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Discussion Paper

Natural (non-)informative priors for skew-symmetric distributions

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Abstract

In this paper, we present an innovative method for constructing proper priors for the skewness parameter in the skew-symmetric family of distributions. The proposed method is based on assigning a prior distribution on the perturbation effect of the skewness parameter, which is quantified in terms of the Total Variation distance. We discuss strategies to translate prior beliefs about the asymmetry of the data into an informative prior distribution of this class. We show that our priors induce posterior distributions with good frequentist properties via a Monte Carlo simulation study. We also propose a scale- and location-invariant prior structure for models with unknown location and scale parameters and provide sufficient conditions for the propriety of the corresponding posterior distribution. Illustrative examples are presented using simulated and real data.

Keywords: Measure of skewness; Prior elicitation; Skew-symmetric distributions; Total variation distance; Wasserstein metric.

1 Introduction

It is a well-known fact that several data sets cannot be modeled by means of symmetric distributions, and hence even less via the normal distribution, due to skewness inherent to the data. Such data are frequently encountered in domains such as biometry, finance, materials sciences or environmetrics, to cite but these. See for instance Ley (2015) for detailed explanations.

Given these needs, there exists a plethora of distinct proposals for skew distributions in the literature; for a recent and extensive overview of the state-of-the-art, we refer the reader to the discussion paper Jones (2015). A popular class of such distributions are the *skew-symmetric densities* of the form

$$s_{f;G}(x;\mu,\sigma,\lambda) = \frac{2}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) G\left(\lambda\omega\left(\frac{x-\mu}{\sigma}\right)\right), \quad x \in \mathbb{R}, \tag{1}$$

with f the symmetric density (to be skewed), G any symmetric, univariate, absolutely continuous cumulative distribution function (cdf), and ω an increasing and odd function (Azzalini and Capitanio, 2003; Wang et al., 2004). In (1), $\mu \in \mathbb{R}$ is a location, $\sigma \in \mathbb{R}_0^+$ a scale, and $\lambda \in \mathbb{R}$ a skewness parameter. These distributions generalise the popular *skew-normal* distribution, corresponding to f and G respectively the density and cdf of the standard normal distribution and ω the identity function, which was introduced in the seminal paper Azzalini (1985). For a recent account on skew-symmetric distributions and, in particular, the skew-normal distribution, we refer the reader to the monograph Azzalini and Capitanio (2014).

Bayesian inference within these families is a challenge. The prior elicitation for λ is complicated given that this parameter controls several features of the density (1) such as asymmetry, the mode, the spread, and the tail behaviour of the density. In the skew-symmetric setting several priors of this type have been proposed by Liseo and Loperfido (2006), Cabras et al. (2012), Branco et al. (2013) and Rubio and Liseo (2014), among others. These references focus on the construction of "noninformative priors" from different viewpoints. However there are several situations where we do have *a priori* information on how the data shall behave, and hence at least we know the sign of the skewness parameter λ. For instance, when modeling BMI (body-mass index) data, we know the data will be right-skewed for biometric reasons, see e.g. Heinz et al. (2003). The same holds true for other biometric indicators and size measurements. Given the popularity of skew-symmetric distributions it is thus of paramount importance to construct informative priors for λ that reflect our *a priori* knowledge of the situation. To our knowledge, only Canale et al. (2016), who proposed the use of normal and skew-normal priors for λ, have studied informative priors. Their main motivation for using these kinds of priors is that they facilitate sampling from the corresponding posterior distribution.

In the present paper we tackle the problem of constructing priors for λ by interpreting it as a *perturbation parameter* turning the initial symmetric density f into a skew-symmetric density of the form (1). Indeed f is modified by multiplication with a "skewing function" $2G(\lambda\,\cdot)$, which is also referred to as "modulation of symmetry" (see, e.g., Azzalini and Capitanio, 2014). This perturbation effect becomes obvious when we consider $\lambda = 0$: only then do we retrieve the initial (symmetric) density f, while any non-zero value of λ induces a perturbation. Viewing λ as perturbation parameter actually reflects its very nature as foreseen by Fernando de Helguero (1880–1908), the early pioneer of skew-symmetric distributions. Quoting him "*But it may happen, and indeed this must often take place, that other perturbation causes join in [...] The curve will be abnormal, asymmetrical*"[1].

With this interpretation of λ as perturbation parameter it is appealing to invoke its perturbation capacity as a principle on which to construct prior distributions. In Section 2 we shall therefore measure this effect of λ by calculating the Total Variation distance between f and its skew-symmetric counterpart (1). Rather than putting a prior on the parameter λ, whose values are difficult to interpret, we shall put a prior on this easily interpretable distance. We opt in Section 3 to assign Beta distributions on the range of values taken by this distance. This allows us, by varying the choice of the Beta hyperparameters, to build informative as well as noninformative priors, which moreover enjoy a clear interpretability. Although our main focus in this paper is on the Total Variation distance, other distances could be used as well, and as an example we briefly discuss the Wasserstein distance in Section 4. In Section 5, we first compare the performance of our priors to existing priors by means of a Monte Carlo simulation study, and then we illustrate their usefulness by analyzing two data sets. Finally some proofs are provided in the Appendix. The present paper is complemented by an online Supplementary Material containing further details on the simulation study and a short application of our methodology to other distributions containing a shape parameter.

2 Measuring the perturbation within skew-symmetric families

There exist several distinct measures for the distance between two distributions. Those are called probability distances (or metrics, if the distance happens to be a true metric, see Gibbs and Su, 2002). Our choice in the present paper for the Total Variation metric has been driven by the fact that this distance allows precisely to measure mass relocation when passing from f to $s_{f;G}$ for a given value of the parameter λ. Moreover, contrary to other distances such as the Hellinger distance or Kullback-Leibler divergence, the Total Variation distance seems tailor-made for the problem at hand as it gives rise to simple expressions which is mostly not the case for other distances but is obviously crucial for our goal of building a prior for λ.

[1] This is a passage from de Helguero (1909) translated to English in Azzalini and Regoli (2012b).

The Total Variation distance between two probability measures $\mu(\cdot)$ and $\nu(\cdot)$ on $\mathbb{R}$ is defined as

$$d_{TV}(\mu,\nu) = \sup_{A\subset\mathbb{R}} |\mu(A) - \nu(A)|,$$

explaining why this distance represents the largest possible difference between the probability assigned to the same event by two such measures. One easily sees that $0 \leq d_{TV}(\mu,\nu) \leq 1$. If the probability measures admit Radon-Nikodym derivatives f_1 and f_2, supported on the interval $\mathbb{R}$, then the definition becomes

$$d_{TV}(f_1, f_2) = \frac{1}{2}\int_{\mathbb{R}} |f_1(x) - f_2(x)|dx.$$

Using this expression, the Total Variation distance between the baseline symmetric density f and its skew-symmetric counterpart $s_{f;G}$ from (1), for fixed $\lambda \in \mathbb{R}$, can be written as

$$d_{TV}(f, s_{f;G}|\lambda) = \frac{1}{2}\int_{\mathbb{R}} |2G(\lambda\omega(x)) - 1|f(x)dx.$$

The symmetry of G implies that $d_{TV}(f, s_{f;G}|\lambda) = d_{TV}(f, s_{f;G}|-\lambda)$, hence this distance is not a one-to-one function of the parameter λ. This suggests the meaningful measure of perturbation

$$M_{TV}(\lambda) = \text{sign}(\lambda)d_{TV}(f, s_{f;G}|\lambda), \tag{2}$$

which enjoys some appealing properties. First, for f and G fixed, $M_{TV}(0) = 0$, which corresponds to the case $s_{f;G} = f$. Since $\lambda \mapsto M_{TV}(\lambda)$ is monotone increasing (see equation (3) below), the largest difference is obtained for $\lambda \to \pm\infty$, when $s_{f;G}$ converges to the positive/negative half-f. This largest difference equals $\pm 1/2$, hence $M_{TV}(\lambda) \in (-1/2, 1/2)$. Given that we only consider the case when f and $s_{f;G}$ have the same location and scale parameters, it follows that this measure is also invariant under affine transformations. By construction, we have that $M_{TV}(\lambda) = -M_{TV}(-\lambda)$. These properties resemble the desirable conditions P.1–P.3 discussed in Arnold and Groeneveld (1995) for a measure of skewness, and indeed, given the skewness nature of the parameter λ, M_{TV} can also be considered a measure of skewness within the skew-symmetric family.

By using the symmetry properties of f and G, we can re-express (2) as

$$M_{TV}(\lambda) = \frac{1 - 2S_{f;G}(0;\lambda)}{2}, \tag{3}$$

where $S_{f;G}$ is the cdf associated with $s_{f;G}$. This expression reveals that, for a fixed choice of f and G, M_{TV} is simply a re-scaling of the difference between the mass cumulated on either side of 0 by the distribution $S_{f;G}$ (note that $1 - 2S_{f;G}(0;\lambda) = \{1 - S_{f;G}(0;\lambda)\} - S_{f;G}(0;\lambda)$). Therefore $M_{TV}(\lambda)$ measures the effect of the parameter λ in terms of the relocation of mass on either side of the symmetry center of f, as desired.

Example 1 For the skew-normal density we use the standard normal probability density function (pdf) ϕ and cdf Φ for f and G in (1), respectively, and $\omega(x) = x$, and obtain from Godoi et al. (2016) and (2) the representation

$$M_{TV}(\lambda) = \frac{\text{ArcTan}(\lambda)}{\pi} \tag{4}$$

for the perturbation measure M_{TV}. For the skew-Laplace density (obtained when f and G are the Laplace pdf and cdf, respectively, and $\omega(x) = x$) we have

$$M_{TV}(\lambda) = \frac{1}{2}\frac{\lambda}{1 + |\lambda|}.$$

Finally, let t_ν and T_ν denote the pdf and cdf of the Student t distribution with $\nu > 0$ degrees of freedom, respectively. The density of the skew-t distribution with ν degrees of freedom proposed by Azzalini and Capitanio (2003) is given by

$$\frac{2}{\sigma} t_\nu\Big(\frac{x-\mu}{\sigma}\Big) T_{\nu+1}\Big(\lambda(x-\mu)\sqrt{\tfrac{\nu+1}{\nu\sigma^2+(x-\mu)^2}}\Big), \;\; x \in \mathbb{R}.$$

This distribution is a special case of the class of densities defined in (1). In the Appendix, we show that its perturbation measure M_{TV} is given by (4) and therefore coincides with the corresponding measure for the skew-normal distribution (which is a special case of the skew-t when $\nu \to \infty$).

3 Proposed objective priors

The proposed perturbation measure $M_{TV}(\lambda)$ allows us to build informative as well as non-informative priors for the skewness parameter λ in skew-symmetric models. Recall that M_{TV} varies in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and is an injective function of λ. Consequently any probability distribution on $\left(-\frac{1}{2}, \frac{1}{2}\right)$ as prior choice for M_{TV} induces a proper prior on λ. For these distributions we choose the very versatile beta distribution with density

$$\frac{1}{B(\alpha,\beta)}\left(u+\frac{1}{2}\right)^{\alpha-1}\left(\frac{1}{2}-u\right)^{\beta-1}, \quad u \in \left(-\frac{1}{2}, \frac{1}{2}\right),$$

where $B(\alpha, \beta)$ represents the beta function and $\alpha, \beta > 0$. We refer to this class of priors as the Beta Total Variation priors $BTV(\alpha, \beta)$ with hyperparameters $\alpha, \beta > 0$. Of course, any other distribution with support $\left(-\frac{1}{2}, \frac{1}{2}\right)$ can be employed instead of the beta distribution, however, this choice facilitates some aspects of our study thanks to its flexibility and interpretability.

