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Christian Eggert, Ralf Gäer, Frank Klinker

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# The general treatment of non-symmetric, non-balanced star circuits: On the geometrization of problems in electrical metrology

Christian Eggert<sup>1</sup>, Ralf Gäer<sup>2</sup>, Frank Klinker<sup>\*3</sup>

<sup>1</sup>ThyssenKrupp Rothe Erde GmbH, Tremoniastraße 5-11, 44137 Dortmund, Germany  
[christian.eggert@thyssenkrupp.com](mailto:christian.eggert@thyssenkrupp.com)

<sup>2</sup>Schniewindt GmbH & Co. KG, Schöntaler Weg 46, 58809 Neuenrade, Germany  
[ralf.gaeer@schniewindt.de](mailto:ralf.gaeer@schniewindt.de)

<sup>3</sup>Faculty of Mathematics, TU Dortmund University, 44221 Dortmund, Germany  
[frank.klinker@math.tu-dortmund.de](mailto:frank.klinker@math.tu-dortmund.de)

**Abstract.** In the present note we provide the general solution of a question concerning non-symmetric AC star circuits that came up in a practical application: Given a non-symmetric AC star circuit, we need the quantities of the line voltages. For technical reasons these quantities cannot be measured directly but the phase-to-phase voltages can be. In this text we present a way to compute the needed quantities from the measured ones. We translate this problem in electrical metrology to a geometric one and present in detail a general solution that is well adapted to the practical problem. Furthermore, we solve the generalization of the problem that discusses the non-symmetric, non-balanced star circuit. In addition, we give some further remarks on the mathematical side of the initial problem.

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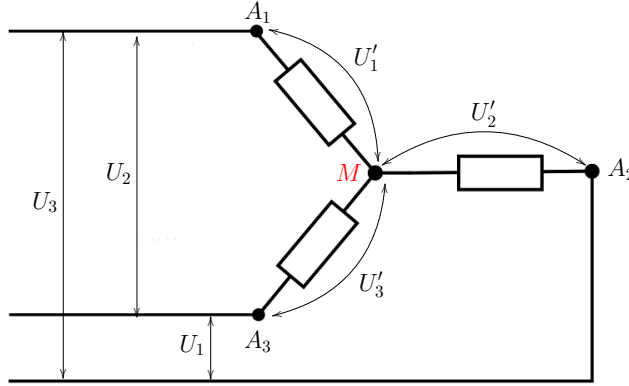
\*Corresponding author  
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## 1 Introducing the initial problem

### 1.1 The technical problem

We consider the situation as drawn in the circuit diagram in Figure 1, i.e. a star circuit. The well known symmetric situation is as follows: Between the points  $A_i$  and  $M$  we have

Figure 1: The basic circuit



AC voltages with same amplitudes but phase differences of  $\psi = 120^\circ$ . Then the voltages can be described in terms of harmonic oscillations in the following way:<sup>3</sup>

$$\begin{aligned} U'_1 &= \hat{U}'_1 \cos(\omega t + \psi_1) = \hat{U}'_1 \Re(e^{i\omega t + i\psi_1}), \\ U'_2 &= \hat{U}'_2 \cos(\omega t + \psi_2) = \hat{U}'_2 \Re(e^{i\omega t + i\psi_2}), \\ U'_3 &= \hat{U}'_3 \cos(\omega t + \psi_3) = \hat{U}'_3 \Re(e^{i\omega t + i\psi_3}), \end{aligned}$$

with  $\hat{U}'_1 = \hat{U}'_2 = \hat{U}'_3 = \hat{U}$  and  $\psi_1 = \psi, \psi_2 = 0, \psi_3 = 2\psi$ . The voltage  $U_3$  between the points  $A_1$  and  $A_2$  is given by the difference of the two voltages  $U'_1$  and  $U'_2$ , i.e.

$$\begin{aligned} U_3 &= \hat{U} \cos(\omega t + \psi_2) - \hat{U} \cos(\omega t + \psi_1) = \hat{U} \Re(e^{i\omega t + \psi_2} - e^{i\omega t + \psi_1}) \\ &= 2\hat{U} \sin \frac{\psi_1 - \psi_2}{2} \cos\left(\omega t + \frac{\psi_1 + \psi_2}{2} - 90^\circ\right). \end{aligned}$$

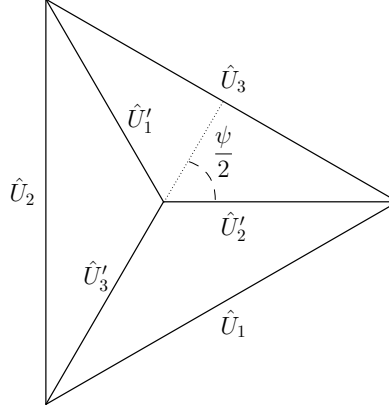
Due to the symmetric situation,  $|\psi_i - \psi_j| \sim 120^\circ$ ,  $\hat{U}'_i = \hat{U}'_j$ , the amplitudes of  $U_1, U_2$ , and  $U_3$  are given by

$$\hat{U}_1 = \hat{U}_2 = \hat{U}_3 = 2\hat{U} \sin 60^\circ = \sqrt{3}\hat{U}. \quad (1)$$

Using the relation between complex numbers and plane geometry, where addition and multiplication are replaced by vector addition and dilatation rotation, we may translate

<sup>3</sup>By  $\Re(z)$  we denote the real part of the complex number  $z$ , i.e.  $\Re(z) = \frac{1}{2}(z + z^*)$ .

Figure 2: The basic circuit translated to the plane



the above circuit into the plane and get the situation from Figure 2. Such diagrams related to AC calculations are called phasor diagrams and a basic introduction can be found in [1, 2], for example. In Figure 2 we only draw the amplitudes of the voltages. To see the vector character note that  $U'_1, U'_2,$  and  $U'_3$  point outwards such that, for example,  $U_3$  points south-east. Now you will find the phase shift of  $\frac{\psi}{2} - 90^\circ \sim \frac{\psi}{2} + 270^\circ = 330^\circ$  as the angle between the horizontal and  $\hat{U}_3$  measured in the upper point and counter clock wise.

We consider a non symmetric variant of the circuit from Figure 1 in the following way: The phase differences of the primed voltages remain  $120^\circ$  but the amplitudes differ.

**Problem 1.** We start with the star circuit as given in Figure 1 with non-symmetric line voltages (primed). The configuration of our system only allows to measure the phase-to-phase voltages (non-primed). Because we need the primed quantities, we are looking for a way to compute them from the non-primed ones.

The question above came up during the testing of high voltage generators. The electrical test-resistances that have been used are not constant but independently deviate in time due to warming of the components and the result is a non-symmetry in the voltages. The schematic setup of such test resistances are shown in Figure 3.

## 1.2 The geometric setup

Given three rays starting from one point  $M$  that pairwise form an angle of  $120^\circ$ . Furthermore, given three points  $A, B, C$  each lying on one ray. These point form a triangle  $\Delta(ABC)$  of which none of its angles  $\alpha, \beta, \gamma$  reach  $120^\circ$ . This is because the angle between one edge of the triangle and a corresponding ray does not reach  $60^\circ$ , see Figure 4.

Figure 3: Schematic setup of a testing-resistance

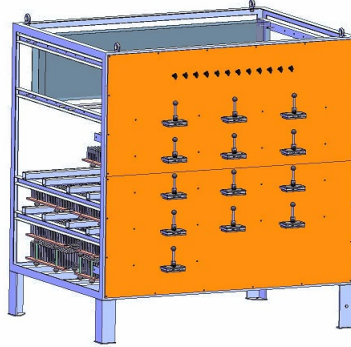
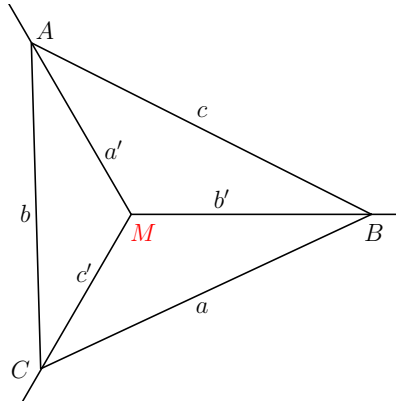


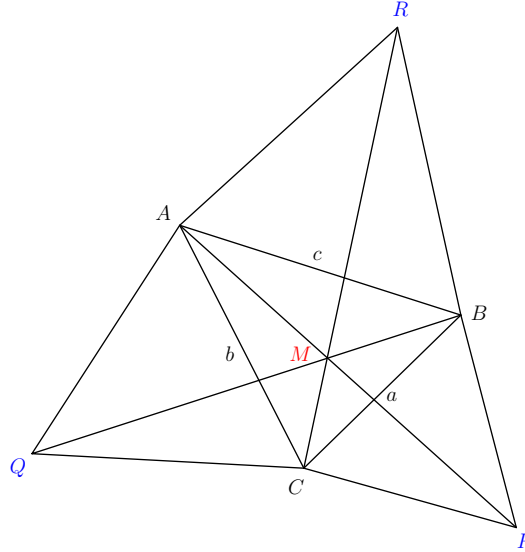
Figure 4: The geometric setup



**Proposition 2.** Given a triangle  $\triangle(ABC)$  with all angles less than  $120^\circ$ . Then there is a unique point  $M$  in the interior for which the lines from  $M$  to the corners form equal angles of  $120^\circ$ . It can be constructed in the following way:

1. Over each edge of the triangle  $\triangle(ABC)$  draw an equilateral triangle:  $\triangle(ARB)$ ,  $\triangle(BPC)$  and  $\triangle(ACQ)$ .
2. Draw straight lines  $\overline{BQ}$ ,  $\overline{AP}$  and  $\overline{CR}$ .
3. These lines intersect in one point, namely  $M$ .

