$$
\begin{aligned}
D_1(t, T_1, T_2)|_{a=0} &= \ \left(e^{-\lambda(T_1 - t)} - e^{-\lambda(T_2 - t)} \right) / \lambda = G_1(t, T_1, T_2), \\
E_1(t, T_1, T_2)|_{a=0} &= \ 0 \\
U_2^1(t, T_1, T_2)|_{a=0} &= \ U_3^1(t, T_1, T_2)
\end{aligned}
$$

and hence

$$
\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} E_Q \left[Z(u) | \mathcal{F}_t \right] \ du = \ \frac{1}{T_2 - T_1} \ \left(Z(t) \cdot G_1(t, T_1, T_2) \ + \ U_3^1(t, T_1, T_2) \right), \tag{5.14}
$$

which completes our proof for $n = 1$. □

Note that our formula for the forward price only depends on deterministic functions and the processes X, Y and Z at time t, which makes it possible to find explicit prices based on the parameters of our model. In the special case of only considering Z we are back to results of [7].

In addition to the previous formula we may derive an alternative one which is useful for more general derivative pricing. We proceed similarly to [7] in the case of classical Ornstein-Uhlenbeck processes.

Lemma 5.3. *Let S be the spot price process defined in (3.2) with building blocks $\widetilde{S}_i$ for $i = 1, \ldots, n$ given in (3.1) with $a_i = b_i$. Assume that the driving processes A_i are of the form (5.9). Assume that the risk neutral measure is given by 5.2, then we obtain for the forward price at $t > 0$ for the delivery of one MWh electricity in the time interval $[T_1, T_2]$ with $T_2 > T_1 > t \geq 0$:*

$$
F(t, T_1, T_2) = \ \frac{1}{T_2 - T_1} \sum_{i=1}^n c_i \ \left(U^i(T_1, T_2) + \int_{-\infty}^t \Sigma_i(s, T_1, T_2) \ d\widetilde{A}(s) \right)
$$

with

$$\Sigma_i(s,T_1,T_2) := \ \omega_1^i E_i(t,T_1,T_2) \ + \ \omega_2^i D_i(t,T_1,T_2) \ + \ \omega_3^i G_i(t,T_1,T_2)$$

and

$$U^i(T_1,T_2) := \ \omega_1^i U_1^i(T_1,T_2) \ + \ \omega_2^i U_2^i(T_1,T_2) \ + \ \omega_3^i U_2^i(T_1,T_2).$$

The functions $D_i(t,T_1,T_2), E_i(t,T_1,T_2)$ and $G_i(t,T_1,T_2)$ are defined as in Theorem 5.2. $U^i(T_1,T_2)$ may be derived from $U_j^i(t,T_1,T_2)$ defined in Theorem 5.2 by

$$U_j^i(T_1,T_2) := U_j^i(-\infty,T_1,T_2) \quad \textit{for } j=1,2,3.$$

Proof. Again we will only look at $n=1$ and spit the calculations into the terms for X, Y and Z, which my be treated similarly. Hence we will only derive the results for X.

$$\begin{aligned}
&\frac{1}{T_2-T_1}\int_{T_1}^{T_2} E_Q\left[X(u)|\mathcal{F}_t\right] du \\
&\qquad =\frac{1}{T_2-T_1}\int_{T_1}^{T_2} E_Q\left[\int_{-\infty}^{u} \sin\left(a((u-s)\right) e^{-\lambda(u-s)} dA(s)|\mathcal{F}_t\right] du \\
&\qquad = \frac{1}{T_2-T_1}\int_{T_1}^{T_2}\left(\int_{-\infty}^{u}\int_0^{\infty} z e^{\lambda(u-s)} \sin\left(a((u-s)\right) \ \widetilde{\nu}(dz,ds)\right) du \\
&\qquad\quad + \frac{1}{T_2-T_1} E_Q\left[\int_{-\infty}^{u} \sin\left(a((u-s)\right) e^{-\lambda(u-s)} d\widetilde{A}(s)|\mathcal{F}_t\right] du \\
&\qquad = \frac{1}{T_2-T_1} U_1(T_1,T_2) \\
&\qquad\quad + \frac{1}{T_2-T_1}\int_{T_1}^{T_2}\int_{\infty}^{t} \sin\left(a((u-s)\right) e^{-\lambda(u-s)} d\widetilde{A}(s) \ du \\
&\qquad =\frac{1}{T_2-T_1}\left(U_1(T_1,T_2) + \int_{\infty}^{t} E_i(t,T_1,T_2) \ d\widetilde{A}(s)\right)
\end{aligned}$$

□

Finally we may use this formula to derive prices for options on futures. More precisely we will derive a formula for the price of a put option on the forward with payoff function $g \in L^1(\mathbb{R})$ and maturity T. Denote by $p(t,T,T_1,T_2)$ the price of such a put option on a forward $F(t,T_1,T_2)$ defined in (5.7) with $0 \le t < T < T_1 \le T_2$, then the following pricing formula holds

$$p(t,T,T_1,T_2) = \ e^{-r(T-t)} \ E_Q\left[g(F(T,T_1,T_2))|\mathcal{F}_t\right], \tag{5.15}$$

cf. e.g. [7]. Here Q denotes a risk neutral measure, e.g. given by Theorem 5.1 and $r > 0$ denotes the interest rate of the risk free bond B.

