

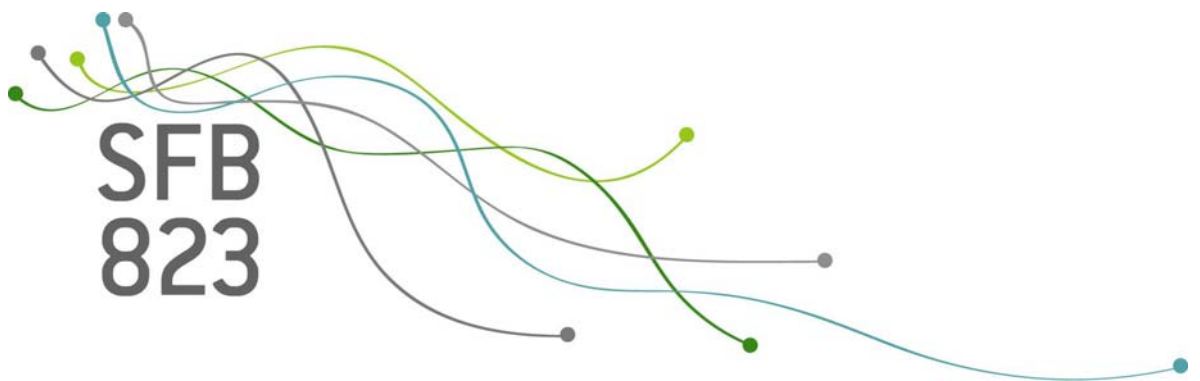
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Testing asymmetry in dependence with copula- coskewness

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Discussion Paper



TESTING ASYMMETRY IN DEPENDENCE WITH COPULA-COSKEWNESS

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ABSTRACT. A new measure of asymmetry in dependence is proposed which is based on taking the difference between the margin-free coskewness parameters of the underlying copula. The new measure and a related test are applied to both a hydrological and a financial market data sample and we show that both samples exhibit systematic asymmetric dependence.

Keywords: asymmetry, coskewness, exchangeability, copula, diversification.

JEL codes: C00, C12, C58.

1. INTRODUCTION

The past two decades have seen a steady increase in research on dependence modeling in general, and on copulas in particular. Possibly the main reason for the interest in copulas is the fact that in contrast to correlation-based models copulas allow for the modeling of different behaviours in the tail of a distribution.¹ For example, while the Gaussian copula is tail-independent and the Student's t copula is symmetrically tail dependent, other copulas like the Clayton or Gumbel–Hougaard copula are characterized by asymmetric tail dependence in the sense that the distribution's behaviour in the upper tail does not need to equal the behaviour in the lower tail. Despite the extensive work done on asymmetry in the tail dependence, however, asymmetry in the copula itself (also called exchangeability) and its importance in applied settings have so far received much less attention.

A copula is said to be asymmetric (or non-exchangeable) if $C(u, v) \neq C(v, u)$ holds for at least one pair $u, v, \in [0, 1]$, i.e., if the distribution exhibits different behaviours in the upper left and the lower right triangle of the unit square. In applications, such a case can easily occur if, for example, there exists a causal relation between the two random variables. Surprisingly, asymmetry in dependence is seldomly studied in economics and financial econometrics despite its obvious usefulness.² This is perhaps even more surprising given the fact that the asymmetry of a copula is closely connected with the coskewness of a bivariate random vector, a concept widely used in the context of asset pricing theory (see, e.g., Kraus and Litzenberger, 1976; Harvey and Siddique, 2000; Ang et al., 2006).

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¹Modeling the tail of a distribution is also relevant for various applications in both risk management and actuarial science, (see, e.g., Peng, 2008; Jong, 2012; Hua and Xia, 2014).

²This is in contrast to asymmetry in tail dependence, which has been extensively studied by, e.g., Demarta and McNeil (2004); Patton (2006); Christoffersen et al. (2012).

In this paper, we exploit the relation between the asymmetry of a copula and the coskewness of a random vector and propose a new measure of asymmetry in dependence that is based on the difference between two coskewness parameters. Consequently, our new measure is easy to interpret and allows for an analysis of the question into which direction the copula is skewed. Just like Spearman's correlation coefficient is related to Pearson's correlation coefficient, our proxy for the asymmetry of a copula is related to coskewness. In particular, no moment conditions are necessary for its definition and it is invariant with respect to strictly monotone transformations of the marginals.

In our empirical study, we illustrate the usefulness of our test for both the modeling of flooding events and losses in asset management by applying our test to two data samples from hydrology and finance. Our results, which should be of particular interest to non-life insurers, show that both samples exhibit strong asymmetry in their dependence that should be taken into account when modeling claims and losses.

The remaining part of the paper is structured as follows. In Section 2, we define our new measure of the asymmetry of a copula and discuss its properties. In Section 3, we illustrate our new measure by applying it to a hydrological and a financial market data sample. Section 4 concludes. All proofs are deferred to an appendix.