Our way of proceeding leads to highly tractable and easily interpretable priors. If, *a priori*, we favour right/left skewness and hence need informative priors, we just need to choose the hyperparameters α and β in such a way that the prior assigns more mass to the appropriate range of values (values of M_{TV} below 0 represent left-skewness, values above 0 represent right-skewness). For those cases where there is no reliable prior information about the asymmetry of the data, we explore the use of two types of noninformative priors, obtained for (i) $\alpha = \beta = 1$, the uniform distribution, which gives equal probability mass to any pair of subintervals of $[0, 1]$ of equal length, and (ii) $\alpha = \beta = 1/2$, corresponding to a U-shape beta density. The second choice is motivated as follows. By assigning a $Beta(\alpha, \beta)$ prior to interpretable measures of perturbation/skewness, we implicitly associate a probability p with values that produce right-skewed distributions, and a probability $1-p$ with values that produce left-skewed distributions. We can interpret this scenario as a Bernoulli trial with parameter p. A noninformative prior that has been widely studied for the parameter p of the Bernoulli distribution is the Jeffreys prior, which is precisely the $Beta(1/2, 1/2)$ prior.

In the remainder of this section, we shall first describe and investigate the resulting $BTV(\alpha, \beta)$ priors for the location-scale-free densities $2f(x)G(\lambda\omega(x))$ (Section 3.1), and then discuss joint location-scale-skewness priors for the skew-symmetric models of interest (1) (Section 3.3). A simple remark on the invariance of these sorts of priors is presented below.

Remark 1 *The $BTV(\alpha, \beta)$ priors are invariant under one-to-one transformations of λ. This implies that the BTV priors associated to a reparameterisation $\alpha = h(\lambda)$, where $h : \mathbb{R} \to D \subset \mathbb{R}$ is a diffeomorphism, can be derived from the corresponding priors on λ using a change of variable.*

3.1 Beta-TV priors

Putting a $Beta(\alpha, \beta)$ prior on $M_{TV}(\lambda)$ induces a prior on the parameter λ with pdf

$$\pi_{TV}^{\alpha,\beta}(\lambda) = \frac{1}{B(\alpha,\beta)}\left(M_{TV}(\lambda)+\frac{1}{2}\right)^{\alpha-1}\left(\frac{1}{2}-M_{TV}(\lambda)\right)^{\beta-1}\frac{d}{d\lambda}M_{TV}(\lambda). \tag{5}$$

In order to analyze the general priors $BTV(\alpha, \beta)$, we first investigate some properties of the simpler $BTV(1, 1)$ prior which reduces to $\pi_{TV}^{1,1}(\lambda) = \frac{d}{d\lambda} M_{TV}(\lambda)$. Sufficient conditions for the well-definiteness of this prior are stated in the following result.

Lemma 1 *Consider the class of skew-symmetric densities of the type* (1). *If g is a bounded pdf and $\int_{-\infty}^{0} \omega(x)f(x)dx < \infty$, the $BTV(1, 1)$ prior is well-defined for all λ and given by*

$$\pi_{TV}(\lambda) = \left| 2 \int_{-\infty}^{0} \omega(x)f(x)g(\lambda\omega(x))dx \right|. \tag{6}$$

In the following we provide some general properties of the prior (6), including a characterisation of its tails in the important case $\omega(x) = x$.

Theorem 2 *Consider the class of skew-symmetric densities of the type* (1), *where g is a bounded pdf and $\int_{-\infty}^{0} \omega(x)f(x)dx < \infty$. Then, the prior* (6) *has the following properties:*

(i) $\pi_{TV}^{1,1}(\lambda)$ *is symmetric about* $\lambda = 0$.

(ii) If g is unimodal, then $\pi_{TV}^{1,1}(\lambda)$ is decreasing in $|\lambda|$.

(iii) For $\omega(x) = x$, and under the assumptions that f is unimodal, $f(0) = M < \infty$ and $\int_{-\infty}^{0} xg(x)dx < \infty$, the tails of $\pi_{TV}^{1,1}(\lambda)$ are of order $O(|\lambda|^{-2})$.

Example 2 Using expression (6) with $\omega(x) = x$ we obtain $\pi_{TV}^{1,1}(\lambda) = \dfrac{1}{\pi\,(1 + \lambda^2)}$ as $BTV(1, 1)$ prior for the skew-normal and skew-t distributions, and $\pi_{TV}^{1,1}(\lambda) = \dfrac{1}{2\,(1 + |\lambda|)^2}$ for the skew-Laplace distribution.

Thanks to (5), any $BTV(\alpha, \beta)$ prior possesses a nice closed-form expression whenever the $BTV(1, 1)$ prior does. The following result describes the tail behaviour of the density $\pi_{TV}^{\alpha,\beta}(\lambda)$ of the $BTV(\alpha, \beta)$ prior and is a consequence of Theorem 2 and the tail behaviour of Beta-transformations of symmetric distributions, see Section 4.5 of Jones (2004).

Corollary 3 *Consider the skew-symmetric densities defined by* (1) *for $\omega(x) = x$, together with the assumptions of Theorem 2(iii). The right tail of $\pi_{TV}^{\alpha,\beta}(\lambda)$ is of order $O(|\lambda|^{-\beta-1})$, while its left tail is of order $O(|\lambda|^{-\alpha-1})$. Moreover, if $\alpha = \beta$, then $\pi_{TV}^{\alpha,\beta}(\lambda)$ is symmetric.*

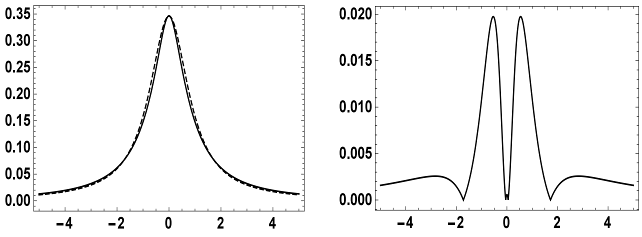
In particular, for the $BTV(1/2, 1/2)$ prior we obtain the following expression:

$$\pi_{TV}^{\frac{1}{2},\frac{1}{2}}(\lambda) = \frac{1}{\pi\sqrt{\frac{1}{4} - M_{TV}^2(\lambda)}} \pi_{TV}^{1,1}(\lambda). \tag{7}$$

This prior is symmetric and, for skew-symmetric models with $\omega(x) = x$, its tails are of order $O(|\lambda|^{-3/2})$, which interestingly coincide with those of the Jeffreys prior (Rubio and Liseo, 2014). However, the prior $\pi_{TV}^{\frac{1}{2},\frac{1}{2}}(\lambda)$ and the Jeffreys prior are not identical. In fact, the Jeffreys prior has no closed-form expression, and moreover it can be ill-defined for certain combinations of f and G due to singularities in the Fisher information matrix in the neighborhood of $\lambda = 0$, see Hallin and Ley (2012).

3.2 Heuristic approximations to the $BTV(1,1)$ priors

In general the expression (6) is not available in closed-form. However, we can appeal to the characterisation of the tail behaviour of these priors in Theorem 2 to come up with tractable approximations. For example, in the case when $\omega(x) = x$ and f and G are the logistic pdf and cdf, respectively, the $BTV(1,1)$ prior is not available in closed-form but can be reasonably well approximated with a Student-t distribution with 1 degree of freedom and scale parameter 0.92. Figure 1 shows the quality of this approximation. The quality of Student-t approximations for $BTV(1,1)$ priors associated to other skew-symmetric models seems to require a case by case analysis.



(a) (b)

Figure 1: (a) Total variation prior of λ (continuous line) and Student-t approximation (dashed line); (b) Absolute difference between the Total variation prior of λ and the Student-t approximation.

3.3 Location-scale-skewness models: partial information priors

Consider now the initial densities of interest (1), which contain unknown location and scale parameters. For this model we adopt the prior structure

$$\pi(\mu, \sigma, \lambda) = \frac{p(\lambda)}{\sigma}, \tag{8}$$

where $p(\lambda)$ is a proper prior on λ, here $BTV(\alpha, \beta)$. This prior structure can be justified as a sort of *partial information prior* (Sun and Berger, 1998) in the sense that we are using the reference prior for the location and scale parameters, $\pi(\mu, \sigma) \propto \sigma^{-1}$, while we allow for using a subjective prior on the skewness parameter λ. Such structures can also be motivated as priors inspired by the structure of the independence Jeffreys prior (Rubio and Steel, 2014, 2015). Theorem 4 below presents sufficient conditions for the propriety of the posterior distribution under the prior structure (8). We restrict our study to the cases when f belongs to scale mixtures of normals. This is a wide family of symmetric distributions which contains many models of practical interest such as the normal, logistic, Laplace, symmetric hyperbolic, Student-t, among many other distributions.

Theorem 4 *Let* $\mathbf{x} = (x_1, \dots, x_n)$ *be an i.i.d. sample from a skew-symmetric model* (1)*. Suppose that* f *is a scale mixture of normals. Then the posterior distribution of* (μ, σ, λ) *associated with the prior structure (8) is proper if* $n \geq 2$ *and if all the observations are different.*

This theorem, proved in the Appendix, guarantees that the priors proposed in the present paper for skew-symmetric densities lead almost surely to proper posterior distributions.

4 Extension of the proposed method: the Wasserstein metric

As mentioned in the Introduction, alternative distances could be used to measure the perturbation effect of the parameter λ. In this section we study exemplarily the Wasserstein metric, which is defined for two distributions F_1 and F_2 on $\mathbb{R}$, with finite first moment, by

$$d_{\mathcal{W}}(F_1, F_2) = \int_{\mathbb{R}} |F_1(x) - F_2(x)| dx,$$

see Vallender (1974). The Wasserstein distance is a minimal distance between two random variables with fixed distributions F_1 and F_2 or the minimal cost of transporting one distribution onto another. It thus measures precisely the perturbation effect turning f into $s_{f;G}$. If the probability laws associated with F_1 and F_2 are stochastically ordered (see *e.g.* Ross, 1996, Chapter 9), then either $F_1 \leq F_2$ or $F_2 \leq F_1$. Assuming the latter, we can rewrite the Wasserstein distance as

$$d_{\mathcal{W}}(F_1, F_2) = \int_{\mathbb{R}} (F_1(x) - F_2(x)) dx = \int_{\mathbb{R}} x(f_2(x) - f_1(x)) dx,$$

provided that $\lim_{x\to\pm\infty} x(F_1(x) - F_2(x)) = 0$. Azzalini and Regoli (2012a) have shown that skew-symmetric densities (1) are stochastically ordered, with $S_{f;G} \leq F$ for $\lambda > 0$ (and $S_{f;G} \geq F$ for $\lambda < 0$). Consequently, if f has finite first moment, the Wasserstein distance between f and $s_{f;G}$ becomes

$$d_{\mathcal{W}}(F, S_{f;G}|\lambda) = \begin{cases} \int_{\mathbb{R}} x(2G(\lambda\omega(x)) - 1) f(x) dx & \text{if } \lambda > 0 \\ \int_{\mathbb{R}} x(1 - 2G(\lambda\omega(x))) f(x) dx & \text{if } \lambda \leq 0, \end{cases}$$

which resembles the Total Variation distance between F and $S_{f;G}$. It is hence easy to see that it satisfies the same properties, and we can analogously define a Wasserstein-based perturbation measure by $M_{\mathcal{W}}(\lambda) = \text{sign}(\lambda) d_{\mathcal{W}}(F, S_{f;G}|\lambda)$. Note that, since $M_{\mathcal{W}}(0) = 0$, the latter expression reflects well the mass relocation measure. It also shows how tractable the Wasserstein-induced measure is.

