Construction of Nonnegatively Curved Invariant Metrics on Homogeneous Disc Bundles

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Chapter 0

Introduction

Riemannian manifolds[1] with positive or more generally nonnegative curvature[2] have been of interest since the very beginnings of Riemannian geometry.
One class of such manifolds is constituted by compact Lie groups G which always admit a biinvariant metric g. These are metrics for which left and right translations are isometries and which are nonnegatively curved because their curvature is given as

$$\frac{1}{4}\left\|[X,Y]\right\|_g \qquad \text{for the plane spanned by an orthonormal pair } X \text{ and } Y$$

w.r.t. to the Levi-Civita connection on G.
By a theorem of O'Neill it is known that Riemannian submersions do not decrease curvature. This theorem can is used to construct manifolds with nonnegative curvature out of ones which are known to admit a nonnegatively curved metric. For instance let (M, g_M) be a nonnegatively curved Riemannian manifold. Consider an isometric and properly discontinuous action by a compact Lie group G on M. Then by O'Neills Theorem it follows that the metric $\tilde{g}$ on M/G for which the quotient map $(M, g_M) \to (M/G, \tilde{g})$ becomes a Riemannian submersion is nonnegatively curved. For example let $K \subset G$ be compact Lie groups with K acting by group multiplication on G. Then a metric on G/K induced by a biinvariant metric on G has nonnegative curvature since a biinvarinat metric has nonnegative curvature as observed above. Such a submersion metric is called normal homogeneous metric.
Apart from the method of using Riemannian submersion to obtain manifolds with nonnegative curvature out of given ones, one can consider the product of two nonnegatively curved manifolds. Furthermore we can glue two nonnegatively curved manifolds with non-

[1]manifolds are assumed to be connected in this work, except for Lie groups
[2]throughout the work the word "curvature" refers to the sectional curvature

negative curvature along their totally geodesic common boundary given that the metrics near boundaries are isometric to obtain a new manifold with nonnegative curvature.
Cheeger used a combination of these three methods to construct new examples of nonnegatively curved manifolds in [C]. Namely he showed that the connected sum of two compact symmetric spaces of rank one admits metrics wih nonnegative curvature.
Before going on to another application of Cheeger's method we describe the notions of homogeneous disc bundles and of a collar metric on these bundles. A homogeneous disc bundle is a bundle of the form

$$G \times_K D \to G/K\,,$$

where D is a disc in some vector space V whose boundary is the sphere S_R^V and the subgroup $K \subset G$ acts transitively on this sphere. The action of K on $G \times D$ is given by the free diagonal action

$$k * (g,p) = (gk^{-1}, k \cdot p)\,.$$

A metric on a homogeneous disc bundle is called collar metric if a neighborhood of the boundary $\partial(G \times_K D) \cong G/H$, where $H \subset K$ denotes the isotropy group of some point in $G \times_K D \setminus \{0\}$, is G-equivariantly isometric to $\big((R-\varepsilon, R) \times G/H, dr^2 + g\big)$ for some $\varepsilon > 0$, where g is an invariant metric on G/H. If g is a normal homogeneous metric we have a normal homogeneous collar metric.
In [GZ1] submersion metrics on homogeneous disc bundles are discussed. Taking a nonnegatively curved left-invariant metric on G which is also K-right-invariant and a nonnegatively curved K-invariant metric on D yields a nonnegatively curved G - invariant metric on $G \times_K D$. In [GZ1] it is shown that if the rank ($\dim_{\mathbb{R}} V$) of the disc bundle is at most 2 it is possible to carry out the construction described above in such a way that the metric is a normal homogeneous collar metric.
Now every closed cohomogeneity one manifold M with two nonprincipal orbits can be described by two homogeneous disc bundles which are glued along their common boundary which is a principal orbit G/H, i.e.

$$M \cong \big(G \times_{K_+} D^+\big) \cup_{G/H} \big(G \times_{K_-} D^-\big)\,.$$

If the codimension of the nonprincipal orbits is at most 2 we can make the above described construction on each half to obtain a G-invariant metric on M. This result led to new examples in [GZ1] such as that every principal $SO(k)$-bundle over S^4 admits an invariant metric with nonnegegative curvature since these bundles carry a cohomogeneity one action by $SO(3) \times SO(k)$. Other examples are the sphere bundles over S^4 and resulting from that 10 out of the 14 (unoriented) exotic 7-spheres.

In that paper it was conjectured that every cohomogeneity one manifold supports an invariant metric with nonnegative curvature, but in [GVWZ] the authors showed that when the ranks of the two halves of a cohomogeneity one manifold is given by a pair of integers (ℓ_1, ℓ_2) with $\ell_1, \ell_2 \geq 2$ and $(\ell_1, \ell_2) \neq (2, 2)$, one can find an infinite family of cohomogeneity one G-manifolds that do not admit a G-invariant metric with nonnegative curvature.
In [STa2] homogeneous disc bundles admitting normal homogeneous collar metrics have been extensively studied. The authors showed that if a normal homogeneous collar metric exists then $G \times_K D$ is either the quotient of a trivial bundle or its rank has to be in $\{2, 3, 4, 6, 8\}$. Furthermore they gave a complete classification for such bundles in the rank 6 and 8 case and a partial one in the rank 3 case. Let Q denote a Ad_G-invariant inner product on $\mathfrak{g}$. Denoting by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{h} \subset \mathfrak{k}$ and by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{h} \subset \mathfrak{g}$ where $\mathfrak{h}$ resp. $\mathfrak{k}$ resp. $\mathfrak{g}$ denote the Lie algebras of the Lie groups H resp. K resp. G given as above the authors showed that a normal homogeneous collar metric exists if there exists a $C > 0$ such that for all $X, Y \in \mathfrak{p}$ we have

$$\|X_{\mathfrak{m}} \wedge Y_{\mathfrak{m}}\| \leq C \, \|[X, Y]\| \,, \tag{1}$$

where subscripted $\mathfrak{m}$'s indicate the projection onto $\mathfrak{m}$.
Denote the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ by $\mathfrak{s}$ and consider the left-invariant metric g_ε on G induced by the inner product

$$Q_\varepsilon|_{\mathfrak{k}} := (1 + \varepsilon) Q|_{\mathfrak{k}}, \quad Q_\varepsilon|_{\mathfrak{s}} := Q|_{\mathfrak{s}} \,.$$

Condition (1) ensures by a result in [STa2] that for sufficiently small ε the metric g_ε has nonnegative curvature for planes which are contained in $\mathfrak{p}$. By the fact that the portion of the horizontal lifts of planes in the base manifold lying in the first component of the tangent space to $G \times_K D$ is in $\mathfrak{p}$ we take the metric g_ε and a nonnegatively curved K-invariant metric on D to yield a nonnegatively curved quotient metric on $G \times_K D$.
Furthermore in the same paper the authors proved that (1) is almost necessary for the existence of a nonnegatively curved normal homogeneous collar metric. In fact $\mathfrak{m}$ replaced by $\mathfrak{m}_1$ in condition (1), for an irreducible subspace $\mathfrak{m}_1 \subset \mathfrak{m}$ is proven to be a necessary condition.

In this thesis we are going to analyze under which conditions metrics with prescribed properties can be constructed on homogeneous disc bundles. In particular we point out the following

Theorem 5.13. *Let $K \subset O(n+1)$ be a Lie subgroup which acts transitively on $S^n \subset \mathbb{R}^{n+1}$. Let g_1 be a K-invariant metric on S^n with positive curvature and let $r(x) := \|x\|$ be the radius function on $\mathbb{R}^{n+1}$.*
Then there exists a K-invariant metric g on the unit ball $B_1(0) \subset \mathbb{R}^{n+1}$ with positive curvature and an $\varepsilon > 0$ such that on $r^{-1}((1-\varepsilon, 1))$ we have $g = dr^2 + \eta(r)^2 g_1$ where $\eta : (1-\varepsilon, 1) \to \mathbb{R}$ satisfies $\eta, \eta' > 0$.

This generalizes Theorem 5.1 in [STu], where this theorem was shown for the case when g_1 is a normal homogeneous metric.

In addition we are going to show

Theorem 6.5. *Suppose that $\mathfrak{m}$ decomposes irreducibly as $\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_r$ w.r.t. the action of H on $\mathfrak{m}$ via the adjoint representation and that we have a chain of Lie groups $H \subset K_1 \subset \ldots \subset K_r = K$ such that the Lie algebra $\mathfrak{k}_i$ of K_i is given as $\mathfrak{k}_i = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_i$.*
Then there exists a nonnegatively curved invariant collar metric on the homogeneous disc bundle $G \times_K D$ with totally geodesic fibers and boundary metric g if there exists a constant $C > 0$ such that

$$R^g(X, Y; Y, X) \geq C\big(\|X_{\mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_{r-1}} \wedge Y_{\mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_{r-1}}\|^2 + \|[X, Y]_{\mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_{r-1}}\|^2\big).$$

Here R^g denotes the curvature tensor of the metric g where g is assumed to be given on $\mathfrak{p}$ as

$$(g)_{[eH]}|_{\mathfrak{m}_i} = f_i^2 \,\mathrm{id}|_{\mathfrak{m}_i}, \qquad (g)_{[eH]}|_{\mathfrak{s}} = \mathrm{id}|_{\mathfrak{s}},$$

with $0 < f_i^2 < 1$.

Here the obstruction for the functions f_i is in fact not restrictive in the sense that otherwise we obtain the same rigidity as in [STa2] where normal homogeneous collar metrics were analyzed as stated above (cf. Corollary 6.3).

A more detailed summary of the contents of this work is given in the following outline of the work.

Outline of the work

In the first chapter we give a brief introduction to invariant metrics on homogeneous manifolds along with the computation of the curvature of such metrics in the special case when an underlying chain of Lie groups is given.

Chapter 2 is devoted to a construction method mentioned above which goes back to Cheeger. After clarifying how the construction is carried out we compute the curvature of Cheeger metrics explicitly.

In Chapter 3 we revisit Cheeger's construction for the special case that it provides a deformation of a G-invariant metric on a manifold M on which a compact Lie group G acts by isometries.
Furthermore we give an equivalent condition when there exists a Cheeger deformation for a G-invariant metric on a homogeneous space G/H such that the deformed metric is nonegatively curved.

Chapter 4 deals with invariant metrics on spheres. At the end of Chapter 4 we use the results from Chapter 3 to show that every positively curved invariant metric g_1 on the sphere can be joined with the round metric by a diagonal path of positively curved invariant metrics on spheres.

In the following Chapter 5 we discuss the curvature of generalized warped product metrics. Here a metric of the form $dt^2 + g_t$ on a product $I \times G/H$ where I is an interval and g_t is a family of G-invariant metrics in G/H is called a generalized warped product metric.
We determine conditions under which there exist a reparametrization for a generalized warped product metric such that the reparametrized metric possesses nonnegative resp. positive curvature.
We use these results on the one hand to show Theorem 5.13 stated above. On the other hand we use the results on reparametrizations to prove Theorem 6.5 in Chapter 6. There we also analyze to what extent metrics on the boundary of a disc in Cheeger's construction can even have negative curvature when taking a biinvariant metric on G and demanding for the quotient metric to be nonnegatively curved.

Finally in Chapter 7 we put the obtained results into the context of cohomogeneity one manifolds.

Chapter 1

Preliminaries

1.1 Invariant Metrics on Homogeneous Manifolds

We will give a brief introduction to invariant metrics on homogeneous manifolds. A more detailed treatment can be found in standard textbooks such as [Be], [CE] or [Pe].
A manifold M is called homogeneous if there is a Lie group K acting transitively on M. Here we will consider left actions. Let $H_p \subset K$ be the isotropy subgroup at some point $p \in M$, i. e. the elements of K fixing the point p. Since H_p is closed it is itself a Lie group. The set $K/H_p := \{kH_p : k \in K\}$ of left cosets modulo H_p inherits a unique manifold structure from K via the canonical projection $\pi : K \to K/H_p$ by the requirement that π is smooth and that $(\pi, K, K/H_p)$ is a principal H_p-bundle. Then the map $\alpha : K/H_p \to M$ with $kH_p \mapsto k{*}p$ is a diffeomorphism with respect to the unique smooth manifold structure on K/H_p. Since the isotropy groups at different points are conjugate in K, this construction does not depend on the point $p \in M$ and we henceforth write $K/H \cong M$
According to the identification of M with K/H the left action of K on M is carried into a transitive left action of K on K/H which can be described by diffeomorphisms.
More precisely we have a map

$$\begin{aligned} \theta : \; K &\to \mathrm{Diff}(K/H) \\ k &\mapsto \theta_k \end{aligned} \quad , \tag{1.1}$$

with

$$\begin{aligned} \theta_k : \; K/H &\to K/H \\ [k_1 H] &\mapsto [k k_1 H] \, . \end{aligned} \tag{1.2}$$

If $k_1 = e$ and $k \in H$ in (1.2) we obtain that $d\theta_k$ is an automorphism of $T_{[eH]}K/H$ which leads to the isotropy representation

$$\rho : H \to \mathrm{Aut}\big(T_{[eH]}K/H\big)\,.$$

A metric $\langle\, ,\,\rangle$ on K/H is called K-invariant or K-homogeneous if

$$\langle d\theta_k v, d\theta_k w\rangle_{[kk_1H]} = \langle v, w\rangle_{[k_1H]} \quad \forall\, v, w \in T_{[k_1H]}K/H\,, \tag{1.3}$$

i.e. K is acting by isometries. If H is the trivial group this metric is nothing else than a left invariant metric on K.
Before dealing with other characterizations and the existence of invariant metrics on homogeneous spaces we note the following.
Let $\langle\, ,\,\rangle$ be a K-invariant metric on K/H and denote by K^* the isometry group of $(K/H, \langle\, ,\,\rangle)$ which by hypothesis acts transitively. Then K^* acts effectively on K/H since it is a subgroup of $\mathrm{Diff}(K/H)$. Let H^* denote the isotropy group of this action so that especially $K^*/H^* \cong K/H$. Consider the isotropy representation

$$\rho^* : H^* \to O\big(T_{[eH]}K/H\big)\,,$$

where we use the identification $T_{[eH^*]}K^*/H^* \cong T_{[eH]}K/H$. The image of H^* under ρ^* is contained $O\big(T_{[eH]}K/H\big) \subset \mathrm{Aut}\big(T_{[eH]}K/H\big)$ since $H^* \subset K^*$ also acts by isometries on K/H. Moreover an isometry on a connected manifold is determined by giving its differential at some point. Therefore ρ^* is injective and we can identify H^* with a subgroup of $O\big(T_{[eH]}K/H\big)$. By the Myers-Steenrod Theorem (cf. [Be], Theorem 1.77), K^* is in fact a Lie group and moreover the isotropy subgroup can be identified with a closed subgroup of $O\big(T_{[eH]}K/H\big)$ and therefore is compact.
Since K also acts by isometries, the image of the map θ in (1.1) is contained in K^*. If θ is injective we can identify K with a subgroup of K^* and the action of K on K/H is also effective.
If θ is not injective, K does not act effectively. Then we make the following construction. Let $H_0 = \ker(\theta) = \ker(\rho)$. Then H_0 is a normal subgroup of K which is contained in H. So we obtain Lie groups

$$\hat{K} := K/H_0 \quad \text{and} \quad \hat{H} := H/H_0\,.$$

It follows that $\hat{K}/\hat{H}$ is diffeomorphic to K/H and that the canonical action of $\hat{K}$ on $\hat{K}/\hat{H}$ is effective. Since every non-effective action can be made effective in this manner we assume from now on w.l.o.g. that the actions we consider are effective and that we can identify K

with a subgroup of K^*.
By abuse of notation we assume from now on that $K = \hat{K} \subset K^*$.
For getting a K-invariant metric on K/H we propagate an inner product on $T_{[eH]}K/H$ such that (1.3) is satisfied. That is we define

$$\langle v, w \rangle_{[kH]} := \left\langle d\theta_k^{-1} v, d\theta_k^{-1} w \right\rangle_{[eH]} \quad \forall\, v, w \in T_{[kH]}K/H\,. \tag{1.4}$$

It follows for $k \in H$ that $\langle\ ,\ \rangle_{[eH]}$ must be invariant under the isotropy representation. Conversely if we have an inner product on $T_{[eH]}K/H$ which is invariant under the isotropy representation we get a K-invariant metric on K/H by defining an inner product according to (1.4) on $T_{[kH]}K/H$.
For another characterization of K-invariant metrics on K/H we make use of the canonical identification of $T_{[eH]}K/H$ with $\mathfrak{k}\,/\,\mathfrak{h}$, where $\mathfrak{h}$ and $\mathfrak{k}$ denote the Lie algebras of H and K which in turn enables us to interpret the differential of π at the identity as the natural projection $d\pi : \mathfrak{k} \to \mathfrak{k}\,/\,\mathfrak{h}$. Since $\mathfrak{h}$ is Ad_H-invariant so is $\mathfrak{k}\,/\,\mathfrak{h}$ and therefore H acts naturally on $\mathfrak{k}\,/\,\mathfrak{h}$ via the adjoint representation. Note that for all $v \in \mathfrak{k}$ and $h \in H$ we have

$$\begin{aligned} \theta_h(\pi(e^{tv})) &= \theta_h([e^{tv}H]) = \theta_h(\theta_{e^{tv}}[eH]) \\ &= \theta_{he^{tv}}([h^{-1}H]) = \theta_{he^{tv}h^{-1}}([eH]) \\ &= \pi(he^{tv}h^{-1})\,. \end{aligned}$$

Differentiating this yields

$$d\theta_h(d\pi(v)) = d\pi(\mathrm{Ad}_h(v)) \quad \forall\, v \in \mathfrak{k},\ h \in H\,. \tag{1.5}$$

Now, a K-invariant metric on K/H is determined by an inner product invariant under the isotropy representation at $[eH]$ as observed above. In view of (1.5) this leads to an inner product on $\mathfrak{k}\,/\,\mathfrak{h}$ invariant under the action of Ad_H. Conversely suppose that we have an Ad_H-invariant inner product on $\mathfrak{k}\,/\,\mathfrak{h}$ which leads to an inner product in $T_{[eH]}K/H$ invariant under the isotropy representation. Then propagating this inner product as in (1.4) gives rise to a well-defined K-invariant metric on K/H.
For being able to formulate some of the upcoming facts more compactly we give

Definition 1.1. A homogeneous space K/H is called reductive, if $\mathfrak{k}$ admits a decomposition $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$ such that $\mathrm{Ad}_H(\mathfrak{m}) \subset \mathfrak{m}$.

For later references we state

Proposition 1.2. *On a reductive honogeneous space K/H, K-invariant metrics on K/H are in 1-1 correspondence with Ad_H-invariant inner products on $\mathfrak{m}$.*

The proof follows from the identification of $\mathfrak{m}$ with $\mathfrak{k}\,/\,\mathfrak{h}$.
Let $(,)$ denote an Ad_H-invariant inner product on $\mathfrak{k}\,/\,\mathfrak{h}$. As Ad_H acts naturally on $\mathfrak{k}\,/\,\mathfrak{h}$ so does $\mathrm{ad}_{\mathfrak{h}}$. Then for $x, y \in \mathfrak{k}\,/\,\mathfrak{h}$ and $v \in \mathfrak{h}$ we have

$$(\mathrm{Ad}_{e^{tv}}x, \mathrm{Ad}_{e^{tv}}y) = (x, y)\,.$$

Differentiating this yields

$$(\mathrm{ad}_v x, y) + (x, \mathrm{ad}_v y) = 0\,.$$

So a Ad_H-invariant inner product on $\mathfrak{k}\,/\,\mathfrak{h}$ is ad_v skew symmetric for all $v \in \mathfrak{h}$.
Conversely suppose that we have an ad_v skew symmetric inner product on $\mathfrak{k}\,/\,\mathfrak{h}$ for all $v \in \mathfrak{h}$. Then for $x, y \in \mathfrak{k}\,/\,\mathfrak{h}$ and $v \in \mathfrak{h}$ we get

$$\begin{aligned}(\mathrm{Ad}_{e^{tv}}x, \mathrm{Ad}_{e^{tv}}y) &= (e^{\mathrm{ad}_{tv}}x, e^{\mathrm{ad}_{tv}}y) = (e^{\mathrm{ad}_{tv}}x, \sum (t^n/n!)(\mathrm{ad}_v)^n y) \\ &= \sum ((-1)^n (t^n/n!)(\mathrm{ad}_v)^n e^{\mathrm{ad}_{tv}}x, y) = (e^{\mathrm{ad}_{-tv}} e^{\mathrm{ad}_{tv}}x, y)\,.\end{aligned}$$

Since elements of the form e^{tv} for $v \in \mathfrak{h}$ generate the identity component of H we get that every ad_v skew symmetric inner product on $\mathfrak{k}\,/\,\mathfrak{h}$ is also Ad_H-invariant if H is connected.

Again for later references we state

Proposition 1.3. *Let H be connected. Then a homogeneous space K/H with $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$ is reductive if and only if $\mathrm{ad}_{\mathfrak{h}}(\mathfrak{m}) \subset \mathfrak{m}$.*

Here again the proof follows from the identification of $\mathfrak{m}$ with $\mathfrak{k}\,/\,\mathfrak{h}$.

But still the question remains when invariant metrics do exist at all. This is answered by the following

Proposition 1.4. *K/H admits a K-invariant metric if and only if $\overline{\mathrm{Ad}_H}$ is compact in* $\mathrm{GL}(\mathfrak{k})$.

Proof. Suppose that $\overline{\mathrm{Ad}_H}$ is compact in $\mathrm{GL}(\mathfrak{k})$. Let ω be a right invariant volume form on $\overline{\mathrm{Ad}_H}$, i.e. we have $R^*_{h^{-1}}\omega(h \cdot h_1) = \omega(h)$ for all $h \in \overline{\mathrm{Ad}_H}$. Such a form can be obtained from a right invariant metric on $\overline{\mathrm{Ad}_H}$. Then for an arbitrary inner product $(,)$ on $\mathfrak{k}$ we can define the inner product

$$\langle x, y\rangle := \int_{\overline{\mathrm{Ad}_H}} (\mathrm{Ad}_h(x), \mathrm{Ad}_h(y))\,\omega(h) \tag{1.6}$$

on $\mathfrak{k}$. Then $\langle\, , \,\rangle$ defines an Ad_H-invariant inner product on $\mathfrak{k}$ which can be seen by the following computation where we make use of the right invariance of ω and the fact that R_k is a diffeomorphism for all $k \in K$. For $h_1 \in H$ and $x, y \in \mathfrak{k}$ compute

$$\begin{aligned}
\langle \mathrm{Ad}_{h_1}(x), \mathrm{Ad}_{h_1}(y)\rangle &= \int_{\overline{\mathrm{Ad}_H}} (\mathrm{Ad}_h \mathrm{Ad}_{h_1}(x), \mathrm{Ad}_h \mathrm{Ad}_{h_1}(y))\, \omega(h) \\
&= \int_{\overline{\mathrm{Ad}_H}} (\mathrm{Ad}_{h\cdot h_1}(x), \mathrm{Ad}_{h\cdot h_1}(y)) R^*_{h_1^{-1}} \omega(h \cdot h_1) \\
&= \int_{\overline{\mathrm{Ad}_H}} (\mathrm{Ad}_h(x), \mathrm{Ad}_h(y))\, \omega(h) \\
&= \langle x, y\rangle \ .
\end{aligned} \tag{1.7}$$

Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{k}$ w.r.t. the metric $\langle\, , \,\rangle$. Then $\langle\, , \,\rangle|_{\mathfrak{m}}$ is Ad_H-invariant as well and by Proposition 1.2 we get the desired result.
Conversely suppose that we have a K-invariant metric $\langle , \rangle$ on K/H. Let K^* denote the isometry group of $(K/H, \langle , \rangle)$ with Lie algebra $\mathfrak{k}^*$ and let H^* denote the isotropy group of some point with Lie algebra $\mathfrak{h}^*$. We have pointed out above that H^* is compact. It follows that Ad_{H^*} is a compact subgroup of $\mathrm{GL}(\mathfrak{k}^*)$. So similar to what we have done above we can define a metric on $\mathfrak{k}^*$ which is Ad_{H^*}-invariant so its restriction to $\mathfrak{k}$ is Ad_H-invariant. Hence with respect to this metric Ad_H operates by isometries and it follows that it can be identified with a subgroup of $\mathrm{O}(\mathfrak{k})$ from which we can deduce that its closure is compact in $\mathrm{GL}(\mathfrak{k})$. □

Remark 1.5. If K is a closed subgroup of K^* then the existence of a K-invariant metric on K/H is guaranteed. Since then H is a closed subgroup of H^* and therefore compact from which the compactness of Ad_H follows and then by Proposition 1.4 we get the statement.

From now on we suppose that K is a closed subgroup of K^*.
The next Proposition links left invariant metrics on K to K-invariant metrics on K/H.

Proposition 1.6. *Let K/H be a reductive homogeneous space with $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$. Then a left invariant metric g on K induces a K-invariant metric on K/H if $g_e|_{\mathfrak{m}}$ is Ad_H-invariant. Conversely a K-invariant metric on K/H induces a left invariant metric g on K which is also right invariant when restricted to H and for which we have $\mathfrak{m} = \mathfrak{h}^{\perp}$.*

Proof. That a left invariant metric g on K induces a K-invariant metric on K/H if $g_e|_{\mathfrak{m}}$ is Ad_H-invariant follows immediately from Proposition 1.2. Conversely if we have a decomposition $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$ with $\mathrm{Ad}_H(\mathfrak{m}) \subset \mathfrak{m}$ and a K-invariant metric on K/H then again by Proposition 1.2 this metric leads to an Ad_H-invariant inner product on $\mathfrak{m}$. We extend this inner product to all of $\mathfrak{k}$ and declare $\mathfrak{h}$ and $\mathfrak{m}$ to be orthogonal and Ad_H-invariant on $\mathfrak{h}$. Then propagating this inner product leads to a left invariant metric on K which is biinvariant when restricted to H. $\square$

For the remainder of this work let K be a compact Lie group. Then the existence of a biinvariant metric on K or equivalently the existence of of a Ad_K-invariant inner product on $\mathfrak{k}$ is guaranteed. Since K is compact so is Ad_K in $O(\mathfrak{k})$. So we can define a metric on $\mathfrak{k}$ as in (1.6) with $\overline{\mathrm{Ad}_H}$ replaced by Ad_K and this metric is Ad_K-invariant. This follows with an analogeous computation as in (1.7).

Remark 1.7. A left invariant metric g on K as in Proposition 1.6 with $g(\mathfrak{h}, \mathfrak{m}) = 0$ not only induces a K-invariant metric on K/H but turns the canonical projection $\pi : K \to K/H$ into a Riemanniann submersion with fiber type H.

To point out a geometric feature of the fibration in the previous remark we need

Proposition 1.8. *Let g be a left invariant metric on K and let $X, Y, Z \in \mathfrak{k}$. Then*

$$\overline{\nabla}_X Y = \frac{1}{2}\,[X, Y] - \overline{U}(X, Y)\,, \tag{1.8}$$

where $\overline{U} : \mathfrak{k} \times \mathfrak{k} \to \mathfrak{k}$ is defined by

$$2\,g(\overline{U}(X, Y), Z) = g([Z, X], Y) + g(X, [Z, Y])$$

and $\overline{\nabla}$ is the Levi-Civita connection of the metric g.

Proof. By left invariance we have

$$\begin{aligned} 0 &= X\,g(Y, Z) = g(\,\overline{\nabla}_X Y, Z) + g(Y, \overline{\nabla}_X Z) \\ 0 &= Y\,g(X, Z) = g(\,\overline{\nabla}_Y X, Z) + g(X, \overline{\nabla}_Y Z) \\ 0 &= Z\,g(X, Y) = g(\,\overline{\nabla}_Z X, Y) + g(X, \overline{\nabla}_Z Y)\,. \end{aligned}$$

Substracting the third equation from the sum of the first two and using that the Levi-Civita connection is torsion-free we get

$$2\,g(\overline{\nabla}_X Y, Z) = g([X, Y], Z) - g(Y, [X, Z]) - g(X, [Y, Z])\,,$$

from which (1.8) follows. $\square$

This leads to the following

Remark 1.9. The fibers of the fibration in in Remark 1.7 are totally geodesic.
To show this fact, let $X, Y \in \mathfrak{h}$. Then on the one hand we have $[X, Y] \in \mathfrak{h}$ since $\mathfrak{h}$ is a Lie algebra. On the other hand we have $\overline{U}(X, Y) \in \mathfrak{h}$. For this let $Z \in \mathfrak{m}$. Then we obtain

$$2\, g(\overline{U}(X, Y), Z) = g([Z, X], Y) + g(X, [Z, Y]) = 0\,,$$

if and only if g is chosen as in Proposition 1.6. Since then we get $[Z, X] \in \mathfrak{m}$ as well as $[Z, Y] \in \mathfrak{m}$. So we get $\overline{\nabla}_X Y \in \mathfrak{h}$ for $X, Y \in \mathfrak{h}$ and the claim follows.

For later use we state

Corollary 1.10. *Let $\langle\ ,\ \rangle$ be a K-invariant metric on K/H and let $X, Y, Z \in \mathfrak{m}$. Then*

$$\nabla_X Y = \frac{1}{2}\,[X, Y]_{\mathfrak{m}} - U(X, Y)\,, \tag{1.9}$$

where $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is defined by

$$2\,\langle U(X, Y), Z\rangle = \langle [Z, X]_{\mathfrak{m}}, Y\rangle + \langle X, [Z, Y]_{\mathfrak{m}}\rangle$$

and ∇ denotes the Levi-Civita connection of the metric $\langle\ ,\ \rangle$.

Proof. Let g be left invariant metric on K which induces the metric $\langle\ ,\ \rangle$ in accordance with Proposition 1.6 and let $\overline{\nabla}$ denote the Levi-Civita connection of g. Then from the general theory of Riemannian submersions we have

$$\overline{\nabla}_{\bar{X}} \bar{Y} = \overline{\nabla_X Y} + \frac{1}{2}\,[\bar{X}, \bar{Y}]^{\mathcal{V}}\,, \tag{1.10}$$

where $X, Y \in \mathfrak{m}$ and $\bar{X}, \bar{Y}$ are the corresponding horizontal lifts and the superscript $\mathcal{V}$ denotes the projection onto the vertical space. Here the submersion is the projection map. Therefore the horizontal lifts coincide with the actual elements and the vertical space is $\mathfrak{h}$. Now using (1.8) yields (1.9). □

For another characterization of invariant metrics on homogeneous spaces let Q denote an Ad_K-invariant inner product on $\mathfrak{k}$ and let g_Q denote the induced biinvariant metric on K. With respect to Q we have a Q-orthogonal decomposition $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$. On $\mathfrak{m}$ define the inner product

$$Q_\psi(X, Y) := Q(\psi X, Y) \quad X, Y \in \mathfrak{m}\,, \tag{1.11}$$

where $\psi : \mathfrak{m} \to \mathfrak{m}$ is some Q-symmetric, positive definite linear map. It follows that Q_ψ is Ad_H-invariant if and only if ψ is Ad_H-equivariant. So by Proposition 1.2, Q_ψ induces a K-invariant metric on K/H in this case. If ψ is Ad_H-equivariant it follows especially that ψ is $\mathrm{ad}_{\mathfrak{h}}$-equivariant as well.
Let Ψ be a Q-symmetric extension of ψ to $\mathfrak{k}$ with $\Psi|_{\mathfrak{m}} = \psi$ and arbitrary on $\mathfrak{h}$. Then we especially have $\Psi(\mathfrak{h}) \subset \mathfrak{h}$. The inner product

$$Q_\Psi(X, Y) := Q(\Psi X, Y) \quad X, Y \in \mathfrak{k}$$

fulfills $Q_\Psi(\mathfrak{h}, \mathfrak{m}) = 0$ and induces a left invariant metric g_Ψ on K such that by Remark 1.7 the projection map $\pi : (K, g_\Psi) \to (K/H, g_\psi)$ is a Riemannian submersion not depending on the choice of the extension Ψ.

We make the following convention for the remainder of this work

Convention Let Q be some Ad_K-invariant inner product on $\mathfrak{k}$.
If it is not further specified we mean a homogeneous metric on K/H induced by an inner product as in (1.11) when writing g_ψ.
Similarily when writing g_Ψ for a left invariant metric on K we mean a metric induced by some Q-symmetric, positive definite map $\Psi : \mathfrak{g} \to \mathfrak{g}$.