2. A NEW MEASURE OF THE ASYMMETRY OF A COPULA

Let (X, Y) be a random vector with joint cumulative distribution function (cdf) F and continuous marginal cdfs F_X and F_Y and copula C . We recall Sklar's theorem which establishes the following relation between the joint distribution's cdf and its marginal cdfs:

$$F(x, y) = C\{F_X(x), F_Y(y)\} \quad \text{for all } x, y \in \mathbb{R},$$

where C is uniquely determined and given by the cdf of the random vector (U, V) , with $U = F_X(X)$, $V = F_Y(Y)$. A copula C is called symmetric or exchangeable if the following holds true:

$$C(u, v) = C(v, u) \quad \text{for all } u, v \in [0, 1].^3$$

While many of the most common copula models allow for radial asymmetry (in particular, the lower and upper tail dependence may be different), they often do not allow for asymmetry in the copula itself.⁴ To detect such asymmetry, one could simply look at the difference of the two functions $(u, v) \mapsto C(u, v)$ and $(u, v) \mapsto C(v, u)$, which is zero for symmetric copulas, and define the following measure of asymmetry:

$$d_\infty(C) = 3 \sup_{(u,v) \in [0,1]^2} |C(u, v) - C(v, u)|.$$

In fact, any distance between the two function $(u, v) \mapsto C(u, v)$ and $(u, v) \mapsto C(v, u)$ may be used to measure asymmetry. It can be shown that $d_\infty(C)$ takes values in $[0, 1]$ (whence the constant 3), and it is equal to zero if and only if C is symmetric (see [Nelsen, 2007](#)). Replacing C by the empirical copula allows to define a test that may detect

³This is not to be confused with the related concept of radial symmetry, $C(u, v) = \bar{C}(u, v)$, where \bar{C} denotes the survival copula associated with C .

⁴An extensive overview of methods how to construct asymmetric copulas from given symmetric ones can be found in [Liebscher \(2008\)](#).

departures from symmetry, which was proposed by [Genest et al. \(2011\)](#) with a recent application in [Siburg et al. \(2016\)](#).

Obviously, a positive coefficient $d_\infty(C)$ is hard to interpret when one is interested in the direction of the asymmetry. However, in many applications it is important to know for which points (u, v) or for which regions of the unit square the symmetry relation does not hold, and in which direction it points. Hence, we propose a simpler coefficient that allows for an easy interpretation and relates to the concept of coskewness: *co-coskewness* (copula-coskewness). Our co-coskewness measure is related to coskewness in a similar way the Spearman correlation is related to Pearson correlation. In particular, no moment conditions are necessary for its definition and it is invariant with respect to strictly monotone transformations of the marginals.⁵

We recall the definition of the two coskewness parameters (see, e.g., [Miller, 2012](#)):

$$\bar{s}_{X,Y} = \frac{\mathbb{E}[(X - \mu_X)^2(Y - \mu_Y)]}{\sigma_X^2 \sigma_Y}, \quad \bar{s}_{Y,X} = \frac{\mathbb{E}[(X - \mu_X)(Y - \mu_Y)^2]}{\sigma_X \sigma_Y^2},$$

where μ_X, μ_Y and σ_X, σ_Y are the mean and standard deviation of X and Y , respectively. The coefficients are defined for any (X, Y) whose third moments exist.

$\bar{s}_{X,Y}$ is positive, when large values of Y tend to occur jointly with either small or large values of X and vice versa for $\bar{s}_{Y,X}$. We define the co-coskewness parameter (copula coskewness) as⁶

$$s_{X,Y} := \bar{s}_{U,V} = 12^{3/2} \times \mathbb{E}[(U - 1/2)^2(V - 1/2)]$$

$$s_{Y,X} := \bar{s}_{V,U} = 12^{3/2} \times \mathbb{E}[(U - 1/2)(V - 1/2)^2].$$

Using simple algebra yields the following result:

$$\begin{aligned} \mathbb{E}[(U - 1/2)^2(V - 1/2)] &= \int (u - \frac{1}{2})^2(v - \frac{1}{2}) dC(u, v) \\ &= \int u^2v - uv + \frac{1}{4}v - \frac{1}{2}u^2 + \frac{1}{2}u - \frac{1}{8} dC(u, v) \\ &= \int u^2v - uv dC(u, v) + \frac{1}{12}. \end{aligned}$$

Here, and throughout, integration is over $[0, 1]^2$ if not otherwise mentioned. The latter formula shows that the co-coskewness parameters are simple functionals of the copula C . Further note that if D is an arbitrary continuous cdf on $[0, 1]^2$ (not necessarily a copula), then $\int D(u, v) dC(u, v) = \int C(u, v) dD(u, v)$ (see, e.g., Lemma 1 in [Remillard, 2010](#)). Applying this twice to the last display, we also have

$$\mathbb{E}[(U - 1/2)^2(V - 1/2)] = \int (2u - 1)C(u, v) d(u, v) + \frac{1}{12}.$$

This second formula expresses the co-coskewness parameters through the copula, but this time the copula appears in the integrand (which is convenient for proofs).

⁵One disadvantage of the co-coskewness measure is that a value of zero does not necessarily imply symmetry in the copula. This, however, also holds for the common skewness parameter of a real-valued distribution. In the same way, independence of two random variables is not implied by a Spearman's rho that equals zero.

⁶Note that $\mu_U = \mu_V = 1/2$ and $\sigma_U^2 = \sigma_V^2 = 1/12$

The co-coskewness parameter $s_{U,V}$ attains values larger than zero when the distribution associated with the copula puts much of its mass to regions close to the points $(0, 1)$ or $(1, 1)$ of the unit square. Similarly, positive values of $s_{V,U}$ occur whenever much mass is concentrated near the points $(1, 0)$ or $(1, 1)$. The (possibly scaled) difference between the two parameters, $a_{X,Y}$, is hence an obvious choice for measuring the asymmetry of C :

$$a_{X,Y} \propto s_{U,V} - s_{V,U}.$$

A positive value of $a_{X,Y}$ indicates that large values of Y occurring simultaneously with small values of X is more likely than large values of X occurring simultaneously with small values of Y , and vice versa for negative values. Clearly, it would be desirable to choose the constant in front of the difference in such a way that $a_{X,Y} \in [-1, 1]$, with $a_{X,Y} = 0$ whenever the copula is symmetric. The choice of the constant is the topic of the subsequent Lemma 2.1.