Example 3 For the skew-normal density we obtain from Ley et al. (2016) the representation

$$M_{\mathcal{W}}(\lambda) = \sqrt{\frac{2}{\pi}} \frac{\lambda}{\sqrt{1+\lambda^2}}.$$

Similarly, straightforward calculations yield the corresponding distances for the skew-Laplace and skew-t density, which are respectively

$$M_{\mathcal{W}}(\lambda) = \frac{\lambda(2+|\lambda|)}{(1+|\lambda|)^2} \quad \text{and} \quad M_{\mathcal{W}}(\lambda) = \frac{\nu^{1/2}\Gamma((\nu-1)/2)}{\pi^{1/2}\Gamma(\nu/2)} \frac{\lambda}{\sqrt{1+\lambda^2}}.$$

We conclude this section by stating some results for Beta-W priors, which are obtained by putting a Beta-prior on the range of $M_{\mathcal{W}}$ and will be called $BW(\alpha, \beta)$-priors. The corresponding densities are denoted by $\pi_{\mathcal{W}}^{\alpha,\beta}$. The $BW(1,1)$ prior is defined as $\pi_{\mathcal{W}}^{1,1}(\lambda) \propto \frac{d}{d\lambda} M_{\mathcal{W}}(\lambda)$, the proportionality symbol being required since the length of the range of $M_{\mathcal{W}}(\lambda)$ is here not necessarily equal to 1 as was the case for M_{TV}. The proofs of the following statements are obtained by similar arguments as for the Total Variation distance and the details are omitted for the sake of brevity.

Theorem 5 *Consider the class of skew-symmetric densities of the type* (1)*, where g is a bounded pdf and $\int_{-\infty}^{0} \omega(x) x f(x) dx < \infty$. Then, the prior*

$$\pi_{\mathcal{W}}^{1,1}(\lambda) \propto \int_{-\infty}^{0} \omega(x) x f(x) g(\lambda x) dx \tag{9}$$

is well-defined and has the following properties:

(i) $\pi_W^{1,1}(\lambda)$ is symmetric about $\lambda = 0$.

(ii) If g is unimodal, then $\pi_W^{1,1}(\lambda)$ *is decreasing in* $|\lambda|$.

(iii) For $\omega(x) = x$*, and under the assumptions that f is unimodal,* $f(0) = M < \infty$ *and* $\int_{-\infty}^{0} x^2 g(x)dx < \infty$*, the tails of* $\pi_W^{1,1}(\lambda)$ *are of order* $O(|\lambda|^{-3})$.

Example 4 For the skew-normal and skew-t distribution we find $\pi_{\mathcal{W}}^{1,1}(\lambda) \propto (1 + \lambda^2)^{-3/2}$ as $BW(1,1)$ prior, which coincides with the noninformative prior $\pi_\rho(\lambda)$ proposed in Canale et al. (2016). For the skew-Laplace distribution we have $\pi_{\mathcal{W}}^{1,1}(\lambda) \propto (1 + |\lambda|)^{-3}$.

Corollary 6 *Consider the skew-symmetric densities defined by* (1) *for* $\omega(x) = x$*, together with the assumptions of Theorem 5(iii). Then the right tail of* $\pi_{\mathcal{W}}^{\alpha,\beta}(\lambda)$ *is of order* $O(|\lambda|^{-2\beta-1})$*, while the left tail is of order* $O(|\lambda|^{-2\alpha-1})$*. Moreover, if* $\alpha = \beta$*, then* $\pi_{\mathcal{W}}^{\alpha,\beta}(\lambda)$ *is symmetric.*

Interestingly, in the skew-normal and skew-t cases the $BTV(1,1)$ and $BW(1/2,1/2)$ priors are identical. However, this coincidence does not occur in other skew-symmetric models such as the skew-Laplace model.

5 Finite sample properties and practical performance

5.1 Monte Carlo simulation study

In this section, we shall conduct a Monte Carlo simulation study wherein we compare the performance of the proposed new priors to other priors from the literature. To this end, we shall first consider noninformative and then informative priors.

Noninformative priors

In order to compare the performance of the priors proposed in Section 3 with that of the Jeffreys prior (Liseo and Loperfido, 2006, Rubio and Liseo, 2014) we have conducted a thorough simulation study, of which we only present certain results here, the others being provided in the Supplementary Material. We have generated $N = 1,000$ samples of sizes $n = 50, 100$ from a skew-normal distribution with location parameter $\mu = 0$, scale parameter $\sigma = 1$, and skewness parameter $\lambda = 0, 2.5, 5$. Results for the sample size $n = 200$, as well as for the skew-logistic and skew-Laplace distributions can be found in the Supplementary Material. For each of these samples, we simulate a posterior sample of size $1,000$ from (μ, σ, λ) using the $BTV(1,1)$, $BTV(1/2,1/2)$, $BW(1,1)$ and Jeffreys priors. We employ a self-adaptive MCMC sampler (Christen and Fox, 2010) to obtain the posterior samples. For each posterior sample, we calculate the coverage proportions of the 95% credible intervals of each parameter (that is, the proportion of credible intervals that contain the true value of the parameter) as well as the 5%, 50% and 95% quantiles of the posterior medians and maximum *a posteriori* (MAP) estimators. In addition, we obtain the median of the Bayes factors (BFs) associated to the hypothesis $H_0 : \lambda = 0$. The Bayes factors are approximated using the Savage-Dickey density ratio.

The BTV and BW priors for the skew-normal model enjoy nice closed-form expressions. In order to facilitate the implementation of the Jeffreys prior, we use the corresponding Student-t approximation proposed in Bayes and Branco (2007) (1/2 degrees of freedom and scale $\pi/2$). The results are reported in Tables 1–2. Overall, we observe that the $BTV(1/2,1/2)$ and Jeffreys priors exhibit the best, and very similar, performance. However, we emphasise that the $BTV(1/2,1/2)$ prior is more tractable than the Jeffreys prior and it is well-defined under less restrictive conditions. These conclusions are further supported by the simulation studies of the Supplementary Material for the skew-Laplace and skew-logistic distributions.

Prior	MAP			Median			Coverage	BF
	5%	50%	95%	5%	50%	95%		
$\lambda = 0$								
BTV(1/2,1/2)								
μ	-1.157	-0.021	1.169	-0.820	-0.005	0.922	0.990	–
σ	0.904	1.103	1.493	0.992	1.195	1.515	0.874	–
λ	-1.597	0.011	1.453	-1.769	0.004	1.484	0.990	1.715
Jeffreys								
μ	-1.170	-0.076	1.229	-0.871	-0.018	1.008	0.983	–
σ	0.923	1.115	1.541	0.999	1.216	1.528	0.858	–
λ	-1.854	0.015	1.663	-1.876	0.017	1.589	0.986	1.824
BTV(1,1)								
μ	-1.059	0.004	1.089	-0.647	-0.007	0.731	0.997	–
σ	0.897	1.081	1.344	0.974	1.163	1.412	0.892	–
λ	-0.712	0.003	0.552	-1.158	-0.011	0.938	0.996	1.245
BW(1,1)								
μ	-0.547	-0.012	0.763	-0.414	-0.005	0.449	0.999	–
σ	0.881	1.056	1.252	0.931	1.116	1.319	0.919	–
λ	-0.243	0.010	0.250	-0.516	0.007	0.426	1.000	1.024
$\lambda = 2.5$								
BTV(1/2,1/2)								
μ	-0.281	0.039	0.921	-0.224	0.189	0.821	0.899	–
σ	0.610	0.832	1.220	0.667	0.880	1.202	0.931	–
λ	-0.273	1.033	5.290	-0.103	1.414	7.759	0.869	0.949
Jeffreys								
μ	-0.283	0.036	0.994	-0.233	0.170	0.837	0.897	–
σ	0.614	0.847	1.220	0.674	0.891	1.213	0.936	–
λ	-0.307	1.342	5.964	-0.119	1.560	8.571	0.877	0.988
BTV(1,1)								
μ	-0.225	0.093	0.891	-0.163	0.308	0.815	0.862	–
σ	0.602	0.782	1.171	0.647	0.845	1.147	0.917	–
λ	-0.163	0.415	4.094	-0.076	1.032	5.345	0.843	0.797
BW(1,1)								
μ	-0.134	0.410	0.903	-0.070	0.505	0.824	0.749	–
σ	0.581	0.733	1.063	0.617	0.786	1.062	0.834	–
λ	-0.156	0.098	2.757	-0.053	0.412	3.195	0.694	0.819
$\lambda = 5$								
BTV(1/2,1/2)								
μ	-0.174	-0.004	0.341	-0.157	0.026	0.576	0.918	–
σ	0.594	0.958	1.197	0.662	0.960	1.193	0.926	–
λ	-15.792	3.132	30.601	0.557	4.759	31.230	0.891	0.140
Jeffreys								
μ	-0.180	-0.007	0.318	-0.153	0.019	0.552	0.919	–
σ	0.595	0.958	1.200	0.666	0.963	1.199	0.925	–
λ	-7.616	3.265	38.095	0.609	4.849	32.032	0.896	0.136
BTV(1,1)								
μ	-0.141	0.028	0.623	-0.114	0.072	0.642	0.895	–
σ	0.581	0.918	1.153	0.639	0.921	1.152	0.909	–
λ	-0.010	2.921	8.071	0.344	3.595	11.755	0.874	0.164
BW(1,1)								
μ	-0.090	0.097	0.839	-0.057	0.202	0.766	0.777	–
σ	0.550	0.788	1.095	0.593	0.823	1.076	0.819	–
λ	-0.075	1.572	4.794	0.139	1.957	6.301	0.736	0.330

Table 1: skew-normal data for noninformative priors: $\mu = 0, \sigma = 1, n = 50$.

