Traveling wave solutions of reaction-diffusion equations with x-dependent combustion type nonlinearities

Dissertation
zur Erlangung des akademischen Grades
eines Doktors der Naturwissenschaften (Dr. rer. nat.)

Der Fakultät für Mathematik
der Technischen Universität Dortmund
vorgelegt von

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im März 2016

Dissertation

Traveling wave solutions of reaction-diffusion equations with x-dependent combustion type nonlinearities

Fakultät für Mathematik
Technische Universität Dortmund

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Tag der mündlichen Prüfung: 27.04.2016

Acknowledgements

I would like to express my gratitude towards the people who supported me in creating this thesis in different ways.

First and foremost, I would like to thank my supervisor Prof. Dr. Ben Schweizer for introducing me to the field of partial differential equations and providing me with the opportunity and guidance to write this thesis.

My gratitude also belongs to the members and former members of Lehrstuhl I and the Biomathematics group at the TU Dortmund, including Prof. Dr. Matthias Röger, Jun.-Prof. Dr. Tomáš Dohnal, Priv.-Doz. Dr. Andreas Rätz, Dr. Agnes Lamacz, Dr. Peter Furlan, Maik Urban and Saskia Stockhaus, for creating such a wonderful working atmosphere. I am particularly grateful to my friends and colleagues Lisa Helfmeier, Stephan Hausberg, Carsten Zwilling, Dr. Jan Koch, Dr. Tobias Kloos and Christian Kühbacher for their support and encouragement.

A special thanks goes to my friend Sebastian Schäfer for his support and encouragement and his assistance with the English language.

Finally, I want to thank my family, in particular my parents Manfred and Jutta Badke, my sister Sandra Karla and my grandmother Rita Henkemeyer. With their unconditional love and support, they contributed a great deal to the creation of this thesis.

Contents

1. Introduction

1.1. Traveling waves in homogeneous media

The notion of traveling waves in homogeneous media and typical nonlinearities

This work is concerned with traveling wave solutions to reaction-diffusion equations. The basic form of a reaction-diffusion equation is

$$u_t = \Delta u + f(u) \text{ in } (0, \infty) \times \mathbb{R}^n. \tag{1.1}$$

According to [5], this equation was first introduced in the articles [10] of Fischer and [14] of Kolmogorov, Petrovsky and Piskunov in 1937 and 1938. The original motivation was to investigate the spreading of advantageous genes. The considered nonlinearities f were of logistic type, e.g. $f(u) = u(1-u)$ or $f(u) = u(1-u^2)$. In both cases there are stationary states 0 and 1 of the equation.

A particular kind of solutions to the equation are traveling wave solutions or traveling front solutions. They can be imagined as front profiles, which connect the stationary states 0 and 1; the time-dependence is a shift into a direction k, $|k| = 1$, as t grows with a speed c. More exactly, this means that u is given by a pair (U, c), with a front profile U and a wavespeed c, such that u is of the form

$$u(t, x) = U(k \cdot x - ct), \tag{1.2}$$

$$0 \leq U \leq 1, \qquad U(-\infty) = 0, \qquad U(+\infty) = 1.$$

The direction k is given and the speed c is an unknown that is to be determined. In some cases there is a unique c, such that there exists a corresponding traveling wave solution (U, c). In other cases there are multiple c (for example an interval of the form $(-\infty, c^*)$ or $(-\infty, c^*]$), such that there exist corresponding traveling wave solutions. The condition (1.2) on u can also be expressed as

$$u\left(t + \frac{k \cdot v}{c}, x + v\right) = u(t, x) \text{ for any vector } v \in \mathbb{R}^n. \tag{1.3}$$

In case of space dimension one, this is perhaps more illustrative:

$$u\left(t + \frac{l}{c}, x + l\right) = u(t, x) \text{ for any } l \in \mathbb{R}.$$