After some simple algebra we obtain

$$\begin{aligned} s_{U,V} - s_{V,U} &\propto \mathbb{E}[(U - 1/2)^2(V - 1/2) - (U - 1/2)(V - 1/2)^2] \\ &= \mathbb{E}[(U - \frac{1}{2})(V - \frac{1}{2})(U - V)] \\ &= \mathbb{E}[U^2V - V^2U] \\ &= \frac{1}{3} \mathbb{E}[(V - U)^3]. \end{aligned}$$

The co-coskewness-based parameter of asymmetry $a_{X,Y}$ is hence a multiple of the third (central) moment of $V - U$. The following Lemma establishes the range of possible values of $\mathbb{E}[(V - U)^3]$.

Lemma 2.1. *Let \mathcal{C} denote the set of all copulas. Then*

$$\max \left\{ \int (v - u)^3 dC(u, v) : C \in \mathcal{C} \right\} = \frac{3^3}{4^4}.$$

The maximum is attained for the (shuffle of $\min(u, v)$) copula C_+ whose support is the union of the lines $\{(u, v) \in (0, 0.25) \times (0.75, 1) : v = 1 - u\}$ and $\{(u, v) \in (0.25, 1) \times (0, 0.75) : v = u - 0.25\}$. Figure 1 provides a picture of the support. Similarly,

$$\min \left\{ \int (v - u)^3 dC(u, v) : C \in \mathcal{C} \right\} = -\frac{3^3}{4^4}.$$

The minimum is attained for the (shuffle of $\min(u, v)$) copula C_- whose support is the union of the lines $\{(u, v) \in (0, 0.75) \times (0.25, 1) : v = u + 0.25\}$ and $\{(u, v) \in (0.75, 1) \times (0, 0.25) : v = 1 - u\}$. Figure 1 shows a picture of the support.

A proof of Lemma 2.1, relying on the theory of mass transportation problems, can be found in Appendix A. Lemma 2.1 suggests to scale the asymmetry parameter with $4^4/3^3$. Hence, from now on, let

$$a_{X,Y} := \frac{4^4}{3^3} \mathbb{E}[(V - U)^3] = \frac{256}{27} \mathbb{E}[(V - U)^3], \quad (2.1)$$

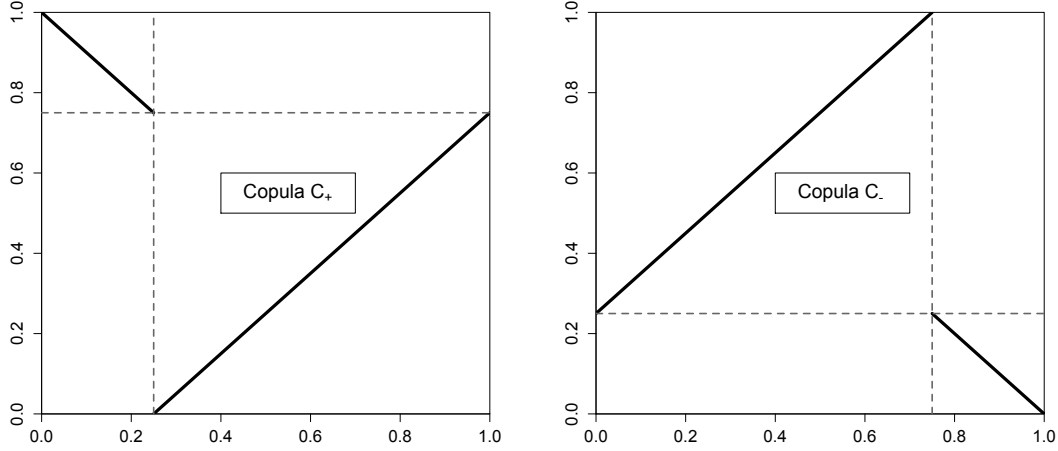


FIGURE 1. Support of the copula C_+ (left) and C_- (right). These are the maximal asymmetric elements with respect to the measure $a_{X,Y}$.

which attains values in $[-1, 1]$ and which is equal to 0 whenever the copula is symmetric. By the preceding calculations, we have:

$$\begin{aligned} a_{X,Y} &= \frac{4^4}{3^3} \cdot 3 \cdot 12^{-3/2} (s_{X,Y} - s_{Y,X}) = \frac{32}{27\sqrt{3}} (s_{X,Y} - s_{Y,X}) \\ &\approx 0.6842 (s_{X,Y} - s_{Y,X}). \end{aligned}$$

Estimation of $a_{X,Y}$ can be based on using an empirical analogue of the expression in (2.1). More precisely, we define

$$\hat{a}_{X,Y} = \frac{256}{27} \times \frac{1}{n(n+1)^3} \sum_{i=1}^n \{\text{rank}(Y_i) - \text{rank}(X_i)\}^3, \quad (2.2)$$

where $\text{rank}(X_i)$ denotes the rank of X_i among X_1, \dots, X_n ; and similarly for $\text{rank}(Y_i)$. Note that in terms of the empirical copula $\hat{C}_n(u, v) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}(\hat{U}_i \leq u, \hat{V}_i \leq v)$, where $\hat{U}_i = \text{rank}(X_i)/(n+1)$ and $\hat{V}_i = \text{rank}(Y_i)/(n+1)$, we can rewrite (2.2) in the following way:

$$\hat{a}_{X,Y} = \frac{256}{27} \times \frac{1}{n} \sum_{i=1}^n (\hat{V}_i - \hat{U}_i)^3 = \frac{256}{27} \times \int (v - u)^3 d\hat{C}_n(u, v). \quad (2.3)$$

Due to the fact that the sample of pseudo-observations is not independent over i , asymptotic normality of $\hat{a}_{X,Y}$ cannot be deduced from a simple application of the central limit theorem. In fact, the asymptotic variance is quite complicated.