We will recall an elementary geometric proof of this statement in Section 5.1. Unfortunately, this geometric proof does not give the position of the Point  $M$  explicitly. Therefore, we will use other techniques to get a solution of Problem 1 in Sections 2 and 3.

Figure 5: Construction of the point  $M$ 

### 1.3 The geometrized problem

We already discussed the geometric properties of Problem 1 before, see Figure 2 and 4. Now we reformulate it in the following way:

**Problem 3.** Starting from the ray configuration as presented in Figure 4 and given the lengths of the three edges  $a = |\overline{BC}|$ ,  $b = |\overline{AC}|$ ,  $c = |\overline{AB}|$  of the triangle, we like to know the lengths  $a'$ ,  $b'$ ,  $c'$  of the segments  $\overline{MA}$ ,  $\overline{MB}$ ,  $\overline{MC}$ .

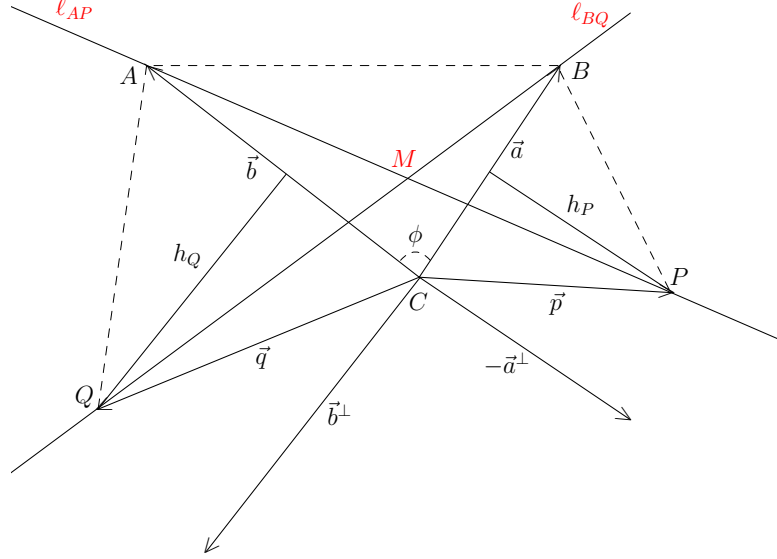
## 2 The solution of the geometrized problem

In this section we will give a proof of Proposition 2 that makes use of the tools from linear algebra. We will calculate the coordinates of the point  $M$  by using its construction. We will then show in Section 3 how formulas for  $a'$ ,  $b'$ ,  $c'$  can be obtained that are symmetric as functions of  $a, b, c$ . This solves our initial Problem 1.

We consider the situation as outlined in Figure 6 where the Point  $C$  is the origin of the plane. Before starting the calculations we recall some basic facts of vectors of the euclidean plane.<sup>1</sup> In particular, we introduce additional vectors denoted by  $^{\perp}$  in Figure 6. For more details on basics in linear algebra see [3], for example.

<sup>1</sup>In the formulation of the facts we always assume the two vectors to be non-vanishing.

Figure 6: The vector formulation



- Consider the two plane vectors  $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  and  $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$  drawing an angle  $\phi = \angle(\vec{v}, \vec{w})$ .<sup>2</sup> The angle  $\phi$  between  $\vec{v}$  and  $\vec{w}$  obeys

$$\cos \phi = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \|\vec{w}\|} \quad (2)$$

with

$$\langle \vec{v}, \vec{w} \rangle = v_1 w_1 + v_2 w_2 \quad (3)$$

and

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{v_1^2 + v_2^2}. \quad (4)$$

In particular the value  $\cos \phi$  is independent of the oriented angle.

- The inner product (3) obeys the *Cauchy-Schwarz inequality*

$$|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \|\vec{w}\| \quad (5)$$

with equality if and only if the two vectors are linearly dependent.

<sup>2</sup>By  $\angle(\vec{v}, \vec{w})$  we will always mean the oriented angle  $0^\circ \leq \phi < 360^\circ$  that goes from  $\vec{v}$  in counterclockwise direction to  $\vec{w}$ . The angle between  $\vec{w}$  and  $\vec{v}$  is then  $\angle(\vec{w}, \vec{v}) = 360^\circ - \phi$  if  $\phi \neq 0^\circ$  and  $\angle(\vec{w}, \vec{v}) = 0^\circ$  if  $\phi = 0^\circ$ . If we don't want to emphasize the orientation we write e.g.  $\angle(\vec{v}, \vec{w}) \sim 45^\circ$ , i.e.  $\angle(\vec{v}, \vec{w}) = 45^\circ$  or  $315^\circ$ .

- For later purposes the following observation on three non-vanishing plane vectors  $\vec{u}, \vec{v}, \vec{w}$  with equal lengths  $r$  turns out to be useful.

On the one hand, consider  $\vec{u} + \vec{v} + \vec{w} = 0$ . This yields  $\|\vec{w}\|^2 = \|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\langle\vec{u}, \vec{v}\rangle$ , i.e.  $r^2 = 2r^2 + 2\langle\vec{u}, \vec{v}\rangle$  or  $\langle\vec{u}, \vec{v}\rangle = -\frac{1}{2}r^2$ . Interchanging the roles of the vectors yields  $\langle\vec{u}, \vec{w}\rangle = \langle\vec{v}, \vec{w}\rangle = -\frac{1}{2}r^2$ , too. On the other hand, consider  $\langle\vec{u}, \vec{v}\rangle = \langle\vec{u}, \vec{w}\rangle = \langle\vec{v}, \vec{w}\rangle = -\frac{1}{2}r^2$ . This yields  $\|\vec{u} + \vec{v} + \vec{w}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\langle\vec{u}, \vec{v}\rangle + 2\langle\vec{u}, \vec{w}\rangle + 2\langle\vec{v}, \vec{w}\rangle = 3r^2 + 2 \cdot 3 \cdot (-\frac{1}{2}r^2) = 0$ , i.e.  $\vec{u} + \vec{v} + \vec{w} = 0$ .

This shows that for all  $\|\vec{u}\| = \|\vec{v}\| = \|\vec{w}\| = r \neq 0$  we have

$$\begin{aligned} \vec{u} + \vec{v} + \vec{w} = 0 &\iff \langle\vec{u}, \vec{v}\rangle = \langle\vec{u}, \vec{w}\rangle = \langle\vec{v}, \vec{w}\rangle = -\frac{1}{2}r^2 \\ &\iff \angle(\vec{v}, \vec{w}) = \angle(\vec{u}, \vec{v}) = \angle(\vec{u}, \vec{w}) \sim 120^\circ. \end{aligned} \quad (6)$$

- For any vector  $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  there exists a unique vector  $\vec{v}^\perp = \begin{pmatrix} v_1^\perp \\ v_2^\perp \end{pmatrix}$  that satisfies

$$\langle\vec{v}, \vec{v}^\perp\rangle = 0 \quad \text{and} \quad v_1 v_2^\perp - v_2 v_1^\perp = \|\vec{v}\|^2 > 0. \quad (7)$$

This is given by  $v_1^\perp = -v_2$  and  $v_2^\perp = v_1$ , i.e.

$$\vec{v}^\perp = \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}. \quad (8)$$

The first condition in (7) fixes the line that is spanned by  $\vec{v}^\perp$  to be perpendicular to the one spanned by  $\vec{v}$ . The second condition fixes the orientation and the length of  $\vec{v}^\perp$  to  $\angle(\vec{v}, \vec{v}^\perp) = 90^\circ$  and  $\|\vec{v}^\perp\| = \|\vec{v}\|$ , respectively.

- For all vectors  $\vec{v}, \vec{w}$  and real numbers  $\alpha$  the map  $\vec{v} \mapsto \vec{v}^\perp$  obeys

$$(\vec{v}^\perp)^\perp = -\vec{v}, \quad (\alpha\vec{v} + \vec{w})^\perp = \alpha\vec{v}^\perp + \vec{w}^\perp. \quad (9)$$

In particular  $\vec{v} \mapsto \vec{v}^\perp$  is a linear map with minimal polynomial  $p(x) = x^2 + 1$ .