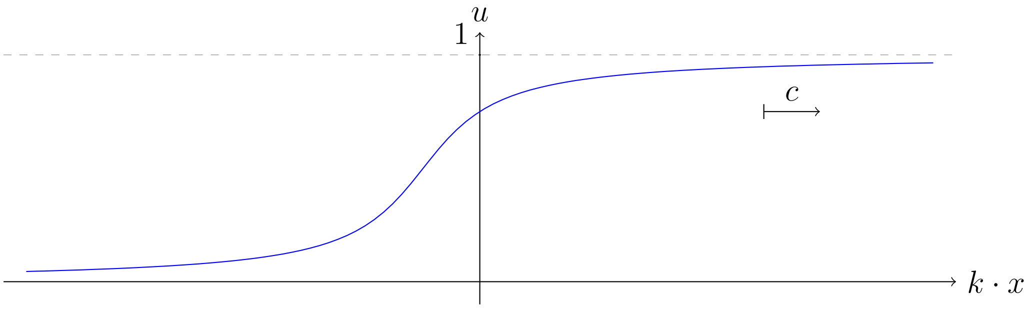


Figure 1.1.: A qualitative view of a front profile in the homogeneous case.

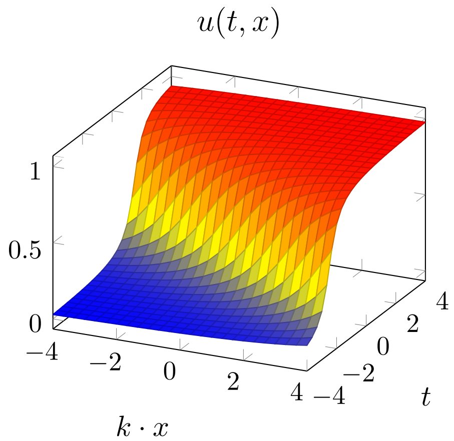


Figure 1.2.: A qualitative picture of a traveling wave solution in the homogeneous case.

The ansatz $u(t,x) = U(s)$ with $s = k \cdot x - ct$ is commonly called the moving frame ansatz. Inserting it into equation (1.1) leads to an ordinary differential equation for U:

$$U_{ss} = -cU_s - f(U). \tag{1.4}$$

Among the usual questions are those of existence, uniqueness, monotonicity and stability of traveling wave solutions (U, c). Also of interest is the long term behavior of solutions to the initial value problem

$$u_t = \Delta u + f(u) \text{ in } (0,\infty) \times \mathbb{R}^n,$$

$$u(0,x) = u_0(x).$$

Of course, the results depend on the type of the nonlinearity f. As far as uniqueness is concerned, a traveling wave solution can only be unique up to a constant shift in s due to the shift-invariance of the equation in s. From (1.4), one can see that the space dimension is irrelevant in the questions of existence and uniqueness of traveling wave solutions.

In the overview article [17], Xin lists the following types of nonlinearities:

1. $f(u) = u(1-u)$ is called the KPP nonlinearity (after [14]) or Fisher nonlinearity (after [10]).

2. $f(u) = u^m(1-u)$ with $m \geq 2$, $m \in \mathbb{N}$ is called the m-th order Fisher nonlinearity or Zeldovich nonlinearity, if $m = 2$.

3. $f(u) = u(1-u)(\mu - u)$ with $\mu \in (0,1)$ is called the bistable nonlinearity.

4. $f(u) = e^{-E/u}(1-u)$ with activation energy $E > 0$ is called Arrhenius combustion nonlinearity or combustion nonlinearity with activation energy E but no ignition temperature cutoff.

5. $f(u) = 0$ in $[0,\theta]$ with $\theta \in (0,1)$, $f(u) > 0$ in $(\theta,1)$ and $f(1) = 0$, f Lipschitz continuous, is called a combustion nonlinearity with ignition temperature θ.

In [17], Xin also lists some of the fields in which these nonlinearities arise: Types 1 and 2 have their origin in chemical kinetics. Type 3 arises in biological applications. Types 4 and 5 arise in combustion science.