Proposition 2.2. *Let $(X_1, X_2), \dots, (X_n, Y_n)$ be independent and identically distributed with continuous margins and copula C . Then, with $(U_i, V_i) = (F(X_i), G(Y_i))$, we have*

the asymptotic expansion

$$\sqrt{n}(\hat{a}_{X,Y} - a_{X,Y}) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \{h(U_i, V_i) - \mathbb{E}[h(U_i, V_i)]\} + o_{\mathbb{P}}(1),$$

where $h(u, v) = (256/27) \cdot \{g_0(u, v) - 3g_1(u) + 3g_2(v)\}$ with $g_0(u, v) = (v - u)^3$ and

$$g_1(u) = \int (x - y)^2 \mathbf{1}(u \leq x) dC(x, y) = \mathbb{E}[(U - V)^2 \mathbf{1}(U \geq u)],$$

$$g_2(v) = \int (x - y)^2 \mathbf{1}(v \leq y) dC(x, y) = \mathbb{E}[(U - V)^2 \mathbf{1}(V \geq v)],$$

and where expectation on the right-hand side is with respect to $(U, V) \sim C$. As a consequence,

$$\sqrt{n}(\hat{a}_{X,Y} - a_{X,Y}) \rightsquigarrow \mathcal{N}(0, \sigma^2),$$

where $\sigma^2 = (256/27)^2 \cdot \text{Var} \{(V - U)^3 - 3g_1(U) + 3g_2(V)\}$.

A proof of the proposition can be found in Appendix A. Note that the result does not require any smoothness assumptions on the copula at all. This is in contrast to, for instance, the test for symmetry of Genest et al. (2011) which relies on the assumption of existing continuous first order partial derivatives of the copula C . The formal test that we are going to derive below can hence be used under far broader conditions than the test of Genest et al. (2011).

Making inference (confidence bands or testing) on basis of Proposition 2.2 requires estimation of the asymptotic variance. Motivated by the asymptotic expansion, we propose to estimate σ^2 by $\hat{\sigma}^2$, defined as the empirical variance of the (observable) sample $\hat{Z}_1, \dots, \hat{Z}_n$, where

$$\hat{Z}_i = (256/27) \cdot \{g_0(\hat{U}_i, \hat{V}_i) - 3\hat{g}_1(\hat{U}_i) + 3\hat{g}_2(\hat{V}_i)\}, \quad i = 1, \dots, n,$$

and

$$\hat{g}_1(\hat{U}_i) = \frac{1}{n-1} \sum_{j=1, j \neq i}^n (\hat{U}_j - \hat{V}_j)^2 \mathbf{1}(\hat{U}_j \geq \hat{U}_i),$$

$$\hat{g}_2(\hat{V}_i) = \frac{1}{n-1} \sum_{j=1, j \neq i}^n (\hat{U}_j - \hat{V}_j)^2 \mathbf{1}(\hat{V}_j \geq \hat{V}_i).$$

It can be shown that $\hat{\sigma}^2$ is consistent for σ^2 . As a consequence, we can derive asymptotic one- or two-sided confidence bands for $a_{X,Y}$ and can formally test the hypothesis of symmetry by rejecting symmetry whenever $n\hat{a}_{X,Y}^2/\hat{\sigma}^2 > \chi_{1-\alpha}^2$, the $1 - \alpha$ -quantile of the χ^2 -distribution. The test asymptotically holds its level and is consistent against any alternative with $a_{X,Y} \neq 0$.

3. APPLICATIONS

In this section, we illustrate our new proxy for asymmetry in dependence by applying the concept of copula-coskewness in two major fields of research in which copulas have become one of the mainstays of dependence models: hydrology and finance. Both applications should yield interesting insights for non-life insurers. In our first study