- For any two vectors  $\vec{v}, \vec{w}$  with  $\phi = \angle(\vec{v}, \vec{w})$  we have  $\phi' = \angle(\vec{v}, \vec{w}^\perp) = \phi + 90^\circ$ . Furthermore,

$$\langle\vec{v}^\perp, \vec{w}^\perp\rangle = \langle\vec{v}, \vec{w}\rangle, \quad (10)$$

$$\langle\vec{v}^\perp, \vec{w}\rangle = -\langle\vec{w}^\perp, \vec{v}\rangle, \quad (11)$$

$$\sin \phi = -\cos \phi' = -\frac{\langle\vec{v}, \vec{w}^\perp\rangle}{\|\vec{v}\|\|\vec{w}\|} = \frac{\langle\vec{v}^\perp, \vec{w}\rangle}{\|\vec{v}\|\|\vec{w}\|}. \quad (12)$$

- Due to the fact that in our situation  $\vec{v}$  and  $\vec{w}$  are linearly independent  $\vec{v}^\perp$  and  $\vec{w}^\perp$  can be expressed as linear combinations of  $\vec{v}$  and  $\vec{w}$ . The approach  $\vec{v}^\perp = \alpha\vec{v} + \beta\vec{w}$  yields

$$\langle\vec{v}^\perp, \vec{w}^\perp\rangle = \alpha\langle\vec{v}, \vec{w}^\perp\rangle \implies \alpha = \frac{\langle\vec{v}, \vec{w}^\perp\rangle}{\langle\vec{w}^\perp, \vec{v}\rangle}, \quad (13)$$



$$\langle \vec{v}^\perp, \vec{v}^\perp \rangle = \beta \langle \vec{w}, \vec{v}^\perp \rangle \implies \beta = \frac{\|\vec{v}\|^2}{\langle \vec{v}^\perp, \vec{w} \rangle}, \quad (14)$$

such that

$$\vec{v}^\perp = \frac{\langle \vec{v}, \vec{w} \rangle}{\langle \vec{w}^\perp, \vec{v} \rangle} \vec{v} - \frac{\|\vec{v}\|^2}{\langle \vec{w}^\perp, \vec{v} \rangle} \vec{w}, \quad (15)$$

and, analogously,

$$\vec{w}^\perp = \frac{\langle \vec{v}, \vec{w} \rangle}{\langle \vec{v}^\perp, \vec{w} \rangle} \vec{w} - \frac{\|\vec{w}\|^2}{\langle \vec{v}^\perp, \vec{w} \rangle} \vec{v}. \quad (16)$$

Let us turn to our situation from Figure 6 and include the additional perpendicular vectors  $\vec{a}^\perp$  and  $\vec{b}^\perp$  into our discussion. Then we can describe the position vectors  $\vec{p}$  and  $\vec{q}$  of  $P$  and  $Q$  as follows: We note that the lengths  $h_P$  and  $h_Q$  of the two heights of the equilateral triangles are given by

$$h_P = \frac{\sqrt{3}}{2}a \quad \text{and} \quad h_Q = \frac{\sqrt{3}}{2}b. \quad (17)$$

This yields

$$\vec{p} = \frac{1}{2}\vec{a} - \frac{\sqrt{3}}{2}\vec{a}^\perp = \frac{\langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}\langle \vec{a}, \vec{b} \rangle}{2\langle \vec{b}^\perp, \vec{a} \rangle} \vec{a} + \frac{\sqrt{3}a^2}{2\langle \vec{b}^\perp, \vec{a} \rangle} \vec{b}, \quad (18)$$

$$\vec{q} = \frac{1}{2}\vec{b} + \frac{\sqrt{3}}{2}\vec{b}^\perp = \frac{\langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}\langle \vec{a}, \vec{b} \rangle}{2\langle \vec{b}^\perp, \vec{a} \rangle} \vec{b} + \frac{\sqrt{3}b^2}{2\langle \vec{b}^\perp, \vec{a} \rangle} \vec{a}. \quad (19)$$

Therefore, the lines  $\ell_{AP}$  and  $\ell_{BQ}$  that contain the segments  $\overline{AP}$  and  $\overline{BQ}$  are given by

$$\ell_{AP}(\tau) = \vec{b} + \tau(\vec{p} - \vec{b}) = \left(1 - \tau + \frac{\sqrt{3}a^2}{2\langle \vec{b}^\perp, \vec{a} \rangle} \tau\right) \vec{b} + \frac{\langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}\langle \vec{a}, \vec{b} \rangle}{2\langle \vec{b}^\perp, \vec{a} \rangle} \tau \vec{a}, \quad (20)$$

and

$$\ell_{BQ}(\sigma) = \vec{a} + \sigma(\vec{q} - \vec{a}) = \left(1 - \sigma + \frac{\sqrt{3}b^2}{2\langle \vec{b}^\perp, \vec{a} \rangle} \sigma\right) \vec{a} + \frac{\langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}\langle \vec{a}, \vec{b} \rangle}{2\langle \vec{b}^\perp, \vec{a} \rangle} \sigma \vec{b}, \quad (21)$$

with real parameters  $\tau$  and  $\sigma$ .

The intersection point of  $\ell_{AP}$  and  $\ell_{BQ}$  is determined by the solution  $(\tau_0, \sigma_0)$  of the equation  $\ell_{AP}(\tau) = \ell_{BQ}(\sigma)$ . We have

$$\begin{aligned} \ell_{AP}(\tau) &= \ell_{BQ}(\sigma) \\ \iff \left( \frac{\langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}\langle \vec{a}, \vec{b} \rangle}{2\langle \vec{b}^\perp, \vec{a} \rangle} \tau - \left(1 - \sigma + \frac{\sqrt{3}b^2}{2\langle \vec{b}^\perp, \vec{a} \rangle} \sigma\right) \right) \vec{a} \end{aligned}$$

$$\begin{aligned}
&= \left( \frac{\langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}\langle \vec{a}, \vec{b} \rangle}{2\langle \vec{b}^\perp, \vec{a} \rangle} \sigma - \left( 1 - \tau + \frac{\sqrt{3}a^2}{2\langle \vec{b}^\perp, \vec{a} \rangle} \tau \right) \vec{b} \right) \\
&\iff \left( (\langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}\langle \vec{a}, \vec{b} \rangle) \tau - 2\langle \vec{b}^\perp, \vec{a} \rangle + (2\langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}b^2) \sigma \right) \vec{a} \\
&= \left( (\langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}\langle \vec{a}, \vec{b} \rangle) \sigma - 2\langle \vec{b}^\perp, \vec{a} \rangle + (2\langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}a^2) \tau \right) \vec{b} \\
&\iff \begin{cases} (\langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}\langle \vec{a}, \vec{b} \rangle) \tau + (2\langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}b^2) \sigma = 2\langle \vec{b}^\perp, \vec{a} \rangle \\ (2\langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}a^2) \tau + (\langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}\langle \vec{a}, \vec{b} \rangle) \sigma = 2\langle \vec{b}^\perp, \vec{a} \rangle \end{cases} \\
&\iff \underbrace{\begin{pmatrix} \langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}\langle \vec{a}, \vec{b} \rangle & 2\langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}b^2 \\ 2\langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}a^2 & \langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}\langle \vec{a}, \vec{b} \rangle \end{pmatrix}}_{= \Omega} \begin{pmatrix} \tau \\ \sigma \end{pmatrix} = \langle \vec{b}^\perp, \vec{a} \rangle \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\end{aligned}$$

The determinant of the coefficient matrix is

$$\begin{aligned}
\det \Omega &= (\langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}\langle \vec{a}, \vec{b} \rangle)(\langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}\langle \vec{a}, \vec{b} \rangle) \\
&\quad - (2\langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}b^2)(2\langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}a^2) \\
&= -3\langle \vec{b}^\perp, \vec{a} \rangle^2 - 3 \underbrace{(a^2b^2 - \langle \vec{a}, \vec{b} \rangle^2)}_{= \langle \vec{b}^\perp, \vec{a} \rangle^2} + 2\sqrt{3}\langle \vec{b}^\perp, \vec{a} \rangle(a^2 + b^2 - \langle \vec{a}, \vec{b} \rangle) \\
&= -6\langle \vec{b}^\perp, \vec{a} \rangle^2 + 2\sqrt{3}\langle \vec{b}^\perp, \vec{a} \rangle(a^2 + b^2 - \langle \vec{a}, \vec{b} \rangle). \tag{22}
\end{aligned}$$

Therefore, the solution  $(\tau_0, \sigma_0)$  is given by

$$\begin{aligned}
\begin{pmatrix} \tau_0 \\ \sigma_0 \end{pmatrix} &= \frac{2\langle \vec{b}^\perp, \vec{a} \rangle}{\det \Omega} \begin{pmatrix} \langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}\langle \vec{a}, \vec{b} \rangle & -2\langle \vec{b}^\perp, \vec{a} \rangle + \sqrt{3}b^2 \\ -2\langle \vec{b}^\perp, \vec{a} \rangle + \sqrt{3}a^2 & \langle \vec{b}^\perp, \vec{a} \rangle - \sqrt{3}\langle \vec{a}, \vec{b} \rangle \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{\sqrt{3}(b^2 - \langle \vec{a}, \vec{b} \rangle) - \langle \vec{b}^\perp, \vec{a} \rangle}{\sqrt{3}(a^2 + b^2 - \langle \vec{a}, \vec{b} \rangle) - 3\langle \vec{b}^\perp, \vec{a} \rangle} \\ \frac{\sqrt{3}(a^2 - \langle \vec{a}, \vec{b} \rangle) - \langle \vec{b}^\perp, \vec{a} \rangle}{\sqrt{3}(a^2 + b^2 - \langle \vec{a}, \vec{b} \rangle) - 3\langle \vec{b}^\perp, \vec{a} \rangle} \end{pmatrix}. \tag{23}
\end{aligned}$$

We introduce the angle  $\phi = \angle(\vec{a}, \vec{b})$  and recall  $\langle \vec{a}, \vec{b} \rangle = ab \cos \phi$  and  $\langle \vec{b}^\perp, \vec{a} \rangle = -ab \sin \phi$ . Due to  $\sin 60^\circ = \frac{\sqrt{3}}{2}$  and  $\cos 60^\circ = \frac{1}{2}$  we get

$$\begin{pmatrix} \tau_0 \\ \sigma_0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}b^2 - 2ab(\sin 60^\circ \cos \phi - \cos 60^\circ \sin \phi)}{\sqrt{3}(a^2 + b^2) - 2\sqrt{3}ab(\cos 60^\circ \cos \phi - \sin 60^\circ \sin \phi)} \\ \frac{\sqrt{3}a^2 - 2ab(\sin 60^\circ \cos \phi - \cos 60^\circ \sin \phi)}{\sqrt{3}(a^2 + b^2) - 2\sqrt{3}ab(\cos 60^\circ \cos \phi - \sin 60^\circ \sin \phi)} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{3}b^2 + 2ab \sin(\phi - 60^\circ)}{\sqrt{3}(a^2 + b^2) - 2\sqrt{3}ab \cos(\phi + 60^\circ)} \\ \frac{\sqrt{3}a^2 + 2ab \sin(\phi - 60^\circ)}{\sqrt{3}(a^2 + b^2) - 2\sqrt{3}ab \cos(\phi + 60^\circ)} \end{pmatrix} \quad (24)$$