Let g_ψ be a K-invariant metric on K/H. Since ψ is Q-symmetric we obtain a Q-orthogonal decomposition $\mathfrak{m} = \tilde{\mathfrak{m}}_1 \oplus \ldots \oplus \tilde{\mathfrak{m}}_s$ into eigenspaces of ψ. So on these eigenspaces the inner product Q_ψ inducing g_ψ must be a multiple of Q. Moreover since ψ is Ad_H-equivariant this decomposition in fact yields an Ad_H-invariant decomposition of $\mathfrak{m}$, which in turn provides a decomposition invariant under the action of the isotropy group via the adjoint representation. This leads to a decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_r$ where H acts trivially on $\mathfrak{m}_1$ and $\mathfrak{m}_2, \cdots, \mathfrak{m}_r$ are irreducible subspaces, where some eigenspaces are possibly subsumed to yield $\mathfrak{m}_1$ and others split further to yield the subspaces $\mathfrak{m}_2, \ldots \mathfrak{m}_r$. Since by our construction each $\mathfrak{m}_i$, $2 \le i \le r$, is contained in some eigenspace $\tilde{\mathfrak{m}}_j$, it follows that

$$\psi|_{\mathfrak{m}_i} = a_i^2 \, \mathrm{Id}_{\mathfrak{m}_i} \quad \text{for} \quad 2 \le i \le r \, . \tag{1.12}$$

Restricted to $\mathfrak{m}_1$ the metric can be arbirary.

Remark 1.11. If $\dim(\mathfrak{m}_1) \le 1$ the metric is determined by (1.12) for $1 \le i \le r$.

1.2 Curvature of Homogeneous Metrics

As explained in the previous chapter homogeneous metrics arise as Riemmannian submersion metrics of certain left invariant metrics by means of the projection map. For the relation between the curvature of submersion metrics and the metric on the total space we have a formula due to O'Neill

Theorem 1.12 (B. O'Neill). *Let $\pi : (B, \bar{g}) \to (M, g)$ be a Riemannian submersion, then we have*

$$R^g\big(X, Y; Y, X\big) = R^{\bar{g}}\big(\bar{X}, \bar{Y}; \bar{Y}, \bar{X}\big) + \frac{3}{4}\left\|[\bar{X}, \bar{Y}]^{\mathcal{V}}\right\|_{\bar{g}}^2, \tag{1.13}$$

where R^g denotes the curvature tensor of g, $R^{\bar{g}}$ the curvature tensor of $\bar{g}$, $\bar{X}, \bar{Y}$ are the horizontal lifts of $X, Y \in \mathfrak{X}(M)$ and the superscript $\mathcal{V}$ denotes the projection onto the vertical part.
The expression $\frac{3}{4}\left\|[\bar{X}, \bar{Y}]^{\mathcal{V}}\right\|_{\bar{g}}^2$ is called O'Neill term.

From Theorem 1.12 it follows that a homogeneous metric on K/H has nonnegative curvature if the left invariant metric on K inducing the homogeneous metric by means of the canonical projection has.

For computing the curvature of a left invariant metric g_Ψ on K we use a result of Püttmann which is stated in the next

Theorem 1.13 (cf. [Pü], p. 344). *Let g_Ψ be a left invariant metric on K. Denote the curvature tensor of g_Ψ by R^{g_Ψ}. Then we have*

$$\begin{aligned} R^{g_\Psi}\big(X, Y; Y, X\big) = \ & \frac{1}{2}\, Q\big([X, \Psi Y] + [\Psi X, Y], [X, Y]\big) \\ & - \frac{3}{4}\, Q\big(\Psi[X, Y], [X, Y]\big) \\ & + \frac{1}{4}\, Q\big([X, \Psi Y] - [\Psi X, Y], \Psi^{-1}\big([X, \Psi Y] - [\Psi X, Y]\big)\big) \\ & - Q\big([X, \Psi X], \Psi^{-1}[Y, \Psi Y]\big)\,, \end{aligned} \tag{1.14}$$

where $X, Y \in \mathfrak{k}$.

Using (1.13) and (1.14) we can give a formula for the cuvature of a homogneous metric

Corollary 1.14. *Let g_ψ be a homogeneous metric on K/H and let g_Ψ be a left invariant metric on K inducing g_ψ by means of the projection map. Then we have*

$$\begin{aligned} R^{g_\psi}\big(X, Y; Y, X\big) &= R^{g_\Psi}\big(\bar{X}, \bar{Y}; \bar{Y}, \bar{X}\big) + \frac{3}{4}\, Q\big(\Psi[\bar{X}, \bar{Y}]_{\mathfrak{h}}\,, [\bar{X}, \bar{Y}]_{\mathfrak{h}}\big) \\ &= R^{g_\Psi}\big(X, Y; Y, X\big) + \frac{3}{4}\, Q\big(\Psi[X, Y]_{\mathfrak{h}}\,, [X, Y]_{\mathfrak{h}}\big)\,, \end{aligned}$$

where $X, Y \in \mathfrak{m}$ and $\bar{X}, \bar{Y} \in \mathfrak{m}$ are the corresponding horizontal lifts for which we have $\bar{X} = X$ and $\bar{Y} = Y$ since the differential of the projection map is the identity and $\frac{3}{4} Q\big(\Psi[X,Y]_{\mathfrak{h}}, [X,Y]_{\mathfrak{h}}\big)$ is the O'Neill term because the vertical space of the projection map is $\mathfrak{h}$. Replacing $R^{g_\Psi}\big(X,Y;Y,X\big)$ with (1.14) *gives an explicit formula for $R^{g_\psi}\big(X,Y;Y,X\big)$.*

Remark 1.15. The curvature formula given in Corollary 1.13 does not depend on the choice of the extension Ψ. This can be seen as follows.
Adding the second line of (1.14) and the O'Neill term gives

$$\begin{aligned}-\frac{3}{4} Q\big(\Psi[X,Y],[X,Y]\big) + \frac{3}{4} Q\big(\Psi[X,Y]_{\mathfrak{h}}\,, [X,Y]_{\mathfrak{h}}\big) &= \frac{3}{4} Q\big(\Psi[X,Y]_{\mathfrak{m}}, [X,Y]_{\mathfrak{m}}\big)\\ &= \frac{3}{4} Q\big(\psi[X,Y]_{\mathfrak{m}}, [X,Y]_{\mathfrak{m}}\big)\,.\end{aligned}$$

Since Ψ is $\mathrm{ad}_{\mathfrak{h}}$-equivariant we obtain

$$[\Psi X, Z] = \Psi[X,Z] \quad \forall\ X \in \mathfrak{m}\,, Z \in \mathfrak{h}\ .$$

From this we deduce for all $X, Y \in \mathfrak{m}$ and $Z \in \mathfrak{h}$,

$$Q\big([\Psi X, Y], Z\big) = -\,Q\big(Y, [\Psi X, Z]\big) = -\,Q\big(Y, \Psi[X,Z]\big) = Q\big([X, \Psi Y], Z\big)\,,$$

Therefore we have

$$[\Psi X, Y]_{\mathfrak{h}} - [X, \Psi Y]_{\mathfrak{h}} = 0 \qquad \forall\ X, Y \in \mathfrak{m}$$

and

$$[\Psi X, X]_{\mathfrak{h}} = 0 \qquad \forall\ X \in \mathfrak{m}\ .$$

So the claim follows.

We compute the curvature just for a special case of homogeneous metrics which will suffice for our purpose.

Suppose that we have an underlying chain of Lie groups

$$H \subset K_1 \subset K_2 \subset \ldots \subset K_r = K$$

and the induced Q-orthogonal decomposition

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_r\,, \tag{1.15}$$

such that the Lie algebra $\mathfrak{k}_i$ of K_i is given by

$$\mathfrak{k}_i = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_i \quad \text{for } 1 \leq i \leq r\,.$$

Define the following expressions which will be used throughout the text

$$\begin{aligned} B^{ij}_k &:= [X_i, Y_j]_k + [X_j, Y_i]_k \quad \text{and} \quad B^{ij} := \sum_{0\leq k\leq r} B^{ij}_k \quad \text{for } 1 \leq i,j \leq r,\ i \neq j,\ 0 \leq k \leq r \\ B^i_k &:= [X_i, Y_i]_k \quad \text{and} \quad B^i := \sum_{0\leq k\leq r} B^i_k \quad \text{for } 1 \leq i \leq r,\ 0 \leq k \leq r\,, \end{aligned}$$

where the subscript on X resp. Y denotes from which $\mathfrak{m}_i$ it is from and the subscript on the bracket denotes the $\mathfrak{m}_k$-part of the bracket if $1 \leq k \leq r$ and the $\mathfrak{h}$-part of the bracket if $k = 0$. Due to the ad-skew-symmetry of Q we obtain immediately

$$\begin{aligned} B^{ij}_k &= 0 \quad \text{for } i < j \text{ and } j \neq k\,, \\ B^i_k &= 0 \quad \text{for } k > i\,. \end{aligned}$$

Furthermore we suppose that the metric on K/H is diagonal with respect to (1.15). We set

$$\psi|_{\mathfrak{m}_i} := a_i^2 \,\mathrm{Id} \quad \text{for} \quad 1 \leq i \leq r\,. \tag{1.16}$$

In this situation we say that the homogeneous metric g_ψ induced by ψ has the parameters $(a_1^2, a_2^2, \ldots, a_r^2)$. We are going to compute $R^{g_\psi}(\psi^{-1}X, \psi^{-1}Y; \psi^{-1}Y, \psi^{-1}X)$ rather than $R^{g_\psi}(X, Y; Y, X)$ because this will be needed later. The actual computations can be found in the appendix. Here we just give the result

Lemma 1.16. *Let a chain of Lie groups be given and let ψ be defined as in* (1.16). *Then we have*

$$\begin{aligned} & R^{g_\psi}\big(\psi^{-1}X, \psi^{-1}Y, \psi^{-1}Y, \psi^{-1}X\big) \\ = \; & \sum_{1\leq i\leq r} a_i^{-6} \big\|B^i_0\big\|^2 + \sum_{1\leq i<j\leq r} a_j^{-4} a_i^{-2} \big(3 - a_i^2 a_j^{-2}\big) Q\big(B^i_0, B^j_0\big) + \frac{1}{4}\, a_r^{-6} \Big\|B^r_r + \sum_{1\leq i<r} B^{ir}\Big\|^2 \\ & + \frac{1}{4} \sum_{1\leq k\leq r-1} a_k^{-6} \Big\|B^k_k + \sum_{1\leq i<k} B^{ik}\Big\|^2 + \frac{1}{4} \sum_{1\leq k<i\leq r} a_i^{-6} \big(4 - 3a_k^2 a_i^{-2}\big) \big\|B^i_k\big\|^2 \\ & + \frac{1}{2} \sum_{1\leq k<j\leq r} a_j^{-4} a_k^{-2} \big(3 - 2\, a_k^2 a_j^{-2}\big) Q\Big(B^j_k, B^k_k + \sum_{1\leq i<k} B^{ik}\Big) \\ & + \frac{1}{2} \sum_{1\leq k<i<j\leq r} a_j^{-4} a_i^{-2} \big(6 - 3\, a_k^2 a_i^{-2} - 2\, a_i^2 a_j^{-2}\big) Q\big(B^i_k, B^j_k\big)\,. \end{aligned}$$

Remark 1.17. For $\psi = \mathrm{Id}$ this is the well-known formula

$$R^{g_Q}\big(X,Y,Y,X\big) = \frac{1}{4}\,\big\|[X,Y]\big\|^2 + \frac{3}{4}\,\big\|[X,Y]_{\mathfrak{h}}\big\|^2$$

for the unnormalized curvature of a normal homogeneous metric.

For $r = 2$ we obtain

$$\begin{aligned}
&R^{g_\psi}\big(\psi^{-1}X,\psi^{-1}Y;\psi^{-1}Y,\psi^{-1}X\big)\\
&= a_1^{-6}\big\|B_0^1\big\|^2 + a_2^{-4}a_1^{-2}\big(3 - a_1^2a_2^{-2}\big)Q\big(B_0^1,B_0^2\big) + a_2^{-6}\big\|B_0^2\big\|^2 + \frac{1}{4}\,a_2^{-6}\big\|B_2^2 + B^{12}\big\|^2\\
&\quad \frac{1}{4}\Big(\,a_1^{-6}\big\|B_1^1\big\|^2 + 2\,a_1^{-2}a_2^{-4}\big(3 - 2\,a_1^2a_2^{-2}\big)Q\big(B_1^1,B_1^2\big) + a_2^{-6}\big(4 - 3\,a_1^2a_2^{-2}\big)\big\|B_1^2\big\|^2\Big) \qquad (1.17)\\
&= \frac{1}{4}\Big(3\,a_1^2\big\|a_1^{-4}B_0^1 + a_2^{-4}B_0^2\big\|^2 + a_1^{-6}\big\|B^1\big\|^2 + a_2^{-6}\big\|B_2^2 + B^{12}\big\|^2\\
&\qquad + 2\,a_1^{-2}a_2^{-4}\big(3 - 2\,a_1^2a_2^{-2}\big)Q\big(B^1,B^2\big) + a_2^{-6}\big(4 - 3\,a_1^2a_2^{-2}\big)\big\|B_0^2 + B_1^2\big\|^2\Big)\,.
\end{aligned}$$

Remark 1.18. The last two lines give exactly the formula of Proposition 3.1 in [STu] when twisting is used there.

For the rest of the chapter let a triple of Lie groups $H \subset K_1 \subset K$ be given. This case has been extensively studied in [STa1]. The following theorem can be found there. The proof presented there uses a power series expansion for the unnomalized curvature whereas the one presented here just uses the formula in (1.17).

Theorem 1.19 (cf.[STa1], Thm 0.1)**.** *Consider the homogeneous metric g_ψ on K/H with*

$$\psi|_{\mathfrak{m}_1} := (1+a)\,\mathrm{id} \qquad \textit{and} \qquad \psi|_{\mathfrak{m}_2} := \mathrm{id}\,.$$

Then g_ψ has nonnegative curvature for small $a > 0$ iff there exists a $C > 0$ such that

$$\big\|B_1^1\big\| \le C\,\big\|[X,Y]\big\| \qquad \forall\, X,Y \in \mathfrak{m}\,. \tag{1.18}$$

In fact if (K_1,H) is a symmetric pair, i.e. $B_1^1 = 0$ then g_ψ has nonnegative curvature for $a \le 1/3$.

Proof. For the proof we consider elements of the form

$$X = (1+a)^{-1}X_1 + X_2 \qquad \text{and} \qquad Y = (1+a)^{-1}Y_1 + Y_2\,.$$

The unnormalized curvature in (1.17) evaluated for ψX and ψY yields

$$R(X,Y;Y,X) = \underbrace{\frac{1}{(1+a)^3}\left\|B_0^1\right\|^2 + \frac{2-a}{1+a}\,Q\big(B_0^1,B_0^2\big) + \left\|B_0^2\right\|^2}_{=:I_0} + \underbrace{\frac{1}{4}\left\|B_2^2 + B^{12}\right\|^2}_{=:I_2}$$
$$+ \underbrace{\frac{1}{4}\left(\frac{1}{(1+a)^3}\left\|B_1^1\right\|^2 + 2\,\frac{1-2a}{1+a}\,Q\big(B_1^1,B_1^2\big) + (1-3a)\left\|B_1^2\right\|^2\right)}_{=:I_1}.$$

Clearly I_2 is nonnegative. For I_0 we claim that

$$I_0 - \frac{1}{8}\left\|B_1^0 + B_0^2\right\|^2 \geq 0\,. \tag{1.19}$$

For this observe that the discriminant

$$4\cdot\left(\frac{1}{(1+a)^3} - \frac{1}{8}\right)\cdot\left(1-\frac{1}{8}\right) - \left(\frac{2-a}{1+a} - \frac{1}{4}\right)^2 = -\frac{(2a-\frac{3}{2})a^2}{(1+a)^3}$$

is nonnegative if $a \leq \frac{3}{4}$. In particular I_0 is nonnegative.
By the Cauchy-Schwarz inquality we have

$$-\left\|B_1^1\right\|\left\|B_1^2\right\| \;\leq\; Q\big(B_1^1,B_1^2\big) \;\leq\; \left\|B_1^1\right\|\left\|B_1^2\right\|.$$

If $Q\big(B_1^1,B_1^2\big)$ is nonegative then it follows immediately that I_1 is nonnegative for sufficiently small a.
If

$$-\left\|B_1^1\right\|\left\|B_1^2\right\| q \;\leq\; Q\big(B_1^1,B_1^2\big) \;<0 \quad \text{with } q^2 \leq \frac{1-3a}{(1-2a)(1+a)}$$

we have

$$\frac{4(1-3a)}{(1+a)^3} - q^2\,\frac{4(1-2a)^2}{(1+a)^2} \geq 0\,.$$

So I_1 is nonnegative in this case.
The last case is

$$-\left\|B_1^1\right\|\left\|B_1^2\right\| \;\leq\; Q\big(B_1^1,B_1^2\big) \;<-\left\|B_1^1\right\|\left\|B_1^2\right\| q \quad \text{with } q^2 \leq \frac{1-3a}{(1-2a)(1+a)}\,.$$

Observe that for sufficiently small a, q is arbitrarily close to 1. This in turn implies that for sufficiently small a we have that B_1^1 is arbitrarily close to $-B_1^2$ which ensures the estimate

$$\left\|B_1^1 + B_1^2\right\| \leq \frac{1}{2C^2}\left\|B_1^1\right\|.$$

Therefore by using the assumption we can perform

$$\begin{aligned}\left\|B_1^1\right\|^2 \leq C^2\|[X,Y]\|^2 &= C^2\Big(\left\|B_0^1 + B_0^2\right\|^2 + \left\|B_1^1 + B_1^2\right\|^2 + \left\|B_2^2 + B^{12}\right\|^2\Big) \\ &\leq C^2\Big(\left\|B_0^1 + B_0^2\right\|^2 + \left\|B_2^2 + B^{12}\right\|^2\Big) + \frac{1}{2}\left\|B_1^1\right\|^2.\end{aligned} \tag{1.20}$$

Moreover it can be checked that we have the estimate

$$I_1 \geq \underbrace{\frac{1-3a-(1-2a)^2(1+a)}{4(1-3a)(1+a)^3}}_{=:g(a)}\left\|B_1^1\right\|^2.$$

Here g is a negative valued function with $g(0) = 0$. By using this, (1.19) and (1.20) we obtain

$$\begin{aligned}R(X,Y;Y,X) &\geq \frac{1}{8}\left\|B_1^0 + B_0^2\right\|^2 + g(a)\left\|B_1^1\right\|^2 + \frac{1}{4}\left\|B_2^2 + B^{12}\right\|^2 \\ &\geq \Big(\frac{1}{8} + 2\,C^2 g(a)\Big)\left\|B_1^0 + B_0^2\right\|^2 + \Big(\frac{1}{4} + 2\,C^2 g(a)\Big)\left\|B_2^2 + B^{12}\right\|^2.\end{aligned}$$

Thus choosing a small enough such that $\frac{1}{8} + 2\,C^2 g(a) \geq 0$ we obtain the desired result. The supplementary statement in the Theorem follows since I_0 and I_2 are nonnegative as mentioned above and the coefficient of $\left\|B_1^2\right\|$ is nonnegative for $a \leq \frac{1}{3}$. The remaining terms vanish since $B_1^1 = 0$ for symmetric pairs. □

As the proof suggests one can in general not hope for a metric g_Ψ on K inducing a homogeneous metric on K/H as in Theorem 1.19 fulfilling (1.18) to have nonnegative curvature as well because the O'Neill tensors contribution to the curvature is essential in the proof. In fact g_Ψ does not even have nonnegative curvature for planes contained in $\mathfrak{m}$ (cf. [S1]).

Nevertheless we have

Theorem 1.20 (cf.[STa1], Prop. 4.2)**.** *Consider the homogeneous metric g_ψ on K/H with*

$$\psi|_{\mathfrak{m}_1} \text{ arbitrary} \qquad \text{and} \qquad \psi|_{\mathfrak{m}_2} := \mathrm{id}\,.$$

Then g_ψ has nonnegative curvature for $\psi|_{\mathfrak{m}_1}$ sufficiently close to id *if there exists a $C > 0$*

such that

$$\|X_{\mathfrak{m}_1} \wedge Y_{\mathfrak{m}_1}\| \leq C\|[X,Y]\| \qquad \forall\, X,Y \in \mathfrak{m}\ . \tag{1.21}$$

In fact, if (1.21) *is fulfilled then every left invariant metric g_Ψ on K which induces g_ψ by means of the canonical projection has nonnegative curvature for all planes contained in $\mathfrak{m}$.*

Remark 1.21. Note that (1.21) implies (1.18).

The proof of Theorem 4 presented in [STa1] is carried out by using a power series expansion for the unnormalized curvature in the case when there is an inverse linear path between a normal homogeneous metric and the metric in case. Here a path is called inverse linear if the inverses of a path between metric inducing linear maps form a straight line.
The power series used there is a result of adding the O'Neill term coming from the projection map to the power series expansion of the unnormalized curvature of a left invariant metric on K first developed in [HT]. In fact the same power series was also used in [STa1] to prove Theorem 1.19.

Chapter 2

Cheeger's Construction

In the upcoming we are going to describe Cheeger's construction introduced in [C].
Let G be a Lie group and $K \subset G$ a compact Lie subgroup acting isometrically on a Riemannian manifold (M, g_M) from the left. Let G be equipped with a left invariant metric $\langle\, , \,\rangle$ which is also right invariant under K, i.e. $\langle\, , \,\rangle$ is biinvariant restricted to K. Then K acts isometrically on $G \times M$ via the diagonal action

$$k * (g, p) := (gk^{-1}, k \cdot p)\,. \tag{2.1}$$

Especially this action is free and automatically properly discontinuous since K is compact, so that the quotient $G \times_K M$ modulo this action is a manifold.
Let $\tilde{g}$ denote the inherited metric on $G \times_K M$ such that the canonical submersion

$$\pi : \big(G \times M,\, \langle\, , \,\rangle \oplus g_M\big) \to \big(G \times_K M, \tilde{g}\big) \tag{2.2}$$

becomes a Riemannian submersion. Then the metric $\tilde{g}$ is G-invariant w.r.t. the canonical action of G on $G \times_K M$ given by

$$g * [g_1, p] := [gg_1, p] \tag{2.3}$$

and the codimension of a principal orbit of this action equals the codimension of a principal orbit of the action of K on M.
Before pointing out a geometric feature of the submersion in (2.2) we need

Lemma 2.1. *Let $\pi : (M, \bar{g}) \to (B, g)$ be a Riemannian submersion and let $S \subset B$ be a submanifold. If $\bar{S} := \pi^{-1}(S) \subset M$ is totally geodesic then S is totally geodesic as well.*

Proof. Let $\overline{\nabla}$ resp. ∇ denote the Levi-Civita connection of $(M, \bar{g})$ resp. (B, g) and let

$X, Y \in TS$ with horizontal lifts $\bar{X}, \bar{Y} \in T\bar{S}$. Then by (1.10) we have

$$\overline{\nabla_X Y} = (\overline{\nabla}_{\bar{X}} \bar{Y})^{\mathcal{H}} .$$

Since $\bar{S}$ is totally geodesic it follows that $\overline{\nabla}_{\bar{X}} \bar{Y} \in T\bar{S}$. Therefore we get $\overline{\nabla_X Y} \in T\bar{S}$ and consequently $\nabla_X Y \in TS$, that is $S \subset B$ is totally geodesic. □

The previous Lemma can be used for the proof of

Proposition 2.2. *The map $(G \times_K M, \tilde{g}) \to (G/K, \langle\,,\rangle)$ given by $(g, p) \mapsto gK$ is a fiber bundle with fiber M. Moreover, it is a Riemannian submersion with totally geodesic fibers. Here by abuse of notation we denote the metric on G/K induced by $\langle\,,\rangle$ by the same symbol.*

Proof. Observe that $G \times_K M$ is the total space of the associated fiber bundle $M \hookrightarrow G \times_K M \to G/K$ to the principal K-bundle $K \hookrightarrow G \to G/K$. Furthermore with the metrics considered the fibers of $K \hookrightarrow G \to G/K$ are totally geodesic (cf. Remark 1.9). Due to the left invariance of the metric on G, the manifolds gK are totally geodesic as well. Therefore the manifolds $gK \times M \subset G \times M$ are totally geodesic since we have a product metric on $G \times M$. Moreover the preimage of a fiber of $M \hookrightarrow G \times_K M \to G/K$ under π can be identified with a submanifold of the form $gK \times M \subset G \times M$. Therefore the fibers are totally geodesic by Lemma 2.1. □

A special situation in the above construction comes into light if $M = V$ is a vector space of dimension $n+1$ and K acts transitively on the unit sphere S_1^n in V by means of a representation $K \to O(V)$. In what follows we assume by abuse of language that K itself acts on V. It is no obstruction to assume that K acts by orthogonal transformations since in [MS] it is shown that transitive actions on spheres are by linear transformations. A vector bundle of the form

$$T := G \times_K V \to G/K \tag{2.4}$$

with the properties described above is called a homogeneous vector bundle. As we assume K to act transitively on the unit sphere in V it leaves all spheres centered at the origin invariant.

Furthermore the norm function $r_V : V \to \mathbb{R}^+$, $v \mapsto \|v\|$ is K-invariant and hence induces a well-defined function $r_T : T \to \mathbb{R}^+$, $K(g, v) \mapsto \|v\|$. For $R \in \mathbb{R}^+$ define T_R by

$$T \supset T_R := G \times_K \overline{B_R(0)} = r_T^{-1}([0, R]) ,$$

where $\overline{B_R(0)}$ denotes the closed ball of radius R in V. T in (2.4) replaced by T_R is called homogeneous disc bundle. The level sets of r_T are precisely the G-orbits of T which in

view of (2.3) is clear.
We write the sphere as a homogeneous space as K/H where $H \subset K$ is the stabilizer of some point $p \in S^n$. By the map $G/H \to G \times_K K/H$, $gH \mapsto [g, eH]$ with inverse $G \times_K K/H \to G/H$, $[g, kH] \mapsto gkH$ we have off the 0-section an identification of the G-Orbits of T with G/H.
This identification map associates to the isometries in (2.3) the usual action map of G on G/H by left translations which are therefore isometries as well. That is, the metric on G/H is homogeneous.
Moreover we get that $G \times_K (\overline{B_R(0)} \setminus \{0\})$ is G-equivariantly diffeomorphic to $(0, R] \times G/H$ w.r.t. to these actions.
From now on let G be a compact Lie group so that the existence of a biinvariant metric is guaranteed. Let Q be an Ad_G-invariant inner product on the Liealgebra $\mathfrak{g}$ of G and denote the induced biinvariant metric by g_Q. We fix a Q-orthogonal decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{m} \oplus \mathfrak{s} = \mathfrak{k} \oplus \mathfrak{s},$$

where $\mathfrak{h}$ resp. $\mathfrak{k}$ are the Lie algebras of H resp. K.
On $V = [0, \infty) \times K/H$ we consider a metric which in polar coordinates can be written in the form

$$g_V = dr^2 + g_{\psi(r)}. \tag{2.5}$$

We call a metric of this form a generalized warped product metric. The $g_{\psi(r)}$ constitute a one-parameter-family of K-invariant metrics on $K/H \cong S^n$ which arise by propagating inner products $Q_{\psi(r)}(X, Y) := Q(\psi(r)X, Y)$ on $\mathfrak{m}$ according to (1.4), where $\psi : \mathfrak{m} \to \mathfrak{m}$ are Q-symmetric, positive definite, Ad_H-equivariant maps. As we have pointed out in the previous chapter there is a one-to-one correspondence between K-invariant metrics on $K/H \cong S^n$ and these maps $\psi(r)$. In addition we demand for ψ_r to depend smoothly on r and to be of the form $dr^2 + f(r)^2 g_{can}$ on $[0, \varepsilon) \times K/H$ for some $\varepsilon > 0$ and some smooth function $f : [0, \varepsilon) \to \mathbb{R}$ where g_{can} denotes the round metric on the sphere. Then the metric in (2.5) is smooth iff $f(0) = 0$ and $f'(0) = 1$ (see [GZ2]). In fact the submersion metric on T_R is smooth if the metric in (2.5) is smooth.
Furthermore we identify $\mathfrak{m}$ with T_pS^n via action fields,

$$\mathfrak{m} \ni X \mapsto X^*(p) = \frac{d}{dt}\Big|_{t=0} \exp(tX)(p) \in T_pS^n.$$

Before stating the next lemma we underline that G-invariant metrics on the G-orbits of T which are diffeomorphic to G/H are in one-to-one correspondence with Q-symmetric, positive definite, Ad_H-equivariant maps $\phi : \mathfrak{p} \to \mathfrak{p}$.