Prior	MAP			Median			Coverage	BF
	5%	50%	95%	5%	50%	95%		
$\lambda = 0$								
BTV(1/2,1/2)								
μ	-1.031	0.018	0.993	-0.797	0.009	0.775	0.983	–
σ	0.943	1.074	1.389	1.000	1.150	1.387	0.861	–
λ	-1.455	0.001	1.481	-1.244	-0.018	1.315	0.984	2.002
Jeffreys								
μ	-1.039	0.028	1.020	-0.864	0.011	0.800	0.982	–
σ	0.946	1.084	1.432	1.008	1.159	1.400	0.839	–
λ	-1.537	-0.021	1.482	-1.392	-0.026	1.401	0.982	2.204
BTV(1,1)								
μ	-0.997	-0.007	0.960	-0.708	0.015	0.683	0.992	–
σ	0.933	1.067	1.355	0.985	1.131	1.346	0.877	–
λ	-1.186	0.008	1.279	-0.992	-0.010	1.075	0.992	1.404
BW(1,1)								
μ	-0.827	0.005	0.823	-0.504	0.022	0.451	0.997	–
σ	0.922	1.049	1.201	0.968	1.101	1.265	0.903	–
λ	-0.360	0.000	0.349	-0.563	-0.008	0.624	0.997	1.087
$\lambda = 2.5$								
BTV(1/2,1/2)								
μ	-0.188	0.014	0.646	-0.170	0.058	0.635	0.880	–
σ	0.650	0.955	1.177	0.704	0.947	1.175	0.898	–
λ	-0.029	2.111	4.318	0.212	2.139	5.127	0.879	0.386
Jeffreys								
μ	-0.202	0.009	0.478	-0.174	0.052	0.638	0.890	–
σ	0.651	0.959	1.180	0.708	0.952	1.177	0.919	–
λ	0.003	2.138	4.308	0.274	2.191	5.028	0.885	0.369
BTV(1,1)								
μ	-0.176	0.031	0.739	-0.148	0.088	0.667	0.870	–
σ	0.645	0.931	1.155	0.691	0.922	1.150	0.897	–
λ	-0.069	1.955	3.867	0.151	1.950	4.504	0.859	0.338
BW(1,1)								
μ	-0.137	0.075	0.815	-0.109	0.185	0.724	0.815	–
σ	0.634	0.844	1.119	0.668	0.864	1.107	0.841	–
λ	-0.101	1.581	3.387	0.071	1.430	3.786	0.789	0.423
$\lambda = 5$								
BTV(1/2,1/2)								
μ	-0.142	-0.006	0.152	-0.120	0.004	0.185	0.922	–
σ	0.806	0.983	1.137	0.814	0.988	1.147	0.946	–
λ	1.724	4.245	10.966	2.289	4.995	14.257	0.931	0.004
Jeffreys								
μ	-0.134	-0.003	0.150	-0.123	0.004	0.184	0.925	–
σ	0.810	0.986	1.147	0.817	0.991	1.147	0.944	–
λ	1.893	4.336	10.033	2.409	5.044	15.041	0.927	0.004
BTV(1,1)								
μ	-0.117	0.011	0.177	-0.103	0.019	0.231	0.922	–
σ	0.788	0.975	1.133	0.780	0.975	1.131	0.935	–
λ	1.767	4.044	8.088	1.945	4.582	10.286	0.932	0.003
BW(1,1)								
μ	-0.094	0.033	0.229	-0.076	0.047	0.402	0.889	–
σ	0.641	0.947	1.105	0.709	0.947	1.109	0.902	–
λ	0.218	3.446	6.433	0.996	3.861	7.616	0.891	0.008

Table 2: skew-normal data for noninformative priors: $\mu = 0, \sigma = 1, n = 100$.

Informative priors

We now explore the use of the proposed informative priors. We simulate $N = 1,000$ samples of size $n = 50$ from a skew-normal distribution with parameters $\mu = 0$, $\sigma = 1$ and $\lambda = 5$. We employ again a self-adaptive MCMC sampler to obtain the posterior samples. For each of these samples, we simulate a posterior sample of size $1,000$ from (μ, σ, λ) using the $BTV(3, 1/2)$ and $BW(3, 1/2)$ priors. These priors assign 5% of the mass to values of $\lambda < 0$ while being vaguely informative about $\lambda > 0$. We also consider the skew-normal prior proposed in Canale et al. (2016) with hyperparameters $(\mu_0, \sigma_0, \lambda_0) = (0, 1, 6.5)$. This prior also assigns 5% of the mass to values of $\lambda < 0$ and is vaguely informative about $\lambda > 0$, however, it has lighter tails than the BTV and BW priors. The shape of these priors is presented in Figure 2c. We calculate the coverage proportions of the 95% credible intervals of each parameter as well as the 5%, 50% and 95% quantiles of the posterior medians and MAP estimators. Results are reported in Table 3. We observe that the $BTV(3, 1/2)$ and $BW(3, 1/2)$ priors exhibit better frequentist properties than their competitor. This, together with the intuitive nature of our priors, underlines the strength of our new approach.

Prior	MAP			Median			Coverage
	5%	50%	95%	5%	50%	95%	
$\lambda = 5$							
BTV(3,1/2)							
μ	-0.185	-0.020	0.241	-0.161	0.001	0.309	0.935
σ	0.680	0.976	1.200	0.722	0.986	1.210	0.945
λ	-22.182	3.348	40.942	1.546	5.269	36.241	0.919
BW(3,1/2)							
μ	-0.147	0.023	0.288	-0.117	0.046	0.338	0.940
σ	0.665	0.939	1.153	0.705	0.944	1.159	0.933
λ	0.927	3.233	8.794	1.349	3.960	12.739	0.919
SN(0,2.5,6.5)							
μ	-0.054	0.078	0.247	-0.045	0.093	0.311	0.890
σ	0.692	0.902	1.087	0.703	0.904	1.084	0.908
λ	1.118	2.962	4.147	1.494	3.091	4.204	0.804

Table 3: skew-normal data for informative priors: $\mu = 0, \sigma = 1, n = 50$.

5.2 The frontier data

We now analyse the frontier data set which is available from the 'sn' R package. It consists of $n = 50$ simulated observations from a skew-normal with parameters $(\mu, \sigma, \lambda) = (0, 1, 5)$. This data set is *infamous* because the related maximum likelihood estimator (MLE) of λ is infinite. This problem is described on the website http://azzalini.stat.unipd.it/SN/index.html of Adelchi Azzalini, with an open request to propose a "reasonable" estimate for λ. We believe our approach does allow a correct treatment of these data.

We calculate the median posterior estimators associated to the prior (8) with five choices for $p(\lambda)$: (i) the $BTV(1, 1)$ prior (6), (ii) the $BTV(1/2, 1/2)$ prior (7), (iii) the $BW(1, 1)$ prior (9), (iv) the Jeffreys prior and (v) the matching prior of λ (Cabras et al., 2012). For each of these models, we simulate, using an adaptive MCMC sampler, a posterior sample of size $N = 10,000$ from (μ, σ, λ) (with a burn-in period of $100,000$ iterations and a thinning period of 100 iterations; this means that we simulated a chain of total length $1,100,000$). Table 4 shows the corresponding posterior medians and the 95% highest posterior density (HPD) credible intervals. All of the HPD intervals contain the true value of the parameters. The length of the HPD intervals for μ and σ associated to the $BW(1, 1)$ prior are the largest, while the corresponding posterior median estimator of λ is the most accurate and the length of the corresponding posterior interval is the shortest. The simulation study shows that this prior does not have good frequentist properties for this sample size and therefore the closeness of the posterior median estimators is just a mere coincidence. Among the other priors, the $BTV(1,1)$ does lead to the best estimation, improving in particular on the Jeffreys prior and the matching prior.

Prior	μ	σ	λ
BTV(1,1)	-0.092 (-0.244,0.143)	1.214 (0.940,1.533)	11.621 (-1.073,151.209)
BW(1,1)	-0.033 (-0.280,0.498)	1.158 (0.799,1.490)	5.610 (-0.986,24.483)
BTV(1/2,1/2)	-0.114 (-0.242,0.037)	1.245 (1.008,1.538)	36.174 (-0.253,4440.252)
Jeffreys	-0.113 (-0.236,0.049)	1.243 (0.995,1.538)	31.526 (-0.112,3648.675)
Matching	-0.120 (-0.236,0.026)	1.258 (0.998,1.527)	40.946 (0.871,4346.058)

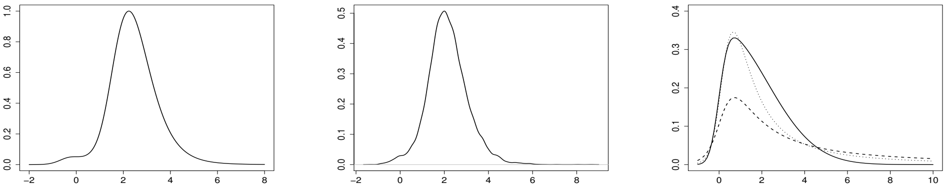
Table 4: Frontier data: posterior median and 95% HPD intervals.

5.3 Body Mass Index

In this application we analyse the Body Mass Index (BMI) of $n = 100$ female Australian athletes, available in the R package 'sn'. Biometric reasons entail that such data is typically asymmetric with a longer right tail. Consequently, we fit a skew-normal distribution to this data set together with the prior (8) and expect informative priors to yield better results. For $p(\lambda)$ we use the following priors: (i) the $BTV(1,1)$ prior (6), (ii) the $BTV(1/2,1/2)$ prior (7), (iii) the $BW(1,1)$ prior (9), (iv) the Jeffreys prior, (v) the informative $BTV(3,1/2)$ prior, (vi) the informative $BW(3,1/2)$ prior, (vii) the informative skew-normal prior of Canale et al. (2016) with hyperparameters $(\mu_0, \sigma_0, \lambda_0) = (0, 2.25, 6.5)$, and (viii) the matching prior of Cabras et al. (2012). The informative priors (v)-(vii) assign 5% of the mass to values of $\lambda < 0$ (see Figure 2). This is, we are assigning little prior probability mass to values of $\lambda < 0$, as suggested by the anthropometric theory (Heinz et al., 2003). For each of these models, we simulate, using an adaptive MCMC sampler, a posterior sample of size $N = 10,000$ from (μ, σ, λ) (with a burn-in period of $100,000$ iterations and a thinning period of 100 iterations). Table 5 shows a summary of the posterior simulations, maximum likelihood estimator of the parameters and the 95% quantile bootstrap-confidence intervals, and the Bayes factors associated to the hypothesis $H_0 : \lambda = 0$ (obtained using the Savage-Dickey density ratio for priors (i)–(vii), and a Laplace approximation for prior (viii)). The posterior inference for μ and σ is similar throughout the different Bayesian models, however, we can observe that the informative priors $BTV(3,1/2)$ and $BW(3,1/2)$ produce credible intervals for λ that do not contain the value $\lambda = 0$. This clearly shows the added value of interpretable informative priors.