We use this result to calculate the position vector  $\vec{m}$  of the intersection point  $M$  in the situation of Figure 6.

$$\vec{m} = \ell_{AP}(\tau_0) = \ell_{BQ}(\sigma_0) = (1 - \tau_0)\vec{b} + \tau_0\vec{p} = (1 - \sigma_0)\vec{a} + \sigma_0\vec{q}. \quad (25)$$

We use

$$\langle \vec{b}, \vec{p} \rangle = \frac{1}{2} \langle \vec{b}, \vec{a} \rangle - \frac{\sqrt{3}}{2} \langle \vec{b}, \vec{a}^\perp \rangle = ab \cos(\phi + 60^\circ), \quad (26)$$

$$\langle \vec{a}, \vec{q} \rangle = \frac{1}{2} \langle \vec{a}, \vec{b} \rangle + \frac{\sqrt{3}}{2} \langle \vec{a}, \vec{b}^\perp \rangle = ab \cos(\phi + 60^\circ), \quad (27)$$

such that in terms of  $\tau_0$

$$\begin{aligned} (c')^2 &= |CM|^2 = \|\vec{m}\|^2 = (1 - \tau_0)^2 b^2 + \tau_0^2 \|\vec{p}\|^2 + 2\tau_0(1 - \tau_0) \langle \vec{b}, \vec{p} \rangle \\ &= (1 - \tau_0)^2 b^2 + \tau_0^2 a^2 + 2\tau_0(1 - \tau_0) ab \cos(\phi + 60^\circ) \end{aligned} \quad (28)$$

or in terms of  $\sigma_0$

$$\begin{aligned} (c')^2 &= (1 - \sigma_0)^2 a^2 + \sigma_0^2 \|\vec{q}\|^2 + 2\sigma_0(1 - \sigma_0) \langle \vec{a}, \vec{q} \rangle \\ &= (1 - \sigma_0)^2 a^2 + \sigma_0^2 b^2 + 2\sigma_0(1 - \sigma_0) ab \cos(\phi + 60^\circ). \end{aligned} \quad (29)$$

Furthermore, we get

$$\begin{aligned} (a')^2 &= |AM|^2 = \|\vec{b} - \vec{m}\|^2 = \tau_0^2 \|\vec{b} - \vec{p}\|^2 = \tau_0^2 (a^2 + \|\vec{p}\|^2 - 2\langle \vec{b}, \vec{p} \rangle) \\ &= \tau_0^2 (a^2 + b^2 - 2ab \cos(\phi + 60^\circ)), \end{aligned} \quad (30)$$

and

$$\begin{aligned} (b')^2 &= |BM|^2 = \|\vec{a} - \vec{m}\|^2 = \sigma_0^2 \|\vec{a} - \vec{q}\|^2 = \sigma_0^2 (a^2 + \|\vec{q}\|^2 - 2\langle \vec{a}, \vec{q} \rangle) \\ &= \sigma_0^2 (a^2 + b^2 - 2ab \cos(\phi + 60^\circ)). \end{aligned} \quad (31)$$

### 3 The symmetrization of the result

Starting from (28)-(31), we will provide the solution of Problem 3 in this Section, see Proposition 5, below. To get formulas for  $a'$ ,  $b'$ , and  $c'$  that are symmetric with respect to  $a$ ,  $b$ , and  $c$  we recall the cosine-theorem that states

$$2ab \cos \phi = a^2 + b^2 - c^2. \quad (32)$$

In the same way the expression  $2ab \sin \phi$  enters into the discussion. Therefore, we give a formula in terms of  $a, b, c$  alone. We have

$$\begin{aligned} 4a^2b^2 \sin^2 \phi &= 4a^2b^2(1 - \cos^2 \phi) \\ &= 4a^2b^2 - (a^2 + b^2 - c^2)^2 \\ &= -(a^4 + c^4 + b^4 - 2a^2c^2 - 2a^2b^2 - 2b^2c^2) \\ &= (c + a + b)(a + c - b)(b + c - a)(a + b - c). \end{aligned}$$

We see that this result is totally invariant under relabeling the three edges of the triangle and we will use the abbreviation

$$\Theta^2 = \sqrt{(c + a + b)(a + c - b)(b + c - a)(a + b - c)}. \quad (33)$$

**Remark 4.** Formula (33) yields the famous Heron formula that gives the area of an triangle in terms of its three edges. In fact, the quantity  $a \sin \phi$  gives the length of the height of the triangle spanned by  $a, b$  and  $\phi$  when  $b$  is the hypotenuse. Then the area is given by

$$\text{Area}(\triangle(ABC)) = \frac{1}{2}ba \sin \phi = \frac{1}{4}\Theta^2.$$

One important ingredient in the calculation of  $a', b'$ , and  $c'$  is the denominator of  $\tau_0$  and  $\sigma_0$ , that is up to a factor of 3 the denominator of  $a', b'$ , and  $c'$ . It can be rewritten in the following way.

$$\begin{aligned} a^2 + b^2 - 2ab \cos(\phi + 60^\circ) &= a^2 + b^2 - ab \cos(\phi) + \sqrt{3}ab \sin \phi \\ &= a^2 + b^2 - \frac{1}{2}(a^2 + b^2 - c^2) + \frac{\sqrt{3}}{2}\Theta^2 \\ &= \frac{1}{2}(a^2 + b^2 + c^2) + \frac{\sqrt{3}}{2}\Theta^2. \end{aligned}$$

Next we rewrite the numerator of  $a'$  that is given by the numerator of  $\tau_0^2$ . It differs from that of  $b'$  or  $\sigma_0^2$  only by interchanging  $a$  and  $b$ . We have

$$\begin{aligned} \left(\sqrt{3}b^2 + 2ab \sin(\phi - 60^\circ)\right)^2 &= \left(\sqrt{3}b^2 + ab \sin(\phi) - \sqrt{3}ab \cos \phi\right)^2 \\ &= \left(\sqrt{3}b^2 + \frac{1}{2}\Theta^2 - \frac{\sqrt{3}}{2}(a^2 + b^2 - c^2)\right)^2 \\ &= \frac{1}{4}\left(\sqrt{3}(b^2 + c^2 - a^2) + \Theta^2\right)^2. \end{aligned}$$

We insert these expressions into (30) and (31) and get expressions for  $a'$  and  $b'$ . A similar but more lengthy calculation yields the remaining length  $c'$  from (28). The results of the calculations from Section 2 and their symmetric reformulations are collected in the following Proposition.

**Proposition 5.** Given a triangle  $\triangle(ABC)$  and the point  $M$  as given in Figure 4. Then  $a', b'$ , and  $c'$  are given in terms of  $a, b$ , and  $c$  by

$$(a')^2 = \frac{1}{6} \cdot \frac{(\sqrt{3}(b^2 + c^2 - a^2) + \Theta^2)^2}{a^2 + b^2 + c^2 + \sqrt{3}\Theta^2} \quad (34)$$

$$(b')^2 = \frac{1}{6} \cdot \frac{(\sqrt{3}(a^2 + c^2 - b^2) + \Theta^2)^2}{a^2 + b^2 + c^2 + \sqrt{3}\Theta^2} \quad (35)$$

and

$$(c')^2 = \frac{1}{6} \cdot \frac{(\sqrt{3}(a^2 + b^2 - c^2) + \Theta^2)^2}{a^2 + b^2 + c^2 + \sqrt{3}\Theta^2} \quad (36)$$

**Example 6.** As a first example and also a first check of our result we consider  $a = b = c$ , i.e. an equilateral triangle. In this case we have  $\Theta^2 = \sqrt{3}a^2$  and  $a' = b' = c' = \frac{1}{\sqrt{3}}a$  which is exactly the result from (1). In particular  $\vec{m} = \frac{1}{3}(\vec{a} + \vec{b})$  is the position vector of the geometric center of the triangle.

**Remark 7.** We could go a step further and do some more calculations.

- We can first show that  $M$  lies on the line that connects the origin and  $R$ , i.e. there exists  $\lambda_0$  such that  $\vec{m} = \ell_{CR}(\lambda_0) = \lambda_0 \vec{r} = \frac{\lambda_0}{2} (\vec{a} + \vec{b} + \sqrt{3}(\vec{a}^\perp - \vec{b}^\perp))$ .
- Then we can show that the angles  $\angle(\vec{p} - \vec{a}, \vec{q} - \vec{b})$ ,  $\angle(\vec{m}, \vec{p} - \vec{a})$ , and  $\angle(\vec{m}, \vec{q} - \vec{b})$  coincide and, therefore, are given by  $120^\circ$ .

That would complete a proof of Proposition 2 that also provides explicit values for the involved terms.

#### 4 A generalization: the non-balanced star circuit

We consider a 3-phase AC star circuit with non-balanced star point, i.e. the line between the null potential  $N$  of the generator and the star point  $M$  of the circuit is missing, see Figure 7. The generator provides three voltages  $U_{10}, U_{20}, U_{30}$  of equal amplitude, say  $\frac{1}{\sqrt{3}}\hat{U}$ , and a phase difference of  $\psi = 120^\circ$ . Then  $U_1, U_2, U_3$  have an amplitude of  $\hat{U}$  and a phase difference of  $120^\circ$ , too.