Some results in the homogeneous case

Let us first list some of the results, which are given in [17]: Let f be any of type 1 - type 5 in the above list with $\mu \in \left(0, \frac{1}{2}\right)$ in case of type 3. By multiplying (1.4) with U_s and integrating over $\mathbb{R}$, it can be seen that

$$c = -\frac{\int_0^1 f(z)\,dz}{\int_{\mathbb{R}} U_s^2\,ds} < 0. \tag{1.5}$$

We want to explain this argument. By multiplying (1.4) with U_s and integrating over an interval $[a,b]$, one obtains

$$\frac{1}{2}U_s^2(b) - \frac{1}{2}U_s^2(a) = -c\int_a^b U_s^2(s)\,ds - \int_{U(a)}^{U(b)} f(z)\,dz.$$

For any fixed $a \in \mathbb{R}$, the right hand side converges for $b \to \infty$ to a value in $\mathbb{R} \cup \{-\infty, \infty\}$. Therefore, the left hand side has to converge as well. It follows that, $U_s^2(b) \to d$ for $b \to \infty$ and some $d \in [0,\infty]$ (since $U_s^2(b) \geq 0$). Consequently, $U_s(b) \to \pm\sqrt{d}$ for $b \to \infty$. But then $d = 0$, because otherwise $U(b) \to 1$ for $b \to \infty$ cannot hold. By the same reasoning $U_s(a) \to 0$ for $a \to -\infty$. Therefore, one obtains

$$c\int_{-\infty}^{\infty} U_s^2(s)\,ds = -\int_0^1 f(z)\,dz < 0.$$

The left hand side cannot be 0, and one obtains (1.5). By this argumentation, one has also obtained $U_s(s) \to 0$ for $s \to \pm\infty$.

We continue with the results given in [17]: It is often useful to rewrite the ordinary differential equation for U as a first order system

$$\begin{aligned} U_s &=: V, \\ V_s &= -cV - f(U). \end{aligned}$$

In this form, one can perform a phase plane analysis. A traveling wave solution of (1.1) corresponds to a trajectory in the (U, V) plane connecting the points $(0, 0)$ and $(1, 0)$. For f of type 1, this method yields the existence of such a trajectory for every $c \le c^* = -2\sqrt{f(0)}$. In contrast, for f of type 3 with $\mu \in (0, \frac{1}{2})$, there exists a unique trajectory for a unique c. In case of a type 2 nonlinearity f, there is c_m such that there is a connecting trajectory for every $c < c_m$.

For nonlinearities of types 4 and 5, Xin describes different methods involving degree theory. If f is of type 4, there is a c^*, such that for every $c < c^*$, there is a traveling wave solution (U, c). In case of a type 5 nonlinearity f, there is a unique traveling wave solution (U, c).

In the work [1], Aronson and Weinberger investigate the one dimensional reaction diffusion equation

$$u_t = u_{xx} + f(u). \tag{1.6}$$

Their nonlinearity f is of the form $f(u) = u(1-u)((\tau_1 - \tau_2)(1-u) - (\tau_3 - \tau_2)u)$. The parameters τ_1, τ_2 and τ_3 stem from a biological model. This leads to three major cases and different relevant properties of f:

Case 1 $\tau_3 \le \tau_2 < \tau_1$. Then $f'(0) > 0$, $f(u) > 0$ in $(0, 1)$.

Case 2 $\tau_2 < \tau_3 \le \tau_1$. Then $f'(0) > 0$, $f'(1) > 0$, $f(u) > 0$ in $(0, \alpha)$, $f(u) < 0$ in $(\alpha, 1)$ for some $\alpha \in (0, 1)$.

Case 3 $\tau_3 \le \tau_1 < \tau_2$. Then $f'(0) < 0$, $f(u) < 0$ in $(0, \alpha)$, $f(u) > 0$ in $(\alpha, 1)$ for some $\alpha \in (0, 1)$, $\int_0^1 f(u)\, du > 0$.

Aronson and Weinberger examine the behavior of solutions of the initial boundary value problem on $\mathbb{R}_+ \times \mathbb{R}_+$ and the pure initial value problem on $\mathbb{R}_+ \times \mathbb{R}$ and also the existence of traveling front solutions. For the results, we refer to the article [1].