Lemma 2.3. *Let $T_R \to G/K$ be a homogeneous disc bundle. Let g_φ be a left invariant metric on G which is induced by a Q-symmetric, positive definite,* Ad_K*-equivariant map $\varphi : \mathfrak{g} \to \mathfrak{g}$. Furthermore let g_V be a K-invariant metric on V as in* (2.5) *and let $\varphi|_\mathfrak{m} \circ \psi(t) = \psi(t) \circ \varphi|_\mathfrak{m}$ for all $t \in [0, R]$. Then the metric on the G-orbit $r_T^{-1}(t_0)$ induced by means of a Riemannian submersion*

$$\pi : (G \times (\overline{B_R(0)} \setminus \{0\}), g_\varphi \oplus (dt^2 + g_{\psi(t)})) \to ((0, R] \times G/H, dt^2 + g_{\phi(t)})$$

is induced by the map $\phi : \mathfrak{p} \to \mathfrak{p}$ given by

$$\phi(t_0)|_\mathfrak{m} = \varphi|_\mathfrak{m}\, \psi(t_0)\, (\varphi|_\mathfrak{m} + \psi(t_0))^{-1} \,, \quad \textit{and} \quad \phi(t_0)|_\mathfrak{s} = \varphi|_\mathfrak{s} \,,$$

and we have

$$g_{\phi(t_0)}(\mathfrak{m}, \mathfrak{s}) = 0 \,.$$

Proof. We decompose $X \in \mathfrak{g}$ as $X = X_\mathfrak{h} + X_\mathfrak{m} + X_\mathfrak{s}$ where the subscripts denote the projection onto the corresponding parts. To determine the vertical space $\mathcal{V}$, that is the tangent space to the fiber at the point $(g, p) \in G \times S^n_{t_0}$ we have to compute for $X \in \mathfrak{k}$

$$\begin{aligned} \frac{d}{ds}\Big|_{s=0} \exp(sX)(g, p) &= \Big(\frac{d}{ds}\Big|_{s=0} g \exp(-sX), \frac{d}{ds}\Big|_{s=0} \big(\exp(sX)p\big) \Big) \\ &= \Big(\frac{d}{ds}\Big|_{s=0} L_g(\exp(-sX)), X^*_\mathfrak{m} \Big) = \big(-dL_g(X), X^*_\mathfrak{m} \big) = \big(-X_g, X^*_\mathfrak{m} \big) \,, \end{aligned}$$

where $X^*_\mathfrak{m}$ denotes the action field of $X_\mathfrak{m} \in \mathfrak{m}$. So

$$\mathcal{V}_{(e,p)} = \{(X_\mathfrak{h}, 0) \,|\, X_\mathfrak{h} \in \mathfrak{h}\} \oplus \{(-X_\mathfrak{m}, X^*_\mathfrak{m}) \,|\, X_\mathfrak{m} \in \mathfrak{m}\} \,.$$

To determine the horizontal space $\mathcal{H}_{(e,p)}$ let $(Y, V) \in T_{(e,p)}\big(G \times S^n_{t_0}\big)$ and compute

$$\begin{aligned} &\big(g_\varphi + g_{\psi(t_0)}\big)\big((-X, X^*_\mathfrak{m}), (Y, V)\big) \\ &= g_\varphi(-X_\mathfrak{h}, Y_\mathfrak{h}) + g_\varphi(-X_\mathfrak{m}, Y_\mathfrak{m}) + g_\varphi(0, Y_s) + g_{\psi(t_0)}(X^*_\mathfrak{m}, V) \\ &= Q(-\varphi|_\mathfrak{h} X_\mathfrak{h}, Y_\mathfrak{h}) + Q(0, \varphi|_\mathfrak{s} Y_s) + Q(X_\mathfrak{m}, \psi(t_0)V - \varphi|_\mathfrak{m} Y_\mathfrak{m}) \,. \end{aligned}$$

So we obtain

$$\mathcal{H}_{(e,p)} = \{(X_\mathfrak{s}, 0) \,|\, X_\mathfrak{s} \in \mathfrak{s}\} \oplus \{(\psi(t_0)X_\mathfrak{m}, (\varphi|_\mathfrak{m} X_\mathfrak{m})^*) \,|\, X_\mathfrak{m} \in \mathfrak{m}\} \,.$$

In order to give the horizontal lift of a tangent vector $X = X_\mathfrak{m} + X_\mathfrak{s} \in \mathfrak{m} \oplus \mathfrak{s} = \mathfrak{p} \cong$

$T_{[eH]}G/H$ we observe that

$$d\pi((X_{\mathfrak{s}},0)) = X_{\mathfrak{s}}, \quad \text{and} \quad d\pi((\psi(t_0)X_{\mathfrak{m}}, (\varphi|_{\mathfrak{m}}X_{\mathfrak{m}})^*)) = (\varphi|_{\mathfrak{m}} + \psi(t_0))X_{\mathfrak{m}}\,.$$

So one can check that the horizontal lift is

$$\bar{X} = \left(\psi(t_0)(\varphi|_{\mathfrak{m}} + \psi(t_0))^{-1}X_{\mathfrak{m}} + X_{\mathfrak{s}}\,, (\varphi|_{\mathfrak{m}}(\varphi|_{\mathfrak{m}} + \psi(t_0))^{-1}X_{\mathfrak{m}})^*\right).$$

Therewith π becomes a Riemannian submersion we get for $X_{\mathfrak{m}} + X_{\mathfrak{s}}\,,\, Y_{\mathfrak{m}} + Y_{\mathfrak{s}} \in \mathfrak{m} \oplus \mathfrak{s}$

$$\begin{aligned}
g_{\phi(t_0)}(X_{\mathfrak{s}}, Y_{\mathfrak{s}}) &= g_{\varphi}(X_{\mathfrak{s}}, Y_{\mathfrak{s}})\\
g_{\phi(t_0)}(X_{\mathfrak{s}}, Y_{\mathfrak{m}}) &= 0\\
g_{\phi(t_0)}(X_{\mathfrak{m}}, Y_{\mathfrak{m}}) &= \left(g_{\varphi} + g_{\psi(t_0)}\right)\left(\bar{X}, \bar{Y}\right)\\
&= \left(g_{\varphi} + g_{\psi(t_0)}\right)\Big(\big(\psi(t_0)(\varphi|_{\mathfrak{m}} + \psi(t_0))^{-1}X_{\mathfrak{m}}, \varphi|_{\mathfrak{m}}(\varphi|_{\mathfrak{m}} + \psi(t_0))^{-1}X_{\mathfrak{m}}^*\big),\\
&\qquad\qquad\qquad \big(\psi(t_0)(\varphi|_{\mathfrak{m}} + \psi(t_0))^{-1}Y_{\mathfrak{m}}, \varphi|_{\mathfrak{m}}(\varphi|_{\mathfrak{m}} + \psi(t_0))^{-1}Y_{\mathfrak{m}}^*\big)\Big)\\
&= Q\left(\varphi|_{\mathfrak{m}}\psi(t_0)(\varphi|_{\mathfrak{m}} + \psi(t_0))^{-1}X_{\mathfrak{m}}, \psi(t_0)(\varphi|_{\mathfrak{m}} + \psi(t_0))^{-1}Y_{\mathfrak{m}}\right)\\
&\quad + Q\left(\varphi|_{\mathfrak{m}}\psi(t_0)(\varphi|_{\mathfrak{m}} + \psi(t_0))^{-1}X_{\mathfrak{m}}, \varphi|_{\mathfrak{m}}(\varphi|_{\mathfrak{m}} + \psi(t_0))^{-1}Y_{\mathfrak{m}}\right)\\
&= Q\left(\varphi|_{\mathfrak{m}}\psi(t_0)(\varphi|_{\mathfrak{m}} + \psi(t_0))^{-1}X_{\mathfrak{m}}, Y_{\mathfrak{m}}\right)\,,
\end{aligned}$$

from which the statements in the Lemma follow. □

Remark 2.4. We especially have the following geometric feature for the metrics in Lemma 2.3. For all $t \in (0, R]$ the fibers of the homogeneous fibration

$$(K/H, g_{\psi(t)}) \hookrightarrow (G/H, g_{\phi(t)}) \to (G/K, g_{\varphi})$$

are totally geodesic (cf. [BB]). We can prove this fact by using Corollary 1.10. The proof itself is analogeous to that of Remark 1.9.

For the unnormalized curvature for a special form of a metric as in the previous lemma we have

Proposition 2.5. *Consider a submersion metric as in 2.3 with $\varphi|_{\mathfrak{m}} = c\,\mathrm{Id}$. Let R^{g_φ} denote the curvature tensor of the metric g_φ and let R^{g_ψ} denote the curvature tensor of the metric*

g_ψ. *Then for all* $X, Y \in \mathfrak{m}$ *we have*

$$\begin{aligned}
&R^{g_\phi}\big(\phi^{-1}X, \phi^{-1}Y; \phi^{-1}Y, \phi^{-1}X\big) \\
=&R^{g_\varphi}\big(\varphi^{-1}X, \varphi^{-1}Y; \varphi^{-1}Y, \varphi^{-1}X\big) + R^{g_\psi}\big(\psi^{-1}X_\mathfrak{m}, \psi^{-1}Y_\mathfrak{m}; \psi^{-1}Y_\mathfrak{m}, \psi^{-1}X_\mathfrak{m}\big) \\
&+ \frac{3}{4}\, Q\big((c+\psi)^{-1}\big(cB^\varphi_\mathfrak{m} + A^\psi_\mathfrak{m} - \psi B^\psi_\mathfrak{m}\big), cB^\varphi_\mathfrak{m} + A^\psi_\mathfrak{m} - \psi B^\psi_\mathfrak{m}\big) \\
&+ \frac{3}{4}\, c\, \big\|B^\varphi_\mathfrak{h} + c^{-1}A^\psi_\mathfrak{h}\big\|^2,
\end{aligned} \tag{2.6}$$

where we use the abbreviations

$$\begin{aligned}
A^\varphi &:= \big[\varphi^{-1}X, Y\big] + \big[X, \varphi^{-1}Y\big] & A^\psi &:= \big[\psi^{-1}X_\mathfrak{k}, Y_\mathfrak{k}\big] + \big[X_\mathfrak{k}, \psi^{-1}Y_\mathfrak{k}\big] \in \mathfrak{k} \\
B^\varphi &:= \big[\varphi^{-1}X, \varphi^{-1}Y\big] & B^\psi &:= \big[\psi^{-1}X_\mathfrak{k}, \psi^{-1}Y_\mathfrak{k}\big] \in \mathfrak{k} \\
C^\varphi &:= \big[\varphi^{-1}X, Y\big] - \big[X, \varphi^{-1}Y\big] \in \mathfrak{s} & C^\psi &:= \big[\psi^{-1}X_\mathfrak{k}, Y_\mathfrak{k}\big] - \big[X_\mathfrak{k}, \psi^{-1}Y_\mathfrak{k}\big] \in \mathfrak{k}
\end{aligned}$$

The actual computations for verifying the validity of the formula given in Proposition 2.5 can be found in the appendix.

Remark 2.6. We work with twisting, i.e. we consider elements of the form $\phi^{-1}X$, $\phi^{-1}Y$ as in Proposition 2.5 rather than X, Y because the horizontal lifts can be handled much easier when twisting is used. For a more detailed treatment of this feature see the appendix.

Chapter 3

Cheeger Deformations

An important special case of Cheeger's construction as in (2.2) occurs if we take $K = G$ because then Cheeger's construction gives a deformation of the initial G-invariant metric on M which is also G-invariant.

To make this process more precise let Q be an Ad_G-invariant inner product on the Liealgebra $\mathfrak{g}$ of G and denote the induced biinvariant metric by g_Q. We consider the diagonal action of G on $G \times M$ as in (2.1). By the map $\{e\} \times M \to G \times_G M$, $(e,p) \mapsto G(e,p)$ with inverse $G \times_G M \to \{e\} \times M$, $G(g,p) \mapsto (e, g(p))$ we have an identification of $G \times_G M$ with $M \cong \{e\} \times M$.

Especially the composition of the submersion as in (2.2) and the diffeomorphism between $G \times_G M$ and M is nothing else than the usual action map of the G-action on M.

Legitimated by this construction we give

Definition 3.1. Let (M, g) be a Riemannian manifold and let the situation be as above. Then for $t > 0$, the metric g_t on M for which the action map

$$\big(G \times M,\, t^{-1} g_Q \oplus g\big) \to \big(M, g_t\big)$$

becomes a Riemannian submersion is called a Cheeger deformation of g.

The next proposition states some properties of Cheeger deformations

Proposition 3.2 (cf. [S2], Proposition 2.2)**.** *Let g be a G-invariant metric on M and let g_t be the Cheeger deformation of g for $t > 0$.*

1. $\lim_{t \to 0} g_t = g$
2. *If $M = G/H$ is a homogeneous space with a G-invariant metric g then* $\lim_{t \to \infty} t g_t = g_Q$

Proof. Let $p \in M$. Decompose the tangent space of M in p orthogonally w.r.t. g as

$$T_p(G \cdot p) \oplus S_p \,,$$

where $G \cdot p$ is the G-orbit through p.
Let $H \subset G$ be the stabilizer of p, so that $G \cdot p = G/H$. Consider the Q-orthogonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and write $\mathfrak{g} \ni X = X_{\mathfrak{h}} + X_{\mathfrak{m}}$, where $X_{\mathfrak{h}}$ resp. $X_{\mathfrak{m}}$ are the projections onto $\mathfrak{h}$ resp. $\mathfrak{m}$. We have the identification $\mathfrak{m} \cong T_p(G \cdot p)$ via action fields $\mathfrak{m} \ni X \mapsto X^*(p)$.
Let $\varphi : \mathfrak{m} \to \mathfrak{m}$ be the Q-symmetric, positive definite, Ad_H - equivariant map such that

$$g\big(X^*(p), Y^*(p)\big) = Q(\varphi X, Y)\,.$$

With analogeous computations as in the proof of Lemma 2.3 we get

$$\mathcal{V}_{(e,p)} = \big\{ \big(-X, X^*_{\mathfrak{m}}(p) \big) \,\big|\, X_{\mathfrak{m}} \in \mathfrak{m} \big\}\,.$$

and

$$\mathcal{H}_{(e,p)} = \big\{ \big(\varphi X_{\mathfrak{m}}, \big(t^{-1} X_{\mathfrak{m}}\big)^*(p) \big) \,\big|\, X_{\mathfrak{m}} \in \mathfrak{m} \big\} \oplus S_p\,.$$

Therefore we get for the horizontal lift of $X^*(p)$,

$$\big(\varphi \left(t^{-1} + \varphi\right)^{-1} X_{\mathfrak{m}}, \big(t^{-1} \big(t^{-1} + \varphi\big)^{-1} X_{\mathfrak{m}} \big)^*(p) \big)\,.$$

So we obtain

$$g_t|_{S_p} = g|_{S_p}, \quad g_t\big(S_p, T_p(G \cdot p)\big) = 0, \quad g_t\big(X^*(p), Y^*(p)\big) = Q\big(\varphi\,(t\varphi + 1)^{-1} X, Y\big)\,. \quad (3.1)$$

From this the statements in the proposition follow. □

Remark 3.3. In view of 1. in Proposition 3.2 we will allow t for technical reasons in the definition of Cheeger deformations to be 0. In this case the initial metric on M will stay unchanged.

Next we are going to deal with the question when it is possible to apply a Cheeger deformation to a G-invariant metric on a homogeneous space G/H such that the Cheeger deformed metric is nonnegatively curved.
We use the same notions as in the previous chapter. Here we have $\mathfrak{s} = 0$ and a Riemannian

submersion

$$\pi : \big(G \times G/H, t^{-1} g_Q \oplus g_\psi\big) \to \big(G/H, g_\phi\big)\,.$$

We are going to use the same abbreviations as in Proposition 2.5 and write A, B, C instead of A^ψ, B^ψ, C^ψ since here we just have one unspecified map ψ. For the unnormalized curvature of g_ϕ we have according to Proposition 2.5

$$\begin{aligned}
&R^{g_\phi}\big(\phi^{-1}X, \phi^{-1}Y; \phi^{-1}Y, \phi^{-1}X\big)\\
=&\frac{1}{4}\,t^3\,\|[X,Y]\|^2 + R^{g_\psi}\big(\psi^{-1}X, \psi^{-1}Y; \psi^{-1}Y, \psi^{-1}X\big)\\
&+\frac{3}{4}\,t^2 Q\big((t^{-1}+\psi)^{-1}\big([X,Y]_{\mathfrak{m}} + t^{-1}A_{\mathfrak{m}} - t^{-1}\psi B_{\mathfrak{m}}\big), [X,Y]_{\mathfrak{m}} + t^{-1}A_{\mathfrak{m}} - t^{-1}\psi B_{\mathfrak{m}}\big)\\
&+\frac{3}{4}\,t^3\,\big\|[X,Y]_{\mathfrak{h}} + t^{-1}A_{\mathfrak{h}}\big\|^2\,,
\end{aligned} \tag{3.2}$$

where $X, Y \in \mathfrak{m}$.
If $[X,Y] = 0$, $A_{\mathfrak{h}} = 0$ and $A_{\mathfrak{m}} = \psi B_{\mathfrak{m}}$ all the terms coming from the Cheeger deformation in (3.2) vanish. Moreover for R^{g_ψ} which is invariant under Cheeger deformations we obtain in this case

$$\begin{aligned}
&R^{g_\psi}\big(\psi^{-1}X, \psi^{-1}Y; \psi^{-1}Y, \psi^{-1}X\big)\\
=&\frac{1}{2}\,Q(A,B) - \frac{3}{4}\,Q(\psi B_{\mathfrak{m}}, B_{\mathfrak{m}}) + \frac{1}{4}\,Q\big(C, \psi^{-1}C\big) - Q\big(\big[\psi^{-1}X, X\big], \psi^{-1}\big[\psi^{-1}Y, Y\big]\big)\\
=&\frac{1}{4}\,Q\big(C_{\mathfrak{m}}, \psi^{-1}C_{\mathfrak{m}}\big) - \frac{1}{4}\,Q\big(A_{\mathfrak{m}}, \psi^{-1}A_{\mathfrak{m}}\big) - Q\big(\big[\psi^{-1}X, X\big], \psi^{-1}\big[\psi^{-1}Y, Y\big]\big)\,,
\end{aligned} \tag{3.3}$$

where the $\mathfrak{h}$-part of C vanishes as shown in Remark 1.15. Moreover by using the definitions of A and C we get by calculation

$$\begin{aligned}
&\frac{1}{4}\,Q\big(C_{\mathfrak{m}}, \psi^{-1}C_{\mathfrak{m}}\big) - \frac{1}{4}\,Q\big(A_{\mathfrak{m}}, \psi^{-1}A_{\mathfrak{m}}\big)\\
=&\frac{1}{4}\,Q\big(\big[\psi^{-1}X, Y\big]_{\mathfrak{m}} - \big[X, \psi^{-1}Y\big]_{\mathfrak{m}}, \psi^{-1}\big(\big[\psi^{-1}X, Y\big]_{\mathfrak{m}} - \big[X, \psi^{-1}Y\big]_{\mathfrak{m}}\big)\big)\\
&-\frac{1}{4}\,Q\big(\big[\psi^{-1}X, Y\big]_{\mathfrak{m}} + \big[X, \psi^{-1}Y\big]_{\mathfrak{m}}, \psi^{-1}\big(\big[\psi^{-1}X, Y\big]_{\mathfrak{m}} + \big[X, \psi^{-1}Y\big]_{\mathfrak{m}}\big)\big)\\
=&-Q\big(\big[\psi^{-1}X, Y\big]_{\mathfrak{m}}, \psi^{-1}\big[X, \psi^{-1}Y\big]_{\mathfrak{m}}\big)\,.
\end{aligned}$$

Taking (3.3) into account we see that

$$-Q\big(\big[\psi^{-1}X, Y\big], \psi^{-1}\big[X, \psi^{-1}Y\big]\big) - Q\big(\big[\psi^{-1}X, X\big], \psi^{-1}\big[\psi^{-1}Y, Y\big]\big) \geq 0$$

is necessary for $\sec_{g_\phi} \geq 0$ for all $X, Y \in \mathfrak{m}$. Nevertheless this need not to be sufficient. The next theorem gives an equivalent condition to the existence of a $c > 0$ such that $\sec_{g_\phi} \geq 0$ for all $X, Y \in \mathfrak{m}$.

Theorem 3.4. *There exists a Cheeger deformation of a homogeneous metric g_ψ on G/H into a metric with nonnegative curvature if and only if there exists a constant r, such that for all $X, Y \in \mathfrak{m}$*

$$R^{g_\psi}\left(\psi^{-1}X, \psi^{-1}Y; \psi^{-1}Y, \psi^{-1}X\right) \geq -r\left(\|[X,Y]\|^2 + \|A_\mathfrak{h}\|^2 + \|A_\mathfrak{m} - \psi B_\mathfrak{m}\|^2\right) \tag{3.4}$$

holds.

Proof. We have to show that (3.4) is equivalent to the nonnegativity of (3.2).
To see that (3.4) is sufficient for nonnegative curvature we first observe that there is a constant $c_1 > 0$ such that

$$\begin{aligned} &Q\big((t^{-1}+\psi)^{-1}\big([X,Y]_\mathfrak{m} + t^{-1}A_\mathfrak{m} - t^{-1}\psi B_\mathfrak{m}\big), [X,Y]_\mathfrak{m} + t^{-1}A_\mathfrak{m} - t^{-1}\psi B_\mathfrak{m}\big) \\ &\geq c_1 t \left\|[X,Y]_\mathfrak{m} + t^{-1}A_\mathfrak{m} - t^{-1}\psi B_\mathfrak{m}\right\|^2 \end{aligned}$$

and that we can write

$$\begin{aligned} &\frac{1}{4}\left\|[X,Y]_\mathfrak{h}\right\|^2 + \frac{3}{4}\left\|[X,Y]_\mathfrak{h} + t^{-1}A_\mathfrak{h}\right\|^2 \\ =\ &\frac{19}{100}\left\|[X,Y]_\mathfrak{h}\right\|^2 + \left\|\frac{9}{10}[X,Y]_\mathfrak{h} + \frac{5}{6}t^{-1}A_\mathfrak{h}\right\|^2 + \frac{1}{18}t^{-2}\left\|A_\mathfrak{h}\right\|^2. \end{aligned}$$

So

$$\begin{aligned} &R^{g_\phi}\left(\phi^{-1}X, \phi^{-1}Y; \phi^{-1}Y, \phi^{-1}X\right) \\ &\geq t^3\Big(t^{-3}R^{g_\psi}\left(\psi^{-1}X, \psi^{-1}Y; \psi^{-1}Y, \psi^{-1}X\right) + \frac{1}{4}\left\|[X,Y]_\mathfrak{m}\right\|^2 \\ &\qquad + \frac{19}{100}\left\|[X,Y]_\mathfrak{h}\right\|^2 + \left\|\frac{9}{10}[X,Y]_\mathfrak{h} + \frac{5}{6}t^{-1}A_\mathfrak{h}\right\|^2 + \frac{1}{18}t^{-2}\left\|A_\mathfrak{h}\right\|^2 \\ &\qquad + \frac{3}{4}c_1\left\|[X,Y]_\mathfrak{m} + t^{-1}A_\mathfrak{m} - t^{-1}\psi B_\mathfrak{m}\right\|^2\Big) \\ &= t^3\Big(t^{-3}R^{g_\psi}\left(\psi^{-1}X, \psi^{-1}Y; \psi^{-1}Y, \psi^{-1}X\right) + \frac{1}{4}\left\|[X,Y]_\mathfrak{m}\right\|^2 \\ &\qquad + \frac{19}{100}\left\|[X,Y]_\mathfrak{h}\right\|^2 + \left\|\frac{9}{10}[X,Y]_\mathfrak{h} + \frac{5}{6}t^{-1}A_\mathfrak{h}\right\|^2 + \frac{1}{18}t^{-2}\left\|A_\mathfrak{h}\right\|^2 \\ &\qquad + \frac{3}{4}c_1\left\|[X,Y]_\mathfrak{m}\right\|^2 + \frac{3}{2}c_1 Q\big(\lambda^{-1}[X,Y]_\mathfrak{m}, \lambda t^{-1}(A_\mathfrak{m} - \psi B_\mathfrak{m})\big) \\ &\qquad + \frac{3}{4}c_1 t^{-2}\left\|A_\mathfrak{m} - \psi B_\mathfrak{m}\right\|^2\Big). \end{aligned}$$

Applying the parallelogram inequality we get

$$\frac{3}{2}\,c_1 Q\big(\lambda^{-1}[X,Y]_{\mathfrak{m}},\lambda t^{-1}(A_{\mathfrak{m}}-\psi B_{\mathfrak{m}})\big) \geq -\frac{3}{4}\,c_1\lambda^{-2}\big\|[X,Y]_{\mathfrak{m}}\big\|^2 - \frac{3}{4}\,c_1\lambda^2 t^{-2}\big\|A_{\mathfrak{m}}-\psi B_{\mathfrak{m}}\big\|^2 .$$

Leaving out the nonnegative term $\big\|\frac{9}{10}[X,Y]_{\mathfrak{h}}+\frac{5}{6}\,t^{-1}A_{\mathfrak{h}}\big\|^2$ and using (3.4) we can state that $R^{g_\phi}\big(\phi^{-1}X,\phi^{-1}Y;\phi^{-1}Y,\phi^{-1}X\big)$ is greater or equal to

$$\begin{aligned}
&t^3\Big(-rt^{-3}\big(\,\big\|[X,Y]\big\|^2+\big\|A_{\mathfrak{h}}\big\|^2+\big\|A_{\mathfrak{m}}-\psi B_{\mathfrak{m}}\big\|^2\,\big)\\
&\qquad+\Big(\frac{1}{4}+\frac{3}{4}\,c_1\Big)\big\|[X,Y]_{\mathfrak{m}}\big\|^2+\frac{19}{100}\,\big\|[X,Y]_{\mathfrak{h}}\big\|^2+\frac{1}{18}\,t^{-2}\,\big\|A_{\mathfrak{h}}\big\|^2\\
&\qquad-\frac{3}{4}\,c_1\lambda^{-2}\,\big\|[X,Y]_{\mathfrak{m}}\big\|^2-\frac{3}{4}\,c_1\lambda^2t^{-2}\,\big\|A_{\mathfrak{m}}-\psi B_{\mathfrak{m}}\big\|^2+\frac{3}{4}\,c_1t^{-2}\,\big\|A_{\mathfrak{m}}-\psi B_{\mathfrak{m}}\big\|^2\Big)\\
&\geq t^3\Big(\Big(\frac{1}{18}\,t^{-2}-rt^{-3}\Big)\,\big\|A_{\mathfrak{h}}\big\|^2+\Big(\frac{19}{100}-rt^{-3}\Big)\,\big\|[X,Y]_{\mathfrak{h}}\big\|^2\\
&\qquad+\Big(\frac{1}{4}+\frac{3}{4}\,c_1\big(1-\lambda^{-2}\big)-rt^{-3}\Big)\,\big\|[X,Y]_{\mathfrak{m}}\big\|^2\\
&\qquad+\Big(\frac{3}{4}\,c_1t^{-2}\big(1-\lambda^2\big)-rt^{-3}\Big)\,\big\|A_{\mathfrak{m}}-\psi B_{\mathfrak{m}}\big\|^2\Big)\,.
\end{aligned}$$

So for an appropriate choice of λ and a sufficiently large t nonnegativity can surely be achieved.

Now suppose that we have nonnegative curvature but (3.4) does not hold. Then there are sequences $r_n\to\infty$, $X_n\to X$ and $Y_n\to Y$ such that for all $n\in\mathbb{N}$ we have

$$R^{g_\psi}\big(\psi^{-1}X_n,\psi^{-1}Y_n;\psi^{-1}Y_n,\psi^{-1}X_n\big) < -\,r_n\big(\,\big\|[X_n,Y_n]\big\|^2+\big\|A^n_{\mathfrak{h}}\big\|^2+\big\|A^n_{\mathfrak{m}}-\psi B^n_{\mathfrak{m}}\big\|^2\big)\,.$$

First observe that

$$\begin{aligned}
&\frac{3}{4}\,Q\big(t^{-1}\big(t^{-1}+\psi\big)^{-1}\big([X,Y]_{\mathfrak{m}}+t^{-1}A_{\mathfrak{m}}-t^{-1}\psi B_{\mathfrak{m}}\big),[X,Y]_{\mathfrak{m}}+t^{-1}A_{\mathfrak{m}}-t^{-1}\psi B_{\mathfrak{m}}\big)\\
&=\frac{3}{4}\,\big\|[X,Y]_{\mathfrak{m}}+t^{-1}A_{\mathfrak{m}}-t^{-1}\psi B_{\mathfrak{m}}\big\|^2-\frac{3}{4}\,t\big\|[X,Y]_{\mathfrak{m}}+t^{-1}A_{\mathfrak{m}}-t^{-1}\psi B_{\mathfrak{m}}\big\|^2_{g_\phi}\\
&\leq\frac{3}{4}\,\big\|[X,Y]_{\mathfrak{m}}+t^{-1}A_{\mathfrak{m}}-t^{-1}\psi B_{\mathfrak{m}}\big\|^2\,.
\end{aligned}$$

Moreover by the parallelogram inequality we have

$$\frac{3}{4}\,\big\|[X,Y]_{\mathfrak{h}}+t^{-1}A_{\mathfrak{h}}\big\|^2\leq\frac{3}{2}\,\big\|[X,Y]_{\mathfrak{h}}\big\|^2+\frac{3}{2}\,t^{-2}\big\|A_{\mathfrak{h}}\big\|^2\,,$$

and

$$\frac{3}{4}\,\big\|[X,Y]_{\mathfrak{m}}+t^{-1}A_{\mathfrak{m}}-t^{-1}\psi B_{\mathfrak{m}}\big\|^2\leq\frac{3}{2}\,\big\|[X,Y]_{\mathfrak{m}}\big\|^2+\frac{3}{2}\,t^{-2}\big\|A_{\mathfrak{m}}-\psi B_{\mathfrak{m}}\big\|^2\,.$$

So we obtain from (3.2),

$$
\begin{aligned}
0 \leq\ & \frac{1}{4}\big\|[X_n,Y_n]\big\|^2 + \frac{3}{4}\,\big\|[X_n,Y_n]_{\mathfrak{h}} + t^{-1}A^n_{\mathfrak{h}}\big\|^2 + t^{-3}R^{g_\psi}\big(\psi^{-1}X_n\psi^{-1}Y_n;\psi^{-1}Y_n,\psi^{-1}X_n\big)\\
&+ \frac{3}{4}\,t^{-1}Q\big((t^{-1}+\psi)^{-1}\big([X_n,Y_n]_{\mathfrak{m}} + t^{-1}A^n_{\mathfrak{m}} - t^{-1}\psi B^n_{\mathfrak{m}}\big), [X_n,Y_n]_{\mathfrak{m}} + t^{-1}A^n_{\mathfrak{m}} - t^{-1}\psi B^n_{\mathfrak{m}}\big)\\
<\ & \frac{1}{4}\,\big\|[X_n,Y_n]\big\|^2 + \frac{3}{2}\,\big\|[X,Y]_{\mathfrak{h}}\big\|^2 + \frac{3}{2}\,t^{-2}\big\|A_{\mathfrak{h}}\big\|^2 + \frac{3}{2}\,\big\|[X,Y]_{\mathfrak{m}}\big\|^2 + \frac{3}{2}\,t^{-2}\big\|A_{\mathfrak{m}} - \psi B_{\mathfrak{m}}\big\|^2\\
&- t^{-3}r_n\big(\big\|[X_n,Y_n]\big\|^2 + \big\|A^n_{\mathfrak{h}}\big\|^2 + \big\|A^n_{\mathfrak{m}} - \psi B^n_{\mathfrak{m}}\big\|^2\big)\\
=\ & \Big(\frac{7}{4} - r_n t^{-3}\Big)\big\|[X_n,Y_n]\big\|^2 + \Big(\frac{3}{2}\,t^{-2} - r_n t^{-3}\Big)\big\|A_{\mathfrak{h}}\big\|^2 + \Big(\frac{3}{2}\,t^{-2} - r_n t^{-3}\Big)\big\|A^n_{\mathfrak{m}} - \psi B^n_{\mathfrak{m}}\big\|^2 .
\end{aligned}
$$

As $r_n \to \infty$ we are provided with a contradiction. □

There are some Lemmas to follow which we prove using Cheeger deformations. Before that we need

Definition 3.5. A path of K-invariant metrics on K/H, given by $g(s) = g_{\psi(s)}$, $s \in [a,b]$ is called diagonal if a fixed decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_r$ provides a decomposition into eigenspaces for each $g_{\psi(s)}$.

Lemma 3.6. *Let g_1 be a K-invariant metric on K/H with positive curvature and suppose that a normal homogeneous metric, denoted by g_Q, is also positively curved. Then there is a path, $g(s)$, $s \in [0,1]$, of positively curved K-invariant metrics in K/H such that $g(0) = g_Q$ and $g(1) = g_1$. In particular this path is digonal.*

Proof. Let $\varphi_1 : \mathfrak{m} \to \mathfrak{m}$ be the self-adjoint, positive definite, Ad_H-equivariant linear map which induces the K-invariant metric g_1 on K/H. Consider the Cheeger deformation

$$\big(K \times K/H, t^{-1}g_Q \oplus g_1\big) \to \big(K/H, g_t\big)\,.$$

The metrics g_t are K-invariant and therefore induced by some self-adjoint, positive definite, Ad_H-equivariant linear map $\varphi_t : \mathfrak{m} \to \mathfrak{m}$. Moreover these metrics are positively curved since we assume that the normal homogeneous metric is and Riemannian submersions do not decrease curvature due to O'Neill's formula. Since φ_1 is self-adjoint and positive definite there exists a Q-orthonormal basis of $\mathfrak{m}$ of eigenvectors $\{e_i\}$ with corresponding positive eigenvalues $\{\lambda_i\}$. The linear maps φ_t are related to φ_1 by

$$\varphi_t = \varphi_1(\mathrm{Id} + t\varphi_1)^{-1}$$

as can be deduced from (3.1). Therefore the eigenvectors $\{e_i\}$ also form a Q-orthonormal basis for the maps φ_t. Define

$$\begin{aligned} g : [0,1] &\to \mathrm{Aut}(\mathfrak{m}) \\ s &\mapsto \left((1-s)\mathrm{Id} + s\varphi_1^{-1}\right)^{-1} . \end{aligned}$$

We have $g(0) = \mathrm{Id}$, $g(1) = \varphi_1$ and g is smooth. Moreover it can be checked that $g(s) = \frac{1}{s}\varphi_{\frac{1-s}{s}}$, so that the curvature of the metric induced by $g(s)$ is positive for all $s \in [0,1]$ and g induces a path as stated in the Lemma. □

For the following statements suppose that a chain

$$H \subset K_1 \subset K_2 \subset \cdots \subset K_r = K$$

of Lie groups is given. Let Q be a Ad_K-invariant inner product on $\mathfrak{k}$ and let

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_r ,$$

be the induced Q-orthogonal decomposition such that the Lie algebra $\mathfrak{k}_i$ of K_i is given by

$$\mathfrak{k}_i = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_i \quad \text{for } 1 \le i \le r .$$

Lemma 3.7. *Let g_φ be a homogeneous metric on K/H with parameters $(a_1^2, a_2^2, \ldots, a_r^2)$ and let $g_{\tilde\varphi}$ be another homogeneous metric on K/H with parameters $(\tilde a_1^2, \tilde a_2^2, , \ldots, \tilde a_r^2)$. Suppose that there exist $t_1, t_2, \ldots, t_r \in \mathbb{R}^{\ge 0}$ such that*

$$\tilde a_j^2 = a_j^2 \left(\left(\sum_{l=j}^{r} t_l \right) a_j^2 + 1 \right)^{-1} , \ 1 \le j \le r .$$

Then we have $\sec_{g_{\tilde\varphi}} \ge \sec_{g_\varphi}$.