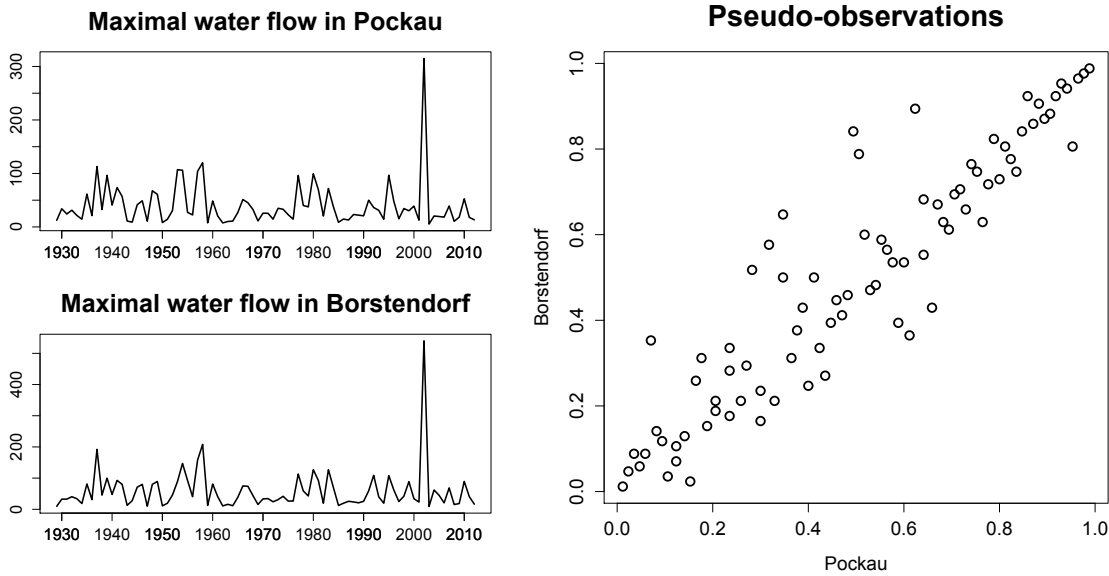


FIGURE 2. Left: Maximal summer flows in m^3/s measured at gauges in Pockau and Borstendorf, Germany. Right: Corresponding pseudo-observations.

on hydrological data, we show how asymmetric dependence structures naturally appear in river systems making our new test of asymmetric a valuable tool for modeling and forecasting floods (and consequently claims). In our second empirical study, we show that portfolios of financial assets are frequently characterized by asymmetric dependence structures. Our new test can thus help asset managers to detect these asymmetries and chose a suiting asymmetric copula model for forecasting losses.

3.1. Hydrology. Consider a simple river system consisting of a main river and a tributary. We are interested in the bivariate dependence between flood events on the main river occurring upstream (X) and downstream (Y) of the confluence of the two rivers. A flood at station X necessarily yields a flood at station Y , whereas a flood at station Y may also well be caused by a flood stemming from the tributary, with a comparably low water level at station X . As a consequence, one would expect asymmetry in the copula between the two variables: mass may be attained to subsets of the unit cube close to the point $(0, 1)$, but not to subsets close to the point $(1, 0)$, resulting in a positive value for the co-coskewness parameter $a_{X,Y}$. The strength of the asymmetry of course depends on the precise location of the gauges and the local climatic conditions.

We consider a data set which reveals that the above heuristic is in fact statistically significant. The data set consists of $n = 84$ bivariate maximal summer water flows (i.e., of flood events, measured in m^3/s) measured between 1929 and 2012 at the river Flöha in Saxony, Germany. The river station Pockau (X) lies approximately 20 km upstream of the river station Borstendorf (Y) and, in between, two smaller rivers join the river Flöha. The observations are illustrated in Figure 2. Note in particular that the marginal distributions are heavy tailed (ML-estimators of the tail index γ under a

GEV-assumption are 0.52 and 0.54 with standard errors of 0.12 and 0.14, respectively) whence regular (second and) third moments and coskewness parameters do not exist; it is really mandatory to switch to (some) standard scale for the margins. From the plot of the pseudo-observations we can in fact observe that, from the subsample of observations far from the main diagonal (say, at distance larger than 0.1), a slightly higher proportion lies above than below the diagonal.

Applying the methodology developed in this paper, the null hypothesis of symmetry gets rejected with a p-value of 0.020. The estimated parameter value is $\hat{a}_{X,Y} = 0.012$, which is positive as expected by the above heuristics. The 95%-two-sided confidence interval based on the normal approximation in Proposition 2.2 is given by $[0.002, 0.024]$. The estimated value may appear rather low on first sight, which, however, is not too surprising, given that the two joining rivers are quite small (not longer than 30km), resulting in very similar climatic conditions at the entire river system. Floods will hence quite often occur simultaneously, as can also be seen from the plot of the pseudo-observations and an estimated value of Spearman's rho of 0.92. Finally, note that the Cramér-von-Mises test for symmetry of the copula by Genest et al. (2011) also rejects the null hypothesis, with a p-value of 0.005 based on $N = 50,000$ multiplier bootstrap repetitions. The latter test, however, does not give any insight into the direction of asymmetry.

3.2. Finance. Recall the interpretation of coskewness $\bar{s}_{X,Y}$ and $\bar{s}_{Y,X}$ of two assets X and Y . Whenever $\bar{s}_{X,Y}$ has positive and high values, large values of Y tend to occur jointly with small and large values of X (which translates into more copula-mass around the points $(0,1)$ and $(1,1)$ of the unit cube). Assume for example that X and Y are daily returns of two financial assets (e.g., stocks or indices). The co-coskewness-based asymmetry parameter $a_{X,Y}$ is defined as the difference of the two co-coskewness parameters and thus, measures which joint movements of the returns are more likely. When $a_{X,Y}$ is zero, high returns of Y tend to occur jointly with higher returns of X in the same way high returns of X tend to occur jointly with lower returns of Y . When $a_{X,Y}$ is unequal to zero, we observe a diversification benefit for investors. When coskewness of the two assets X and Y is high, we expect higher returns of asset Y to be associated with small and large values of X . On the other hand, mediocre returns on asset X do not necessarily occur jointly with high returns of asset Y . In this way, Y provides an opportunity to hedge extreme (negative) returns of asset X while normal returns of X are associated with mediocre returns of Y .

This kind of asymmetry could be found in returns of different markets, e.g., a thriving oil market may induce an increase in profits for industrial companies that provide respective equipment. In contrast, however, a successful industrial firm may not affect the oil market in the same way. Further, one might find asymmetry in the dependence of returns on stocks in a sub-sector and returns on a diversified market index (e.g., financial firms' stocks as part of the S&P 500 equity index) (see also Siburg et al., 2016, for a related interpretation of asymmetry in bivariate financial data).