In [9], Fife and McLeod treat the equation

$$u_t = u_{xx} + f(u) \tag{1.7}$$

in one space dimension for a broad class of nonlinearities. In the existence part, it is assumed that $f \in C^1[0, 1]$, $f(0) = f(1) = 0$ and for some $\alpha \in (0, 1)$ one of the following three cases holds:

(a) $f \le 0$ in $(0,\alpha)$, $\quad f > 0$ in $(\alpha,1)$, $\quad \int_0^1 f(z)\,dz > 0$.

(b) $f < 0$ in $(0,\alpha)$, $\quad f \ge 0$ in $(\alpha,1)$, $\quad \int_0^1 f(z)\,dz < 0$.

(c) $f < 0$ in $(0,\alpha)$, $\quad f > 0$ in $(\alpha,1)$.

In each of these cases, Fife and McLeod show the existence of traveling wave solutions.

They also investigate the asymptotic behavior of the initial value problem (1.7) on $(0,\infty)\times\mathbb{R}$, $u(0,x) = \phi(x)$ under similar conditions on f. If the initial values are pulselike and if there exists a traveling wave solution, the solution to the initial value problem converges exponentially to a shift of the traveling front solution. The exact result in the frontlike case is the following: Assume $f \in C^1[0,1]$, $f(0) = f(1) = 0$, $f'(0) < 0$ and $f'(1) < 0$. Furthermore, assume $f(u) < 0$ for $0 < u < \alpha_0$ and $f(u) > 0$ for $\alpha_1 < u < 1$, where $0 < \alpha_0 \le \alpha_1 < 1$. Suppose that there is a traveling wave solution (U,c). Let the initial values satisfy $0 \le \phi \le 1$ and $\limsup_{x\to-\infty}\phi(x) < \alpha_0$, as well as $\liminf_{x\to\infty}\phi(x) > \alpha_1$. Then, for some constants z_0, $\omega > 0$ and $K > 0$, there holds

$$|u(t,x) - U(x - ct - z_0)| < Ke^{-\omega t}.$$

The result for pulselike initial values is of similar nature. Under similar assumptions on f as in the frontlike case and pulselike initial data, the solution u to the initial value problem converges exponentially to a shifted traveling wave solution for $x < 0$ and to a traveling wave moving in the opposite direction for $x > 0$. For the exact result in the pulselike case we refer to the article [9].

1.2. Traveling waves in periodic heterogeneous media

The notion of traveling waves in periodic heterogeneous media

We now consider the reaction-diffusion equation in a periodic heterogeneous medium. A heterogeneous medium is modeled with x-dependent coefficients a_{ij}. We assume that $(a_{ij})_{i,j=1} = (a_{ij}(x))_{i,j=1}^n$ is a smooth, symmetric and uniformly elliptic matrix field, which is 1-periodic in every component of x. The periodicity can also be expressed by defining the matrix field on T^n, the n-dimensional torus. Prototypes of the reaction-diffusion equation in heterogeneous media are

$$u_t = \sum_{i,j=1}^n (a_{ij}(x)u_{x_i})_{x_j} + f(u) \qquad \text{(divergence form)} \tag{1.8}$$

and

$$u_t = \sum_{i,j=1}^n a_{ij}(x)u_{x_ix_j} + f(u) \qquad \text{(non-divergence form).} \tag{1.9}$$

Of particular interest is the case where the nonlinearity f satisfies $f(0) = f(1) = 0$. In this case, the equations have the stationary states 0 and 1.

Neither for (1.8) nor for (1.9) we can expect to find a traveling wave solution of the form (1.2). Instead, the moving frame ansatz has to be extended. Consider the moving frame coordinates

$$s = k \cdot x - ct, \qquad y = x. \tag{1.10}$$

A (pulsating) traveling wave solution is a solution u given by a pair (U, c) with

$$\left.\begin{array}{c} u(t,x) = U(s,y), \qquad 0 \le U \le 1, \\ U(s,\cdot) \text{ 1-periodic in each component of } y, \\ \lim_{s\to-\infty} U(s,y) = 0 \text{ and } \lim_{s\to\infty} U(s,y) = 1 \text{ uniformly in } y. \end{array}\right\} \tag{1.11}$$