Proof. Consider the following iteration of Cheeger deformations.
Set $\varphi_r := \varphi$. For $1 \le m \le r$ do the Cheeger deformation

$$\left(K_{r-m+1} \times K/H, t_{r-m+1}^{-1} g_Q \oplus g_{\varphi_{r-m+1}}\right) \longrightarrow \left(K/H, g_{\varphi_{r-m}}\right),$$

where we use the constants $t_1, t_2, \ldots, t_r \in \mathbb{R}^{\ge 0}$ given in the lemma.
Since $g_{\varphi_{r-m}}$ is constructed by means of a Cheeger deformation from $g_{\varphi_{r-m+1}}$ we obtain $\sec_{g_{\varphi_{r-m}}} \ge \sec_{g_{\varphi_{r-m+1}}}$ because Cheeger deformations are curvature nondecreasing.
Note that with this iteration we especially get $\varphi_0 = \tilde\varphi$ from which the statement in the lemma follows. □

Corollary 3.8. *Let g_φ be a homogeneous metric on K/H with parameters $(a_1^2, a_2^2, \ldots, a_r^2)$ satisfying*

$$a_1^2 \leq a_2^2 \leq \ldots \leq a_r^2 \,.$$

Then $\sec_{g_\varphi} \geq 0$.

Proof. The metric $a_r^2 g_Q$ has nonnegative curvature. Due to our assumption there exist $t_1, t_2, \ldots, t_r$ such that

$$a_j^2 = a_r^2 \Bigg(\Big(\sum_{l=j}^{r} t_l \Big) a_r^2 + 1 \Bigg)^{-1} , \; 1 \leq j \leq r \,.$$

So the homogeneous metric g_φ can be constructed by an iteration of Cheeger deformations from the nonnegatively curved metric $a_r^2 g_Q$ and so has nonnegative curvature by Lemma 3.7. □

Remark 3.9. If a biinvariant metric on K/H is positively curved, then a metric as in the previous lemma is also positively curved.

Corollary 3.10. *For every homogeneous metric g_φ on K/H with parameters $(a_1^2, a_2^2, \ldots, a_r^2)$ there exists an iteration of Cheeger deformations such that the resulting metric has nonnegative curvature.*

Proof. Choose $t_1, t_2, \ldots, t_r \in \mathbb{R}^{\geq 0}$ such that

$$\tilde{a}_j^2 := a_j^2 \Bigg(\Big(\sum_{l=j}^{r} t_l \Big) a_j^2 + 1 \Bigg)^{-1}$$

satisfy

$$\tilde{a}_1^2 \leq \tilde{a}_2^2 \leq \ldots \leq \tilde{a}_r^2 \,.$$

Then by Corollary 3.8 the metric with parameters $(\tilde{a}_1^2, \tilde{a}_2^2, \ldots, \tilde{a}_r^2)$ has nonnegative curvature. □

Lemma 3.11. *Let g_φ be a homogeneous metric on K/H with parameters $(a_1^2, a_2^2, \ldots, a_r^2)$ satisfying $a_1^2 \leq a_2^2 \leq \ldots \leq a_r^2$. Let $g_{\tilde{\varphi}}$ be another homogeneous metric on K/H with parameters $(\tilde{a}_1^2, \tilde{a}_2^2, \ldots, \tilde{a}_r^2)$. Suppose that*

$$\tilde{a}_j^2 \leq a_j^2, \quad \textit{for } 1 \leq j \leq r$$

and that

$$a_i^2 - \tilde{a}_i^2 \;\geq\; a_j^2 - \tilde{a}_j^2 \quad \textit{for } 1 \leq i < j \leq r\,.$$

Then $\sec_{g_{\tilde{\varphi}}} \geq \sec_{g_{\varphi}}$.

Proof. With the conditions imposed we can find an iteration of Cheeger deformations as in the proof of Lemma 3.7 such that $(\tilde{a}_1^2, \tilde{a}_2^2, \ldots, \tilde{a}_r^2)$ are the parameters of the resulting metric. Namely we use the constants

$$t_j = \frac{a_j^2 - \tilde{a}_j^2}{a_j^2\, \tilde{a}_j^2} - \frac{a_{j+l}^2 - \tilde{a}_{j+l}^2}{a_{j+l}^2\, \tilde{a}_{j+l}^2} > 0 \quad , \; 1 \leq j < l \leq r,$$

for the Cheeger deformations. □

Chapter 4

Homogeneous Sphere Metrics

There are various ways to describe a sphere as a homogeneous manifold. A classification can be found in [MS]. We give the following complete list of almost effective transitive actions by connected Lie groups on spheres in which it is also listed how the isotropy representation decomposes where the notions coincide with that of Section 1.1.

	K	H	$\dim(K/H)$	isotropy representation
1	$SO(n+1)$	$SO(n)$	n	$\mathfrak{m} = \mathfrak{m}_1 \cong \mathbb{R}^n$
2	G_2	$SU(3)$	6	$\mathfrak{m} = \mathfrak{m}_1 \cong \mathbb{C}^3$
3	$Spin(7)$	G_2	7	$\mathfrak{m} = \mathfrak{m}_1 \cong \mathbb{C}a$
4	$SU(n+1)$	$SU(n)$	$2n+1$	$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \cong \mathbb{R} \oplus \mathbb{C}^n$
5	$U(n+1)$	$U(n)$	$2n+1$	$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \cong \mathbb{R} \oplus \mathbb{C}^n$
6	$Sp(n+1)$	$Sp(n)$	$4n+3$	$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \cong Im(\mathbb{H}) \oplus \mathbb{H}^n$
7	$Sp(1) \cdot Sp(n+1)$	$Sp(1) \cdot Sp(n)$	$4n+3$	$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \cong Im(\mathbb{H}) \oplus \mathbb{H}^n$
8	$U(1) \cdot Sp(n+1)$	$U(1) \cdot Sp(n)$	$4n+3$	$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \cong \mathbb{R} \oplus \mathbb{C} \oplus \mathbb{H}^n$
9	$Spin(9)$	$Spin(7)$	15	$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \cong \mathbb{R}^7 \oplus \mathbb{R}^8$

For the rest of this work we will refer to the enumeration in this table when talking about transitive action on spheres.
The following table lists the action of Ad_H on $\mathfrak{m}$,

	$\mathrm{Ad}_H\|_{\mathfrak{m}_1}$	$\mathrm{Ad}_H\|_{\mathfrak{m}_2}$	$\mathrm{Ad}_H\|_{\mathfrak{m}_3}$
1	can		
2	can		
3	can		
4	id	can	
5	id	can	
6	id	can	
7	$v \mapsto qvq^{-1}$	$X \mapsto AXq^{-1}$	
8	id	$v \mapsto zvz^{-1}$	$X \mapsto Avz^{-1}$
9	ρ_7	Δ_8	

where in 7 we have $v \in Im(\mathbb{H}), X \in \mathbb{H}^n, q \in Sp(1)$ and $A \in Sp(n)$. In 8 we have $v \in Im(\mathbb{H})\backslash\{i\mathbb{R}\}, X \in \mathbb{H}^n, z \in U(1)$ and $A \in Sp(n)$. Furthermore ρ_7 resp. Δ_8 denote the standard representations of $Spin(7)$ on $\mathbb{R}^7$ resp. $\mathbb{R}^8$.

Remark 4.1. Except in case 9 the action of Ad_H on the last summand of the isotropy decomposition in fact coincides with the actual action of K on the corresponding sphere. Furthermore we point out that in the cases 7 and 8 the action is not effective. In each case the determination of the ineffective kernel is immediate which is isomorphic to $\mathbb{Z}_2$. For the matters of clarity we decided to include the discrete ineffective kernel in the exposition above.

As one sees the isotropy representation on the $\mathfrak{m}_i$'s is inequivalent in each case.
We want to work with homogeneous sphere metrics described via Ad_H-equivariant, positive definite Q-symmetric maps $\psi : \mathfrak{m} \to \mathfrak{m}$ as described in (1.11), where Q is a fixed Ad_K-invariant inner product on the Lie algebra $\mathfrak{k}$ of K. In case 6 however the isotropy representation acts trivially on $\mathfrak{m}_1 = Im(\mathbb{H})$ and so on this 3-dimensional subspace the metric can depend on six parameters in general. In the other cases all homogeneous metrics can be decribed via metric inducing maps which are multiples of the identity restricted to the irreducible summands, i.e.

$$\psi|_{\mathfrak{m}_i} = a_i^2 \,\mathrm{Id}|_{\mathfrak{m}_i} \ .$$

In the upcoming we will describe homogeneous metrics on spheres just by giving the constants a_i^2. When doing so in case 6 we will assume that the metric is a multiple of the identity on $\mathfrak{m}_1$. This clearly just covers a small portion of homogeneous metrics on $Sp(n+1)/Sp(n)$, namely the ones which are even invariant under the bigger group $Sp(1) \cdot Sp(n+1)$ but this will be enough for the purpose of this work.

It follows especially that in the cases $1, 2, 3$ any invariant metric on the sphere is homothetic to the round metric and threrefore has positive curvature. In what follows we will not mention these cases anymore.
In the other cases except in case 6 we have an underlying chain of Lie groups such that the isotropy decomposition happens to be exactly the decomposition induced by the chain. These chains are the content of the following table

	H	K_1	K_2	K
4	$SU(n)$	$S(U(1)U(n))$		$SU(n+1)$
5	$U(n)$	$U(1)U(n)$		$U(n+1)$
7	$Sp(1)Sp(n)$	$Sp(1)Sp(1)Sp(n)$		$Sp(1)Sp(n+1)$
8	$U(1)Sp(n)$	$U(1)U(1)Sp(n)$	$U(1)Sp(1)Sp(n)$	$U(1)Sp(n+1)$
9	$Spin(7)$	$Spin(8)$		$Spin(9)$

These chains enable us to apply the results of Chapter 3 where an underlying chain of Lie groups is supposed to exist to homogeneous sphere metrics.

Furthermore these chains give rise to a geometric description of homogeneous sphere metrics.
In the cases 4 and 5 we can associate to $K_1/H \hookrightarrow K/H \to K/K_1$ the Hopf fibration

$$S^1 \hookrightarrow S^{2n+1} \to \mathbb{CP}^n \,. \tag{4.1}$$

In case 7 we can associate to $K_1/H \hookrightarrow K/H \to K/K_1$ the Hopf fibration

$$S^3 \hookrightarrow S^{4n+3} \to \mathbb{HP}^n \,. \tag{4.2}$$

In case 9 we can associate to $K_1/H \hookrightarrow K/H \to K/K_1$ the Hopf fibration

$$S^7 \hookrightarrow S^{15} \to S^8 \,.$$

In case 8 we can associate to $K_2/H \hookrightarrow K/H \to K/K_2$ the Hopf fibration (4.2) and to $K_1/H \hookrightarrow K/H \to K/K_1$ the Hopf fibration (4.1) for $n = 2m+1$.
The fibers of the Hopf fibrations are in fact totally geodesic which follows from Remark 2.4.

For the homogeneous metrics on the spheres $5, 6, 7, 9$ we can interpret a shrinking resp. enlarging of a_1^2 with shrinking resp. enlarging the fibers of the corresponding Hopf fibration. For the homogeneous metric on sphere number 8 with 3 irreducible summands a shrinking resp. enlarging of a_1^2 can be identified with shrinking resp. enlarging the fibers of the Hopf fibration (4.1) and a simultaneous shrinking resp. enlarging of a_1^2 and a_2^2 can be identified with shrinking resp. enlarging the fibers of the Hopf fibration (4.2).

It is likely to take a Ad_K-invariant inner product on $\mathfrak{k}$ as a basis for investigating invariant metrics on homogeneous spaces. But this point of view comes with the disadvantage that K can support more than one biinvariant metric. When dealing with spheres we have a distinguished metric in the round metric so it is of natural interest what parameters we have to take to obtain the round metric. The following proposition deals with that issue.

Proposition 4.2 (cf.[GZ2], Lemma 2.4). *In the table the parameters for obtaining the round metric on the sphere of radius* 1 *are listed, where in the first column we give the only* Ad_K*-invariant inner product on* $\mathfrak{k}$ *in each case up to scaling with a positive constant.*

	Ad_K-*invariant inner product on* $\mathfrak{k}$	*round metric*
4	$Q(A,B) = -\frac{1}{2}\, Re\,(\mathrm{tr}(AB))$	$\big(a_1^2, a_2^2\big) = \big(\frac{2n}{n+1}, 1\big)$
5	$Q(A_1+A_2, B_1+B_2) = -\frac{1}{2}\, Re\,\big(\mathrm{tr}\big(\frac{\lambda}{n} A_1 B_1 + A_2 B_2\big)\big)$	$\big(a_1^2, a_2^2\big) = \big(\frac{\lambda+1}{\lambda}\frac{2n}{n+1}, 1\big)$
6	$-\frac{1}{2}\, Re\,(\mathrm{tr}(AB))$	$\big(a_1^2, a_2^2\big) = (2, 1)$
7	$Q((v,A),(w,B)) = -\frac{1}{2}\big(\lambda Re\,(vw) + Re\,(\mathrm{tr}(AB))\big)$	$\big(a_1^2, a_2^2\big) = \big(2\,\frac{\lambda+1}{\lambda}, 1\big)$
8	$Q((ix,A),(iy,B)) = \frac{1}{2}\big(\lambda xy - Re\,(\mathrm{tr}(AB))\big)$	$\big(a_1^2, a_2^2, a_3^2\big) = \big(2\,\frac{\lambda+1}{\lambda}, 2, 1\big)$
9	$Q(A,B) = -\frac{1}{2}\, Re\,(\mathrm{tr}(AB))$	$\big(a_1^2, a_2^2\big) = (4, 1)$

where in case 5 the metric is defined according to the decomposition $\mathfrak{u}(n+1) = \mathrm{span}\{i\,\mathrm{Id}\} \oplus \mathfrak{su}(n+1)$*, where* $A_1, B_1 \in \mathrm{span}\{i\,\mathrm{Id}\}$ *and* $A_2, B_2 \in \mathfrak{su}(n+1)$*.*

The computation of the parameters can be found in the appendix.

Remark 4.3. The sequence of parameters for obtaining the round metric is decreasing in each case. In the cases 4, 5, 6, 7, 9 there exists a $t \in \mathbb{R}^{>0}$ such that

$$a_1^2\big(t\,a_1^2 + 1\big)^{-1} = 1\,.$$

In case 8 there exist $t_1, t_2 \in \mathbb{R}^{>0}$ such that

$$a_1^2\big((t_1+t_2)\,a_1^2 + 1\big)^{-1} = 1 \qquad a_2^2\big(t_2\,a_2^2 + 1\big)^{-1} = 1\,.$$

That is in each case we obtain the parameters for the normal homogeneous metric induced by the given biinvariant metric.

From Lemma 3.7 it follows that $\sec_{g_Q} \geq \sec_{g_{can}} > 0$, i.e. a normal homogeneous metric on the sphere has positive curvature. In particular we use the chain $Sp(n) \subset Sp(1)Sp(n) \subset Sp(n+1)$ in case 6 for being able to apply Lemma 3.7 in this case.
This is a result first obtained by Berger in [B].

In [VZ] homogeneous sphere metrics with positive curvature have been analyzed. The authors showed for the cases $4, 5, 7$ that when taking the round metric on the total space of the corresponding Hopf fibration the metric stays positively curved if and only if the fibers are scaled with a positive factor less than $4/3$.
Invariant metrics on sphere 6 have also been analyzed. We are not going to deal with this case but as a special case we get the situation on sphere 8 which is the content of the following

Proposition 4.4. *Let the parameters for a homogeneous metric on sphere 8 be given as*

$$a_1^2 = 2\mu_1 \frac{\lambda+1}{\lambda}, \quad a_2^2 = 2\mu_2, \quad a_3^2 = \mu_3$$

and let

$$q_{12} = \frac{\mu_1}{\mu_2}, \quad q_{13} = \frac{\mu_1}{\mu_3}, \quad q_{23} = \frac{\mu_2}{\mu_3}\,.$$

Then the metric has positive curvature if and only if

1. $q_{12},\ q_{13},\ q_{23} < \dfrac{4}{3}$
2. $3\,|q_{23}^2 - 2q_{23} + q_{13}| < q_{23}^2 + \sqrt{(4q_{23} - 3q_{13})(4 - 3q_{13})}$

The next Proposition generalizes Proposition 3.3 in [STu]

Proposition 4.5. *Let $K \subset O(n+1)$ be a Lie subgroup acting transitively on $S^n \subset \mathbb{R}^{n+1}$ and let $H \subset K$ be the isotropy group at some point. Let g_1 be a K-invariant metric on $K/H \cong S^n$ with positive curvature. Then there is a diagonal path, $g(t)$, $t \in [0,1]$, through positively curved homogeneous metrics on K/H such that $g(0) = g_0$ and $g(1) = g_1$, where g_0 denotes the round metric.*

Proof. Let g_Q denote a normal homogeneous metric on the sphere which has positive curvature as stated in Remark 4.3. The round metric on the sphere has also positive curvature. Thus by Lemma 3.6 we can find a path of positively curved metrics between g_0 and g_Q and between g_Q and g_1. Concatenating these paths gives a path of positively curved metrics between g_0 and g_1 as stated in the Proposition. We obtain a diagonal path

if we can choose the same orthonormal basis in $\mathfrak{m}$ of eigenvectors for the inducing linear maps appearing in the proof of Lemma 3.6 for g_0 and g_1.
Except for case 6 the isotropy representations decompose irreducibly so the $\mathfrak{m}_i' s$ in the list above have to be eigenspaces for the inducing linear maps by Schur's Lemma so we can choose the same basis of eigenvectors in these cases.
In case 6, note that for g_0, the normalizer $Norm_{O(4n+4)}H = Sp(1) \cdot Sp(n)$ operates by isometries on the tangent space at $[eH]$, whence the inducing linear map φ_1 must be a multiple of the identity on $\mathfrak{m}_1$ and therefore $\mathfrak{m}_1$ is an eigenspace for φ_1. So we can choose the same eigenvectors which diagonalize the inducing map of g_1 as an orthonormal basis of eigenvectors in $\mathfrak{m}_1$ for φ_1 and concatenate these paths to yield a diagonal path. $\square$

Chapter 5

Curvature of Generalized Warped Product Metrics

On $I \times K/H$ consider the metric $g := dt^2 + g_{\varphi(t)}$, where $I \subset \mathbb{R}$ is an interval and $g_{\varphi(t)}$ is a K-invariant metric on K/H which is constructed by propagating the inner product

$$Q_{\varphi(t)}(X,Y) := Q(\varphi(t)X,Y) \quad \text{for } X,Y \in \mathfrak{m}\,,$$

where $\varphi(t) : \mathfrak{m} \to \mathfrak{m}$ is a self-adjoint, positive definite, Ad_H-equivariant map. We call such a metric a generalized warped product metric. The parameter t is omitted from now on for the matters of clarity.

For analyzing the curvature of $g = dt^2 + g_{\varphi(t)}$ we use a formula developed in [STu] which is the content of the next

Proposition 5.1. *Let $g = dt^2 + g_{\varphi(t)}$ be a K-invariant metric on $I \times K/H$ as above and let $u \in \mathbb{R}$, $X, Y \in T_{[eH]}K/H \cong \mathfrak{m}$. Then*

$$\begin{aligned}
&R^g\big(u\partial_t + X, Y; Y, u\partial_t + X\big)\\
&= R^g\big(X,Y;Y,X\big) + 2uR^g\big(\partial_t, Y; Y, X\big) + u^2 R^g\big(\partial_t, Y; Y, \partial_t\big)\\
&= R^{g_\varphi}\big(X,Y;Y,X\big) + \frac{1}{4}\Big(Q\big(\dot\varphi X, Y\big)^2 - Q\big(\dot\varphi X, X\big)Q\big(\dot\varphi Y, Y\big)\Big)\\
&\qquad + \frac{1}{2}u\Big(3Q\big(\dot\varphi[X,Y],Y\big) + Q\big(\varphi^{-1}Y, \dot\varphi\big([X,\varphi Y] + [Y,\varphi X]\big)\big) - 2\,Q\big(\varphi^{-1}X, \dot\varphi[Y,\varphi Y]\big)\Big)\\
&\qquad - \frac{1}{4}u^2\,Q\big(\big(2\ddot\varphi - \dot\varphi\varphi^{-1}\dot\varphi\big)Y, Y\big)\,,
\end{aligned}$$

where R^g denotes the curvature tensor of g and R^{g_φ} denotes the curvature tensor of $g_{\varphi(t)}$.

Remark 5.2. Note that it is necessary that $\dot{\varphi}$ is bounded in order to get a nonnegatively curved metric. We will assume this from now on. Furthermore we will assume that $\dot{\varphi}$ is positive semidefinite.

The curvature formula for a more special metric is stated in the next

Corollary 5.3. *Let $\tilde{g} = dt^2 + f(t)^2 g$ be a K-invariant metric on $I \times K/H$ and let $u \in \mathbb{R}$, $X, Y \in T_{[eH]}K/H \cong \mathfrak{m}$. Then*

$$\tilde{R}\big(u\partial_t + X, Y; Y, u\partial_t + X\big) = -u^2 \ddot{f} f \, \|Y\|_g^2 + f^2 \left(R(X,Y;Y,X) - (\dot{f})^2 \, \|X \wedge Y\|_g^2 \right) ,$$

where $\tilde{R}$ denotes the curvature tensor of $\tilde{g}$ and R denotes the curvature tensor of g. It follows that the curvature of $(I \times K/H, \tilde{g})$ is positive (nonnegative) iff $\ddot{f} < 0$ and $(\dot{f})^2 < \inf(\sec(K/H, g))$, $\big(\ddot{f} \leq 0$ and $(\dot{f})^2 \leq \inf(\sec(K/H, g))\big)$.

Some of the following results will deal with metrics if we have an underlying chain of Lie groups and which are diagonal, i.e. we assume $\varphi(t)$ is determined by

$$\varphi(t)|_{\mathfrak{m}_i} = f_i^2(t) \, \mathrm{Id}|_{\mathfrak{m}_i} \quad 1 \leq i \leq r \, , \tag{5.1}$$

for some fixed Ad_H-invariant Q-orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_r$ and where the f_i's are smooth functions on I. For the first and second derivative of φ as in (5.1) we have

$$\dot{\varphi}|_{\mathfrak{m}_i} = 2 f_i \dot{f}_i \, \mathrm{Id}|_{\mathfrak{m}_i} \quad \text{and} \quad \ddot{\varphi}|_{\mathfrak{m}_i} = 2\left((\dot{f}_i)^2 + f_i \ddot{f}_i \right) \mathrm{Id}|_{\mathfrak{m}_i} \quad 1 \leq i \leq r \, .$$

Note that by Remark 5.2 the $\dot{f}_i$'s have to be bounded.
We write X and Y in terms of an orthonormal basis $\{E_1, \ldots, E_r\}$ as $X = \sum X_i$ and $Y = \sum Y_j$ where $X_i = a_i E_i$ and $Y_j = b_j E_j$. We have

Proposition 5.4 (cf. [STu], page 11 Prop 4.3)**.** *Let the situation be as above. Then*

$$\begin{aligned} & R^g\big(u\partial_t + X, Y; Y, u\partial_t + X\big) \\ = & R^{g_\varphi}\big(X, Y; Y, X\big) - \sum_i f_i^2 (\dot{f}_i)^2 \, \|X_i \wedge Y_i\|^2 - \sum_{i<j} f_i \dot{f}_i f_j \dot{f}_j \, \|X_i \wedge Y_j + X_j \wedge Y_i\|^2 \\ & + 3u \sum_{1 \leq k < i \leq r} f_k^2 \left(\frac{\dot{f}_k}{f_k} - \frac{\dot{f}_i}{f_i} \right) Q\big(B_k^i, Y_k\big) - u^2 \sum_i f_i \ddot{f}_i \, Q\big(Y_i, Y_i\big) \, . \end{aligned}$$

Our main goal in the upcoming is to develop conditions under which there exists a reparametrization $\sigma(s) := \varphi(t(s))$ such that we obtain nonnegative or rather positive curvature for a generalized warped product metric. Let $R^{\tilde{g}}$ denote the curvature tensor of the metric $\tilde{g} := ds^2 + g_{\sigma(s)}$. Denoting differentiation w.r.t. t with $\dot{}$ and w.r.t. s with $'$ we get by calculation

$$\begin{aligned} &R^{\tilde{g}}\big(u\partial_s + X, Y; Y, u\partial_s + X\big) \\ &= R^{g_\varphi}\big(X, Y; Y, X\big) - \frac{1}{4}\,(t')^2\,\|X \wedge Y\|^2_{g_{\dot\varphi}} + 2\,ut'R^g\big(\partial_t, Y; Y, X\big) \\ &\quad + u^2\Big((t')^2 R^g\big(\partial_t, Y; Y, \partial_t\big) - \frac{1}{2}\,t''Q\big(\dot\varphi Y, Y\big)\Big)\,. \end{aligned} \tag{5.2}$$

Given an underlying chain of Lie groups and a diagonal metric this reads

$$\begin{aligned} &R^{\tilde{g}}\big(u\partial_s + X, Y; Y, u\partial_s + X\big) \\ &= R^{g_\varphi}\big(X, Y; Y, X\big) - \frac{1}{4}\,(t')^2\Big(\sum_i f_i^2(\dot f_i)^2\,\|X_i \wedge Y_i\|^2 + \sum_{i<j} f_i\dot f_i f_j \dot f_j\,\|X_i \wedge Y_j + X_j \wedge Y_i\|^2\Big) \\ &\quad + 3ut'\sum_{1\le k<i\le r} f_k^2\Big(\frac{\dot f_k}{f_k} - \frac{\dot f_i}{f_i}\Big)Q\big(B^i_k, Y_k\big) - u^2\Big(\sum_i f_i\big((t')^2\ddot f_i + t''\dot f_i\big)Q\big(Y_i, Y_i\big)\Big)\,. \end{aligned}$$

Proposition 5.5. *Suppose that there are constants $C_1, C_2, C_3 > 0$ such that*

$$R^{g_\varphi} \ge C_1\big(\,\|X \wedge Y\|^2_{g_{\dot\varphi}} + \big(H(X,Y)\big)^2\,\big) \tag{5.3}$$

$$\big|R^g\big(\partial_t, Y; Y, X\big)\big| \le C_2\,\|Y\|_{g_{\dot\varphi}} \cdot \big(\|X \wedge Y\|_{g_{\dot\varphi}} + H(X,Y)\big) \tag{5.4}$$

$$R^g\big(\partial_t, Y; Y, \partial_t\big) \ge -\,C_3\,\|Y\|^2_{g_{\dot\varphi}}\,, \tag{5.5}$$

where $H : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}_{\ge 0}$ is some function. Then there is a reparametrization $\sigma(s) = \varphi(t(s))$ such that $ds^2 + g_{\sigma(s)}$ has nonnegative curvature.

Proof. Choosing a reparametrization where $|t'|$ is sufficiently small we can surely guarantee $C_1 - \frac{1}{4}(t')^2 \ge \varepsilon > 0$ for some $\varepsilon > 0$. Therefore by using (5.3) we can estimate

$$\begin{aligned} R^{g_\varphi}\big(X, Y; Y, X\big) - \frac{1}{4}\,(t')^2\,\|X \wedge Y\|^2_{g_{\dot\varphi}} &\ge \varepsilon\,\|X \wedge Y\|^2_{g_{\dot\varphi}} + C_1\big(H(X,Y)\big)^2 \\ &\ge C\Big(\,\|X \wedge Y\|^2_{g_{\dot\varphi}} + \big(H(X,Y)\big)^2\,\Big)\,, \end{aligned}$$

where $C := \min\{\varepsilon, C_1\}$. Furthermore with condition (5.5) we have

$$\Big((t')^2 R^g\big(\partial_t, Y; Y, \partial_t\big) - \frac{1}{2}\,t''Q\big(\dot\varphi Y, Y\big)\Big) \ge \Big(-(t')^2 C_3 - \frac{1}{2}\,t''\Big)\|Y\|^2_{g_{\dot\varphi}}\,.$$

Thus the coefficient of u^2 is nonnegative when choosing a reparametrization satisfying

$$-\,(t')^2 C_3 - \frac{1}{2}\,t'' > 0 \quad\iff\quad \frac{-t''}{(t')^2} > 2\,C_3 \quad\iff\quad \left(\frac{1}{t'}\right)' > 2\,C_3\,. \tag{5.6}$$

Moreover by this demand the coefficient of u^2 is zero if and only if $\|Y\|_{g_{\dot\varphi}} = 0$ but then the coefficient of u is zero as well because of condition (5.4). For the time being consider that (5.6) is given. Using the observations above and condition (5.4) we can state that the discriminant of (5.2) is greater or equal to

$$\left(\left(-\,(t')^2 C_3 - \frac{1}{2}\,t''\right)C - (t')^2\tilde{C}^2\right)\left(\|X\wedge Y\|^2_{g_{\dot\varphi}} + \big(H(X,Y)\big)^2\right)\|Y\|^2_{g_{\dot\varphi}}\,,$$

where $\tilde{C}$ satisfies $C_2\,\|Y\|_{g_{\dot\varphi}}\cdot\big(\|X\wedge Y\|_{g_{\dot\varphi}} + H(X,Y)\big) \le \tilde{C}\,\|Y\|^2_{g_{\dot\varphi}}\big(\|X\wedge Y\|^2_{g_{\dot\varphi}} + (H(X,Y))^2\big)$. By performing similar manipulations to that in (5.6) we can state that this expression is nonnegative if we demand for the reparametrization to satisfy

$$\left(\frac{1}{t'}\right)' > \frac{2\left(C_3 C + \tilde{C}^2\right)}{C}\,. \tag{5.7}$$

Note that (5.6) is satisfied if (5.7) is. Thus demanding for the reparametrization to have a sufficiently small first derivative and to satisfy (5.7) gives the desired result. □

Corollary 5.6. *Let a chain of Lie groups be given and let φ induce a diagonal metric. Moreover let $p \in \{1,\dots,r\}$. Then there is a reparametrization $\sigma(s) = \varphi(t(s))$ such that $ds^2 + g_{\sigma(s)}$ has nonnegative curvature if the following conditions are satisfied*

$$R^{g_\varphi}\big(X,Y;Y,X\big) \ge C\Big(\big\|X_{\mathfrak{k}_p\cap\mathfrak{m}}\wedge Y_{\mathfrak{k}_p\cap\mathfrak{m}}\big\|^2 + \big\|[X,Y]_{\mathfrak{k}_p\cap\mathfrak{m}}\big\|^2\Big) \quad \text{for some } C>0 \tag{5.8}$$

$$\dot f_k \ge \xi \quad \text{for } 1\le k\le p \quad \text{for some } \xi > 0 \tag{5.9}$$

$$\dot f_k = 0 \quad \text{for } p+1\le k\le r \tag{5.10}$$

$$\ddot f_i \le \delta \dot f_i \quad \text{for some } \delta>0 \tag{5.11}$$

Proof. We have to show that the conditions of Proposition 5.5 are satisfied. Note that by (5.9), $\dot\varphi$ induces a metric on K_p . Therefore there are constants $c_1, c_2 \in R^+_*$ such that

$$c_1\|Z\|_{g_{\dot\varphi}} \le \|Z\|_{g_\varphi} \le c_2\|Z\|_{g_{\dot\varphi}} \quad \forall Z\in\mathfrak{k}_p\,. \tag{5.12}$$

Thus by (5.8) and (5.10) we have

$$R^{g_\varphi}\big(X,Y;Y,X\big) \geq C_1\Big(\big\|X\wedge Y\big\|^2_{g_{\dot\varphi}} + \big\|[X,Y]_{\mathfrak{k}_p\cap\mathfrak{m}}\big\|^2\Big)\,,$$

for some constant $C_1>0$.
Since we have an underlying chain of Lie groups the coefficient of u is given as in Proposition 5.4 and we obtain by using (5.10)

$$\begin{aligned}
&3\sum_{1\leq k<i\leq r} f_k^2\bigg(\frac{\dot f_k}{f_k}-\frac{\dot f_i}{f_i}\bigg)Q\big(B_k^i,Y_k\big)\\
=&3\bigg(\sum_{1\leq k\leq p}\sum_{k<i\leq r} f_k^2\bigg(\frac{\dot f_k}{f_k}-\frac{\dot f_i}{f_i}\bigg)Q\big(B_k^i,Y_k\big)+\sum_{p<k\leq r}\sum_{k<i<r} f_k^2\bigg(\frac{\dot f_k}{f_k}-\frac{\dot f_i}{f_i}\bigg)Q\big(B_k^i,Y_k\big)\bigg)\\
=&3\bigg(\sum_{1\leq k<i\leq p} f_k^2\bigg(\frac{\dot f_k}{f_k}-\frac{\dot f_i}{f_i}\bigg)Q\big(B_k^i,Y_k\big)+\sum_{1\leq k\leq p}\dot f_k f_k\sum_{p<i\leq r} Q\big(B_k^i,Y_k\big)\bigg)\,. \qquad (5.13)
\end{aligned}$$

We set $D_{\inf}:=\inf\{f_i\,|\,1\leq i\leq p\}$ and $D_{\sup}:=\sup\{f_i\,|\,1\leq i\leq p\}$.
Because of (5.9) and the general assumption that the $\dot f_i$'s are bounded we can define $\hat C:=\sup\Big\{\frac{\dot f_i}{\dot f_k}\,\big|\,1\leq k<i\leq p\Big\}$ and so we obtain

$$f_k^2\left|\frac{\dot f_k}{f_k}-\frac{\dot f_i}{f_i}\right| \leq \dot f_k f_k\bigg(1+\left|\frac{\dot f_i}{\dot f_k}\right|\left|\frac{f_k}{f_i}\right|\bigg)\leq \dot f_k f_k\bigg(1+\hat C\,\frac{D_{\sup}}{D_{\inf}}\bigg)\leq \dot f_k f_k C \quad \text{for } k<i\leq p\,, \qquad (5.14)$$

where we have set $C:=1+\hat C\frac{D_{\sup}}{D_{\inf}}$.
Moreover observe that we have $\big\|B_k^i\big\|_{g_\varphi}\leq\big\|B^i\big\|_{g_\varphi}\leq\lambda\big\|X_i\wedge Y_i\big\|_{g_\varphi}\leq\lambda\big\|X\wedge Y\big\|_{g_\varphi}$, where λ is the norm of the linear map $[,]:\Lambda^2\mathfrak{m}\to\mathfrak{k}$ arising from the universal property of tensors. And since there is a constant $c_1>0$ such that $\big\|B_k^i\big\|_{g_{\dot\varphi}}\leq c_1\big\|B_k^i\big\|_{g_\varphi}$ we have especially $\big\|B_k^i\big\|_{g_{\dot\varphi}}\leq\lambda_1\big\|X\wedge Y\big\|_{g_\varphi}$ for some $\lambda_1>0$. So by using (5.12), (5.14) and the Cauchy-Schwarz inequality we can surely estimate

$$\begin{aligned}
&\bigg|\sum_{1\leq k<i\leq r} f_k^2\bigg(\frac{\dot f_k}{f_k}-\frac{\dot f_i}{f_i}\bigg)Q\big(B_k^i,Y_k\big)\bigg| \leq \sum_{1\leq k<i\leq p}\dot f_k f_k C\big|Q\big(B_k^i,Y_k\big)\big|\\
=&\frac{C}{2}\sum_{1\leq k<i\leq p}\big|Q_{g_{\dot\varphi}}\big(B_k^i,Y_k\big)\big|\leq\frac{C}{2}\sum_{1\leq k<i\leq p}\big\|B_k^i\big\|_{g_{\dot\varphi}}\big\|Y_k\big\|_{g_{\dot\varphi}}\leq\bar C_2\big\|X\wedge Y\big\|_{g_{\dot\varphi}}\big\|Y\big\|_{g_{\dot\varphi}}\,,
\end{aligned}$$

where the constant $\bar C$ fulfilling the last estimate exists because of (5.12).