Prior	μ	σ	λ	BF
Jeffreys	19.391 (18.178,22.427)	3.692 (2.570,4.667)	1.937 (-0.447,4.094)	0.473
BTV(1/2,1/2)	19.411 (18.254,22.998)	3.682 (2.537,4.605)	1.896 (-0.650,3.985)	0.433
BTV(1,1)	19.529 (18.287,23.044)	3.605 (2.513,4.568)	1.735 (-0.777,3.827)	0.406
BW(1,1)	19.796 (18.467,23.066)	3.404 (2.389,4.358)	1.413 (-0.732,3.151)	0.425
BTV(3,1/2)	19.295 (18.149,20.751)	3.768 (2.814,4.781)	2.097 (0.410,4.118)	0.278
BW(3,1/2)	19.361 (18.263,20.784)	3.707 (2.741,4.659)	1.960 (0.292,3.669)	0.175
SN	19.288 (18.275,20.556)	3.769 (2.884,4.733)	2.087 (0.668,3.845)	0.119
Matching	19.190 (18.313,20.219)	3.858 (2.991,4.735)	2.265 (0.895,4.035)	0.370
MLE	19.229 (18.445,20.876)	3.810 (2.625,4.634)	2.233 (0.597,4.248)	–

Table 5: BMI data: posterior median, 95% HPD intervals, and Bayes factors associated to $\mathcal{H}_0 : \lambda = 0$.



(a) (b) (c)

Figure 2: (a) Profile likelihood of λ; (b) Marginal posterior of λ using the informative prior $BTV(3, 1/2)$; (c) Informative priors: $BTV(3, 1/2)$ prior (continuous line), $BW(3, 1/2)$ prior (dashed line), and SN prior (dotted line).

6 Discussion

The construction of meaningful priors, either informative or noninformative, is of central importance in Bayesian inference. Prior elicitation is particularly challenging when the model parameters control several features. Such is the case of the skewness parameter in skew-symmetric distributions, where this parameter controls the mode, asymmetry, tail behaviour, and spread of the pdf. We proposed a new method for constructing priors for this parameter based on its overall effect on the shape of the density. For this purpose, we studied the perturbation effect of the skewness parameter through the Total Variation and Wasserstein distances. We showed that the priors induced by the Total Variation distance are very intuitive and hence user-friendly, have very good frequentist properties and enjoy tractable expressions, especially compared to the popular Jeffreys prior which moreover can have singularities.

The constructive strategy proposed in this paper can be extended to shape parameters in other distributions. In the Supplementary Material, we provide a brief study on the construction of priors using the Total Variation and Wasserstein distances for log-skew-symmetric distributions and two-piece distributions. It is shown that the priors for the entire family of two-piece distributions have closed-form expressions, which are linked to a family of priors proposed in Rubio and Steel (2014). Applying this new strategy of prior construction to various other families with shape parameters represents a promising research direction.

Appendix: Proofs

Proof of the representation (4)

Let $X_{\lambda,\nu}$ be a random variable following a skew-t distribution. By using the stochastic representation of the skew-t distribution as a scale mixture of skew-normal distributions (Azzalini and Capitanio, 2003) it follows that

$$S_{t_\nu,T_{\nu+1}}(0;\lambda,\nu) \quad = \quad \mathbb{P}\left(X_{\lambda,\nu} \leq 0\right) = \mathbb{P}\left(V_\nu^{-1/2} Z_\lambda \leq 0\right) = \mathbb{P}\left(Z_\lambda \leq 0\right),$$

where $V_\nu \sim \chi^2_\nu/\nu$, and Z_λ is a skew-normal random variable with location 0, unit scale, and skewness parameter λ. The result follows from this relationship together with equation (3).

Proof of Theorem 2

(i) The symmetry property is immediate from expression (6).

(ii) It is easily seen that $\pi_{TV}(\lambda) = \int_0^\infty 2\omega(u)f(u)g(\lambda\omega(u))du$. For $u > 0$ and $|\lambda_1| > |\lambda_2| > 0$, it follows that $\omega(u)f(u)g(\lambda_1\omega(u)) < \omega(u)f(u)g(\lambda_2\omega(u))$ thanks to the unimodality and symmetry of g. Thus

$$\int_0^\infty \omega(u)f(u)g(\lambda_1\omega(u))du \leq \int_0^\infty \omega(u)f(u)g(\lambda_2\omega(u))du$$

and hence the prior is decreasing in $|\lambda|$.

(iii) By using the change of variable $u = \lambda x$ and the maximality of f at 0, it follows that

$$\left|\int_{-\infty}^0 xf(x)g(\lambda x)dx\right| \leq \int_{-\infty}^0 |x|f(x)g(\lambda x)dx \leq M\int_{-\infty}^0 |x|g(\lambda x)dx = \frac{M}{\lambda^2}\int_0^\infty ug(u)du.$$

Now, let $|\lambda| \geq L > 0$. Then, the unimodality and symmetry of f yield $f\left(\frac{u}{\lambda}\right) \geq f\left(\frac{u}{L}\right)$ for $u > 0$. By using the change of variable $u = -\lambda x$ we find

$$\begin{aligned}\left|\int_{-\infty}^0 xf(x)g(\lambda x)dx\right| &= \frac{1}{\lambda^2}\int_0^\infty uf\left(-\frac{u}{\lambda}\right)g(-u)du = \frac{1}{\lambda^2}\int_0^\infty uf\left(\frac{u}{\lambda}\right)g(u)du \\ &\geq \frac{1}{\lambda^2}\int_0^\infty uf\left(\frac{u}{L}\right)g(u)du.\end{aligned}$$

The result follows by combining the previous inequalities.

Proof of Theorem 4

The proof is based on that of Theorem 3 from Rubio and Liseo (2014). Recall that a posterior distribution is proper whenever the marginal distribution $P(x_1,\dots,x_n) < \infty$ (Fernández and Steel, 1999). Now note that $s_{f;G}(x;\mu,\sigma,\lambda) \leq \frac{2}{\sigma}f\left(\frac{x-\mu}{\sigma}\right)$, which entails that

$$\begin{aligned}P(x_1,\dots,x_n) &= \int_{\mathbb{R}}\int_{\mathbb{R}_+}\int_{\mathbb{R}}\left[\prod_{j=1}^n s(x_j;\mu,\sigma,\lambda)\right]\frac{p(\lambda)}{\sigma}d\mu d\sigma d\lambda \\ &\leq \int_{\mathbb{R}_+}\int_{\mathbb{R}}\left[\prod_{j=1}^n \frac{2}{\sigma}f\left(\frac{x_j-\mu}{\sigma}\right)\right]\frac{1}{\sigma}d\mu d\sigma\int_{\mathbb{R}}p(\lambda)d\lambda.\end{aligned}$$

Given that $p(\lambda)$ is proper, it follows that the posterior distribution of (μ,σ,λ) exists whenever the posterior distribution of (μ,σ) exists for a scale mixture of normals sampling model and the prior $\pi(\mu,\sigma) \propto \sigma^{-1}$. The propriety of the latter, for $n \geq 2$ and when all the observations are different, follows by Theorem 1 of Fernández and Steel (1999).

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Supplementary Material: Extensions and Monte Carlo simulation study

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1 Extension to other distributions

We here briefly show how our method applies to other types of flexible distributions where a (skewness) parameter has a perturbation effect on the original distribution.

Log-skew-symmetric distributions

The proposed priors from our main paper have the same interpretation if they are used for the shape parameter in log-skew-symmetric distributions. Recall that a positive random variable Y is said to be distributed according to a log-skew-symmetric distribution if it is distributed according to (1) below. This sort of distributions have been used for modelling environmental, medical, biological, and financial data (see Marchenko and Genton, 2010 and the references therein). The pdf of Y is given by

$$s_l(y;\lambda) = \frac{2}{y} f(\log y) G(\lambda \log y), \quad y > 0. \tag{1}$$

It follows that the TV and Wasserstein distances between (1) and the corresponding baseline log-symmetric density $f_l(y) = \dfrac{1}{y} f(\log y)$ satisfy:

$$\begin{aligned} d_{TV}(s_l, f_l|\lambda) &= \frac{1}{2}\int_0^\infty |s_l(y;\lambda) - f_l(y)|\, dy = \frac{1}{2}\int_{-\infty}^\infty |s(x;\lambda) - f(x)| dx = d_{TV}(s, f|\lambda), \\ d_{\mathcal{W}}(S_l, F_l|\lambda) &= \int_0^\infty |S_l(y;\lambda) - F_l(y)|\, dy = \int_{-\infty}^\infty |S(x;\lambda) - F(x)| dx = d_{\mathcal{W}}(S, F|\lambda). \end{aligned}$$

Consequently, the priors proposed in Section 3 of the main paper for the skew-symmetric family coincide with those obtained for the log-skew-symmetric family. It is also clear that one could use any other increasing diffeomorphism from $\mathbb{R}_+$ to $\mathbb{R}$ instead of the logarithmic transformation.

Two-piece distributions

Consider the family of two-piece distributions with the following parameterisation (see Rubio and Steel, 2014 for a general overview):

$$s_{tp}(x;\gamma) = f\left(\frac{x}{1-\gamma}\right) I(x < 0) + f\left(\frac{x}{1+\gamma}\right) I(x \geq 0), \quad x \in \mathbb{R}, \tag{2}$$

where $\gamma \in (-1, 1)$, and f is a unimodal symmetric pdf with mode at 0. The parameter γ controls the mass cumulated on either side of the mode ($x = 0$) while preserving the tail behaviour of

f. The density (2) is asymmetric for $\gamma \neq 0$ and it reduces to f for $\gamma = 0$. The TV distance between s_{tp} and the baseline pdf f is given by:

$$d_{TV}(s_{tp}, f|\gamma) = \frac{1}{2}\int_{-\infty}^{0} \left| f\left(\frac{x}{1-\gamma}\right) - f(x) \right| dx + \frac{1}{2}\int_{0}^{\infty} \left| f\left(\frac{x}{1+\gamma}\right) - f(x) \right| dx = \frac{|\gamma|}{2}.$$

If we define the measure of asymmetry $M_{TV}(\gamma) = \gamma/2$, this coincides, up to a proportionality constant, with the AG measure of skewness proposed in Arnold and Groeneveld (1995) (see Rubio and Steel, 2014). Consequently, if we assume that $\dfrac{1 + 2M_{TV}(\gamma)}{2} = \dfrac{1+\gamma}{2} \sim Beta(\alpha, \beta)$, we obtain the AG-Beta priors proposed in Rubio and Steel (2014) for this family of distributions.

Regarding the Wasserstein distance, and assuming that the first moment of f is finite, we have

$$\begin{aligned} d_{\mathcal{W}}(s_{tp}, f|\gamma) &= \left| \int_{-\infty}^{0} x \left[f\left(\frac{x}{1-\gamma}\right) - f(x) \right] dx + \int_{0}^{\infty} x \left[f\left(\frac{x}{1+\gamma}\right) - f(x) \right] dx \right| \\ &= 2|\gamma|\omega_1, \end{aligned}$$

where $\omega_1 = \int_{-\infty}^{\infty} |x| f(x) dx$. Inspired by this distance, we can define the measure of asymmetry $M_{\mathcal{W}}(\gamma) = 2\omega_1\gamma$, which again coincides, up to a proportionality constant, with the AG measure of skewness. Consequently, we can also obtain the family of AG-Beta priors by assuming that $\dfrac{1 + M_{\mathcal{W}}(\gamma)/(2\omega_1)}{2} = \dfrac{1+\gamma}{2} \sim Beta(\alpha, \beta)$.