This can also be expressed without changing the coordinate system. A solution u of (1.8) or (1.9) (more precisely the pair (u, c)) is called a traveling wave solution if

$$\begin{gathered} 0 \le u \le 1, \\ u(t,x) = u\left(t + \frac{k \cdot e_i}{c}, x + e_i\right) \text{ for the unit vectors } e_i,\ i = 1, ..., n, \\ \lim_{t\to-\infty} u(t,x) = 0 \text{ and } \lim_{t\to\infty} u(t,x) = 1 \text{ locally uniformly in } x \in \mathbb{R}^n. \end{gathered}$$

We emphasize that this description extends the formulation of (1.3). Of course, there are variants of the notion of a traveling wave solution. For example, other periodic lengths can be considered. The nonlinearity f might depend explicitly on x. The stationary states that are to be connected could be different from 0 and 1. The stationary states might even be nonconstant. Then, the notion of a traveling wave solution has to be properly adapted.

A prototypical result of Xin

We now come to a result of Xin [21], which is prototypical for our work. Consider the equation

$$u_t = \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + g(u). \tag{1.12}$$

The matrix $(a_{ij}(x))_{i,j=1}^n$ is smooth, positive definite and 1-periodic in every component of $x \in \mathbb{R}^n$. The nonlinearity g is a $C^1[0,1]$ combustion nonlinearity.

Xin shows the existence of a traveling wave solution U of the form (1.11) with $0 < U < 1$ and $U_s > 0$, as well as $c < 0$. The proof uses the method of elliptic regularization. Under the additional assumption $g'(1) < 0$, Xin proves that every traveling wave solution U satisfies the monotonicity $U_s > 0$. Under the same assumption, he also proves the uniqueness of (U, c) up to a constant shift in s. He uses the sliding domain method in the proofs of both the monotonicity and the uniqueness result. Since our work is essentially based on [21], we will discuss that article in more detail in chapter 2.

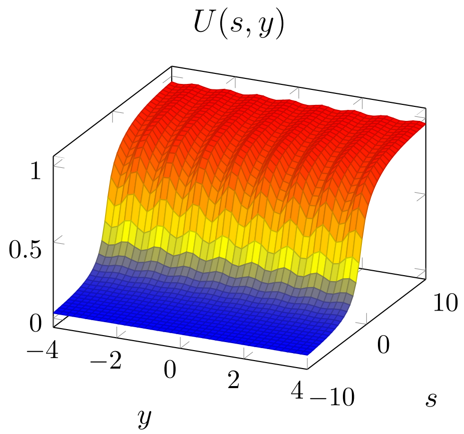


Figure 1.3.: A qualitative view of a pulsating traveling wave solution in moving frame coordinates.

A review of results in the periodic heterogeneous case

A result from Xin which is similar to the previously discussed prototypical result is given in [18]. Consider the equation

$$u_t = \sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + g(u).$$

It is assumed that $(a_{ij})_{i,j=1}^n$ is a smooth, positive definite matrix field on T^n. Moreover, $(b_i)_{i=1}^n$ is a smooth vector field on T^n, which is divergence free and has zero mean over T^n. The function g is a combustion nonlinearity, which is C^1 in a neighborhood of 1 and satisfies $g'(1) < 0$.

Under these assumptions, Xin proves the existence of a traveling wave solution (U, c) with $c < 0$ and U strictly increasing in s. In the proof, he uses the method of continuation. Under the same assumptions, Xin also shows the strict monotonicity of any traveling wave solution (if $g \in C^1[0, 1]$, even $U_s > 0$) and the uniqueness of (U, c) up to constant shifts in s. As opposed to the case of equation (1.12), the condition $g'(1) < 0$ is also used in the existence part.