The second sum in (5.13) can be estimated as

$$\begin{aligned}
&\Big|\sum_{1\le k\le p}\dot f_k f_k\sum_{p<i\le r}Q\big(B^i_k,Y_k\big)\Big| = \Big|\sum_{1\le k\le p}\dot f_k f_k\sum_{p<i\le r}Q\big(B^i,Y_k\big)\Big| \\
&= \Big|Q\Big(\sum_{p<i\le r}B^i,\sum_{1\le k\le p}\dot f_k f_k Y_k\Big)\Big| = \Big|Q\Big(\big[X_{\mathfrak{k}_p^\perp\cap\mathfrak{m}},Y_{\mathfrak{k}_p^\perp\cap\mathfrak{m}}\big]_{\mathfrak{k}_p\cap\mathfrak{m}},\sum_{1\le k\le p}\dot f_k f_k Y_k\Big)\Big| \\
&= \frac{1}{2}\,\big|Q_{g_{\dot\varphi}}\big(\big[X_{\mathfrak{k}_p^\perp\cap\mathfrak{m}},Y_{\mathfrak{k}_p^\perp\cap\mathfrak{m}}\big]_{\mathfrak{k}_p\cap\mathfrak{m}},Y\big)\big| \le \frac{1}{2}\,\|Y\|_{g_{\dot\varphi}}\cdot\big\|\big[X_{\mathfrak{k}_p^\perp\cap\mathfrak{m}},Y_{\mathfrak{k}_p^\perp\cap\mathfrak{m}}\big]_{\mathfrak{k}_p\cap\mathfrak{m}}\big\|_{g_{\dot\varphi}} \\
&= \frac{1}{2}\,\|Y\|_{g_{\dot\varphi}}\cdot\big\|\big[X,Y\big]_{\mathfrak{k}_p\cap\mathfrak{m}}-\big[X_{\mathfrak{k}_p\cap\mathfrak{m}},Y_{\mathfrak{k}_p\cap\mathfrak{m}}\big]_{\mathfrak{k}_p\cap\mathfrak{m}}\big\|_{g_{\dot\varphi}} \\
&\le \tilde C_2\,\|Y\|_{g_{\dot\varphi}}\cdot\big(\|[X,Y]_{\mathfrak{k}_p\cap\mathfrak{m}}\|+\|X\wedge Y\|_{g_{\dot\varphi}}\big)\,,
\end{aligned}$$

where we used the triangle inequality and the above mentioned identities in the last estimate and $\tilde C$ is an apprpriate constant fulfilling the last estimate.
So taking $C_2:=\max\{\tilde C_2,\bar C_2\}$ and

$$H(X,Y):=\big\|[X,Y]_{\mathfrak{k}_p\cap\mathfrak{m}}\big\|$$

we see that conditions (5.3) and (5.4) of Proposition 5.5 are satisfied. The validity of (5.5) is guaranteed by (5.11). □

Remark 5.7. If we make the above considerations modulo Cheeger deformations we can replace condition (5.8) in Corollary 5.6 by the weaker condition

$$R^\varphi\big(X,Y;Y,X\big)\ge C\,\big\|X_{\mathfrak{k}_p\cap\mathfrak{m}}\wedge Y_{\mathfrak{k}_p\cap\mathfrak{m}}\big\|^2\,. \tag{5.15}$$

For this consider the Cheeger deformation

$$\big(K\times K/H,\lambda^{-1}g_Q\oplus g_\varphi\big)\to\big(K/H,g_\phi\big)\,.$$

In Chapter 3 we have already computed the curvature of the metric g_ϕ to be

$$\begin{aligned}
&R^{g_\phi}\big(\phi^{-1}X,\phi^{-1}Y;\phi^{-1}Y,\phi^{-1}X\big) \\
&= \frac{1}{4}\,\lambda^3\|[X,Y]\|^2+R^{g_\varphi}\big(\psi^{-1}X,\psi^{-1}Y;\psi^{-1}Y,\psi^{-1}X\big) \\
&\quad+\frac{3}{4}\,\lambda^2\,Q\big(\big(\lambda^{-1}+\psi\big)^{-1}\big([X,Y]_\mathfrak{m}+\lambda^{-1}A_\mathfrak{m}-\lambda^{-1}\psi B_\mathfrak{m}\big),[X,Y]_\mathfrak{m}+\lambda^{-1}A_\mathfrak{m}-\lambda^{-1}\psi B_\mathfrak{m}\big) \\
&\quad+\frac{3}{4}\,\lambda^3\big\|[X,Y]_\mathfrak{h}+\lambda^{-1}A_\mathfrak{h}\big\|^2\,.
\end{aligned}$$

For the argumentation here the detailed expansion of the O'Neill term, i.e. the last two

lines in the above formula, is not important, just that the O'Neill term is of course non-negative. The claim follows when choosing λ approriate since the the missing term in (5.8) is contributed by the Cheeger deformation.
Furthermore when condition (5.8) is replaced by (5.15) the conditions in Corollary 5.6 are equivalent for nonnegative curvature since if (5.15) is not satisfied we can not even have nonnegative curvature for planes tangential to the orbits (the case when $u = 0$).

If we are seeking for positive rather than nonnegative curvature, condition (5.3) of Proposition 5.5 is not sufficient because if the right hand side of (5.3) vanishes we can not even guarantee positive curvature for planes tangential to the orbits.
Instead we have

Proposition 5.8. *Suppose that there are constants $C_1, C_2, C_3 > 0$ such that*

$$R^{g_\varphi} \geq C_1 \|X \wedge Y\|^2_{g_\varphi} \tag{5.16}$$
$$\left|R^g\big(\partial_t, Y; Y, X\big)\right| \leq C_2 \|X \wedge Y\|_{g_\varphi} \|Y\|_{g_{\dot\varphi}} \tag{5.17}$$
$$R^g\big(\partial_t, Y; Y, \partial_t\big) \geq -C_3 \|Y\|^2_{g_{\dot\varphi}} \,. \tag{5.18}$$

Then there is a reparametrization $\sigma(s) = \varphi(t(s))$ such that $ds^2 + g_{\sigma(s)}$ has positive curvature.

Proof. As in the proof of Proposition 5.5 we have show that there is a reparametrization such that the discriminant of (5.2) is positive. Using the conditions in the proposition we can make analogeous estimates as in the proof of Proposition 5.5 to get the desired result. □

Corollary 5.9. *Let I be compact. Suppose that $\dot\varphi$ induces a metric and that*

$$R^\varphi(X, Y; Y, X) \geq C_1 \|X \wedge Y\|^2_{g_{\dot\varphi}} \tag{5.19}$$

for some constant C_1. Then there is a reparametrization $\sigma(s) = \varphi(t(s))$ such that $ds^2 + g_{\sigma(s)}$ has positive curvature.

Proof. We have to show that the conditions of Proposition 5.8 are satisfied. Since we assume that I is compact and $\dot\varphi$ induces a metric there are constants $c_1, c_2 \in \mathbb{R}^+_*$ such that for all $t \in I$

$$c_1 \|Z\|_{g_\varphi} \leq \|Z\|_{g_{\dot\varphi}} \leq c_2 \|Z\|_{g_\varphi} \quad \forall Z \in \mathfrak{k}\,.$$

So the validity of condition (5.16) follows from (5.19). Moreover (5.17) follows once we observe that we have

$$2\left|R^g\big(\partial_t, Y; Y, X\big)\right| \leq C_2 \, \|X \wedge Y\|_{g_\varphi} \|Y\|_{g_\varphi} ,$$

where C_2 is the norm of the linear map

$$\begin{aligned} \Lambda^2 \, \mathfrak{m} \otimes \mathfrak{m} &\to \mathbb{R} \\ (X \wedge Y) \otimes Z &\mapsto R^g(\partial_t, Y; Y, X) , \end{aligned}$$

arising from the universal property of tensors. Besides condition (5.18) follows from the assumption that I is compact. □

Corollary 5.10. *Let a chain of Lie groups be given and let φ induce a diagonal metric. Then there is a reparametrization $\sigma(s) = \varphi(t(s))$ such that $ds^2 + g_{\sigma(s)}$ has positive curvature if the following conditions hold*

$$\sec_{g_\varphi} \geq \varepsilon > 0 \tag{5.20}$$

$$\ddot{f}_i \leq \delta \dot{f}_i \quad \textit{for some } \delta > 0 \tag{5.21}$$

$$\frac{\dot{f}_i}{\dot{f}_j} \geq c_{ij} \quad \textit{for } i < j \textit{ and some constants } c_{ij} > 0 \tag{5.22}$$

Proof. The statement in the corollary is proven if we can show that with the properties demanded for the f_i's the conditions of Proposition 5.8 are satisfied. (5.20) is equivalent to (5.16). To show that (5.17) is satisfied we have to show that there is a constant $C_2 > 0$ such that

$$\left| \frac{3}{2} \sum_{1 \leq k < i \leq r} f_k^2 \left(\frac{\dot{f}_k}{f_k} - \frac{\dot{f}_i}{f_i} \right) Q\big(B_k^i, Y_k\big) \right| \leq C_2 \, \|X \wedge Y\|_{g_\varphi} \|Y\|_{g_{\dot{\varphi}}} .$$

This is done analogeously to the estimate of the first sum in (5.13) in the proof of Corollary 5.6 once we observe that an analogue to the constant C there can also be defined here due to assumption (5.22).
The validity of (5.18) is guaranteed by (5.21). □

Proposition 5.11. *Suppose that g has nonnegative curvature. Then g remains nonnegatively curved if we reparametrize g with a function satisfying $|t'(s)| \leq 1$ and $t''(s) \leq 0$.*

Proof. With the conditions imposed on the reparametrization it is easy to see that the discriminant of (5.2) gets greater than if $t(s) = s$ in which case the discriminant is nonnegative because of our assumption. □

Remark 5.12. If g has nonnegative curvature then there is a reparametrization such that the reparametrized metric is nonnegatively curved and the principal orbit on the boundary is totally geodesic. For this we just have to choose a reparametrization fulfilling the conditions of Proposition 5.11 with the additional property $\lim_{s\to b} t^{(k)}(s) = 0$, $k \in \mathbb{N}$.

The following theorem generalizes Theorem 5.1 in [STu], where it was shown that there is an extension of a normal homogeneous metric on the sphere to its interior, i.e. to the ball with boundary the sphere in such a way that the metric is a warped product metric near the boundary. Here a normal homogeneous metric is replaced by an arbitrary positively curved metric on the sphere.

Theorem 5.13. *Let $K \subset O(n+1)$ be a Lie subgroup which acts transitively on $S^n \subset \mathbb{R}^{n+1}$ and let $H \subset K$ be the isotropy group at some point. Let g_1 be a K-invariant metric on K/H with positive curvature and let $r(x) := \|x\|$ be the radius function on $\mathbb{R}^{n+1}$.*
Then there exists a K-invariant metric g on the unit ball $B_1(0) \subset \mathbb{R}^{n+1}$ with positive curvature and an $\varepsilon > 0$ such that on $r^{-1}((1-\varepsilon, 1))$ we have $g = dr^2 + \eta(r)^2 g_1$ where $\eta : (1-\varepsilon, 1) \to \mathbb{R}$ satisfies $\eta, \eta' > 0$.

Proof. Let $[a, b] \subset \mathbb{R}$ and consider a diagonal path $\varphi(t) = \operatorname{diag}\{f_i^2(t)\}$ through positively curved metrics with $g_{\varphi(a)} = g_0$ and $g_{\varphi(b)} = g_1$ which exists by Proposition 4.5. Impose the additional condition

$$f_i(t)|_{[a,a+\varepsilon)} = const\,, \qquad f_i(t)|_{(b-\varepsilon,b]} = const\,,$$

for some $0 < \varepsilon < \frac{a+b}{2}$. This additional condition does not change the fact that we have a diagonal path through positively curved metrics. We consider the metric $g = dt^2 + g_{\varphi(t)}$ which on $[a, a+\varepsilon) \times K/H$ takes the form $dt^2 + g_0$ and on $(b-\varepsilon, b] \times K/H$ it takes the form $dt^2 + g_1$. For some $\delta > 0$ define $\tilde\varphi(t) := \operatorname{diag}\{\exp(2h(t)) f_i^2(t)\}$ with a function $h : [a, b] \to \mathbb{R}$ satisfying

$$h'(t) \geq \delta - \frac{f_i'(t)}{f_i(t)} \quad \text{for all } 1 \leq i \leq r\,. \tag{5.23}$$

Then $\tilde f_i(t) := \exp(h(t)) f_i(t)$ satisfy $\tilde f_i'(t) \geq \delta \tilde f_i(t)$ and the condition of Corollary 5.9 is satisfied by the metric $\tilde g := dt^2 + g_{\tilde\varphi(t)}$. Especially the metric $\tilde g$ is of the form $dt^2 + \exp(2h(t)) g_0$ on $[a, a+\varepsilon) \times K/H$ and on $(b-\varepsilon, b] \times K/H$ we have $dt^2 + \exp(2h(t)) g_1$. Now an application of Corollary 5.9 yields a metric $ds^2 + g_{\sigma(s)}$ on $[\tilde a, \tilde b] \times K/H$ where $[\tilde a, \tilde b]$ is in accordance with the reparametrization $\sigma(s) = \tilde\varphi(t(s))$ which has the form $ds^2 + \exp(2h(t(s))) g_0$ on $[\tilde a, \tilde a + \tilde\varepsilon) \times K/H$ and the form $ds^2 + \exp(2h(t(s))) g_1$ on $(\tilde b - \tilde\varepsilon, \tilde b] \times K/H$, where $\tilde\varepsilon > 0$. By virtue of Corollary 5.9 this metric has positive curvature. Therefore we must have by Corollary 5.3 that $\frac{d^2}{ds^2} \exp(h(t(s))) < 0$ on the interval $[\tilde a, \tilde a + \tilde\varepsilon)$. Moreover

we can choose h such that $\frac{d}{ds}\exp(h(t(s))) > 0$ because the obstruction (5.23) does not conflict this.
Let $s_0 \in [\tilde{a}, \tilde{a} + \tilde{\varepsilon})$. For an appropriate choice of $\tilde{a}$ the existence of a smooth function $\zeta : (0, \tilde{a} + \tilde{\varepsilon}) \to \mathbb{R}$ with the following properties

$$\zeta|_{(0,\mu)} = \sin(s), \qquad \zeta|_{(s_0,\tilde{a}+\tilde{\varepsilon})} = \exp(h(t(s))), \qquad \zeta'' < 0\,,$$

where $\mu > 0$ is sufficiently small, is guaranteed. Now define the $O(n+1)$-invariant metric $g := ds^2 + \zeta(s)^2 g_0$ on the Ball $B_{\tilde{a}+\tilde{\varepsilon}}(0) \subset \mathbb{R}^{n+1}$. The standard metric on S^{n+1} with respect to the normal coordinate chart is given by $ds^2 + \sin^2(s) g_0$. Therefore the germ of g at $s = 0$ is a smooth metric of constant curvature 1. Moreover since $\zeta'' < 0$ and $\zeta'(s) < \zeta'(0) = 1 = \min(\sec(S^n, g_0))$ for all $s > 0$ we have by Corollary 5.3 that g has positive curvature. Furthermore the germ of g coincides with the germ of the metric $ds^2 + g_{\sigma(s)}$ at $s = \tilde{a} + \tilde{\varepsilon}$ by construction. So we can glue these metrics to obtain a metric with positive curvature on $B_{\tilde{b}}(0)$ which on $(\tilde{b} - \tilde{\varepsilon}, \tilde{b}\,] \times K/H$ has the form $ds^2 + \exp(2h(t(s)))g_1$. Finally a scaling process gives the statement in the theorem. □

Remark 5.14. Observe that in the construction of the function η in the proof of Theorem 5.13 we are free to prescribe its derivative by changing the auxiliary function h or the reparametrization t.
Therefore given a positively curved metric $g = dr^2 + \eta(r)^2 g_1$ on $r^{-1}((1-\varepsilon, 1))$ where $\eta : (1-\varepsilon, 1) \to \mathbb{R}$ satisfies $\eta, \eta' > 0$ for some $\varepsilon > 0$, we can extend this metric to a positively curved metric on $B_1(0) \subset \mathbb{R}^{n+1}$

Remark 5.15. We can extend the resulting metric of Theorem 5.13 to a nonnegatively curved metric on $\mathbb{R}^{n+1}$ which outside a compact set $\overline{B_R(0)} \subset \mathbb{R}^{n+1}$, $R > 1$, is of the form $dt^2 + c_0^2 g_1$ for some arbitrary large constant c_0^2. For this observe that when extending η to the interval $(1-\varepsilon, \infty)$ by demanding η to be concave, the metric will stay nonnegatively curved by Corollary 5.3. So given a constant $c_0^2 > 1$ we obtain the statement claimed above when we demand in addition for η to satisfy $\lim_{r \to R_1} \eta(r) = c_0^2$ for some $1 < R_1 < R$.

Chapter 6

Nonnegatively Curved Invariant Metrics on Homogeneous Disc Bundles

Next we are going to deal with nonnegatively curved invariant metrics on homogeneous disc bundles $G \times_K \overline{B_R(0)}$ with a totally geodesic principal orbit on the boundary. We will work with the notions of Chapter 2.
Once we have constructed a nonnegatively curved invariant metric on $G \times_K \overline{B_R(0)}$ we can make the principal orbit on the boundary totally geodesic by Remark 5.12.

In [STa2] the authors considered nonnegatively curved invariant metrics on homogeneous disc bundles with normal homogeneous collars. These are metrics that are G-equivariantly isometric to $\big((R-\varepsilon, R] \times G/H, dt^2 + g_Q\big)$ near the boundary for some $\varepsilon > 0$. They proved the following result

Theorem 6.1 (cf. [STa2], p.5, Thm. 2.1)**.** *If there exists a $C > 0$ such that for all $X = X_{\mathfrak{m}} + X_{\mathfrak{s}}$, $Y = Y_{\mathfrak{m}} + Y_{\mathfrak{s}} \in \mathfrak{p}$ we have the inequality*

$$\big\| X_{\mathfrak{m}} \wedge Y_{\mathfrak{m}} \big\| \leq C \cdot \big\| [X, Y] \big\| \,, \tag{6.1}$$

then $G \times_K \overline{B_R(0)}$ admits a nonnegatively curved G-invariant metric with normal homogeneous collar.

Proof. The metric in question is constructed via Cheeger's method as a submersion metric. Let g_a be the Ad_K-invariant metric on G which is induced by the inner product

$$Q_a := (1 + a)\, Q|_{\mathfrak{k}} + Q_{\mathfrak{s}} \,,$$

where $a > 0$. By Theorem 1.20, g_a has nonnegative curvature for all planes contained in $\mathfrak{p}$ for sufficiently small a

Consider $\left(G \times \mathbb{R}^n, g_a + \tilde{g}\right)$, where $\tilde{g}$ is the metric of Remark 5.15 with g_1 replaced by the biinvariant metric g_Q. Denote by $\hat{g}$ the metric on $G \times_K \mathbb{R}^n$ arising from Cheeger's construction. Then $\hat{g}$ has nonnegative curvature since the horizontal lifts of all planes in the base have nonnegative curvature so the planes in the base also have nonnegative curvature due to O'Neill's formula.

Moreover we choose $c_0^2 = \frac{a+1}{a}$ so that $(a^{-1} + c_0^{-2})^{-1} = 1$. Thus from Lemma 2.3 it follows that outside a compact set $\hat{g}$ has the form $dt^2 + g_Q$ and the statement follows. □

In fact in the same paper the authors show that (6.1) is almost necessary. Namely we have

Theorem 6.2 (cf. [STa2], p.7, Thm. 3.1)**.** *Let $\mathfrak{m}_1 \subset \mathfrak{m}$ be a non-trivial Ad_H-irreducible subspace such that $\mathfrak{m}$ contains no irreducible summand equivalent to $\mathfrak{m}_1$ and let $dt^2 + g_{\phi(t)}$ be an invariant metric on $(0, R] \times G/H$ with nonnegative curvature and $\phi(R)|_{\mathfrak{m}_1 \oplus \mathfrak{s}} = \mathrm{Id}$. Then there exists a $C > 0$ such that for all $X = X_{\mathfrak{m}_1} + X_{\mathfrak{s}}, Y = Y_{\mathfrak{m}_1} + Y_{\mathfrak{s}} \in \mathfrak{m}_1 \oplus \mathfrak{s}$ we have:*

$$\left\| X_{\mathfrak{m}_1} \wedge Y_{\mathfrak{m}_1} \right\| \leq C \cdot \left\| [X, Y] \right\| . \tag{6.2}$$

As a consequence we obtain

Corollary 6.3. *Suppose that $\mathfrak{m}$ decomposes irreducibly and that we have an underlying chain of Lie groups. Furthermore suppose that the metric $dt^2 + g_{\phi(t)}$ on $(0, R] \times G/H$ has nonnegative curvature. If $\phi(R)|_{\mathfrak{m}_i} = a \cdot \mathrm{Id}$, $a > 1$, for an irreducible summand $\mathfrak{m}_i$ and $\phi(R)|_{\mathfrak{s}} = \mathrm{Id}$ then for all $X = X_{\mathfrak{m}_i} + X_{\mathfrak{s}}, Y = Y_{\mathfrak{m}_i} + Y_{\mathfrak{s}} \in \mathfrak{m}_i \oplus \mathfrak{s}$ we have* (6.2)*.*

Proof. Let $dt^2 + g_{\phi(t)}$ be the metric with the properties considered in the corollary. Consider the Cheeger deformation

$$\left(K_i \times G/H, \lambda^{-1} g_Q|_{K_i} \oplus g_{\phi(R)}\right) \to \left(G/H, g_\lambda\right),$$

where K_i is the Lie group such that its Lie algebra $\mathfrak{k}_i$ contains $\mathfrak{m}_i$. Then g_λ is induced by a linear map which on $\mathfrak{m}_i$ is given by $a(\lambda a + 1)^{-1}\mathrm{Id}$. Thus choosing $\lambda = \frac{a-1}{a} > 0$ yields a metric on G/H which is given as in Theorem 6.2 and therefore we get (6.2). □

The triples of Lie groups satisfying (6.2) have been (partially) classified in [STa2].

We are going to deal with diagonal metrics $dt^2 + g_{\phi(t)}$ on $(0, R] \times G/H$ with $\phi|_{\mathfrak{s}} = \mathrm{Id}$ where neither the situation of Theorem 6.2 nor the situation of Corollary 6.3 is given.

Furthermore we assume that the metrics can be realized as submersion metrics as follows.

On G we take a biinvariant metric and on $(0,R]\times K/H$ we take a metric $dt^2+g_{\psi(t)}$ where $\psi(t)$ is as in (1.16). By Lemma 2.3 we have

$$\phi(t)|_{\mathfrak{m}_i} := \frac{a_i^2(t)}{1+a_i^2(t)}\,\mathrm{Id}|_{\mathfrak{m}_i} \qquad \text{for } 1\le i\le r\,, \qquad\qquad \phi(t)|_{\mathfrak{s}} := \mathrm{Id}\,.$$

So for given $\phi(t):\mathfrak{p}\to\mathfrak{p}$ with

$$\phi(t)|_{\mathfrak{m}_i} := f_i^2(t)\,\mathrm{Id}|_{\mathfrak{m}_i},\quad f_i^2(t)<1 \qquad \text{for } 1\le i\le r\,, \qquad\qquad \phi(t)|_{\mathfrak{s}} := \mathrm{Id} \tag{6.3}$$

we can always find functions $a_i^2(t)$, $1\le i\le r$ such that

$$f_i^2(t) = \frac{a_i^2(t)}{1+a_i^2(t)} \qquad \text{for } 1\le i\le r\,. \tag{6.4}$$

For technical reasons we set

$$f_{r+1}^2(t)\equiv 1\,,$$

therewith $\phi(t)|_{\mathfrak{s}} = f_{r+1}^2(t)\,\mathrm{Id}$.

Remark 6.4. Note that by the relation (6.4) the functions f_i are monotoneous increasing resp. decreasing if the functions a_i are monotoneous increasing resp. decreasing and we have

$$a_i^2(t) = \frac{f_i^2(t)}{1-f_i^2(t)}\,.$$

The results of Chapter 5 can be applied to the metric $dt^2+g_{\psi(t)}$ on $I\times K/H$ with functions $a_i^2(t)$, $1\le i\le r$, as well as to the metric $dt^2+g_{\phi(t)}$ on $I\times G/H$ with functions $f_i^2(t)$, $1\le i\le r+1$.
We have

Theorem 6.5. *Suppose that $\mathfrak{m}$ decomposes irreducibly as $\mathfrak{m}=\mathfrak{m}_1\oplus\ldots\oplus\mathfrak{m}_r$ and that we have an underlying chain of Lie groups. Let ϕ be as in* (6.3)*. Furthermore suppose that there exists a constant $C>0$ such that*

$$R^{g_{\phi(R)}}(X,Y;Y,X)\ge C\big(\|X_{\mathfrak{m}_1\oplus\ldots\oplus\mathfrak{m}_{r-1}}\wedge Y_{\mathfrak{m}_1\oplus\ldots\oplus\mathfrak{m}_{r-1}}\|^2+\|[X,Y]_{\mathfrak{m}_1\oplus\ldots\oplus\mathfrak{m}_{r-1}}\|^2\big)\,. \tag{6.5}$$

Then there exists a metric on $G\times_K\overline{B_R(0)}$ with nonnegative curvature and a totally geodesic principal orbit on the boundary.

Proof. We have to prove the theorem in the cases $r = 2$ and $r = 3$, since the case $r = 1$ is clear.
By abuse of notation we will not change the parameter or the interval from where a parameter is when we reparametrize a metric.
We begin with the case $r = 3$.
For $R_2 < R$ define auxiliary smooth functions λ_1, λ_2 with the following properties

For $j \in \{1, 2\}$ let $\lambda_j : [R_2, R] \to \mathbb{R}^{\geq 0}$ be with

(a) $\lambda_j(R) = 0$

(b) $\lambda_j' \leq \eta < 0$

(c) $\lambda_j'' \geq \dfrac{3}{2} \dfrac{a_j^2(R)(\lambda_j')^2}{1 + a_j^2(R)(1 + \lambda_j)} + \delta\lambda_j'$ for some $\delta > 0$

We are going to construct the functions as in (6.4) defining the desired metric in two steps.

Step 1. Extend on $[R_2, R]$ according to

$$\begin{aligned}
a_1^2(t) &= a_1^2(R)\big((\lambda_1(t) + \lambda_2(t))\, a_1^2(R) + 1\big)^{-1} \\
a_2^2(t) &= a_2^2(R)\big(\lambda_2(t)\, a_2^2(R) + 1\big)^{-1} \\
a_3^2(t) &= const\,,
\end{aligned}$$

such that

(i) $a_1^2(R_2) \leq a_2^2(R_2) \leq a_3^2(R_2)$

(ii) $a_1'(R_2) \geq a_2'(R_2)$

Note that by Corollary 3.8 and Remark 3.9 we have $\sec_{g_{\psi(R_2)}} > 0$ and assuming (i) is legitimated by Corollary 3.10. Condition (ii) is equivalent to

$$\frac{\lambda_2'(R_2)}{\lambda_1'(R_2) + \lambda_2'(R_2)} \leq \left(\frac{a_1(R_2)}{a_2(R_2)}\right)^3$$

and the functions λ_1 and λ_2 can surely be chosen such that this is valid.
With this extension condition (6.5) is valid for all $t \in [R_2, R]$. This follows by construction because by Lemma 3.7 we obtain $\sec_{g_{\psi(t)}} \geq \sec_{g_{\psi(R)}}$ and therefore $\sec_{g_{\phi(t)}} \geq \sec_{g_{\phi(R)}}$ for all $t \in [R_2, R]$.
Moreover since the functions λ_1, λ_2 are demanded to satisfy $\lambda_1', \lambda_2' \leq \eta < 0$ we get $a_1', a_2' \geq \theta > 0$ and with it $f_1', f_2' \geq \xi > 0$. Furthermore the condition $\lambda_j'' \geq \frac{3}{2} \frac{a_j^2(R)(\lambda_j')^2}{1 + a_j^2(R)(1 + \lambda_j)} + \delta\lambda_j'$ for some $\delta > 0$ yields $f_j'' \leq \delta f_j'$ for $j \in \{1, 2\}$.

So the conditions of Corollary 5.6 are satisfied and there is a reparametrization such that the reparametrized metric has nonnegative curvature.