We apply our co-coskewness-based asymmetry measure to a variety of financial market indices covering returns/yields on equity (e.g., S&P 500 Composite, MSCI World), commodities (e.g., TOPIX Oil & Coal, Gold Bullion LBM, Raw Sugar, Cotton), and U.S. treasury bonds. In total, we compile 21 financial time series from *Thomson Reuters*

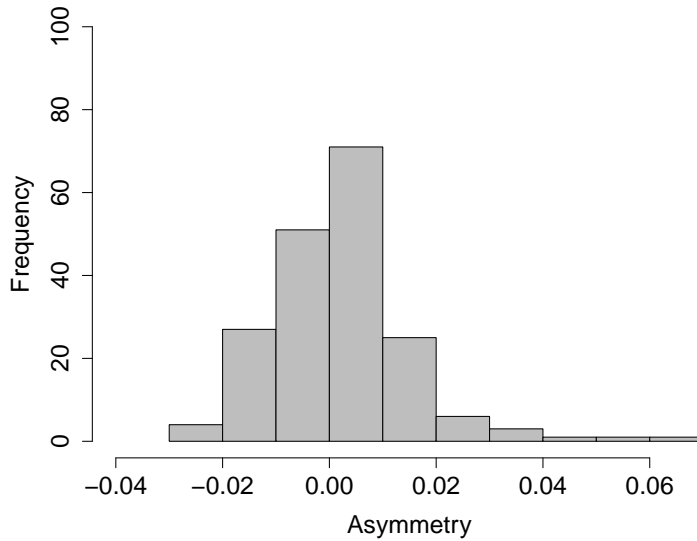


FIGURE 3. Histogram of asymmetry parameters.

Financial Datastream that cover the time period from January 4, 1990 to December 31, 2015 (6,780 observations). We employ AR(3)- and GJR-GARCH(1,1)-processes and filter all univariate time series before computing rank-transformed pseudo-observations. These are then used to estimate our asymmetry measure for all bivariate pairs of the indexes in our sample. In Figure 3, we plot a histogram of the asymmetry parameters of all pairs of indexes. We can observe that the distribution of the asymmetry measure is slightly skewed towards negative values among the full sample. However, most of the values lie between zero and 0.02, with an average of about 0.002 across all pairs.

In total, we test our null hypothesis of symmetry for all 190 bivariate combinations of the 21 indices covering several markets using the co-coskewness-based test and the asymmetry-test proposed in [Genest et al. \(2011\)](#). Using co-coskewness, for 23 out of 190 pairs the null hypothesis of symmetric dependence is rejected at the 5% level. Thus, our test suggests asymmetry in dependence for about 12.1% of all combinations in our sample. In comparison, testing exchangeability using the test of [Genest et al. \(2011\)](#) yields 26 out of 190 pairs (13.7%) for which the null hypothesis is rejected at the 5% level. In Figure 4, we plot the resulting p-values of both tests against the asymmetry measure calculated for all pairs. Both pictures reveal that most of the p-values below the 5% level correspond with a positive value of the asymmetry measure $a_{X,Y}$. This is even more pronounced for our new test. As to be expected, higher p-values for the co-coskewness-based test are associated with lower absolute values of $a_{X,Y}$, whereas the other test does not reveal an obvious pattern of test results being associated with asymmetry in dependence. Further, using the test of [Genest et al. \(2011\)](#), there are significantly more data points associated with a p-value below 10% than for our approach, which shows that our test is somewhat stricter when identifying asymmetry in dependence. Finally,

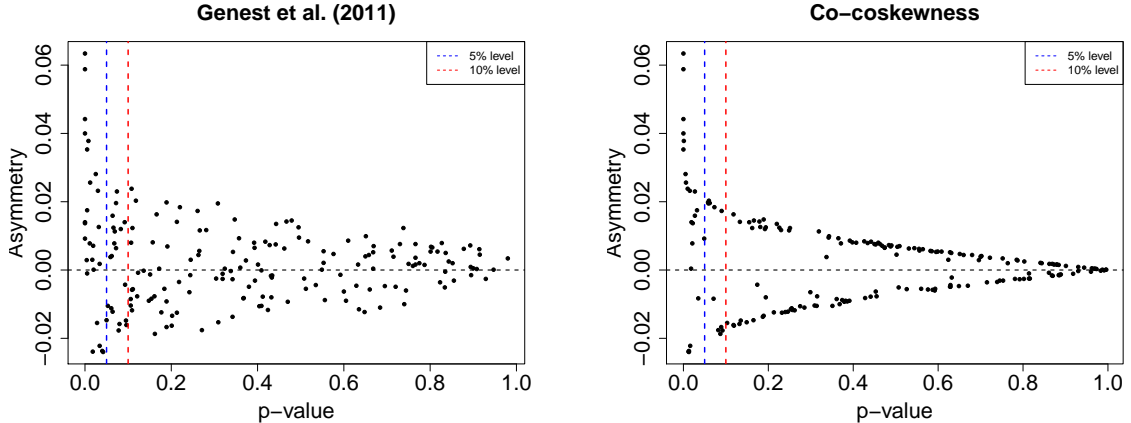


FIGURE 4. Left: Asymmetry parameters and p-values of the test of [Genest et al. \(2011\)](#) tests. Right: Asymmetry parameters and corresponding p-values of co-coskewness-based test.

unreported results on the computational time needed by both tests shows that our test based on co-coskewness is significantly faster than the alternative test of [Genest et al. \(2011\)](#) which requires about 34 times more computation time (with 1,000 multiplier iterations) than our test.