The non-balanced configuration typically yields  $\hat{U}'_1 \neq \hat{U}'_2 \neq \hat{U}'_3$  and is reflected in the phasor diagram in such a way that the star point is displaced in the equilateral triangle defined by  $U_1, U_2, U_3$ , see Figure 8. Such non-balanced star circuits have been considered in [13], but for special almost symmetric configurations only, e.g.  $\hat{U}'_1 = \hat{U}'_2$ .

We now ask a question that is similar to Problem 1 we answered before: Known the phase-to-phase voltages we want to recover the primed voltages. Of course, the phase to

Figure 7: The non-balanced star circuit

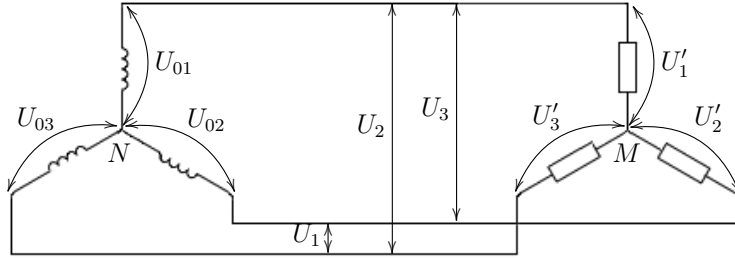
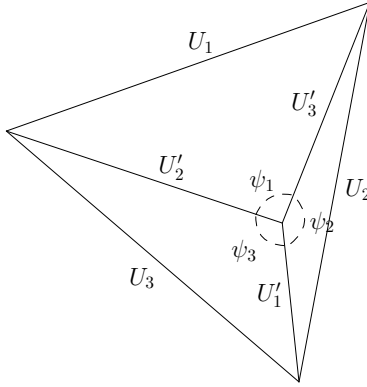


Figure 8: The phasor diagram of the non-balanced star circuit



phase voltages alone do not contain enough information to obtain a solution. But also in the case of Problem 1 we had more information: We knew the amplitudes and we knew about the phase differences of the primed voltages. Therefore, the question is as follows:

**Problem 8.** Given the phase-to-phase-voltages  $U_1, U_2, U_3$  with  $\hat{U}_1 = \hat{U}_2 = \hat{U}_3 = \hat{U}$  and phase difference  $120^\circ$  as well as the phase-differences  $\psi_1, \psi_2, \psi_3$  of the primed line voltages<sup>1</sup>: What are the values of  $\hat{U}'_1, \hat{U}'_2$ , and  $\hat{U}'_3$ ?

We will not address this question but consider the completely non-symmetric situation that covers Problem 1 and Problem 8. The geometric version can be formulated as follows.

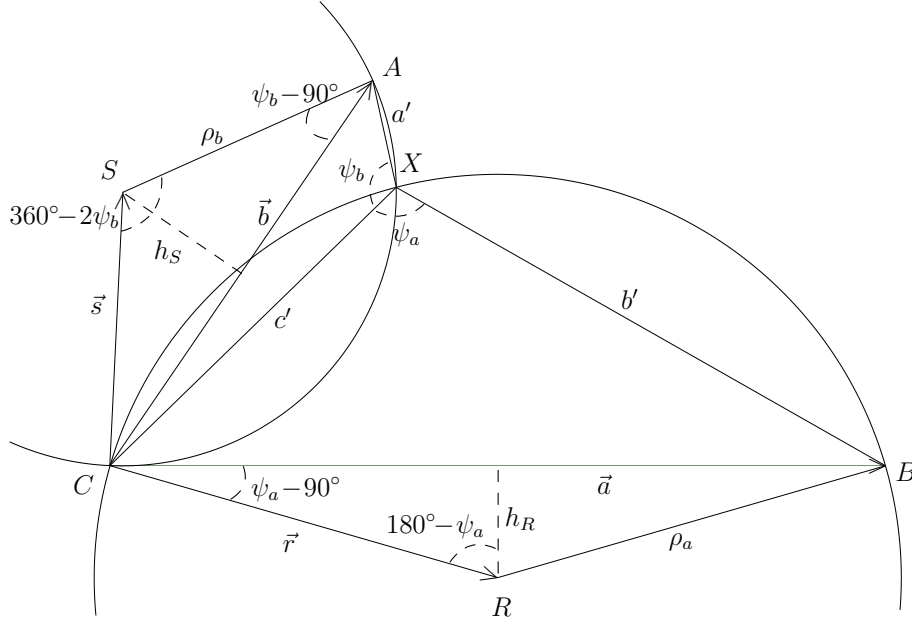
**Problem 9.** Given a plane triangle  $\triangle(ABC)$  which lengths  $a, b, c$  of its edges. Furthermore, given a point  $X$  in the interior of  $\triangle(ABC)$  and the angles  $\psi_a, \psi_b$ : What are the lengths of the connecting edges  $a' = |AX|, b' = |BX|$ , and  $c' = |CX|$ ?

So we consider the general situation from Figure 8 and translate it to Figure 9. We added a few more objects that we describe next:

<sup>1</sup>We note that the angle  $\psi_3$  is fixed by  $\psi_3 = 360^\circ - \psi_1 - \psi_2$ .

- Due to the inscribed angle Theorem 11 below, all points  $X$  that draw an angle  $\psi_b$  with the endpoints of the segment  $\overline{AC}$  lie on a circle with center  $S$ . Suppose  $\psi_b \geq 90^\circ$  then  $S$  lies on the opposite side of  $\overline{AC}$  than  $X$  and the central angle is given by  $2 \cdot (180^\circ - \psi_b) = 360^\circ - 2\psi_b$ .
- If the angle  $\psi_a$  obeys the restriction  $\psi_a \geq 90^\circ$ , too, the point  $X$  is the intersection of the two circles with centers  $R$  and  $S$  that contain the two chords  $\overline{AC}$  and  $\overline{BC}$ , respectively.
- The restriction on the two angles  $\psi_a, \psi_b$  before is actually no restriction, because due to  $\psi_a, \psi_b, \psi_c < 180^\circ$  at least two of the three angles  $\psi_a, \psi_b$ , and  $\psi_c = 360^\circ - \psi_a - \psi_b$  are of this form. Therefore, Figure 9 describes the general situation, at least after renaming the points and edges of the triangle.

Figure 9: The general geometric Situation



First we collect some facts on the geometric quantities given in Figure 9:

- $\rho_a = \frac{a}{2 \cos(\psi_a - 90^\circ)} = \frac{a}{2 \sin \psi_a} = \frac{a}{2} \sqrt{1 + \cot^2 \psi_a}$  and  $\rho_b = \frac{b}{2 \sin \psi_b} = \frac{b}{2} \sqrt{1 + \cot^2 \psi_b}$ .
- $h_R = \frac{a}{2} \tan(\psi_a - 90^\circ) = -\frac{a}{2} \cot \psi_a$  and  $h_S = -\frac{b}{2} \cot \psi_b$ .

$$\bullet \quad \vec{r} = \frac{1}{2}\vec{a} - \frac{h_R}{a}\vec{a}^\perp = \frac{1}{2}\vec{a} + \frac{1}{2}\cot\psi_a\vec{a}^\perp \quad \text{and} \quad \vec{s} = \frac{1}{2}\vec{b} - \frac{1}{2}\cot\psi_b\vec{b}^\perp.$$

We will calculate the position vector  $\vec{x}$  of  $X$  whose length is given by  $c'$ . For this, we write  $\vec{x}$  as a linear combination of the two vectors that span the triangle:

$$\vec{x} = \frac{1}{2}\alpha\vec{a} + \frac{1}{2}\beta\vec{b}.$$

$X$  is given as an intersection point of the two circles

$$\left\{ \vec{y} \mid \|\vec{y} - \vec{r}\|^2 = \rho_a^2 \right\}, \quad \left\{ \vec{y} \mid \|\vec{y} - \vec{s}\|^2 = \rho_b^2 \right\}$$

such that the coefficients of  $\vec{x}$  obey

$$\begin{aligned} & \begin{cases} \|(\alpha - 1)\vec{a} - \cot\psi_a\vec{a}^\perp + \beta\vec{b}\|^2 = a^2(1 + \cot^2\psi_a) \\ \|(\beta - 1)\vec{b} + \cot\psi_b\vec{b}^\perp + \alpha\vec{a}\|^2 = b^2(1 + \cot^2\psi_b) \end{cases} \\ \Leftrightarrow & \begin{cases} 0 = \alpha^2 a^2 + \beta^2 b^2 + 2\alpha\beta\langle\vec{a}, \vec{b}\rangle - 2\alpha a^2 - 2\beta\langle\vec{a}, \vec{b}\rangle - 2\beta\cot\psi_a\langle\vec{a}^\perp, \vec{b}\rangle \\ 0 = \alpha^2 a^2 + \beta^2 b^2 + 2\alpha\beta\langle\vec{a}, \vec{b}\rangle - 2\beta b^2 - 2\alpha\langle\vec{a}, \vec{b}\rangle - 2\alpha\cot\psi_b\langle\vec{a}^\perp, \vec{b}\rangle \end{cases} \end{aligned}$$

We subtract the two equations and get

$$\alpha\left(a^2 - \langle\vec{a}, \vec{b}\rangle - \cot\psi_b\langle\vec{a}^\perp, \vec{b}\rangle\right) - \beta\left(b^2 - \langle\vec{a}, \vec{b}\rangle - \cot\psi_a\langle\vec{a}^\perp, \vec{b}\rangle\right) = 0.$$