Step 2. Extend smoothly on $[R_1, R]$ such that

(i) $a_3'|_{[R_1,R_2)} > 0$

(ii) $a_i''|_{[R_1,R_2)} \leq \delta a_i'|_{[R_1,R_2)}$ for $1 \leq i \leq 3$ and some $\delta > 0$

(iii) $a_1' \geq a_2' \geq a_3'$

(iv) $a_1'|_{[R_1,R_1+\eta)} = a_2'|_{[R_1,R_1+\eta)} = a_3'|_{[R_1,R_1+\eta)}$ for some $\eta > 0$

By construction, it follows from Lemma 3.11 that $\sec_{g_{\psi(t)}} > 0$ on $[R_1, R_2]$ since we have $a_1^2(R_2) \leq a_2^2(R_2) \leq a_3^2(R_2)$ and the third condition implies pointwise the conditions of Lemma 3.11.

Moreover the conditions are made up to satisfy the conditions of Corollary 5.10. Hence there is a reparametrization such that the metric $dt^2 + g_{\psi(t)}$ has positive curvature on $[R_1, R_2] \times K/H$ and therefore the metric $dt^2 + g_{\phi(t)}$ has nonnegative curvature on $[R_1, R_2] \times G/H$. Moreover when applying this reparametrization on the whole interval $[R_1, R]$ observe that by the properties of the reparametrization, i.e. $|t'| < 1$ and $t'' < 0$, we do not lose the nonnegativity of the curvature on $[R_2, R]$ by Proposition 5.11. Furthermore by the demand that $a_1'|_{[R_1,R_1+\eta)} = a_2'|_{[R_1,R_1+\eta)} = a_3'|_{[R_1,R_1+\eta)}$ the metric has the form $dt^2 + f^2(t)g_{\psi(R_1+\eta)}$ on $[R_1, R_1 + \eta] \times K/H$ for some smooth function $f : (R_1, R_1 + \eta) \to \mathbb{R}$ with $f, f' > 0$ and can therefore be extended smoothly to a positively curved metric on $B_{R_1+\eta}(0)$ by virtue of Theorem 5.13 taking Remark 5.14 into account.

Finally an application of Remark 5.12 yields the statement in the theorem.

Note that in the case $r = 2$, (6.5) is exactly condition (5.8) of Proposition 5.5 for $t = b$ and $p = 1$.

The proof is analogeous to the proof in the case $r = 3$ when letting f_2 take the part of f_3 and f_1 that of f_2. □

Next we are going to analyze to what extent a metric on the boundary of $\overline{B_R(0)}$ can even have negative curvature such that condition (6.5) is satisfied in special cases.

If $\varphi = \mathrm{id}$ in Lemma 2.3 then (2.6) becomes

$$
\begin{aligned}
& R^{g_\phi}\big(\phi^{-1}X, \phi^{-1}Y; \phi^{-1}Y, \phi^{-1}X\big) \\
={}& \frac{1}{4}\,\big\|[X,Y]\big\|^2 + R^{g_\psi}\big(\psi^{-1}X_{\mathfrak{m}}, \psi^{-1}Y_{\mathfrak{m}}; \psi^{-1}Y_{\mathfrak{m}}, \psi^{-1}X_{\mathfrak{m}}\big) \\
&+ \frac{3}{4}\,Q\big((1+\psi)^{-1}\big([X,Y]_{\mathfrak{m}} + A^{\psi}_{\mathfrak{m}} - \psi B^{\psi}_{\mathfrak{m}}\big), [X,Y]_{\mathfrak{m}} + A^{\psi}_{\mathfrak{m}} - \psi B^{\psi}_{\mathfrak{m}}\big) \\
&+ \frac{3}{4}\,\big\|[X,Y]_{\mathfrak{h}} + A^{\psi}_{\mathfrak{h}}\big\|^2 .
\end{aligned}
\tag{6.6}
$$

Let $r = 2$. The second line of the right hand side of (6.6) can be computed to be

$$
\begin{aligned}
& \frac{3}{4}\,\frac{1}{1+a_1^2}\,\big\|\big(1+a_1^{-2}\big)B_1^1 + \big(1 + a_2^{-2}\big(2 - a_1^2 a_2^{-2}\big)\big)B_1^2 + B_1^{\mathfrak{s}}\,\big\|^2 \\
&+ \frac{3}{4}\,\frac{1}{1+a_2^2}\,\big\|\big(1+a_2^{-2}\big)\big(B_2^2 + B^{12}\big) + B_2^{\mathfrak{s}}\,\big\|^2 ,
\end{aligned}
$$

and for the third line of the right hand side of (6.6) we have

$$
\frac{3}{4}\,\big\|[X,Y]_{\mathfrak{h}} + A^{\psi}_{\mathfrak{h}}\,\big\|^2 = \frac{3}{4}\,\big\|B_0^1 + B_0^2 + B_0^{\mathfrak{s}} + 2\,a_1^{-2}B_0^1 + 2a_2^{-2}B_0^2\,\big\|^2 .
$$

Summarizing we obtain using (1.17)

$$
\begin{aligned}
& R^{g_\phi}\big(\phi^{-1}X, \phi^{-1}Y; \phi^{-1}Y, \phi^{-1}X\big) \\
={}& \frac{1}{4}\,\big\|[X,Y]\big\|^2 + \frac{3}{4}\,a_1^2\,\big\|a_1^{-4}B_0^1 + a_2^{-4}B_0^2\big\|^2 + \frac{1}{4}\,a_1^{-6}\,\big\|B^1\big\|^2 + \frac{1}{4}\,a_2^{-6}\,\big\|B_2^2 + B^{12}\big\|^2 \\
&+ \frac{1}{2}\,a_1^{-2}a_2^{-4}\,\big(3 - 2\,a_1^2 a_2^{-2}\big)Q\big(B^1, B^2\big) + a_2^{-6}\Big(1 - \frac{3}{4}\,a_1^2 a_2^{-2}\Big)\big\|B_0^2 + B_1^2\big\|^2 \\
&+ \frac{3}{4}\,\frac{1}{1+a_1^2}\,\big\|\big(1+a_1^{-2}\big)B_1^1 + \big(1 + a_2^{-2}(2 - a_1^2 a_2^{-2})\big)B_1^2 + B_1^{\mathfrak{s}}\big\|^2 \\
&+ \frac{3}{4}\,\frac{1}{1+a_2^2}\,\big\|\big(1+a_2^{-2}\big)\big(B_2^2 + B^{12}\big) + B_2^{\mathfrak{s}}\big\|^2 \\
&+ \frac{3}{4}\,\big\|B_0^1 + B_0^2 + B_0^{\mathfrak{s}} + 2a_1^{-2}B_0^1 + 2a_2^{-2}B_0^2\big\|^2 .
\end{aligned}
\tag{6.7}
$$

Proposition 6.6. *Consider a homogeneous metric g_ϕ on $K/H = SU(n+1)/SU(n)$ for $n \geq 2$.*
Then $R^{g_\phi}\big(\phi^{-1}X, \phi^{-1}Y; \phi^{-1}Y, \phi^{-1}X\big)$ is nonnegative for

$$\begin{array}{llll} \dfrac{a_1^2}{a_2^2} \leq \dfrac{2n(4+3a_2^2)}{3(1-n)a_2^4 + 2(3-n)a_2^2 + 3(n+1)} & \text{if} & 0 < a_2^2 < \dfrac{-(n-3)+\sqrt{10n^2-6n}}{3(n-1)} \\ a_1^2 \quad \textit{arbitrary} & \text{if} & a_2^2 \geq \dfrac{-(n-3)+\sqrt{10n^2-6n}}{3(n-1)} . \end{array}$$

Proof. Since $\mathfrak{m}_1$ is abelian in the considered case, we obtain for (6.7)

$$\begin{aligned} & R^\phi\big(\phi^{-1}X, \phi^{-1}Y; \phi^{-1}Y, \phi^{-1}X\big) \\ = \; & \frac{1}{4}\left\|B_0^2 + B_0^{\mathfrak{s}}\right\|^2 + \frac{1}{4}\left\|B_1^2 + B_1^{\mathfrak{s}}\right\|^2 + \frac{1}{4}\left\|[X,Y]_{\mathfrak{m}_2}\right\|^2 + \frac{1}{4}\left\|[X,Y]_{\mathfrak{s}}\right\|^2 \\ & + a_2^{-6}\left\|B_0^2\right\|^2 + \frac{1}{4}\,a_2^{-6}\left\|B_2^2 + B^{12}\right\|^2 + a_2^{-6}\Big(1 - \frac{3}{4}\,a_1^2\,a_2^{-2}\Big)\left\|B_1^2\right\|^2 \\ & + \frac{3}{4}\,\frac{1}{1+a_1^2}\left\|\big(1 + a_2^{-2}(2 - a_1^2 a_2^{-2})\big)B_1^2 + B_1^{\mathfrak{s}}\right\|^2 \\ & + \frac{3}{4}\,\frac{1}{1+a_2^2}\left\|\big(1 + a_2^{-2}\big)\big(B_2^2 + B^{12}\big) + B_2^{\mathfrak{s}}\right\|^2 \\ & + \frac{3}{4}\left\|\big(1 + 2\,a_2^{-2}\big)B_0^2 + B_0^{\mathfrak{s}}\right\|^2 . \end{aligned}$$

This is nonnegative if

$$\begin{aligned} & \frac{1}{4}\left\|B_0^2 + B_0^{\mathfrak{s}}\right\|^2 + \frac{1}{4}\left\|B_1^2 + B_1^{\mathfrak{s}}\right\|^2 + a_2^{-6}\left\|B_0^2\right\|^2 + a_2^{-6}\Big(1 - \frac{3}{4}a_1^2 a_2^{-2}\Big)\left\|B_1^2\right\|^2 \\ + & \frac{3}{4}\,\frac{1}{1+a_1^2}\left\|\big(1 + a_2^{-2}(2 - a_1^2 a_2^{-2})\big)B_1^2 + B_1^{\mathfrak{s}}\right\|^2 + \frac{3}{4}\left\|\big(1 + 2a_2^{-2}\big)B_0^2 + B_0^{\mathfrak{s}}\right\|^2 \;\geq 0 . \end{aligned} \tag{6.8}$$

Decompose $B_{\mathfrak{k}_1}^{\mathfrak{s}} = \mu B_{\mathfrak{k}_1}^2 + Z$, where $Z \in (B_{\mathfrak{k}_1}^2)^\perp \cap \mathfrak{k}_1$.
So (6.8) is valid if

$$\begin{aligned} & \frac{1}{4}\,(1+\mu)^2\left\|B_0^2\right\|^2 + \frac{1}{4}\,(1+\mu)^2\left\|B_1^2\right\|^2 + a_2^{-6}\left\|B_0^2\right\|^2 + a_2^{-6}\Big(1 - \frac{3}{4}a_1^2 a_2^{-2}\Big)\left\|B_1^2\right\|^2 \\ + & \frac{3}{4}\,\frac{1}{1+a_1^2}\,\big(1 + a_2^{-2}(2 - a_1^2 a_2^{-2}) + \mu\big)^2\left\|B_1^2\right\|^2 + \frac{3}{4}\big(1 + 2a_2^{-2} + \mu\big)^2\left\|B_0^2\right\|^2 \\ = \; & \Big(a_2^{-6} + \frac{1}{4}\,(1+\mu)^2 + \frac{3}{4}\,(1 + 2a_2^{-2} + \mu)^2\Big)\left\|B_0^2\right\|^2 \\ + & \Big(a_2^{-6}\Big(1 - \frac{3}{4}\,a_1^2 a_2^{-2}\Big) + \frac{1}{4}\,(1+\mu)^2 + \frac{3}{4}\,\frac{1}{1+a_1^2}\big(1 + a_2^{-2}(2 - a_1^2 a_2^{-2}) + \mu\big)^2\Big)\left\|B_1^2\right\|^2 \geq 0 . \end{aligned}$$

Let

$$X_2 := \begin{pmatrix} 0 & z \\ -z^* & 0 \end{pmatrix} \qquad Y_2 := \begin{pmatrix} 0 & w \\ -w^* & 0 \end{pmatrix}.$$

For sphere number 4 with the metric induced by the inner product $Q(X,Y) = -\frac{1}{2} Re(\mathrm{tr}(XY))$ it can be computed that

$$\begin{aligned} \left\|B_0^2\right\|^2 &= |z|^2|w|^2 - Re(zw^*)^2 + Im(zw^*)^2 - \frac{2}{n}\, Im(zw^*)^2 \\ \left\|B_1^2\right\|^2 &= \frac{2(n+1)}{n}\, Im(zw^*)^2 . \end{aligned}$$

So for $a > 0$ and $b \in \mathbb{R}$ we have

$$a\left\|B_0^2\right\|^2 + b\left\|B_1^2\right\|^2 = a\bigg(|z|^2|w|^2 - Re(zw^*)^2 + \left(1 - \frac{2}{n} + \frac{b}{a}\,\frac{2\,(n+1)}{n}\right) Im(zw^*)^2\bigg).$$

This is surely nonnegative if we can guarantee

$$1 - \frac{2}{n} + \frac{b}{a}\,\frac{2(n+1)}{n} \geq -1 \iff \frac{b}{a} \geq \frac{1-n}{1+n}.$$

It follows that (6.8) is valid if

$$\begin{aligned} & a_2^{-6}\Big(1 - \frac{3}{4}\, a_1^2 a_2^{-2}\Big) + \frac{1}{4}\,(1+\mu)^2 + \frac{3}{4}\,\frac{1}{1+a_1^2}\,\big(1 + a_2^{-2}\big(2 - a_1^2 a_2^{-2}\big) + \mu\big)^2 \\ \geq\ & \frac{1-n}{1+n}\left(a_2^{-6} + \frac{1}{4}\,(1+\mu)^2 + \frac{3}{4}\,(1 + 2a_2^{-2} + \mu)^2\right). \end{aligned}$$

After some elementary modifications this is seen to be equivalent to

$$\frac{a_1^2}{1+a_1^2} \leq \frac{4}{3}\,\frac{2n}{1+n}\,\frac{a_2^2}{1+a_2^2}\left(1 + \frac{\mu a_2^4(a_2^2(\mu+1)+1)}{((1+a_2^2)^2 + \mu a_2^4)^2}\right).$$

Now the function $f_a : \mathbb{R}\backslash\{-(1+a^2)^2/a^4\}$ defined by

$$f_a(\mu) := 1 + \frac{\mu a_2^4(a_2^2(\mu+1)+1)}{((1+a_2^2)^2 + \mu a_2^4)^2}$$

attains its minimum for $\mu = -\frac{(a^2+1)^2}{a^2(a^2+2)}$ with $f_a(-\frac{(a^2+1)^2}{a^2(a^2+2)}) = 1 - \frac{a^2}{4(1+a^2)}$. So we can surely guarantee the validity of (6.8) if

$$\frac{a_1^2}{1+a_1^2} \leq \frac{4}{3}\,\frac{2n}{1+n}\,\frac{a_2^2}{1+a_2^2}\left(1 - \frac{1}{4}\,\frac{a_2^2}{(1+a_2^2)}\right) \Leftrightarrow \frac{a_1^2}{1+a_1^2} \leq \frac{1}{3}\,\frac{2n}{1+n}\,\frac{a_2^2(4+3a_2^2)}{(1+a_2^2)^2}.$$

So demanding for a_1^2 to satisfy

$$\frac{1}{a_1^2} \geq \frac{3(n+1)}{2n}\frac{\left(1+a_2^2\right)^2}{a_2^2\left(4+3a_2^2\right)} - 1 = \frac{3(1-n)a_2^4 + 2(3-n)a_2^2 + 3(n+1)}{2na_2^2\left(4+3a_2^2\right)} \tag{6.9}$$

for given a_2^2 yields (6.8). The right hand side of (6.9) is nonpositive for

$$a_2^2 \geq \frac{-(n-3) + \sqrt{10n^2 - 6n}}{3(n-1)}.$$

In this case we can choose a_1^2 arbitrary. Otherwise if

$$0 < a_2^2 < \frac{-(n-3) + \sqrt{10n^2 - 6n}}{3(n-1)}$$

the right hand side of (6.9) is positive and we have to demand for a_1^2 to satisfy

$$a_1^2 \leq \frac{2na_2^2(4+3a_2^2)}{3(1-n)a_2^4 + 2(3-n)a_2^2 + 3(n+1)}.$$

□

Remark 6.7. If $\mathfrak{m}_1$ is abelian in the case $r = 2$, condition (6.5) in Theorem 6.5 becomes

$$R^{g_{\phi(R)}}\left(\phi_R^{-1}X, \phi_R^{-1}Y; \phi_R^{-1}Y, \phi_R^{-1}X\right) \geq C \cdot \left\|[X,Y]_{\mathfrak{m}_1}\right\|^2.$$

When choosing

$$\begin{aligned} &\frac{a_1^2(R)}{a_2^2(R)} < \frac{2n(4+3a_2^2(R))}{3(1-n)a_2^4(R) + 2(3-n)a_2^2(R) + 3(n+1)} \\ \text{for}\quad &0 < a_2^2(R) < \frac{-(n-3) + \sqrt{10n^2 - 6n}}{3(n-1)} \end{aligned}$$

we can surely guarantee this. Moreover if we choose

$$\begin{aligned} &\frac{4}{3}\frac{2n}{n+1} < \frac{a_1^2(R)}{a_2^2(R)} < \frac{2n\left(4+3a_2^2(R)\right)}{3(1-n)a_2^4(R) + 2(3-n)a_2^2(R) + 3(n+1)} \\ \text{for}\quad &0 < a_2^2(R) < \frac{-(n-3) + \sqrt{10n^2 - 6n}}{3(n-1)}, \end{aligned}$$

we can construct a nonnegatively curved metric on $(0, R] \times G/H$ although $R^{g_{\psi(R)}}\left(\psi_R^{-1}X, \psi_R^{-1}Y; \psi_R^{-1}Y, \psi_R^{-1}X\right) \ngeq 0$.

When choosing

$$a_2^2 > \frac{-(n-3)+\sqrt{10n^2-6n}}{3(n-1)}$$

we can even construct a nonnegatively metric on $(0,R]\times G/H$ where $R^{g_{\psi(R)}}\left(\psi_R^{-1}X,\psi_R^{-1}Y;\psi_R^{-1}Y,\psi_R^{-1}X\right)$ can take arbitrary negative values.

The next proposition is an analogue of Proposition 6.6 in the case of sphere number 5,

Proposition 6.8. *Consider a homogeneous metric g_ϕ on $K/H = U(n{+}1)/U(n)$ for $n\geq 2$. Define*

$$D := \frac{-\lambda(n-3)-4n+\sqrt{2n}\sqrt{\lambda^2(5n-3)+\lambda(13n-3)+8n}}{3(\lambda(n-1)+2n)}$$

Then $R^{g_\phi}\left(\phi^{-1}X,\phi^{-1}Y;\phi^{-1}Y,\phi^{-1}X\right)$ is nonnegative for

$$\frac{a_1^2}{a_2^2} \leq \frac{2(\lambda+1)n(4+3a_2^2)}{(3(1-n)\lambda-6n)a_2^4+2((3-n)\lambda-4n)a_2^2+3\lambda(n+1)} \qquad \text{if} \quad 0<a_2^2<D$$

$$a_1^2 \qquad \text{arbitrary} \qquad \text{if} \quad a_2^2\geq D\ .$$

Proof. Following from similar argumentations as before we achieve nonnegative curvature if we can guarantee

$$\begin{aligned}&a_2^{-6}\left(1-\frac{3}{4}\,a_1^2a_2^{-2}\right)+\frac{1}{4}\,(1+\mu)^2+\frac{3}{4}\,\frac{1}{1+a_1^2}\,\left(1+a_2^{-2}(2-a_1^2a_2^{-2})+\mu\right)^2\\ \geq{}&\left(\frac{1-n}{1+n}-\frac{1}{\lambda}\,\frac{2n}{n+1}\right)\left(a_2^{-6}+\frac{1}{4}(1+\mu)^2+\frac{3}{4}\,(1+2a_2^{-2}+\mu)^2\right).\end{aligned}$$

From here on the proof of Proposition 6.6 carries out over verbatim with the obvious replacement and we obtain the statement in Proposition 6.8 □

Remark 6.9. The limit in Proposition 6.8 is "better" than that of Proposition 6.6 as was to be expected. We get an analogeous result to that of Remark 6.7 in the case of Proposition 6.8.

Chapter 7

Cohomogeneity One Manifolds

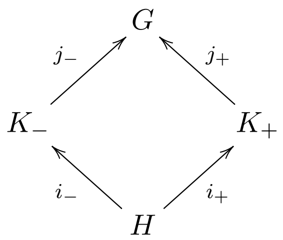
7.1 Topology of Cohomogeneity One Manifolds

A connected manifold M is said to be a cohomogeneity one G-manifold if there is a compact Lie group G acting on M such that $\dim(M/G) = 1$.
So the orbit space can in fact be identified with either an interval I or the sphere S^1. Denote the principal orbits by G/H, where H is the principal isotropy group of such an orbit up to conjugacy. Let $\pi : M \to M/G$ denote the projection map. From the general theory (see [Br]) it is known that the union of the principal orbits forms a dense open subset of M which we denote by M^0. In fact $M^0 = \pi^{-1}(S^1)$ in which case all orbits are principal and π is a bundlle map, or M^0 is the preimage of the interior of the interval I w.r.t. π.
Furthermore the principal isotropy group H is conjugated to a subgroup of any other isotropy group K of a nonprincipal orbit. More than that in [M] it is shown that K/H is a sphere.
The nonprincipal orbits are the preimages of the endpoints of I w.r.t. π and their tubular neighborhoods are homogeneous disc bundles. Denoting the nonprincipal isotropy groups by $K_\pm$ in the case when we have two nonprincipal orbits we obtain the following list for the shape of cohomogeneity one manifolds

(i)	$M/G = \mathbb{R}$	$M = \mathbb{R} \times G/H$
(ii)	$M/G = [a, \infty)$	$M = G \times_K V$
(iii)	$M/G = S^1$	$M = \mathbb{R} \times_{\mathbb{Z}} G/H$
(iv)	$M/G = [a, b]$	$M = G \times_{K_-} D_- \cup_{G/H} G \times_{K_+} D_+$,

where in case (iv) we glue the two tubular neighborhoods of the nonprincipal orbits along a common principal orbit. For the rest of this chapter we will refer to this list when a cohomogeneity one manifold is said to be in some case.

Conversely suppose that we are given compact Lie groups $H \subset \{K_-, K_+\} \subset G$ where $K_\pm/H$ are spheres such that the diagram of inclusions



commutes. Then we can construct a cohomogeneity one G-manifold with principal orbit type G/H and nonprincipal orbits $G/K_\pm$. Here the cohomogeneity one G-action is given by the canonical action of G on the homogeneous disc bundles $G \times_{K_-} D_-$ and $G \times_{K_+} D_+$, see Chapter 2, action in (2.3).

Example 7.1 (Brieskorn varieties)**.** Consider the the real algebraic submanifolds of $\mathbb{C}^{n+1}$ given by

$$W_d^{2n-1} := \left\{ z \in \mathbb{C}^{n+1} \,\middle|\, z_0^d + z_1^2 + \cdots + z_n^2 = 0,\ |z_0|^2 + \cdots + |z_n|^2 = 1 \right\}.$$

These $2n-1$ dimensional manifolds are called Brieskorn varieties. We write elements of W_d^{2n-1} as tupels $(z_0, Z) \in \mathbb{C} \oplus \mathbb{C}^n$.
As was observed in [HH] the Brieskorn varieties carry a cohomogeneity one action by $G := U(1) \times O(n)$ defined by

$$(e^{i\theta}, A) * (z_0, Z) := \left(e^{2i\theta} z_0, e^{id\theta} A \cdot Z\right)$$

Since $\left|e^{2i\theta}\right| = 1$ it follows that $|z_0|$ is invariant under this action. So we obtain immediately that if (z_0, Z) and (w_0, W) are in the same orbit then $|z_0| = |w_0|$.
Conversely if we have $|z_0| = |w_0|$, then $z_0 = e^{2i\theta}\, w_0$ and for $(z_0, Z), (w_0, W) \in W_d^{2n-1}$ we obtain

$$Z^t \cdot Z = e^{2di\theta}\, W^t \cdot W \qquad \text{and} \qquad |Z|^2 = |W|^2.$$

From the first equation it follows that $Z = e^{di\theta}\, B \cdot W$ for some $B \in O(n; \mathbb{C})$ and from the second equation we can extract $Z = \tilde{B} \cdot W$ for some $\tilde{B} \in U(n)$ whence we obtain $Z = e^{\frac{di\theta}{2}} A \cdot W$ for some $A \in O(n)$ since $O(n; \mathbb{C}) \cap U(n) = O(n)$.

So we can deduce that G acts with cohomogeneity one on W_d^{2n-1}.
Observe that $|z_0| \in [0, t_0]$ where t_0 is a real solution of $t^d + t^2 = 1$. That the minimal value which $|z_0|$ can attain is zero follows from the fact that there are elements in W_d^{2n-1} with $z_0 = 0$. For determining the upper bound let $t = |z_0|$. We have

$$t^d = |Z^t \cdot Z| \leq |Z|^2 = 1 - t^2 \iff t^d + t^2 \leq 1\,.$$

Since there are elements in W_d^{2n-1} fulfilling $t^d + t^2 = 1$ the positive real solution of this equation is the upper bound for $|z_0|$.
Hence $W_d^{2n-1}/G \cong [0, t_0]$ with t_0 as above. Denote by $\pi : W_d^{2n-1} \to [0, t_0]$ the projection onto the orbit space. Then the preimages $\pi^{-1}(t)$ for $t \in (0, t_0)$ constitute the principal orbits whereas the preimages $\pi^{-1}(0)$ and $\pi^{-1}(t_0)$ constitute the singular orbits.
Next we are going to determine the groups $H \subset \{K_-, K_+\}$ for the group diagram. For the principal isotropy group we determine the isotropy group of $(z_0, z_1, z_2, 0, \ldots, 0)$ with $|z_0| \in (0, t_0)$.

$$H = \begin{cases} \big(\epsilon, \operatorname{diag}\big(1, 1, B\big)\big) & \text{for } d \text{ even} \\ \big(\epsilon, \operatorname{diag}\big(-\epsilon, -\epsilon, B\big)\big) & \text{for } d \text{ odd}\,, \end{cases}$$

where $\epsilon = \pm 1$ and $B \in O(n-2)$. So we have $H \cong \mathbb{Z}_2 \times O(n-2)$.
The isotropy group K_- of an element in $\pi^{-1}(0)$, say $(0, z_1, z_2, \ldots, 0)$ is given by

$$K_- = \big(e^{-i\theta}, \operatorname{diag}\big(e^{i\theta}, e^{i\theta}, B\big)\big)\,,$$

where $B \in O(n-2)$. Therefore we have $K_- \cong U(1) \times O(n-2)$.
Finally the isotropy group K_+ of an element in $\pi^{-1}(t_0)$, say $(q, i\sqrt{q^d}, 0, \ldots, 0)$ can be computed to be

$$K_+ = \begin{cases} \big(\epsilon, \operatorname{diag}(1, B)\big) & \text{for } d \text{ even} \\ \big(\epsilon, \operatorname{diag}(-\epsilon, B)\big) & \text{for } d \text{ odd}\,, \end{cases}$$

where $\epsilon = \pm 1$ and $B \in O(n-1)$. So we can deduce that $K_+ \cong O(n-1)$.
The dimension of the singular orbit G/K_- is $2n-3$ therefore its codimension is 2 whereas dimension of the singular orbit G/K_+ is n therefore its codimension is $n-1$.
For n and d odd the Briekorn manifolds are known to be homeomorphic to spheres and even diffeomorphic if $d = \pm 1 \bmod 8$. If otherwise $d = \pm 3 \bmod 8$ they are diffeomorphic to the Kervaire sphere, which is an exotic sphere.

We will come back to this example later.

7.2 Nonnegatively Curved Invariant Metrics on Cohomogeneity One Manifolds

In the cases (i), (ii) and (iii) the existence of nonnegatively curved metrics is guaranteed. This statement in obvious in the cases (i) and (iii) and for putting a nonnegatively curved metric on a cohomogeneity one manifold as in (ii) we can use the results of Chapter 6.
Case (iv) is the most interesting one.
Here one way of putting a nonnegatively curved invariant metric on the cohomogeneity one manifold is to define nonnegatively curved metrics on each half which are G-equivariantly isometric to each other near the boundary so that they can be glued along their common boundary
So it seems likely to seek for the metrics being G-equivariantly isometric to $\big((R-\varepsilon, R] \times G/H, dt^2+g_Q\big)$ near the boundary for some $\varepsilon > 0$, where g_Q denotes a normal homogeneous metrics on G/H and to construct the metrics on each half with Cheeger's construction.

In [GZ1] it is shown that this is possible when the codimension of the singular orbits is at most 2.
Observe that in this case we have $\dim K_\pm/H \leq 1$ so that especially the tangent space to $K_\pm/H$ can be identified with a at most 1-dimensional subspace of the Lie algebra $\mathfrak{g}$ of G. Then the condition of Theorem 6.1 is satisfied since the left hand side of (6.1) vanishes and the existence of a nonnegatively curved normal homogeneous collar metric on each half is guaranteed.
This situation is especially apparent for the Brieskorn varieties for $n = 3$. Therefore we get the existence of a $U(1) \times O(3)$-invariant metric on W_d^5.

If the codimension of at least one singular orbit is greater than 2 the existence of a nonnegatively curved invariant metric is not ensured. Let ℓ_- resp. ℓ_+ denote the codimension of the singular orbit G/K_- resp. G/K_+.
In fact we have

Theorem 7.2. *For each pair (ℓ_-, ℓ_+) with $(\ell_-, \ell_+) \neq (2, 2)$ and $\ell_\pm \geq 2$ there exist infinitely many cohomogeneity one G-manifolds that do not admit a G-invariant metric with nonnegative curvature.*

For instance the Brieskorn varieties do not support an $U(1) \times O(n)$-invariant metric with nonnegative curvature for $n \geq 4$ and $d \geq 3$, because in this case the codimension of one of the singular orbits is greater than 2.