Table 1 provides an overview of the 23 pairs of assets that exhibit asymmetry in dependence at a 5% significance level. First, we notice that our measure of asymmetry is large and positive for five equity indexes paired with the LME (London Metal Exchange) index. Referring to our interpretations above, the equity indexes provide a diversification strategy for commodity markets. The combination of a broad equity index and smaller (bank) stock indexes exhibits positive and significant asymmetric dependence. This pattern is consistent across all combinations of a larger and smaller equity index in our sample.

Our measure of asymmetry in dependence is positive and statistically significant for oil or a general commodity index and an aluminum price-index. We also find some negative values of $a_{X,Y}$ when comparing U.S. treasury bonds and an index for cotton price and comparing a gold price index with oil and the treasury bond yields. These results suggest that extreme values of the gold price index can be hedged by owning oil or government bonds.

4. CONCLUSION

In this paper, we have proposed to measure asymmetry in dependence by looking at differences in the coskewness of standardized bivariate random vectors. Our proxy for a data sample's degree of asymmetry of the copula is easy to interpret, signals the direction into which the probability mass of the copula is skewed, and the related test allows for a fast testing of the null hypothesis of symmetric dependence. In our two application studies, we have shown that both hydrological and financial market data may exhibit asymmetry in the underlying copulas. Both the interpretations of asymmetry in dependence being due to a causal relation between two random variables

Co-coskewness	Index i	Index j
0.0634	S&P 500 COMPOSITE	LMEX Index
0.0588	S&P 500 BANKS	LMEX Index
0.0442	G7-DS Banks	LMEX Index
0.0400	RUSSELL 2000	LMEX Index
0.0378	G12-DS Banks	LMEX Index
0.0353	MSCI WORLD EX US	Gold Bullion LBM
0.0281	MSCI WORLD EX US	LME-Aluminium
0.0256	Crude Oil-Brent Cur. Month	LME-Aluminium
0.0238	Moody's Commodities Index	LME-Aluminium
0.0232	US T-Bill 10 Year	LME-Aluminium
0.0230	EU-DS Banks	Gold Bullion LBM
0.0175	MSCI WORLD EX US	TOPIX OIL & COAL PRDS.
0.0159	S&P 500 COMPOSITE	EU-DS Banks
0.0140	S&P 500 COMPOSITE	G12-DS Banks
0.0137	S&P 500 COMPOSITE	G7-DS Banks
0.0092	Moody's Commodities Index	Cotton
0.0078	S&P 500 COMPOSITE	RUSSELL 2000
0.0004	G12-DS Banks	G7-DS Banks
-0.0083	G7-DS Banks	MSCI WORLD EX US
-0.0222	Crude Oil-Brent Cur. Month	Cotton
-0.0237	Gold Bullion LBM	Crude Oil-Brent Cur. Month
-0.0239	US T-Bill 10 Year	Cotton
-0.0240	Gold Bullion LBM	US T-Bill 10 Year

TABLE 1. Asymmetry parameters for pairs with significant asymmetry in dependence (5% level)

X and Y as well as asymmetry signaling diversification benefits during bearish market phases underline the importance of accounting for asymmetric dependence structures in financial applications. Thus, our test should constitute a helpful tool for non-life insurers for both the modeling of insurance claims and portfolio losses.

Future research could try to investigate the systematic nature of the found asymmetric dependence structures in more detail.

APPENDIX A. PROOFS

Proof of Lemma 2.1. We only consider the claim for the maximum, the one for the minimum follows from symmetry. A simple calculation shows that $\int (v - u)^3 dC_+ = 3^3/4^4$, whence it remains to be shown that C_+ is a maximizer of the map $C \mapsto \int (v - u)^3 dC(u, v)$.

Let $B([0, 1])$ denote the set of bounded Borel-measurable functions on $[0, 1]$. Proposition 2 in Gaffke and Rüschendorf (1981) shows that the maximum in Lemma 2.1 is attained, and Corollary 3 in the same reference characterizes that maximum: C^* is a maximizer of $C \mapsto \int (v - u)^3 dC(u, v)$ if and only if there exist functions $f, g \in B([0, 1])$ such that $f(u) + g(v) \geq (v - u)^3$ and such that $f(u) + g(v) = (v - u)^3$ almost surely with respect to the measure induced by C^* .

An argumentation similar to the one in Example 1(a) of [Gaffke and Rüschemdorf \(1981\)](#) suggests to define

$$f(u) = \begin{cases} -\frac{6}{64} - \frac{1}{2}(2u-1)^3, & u < \frac{1}{4} \\ \frac{1}{64} - \frac{3}{16}u, & u \geq \frac{1}{4} \end{cases}$$

and

$$g(v) = \begin{cases} \frac{1}{64} + \frac{3}{16}v, & v \leq \frac{3}{4} \\ \frac{6}{64} + \frac{1}{2}(2v-1)^3, & v > \frac{3}{4}. \end{cases}$$

The proof of Lemma 2.1 is finished once we have shown that $f(u) + g(v) = (v-u)^3$ whenever $(u, v) \in \text{supp}(C_+)$, and that $f(u) + g(v) \geq (v-u)^3$ for all $(u, v) \in [0, 1]^2$.