We write this as  $\beta = t\alpha$  with  $t = \frac{a^2 - \langle\vec{a}, \vec{b}\rangle - \cot\psi_b\langle\vec{a}^\perp, \vec{b}\rangle}{b^2 - \langle\vec{a}, \vec{b}\rangle - \cot\psi_a\langle\vec{a}^\perp, \vec{b}\rangle}$ . We introduce the length of the third edge of the triangle,  $c = \|\vec{a} - \vec{b}\|$ , and use  $2\langle\vec{a}, \vec{b}\rangle = a^2 + b^2 - c^2$  as well as  $2\langle\vec{a}^\perp, \vec{b}\rangle = \Theta^2$ , see (32) and (33), and write

$$t = \frac{c^2 + a^2 - b^2 - \cot\psi_b\Theta^2}{c^2 + b^2 - a^2 - \cot\psi_a\Theta^2}, \quad t_* = \frac{c^2 + b^2 - a^2 - \cot\psi_a\Theta^2}{c^2 + a^2 - b^2 - \cot\psi_b\Theta^2}. \quad (37)$$

We insert this into the quadratic equations and get for  $\alpha, \beta \neq 0$

$$\begin{aligned} \alpha &= \frac{2a^2 + t(a^2 + b^2 - c^2) + t\cot(\psi_a)\Theta^2}{a^2 + t^2b^2 + t(a^2 + b^2 - c^2)}, \\ \beta &= \frac{2b^2 + t_*(a^2 + b^2 - c^2) + t_*\cot(\psi_b)\Theta^2}{a^2 + t_*^2(a, b, c)b^2 + t_*(a^2 + b^2 - c^2)}. \end{aligned} \quad (38)$$

The length

$$(c')^2 = \|\vec{x}\|^2 = \frac{1}{4}\left(\alpha^2 a^2 + \beta^2 b^2 + 2\alpha\beta\langle\vec{a}, \vec{b}\rangle\right) = \frac{1}{4}\left(\alpha^2 a^2 + \beta^2 b^2 + \alpha\beta(a^2 + b^2 - c^2)\right)$$

is now obtained by a lengthy calculation. In particular, we use

$$1 - \cot\psi_a\cot\psi_b = -\cot(\psi_a + \psi_b)(\cot\psi_a + \cot\psi_b) = \cot\psi_c(\cot\psi_a + \cot\psi_b).$$

The result is formulated in the next Proposition.



**Proposition 10.** We consider the situation from Problem 9. Then the length  $c'$  of the connecting edge is given by

$$(c')^2 = \frac{\frac{1}{4} \left( (1 - \cot \psi_a \cot \psi_b) \Theta^2 - (\cot \psi_a + \cot \psi_b)(a^2 + b^2 - c^2) \right)^2}{(c^2 + a^2 \cot^2 \psi_a + b^2 \cot^2 \psi_b) + \cot \psi_a \cot \psi_b (a^2 + b^2 - c^2) - (\cot \psi_a + \cot \psi_b) \Theta^2} \quad (39)$$

$$= \frac{\frac{1}{4} (\cot \psi_a + \cot \psi_b)^2 \left( (a^2 + b^2 - c^2) - \Theta^2 \cot \psi_c \right)^2}{a^2(1 + \cot^2 \psi_a) + b^2(1 + \cot^2 \psi_b) - (\cot \psi_a + \cot \psi_b) \left( (a^2 + b^2 - c^2) \cot \psi_c + \Theta^2 \right)}. \quad (40)$$

By interchanging the roles of  $a, b$ , and  $c$  we get the analogous results for  $a'$  and  $b'$ , see the summarizing Section 6. To end up this section we will check our result by discussing some special examples:

- The equilateral triangle:  $a = b = c$  yields  $\Theta^2 = \sqrt{3}a^2$  and

$$\begin{aligned} c' &= \frac{a}{2} \cdot \frac{(1 - \cot \psi_a \cot \psi_b) \sqrt{3} - (\cot \psi_a + \cot \psi_b)}{\sqrt{1 + \cot^2 \psi_a + \cot^2 \psi_b + \cot \psi_a \cot \psi_b - \sqrt{3}(\cot \psi_a + \cot \psi_b)}} \\ &= \frac{a}{2} \cdot \frac{(\cot \psi_a + \cot \psi_b)(\sqrt{3} \cot \psi_c - 1)}{\sqrt{2 + \cot^2 \psi_a + \cot^2 \psi_b - (\cot \psi_a + \cot \psi_b)(\cot \psi_c + \sqrt{3})}}. \end{aligned}$$

This solves Problem 8.

- The situation of our initial Problem 1:  $\psi_a = \psi_b = 120^\circ$ ,  $\cot \psi_a = -\frac{\sqrt{3}}{3}$

$$(c')^2 = \frac{\frac{1}{6} (\Theta^2 + \sqrt{3}(a^2 + b^2 - c^2))^2}{c^2 + a^2 + b^2 + \sqrt{3}\Theta^2}$$

which is exactly the result we obtained in (36).

- The isosceles triangle:  $a = b$ ,  $\psi_a = \psi_b$ ,  $\Theta^2 = c\sqrt{4a^2 - c^2}$ . This is the case mainly discussed in [13]. In this case the formulas for  $a'$  and  $b'$  analog to (39) coincide such that  $a' = b'$ . Moreover, we have

$$c' = \frac{c}{2} \cdot \frac{(1 - \cot^2 \psi_a) \Theta^2 - 2 \cot \psi_a (2a^2 - c^2)}{c^2 - \Theta^2 \cot \psi_a}.$$

The limiting situation  $c' = 0$  is obtained if and only if  $\cot^2 \psi_a + \frac{2(2a^2 - c^2)}{\Theta^2} \cot \psi_a - 1 = 0$ . For the angle to be  $180^\circ > \psi_a \geq 90^\circ$  we have  $\cot \psi_a \leq 0$  such that we may exclude the positive solution of this quadratic equation. The remaining negative solution is

$$\cot \psi_a = -\frac{\Theta^2}{c^2} = -\frac{\sqrt{4a^2 - c^2}}{c} = -\frac{h_c}{c/2}.$$

where  $h_c$  is length of  $\triangle(ABC)$  over its edge  $c$ . If we denote by  $\phi$  half the angle of  $\triangle(ABC)$  at  $C$  then  $\cot \phi = \frac{h_c}{c/2}$  such that  $\psi_a = 180^\circ - \phi$ . As expected, we see that in the limiting case  $\psi_a$  coincides with the angle between the lines extending  $a$  and  $h_c$ . Moreover, again as expected,  $\psi_c = 2\phi$  and  $a' = b' = a$ .

## 5 Two further aspects of the initial problem

In this section we will present some further interesting facts on the point  $M$  that is related to our initial Problem 1 and that is the topic of Proposition 2. This special point  $M$  is called *Fermat-Point* of the triangle  $\triangle(ABC)$ .

Beyond that, we will also recall the formulation of one important tool that entered in the discussion of Problem 9, namely Proposition 11.

### 5.1 The proof for the existence of the Fermat point

In this first section we will present a very elementary proof of Proposition 2. The only things that are used are basic geometric ideas such as congruences of triangles and equality of certain angles. This is the proof presented by Evangelista Torricelli, see [4]. It is also published briefly in the English Wikipedia<sup>1</sup> and mentioned in [5].

We recall the main tool for the proof: the inscribed angle Theorem.

**Proposition 11** (Inscribed Angle Theorem). 1. Let  $K$  be a circle with five distinguished points  $A_1, A_2, B_1, B'_1, B_2$  and center  $O$ . Two of the points, say  $A_1, A_2$  build a chord of the circle and the two points  $B_1, B'_1$  lie on the same side of the chord as  $O$ , whereas  $B_2$  lays on the opposite side. For the angles in Figure 10 we have

$$\beta_1 = \beta'_1 = 180^\circ - \beta_2 = \frac{1}{2}\rho.$$

2. On the other hand, given a convex quadrilateral of which the two opposite angles add up to  $180^\circ$  then there exists a circle that contains all four vertices.

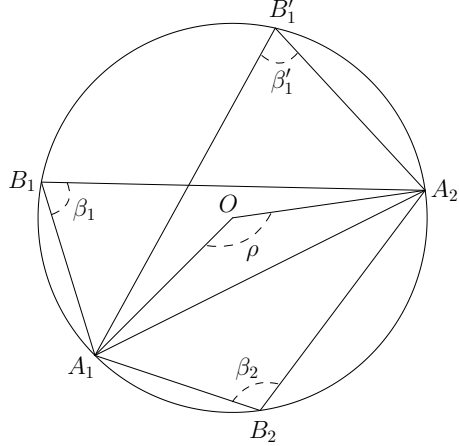
For a proof of the inscribed angle Theorem see [6], for example.

**Remark 12.** A special case of the inscribed angle Theorem is given when the chord is a diameter of the circle. Then all inscribed angles are right angles, which is the content of the famous *Thales' Theorem*.

*Proof.* (Proposition 2). We consider the situation from Figure 11 and make use of the notation introduced there. The first observation we make is that the red and blues triangles  $\triangle(APC)$  and  $\triangle(QBC)$  are congruent: the one is obtained from the other by

<sup>1</sup>[http://en.wikipedia.org/wiki/Fermat\\_point](http://en.wikipedia.org/wiki/Fermat_point).

Figure 10: The interior angle theorem



a rotation by  $60^\circ$  about the center  $C$ . Therefore, the angles denoted by  $\alpha$  at  $B$  and  $P$  coincide and the same for the angles denoted by  $\alpha'$  at  $A$  and  $Q$ .