2 Simulation study

This section is a complement to the Monte Carlo simulation study in Section 5.1 of the main paper. It is a performance comparison between noninformative priors built according to our new method and the Jeffreys prior.

We simulate $N = 1,000$ samples of size $n = 200$ from the skew-normal distribution, and $N = 1,000$ samples of sizes $n = 50, 100, 200$ from skew-logistic and skew-Laplace distributions, in each case with location parameter $\mu = 0$, scale parameter $\sigma = 1$, and skewness parameter $\lambda = 0, 2.5, 5$. For each of these samples, we simulate a posterior sample of size 1,000 from (μ, σ, λ) using the $BTV(1,1)$, $BTV(1/2,1/2)$, $BW(1,1)$, $BW(1/2,1/2)$ and Jeffreys priors. We employ a self-adaptive MCMC sampler (Christen and Fox, 2010) to obtain the posterior samples. For each posterior sample, we calculate the coverage proportions of the 95% credible intervals of each parameter (this is, the proportion of credible intervals that contain the true value of the parameter) as well as the 5%, 50% and 95% quantiles of the posterior medians and maximum *a posteriori* (MAP) estimators. In addition, we obtain the median of the Bayes factors (BFs) associated to the hypothesis $H_0 : \lambda = 0$. The Bayes factors are approximated using the Savage-Dickey density ratio.

The BTV and BW priors for the skew-normal and skew-Laplace distribution are available under closed-form, see Example 2 and Example 3 of the main paper. For the skew-logistic model, we employ the Student-t approximation for the $BTV(1,1)$ prior described in Section 3.2 of the main paper, while the $BW(1,1)$ prior associated to the skew-logistic model is approximated using a Student-t distribution with 2 degrees of freedom and scale parameter 0.6. The $BTV(1/2,1/2)$ and $BW(1/2,1/2)$ priors are readily obtained via the Beta transformation as indicated in expression (5) of the main paper. For the Jeffreys priors associated to the skew-logistic model, we employ the Student-t approximation proposed in Rubio and Liseo (2014) (1/2 degrees of freedom and scale 4/3). For the skew-Laplace model, we propose a new approximation to the Jeffreys prior:

$$\pi_J(\lambda) = \frac{1}{4s_0(1 + |x/s_0|)^{3/2}},$$

where $s_0 = 0.77$. Results are reported in Tables 2S–7S. We attract the reader's attention to the fact that the $BW(1/2,1/2)$ prior is left out of Table 2S since, in the skew-normal setting, it coincides with the $BTV(1,1)$ prior. Overall, we observe that the Jeffreys and $BTV(1/2,1/2)$ priors exhibit the best, and very similar, performance.

Prior	MAP			Median			Coverage	BF
	5%	50%	95%	5%	50%	95%		
$\lambda = 0$								
BTV(1/2,1/2)								
μ	-0.886	-0.035	0.869	-0.682	-0.009	0.692	0.994	–
σ	0.960	1.054	1.330	1.008	1.108	1.296	0.846	–
λ	-1.286	0.028	1.295	-1.028	-0.004	1.037	0.993	2.306
Jeffreys								
μ	-0.890	0.032	0.890	-0.718	0.006	0.720	0.991	–
σ	0.961	1.058	1.351	1.009	1.115	1.319	0.844	–
λ	-1.335	-0.002	1.297	-1.082	-0.019	1.052	0.991	2.557
BTV(1,1)								
μ	-0.857	0.002	0.860	-0.602	-0.007	0.652	0.995	–
σ	0.952	1.048	1.285	0.999	1.098	1.275	0.862	–
λ	-1.207	-0.001	1.180	-0.992	-0.004	0.851	0.995	1.570
BW(1,1)								
μ	-0.728	0.007	0.780	-0.436	-0.002	0.502	1.000	–
σ	0.949	1.040	1.156	0.988	1.082	1.210	0.889	–
λ	-0.993	0.003	0.661	-0.704	-0.014	0.597	1.000	1.172
$\lambda = 2.5$								
BTV(1/2,1/2)								
μ	-0.124	0.008	0.211	-0.119	0.021	0.317	0.905	–
σ	0.699	0.981	1.123	0.760	0.980	1.124	0.909	–
λ	1.158	2.283	3.671	0.924	2.362	3.814	0.907	0.012
Jeffreys								
μ	-0.127	0.006	0.202	-0.119	0.019	0.321	0.914	–
σ	0.709	0.984	1.120	0.772	0.982	1.125	0.911	–
λ	1.191	2.319	3.645	0.922	2.369	3.886	0.916	0.006
BTV(1,1)								
μ	-0.123	0.015	0.225	-0.110	0.031	0.401	0.898	–
σ	0.687	0.976	1.123	0.743	0.972	1.117	0.902	–
λ	0.787	2.231	3.541	0.722	2.294	3.666	0.897	0.014
BW(1,1)								
μ	-0.105	0.035	0.303	-0.097	0.056	0.514	0.866	–
σ	0.663	0.958	1.099	0.711	0.950	1.102	0.875	–
λ	0.174	2.072	3.301	0.438	2.098	3.478	0.855	0.050
$\lambda = 5$								
BTV(1/2,1/2)								
μ	-0.090	0.000	0.097	-0.082	0.003	0.098	0.938	–
σ	0.874	0.992	1.108	0.874	0.995	1.107	0.935	–
λ	2.948	4.603	7.795	3.129	4.939	8.662	0.923	5×10^{-11}
Jeffreys								
μ	-0.087	-0.001	0.095	-0.081	0.004	0.098	0.940	–
σ	0.873	0.989	1.099	0.874	0.995	1.108	0.937	–
λ	2.922	4.615	7.907	3.134	4.947	8.943	0.926	4×10^{-11}
BTV(1,1)								
μ	-0.080	0.004	0.098	-0.076	0.008	0.106	0.945	–
σ	0.867	0.987	1.096	0.869	0.990	1.101	0.944	–
λ	2.853	4.457	7.475	3.048	4.748	8.275	0.929	5×10^{-11}
BW(1,1)								
μ	-0.072	0.014	0.113	-0.067	0.017	0.123	0.933	–
σ	0.857	0.978	1.088	0.855	0.980	1.089	0.925	–
λ	2.593	4.173	6.750	2.825	4.450	7.376	0.920	8×10^{-11}

Table 1S: skew-normal data: $n = 200$.

Prior	MAP			Median			Coverage	BF
	5%	50%	95%	5%	50%	95%		
$\lambda = 0$								
BTV(1/2,1/2)								
μ	-1.513	-0.006	1.624	-1.260	0.042	1.368	0.964	–
σ	0.871	1.080	1.350	0.922	1.148	1.432	0.905	–
λ	-0.771	0.005	0.753	-1.278	-0.027	1.099	0.972	2.049
BW(1/2,1/2)								
μ	-1.324	0.009	1.334	-1.161	0.039	1.155	0.984	–
σ	0.857	1.065	1.317	0.906	1.117	1.371	0.927	–
λ	-0.599	0.002	0.565	-0.947	-0.013	0.846	0.985	1.410
Jeffreys								
μ	-1.524	0.017	1.580	-1.279	0.046	1.384	0.966	–
σ	0.871	1.089	1.372	0.927	1.153	1.433	0.901	–
λ	-0.862	-0.002	0.802	-1.327	-0.026	1.107	0.967	2.138
BTV(1,1)								
μ	-1.378	0.030	1.397	-1.163	0.031	1.187	0.980	–
σ	0.857	1.065	1.307	0.913	1.126	1.379	0.923	–
λ	-0.675	0.002	0.629	-1.009	-0.022	0.920	0.983	1.436
BW(1,1)								
μ	-0.942	0.004	0.951	-0.890	0.017	0.882	0.992	–
σ	0.842	1.045	1.266	0.884	1.086	1.312	0.940	–
λ	-0.426	-0.008	0.437	-0.618	-0.011	0.577	0.995	1.136
$\lambda = 2.5$								
BTV(1/2,1/2)								
μ	-0.459	0.080	0.810	-0.390	0.175	0.924	0.913	–
σ	0.617	0.871	1.252	0.650	0.909	1.266	0.923	–
λ	0.022	1.172	7.476	0.314	1.809	11.010	0.912	0.507
BW(1/2,1/2)								
μ	-0.365	0.182	0.946	-0.287	0.300	0.998	0.895	–
σ	0.595	0.821	1.198	0.636	0.866	1.205	0.897	–
λ	0.138	0.863	4.346	0.211	1.354	6.077	0.874	0.470
Jeffreys								
μ	-0.476	0.065	0.815	-0.378	0.162	0.935	0.912	–
σ	0.624	0.875	1.269	0.658	0.917	1.266	0.919	–
λ	0.073	1.321	7.835	0.316	1.894	9.999	0.905	0.518
BTV(1,1)								
μ	-0.357	0.167	0.859	-0.286	0.276	0.989	0.899	–
σ	0.606	0.835	1.204	0.641	0.875	1.196	0.899	–
λ	0.209	0.937	4.343	0.252	1.439	5.855	0.879	0.463
BW(1,1)								
μ	-0.217	0.446	1.073	-0.119	0.497	1.088	0.780	–
σ	0.587	0.763	1.079	0.612	0.803	1.098	0.818	–
λ	0.133	0.513	2.584	0.183	0.828	3.261	0.738	0.520
$\lambda = 5$								
BTV(1/2,1/2)								
μ	-0.316	0.011	0.524	-0.263	0.049	0.621	0.921	–
σ	0.614	0.938	1.230	0.656	0.955	1.236	0.919	–
λ	-7.980	3.013	33.228	0.995	4.600	33.555	0.915	0.111
BW(1/2,1/2)								
μ	-0.243	0.062	0.667	-0.194	0.129	0.738	0.893	–
σ	0.595	0.896	1.190	0.634	0.905	1.195	0.904	–
λ	0.478	2.520	8.610	0.761	3.370	13.059	0.886	0.113
Jeffreys								
μ	-0.307	0.010	0.486	-0.262	0.052	0.600	0.921	–
σ	0.618	0.938	1.230	0.662	0.958	1.242	0.919	–
λ	-10.231	3.015	33.053	1.099	4.535	33.336	0.906	0.115
BTV(1,1)								
μ	-0.238	0.060	0.632	-0.204	0.124	0.729	0.897	–
σ	0.594	0.894	1.181	0.640	0.913	1.185	0.902	–
λ	0.543	2.585	8.449	0.826	3.483	13.135	0.894	0.109
BW(1,1)								
μ	-0.141	0.196	0.908	-0.088	0.313	0.895	0.789	–
σ	0.557	0.791	1.096	0.591	0.819	1.100	0.815	–
λ	0.362	1.037	4.632	0.516	1.904	6.230	0.730	0.202

Table 2S: skew-logistic data: $n = 50$.