In [20], Xin examines the reaction diffusion equation in the form

$$u_t = \sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + f(x, u). \tag{1.13}$$

It is assumed that the coefficients are smooth and 2π-periodic in every component of $x \in \mathbb{R}^n$. The matrix field $(a_{ij})_{i,j=1}^n$ is assumed to be uniformly positive definite. Furthermore, it is assumed to be a perturbation of the unit matrix: $(a_{ij}(x))_{i,j=1}^n = I + \delta(\tilde{a}_{ij}(x))_{i,j=1}^n$, where $(\tilde{a}_{ij})_{i,j=1}^n$ is smooth and 2π-periodic in every component of x and δ is small. The nonlinearity f is assumed to be a cubic bistable nonlinearity. For example, $f(u) = u(1-u)(\mu - u)$ with $\mu \in (0, \frac{1}{2})$. In the proofs, Xin assumes $b_i = 0$ and that f does not depend on x.

In order to prove the existence of traveling wave solutions, he uses a perturbation ansatz: The case $\delta = 0$ is the homogeneous case. For this case, the existence of a traveling wave solution (ϕ, c_0) is known. Xin's ansatz for a traveling wave solution (U, c) in moving frame coordinates is $U = \phi + \delta v$ and $c = c_0 + \delta c_1$. This leads to a problem for (v, c_1), which we call the perturbation problem for the moment. After imposing an additional normalization condition, Xin obtains a unique solution v, if δ is sufficiently small. To obtain this solution, Xin uses methods of Fourier and spectral analysis.

More precisely, the result is as follows. Let $k \in \mathbb{R}^n$ be a unit vector, $m \in \mathbb{N}^+$ and $m - [m/2] > (n+1)/2$. Consider the space

$$X_m = \{v \in H^{m+1}(\mathbb{R} \times T^n) : (\nabla_y + k\partial_s)^2 v \in H^m(\mathbb{R} \times T^n)\}.$$

Then there exists $\delta_0 = \delta_0(c_0, n, m) > 0$, such that for $\delta < \delta_0$, there exist unique $v \in X_m$ and $c_1 \in \mathbb{R}$, which solve the perturbation problem. Moreover, $(U, c) = (\phi + \delta v, c_0 + \delta c_1)$ is a traveling wave solution of (1.13). If (V, C) is another traveling wave solution of (1.13), then $c = C$ and $U(s - s_0, y) = V(s, y)$ for some $s_0 \in \mathbb{R}$ and $(s, y) \in \mathbb{R} \times T^n$.

In [20], Xin also proves a stability result in space dimension one: Consider the equation

$$u_t = (a(x)u_x)_x + f(u).$$

Xin assumes $a(x) = 1 + \delta\tilde{a}(x)$, where $\tilde{a}$ is a smooth and 2π-periodic function. Moreover, it is assumed that $f(u) = u(1-u)(u-\mu)$. For δ sufficiently small, there exists a traveling wave solution U of the equation, as Xin has proved in the existence part. Consider now the initial value problem

$$\begin{aligned} u_t &= (a(x)u_x)_x + f(u), \\ u(0, x) &= U(x, x) + \varepsilon u_0(x). \end{aligned}$$

The initial values are the perturbation of the traveling wave solution for $t = 0$. Xin's stability result is as follows. Assume $u_0 \in H^1(\mathbb{R})$ and let u be the solution of the initial value problem. Assume further $U \in C^3(\mathbb{R} \times T^n)$ and $U_s \in H^3(\mathbb{R} \times T^n)$. If ε is sufficiently small, there is a function $\gamma = \gamma(\varepsilon)$ and a constant $K = K(\rho_0)$ for some $\rho_0 \in (0, 1)$ such that

$$||u(t, \cdot + ct) - U(\cdot + \gamma(\varepsilon), \cdot + ct)||_{H^1(\mathbb{R})} \leq K\rho_0^t \text{ for all } t \geq 0.$$

Moreover, $\gamma(\varepsilon) = \varepsilon h(\varepsilon)$, with a C^1 function h. Xin also gives a characterization of $h(0)$ with an adjoint problem. For more details, we refer to [20].

In a two paper series [5] and [6], Berestycki, Hamel and Roques investigate a biological model for the persistence of species and propagation phenomena in a periodically fragmented environment model. The underlying equation is

$$u_t = \nabla \cdot (A(x)\nabla u) + f(x, u).$$

The function $f(x,u) = u(\mu(x) - \nu(x)u)$ with a saturation coefficient ν and a growth rate μ is a prototypical example for the class of nonlinearities treated in [5].