We consider the green circumcircle  $K$  of the equilateral triangle  $\triangle(BPC)$ . Then the two angles  $\alpha$  at  $P$  and  $B$  are inscribed angles for  $K$  subject to the chord  $\overline{CM}$  such that  $K$  contains  $M$ , too. Next, we consider the chord  $\overline{BC}$  of  $K$ , instead, and the inscribed angle  $\psi$  at  $M$ . It lies on the opposite side compared to  $P$  such that it is given by  $\psi = 180^\circ - 60^\circ = 120^\circ$ .

The same arguments hold for the angles  $\alpha', \psi'$  in the blue circumcircle  $K'$  of  $\triangle(CQA)$ .

Therefore, from  $\psi + \psi' + \psi'' = 360^\circ$  we get  $\psi = \psi' = \psi'' = 120^\circ$ . The construction also yields that  $M$  is the intersection of the lines  $\overline{AP}$  and  $\overline{BQ}$ .

From the second part of the inscribed angle theorem applied to  $\psi'' = 120^\circ$  and the angle  $60^\circ$  at  $R$  we get that  $M$  also lies on the black circumcircle  $K''$  of the  $\triangle(ARB)$ . Therefore, the angle  $\beta$  at  $M$  indeed coincide with the angle  $60^\circ$  at  $A$  as inscribed angles over the chord  $\overline{BR}$ . From  $\beta + \psi = 180^\circ$  we get that  $\overline{CM} \cup \overline{MR}$  is a straight line, too.

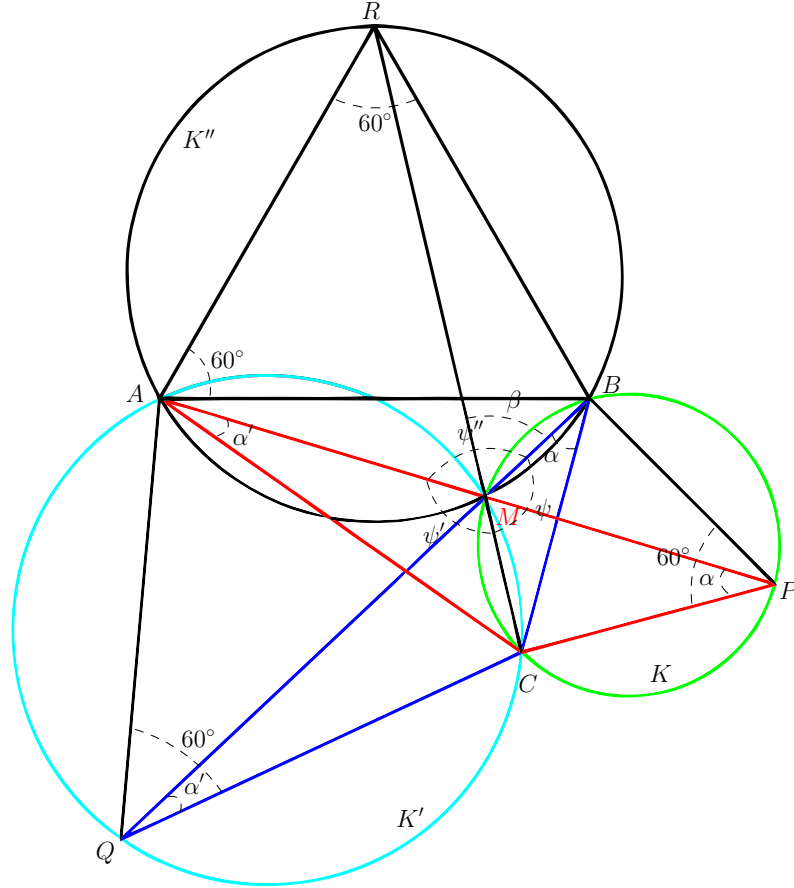
This observation completes the proof of Proposition 2.  $\square$

**Remark 13.** From part 2. of Proposition 11 we also get that  $M$  is the intersection point of all three circumcircles  $K$ ,  $K'$ , and  $K''$ .

## 5.2 The Fermat point as the solution of a minimizing problem

In this section we will present a minimizing property of the Fermat point that is not so obvious although there is a very short geometric proof. In addition we present an analytic

Figure 11: The proof of Proposition 2



description of this property that makes use of our results from Proposition 5.

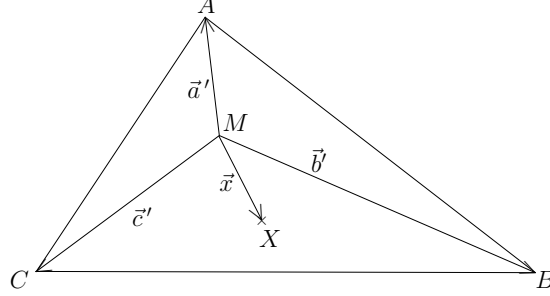
The formulation of the result is as follows:

**Proposition 14.** For the Fermat point  $M$  the sum of the distances to the vertices of the triangle  $\triangle(ABC)$  attains its minimum.

For a historical survey of the geometric treatment of this problem see [7] and the wonderful books [8, 9]. In [8, 9] and in [10] the authors also explain the mechanical content of the minimizing property that describes the Fermat point as a point of equilibrium, see also Example 6 for the special situation of an equilateral triangle.

As promised, we will first present a short, purely geometric proof. We will use the notation from Figure 12 and show that  $a' + b' + c' \leq |AX| + |BX| + |CX|$  for all points  $X$ . Indeed,

Figure 12: Minimum property of the Fermat point I



for all  $\vec{x}$  we have by the Cauchy-Schwarz inequality (5)

$$a' = \left\langle \vec{a}', \frac{\vec{a}'}{a'} \right\rangle = \left\langle \vec{a}' - \vec{x}, \frac{\vec{a}'}{a'} \right\rangle + \left\langle \vec{x}, \frac{\vec{a}'}{a'} \right\rangle \leq \|\vec{a}' - \vec{x}\| + \left\langle \vec{x}, \frac{\vec{a}'}{a'} \right\rangle$$

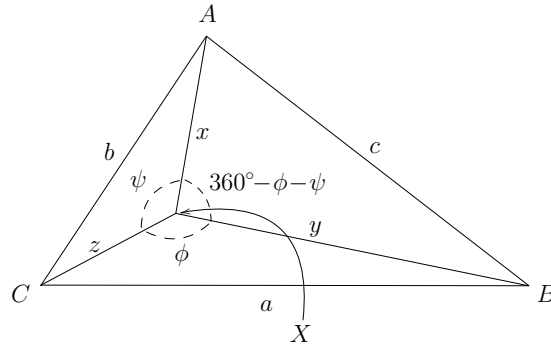
and the same for  $\vec{b}'$  and  $\vec{c}'$ . Indeed, summing up the three relations yields

$$a' + b' + c' \leq \|\vec{a}' - \vec{x}\| + \|\vec{b}' - \vec{x}\| + \|\vec{c}' - \vec{x}\| + \left\langle \vec{x}, \underbrace{\frac{\vec{a}'}{a'} + \frac{\vec{b}'}{b'} + \frac{\vec{c}'}{c'}}_{=0, \text{ see (6)}} \right\rangle$$

and, therefore, proves Proposition 14 due to  $|AX| = \|\vec{a}' - \vec{x}\|$ ,  $|BX| = \|\vec{b}' - \vec{x}\|$ , and  $|CX| = \|\vec{c}' - \vec{x}\|$ .

We will next show how the minimum property from Proposition 14 can be formulated in an analytic way. Therefore, we consider Figure 13 and consider the function  $f : \mathbb{R}^5 \rightarrow \mathbb{R}$  that is given by  $f(x, y, z, \phi, \psi) = x + y + z$ . We look for critical points of this function

Figure 13: Minimum property of the Fermat point II



subject to the three constraints<sup>1</sup>

$$\begin{aligned} g_1(x, y, z, \phi, \psi) &= y^2 + z^2 - 2yz \cos \phi - a^2 = 0, \\ g_2(x, y, z, \phi, \psi) &= x^2 + z^2 - 2xz \cos \psi - b^2 = 0, \\ g_3(x, y, z, \phi, \psi) &= x^2 + y^2 - 2xy \cos(\phi + \psi) - c^2 = 0. \end{aligned} \quad (41)$$

We introduce three Lagrange multipliers  $\lambda_1, \lambda_2, \lambda_3$  to combine  $f, g_1, g_2, g_3$  to the Lagrange function  $L = f + \lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3$ , i.e.

$$\begin{aligned} L(x, y, z, \phi, \psi, \lambda_1, \lambda_2, \lambda_3) &= x + y + z + \lambda_1(y^2 + z^2 - 2yz \cos \phi - a^2) \\ &\quad + \lambda_2(x^2 + z^2 - 2xz \cos \psi - b^2) + \lambda_3(x^2 + y^2 - 2xy \cos \eta - c^2). \end{aligned}$$

The critical points of  $f$  subject to  $g_1 = g_2 = g_3 = 0$  are given by the critical points of  $L$ , see [11], for example. Therefore, we have to solve the system

$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial z} = \frac{\partial L}{\partial \phi} = \frac{\partial L}{\partial \psi} = \frac{\partial L}{\partial \lambda_1} = \frac{\partial L}{\partial \lambda_2} = \frac{\partial L}{\partial \lambda_3} = 0 \quad (42)$$

which is

$$\begin{aligned} 0 &= 1 + 2\lambda_2(x - z \cos \psi) + 2\lambda_3(x - y \cos(\phi + \psi)) \\ 0 &= 1 + 2\lambda_1(y - z \cos \phi) + 2\lambda_3(y - x \cos(\phi + \psi)) \\ 0 &= 1 + 2\lambda_1(z - y \cos \phi) + 2\lambda_2(z - x \cos \psi) \\ 0 &= 2\lambda_1 y z \sin \phi + 2\lambda_3 x y \sin(\phi + \psi) \\ 0 &= 2\lambda_2 x z \sin \psi + 2\lambda_3 x y \sin(\phi + \psi) \\ 0 &= a^2 - y^2 - z^2 + 2yz \cos \phi \\ 0 &= b^2 - x^2 - z^2 + 2xz \cos \psi \\ 0 &= c^2 - x^2 - y^2 + 2xy \cos(\phi + \psi). \end{aligned}$$

Among these equations the last three are most hard to solve. In fact, there are many points that solve these equations. Fortunately, we did this work in our calculations in Sections 2 and 3. In fact,  $g_1 = g_2 = g_3 = 0$  are given by the cosine theorem in the triangles  $\triangle(XBC)$ ,  $\triangle(XCA)$ , and  $\triangle(XAB)$ . Now we have to convince ourself that a solution of the whole system is provided by  $(x, y, z, \phi, \psi) = (a', b', c', 120^\circ, 120^\circ)$  for suitable values of  $\lambda_1, \lambda_2, \lambda_3$ .