Appendix

In the upcoming we are going to present the computations to obtain the curvature formula stated in Lemma 1.16. For $X, Y \in \mathfrak{m}$ we have

$$\begin{aligned}
&\frac{1}{2}\, Q\big([\psi^{-1}X, Y] + [X, \psi^{-1}Y], [\psi^{-1}X, \psi^{-1}Y]\big)\\
&- \frac{3}{4}\, Q\big(\psi[\psi^{-1}X, \psi^{-1}Y], [\psi^{-1}X, \psi^{-1}Y]\big)\\
&+ \frac{1}{4}\, Q\big([\psi^{-1}X, Y] - [X, \psi^{-1}Y], \psi^{-1}\big([\psi^{-1}X, Y] - [X, \psi^{-1}Y]\big)\big)\\
&- Q\big([\psi^{-1}X, X], \psi^{-1}[\psi^{-1}Y, Y]\big)\\
= \sum_{0\le k\le r} \Bigg(&\frac{1}{2}\, Q\Big(2 \sum_{1\le i\le r} a_i^{-2} B_k^i + \sum_{1\le i<j\le r} (a_i^{-2} + a_j^{-2}) B_k^{ij}, \sum_{1\le i\le r} a_i^{-4} B_k^i + \sum_{1\le i<j\le r} a_i^{-2} a_j^{-2} B_k^{ij}\Big)\\
&- \frac{3}{4}\, a_k^2 Q\Big(\sum_{1\le i\le r} a_i^{-4} B_k^i + \sum_{1\le i<j\le r} a_i^{-2} a_j^{-2} B_k^{ij}, \sum_{1\le i\le r} a_i^{-4} B_k^i + \sum_{1\le i<j\le r} a_i^{-2} a_j^{-2} B_k^{ij}\Big)\\
&+ \frac{1}{4}\, a_k^{-2} Q\Big(\sum_{1\le i<j\le r} (a_i^{-2} - a_j^{-2})\big([X_i, Y_j]_k - [X_j, Y_i]_k\big),\\
&\qquad \sum_{1\le i<j\le r} (a_i^{-2} - a_j^{-2})\big([X_i, Y_j]_k - [X_j, Y_i]_k\big)\Big)\\
&- a_k^{-2} Q\Big(\sum_{1\le i<j\le r} (a_i^{-2} - a_j^{-2})\,[X_i, X_j]_k, \sum_{1\le i<j\le r} (a_i^{-2} - a_j^{-2})\,[Y_i, Y_j]_k\Big)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
= \sum_{0\le k\le r} \Big(& \frac{1}{2}\, Q\Big(2 \sum_{k\le i\le r} a_i^{-2} B_k^i + \sum_{1\le i<k} (a_i^{-2} + a_k^{-2}) B^{ik}, \sum_{k\le i\le r} a_i^{-4} B_k^i + \sum_{1\le i<k} a_i^{-2} a_k^{-2} B^{ik}\Big) \\
& - \frac{3}{4}\, a_k^2 Q\Big(\sum_{k\le i\le r} a_i^{-4} B_k^i + \sum_{1\le i<k} a_i^{-2} a_k^{-2} B^{ik}, \sum_{k\le i\le r} a_i^{-4} B_k^i + \sum_{1\le i<k} a_i^{-2} a_k^{-2} B^{ik}\Big) \\
& + \frac{1}{4}\, a_k^{-2} Q\Big(\sum_{1\le i<k} (a_i^{-2} - a_k^{-2})\big([X_i, Y_k] - [X_k, Y_i]\big), \\
& \qquad \sum_{1\le i<k} (a_i^{-2} - a_k^{-2})\big([X_i, Y_k] - [X_k, Y_i]\big)\Big) \\
& - a_k^{-2} Q\big(\sum_{1\le i<k} (a_i^{-2} - a_k^{-2})[X_i, X_k], \sum_{1\le i<k} (a_i^{-2} - a_k^{-2})[Y_i, Y_k]\big)\Big)
\end{aligned}
$$

For fixed k we obtain for the expression in the last line

$$
\begin{aligned}
& - Q\Big(\sum_{1\le i<k} (a_i^{-2} - a_k^{-2})[X_i, X_k], \sum_{1\le i<k} (a_i^{-2} - a_k^{-2})[Y_i, Y_k]\Big) \\
= & - \sum_{1\le i<k} \sum_{1\le l<k} (a_i^{-2} - a_k^{-2})(a_l^{-2} - a_k^{-2}) Q\big([X_i, X_k], [Y_l, Y_k]\big) \\
= & - \sum_{1\le i<k} \sum_{1\le l<k} (a_i^{-2} - a_k^{-2})(a_l^{-2} - a_k^{-2})\big(-Q\big([X_i, Y_k], [X_k, Y_l]\big) + Q\big([X_i, Y_l], [X_k, Y_k]\big)\big) \\
= & \; Q\Big(\sum_{1\le i<k} (a_i^{-2} - a_k^{-2})[X_i, Y_k], \sum_{1\le i<k} (a_i^{-2} - a_k^{-2})[X_k, Y_i]\Big) \\
& - Q\Big(\sum_{1\le i<j\le k} (a_i^{-2} - a_k^{-2})(a_j^{-2} - a_k^{-2}) B^{ij} + \sum_{1\le i<k} (a_i^{-2} - a_k^{-2})^2 B^i, B^k\Big),
\end{aligned}
$$

where we have used the ad-skew-symmetry of Q and the the Jacobi identity from the third to the forth line as

$$
\begin{aligned}
Q\big([X_i, X_j], [Y_k, Y_l]\big) &= -Q\big(X_j, [X_i, [Y_k, Y_l]]\big) \\
&= Q\big(X_j, [Y_k, [Y_l, X_i]]\big) + Q\big(X_j, [Y_l, [X_i, Y_k]]\big) \\
&= -Q\big([X_i, Y_l], [X_j, Y_k]\big) + Q\big([X_i, Y_k], [X_j, Y_l]\big)\,.
\end{aligned}
$$

With this manipulation at hand we get

$$
\begin{aligned}
&\frac{1}{4}\, a_k^{-2} Q\Big(\sum_{1\le i<k} \big(a_i^{-2}-a_j^{-2}\big)\big([X_i,Y_k]-[X_k,Y_i]\big),\ \sum_{1\le i<k} \big(a_i^{-2}-a_k^{-2}\big)\big([X_i,Y_k]-[X_k,Y_i]\big)\Big)\\
&- a_k^{-2} Q\Big(\sum_{1\le i<k} (a_i^{-2}-a_k^{-2})[X_i,X_k],\ \sum_{1\le i<k} (a_i^{-2}-a_k^{-2})[Y_i,Y_k]\Big)\\
=\ &\frac{1}{4} a_k^{-2} Q\Big(\sum_{1\le i<k} \big(a_i^{-2}-a_k^{-2}\big) B^{ik},\ \sum_{1\le i<k} \big(a_i^{-2}-a_k^{-2}\big) B^{ik}\Big)\\
&- a_k^{-2} Q\Big(\sum_{1\le i<j<k} \big(a_i^{-2}-a_k^{-2}\big)\big(a_j^{-2}-a_k^{-2}\big) B^{ij} + \sum_{1\le i<k} \big(a_i^{-2}-a_k^{-2}\big)^2 B^i, B^k\Big)\,.
\end{aligned}
$$

We continue computing

$$
\begin{aligned}
&R^{\bar g_\psi}\big(\psi^{-1}X,\psi^{-1}Y,\psi^{-1}Y,\psi^{-1}X\big)\\
=&\sum_{0\le k\le r}\Bigg(Q\Big(\sum_{k\le i\le r} a_i^{-2}B_k^i,\ \sum_{k\le i\le r} a_i^{-4}B_k^i\Big) - \frac{3}{4}\, a_k^2 Q\Big(\sum_{k\le i\le r} a_i^{-4}B_k^i,\ \sum_{k\le i\le r} a_i^{-4}B_k^i\Big)\\
&\qquad + a_k^{-2} Q\Big(\sum_{k\le i\le r} a_i^{-2}B_k^i,\ \sum_{1\le i<k} a_i^{-2}B^{ik}\Big) + \frac{1}{2}\, Q\Big(\sum_{1\le i<k} \big(a_i^{-2}+a_k^{-2}\big)B^{ik},\ \sum_{k\le i\le r} a_i^{-4}B_k^i\Big)\\
&\qquad + \frac{1}{2}\, a_k^{-2} Q\Big(\sum_{1\le i<k} \big(a_i^{-2}+a_k^{-2}\big)B^{ik},\ \sum_{1\le i<k} a_i^{-2}B^{ik}\Big) - \frac{3}{2}\, Q\Big(\sum_{k\le i\le r} a_i^{-4}B_k^i,\ \sum_{1\le i<k} a_i^{-2}B^{ik}\Big)\\
&\qquad - \frac{3}{4}\, a_k^{-2} Q\Big(\sum_{1\le i<k} a_i^{-2}B^{ik},\ \sum_{1\le i<k} a_i^{-2}B^{ik}\Big) + \frac{1}{4}\, a_k^{-2} Q\Big(\sum_{1\le i<k} a_i^{-2}B^{ik},\ \sum_{1\le i<k} a_i^{-2}B^{ik}\Big)\\
&\qquad - \frac{1}{2}\, a_k^{-4} Q\Big(\sum_{1\le i<k} a_i^{-2}B^{ik},\ \sum_{1\le i<k} B^{ik}\Big) + \frac{1}{4}\, a_k^{-6} Q\Big(\sum_{1\le i<k} B^{ik},\ \sum_{1\le i<k} B^{ik}\Big)\\
&\qquad - a_k^{-2}\, Q\Big(\sum_{1\le i<j<k} \big(a_i^{-2}-a_k^{-2}\big)\big(a_j^{-2}-a_k^{-2}\big)B^{ij} + \sum_{1\le i<k} \big(a_i^{-2}-a_k^{-2}\big)^2 B^i, B^k\Big)\Bigg)\\
=&\sum_{0\le k\le r}\Bigg(\frac{1}{4}\, Q\Big(\sum_{k\le i\le r} a_i^{-4}B_k^i,\ \sum_{k\le i\le r} a_i^{-2}\big(4-3a_k^2a_i^{-2}\big)B_k^i\Big)\\
&\qquad + Q\Big(\sum_{k<i\le r} a_i^{-2}\big(a_k^{-2}-a_i^{-2}\big)B_k^i,\ \sum_{1\le i<k} a_i^{-2}B^{ik}\Big)\\
&\qquad + \frac{1}{2}\, a_k^{-2} Q\Big(\sum_{k\le i\le r} a_i^{-4}B_k^i,\ \sum_{1\le i<k} B^{ik}\Big) + \frac{1}{4}\, a_k^{-6} Q\Big(\sum_{1\le i<k} B^{ik},\ \sum_{1\le i<k} B^{ik}\Big)\\
&\qquad - a_k^{-2} Q\Big(\sum_{1\le i<j<k} \big(a_i^{-2}-a_k^{-2}\big)\big(a_j^{-2}-a_k^{-2}\big)B^{ij} + \sum_{1\le i<k} \big(a_i^{-2}-a_k^{-2}\big)^2 B^i, B^k\Big)\Bigg)\,.
\end{aligned}
$$

After some simplifications and taking the O'Neill term of the canonical submersion into account we finally get

$$
\begin{aligned}
& R^{g_\psi}\big(\psi^{-1}X, \psi^{-1}Y, \psi^{-1}Y, \psi^{-1}X\big) \\
= \; & \sum_{1\le i\le r} a_i^{-6}\big\|B_0^i\big\|^2 + \sum_{1\le i<j\le r} a_j^{-4}a_i^{-2}\big(3 - a_i^2 a_j^{-2}\big) Q\big(B_0^i, B_0^j\big) \\
& + \sum_{1\le k\le r} \Big(\frac{1}{4}\, a_k^{-6}\big\|B_k^k + \sum_{1\le i<k} B^{ik}\big\|^2 + \frac{1}{2}\, Q\Big(\sum_{k<j\le r} a_j^{-4}a_k^{-2}\big(3 - 2a_k^2 a_j^{-2}\big) B_k^j, B_k^k + \sum_{1\le i<k} B^{ik}\Big) \\
& \qquad + \frac{1}{4} \sum_{k<i\le r} a_i^{-6}\big(4 - 3a_k^2 a_i^{-2}\big)\big\|B_k^i\big\|^2 \\
& \qquad + \frac{1}{2} \sum_{k<i<j\le r} a_j^{-4}a_i^{-2}\big(6 - 3a_k^2 a_i^{-2} - 2a_i^2 a_j^{-2}\big) Q\big(B_k^i, B_k^j\big)\Big) \\
= \; & \sum_{1\le i\le r} a_i^{-6}\big\|B_0^i\big\|^2 + \sum_{1\le i<j\le r} a_j^{-4}a_i^{-2}\big(3 - a_i^2 a_j^{-2}\big) Q\big(B_0^i, B_0^j\big) + \frac{1}{4}\, a_r^{-6}\Big\|B_r^r + \sum_{1\le i<r} B^{ir}\Big\|^2 \\
& + \sum_{1\le k\le r-1} \frac{1}{4}\, a_k^{-6}\Big\|B_k^k + \sum_{1\le i<k} B^{ik}\Big\|^2 + \frac{1}{2} \sum_{1\le k<j\le r} a_j^{-4}a_k^{-2}\big(3 - 2a_k^2 a_j^{-2}\big) Q\Big(B_k^j, B_k^k + \sum_{1\le i<k} B^{ik}\Big) \\
& \qquad + \frac{1}{4} \sum_{1\le k<i\le r} a_i^{-6}\big(4 - 3a_k^2 a_i^{-2}\big)\big\|B_k^i\big\|^2 \\
& \qquad + \frac{1}{2} \sum_{1\le k<i<j\le r} a_j^{-4}a_i^{-2}\big(6 - 3a_k^2 a_i^{-2} - 2a_i^2 a_j^{-2}\big) Q\big(B_k^i, B_k^j\big) \\
= \; & \frac{3}{4}\, a_1^2 \Big\| \sum_{1\le i\le r} a_i^{-4} B_0^i \Big\|^2 + \frac{1}{4} \sum_{1\le i\le r} a_i^{-6}\big(4 - 3a_1^2 a_i^{-2}\big)\big\|B_0^i + B_1^i\big\|^2 \\
& + \frac{1}{2} \sum_{1\le i<j\le r} a_j^{-4}a_i^{-2}\big(6 - 3a_1^2 a_i^{-2} - a_i^2 a_j^{-2}\big) Q\big(B_0^i + B_1^i, B_0^j + B_1^j\big) \\
& + \sum_{2\le k\le r} \Big(\frac{1}{4}\, a_k^{-6}\Big\|B_k^k + \sum_{1\le i<k} B^{ik}\Big\|^2 + \frac{1}{4} \sum_{k<i\le r} a_i^{-6}\big(4 - 3a_k^2 a_i^{-2}\big)\big\|B_k^i\big\|^2 \\
& \qquad + \frac{1}{2}\, Q\Big(\sum_{k<j\le r} a_j^{-4}a_k^{-2}\big(3 - 2a_k^2 a_j^{-2}\big) B_k^j, B_k^k + \sum_{1\le i<k} B^{ik}\Big) \\
& \qquad + \frac{1}{2} \sum_{k<i<j\le r} a_j^{-4}a_i^{-2}\big(6 - 3a_k^2 a_i^{-2} - a_i^2 a_j^{-2}\big) Q\big(B_k^i, B_k^j\big)\Big)
\end{aligned}
$$

and this is the formula given in Lemma 1.16.

For verifying that the given formula for the unnormalized curvature in Proposition 2.5 is right we first consider the submersion

$$\pi : (G \times K, g_\varphi \oplus \tilde{g}_\psi) \to (G, \tilde{g}_\phi)\,, \tag{7.1}$$

where $\tilde{g}_\psi$ denotes the K-invariant metric on K inducing the metric g_ψ by means of the canonical projection. Then we look at the canonical projection

$$(G, \tilde{g}_\phi) \to (G/H, g_\phi)\,. \tag{7.2}$$

This twofolded approach is possible because we get the same inducing map ϕ as if we had taken pointwise the submersion from Lemma 2.3 as can be checked. We point out that the canonical projection only affects ϕ in restricting on $\mathfrak{p}$. That the range is also restricted to $\mathfrak{p}$ is due to the general properties inducing maps have to fulfill when inducing homogeneous metrics.

The computation of the vertical space, the horizontal space and the horizontal lifts of elements in $\mathfrak{g}$ for the submersion in (7.1) carry out over verbatim from the proof of Lemma 2.3. The tangent space to a fiber (vertical space) at (g, k) is given as

$$\mathcal{V}_{(g,k)} = \{(-dL_g(X), dR_k(X))\,|\, X \in \mathfrak{k}\} \subset T_{(g,k)}(G \times K) \cong T_gG \oplus T_kK\,.$$

So we have

$$\mathcal{V}_{(e,e)} = \{(-X, X)\,|\, X \in \mathfrak{k}\} \subset T_{(e,e)}(G \times K) \cong \mathfrak{g} \oplus \mathfrak{k}\,.$$

The horizontal space is given by

$$\mathcal{H}_{(e,e)} = (\mathfrak{s}, 0) \oplus \{(\psi X, cX)\,|\, X \in \mathfrak{k}\}\,.$$

Because of the form of the elements in $\mathcal{V}_{(g,k)}$ and the fact that we obtain a left invariant metric by Cheeger's construction it follows that we have to extend the G-coordinate to a left invariant vector field and the K-coordinate to right invariant vector field of an element in $\mathcal{H}_{(e,e)}$ in order to describe $\mathcal{H}_{(g,k)}$.

The horizontal lift of $X = X_\mathfrak{k} + X_\mathfrak{s}$, where the subscripts denote the projections onto the corresponding subspaces of $\mathfrak{g}$, is given by

$$\bar{X} = \left(\psi(c+\psi)^{-1}X_\mathfrak{k} + X_\mathfrak{s}, c(c+\psi)^{-1}X_\mathfrak{k}\right),$$

and the inducing endomorphism for the metric $\tilde{g}_\phi$ is given as

$$\phi|_{\mathfrak{k}} = c\psi(c+\psi)^{-1} \qquad \text{and} \qquad \phi|_{\mathfrak{s}} = \varphi|_{\mathfrak{s}}\,.$$

Especially we can write the horizontal lift of $X = X_{\mathfrak{k}} + X_{\mathfrak{s}}$ in terms of ϕ as follows

$$\bar{X} = \big(c^{-1}\phi\, X_{\mathfrak{k}} + X_{\mathfrak{s}}, \psi^{-1}\phi\, X_{\mathfrak{k}}\big)$$

and in particular we have

$$\overline{\phi^{-1}X} = \big(\varphi^{-1}X, \psi^{-1}X_{\mathfrak{k}}\big)\,.$$

Since the horizontal lift of $\phi^{-1}X$ is easier to handle with in the computations we are going to work with twisted elements.
On the one hand we have by the O'Neill formula

$$\begin{aligned}
&R^{\tilde{g}_\phi}\big(\phi^{-1}X, \phi^{-1}Y; \phi^{-1}Y, \phi^{-1}X\big)\\
&= R^{g_\varphi \oplus \tilde{g}_\psi}\big(\,\overline{\phi^{-1}X}, \overline{\phi^{-1}Y}; \overline{\phi^{-1}Y}, \overline{\phi^{-1}X}\,\big) + \frac{3}{4}\,\big\|\,[\,\overline{\phi^{-1}X}, \overline{\phi^{-1}Y}\,]^v\big\|^2_{g_\varphi \oplus \tilde{g}_\psi}\\
&= R^{g_\varphi \oplus \tilde{g}_\psi}\big((\varphi^{-1}X, \psi^{-1}X_{\mathfrak{k}}), (\varphi^{-1}Y, \psi^{-1}Y_{\mathfrak{k}}); (\varphi^{-1}Y, \psi^{-1}Y_{\mathfrak{k}}), (\varphi^{-1}X, \psi^{-1}X_{\mathfrak{k}})\big)\\
&\qquad + \frac{3}{4}\,\big\|\,[\,(\varphi^{-1}X, \psi^{-1}X_{\mathfrak{k}}), (\varphi^{-1}Y, \psi^{-1}Y_{\mathfrak{k}})\,]^v\big\|^2_{g_\varphi \oplus \tilde{g}_\psi}\\
&= R^{g_\varphi}\big(\varphi^{-1}X, \varphi^{-1}Y; \varphi^{-1}Y, \varphi^{-1}X\big) + R^{\tilde{g}_\psi}\big(\psi^{-1}X_{\mathfrak{k}}, \psi^{-1}Y_{\mathfrak{k}}; \psi^{-1}Y_{\mathfrak{k}}, \psi^{-1}X_{\mathfrak{k}}\big)\\
&\qquad + \frac{3}{4}\,\big\|\,[\,(\varphi^{-1}X, \psi^{-1}X_{\mathfrak{k}}), (\varphi^{-1}Y, \psi^{-1}Y_{\mathfrak{k}})\,]^v\big\|^2_{g_\varphi \oplus \tilde{g}_\psi}\,.
\end{aligned}$$

On the other hand we have by Püttmann's formula

$$\begin{aligned}
&R^{\tilde{g}_\phi}\big(\phi^{-1}X, \phi^{-1}Y; \phi^{-1}Y, \phi^{-1}X\big)\\
=\ &\frac{1}{2}\,Q\big([\phi^{-1}X, Y] + [X, \phi^{-1}Y], [\phi^{-1}X, \phi^{-1}Y]\big) &(7.3)\\
&- \frac{3}{4}\,Q\big(\phi\,[\phi^{-1}X, \phi^{-1}Y], [\phi^{-1}X, \phi^{-1}Y]\big) &(7.4)\\
&+ \frac{1}{4}\,Q\big([\phi^{-1}X, Y] - [X, \phi^{-1}Y], \phi^{-1}\big([\phi^{-1}X, Y] - [X, \phi^{-1}Y]\big)\big) &(7.5)\\
&- Q\big([\phi^{-1}X, X], \phi^{-1}[\phi^{-1}Y, Y]\big)\,. &(7.6)
\end{aligned}$$

For the matters of clarity we use the abbreviations

$$
\begin{aligned}
A^\varphi &:= \big[\varphi^{-1}X, Y\big] + \big[X, \varphi^{-1}Y\big] & A^\psi &:= \big[\psi^{-1}X_{\mathfrak{k}}, Y_{\mathfrak{k}}\big] + \big[X_{\mathfrak{k}}, \psi^{-1}Y_{\mathfrak{k}}\big] \in \mathfrak{k} \\
B^\varphi &:= \big[\varphi^{-1}X, \varphi^{-1}Y\big] & B^\psi &:= \big[\psi^{-1}X_{\mathfrak{k}}, \psi^{-1}Y_{\mathfrak{k}}\big] \in \mathfrak{k} \\
C^\varphi &:= \big[\varphi^{-1}X, Y\big] - \big[X, \varphi^{-1}Y\big] \in \mathfrak{s} & C^\psi &:= \big[\psi^{-1}X_{\mathfrak{k}}, Y_{\mathfrak{k}}\big] - \big[X_{\mathfrak{k}}, \psi^{-1}Y_{\mathfrak{k}}\big] \in \mathfrak{k}\,,
\end{aligned}
$$

which are also given in the formulation of Proposition 2.5 and point out that we have

$$
\phi^{-1}X = \varphi^{-1}X + \psi^{-1}X_{\mathfrak{k}}\,.
$$

Furthermore by Püttmann's formula we have using the above abbreviations

$$
\begin{aligned}
& R^{g_\varphi}\big(\varphi^{-1}X, \varphi^{-1}Y; \varphi^{-1}Y, \varphi^{-1}X\big) \\
= \;& \frac{1}{2}\,Q\big(A^\varphi, B^\varphi\big) - \frac{3}{4}\,Q\big(\varphi B^\varphi, B^\varphi\big) \\
& + \frac{1}{4}\,Q\big(C^\varphi, \varphi^{-1}C^\varphi\big) - Q\big(\big[\varphi^{-1}X, X\big], \varphi^{-1}\big[\varphi^{-1}Y, Y\big]\big)
\end{aligned}
\tag{7.7}
$$

and

$$
\begin{aligned}
& R^{\tilde{g}_\psi}\big(\psi^{-1}X_{\mathfrak{k}}, \psi^{-1}Y_{\mathfrak{k}}; \psi^{-1}Y_{\mathfrak{k}}, \psi^{-1}X_{\mathfrak{k}}\big) \\
= \;& \frac{1}{2}\,Q\big(A^\psi, B^\psi\big) - \frac{3}{4}\,Q\big(\psi B^\psi, B^\psi\big) \\
& + \frac{1}{4}\,Q\big(C^\psi, \psi^{-1}C^\psi\big) - Q\big(\big[\psi^{-1}X_{\mathfrak{k}}, X_{\mathfrak{k}}\big], \psi^{-1}\big[\psi^{-1}Y_{\mathfrak{k}}, Y_{\mathfrak{k}}\big]\big)\,.
\end{aligned}
\tag{7.8}
$$

Note that by the Ad_K-equivariance of φ we have

$$
[\varphi X_{\mathfrak{s}}, Y_{\mathfrak{k}}] = \varphi\,[X_{\mathfrak{s}}, Y_{\mathfrak{k}}] \quad \forall X_{\mathfrak{s}} \in \mathfrak{s},\; Y_{\mathfrak{k}} \in \mathfrak{k}\,.
$$

This property of the Lie bracket will be needed in the upcoming calculations. Especially we point out that we obtain

$$
[\varphi X_{\mathfrak{s}}, Y_{\mathfrak{s}}]_{\mathfrak{k}} = [X_{\mathfrak{s}}, \varphi Y_{\mathfrak{s}}]_{\mathfrak{k}}\,.
$$

For (7.3) we get

$$
\begin{aligned}
&\frac{1}{2}\,Q\big([\phi^{-1}X,Y]+[X,\phi^{-1}Y],[\phi^{-1}X,\phi^{-1}Y]\big)\\
=&\frac{1}{2}\,Q\big([\varphi^{-1}X+\psi^{-1}X_{\mathfrak{k}},Y]+[X,\varphi^{-1}Y+\psi^{-1}Y_{\mathfrak{k}}],[\varphi^{-1}X+\psi^{-1}X_{\mathfrak{k}},\varphi^{-1}Y+\psi^{-1}Y_{\mathfrak{k}}]\big)\\
=&\frac{1}{2}\,Q\big(A^{\varphi}+[\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}]+[X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]+A^{\psi},B^{\varphi}+[\varphi^{-1}X,\psi^{-1}Y_{\mathfrak{k}}]+[\psi^{-1}X_{\mathfrak{k}},\varphi^{-1}Y]+B^{\psi}\big)\\
=&\frac{1}{2}\,\Big(Q\big(A^{\varphi},B^{\varphi}\big)+Q\big(A^{\psi},B^{\psi}\big)+Q\big(A^{\varphi},[\varphi^{-1}X,\psi^{-1}Y_{\mathfrak{k}}]+[\psi^{-1}X_{\mathfrak{k}},\varphi^{-1}Y]+B^{\psi}\big)\\
&\quad+Q\big([\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}]+[X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]+A^{\psi},B^{\varphi}+[\varphi^{-1}X,\psi^{-1}Y_{\mathfrak{k}}]+[\psi^{-1}X_{\mathfrak{k}},\varphi^{-1}Y]\big)\Big)\,.
\end{aligned}
$$

For (7.4) we have

$$
\begin{aligned}
&-\frac{3}{4}\,Q\big(\phi[\phi^{-1}X,\phi^{-1}Y],[\phi^{-1}X,\phi^{-1}Y]\big)\\
=&-\frac{3}{4}\,Q\big(\phi[\varphi^{-1}X+\psi^{-1}X_{\mathfrak{k}},\varphi^{-1}Y+\psi^{-1}Y_{\mathfrak{k}}],[\varphi^{-1}X+\psi^{-1}X_{\mathfrak{k}},\varphi^{-1}Y+\psi^{-1}Y_{\mathfrak{k}}]\big)\\
=&-\frac{3}{4}\,Q\big(\varphi\big(B^{\varphi}_{\mathfrak{s}}+\varphi^{-1}\big([X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]+[\psi^{-1}X_{\mathfrak{k}},Y_s]\big)\big),B^{\varphi}_{\mathfrak{s}}+\varphi^{-1}\big([X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]+[\psi^{-1}X_{\mathfrak{k}},Y_s]\big)\big)\\
&-\frac{3}{4}\,Q\big(c\psi(c+\psi)^{-1}\big(B^{\varphi}_{\mathfrak{k}}+B^{\psi}+c^{-1}A^{\psi}\big),B^{\varphi}_{\mathfrak{k}}+B^{\psi}+c^{-1}A^{\psi}\big)\,.
\end{aligned}
$$

For (7.5) we obtain

$$
\begin{aligned}
&\frac{1}{4}\,Q\big([\phi^{-1}X,Y]-[X,\phi^{-1}Y],\phi^{-1}\big([\phi^{-1}X,Y]-[X,\phi^{-1}Y]\big)\big)\\
=&\frac{1}{4}\,Q\big([\varphi^{-1}X+\psi^{-1}X_{\mathfrak{k}},Y]-[X,\varphi^{-1}Y+\psi^{-1}Y_{\mathfrak{k}}],\\
&\qquad\phi^{-1}\big([\varphi^{-1}X+\psi^{-1}X_{\mathfrak{k}},Y]-[X,\varphi^{-1}Y+\psi^{-1}Y_{\mathfrak{k}}]\big)\big)\\
=&\frac{1}{4}\,Q\Big(C^{\varphi}+[\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}]-[X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]+C^{\psi},\\
&\qquad\varphi^{-1}C^{\varphi}+\varphi^{-1}\big([\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}]-[X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]\big)+c^{-1}C^{\psi}+\psi^{-1}C^{\psi}\Big)\\
=&\frac{1}{4}\,\Big(Q\big(C^{\varphi},\varphi^{-1}C^{\varphi}\big)+Q\big(C^{\psi},\psi^{-1}C^{\psi}\big)+2\,Q\big(C^{\varphi},\varphi^{-1}\big([\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}]-[X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]\big)\big)\\
&\quad+Q\big([\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}]-[X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}],\varphi^{-1}\big([\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}]-[X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]\big)\big)+c^{-1}\big\|C^{\psi}\big\|^2\Big)\,.
\end{aligned}
$$

And for (7.6) we have

$$
\begin{aligned}
&-Q\big([\phi^{-1}X,X],\phi^{-1}[\phi^{-1}Y,Y]\big)\\
=&-Q\big([\varphi^{-1}X+\psi^{-1}X_{\mathfrak{k}},X],\phi^{-1}[\varphi^{-1}Y+\psi^{-1}Y_{\mathfrak{k}},Y]\big)\\
=&-Q\big([\varphi^{-1}X,X]+[\psi^{-1}X_{\mathfrak{k}},X_{\mathfrak{s}}]+[\psi^{-1}X_{\mathfrak{k}},X_{\mathfrak{k}}],\\
&\qquad\varphi^{-1}[\varphi^{-1}Y,Y]+\varphi^{-1}[\psi^{-1}Y_{\mathfrak{k}},Y_{\mathfrak{s}}]+c^{-1}[\psi^{-1}Y_{\mathfrak{k}},Y_{\mathfrak{k}}]+\psi^{-1}[\psi^{-1}Y_{\mathfrak{k}},Y_{\mathfrak{k}}]\big)\\
=&-Q\big([\varphi^{-1}X,X],\varphi^{-1}[\varphi^{-1}Y,Y]\big)-Q\big([\psi^{-1}X_{\mathfrak{k}},X_{\mathfrak{k}}],\psi^{-1}[\psi^{-1}Y_{\mathfrak{k}},Y_{\mathfrak{k}}]\big)\\
&-Q\big([\varphi^{-1}X,X],\varphi^{-1}[\psi^{-1}Y_{\mathfrak{k}},Y_{\mathfrak{s}}]\big)-Q\big([\psi^{-1}X_{\mathfrak{k}},X_{\mathfrak{s}}],\varphi^{-1}[\varphi^{-1}Y,Y]\big)\\
&-Q\big([\psi^{-1}X_{\mathfrak{k}},X_{\mathfrak{s}}],\varphi^{-1}[\psi^{-1}Y_{\mathfrak{k}},Y_{\mathfrak{s}}]\big)-c^{-1}Q\big([\psi^{-1}X_{\mathfrak{k}},X_{\mathfrak{k}}],[\psi^{-1}Y_{\mathfrak{k}},Y_{\mathfrak{k}}]\big)\,.
\end{aligned}
$$

For relating this with the former results we have to modify some expressions. For this we use the Jacobi identity and ad-skew-symmetry of Q to perform

$$
\begin{aligned}
Q([X_1,X_2],[Y_1,Y_2])&=-Q(X_2,[X_1,[Y_1,Y_2]])\\
&=Q(X_2,[Y_1,[Y_2,X_1]])+Q(X_2,[Y_2,[X_1,Y_1]])\\
&=-Q([X_1,Y_2],[X_2,Y_1])+Q([X_1,Y_1],[X_2,Y_2])\,.
\end{aligned}
$$