First, let $(u, v) \in \text{supp}(C_+) = A_1 \cup A_2$, where $A_1 = \{(u, v) \in (0, 0.25) \times (0.75, 1) : v = 1-u\}$ and where $A_2 = \{(u, v) \in (0.25, 1) \times (0, 0.75) : v = u-0.25\}$. If $(u, v) \in A_1$, then

$$\begin{aligned} f(u) + g(v) &= f(u) + g(1-u) = -\frac{6}{64} - \frac{1}{2}(2u-1)^3 + \frac{6}{64} + \frac{1}{2}(1-2u)^3 = (1-2u)^3 \\ &= (v-u)^3. \end{aligned}$$

Similarly, if $(u, v) \in A_2$, then

$$\begin{aligned} f(u) + g(v) &= f(u) + g(u - \frac{1}{4}) = \frac{1}{64} - \frac{3}{16}u + \frac{1}{64} + \frac{3}{16}(u - \frac{1}{4}) = -\frac{1}{64} \\ &= (v-u)^3 \end{aligned}$$

It remains to be shown that $h(u, v) := f(u) + g(v) - (v-u)^3$ is nonnegative for all $(u, v) \in [0, 1]^2$. Four cases need to be distinguished, we begin by $u \geq 1/4$ and $v \leq 3/4$. Let $x = (v-u)$, a number in $[-1, 1/2]$. Then

$$h(u, v) = \frac{1}{64} - \frac{3}{16}u + \frac{1}{64} + \frac{3}{16}v - x^3 = \frac{1}{32} + \frac{3}{16}x - x^3.$$

The polynomial on the right-hand side can be easily seen to be nonnegative on $[-1, 1/2]$ (with two zeros at $x = -1/4$ and $x = 1/2$).

Now, consider $u < 1/4$ and $v > 3/4$. Let $x = 2v-1 \in (1/2, 1]$ and $y = 1-2u \in (1/2, 1]$, such that $v-u = (2v-1+1-2u)/2 = (x+y)/2$. By convexity of $t \mapsto t^3$ on the nonnegative numbers, we have

$$(v-u)^3 = \left\{ \frac{1}{2}(x+y) \right\}^3 \leq \frac{1}{2}x^3 + \frac{1}{2}y^3 = f(u) + g(v),$$

whence $h(u, v) \geq 0$.

Third, let $u < 1/4$ and $v \leq 3/4$, then

$$h(u, v) = -\frac{5}{64} - \frac{1}{2}(2u-1)^3 + \frac{3}{16}v - (v-u)^3.$$

Since $\frac{\partial}{\partial u} h(u, v) = 3\{(v-u)^2 - (1-2u)^2\} = 3(v+1-3u)(v+u-1)$ is nonpositive on $[0, 1/4] \times [0, 3/4]$, the function $u \mapsto h(u, v)$ is nonincreasing for any v , whence

$$h(u, v) \geq h\left(\frac{1}{4}, v\right) = -\frac{1}{64} + \frac{3}{16} - (v - \frac{1}{4})^3.$$

The polynomial on the right-hand side can be easily seen to be nonnegative on $[0, 3/4]$ (with two zeros at $v = 0$ and $v = 3/4$).

Finally, let $u \geq 1/4$ and $v > 3/4$, then

$$h(u, v) = \frac{7}{64} - \frac{3}{16}u + \frac{1}{2}(2v-1)^3 - (v-u)^3.$$

Since $\frac{\partial}{\partial v}h(u, v) = 3\{(2v - 1)^2 - (v - u)^2\} = 3(2v - u + 2)(3v + u - 1)$ is nonnegative on $[1/4, 1] \times [3/4, 1]$, the function $v \mapsto h(u, v)$ is nondecreasing for any v , whence

$$h(u, v) \geq h(u, \frac{3}{4}) = \frac{11}{64} - \frac{3}{16} - (\frac{3}{4} - u)^3.$$

The polynomial on the right-hand side can be easily seen to be nonnegative on $[1/4, 1]$ (with two zeros at $u = 1/4$ and $u = 1$). The proof is finished. \square

Proof of Proposition 2.2. Let $\mathbf{C}_n(u, v) = \sqrt{n}(\hat{C}_n(u, v) - C(u, v))$ denote the empirical copula process. As a consequence of (2.3), we can write $\sqrt{n}(\hat{a}_{X,Y} - a_{X,Y}) = (256/27) \cdot \int (v - u)^3 d\mathbf{C}_n(u, v)$. Random variables of the form $\int g(u, v) d\mathbf{C}_n(u, v)$ are considered in Theorem 6 in Radulovic et al. (2014). For $g(u, v) = g_0(u, v) = (v - u)^3$, the functions $T_1(g)$ and $T_2(g)$ defined in formula (21) of that reference are given by

$$T_1(g) = -3g_1, \quad T_2(g) = 3g_2.$$

It now follows from the proof of Theorem 6 in Radulovic et al. (2014) (in their notation: from the asymptotic equivalence of $\bar{\mathbf{Z}}_n$ and $\tilde{\mathbf{Z}}_n$) that

$$\int (v - u)^3 d\mathbf{C}_n(u, v) = \int \{g_0(u, v) - 3g_1(u, v) + 3g_2(u, v)\} d\alpha_n(u, v) + o_{\mathbb{P}}(1),$$

where $\alpha_n(u, v) = n^{-1/2} \sum_{i=1}^n \{\mathbf{1}(U_i \leq u, V_i \leq v) - C(u, v)\}$. The integral on the right-hand side of this display can be written as

$$n^{-1/2} \sum_{i=1}^n (27/256) \cdot \{h(U_i, V_i) - \mathbb{E}[h(U_i, V_i)]\},$$

which implies the assertion. \square

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