Prior	MAP			Median			Coverage	BF
	5%	50%	95%	5%	50%	95%		
$\lambda = 0$								
BTV(1/2,1/2)								
μ	-1.139	0.048	1.191	-0.994	0.055	1.046	0.957	–
σ	0.906	1.060	1.244	0.936	1.099	1.282	0.911	–
λ	-0.682	-0.014	0.641	-0.842	-0.030	0.820	0.952	2.794
BW(1/2,1/2)								
μ	-1.058	0.005	1.082	-0.910	0.035	0.966	0.963	–
σ	0.904	1.057	1.229	0.933	1.091	1.265	0.921	–
λ	-0.560	-0.009	0.561	-0.736	-0.029	0.729	0.963	1.808
Jeffreys								
μ	-1.190	0.047	1.227	-1.044	0.059	1.097	0.952	–
σ	0.913	1.070	1.254	0.944	1.106	1.295	0.910	–
λ	-0.783	-0.014	0.719	-0.899	-0.044	0.865	0.946	2.913
BTV(1,1)								
μ	-1.030	0.029	1.082	-0.932	0.051	0.965	0.965	–
σ	0.901	1.054	1.230	0.938	1.092	1.270	0.916	–
λ	-0.614	-0.016	0.608	-0.766	-0.035	0.739	0.961	1.901
BW(1,1)								
μ	-0.802	0.030	0.880	-0.781	0.025	0.818	0.982	–
σ	0.895	1.043	1.213	0.918	1.071	1.242	0.932	–
λ	-0.476	-0.006	0.452	-0.589	-0.025	0.552	0.981	1.344
$\lambda = 2.5$								
BTV(1/2,1/2)								
μ	-0.312	0.037	0.546	-0.278	0.090	0.602	0.922	–
σ	0.706	0.947	1.196	0.734	0.950	1.203	0.923	–
λ	0.540	1.929	4.478	0.766	2.141	5.002	0.904	0.122
BW(1/2,1/2)								
μ	-0.285	0.077	0.610	-0.243	0.140	0.674	0.914	–
σ	0.685	0.914	1.167	0.719	0.925	1.171	0.912	–
λ	0.457	1.715	3.831	0.628	1.896	4.519	0.892	0.111
Jeffreys								
μ	-0.305	0.031	0.535	-0.277	0.089	0.604	0.925	–
σ	0.704	0.949	1.203	0.737	0.955	1.204	0.921	–
λ	0.565	1.951	4.205	0.742	2.137	5.028	0.911	0.115
BTV(1,1)								
μ	-0.264	0.074	0.642	-0.237	0.134	0.676	0.918	–
σ	0.685	0.917	1.174	0.718	0.926	1.180	0.913	–
λ	0.496	1.725	3.919	0.675	1.917	4.491	0.894	0.099
BW(1,1)								
μ	-0.216	0.182	0.768	-0.171	0.264	0.773	0.858	–
σ	0.662	0.847	1.129	0.691	0.873	1.130	0.852	–
λ	0.391	1.109	3.262	0.516	1.425	3.651	0.821	0.143
$\lambda = 5$								
BTV(1/2,1/2)								
μ	-0.222	-0.003	0.260	-0.198	0.018	0.294	0.931	–
σ	0.780	0.973	1.168	0.792	0.980	1.175	0.942	–
λ	1.649	4.077	10.223	2.174	4.733	13.117	0.916	0.006
BW(1/2,1/2)								
μ	-0.185	0.023	0.287	-0.165	0.044	0.338	0.932	–
σ	0.760	0.959	1.147	0.777	0.964	1.155	0.928	–
λ	1.591	3.794	8.276	1.922	4.303	10.006	0.916	0.005
Jeffreys								
μ	-0.210	0.001	0.264	-0.200	0.015	0.287	0.933	–
σ	0.788	0.975	1.168	0.796	0.981	1.172	0.943	–
λ	1.507	4.109	9.910	2.178	4.766	13.401	0.907	0.006
BTV(1,1)								
μ	-0.182	0.020	0.303	-0.170	0.040	0.330	0.930	–
σ	0.761	0.961	1.156	0.774	0.967	1.155	0.932	–
λ	1.601	3.792	7.933	1.964	4.338	10.000	0.909	0.005
BW(1,1)								
μ	-0.141	0.067	0.406	-0.124	0.098	0.465	0.895	–
σ	0.700	0.927	1.122	0.725	0.927	1.124	0.891	–
λ	0.987	3.171	6.343	1.432	3.544	7.643	0.858	0.010

Table 3S: skew-logistic data: $n = 100$.

Prior	MAP			Median			Coverage	BF
	5%	50%	95%	5%	50%	95%		
$\lambda=0$								
BTV(1/2,1/2)								
μ	-0.907	-0.022	0.904	-0.824	-0.010	0.807	0.938	–
σ	0.931	1.037	1.168	0.950	1.059	1.187	0.913	–
λ	-0.529	-0.002	0.540	-0.594	-0.002	0.632	0.938	3.838
BW(1/2,1/2)								
μ	-0.826	-0.020	0.828	-0.785	-0.008	0.766	0.946	–
σ	0.927	1.033	1.157	0.947	1.055	1.178	0.921	–
λ	-0.454	0.007	0.505	-0.541	0.005	0.585	0.947	2.437
Jeffreys								
μ	-0.913	-0.012	0.902	-0.826	-0.007	0.820	0.936	–
σ	0.932	1.038	1.165	0.949	1.061	1.191	0.915	–
λ	-0.531	0.011	0.566	-0.606	0.002	0.662	0.931	4.186
BTV(1,1)								
μ	-0.879	-0.013	0.837	-0.780	-0.012	0.755	0.945	–
σ	0.926	1.035	1.154	0.946	1.058	1.181	0.921	–
λ	-0.481	0.007	0.542	-0.531	0.003	0.618	0.942	2.574
BW(1,1)								
μ	-0.717	-0.004	0.710	-0.694	-0.006	0.683	0.965	–
σ	0.922	1.029	1.143	0.941	1.046	1.162	0.931	–
λ	-0.402	0.001	0.426	-0.465	0.000	0.507	0.964	1.722
$\lambda=2.5$								
BTV(1/2,1/2)								
μ	-0.225	0.011	0.363	-0.207	0.037	0.382	0.930	–
σ	0.787	0.975	1.142	0.804	0.975	1.142	0.930	–
λ	1.069	2.233	3.760	1.187	2.330	3.999	0.931	0.001
BW(1/2,1/2)								
μ	-0.215	0.028	0.372	-0.196	0.059	0.424	0.915	–
σ	0.777	0.964	1.135	0.791	0.964	1.132	0.917	–
λ	0.923	2.146	3.564	1.094	2.224	3.793	0.910	0.002
Jeffreys								
μ	-0.228	0.014	0.334	-0.211	0.034	0.392	0.924	–
σ	0.790	0.974	1.143	0.804	0.977	1.145	0.926	–
λ	1.042	2.253	3.745	1.188	2.349	3.936	0.927	0.001
BTV(1,1)								
μ	-0.207	0.029	0.370	-0.194	0.052	0.427	0.918	–
σ	0.777	0.966	1.128	0.790	0.968	1.128	0.918	–
λ	0.954	2.174	3.617	1.088	2.236	3.800	0.911	0.002
BW(1,1)								
μ	-0.187	0.070	0.476	-0.162	0.106	0.507	0.890	–
σ	0.738	0.941	1.111	0.761	0.938	1.114	0.880	–
λ	0.749	1.958	3.315	0.919	2.006	3.491	0.874	0.004
$\lambda=5$								
BTV(1/2,1/2)								
μ	-0.141	0.003	0.157	-0.132	0.009	0.172	0.934	–
σ	0.859	0.989	1.125	0.859	0.992	1.128	0.940	–
λ	2.773	4.640	7.651	3.013	4.942	8.602	0.942	9×10^{-9}
BW(1/2,1/2)								
μ	-0.132	0.011	0.182	-0.121	0.019	0.184	0.941	–
σ	0.846	0.982	1.122	0.849	0.985	1.122	0.937	–
λ	2.742	4.436	7.435	2.893	4.746	8.078	0.934	9×10^{-9}
Jeffreys								
μ	-0.141	0.004	0.169	-0.138	0.010	0.174	0.940	–
σ	0.854	0.987	1.126	0.859	0.992	1.132	0.933	–
λ	2.811	4.615	7.605	3.023	4.957	8.638	0.945	8×10^{-9}
BTV(1,1)								
μ	-0.131	0.012	0.179	-0.123	0.018	0.184	0.936	–
σ	0.851	0.983	1.114	0.851	0.985	1.123	0.932	–
λ	2.704	4.429	7.301	2.883	4.754	8.134	0.939	8×10^{-9}
BW(1,1)								
μ	-0.112	0.029	0.201	-0.101	0.038	0.219	0.928	–
σ	0.830	0.966	1.101	0.835	0.972	1.104	0.921	–
λ	2.390	4.116	6.653	2.576	4.408	7.197	0.908	3×10^{-8}

Table 4S: skew-logistic data: $n=200$.

Prior	MAP			Median			Coverage	BF
	5%	50%	95%	5%	50%	95%		
$\lambda = 0$								
BTV(1/2,1/2)								
μ	-0.585	0.000	0.587	-0.524	-0.005	0.562	0.946	–
σ	0.803	1.032	1.293	0.835	1.067	1.332	0.960	–
λ	-0.369	0.001	0.385	-0.621	0.001	0.582	0.946	3.921
BW(1/2,1/2)								
μ	-0.535	-0.010	0.513	-0.479	-0.014	0.511	0.961	–
σ	0.788	1.027	1.267	0.823	1.061	1.320	0.966	–
λ	-0.332	0.004	0.311	-0.539	0.001	0.497	0.963	2.279
Jeffreys								
μ	-0.591	-0.007	0.629	-0.523	-0.009	0.575	0.948	–
σ	0.795	1.034	1.290	0.835	1.072	1.333	0.959	–
λ	-0.415	-0.001	0.388	-0.654	-0.000	0.574	0.948	3.759
BTV(1,1)								
μ	-0.549	-0.003	0.554	-0.509	-0.011	0.545	0.957	–
σ	0.798	1.031	1.286	0.824	1.065	1.318	0.958	–
λ	-0.347	-0.001	0.329	-0.546	-0.002	0.529	0.954	2.567
BW(1,1)								
μ	-0.461	-0.008	0.459	-0.428	-0.006	0.441	0.960	–
σ	0.793	1.020	1.261	0.813	1.050	1.299	0.961	–
λ	-0.270	0.000	0.248	-0.436	0.000	0.398	0.971	1.610
$\lambda = 2.5$								
BTV(1/2,1/2)								
μ	-0.273	0.014	0.381	-0.230	0.035	0.375	0.932	–
σ	0.661	0.939	1.267	0.706	0.972	1.299	0.936	–
λ	0.280	1.558	6.824	0.697	2.234	10.721	0.926	0.137
BW(1/2,1/2)								
μ	-0.221	0.048	0.456	-0.191	0.070	0.425	0.929	–
σ	0.663	0.909	1.235	0.691	0.942	1.261	0.937	–
λ	0.308	1.326	4.846	0.567	1.880	6.993	0.933	0.094
Jeffreys								
μ	-0.272	0.016	0.416	-0.239	0.038	0.385	0.930	–
σ	0.668	0.933	1.266	0.710	0.967	1.296	0.932	–
λ	0.270	1.493	6.402	0.684	2.194	11.039	0.922	0.136
BTV(1,1)								
μ	-0.229	0.041	0.421	-0.192	0.063	0.402	0.931	–
σ	0.656	0.917	1.230	0.694	0.949	1.254	0.933	–
λ	0.338	1.438	4.853	0.614	1.963	7.188	0.936	0.106
BW(1,1)								
μ	-0.165	0.109	0.509	-0.128	0.135	0.475	0.900	–
σ	0.634	0.871	1.172	0.660	0.901	1.192	0.909	–
λ	0.237	0.981	3.066	0.419	1.390	4.250	0.884	0.092
$\lambda = 5$								
BTV(1/2,1/2)								
μ	-0.187	0.000	0.244	-0.164	0.015	0.262	0.931	–
σ	0.667	0.942	1.234	0.697	0.967	1.263	0.932	–
λ	-3.128	3.112	30.605	1.445	4.790	34.839	0.922	0.054
BW(1/2,1/2)								
μ	-0.153	0.023	0.299	-0.123	0.042	0.299	0.940	–
σ	0.654	0.913	1.208	0.684	0.941	1.232	0.931	–
λ	0.739	2.851	9.570	1.214	3.971	15.015	0.920	0.035
Jeffreys								
μ	-0.182	0.002	0.257	-0.166	0.019	0.264	0.929	–
σ	0.668	0.942	1.246	0.701	0.971	1.267	0.933	–
λ	-6.021	3.107	30.692	1.424	4.824	34.424	0.926	0.053
BTV(1,1)								
μ	-0.157	0.021	0.291	-0.124	0.041	0.296	0.935	–
σ	0.658	0.914	1.211	0.688	0.944	1.237	0.933	–
λ	0.775	2.897	9.682	1.281	4.018	14.715	0.927	0.041
BW(1,1)								
1	-0.099	0.076	0.378	-0.079	0.098	0.377	0.888	–
σ	0.625	0.863	1.141	0.651	0.892	1.168	0.901	–
λ	0.547	1.878	4.771	0.899	2.737	7.180	0.861	0.038