So

$$
\begin{aligned}
&-Q\big([\varphi^{-1}X,X],\varphi^{-1}[\psi^{-1}Y_{\mathfrak{k}},Y_{\mathfrak{s}}]\big)\\
=&-Q\big(c^{-1}[X_{\mathfrak{k}},X_{\mathfrak{s}}]+[\varphi^{-1}X_{\mathfrak{s}},X_{\mathfrak{k}}]+[\varphi^{-1}X_{\mathfrak{s}},X_{\mathfrak{s}}],[\psi^{-1}Y_{\mathfrak{k}},\varphi^{-1}Y_{\mathfrak{s}}]\big)\\
=&\,c^{-1}Q\big(\varphi^{-1}[X_{\mathfrak{k}},Y_{\mathfrak{s}}],[X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]\big)-c^{-1}Q\big([X_{\mathfrak{k}},\psi^{-1}Y_{\mathfrak{k}}],[X_{\mathfrak{s}},\varphi^{-1}Y_{\mathfrak{s}}]\big)\\
&+Q\big([X_{\mathfrak{k}},\psi^{-1}Y_{\mathfrak{k}}],[\varphi^{-1}X_{\mathfrak{s}},\varphi^{-1}Y_{\mathfrak{s}}]\big)-Q\big([X_{\mathfrak{k}},\varphi^{-1}Y_{\mathfrak{s}}],[\varphi^{-1}X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]\big)\\
&+Q\big([\varphi^{-1}X_{\mathfrak{s}},\varphi^{-1}Y_{\mathfrak{s}}],[X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]\big)-Q\big([\varphi^{-1}X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}],[X_{\mathfrak{s}},\varphi^{-1}Y_{\mathfrak{s}}]\big)
\end{aligned}
$$

as well as

$$
\begin{aligned}
&-Q\big([\psi^{-1}X_{\mathfrak{k}},X_{\mathfrak{s}}],\varphi^{-1}[\varphi^{-1}Y,Y]\big)\\
=&-Q\big([\psi^{-1}X_{\mathfrak{k}},\varphi^{-1}X_{\mathfrak{s}}],c^{-1}[Y_{\mathfrak{k}},Y_{\mathfrak{s}}]+[\varphi^{-1}Y_{\mathfrak{s}},Y_{\mathfrak{k}}]+[\varphi^{-1}Y_{\mathfrak{s}},Y_{\mathfrak{s}}]\big)\\
=&\,c^{-1}Q\big([\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}],[\varphi^{-1}X_{\mathfrak{s}},Y_{\mathfrak{k}}]\big)-c^{-1}Q\big([\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{k}}],[\varphi^{-1}X_{\mathfrak{s}},Y_{\mathfrak{s}}]\big)\\
&+Q\big([\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{k}}],[\varphi^{-1}X_{\mathfrak{s}},\varphi^{-1}Y_{\mathfrak{s}}]\big)-Q\big([\psi^{-1}X_{\mathfrak{k}},\varphi^{-1}Y_{\mathfrak{s}}],[\varphi^{-1}X_{\mathfrak{s}},Y_{\mathfrak{k}}]\big)\\
&+Q\big([\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}],[\varphi^{-1}X_{\mathfrak{s}},\varphi^{-1}Y_{\mathfrak{s}}]\big)-Q\big([\psi^{-1}X_{\mathfrak{k}},\varphi^{-1}Y_{\mathfrak{s}}],[\varphi^{-1}X_{\mathfrak{s}},Y_{\mathfrak{s}}]\big)
\end{aligned}
$$

and

$$
\begin{aligned}
&-Q\big([\psi^{-1}X_{\mathfrak{k}},X_{\mathfrak{s}}],[\psi^{-1}Y_{\mathfrak{k}},\varphi^{-1}Y_{\mathfrak{s}}]\big)\\
&=Q\big([\psi^{-1}X_{\mathfrak{k}},\varphi^{-1}Y_{\mathfrak{s}}],[X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]\big)-Q\big([\psi^{-1}X_{\mathfrak{k}},\psi^{-1}Y_{\mathfrak{k}}],[X_{\mathfrak{s}},\varphi^{-1}Y_{\mathfrak{s}}]\big)\,,\\
&-c^{-1}Q\big([\psi^{-1}X_{\mathfrak{k}},X_{\mathfrak{k}}],[\psi^{-1}Y_{\mathfrak{k}},Y_{\mathfrak{k}}]\big)\\
&=c^{-1}Q\big([\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{k}}],[X_{\mathfrak{k}},\psi^{-1}Y_{\mathfrak{k}}]\big)-c^{-1}Q\big([\psi^{-1}X_{\mathfrak{k}},\psi^{-1}Y_{\mathfrak{k}}],[X_{\mathfrak{k}},Y_{\mathfrak{k}}]\big)\,.
\end{aligned}
$$

As a consequence we arrive at

$$
\begin{aligned}
&-Q\big([\phi^{-1}X,X],\phi^{-1}[\phi^{-1}Y,Y]\big)\\
=&-Q\big([\varphi^{-1}X,X],\varphi^{-1}[\varphi^{-1}Y,Y]\big)-Q\big([\psi^{-1}X_{\mathfrak{k}},X_{\mathfrak{k}}],\psi^{-1}[\psi^{-1}Y_{\mathfrak{k}},Y_{\mathfrak{k}}]\big)\\
&+c^{-1}Q\big(\varphi^{-1}[X_{\mathfrak{k}},Y_{\mathfrak{s}}],[X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]\big)+c^{-1}Q\big([\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}],\varphi^{-1}[X_{\mathfrak{s}},Y_{\mathfrak{k}}]\big)\\
&+Q\big([\varphi^{-1}X_{\mathfrak{s}},\varphi^{-1}Y_{\mathfrak{s}}],[\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}]+[X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]\big)+Q\big(\varphi^{-1}[\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}],[X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]\big)\\
&-Q\big([X,\varphi^{-1}Y_{\mathfrak{s}}],\varphi^{-1}[X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]\big)-Q\big([\varphi^{-1}X_{\mathfrak{s}},Y],\varphi^{-1}[\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}]\big)\\
&-c^{-1}Q\big(A^{\psi},[X_{\mathfrak{s}},\varphi^{-1}Y_{\mathfrak{s}}]\big)-c^{-1}Q\big(B^{\psi},[X_{\mathfrak{k}},Y_{\mathfrak{k}}]\big)+c^{-1}Q\big([\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{k}}],[X_{\mathfrak{k}},\psi^{-1}Y_{\mathfrak{k}}]\big)\\
&+Q\big(A^{\psi},[\varphi^{-1}X_{\mathfrak{s}},\varphi^{-1}Y_{\mathfrak{s}}]\big)-Q\big(B^{\psi},[X_{\mathfrak{s}},\varphi^{-1}Y_{\mathfrak{s}}]\big)\,.
\end{aligned}
$$

Collecting all expressions coming from the $\mathfrak{s}$-part and neither arising in (7.7) nor in (7.8) we get

$$
\begin{aligned}
&\frac{1}{2}\,Q\big(A^{\varphi}_{\mathfrak{s}},\varphi^{-1}\big([X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]+[\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}]\big)\big)\\
&+\frac{1}{2}\,Q\big([\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}]+[X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}],B^{\varphi}_{\mathfrak{s}}+\varphi^{-1}\big([X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]+[\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}]\big)\big)\\
&-\frac{3}{4}\,Q\big(\varphi\left(B^{\varphi}_{\mathfrak{s}}+\varphi^{-1}\big([X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]+[\psi^{-1}X_{\mathfrak{k}},Y_{s}]\big)\right),B^{\varphi}_{\mathfrak{s}}+\varphi^{-1}\big([X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]+[\psi^{-1}X_{\mathfrak{k}},Y_{s}]\big)\big)\\
&+\frac{1}{2}\,Q\big(C^{\varphi},\varphi^{-1}\big([\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}]-[X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]\big)\big)\\
&+\frac{1}{4}\,Q\big([\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}]-[X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}],\varphi^{-1}\big([\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}]-[X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]\big)\big)\\
&+c^{-1}Q\big(\varphi^{-1}[X_{\mathfrak{k}},Y_{\mathfrak{s}}],[X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]\big)+c^{-1}Q\big([\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}],\varphi^{-1}[X_{\mathfrak{s}},Y_{\mathfrak{k}}]\big)\\
&+Q\big([\varphi^{-1}X_{\mathfrak{s}},\varphi^{-1}Y_{\mathfrak{s}}],[\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}]+[X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]\big)+Q\big(\varphi^{-1}[\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}],[X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]\big)\\
&-Q\big([X,\varphi^{-1}Y_{\mathfrak{s}}],\varphi^{-1}[X_{\mathfrak{s}},\psi^{-1}Y_{\mathfrak{k}}]\big)-Q\big([\varphi^{-1}X_{\mathfrak{s}},Y],\varphi^{-1}[\psi^{-1}X_{\mathfrak{k}},Y_{\mathfrak{s}}]\big)=0\,.
\end{aligned}
$$

For the expressions coming from the $\mathfrak{k}$-part and neither arising in (7.7) nor in (7.8) we obtain

$$
\begin{aligned}
&\frac{1}{2}\,Q\big(A^{\varphi}_{\mathfrak{k}}, c^{-1}\big(\big[X_{\mathfrak{k}}, \psi^{-1}Y_{\mathfrak{k}}\big] + \big[\psi^{-1}X_{\mathfrak{k}}, Y_{\mathfrak{k}}\big]\big) + B^{\psi}\big)\\
&+\frac{1}{2}\,Q\big(A^{\psi}, B^{\varphi}_{\mathfrak{k}} + c^{-1}\big(\big[X_{\mathfrak{k}}, \psi^{-1}Y_{\mathfrak{k}}\big] + \big[\psi^{-1}X_{\mathfrak{k}}, Y_{\mathfrak{k}}\big]\big)\big)\\
&-\frac{3}{4}\,Q\big(c\psi(c+\psi)^{-1}\big(B^{\varphi}_{\mathfrak{k}} + B^{\psi} + c^{-1}A^{\psi}\big), B^{\varphi}_{\mathfrak{k}} + B^{\psi} + c^{-1}A^{\psi}\big)\\
&+\frac{1}{4}\,c^{-1}\big\|C^{\psi}\big\|^2 - c^{-1}Q\big(A^{\psi}, \big[X_{\mathfrak{s}}, \varphi^{-1}Y_{\mathfrak{s}}\big]\big) - c^{-1}Q\big(B^{\psi}, [X_{\mathfrak{k}}, Y_{\mathfrak{k}}]\big)\\
&+c^{-1}Q\big(\big[\psi^{-1}X_{\mathfrak{k}}, Y_{\mathfrak{k}}\big], \big[X_{\mathfrak{k}}, \psi^{-1}Y_{\mathfrak{k}}\big]\big) + Q\big(A^{\psi}, \big[\varphi^{-1}X_{\mathfrak{s}}, \varphi^{-1}Y_{\mathfrak{s}}\big]\big) - Q\big(B^{\psi}, \big[X_{\mathfrak{s}}, \varphi^{-1}Y_{\mathfrak{s}}\big]\big)\\
={}&\frac{3}{4}\,c\Big(- \big\|B^{\varphi}_{\mathfrak{k}}\big\|^2 - c^{-1}Q\big(\psi B^{\psi}, B^{\psi}\big)\\
&\qquad + c^{-1}Q\big((c+\psi)^{-1}\big(cB^{\varphi}_{\mathfrak{k}} + A^{\psi} - \psi B^{\psi}\big), cB^{\varphi}_{\mathfrak{k}} + A^{\psi} - \psi B^{\psi}\big)\Big)\,,
\end{aligned}
$$

where $-\frac{3}{4}\,c\,\big\|B^{\varphi}_{\mathfrak{k}}\big\|^2$ in fact belongs to (7.7) and $-\frac{3}{4}\,Q\big(\psi B^{\psi}, B^{\psi}\big)$ belongs to (7.8). This is why the O'Neill term is given as

$$
\frac{3}{4}\,Q\big((c+\psi)^{-1}\big(cB^{\varphi}_{\mathfrak{k}} + A^{\psi} - \psi B^{\psi}\big), cB^{\varphi}_{\mathfrak{k}} + A^{\psi} - \psi B^{\psi}\big)\,.
$$

So

$$
\begin{aligned}
&R^{\tilde{g}_{\phi}}\big(\phi^{-1}X, \phi^{-1}Y; \phi^{-1}Y, \phi^{-1}X\big)\\
&= R^{g_{\varphi}}\big(\varphi^{-1}X, \varphi^{-1}Y; \varphi^{-1}Y, \varphi^{-1}X\big) + R^{\tilde{g}_{\psi}}\big(\psi^{-1}X_{\mathfrak{k}}, \psi^{-1}Y_{\mathfrak{k}}; \psi^{-1}Y_{\mathfrak{k}}, \psi^{-1}X_{\mathfrak{k}}\big)\\
&\qquad + \frac{3}{4}\,Q\big((c+\psi)^{-1}\big(cB^{\varphi}_{\mathfrak{k}} + A^{\psi} - \psi B^{\psi}\big), cB^{\varphi}_{\mathfrak{k}} + A^{\psi} - \psi B^{\psi}\big)\,.
\end{aligned}
$$

Now consider the submersion (7.2). We again have by O'Neill's formula

$$
\begin{aligned}
&R^{g_{\phi}}\big(\phi^{-1}X, \phi^{-1}Y; \phi^{-1}Y, \phi^{-1}X\big)\\
&= R^{\tilde{g}_{\phi}}\big(\phi^{-1}X, \phi^{-1}Y; \phi^{-1}Y, \phi^{-1}X\big) + \frac{3}{4}\,\big\|[\phi^{-1}X, \phi^{-1}Y]_{\mathfrak{h}}\big\|^2_{\tilde{g}_{\phi}}\,.
\end{aligned}
$$

where $\frac{3}{4}\,\|[\phi^{-1}X, \phi^{-1}Y]_{\mathfrak{h}}\|^2_{\tilde{g}}$ is the O'Neill term because the vertical space of the canonical submersion from G to the homogeneous space G/H is $\mathfrak{h}$. In the upcoming calculations we will still work with the introduced abbreviations but we have to pay attention that now $X_{\mathfrak{k}}$ resp. $Y_{\mathfrak{k}}$ has to be replaced by $X_{\mathfrak{m}}$ resp. $Y_{\mathfrak{m}}$. We can compute the O'Neill term analogeous to the computation of (7.4) to be

$$
\begin{aligned}
&\frac{3}{4}\,Q\big(\phi\big[\phi^{-1}X,\phi^{-1}Y\big]_{\mathfrak{h}},\big[\phi^{-1}X,\phi^{-1}Y\big]_{\mathfrak{h}}\big)\\
&=\frac{3}{4}\,Q\big(c\psi(c+\psi)^{-1}\big(B^{\varphi}_{\mathfrak{h}}+B^{\psi}_{\mathfrak{h}}+c^{-1}A^{\psi}_{\mathfrak{h}}\big),B^{\varphi}_{\mathfrak{h}}+B^{\psi}_{\mathfrak{h}}+c^{-1}A^{\psi}_{\mathfrak{h}}\big)\,.
\end{aligned}
$$

We have

$$
\begin{aligned}
&\frac{3}{4}\,Q\big((c+\psi)^{-1}\big(cB^{\varphi}_{\mathfrak{h}}+A^{\psi}_{\mathfrak{h}}-\psi B^{\psi}_{\mathfrak{h}}\big),cB^{\varphi}_{\mathfrak{h}}+A^{\psi}_{\mathfrak{h}}-\psi B^{\psi}_{\mathfrak{h}}\big)\\
&+\frac{3}{4}\,Q\big(c\psi(c+\psi)^{-1}\big(B^{\varphi}_{\mathfrak{h}}+B^{\psi}_{\mathfrak{h}}+c^{-1}A^{\psi}_{\mathfrak{h}}\big),B^{\varphi}_{\mathfrak{h}}+B^{\psi}_{\mathfrak{h}}+c^{-1}A^{\psi}_{\mathfrak{h}}\big)\\
&=\frac{3}{4}\,c\,\big\|B^{\varphi}_{\mathfrak{h}}+c^{-1}A^{\psi}_{\mathfrak{h}}\big\|^2+\frac{3}{4}\,Q\big(\psi B^{\psi}_{\mathfrak{h}},B^{\psi}_{\mathfrak{h}}\big)\,,
\end{aligned}
$$

where $\frac{3}{4}Q(\psi B^{\psi}_{\mathfrak{h}},B^{\psi}_{\mathfrak{h}})$ is especially is the O'Neill term of the canonical submersion

$$(K,\tilde{g}_{\psi})\to(K/H,g_{\psi})\,.$$

Putting all together yields the formula given in Proposition 2.5.

Proof of Proposition 4.2. Up to scaling $Q(A,B)=-\frac{1}{2}Re(\mathrm{tr}(AB))$ is the only $\mathrm{Ad}_{SU(n+1)}$-invariant inner product on $\mathfrak{su}(n+1)$ because there are no nontrivial $\mathrm{Ad}_{SU(n+1)}$-invariant subspaces of $\mathfrak{su}(n+1)$ and since $Q(A,B)=-\frac{1}{2}Re(\mathrm{tr}(AB))$ is a $\mathrm{Ad}_{SU(n+1)}$-invariant inner product on $\mathfrak{su}(n+1)$ it follows by Schur's Lemma that it is the only one up to scaling. The Lie algebra of the isotropy group

$$H=\left\{\begin{pmatrix}1&0\\0&M\end{pmatrix}\in SU(n+1)\,\middle|\,M\in SU(n)\right\}$$

in $p_0=(1,0,\cdots,0)\in S^{2n+1}\subset\mathbb{C}^{n+1}$ is given by

$$\mathfrak{h}=\left\{\begin{pmatrix}0&0\\0&A\end{pmatrix}\in\mathfrak{su}(n+1)\,\middle|\,A\in\mathfrak{su}(n)\right\}.$$

It can be checked that

$$\mathfrak{h}^{\perp}=\underbrace{\mathrm{span}\,\{\mathrm{diag}(-ni,i,\cdots,i)\}}_{=:\mathfrak{m}_1}\overset{\perp}{\oplus}\underbrace{\left\{\begin{pmatrix}0&z\\-z^*&0\end{pmatrix}\in\mathfrak{su}(n+1)\,\middle|\,z\in\mathbb{C}^n\right\}}_{=:\mathfrak{m}_2},$$

where $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are the Ad_H-invariant subspaces with $\mathfrak{m}_1\perp\mathfrak{m}_2$.

We compute

$$\|\mathrm{diag}(-ni, i, \cdots, i)\|_Q^2 = \frac{n(n+1)}{2} \quad \text{and} \quad \left\| \begin{pmatrix} 0 & z \\ -z^* & 0 \end{pmatrix} \right\|_Q^2 = |z|^2 \,. \tag{7.9}$$

Now consider $T_{p_0}S^{2n+1}$. According to the identification of $T_{p_0}S^{2n+1}$ with $\mathfrak{m}_1 \oplus \mathfrak{m}_2$ via action fields we get

$$\begin{aligned} & \frac{d}{dt}\Big|_{t=0} \exp(t\,\mathrm{diag}(-ni, i, \cdots, i))(p_0) = (-ni, 0, \cdots, 0) \\ \text{and} \quad & \frac{d}{dt}\Big|_{t=0} \exp\left(t \begin{pmatrix} 0 & z \\ -z^* & 0 \end{pmatrix}\right)(p_0) = (0, z)\,. \end{aligned}$$

We have

$$\|(-ni, 0, \cdots, 0)\|_{g_{can}}^2 = n^2 \quad \text{and} \quad \|(0, z)\|_{g_{can}}^2 = |z|^2\,.$$

Comparing this with (7.9) yields the parameters $a_1^2 = \frac{2n}{n+1}$ and $a_2^2 = 1$ for obtaining the round metric.

For determining the $\mathrm{Ad}_{U(n+1)}$-invariant inner products on $\mathfrak{u}(n+1)$ we observe that

$$\mathrm{span}\,\{i\,\mathrm{Id}\} \quad \text{and} \quad \mathfrak{su}(n+1)$$

are $\mathrm{Ad}_{U(n+1)}$-invariant subspaces of $\mathfrak{u}(n+1)$ and there are no further $\mathrm{Ad}_{U(n+1)}$-invariant subspaces. So we obtain by Schur's Lemma that

$$\begin{aligned} & Q(A_1 + A_2, B_1 + B_2) = -\tfrac{1}{2} Re\big(\mathrm{tr}\big(\tfrac{\lambda}{n} A_1 B_1 + A_2 B_2\big)\big) \\ \text{for} \quad & A_1 + A_2, B_1 + B_2 \in \mathrm{span}\{i\,\mathrm{Id}\} \oplus \mathfrak{su}(n+1),\ \lambda \in \mathbb{R}^{>0} \end{aligned}$$

is up to scaling the only $\mathrm{Ad}_{U(n+1)}$-invariant inner product on $\mathfrak{u}(n+1)$.
The Lie algebra of the isotropy group in $p_0 = (1, 0, \cdots, 0) \in S^{2n+1} \subset \mathbb{C}^{n+1}$,

$$H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} \in U(n+1) \,\Big|\, M \in U(n) \right\}$$

is given by

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \in \mathfrak{u}(n+1) \,\Big|\, A \in \mathfrak{u}(n) \right\}.$$

For computing $\mathfrak{h}^\perp$ we first observe that an element in $\mathfrak{h}$ decomposes as

$$\mathfrak{h} \ni \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} - \frac{\mathrm{tr}(A)}{n+1}\,\mathrm{Id}}_{\in \mathfrak{su}(n+1)} - \underbrace{\frac{i\,\mathrm{tr}(A)}{n+1}\,\mathrm{Id}}_{\in \mathrm{span}\,\{i\,\mathrm{Id}\}} .$$

Now let $B_1 + B_2 \in \mathrm{span}\,\{i\,\mathrm{Id}\} \oplus \mathfrak{su}(n+1)$ and decompose $B_2 \in \mathfrak{su}(n+1)$ as

$$B_2 = b_2 \underbrace{\mathrm{diag}(-ni, i, \cdots, i)}_{=:D} + \begin{pmatrix} 0 & z \\ -z^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \quad \text{with } b_2 \in \mathbb{R},\ z \in \mathbb{C}^n,\ B \in \mathfrak{su}(n) .$$

So we get

$$\begin{aligned} &Q\left(-\frac{i\,\mathrm{tr}(A)}{n+1}\, i\,\mathrm{Id} + \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} - \frac{\mathrm{tr}(A)}{n+1}\,\mathrm{Id}\,,\, b_1 i\,\mathrm{Id} + b_2 D + \begin{pmatrix} 0 & z \\ -z^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \right) \\ &= -\frac{1}{2n}\left((\lambda\, b_1 + b_2 n)\, i\,\mathrm{tr}(A) + nRe\big(\mathrm{tr}(AB)\big) \right) \quad . \end{aligned}$$

This vanishes for all $b_1 \in \mathbb{R}$ and $A \in \mathfrak{u}(n)$ if and only if $b_1 = -\frac{n}{\lambda}\, b_2$ and $B = 0$. Therefore we get

$$\mathfrak{h}^\perp = \underbrace{\mathrm{span}\,\{ni\,\mathrm{Id} - \lambda\,\mathrm{diag}(-ni, i, \cdots, i)\}}_{=:\mathfrak{m}_1} \overset{\perp}{\oplus} \underbrace{\left\{ \begin{pmatrix} 0 & z \\ -z^* & 0 \end{pmatrix} \in \mathfrak{u}(n+1) \,\middle|\, z \in \mathbb{C}^n \right\}}_{=:\mathfrak{m}_2},$$

where $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are the Ad_H-invariant subspaces with $\mathfrak{m}_1 \perp \mathfrak{m}_2$.
We compute

$$\|ni\,\mathrm{Id} - \lambda\,\mathrm{diag}(-ni, i, \cdots, i)\|_Q^2 = \frac{n(n+1)\lambda(\lambda+1)}{2} \quad \text{and} \quad \left\| \begin{pmatrix} 0 & z \\ -z^* & 0 \end{pmatrix} \right\|_Q^2 = |z|^2 .$$

Now consider $T_{p_0}S^{2n+1}$. According to the identification of $T_{p_0}S^{2n+1}$ with $\mathfrak{m}_1 \oplus \mathfrak{m}_2$ via action fields we get

$$\begin{aligned} &\frac{d}{dt}\Big|_{t=0} \exp\big(t\,(ni\,\mathrm{Id} - \lambda\,\mathrm{diag}(-ni, i, \cdots, i))\big)(p_0) = \big(n(\lambda+1)i, 0, \cdots, 0\big) \\ \text{and} \quad &\frac{d}{dt}\Big|_{t=0} \exp\left(t \begin{pmatrix} 0 & z \\ -z^* & 0 \end{pmatrix} \right)(p_0) = (0, z) . \end{aligned}$$

We have

$$\big(n(\lambda+1)i,0,\cdots,0\big)\|^2_{g_{can}} = n^2(\lambda+1) \quad \text{and} \quad \|(0,z)\|^2_{g_{can}} = |z|^2\,.$$

Comparing this with the corresponding norms in the metric Q yields the parameters $a_1^2 = \frac{\lambda+1}{\lambda}\frac{2n}{n+1}$ and $a_2^2 = 1$ for obtaining the round metric.

For determining the $\mathrm{Ad}_{U(1)Sp(n+1)}$-invariant inner products on $\mathfrak{u}(1)\mathfrak{sp}(n+1)$ we observe that

$$\mathfrak{u}(1) \quad \text{and} \quad \mathfrak{sp}(n+1)$$

are naturally $\mathrm{Ad}_{U(1)Sp(n+1)}$-invariant subspaces of $\mathfrak{u}(1)\mathfrak{sp}(n+1)$ and there are no further $\mathrm{Ad}_{U(1)Sp(n+1)}$-invariant subspaces. Hence by Schur's Lemma it follows that

$$\begin{aligned} Q\big((ix,A),(iy,B)\big) &= -\frac{1}{2}\left(\lambda\, xy + Re(\mathrm{tr}(AB))\right) \\ \text{for} \quad (ix,A),(iy,B) &\in \mathfrak{u}(1)\mathfrak{sp}(n+1),\ \lambda \in \mathbb{R}^{>0} \end{aligned}$$

is up to scaling the only $\mathrm{Ad}_{U(1)Sp(n+1)}$-invariant inner product on $\mathfrak{u}(1)\mathfrak{sp}(n+1)$.

The Lie algebra of the isotropy group in $p_0 = (1,0,\cdots,0) \in S^{4n+3} \subset \mathbb{H}^{n+1}$,

$$H = \left\{ \left(z, \begin{pmatrix} z & 0 \\ 0 & M \end{pmatrix}\right) \in U(1)Sp(n+1) \,\middle|\, z \in U(1),\ M \in Sp(n) \right\} \cong U(1)Sp(n)$$

is given by

$$\mathfrak{h} = \left\{ \left(ix, \begin{pmatrix} ix & 0 \\ 0 & A \end{pmatrix}\right) \in \mathfrak{u}(1)\mathfrak{sp}(n+1) \,\middle|\, x \in \mathbb{R},\ A \in \mathfrak{sp}(n) \right\}.$$

Decompose $\tilde{B} \in \mathfrak{sp}(n+1)$ as

$$\tilde{B} = \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & z \\ -z^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \quad \text{with } v = iv_1 + jv_2 + kv_3 \in \mathfrak{sp}(1),$$
$$z \in \mathbb{H}^n,\ B \in \mathfrak{sp}(n)\,.$$

Then

$$Q\left(\left(ix, \begin{pmatrix} ix & 0 \\ 0 & A \end{pmatrix}\right), \left(iy, \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & z \\ -z^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}\right)\right)$$
$$= \frac{1}{2}\left(\lambda xy + xv_1 - Re\big(\mathrm{tr}(AB)\big)\right) \quad .$$

This vanishes for all $x \in \mathbb{R}$ and $A \in \mathfrak{sp}(n)$ if and only if $y = -\frac{1}{\lambda} v_1$ and $B = 0$. Therefore we get

$$\mathfrak{h}^\perp = \left\{ \left(-\tfrac{1}{\lambda}\, iv_1, \begin{pmatrix} v & z \\ -z^* & 0 \end{pmatrix}\right) \in \mathfrak{u}(1)\mathfrak{sp}(n+1) \,\middle|\, v = iv_1 + jv_2 + kv_3 \in \mathfrak{sp}(1),\ z \in \mathbb{H}^n \right\}.$$

An Ad_H-invariant decomposition of $\mathfrak{m} := \mathfrak{h}^\perp$ is given by $\mathfrak{m} = \mathfrak{m}_1 \overset{\perp}{\oplus} \mathfrak{m}_2 \overset{\perp}{\oplus} \mathfrak{m}_3$ with

$$\mathfrak{m}_1 = \left\{ \left(-\tfrac{1}{\lambda}\, iv_1, \begin{pmatrix} iv_1 & 0 \\ 0 & 0 \end{pmatrix}\right) \in \mathfrak{u}(1)\mathfrak{sp}(n+1) \,\middle|\, v_1 \in \mathbb{R} \right\},$$
$$\mathfrak{m}_2 = \left\{ \left(0, \begin{pmatrix} jv_2 + kv_3 & 0 \\ 0 & 0 \end{pmatrix}\right) \in \mathfrak{u}(1)\mathfrak{sp}(n+1) \,\middle|\, jv_2 + kv_3 \in Im(\mathbb{H}) \setminus \{i\,\mathbb{R}\} \right\},$$
$$\mathfrak{m}_3 = \left\{ \left(0, \begin{pmatrix} 0 & z \\ -z^* & 0 \end{pmatrix}\right) \in \mathfrak{u}(1)\mathfrak{sp}(n+1) \,\middle|\, z \in \mathbb{H}^n \right\}.$$

We compute

$$\left\| \left(-\tfrac{1}{\lambda}\, iv_1, \begin{pmatrix} iv_1 & 0 \\ 0 & 0 \end{pmatrix}\right) \right\|_Q^2 = \frac{1}{2}\,\frac{\lambda+1}{\lambda}\, v_1^2, \quad \left\| \left(0, \begin{pmatrix} jv_2 + kv_3 & 0 \\ 0 & 0 \end{pmatrix}\right) \right\|_Q^2 = \frac{1}{2}\,(v_2^2 + v_3^2),$$

and $\left\| \begin{pmatrix} 0 & z \\ -z^* & 0 \end{pmatrix} \right\|_Q^2 = |z|^2\,.$

Consider $T_{p_0}S^{4n+3}$. According to the identification of $T_{p_0}S^{4n+3}$ with $\mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ via action fields we get

$$\frac{d}{dt}\Big|_{t=0}\exp\Big(t\Big(-\tfrac{1}{\lambda}\,iv_1,\begin{pmatrix} iv_1 & 0\\ 0 & 0\end{pmatrix}\Big)\Big)(p_0) = \frac{d}{dt}\Big|_{t=0}\exp\Big(t\begin{pmatrix} iv_1 & 0\\ 0 & 0\end{pmatrix}\Big)(p_0)\Big(\exp\big(t\,(-\tfrac{1}{\lambda}iv_1)\big)\Big)^{-1}$$
$$= \Big(\frac{\lambda+1}{\lambda}\,iv_1, 0, \cdots, 0\Big)$$
$$\frac{d}{dt}\Big|_{t=0}\exp\Big(t\Big(0,\begin{pmatrix} jv_2+kv_3 & 0\\ 0 & 0\end{pmatrix}\Big)\Big)(p_0) = \big(0, jv_2, kv_3, 0, \cdots, 0\big)$$
$$\text{and}\quad \frac{d}{dt}\Big|_{t=0}\exp\Big(t\begin{pmatrix} 0 & z\\ -z^* & 0\end{pmatrix}\Big)(p_0) = (0, z)\,.$$

Computing

$$\|\big(\tfrac{\lambda+1}{\lambda}\,iv_1, 0, \cdots, 0\big)\|^2_{g_{can}} = v_1^2\big(\tfrac{\lambda+1}{\lambda}\big)^2, \quad \|\big(0, jv_2, kv_3, 0, \cdots, 0\big)\|^2_{g_{can}} = v_2^2 + v_3^2$$
$$\text{and}\quad \|(0,z)\|^2_{g_{can}} = |z|^2$$

and comparing this with the corresponding norms in the metric Q yields the parameters $a_1^2 = 2\,\frac{\lambda+1}{\lambda}$, $a_2^2 = 2$ and $a_3^2 = 1$ for obtaining the round metric.

The computations for determining the constants for the round metric in the cases $6, 7$ are similar to the computuations above and are therefore left out.

Nevertheless we observe that for sphere number 7 we have the isotropy group

$$H = \Big\{\Big(q,\begin{pmatrix} q & 0\\ 0 & M\end{pmatrix}\Big) \in Sp(1)Sp(n+1)\,\big|\, q\in Sp(1),\ M\in Sp(n)\Big\} \cong Sp(1)Sp(n)$$

in $p_0 = (1, 0, \cdots, 0) \in S^{4n+3} \subset \mathbb{H}^{n+1}$ with Lie algebra

$$\mathfrak{h} = \Big\{\Big(v,\begin{pmatrix} v & 0\\ 0 & A\end{pmatrix}\Big) \in \mathfrak{sp}(1)\mathfrak{sp}(n+1)\,\big|\, v\in \mathfrak{sp}(1),\ A\in \mathfrak{sp}(n)\Big\}.$$

Then an Ad_H-invariant decomposition of $\mathfrak{m} := \mathfrak{h}^\perp$ is given by $\mathfrak{m} = \mathfrak{m}_1 \overset{\perp}{\oplus} \mathfrak{m}_2$ with

$$\mathfrak{m}_1 = \Big\{\Big(-v,\begin{pmatrix} \lambda v & 0\\ 0 & 0\end{pmatrix}\Big) \in \mathfrak{sp}(1)\mathfrak{sp}(n+1)\,\big|\, v\in \mathfrak{sp}(1)\Big\} \cong Im(\mathbb{H})\,,$$
$$\mathfrak{m}_2 = \Big\{\Big(0,\begin{pmatrix} 0 & z\\ -z^* & 0\end{pmatrix}\Big) \in \mathfrak{sp}(1)\mathfrak{sp}(n+1)\,\big|\, z\in \mathbb{H}^n\Big\} \cong \mathbb{H}^n\,.$$

For a detailed treatment of case 9 see [GZ2]. □

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