In the first paper, the authors are concerned with the existence of a positive periodic stationary state p of the equation. Under appropriate assumptions on A and f, the existence of p is decided by the periodic principle eigenvalue of the linearized elliptic operator. The periodic principle eigenvalue is the unique $\lambda_1 \in \mathbb{R}$, such that there exists a periodic solution ϕ with $\phi > 0$ of the equation

$$-\nabla \cdot (A(x)\nabla\phi) - f_u(x,0)\phi = \lambda_1\phi \text{ in } \mathbb{R}^n.$$

If $\lambda_1 < 0$, the stationary state 0 is called unstable; in this case a stationary state p exists. If $\lambda_1 \geq 0$, then 0 is the only bounded stationary state. Under slightly different assumptions on f, it is shown that in the case $\lambda_1 < 0$, there exists at most one bounded positive stationary state.

The authors also consider solutions u to the initial value problem with certain non-trivial initial values. Under appropriate conditions, the authors show that in the case $\lambda_1 < 0$, the solution $u(t,x)$ converges to p in $C^2_{loc}(\mathbb{R}^n)$ as $t \to \infty$ and in the case $\lambda_1 \geq 0$, $u(t,x)$ converges to 0 as $t \to \infty$.

In the second paper, the existence of traveling wave solutions is shown in the case that the stationary solution p exists. The proof uses an elliptic regularization method. There are traveling wave solutions for a continuum of wave speeds $c \geq c^*$.

One can also study a domain Ω with $\Omega \neq \mathbb{R}^n$ which is periodic in some of the variables, namely in $x = (x_1, ..., x_d)$, and bounded in the rest of the variables $y = (x_{d+1}, ..., x_n)$. Such a case is treated in the work [3]. In this article, the notion of a pulsating traveling wave solution has been carried over to the equation

$$u_t - \nabla_{x,y} \cdot (A(x,y)\nabla_{x,y}u) + q \cdot \nabla_{x,y}u = f(x,y,u).$$

Periodicity conditions are only imposed in x. Two types of nonlinearities f are treated. We want to mention the first type, which is a combustion type with ignition temperature θ and a monotonicity condition in the vicinity of $u = 1$. This monotonicity condition is similar to $g'(1) < 0$ in [18], but less restrictive. For the precise results, we refer the reader to [3].

In [19], Xin is concerned with the asymptotic behavior of the initial value problem

$$u_t = \sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + f(u),$$

$$u(0,x) = u_0(x), \qquad x \in \mathbb{R}^n.$$

It is assumed that $(a_{ij})_{i,j=1}^n$ is a smooth, positive definite matrix field on T^n. Moreover, $(b_i)_{i=1}^n$ is a smooth vector field on T^n, which is divergence free and has zero mean over T^n. Two types of nonlinearities are treated, namely combustion nonlinearities and bistable nonlinearities. The initial values u_0 are either frontlike or pulselike. What the terms frontlike and pulselike mean, depends on the nonlinearity. The asymptotic stability of pulsating traveling wave solutions are unclear. However, it can be shown in the case of frontlike data, that the solution to the initial value problem propagates with the speed of the traveling wave solution, if one exists. As an example, we sketch Xin's result for the case of a bistable nonlinearity $f(u) = u(1-u)(\mu-u)$ and frontlike data. Consider initial data $0 \leq u_0 \leq 1$. For a unit vector $k \in \mathbb{R}^n$, let $S = \{y \in \mathbb{R}^n : y = x - (k \cdot x)k, x \in \mathbb{R}^n\}$. The data u_0 are frontlike, if

$$\limsup_{k\cdot x \to -\infty} u_0(x) < \mu \text{ and } \limsup_{k\cdot x\to +\infty} u_0(x) > \mu$$

uniformly in S. Assume that a traveling wave solution $(U, c(k))$ exists. Then there exist smooth function $\xi_i = \xi_i(t)$ and $q_i = q_i(t)$, $i = 1, 2$, such that

$$U(k \cdot x - c(k)t - \xi_1(t), x) - q_1(t) \leq u(t, x) \leq U(k \cdot x - c(k)t + \xi_2(t), x) + q_2(t)$$