Table 5S: skew-Laplace data: $n = 50$.

Prior	MAP			Median			Coverage	BF
	5%	50%	95%	5%	50%	95%		
$\lambda = 0$								
BTV(1/2,1/2)								
μ	-0.342	-0.007	0.347	-0.341	0.006	0.356	0.936	–
σ	0.850	1.021	1.197	0.868	1.038	1.213	0.950	–
λ	-0.240	0.001	0.226	-0.335	0.001	0.304	0.934	6.549
BW(1/2,1/2)								
μ	-0.337	-0.005	0.313	-0.330	0.002	0.324	0.950	–
σ	0.852	1.016	1.195	0.868	1.035	1.212	0.954	–
λ	-0.218	0.002	0.207	-0.313	0.001	0.272	0.942	3.607
Jeffreys								
μ	-0.343	-0.007	0.350	-0.347	0.006	0.349	0.939	–
σ	0.851	1.021	1.199	0.869	1.038	1.210	0.953	–
λ	-0.244	0.001	0.221	-0.330	0.004	0.303	0.936	6.333
BTV(1,1)								
μ	-0.354	-0.003	0.328	-0.338	0.003	0.333	0.944	–
σ	0.853	1.021	1.200	0.868	1.035	1.212	0.952	–
λ	-0.243	0.001	0.214	-0.319	0.001	0.292	0.934	4.277
BW(1,1)								
μ	-0.308	-0.000	0.298	-0.300	0.004	0.290	0.958	–
σ	0.850	1.016	1.182	0.866	1.030	1.205	0.946	–
λ	-0.211	0.001	0.183	-0.277	0.002	0.254	0.954	2.437
$\lambda = 2.5$								
BTV(1/2,1/2)								
μ	-0.173	0.002	0.237	-0.157	0.014	0.240	0.945	–
σ	0.781	0.969	1.195	0.799	0.983	1.216	0.952	–
λ	0.834	2.039	4.492	1.068	2.377	5.605	0.944	0.011
BW(1/2,1/2)								
μ	-0.162	0.016	0.258	-0.146	0.028	0.260	0.937	–
σ	0.768	0.952	1.191	0.784	0.971	1.202	0.944	–
λ	0.786	1.956	4.344	0.996	2.224	5.101	0.936	0.006
Jeffreys								
μ	-0.173	0.001	0.249	-0.155	0.012	0.238	0.938	–
σ	0.771	0.971	1.203	0.788	0.985	1.217	0.953	–
λ	0.841	2.065	4.552	1.060	2.386	5.598	0.940	0.010
BTV(1,1)								
μ	-0.163	0.012	0.257	-0.145	0.026	0.251	0.938	–
σ	0.772	0.958	1.192	0.788	0.974	1.200	0.944	–
λ	0.778	1.951	4.391	0.991	2.241	5.094	0.946	0.008
BW(1,1)								
μ	-0.131	0.043	0.300	-0.115	0.062	0.297	0.926	–
σ	0.746	0.931	1.163	0.765	0.947	1.169	0.933	–
λ	0.630	1.646	3.630	0.811	1.924	4.242	0.911	0.007
$\lambda = 5$								
BTV(1/2,1/2)								
μ	-0.120	0.001	0.147	-0.113	0.008	0.157	0.938	–
σ	0.786	0.974	1.187	0.801	0.983	1.194	0.947	–
λ	1.943	4.172	9.410	2.427	4.877	12.860	0.936	0.002
BW(1/2,1/2)								
μ	-0.107	0.009	0.161	-0.099	0.018	0.175	0.945	–
σ	0.775	0.966	1.168	0.789	0.972	1.173	0.945	–
λ	1.772	3.858	8.530	2.228	4.492	11.127	0.934	0.002
Jeffreys								
μ	-0.128	-0.000	0.153	-0.117	0.009	0.159	0.943	–
σ	0.787	0.973	1.182	0.798	0.985	1.194	0.945	–
λ	1.950	4.128	10.446	2.381	4.868	12.906	0.936	0.003
BTV(1,1)								
μ	-0.109	0.009	0.164	-0.099	0.017	0.177	0.947	–
σ	0.772	0.962	1.159	0.787	0.974	1.182	0.947	–
λ	1.852	3.848	8.546	2.224	4.479	10.951	0.942	0.002
BW(1,1)								
μ	-0.085	0.031	0.197	-0.079	0.041	0.209	0.931	–
σ	0.754	0.938	1.147	0.770	0.948	1.151	0.928	–
λ	1.552	3.273	6.765	1.897	3.767	8.302	0.916	0.002

Table 6S: skew-Laplace data: $n = 100$.

Prior	MAP			Median			Coverage	BF
	5%	50%	95%	5%	50%	95%		
$\lambda = 0$								
BTV(1/2,1/2)								
μ	-0.204	-0.002	0.225	-0.209	-0.002	0.223	0.941	–
σ	0.889	1.008	1.136	0.898	1.013	1.142	0.942	–
λ	-0.147	-0.000	0.149	-0.183	0.000	0.185	0.947	10.585
BW(1/2,1/2)								
μ	-0.198	0.000	0.213	-0.202	-0.001	0.206	0.948	–
σ	0.886	1.004	1.133	0.895	1.012	1.138	0.943	–
λ	-0.132	0.001	0.143	-0.171	0.001	0.178	0.956	5.739
Jeffreys								
μ	-0.206	-0.001	0.224	-0.211	-0.000	0.214	0.939	–
σ	0.884	1.007	1.133	0.895	1.014	1.141	0.939	–
λ	-0.148	0.001	0.154	-0.180	0.001	0.190	0.945	10.440
BTV(1,1)								
μ	-0.208	-0.002	0.211	-0.206	0.001	0.210	0.941	–
σ	0.887	1.007	1.132	0.898	1.014	1.140	0.942	–
λ	-0.138	0.000	0.148	-0.180	-0.000	0.189	0.945	6.852
BW(1,1)								
μ	-0.192	-0.002	0.203	-0.202	-0.001	0.202	0.950	–
σ	0.889	1.006	1.131	0.896	1.013	1.141	0.943	–
λ	-0.133	0.001	0.136	-0.163	0.002	0.169	0.953	3.770
$\lambda = 2.5$								
BTV(1/2,1/2)								
μ	-0.126	-0.004	0.155	-0.119	0.003	0.155	0.945	–
σ	0.840	0.987	1.147	0.845	0.994	1.159	0.936	–
λ	1.307	2.311	3.849	1.429	2.469	4.217	0.941	7×10^{-6}
BW(1/2,1/2)								
μ	-0.119	0.005	0.168	-0.114	0.011	0.163	0.947	–
σ	0.831	0.978	1.136	0.843	0.986	1.148	0.939	–
λ	1.204	2.243	3.718	1.356	2.392	4.049	0.948	4×10^{-6}
Jeffreys								
μ	-0.127	-0.001	0.157	-0.118	0.004	0.161	0.940	–
σ	0.836	0.982	1.148	0.851	0.992	1.154	0.939	–
λ	1.289	2.291	3.830	1.403	2.460	4.176	0.944	6×10^{-6}
BTV(1,1)								
μ	-0.119	0.004	0.159	-0.110	0.009	0.160	0.945	–
σ	0.828	0.978	1.140	0.843	0.988	1.145	0.938	–
λ	1.263	2.210	3.738	1.382	2.385	4.045	0.945	5×10^{-6}
BW(1,1)								
μ	-0.103	0.020	0.176	-0.095	0.024	0.180	0.936	–
σ	0.823	0.967	1.120	0.828	0.974	1.133	0.939	–
λ	1.160	2.055	3.522	1.276	2.194	3.715	0.942	4×10^{-6}
$\lambda = 5$								
BTV(1/2,1/2)								
μ	-0.084	-0.001	0.097	-0.080	0.002	0.096	0.951	–
σ	0.849	0.984	1.137	0.855	0.991	1.142	0.939	–
λ	2.817	4.596	7.956	3.036	4.900	8.621	0.952	7×10^{-8}
BW(1/2,1/2)								
μ	-0.079	0.004	0.101	-0.075	0.006	0.105	0.956	–
σ	0.841	0.977	1.126	0.848	0.983	1.134	0.935	–
λ	2.669	4.379	7.501	2.925	4.702	8.301	0.951	6×10^{-8}
Jeffreys								
μ	-0.085	-0.002	0.095	-0.079	0.002	0.097	0.951	–
σ	0.847	0.987	1.142	0.853	0.990	1.142	0.940	–
λ	2.784	4.567	7.767	3.033	4.879	8.693	0.945	1×10^{-7}
BTV(1,1)								
μ	-0.080	0.004	0.099	-0.074	0.006	0.102	0.951	–
σ	0.840	0.978	1.128	0.849	0.986	1.135	0.932	–
λ	2.615	4.427	7.547	2.859	4.744	8.132	0.943	7×10^{-8}
BW(1,1)								
μ	-0.070	0.013	0.114	-0.067	0.018	0.119	0.943	–
σ	0.834	0.970	1.121	0.841	0.974	1.124	0.926	–
λ	2.530	4.044	6.888	2.702	4.344	7.487	0.924	8×10^{-8}

Table 7S: skew-Laplace data: $n = 200$.

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