Based on classical Fourier methods we can evaluate this formula along the lines of [7]. Let

$$\Psi_i^{t,T}(\theta(\cdot)) := \ln\left(E_Q\left[\exp\left(\mathrm{i}\int_t^T \theta(s)dA_i(s)\right)\right]\right).$$

denote the cumulant function of the deterministic function $\theta(\cdot)$. In our case of an additive process A we obtain by (5.9)

$$\Psi_i^{t,T}(\theta(\cdot)) = \int_t^T \int_0^\infty \left(e^{\mathrm{i}\theta(s)z} - 1\right)\widetilde{\nu_i}(ds,dz).$$

Now we can state the theorem providing the pricing formula for the put option.

Theorem 5.2. *For $0 \le t < T < T_1 \le T_2$ the price of a put option based on a forward F (5.7) with payoff function g, such that $g(F(T,T_1,T_2)) \in L^1(Q)$ is given by*

$$p(t,T,T_1,T_2) = e^{-r(T-t)} \left(g \star \Phi^{t,T}\right)(F(t,T_1,T_2)).$$

Here $\star$ denotes the convolution and $\Phi^{t,T}$ is given via its characteristic function

$$\widehat{\Phi}^{t,T}(y) := \exp\left(\frac{1}{T_2 - T_1}\sum_{i=1}^n c_i\Psi_i^{t,T}\left(y\Sigma_i\left(\cdot, T_1, T_2\right)\right)\right),$$

where Σ_i is defined in Lemma 5.3.

Proof. We proceed similarly to [7]. Applying the inversion formula of characteristic functions, we obtain

$$g(x) = \frac{1}{2\pi}\int \widehat{g}(y)e^{\mathrm{i}yx}\,dy. \tag{5.16}$$

This yields

$$
\begin{aligned}
&E_Q[g(F(t,T_1,T_2))|\mathcal{F}_t] \\
&= \frac{1}{2\pi}\int \ \widehat{g}(y) \ \ E_Q\left[e^{\mathrm{i}yF(T,T_1,T_2)}|\mathcal{F}_t\right] \ dy \\
&= \frac{1}{2\pi}\int \ \widehat{g}(y) exp\left(\frac{\mathrm{i}y}{T_2-T_1}\sum_{i=1}^{n} c_i U^i(T_1,T_2)\right) \\
&\qquad \cdot E_Q\left[\exp\left(\frac{\mathrm{i}y}{T_2-T_1}\sum_{i=1}^{n} c_i \int_{-\infty}^{t} \Sigma_i(s,T_1,T_2)d\widetilde{A}(s)\right)|\mathcal{F}_t\right] \\
&\qquad \cdot E_Q\left[\exp\left(\frac{\mathrm{i}y}{T_2-T_1}\sum_{i=1}^{n} c_i \int_{t}^{T} \Sigma_i(s,T_1,T_2)d\widetilde{A}(s)\right)|\mathcal{F}_t\right] \ dy \\
&= \frac{1}{2\pi}\int \ \widehat{g}(y)\exp\left(\frac{\mathrm{i}y}{T_2-T_1}\sum_{i=1}^{n} c_i \left(U^i(T_1,T_2)+\int_{-\infty}^{t} \Sigma_i(s,T_1,T_2)d\widetilde{A}_i(s)\right)\right) \\
&\qquad \cdot E_Q\left[exp\left(\frac{\mathrm{i}y}{T_2-T_1}\sum_{i=1}^{n} c_i \int_{t}^{T} \Sigma_i(s,T_1,T_2)d\widetilde{A}(s)\right)|\mathcal{F}_t\right] \ dy \\
&= \frac{1}{2\pi}\int \ \widehat{g}(y)\exp\left(\mathrm{i}y \ F(t,T_1,T_2)\right) \\
&\qquad \cdot E_Q\left[\exp\left(\left(\frac{\mathrm{i}y}{T_2-T_1}\sum_{i=1}^{n} c_i \int_{t}^{T} \Sigma_i(s,T_1,T_2)d\widetilde{A}(s)\right)\right)|\mathcal{F}_t\right] \ dy.
\end{aligned}
$$

Since $\widetilde{A}$ possesses independent increments, we obtain

$$
\begin{aligned}
&E_Q[g(F(t,T_1,T_2))|\mathcal{F}_t] \\
&= \frac{1}{2\pi}\int \ \widehat{g}(y)\cdot \exp\left(\mathrm{i}y \ F(t,T_1,T_2)\right) \\
&\qquad \cdot E_Q\left[exp\left(\mathrm{i}y\left(\frac{1}{T_2-T_1}\sum_{i=1}^{n} c_i \int_{t}^{T} \Sigma_i(s,T_1,T_2)d\widetilde{A}(s)\right)\right)\right] \ dy \\
&= \frac{1}{2\pi}\int \ \widehat{g}(y)\cdot \exp\left(\mathrm{i}y \ F(t,T_1,T_2)+\frac{1}{T_2-T_1}\sum_{i=1}^{n} \ c_i\Psi_i^{t,T}\left(y\Sigma_i\left(\cdot,T_1,T_2\right)\right)\right) \ dy.
\end{aligned}
$$

Finally (5.16) yields the desired result. □

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