We plug in  $\phi = \psi = 120^\circ$  into the first five equations. This yields

$$\begin{aligned} 0 &= 1 + \lambda_2(2x + z) + \lambda_3(2x + y) \\ 0 &= 1 + \lambda_1(2y + z) + \lambda_3(2y + x) \\ 0 &= 1 + \lambda_1(2z + y) + \lambda_2(2z + x) \end{aligned}$$

<sup>1</sup>Strictly speaking, we have one additional angle variable  $\eta$  with corresponding constrained  $\eta + \phi + \psi = 360^\circ$ . This constraint has been erased and inserted into the third one of (41).

$$\begin{aligned} 0 &= \lambda_1 z - \lambda_3 x \\ 0 &= \lambda_2 z - \lambda_3 y \end{aligned}$$

such that in particular  $\lambda_1 = \lambda_3 \frac{x}{z}$  and  $\lambda_2 = \lambda_3 \frac{y}{z}$ . This reduces the first three equations to a single one, namely

$$0 = z + 2\lambda_3(xy + yz + xz),$$

such that

$$\lambda_1 = -\frac{x}{2(xy + yz + xz)}, \quad \lambda_2 = -\frac{y}{2(xy + yz + xz)}, \quad \lambda_3 = -\frac{z}{2(xy + yz + xz)}. \quad (43)$$

Thus, if we restrict to  $\phi = \psi = 120^\circ$  any choice of  $(x, y, z)$  yields unique solutions  $\lambda_1, \lambda_2, \lambda_3$  such that the first five equations of (42) are fulfilled. Therefore, our choice  $(x, y, z, \phi, \psi) = (a', b', c', 120^\circ, 120^\circ)$  is a solution of the initial system with  $\lambda_1, \lambda_2, \lambda_3$  given by (43) with  $(x, y, z)$  replaced by  $(a', b', c')$ .

We have to check if this critical value is indeed a minimum. This can be done by using the Hessian  $HL = \begin{pmatrix} \text{H}f + \lambda \text{H}g & \text{D}g \\ \text{D}g^t & \end{pmatrix}$  of the Lagrange function  $L$  evaluated at the critical point. It is given by

$$HL = \begin{pmatrix} \frac{-2(b' + c')}{\Sigma} & -\frac{c'}{\Sigma} & -\frac{b'}{\Sigma} & \frac{\sqrt{3}b'c'}{\Sigma} & 0 & 0 & 2a' + c' & 2a' + b' \\ -\frac{c'}{\Sigma} & \frac{-2(a' + c')}{\Sigma} & -\frac{a'}{\Sigma} & 0 & \frac{\sqrt{3}a'c'}{\Sigma} & 2b' + c' & 0 & 2b' + a' \\ -\frac{b'}{\Sigma} & \frac{a'}{\Sigma} & \frac{-2(a' + b')}{\Sigma} & -\frac{\sqrt{3}a'b'}{\Sigma} & -\frac{\sqrt{3}a'b'}{\Sigma} & 2c' + b' & 2c' + a' & 0 \\ \frac{\sqrt{3}b'c'}{\Sigma} & 0 & -\frac{\sqrt{3}a'b'}{\Sigma} & \frac{2a'b'c'}{\Sigma} & \frac{a'b'c'}{\Sigma} & \sqrt{3}b'c' & 0 & -\sqrt{3}a'b' \\ 0 & \frac{\sqrt{3}a'c'}{\Sigma} & -\frac{\sqrt{3}a'b'}{\Sigma} & \frac{a'b'c'}{\Sigma} & \frac{2a'b'c'}{\Sigma} & 0 & \sqrt{3}a'c' & -\sqrt{3}a'b' \\ 0 & 2b' + c' & 2c' + b' & \sqrt{3}b'c' & 0 & 0 & 0 & 0 \\ 2a' + c' & 0 & 2c' + a' & 0 & \sqrt{3}a'c' & 0 & 0 & 0 \\ 2a' + b' & 2b' + a' & 0 & -\sqrt{3}a'b' & -\sqrt{3}a'b' & 0 & 0 & 0 \end{pmatrix}$$

where we used  $\Sigma := 2(a'b' + a'c' + b'c')$ . We have to check the sign of the determinant of certain matrices associated to  $HL$ . If we denote by  $HL_{(\mu)}$  the matrix obtained from  $HL$  by erasing the first  $\mu$  rows and columns we have to consider

$$\begin{aligned} \det HL_{(0)} &= \det HL = -9a'b'c'(a' + b' + c')\Sigma^2, \\ \det HL_{(1)} &= -3a'b'c'(a'b' + a'c' + 4b'c')\Sigma^2, \\ \det HL_{(2)} &= -9a'^2b'^2c'^2\Sigma^2. \end{aligned}$$

All give the same sign, namely  $(-1)^3$  where 3 is the number of constraints. Therefore, the point under consideration is indeed a minimum. See [12] for a detailed explanation of this sufficient condition.

## 6 Conclusion: The solutions of Problems 1 and 9

- The technical formulation of the solution of the general Problem 9 is as follows: Given a non-symmetric, non-balanced star circuit according to Figure 7 with phase differences  $\psi_1, \psi_2, \psi_3 = 360^\circ - \psi_1 - \psi_2$  and phase-to-phase voltages  $U_1, U_2, U_3$  then the line voltages  $U'_1, U'_2, U'_3$  are given by<sup>1</sup>

$$\begin{aligned}
 U'_1 &= \frac{1}{2} \cdot \frac{|(\cot \psi_2 + \cot \psi_3)(U_2^2 + U_3^2 - U_1^2 - \Theta^2 \cot \psi_1)|}{\sqrt{U_2^2(1 + \cot^2 \psi_2) + U_3^2(1 + \cot^2 \psi_3) - (\cot \psi_2 + \cot \psi_3)((U_2^2 + U_3^2 - U_1^2) \cot \psi_1 + \Theta^2)}} \\
 U'_2 &= \frac{1}{2} \cdot \frac{|(\cot \psi_1 + \cot \psi_3)(U_1^2 + U_3^2 - U_2^2 - \Theta^2 \cot \psi_2)|}{\sqrt{U_1^2(1 + \cot^2 \psi_1) + U_3^2(1 + \cot^2 \psi_3) - (\cot \psi_1 + \cot \psi_3)((U_1^2 + U_3^2 - U_2^2) \cot \psi_2 + \Theta^2)}} \\
 U'_3 &= \frac{1}{2} \cdot \frac{|(\cot \psi_1 + \cot \psi_2)(U_1^2 + U_2^2 - U_3^2 - \Theta^2 \cot \psi_3)|}{\sqrt{U_1^2(1 + \cot^2 \psi_1) + U_2^2(1 + \cot^2 \psi_2) - (\cot \psi_1 + \cot \psi_2)((U_1^2 + U_2^2 - U_3^2) \cot \psi_3 + \Theta^2)}}
 \end{aligned}$$

- As a special – but more manageable – case we consider the situation from our initial Problem 1: Given a non-symmetric star circuit according to Figure 1 with phase differences  $120^\circ$  and phase-to-phase voltages  $U_1, U_2, U_3$  then the line voltages  $U'_1, U'_2, U'_3$  are given by

$$\begin{aligned}
 U'_1 &= \frac{1}{\sqrt{6}} \cdot \frac{\sqrt{3}(U_2^2 + U_3^2 - U_1^2) + \Theta^2}{\sqrt{U_1^2 + U_2^2 + U_3^2 + \sqrt{3}\Theta^2}} \\
 U'_2 &= \frac{1}{\sqrt{6}} \cdot \frac{\sqrt{3}(U_1^2 + U_3^2 - U_2^2) + \Theta^2}{\sqrt{U_1^2 + U_2^2 + U_3^2 + \sqrt{3}\Theta^2}} \\
 U'_3 &= \frac{1}{\sqrt{6}} \cdot \frac{\sqrt{3}(U_1^2 + U_2^2 - U_3^2) + \Theta^2}{\sqrt{U_1^2 + U_2^2 + U_3^2 + \sqrt{3}\Theta^2}}
 \end{aligned}$$

Here we used the the abbreviation

$$\Theta^2 = \sqrt{(U_1 + U_2 + U_3)(U_2 + U_3 - U_1)(U_1 + U_3 - U_2)(U_1 + U_2 - U_3)}$$

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<sup>1</sup>For the sake of simplicity we omit the  $\hat{\cdot}$  at the quantities.



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