for $(t, x) \in \mathbb{R}^{n+1}$. The functions ξ_i and q_i satisfy for $i = 1, 2$:

$$\begin{aligned} \xi_i' > 0, \qquad \xi_i > 0, \qquad \sup_{t>0} |\xi_i(t)| < +\infty, \\ q_i > 0, \qquad q_i' \leq 0, \qquad q_i(t) \leq Ce^{-\gamma t} \end{aligned}$$

for some $\gamma > 0$. The results for pulselike data are similar but with a pair of traveling wave solutions U_- and U_+ going in opposite directions. The results for combustion nonlinearities are also similar. For the precise results, we refer to the article [19].

We want to mention a homogenization result by Heinze [13]. For $\varepsilon > 0$ he considers the equation

$$\partial_t u = \nabla \cdot (A^\varepsilon(x)\nabla u) + \frac{1}{\varepsilon} b^\varepsilon(x)\nabla u + f(u) \text{ for } (t, x) \in \mathbb{R}^{n+1}. \tag{1.14}$$

It is assumed that $A^\varepsilon(x) = A(\frac{x}{\varepsilon})$ with a matrix field $A \in C^1(\mathbb{R}^n; \mathbb{R}^{n\times n})$, which is symmetric, elliptic and 1-periodic in every component of x. Furthermore, it is assumed that $b^\varepsilon(x) = b(\frac{x}{\varepsilon})$ with a vector field $b \in C^1(\mathbb{R}^n; \mathbb{R}^n)$, which is 1-periodic in every component, divergence free and has zero mean over T^n. The nonlinearity f is supposed to be a combustion nonlinearity as used in Xin [18]. The existence of a traveling wave solution in this setting has been established by Xin in [18] as described above. Heinze reasons that by the assumptions on b, there exists a skewsymmetric 1-periodic matrix B, such that with $B^\varepsilon(y) = B(\frac{y}{\varepsilon})$, there holds $\nabla \cdot (B^\varepsilon \nabla u) = \frac{1}{\varepsilon} b^\varepsilon \nabla u$. He defines $\tilde{A}^\varepsilon := A^\varepsilon + B^\varepsilon$, such that (1.14) becomes

$$\partial_t u = \nabla \cdot (\tilde{A}^\varepsilon(x)\nabla u) + f(u). \tag{1.15}$$

The traveling wave solution $(U^\varepsilon, c^\varepsilon)$ is given in moving frame coordinates $s = k \cdot x + c^\varepsilon t$, $z = \frac{x}{\varepsilon}$, where $c^\varepsilon > 0$ is the wavespeed of the traveling wave solution. The function $U^\varepsilon =$

$U^\varepsilon(s,z)$ is 1-periodic in every component of z and normalized by $\max_{z\in T^n} U(0,z) = \theta$. Note that with these moving frame coordinates, the sign of the wavespeed is reversed in comparison to the moving frame ansatz (1.10). With $\nabla^\varepsilon := \frac{1}{\varepsilon}\nabla_z + k\partial_s$ and $\tilde{A}(z) := \tilde{A}^\varepsilon(\varepsilon z)$, the equation for U^ε reads

$$\nabla^\varepsilon \cdot (\tilde{A}\nabla^\varepsilon U^\varepsilon) - c^\varepsilon \partial_s U^\varepsilon + f(U^\varepsilon) = 0. \tag{1.16}$$

Heinze proves that c^ε converges to some $c > 0$ and that u^ε converges weakly in $H^1(\mathbb{R}\times T^n)$ and strongly in $L^2(\mathbb{R}\times T^n)$ to a function $U \in H^1(\mathbb{R}\times T^n)$ with $U(0) = \theta$. With the homogenized matrix A^h, the pair (U,c) solves the traveling wave problem

$$\begin{aligned} &k^T A^h k U_{ss} - cu_s + f(U) = 0,\\ &U(-\infty) = 0, \qquad U(\infty) = 1. \end{aligned}$$

Organization of this thesis

We are interested in generalizing the results of [21] (see the description regarding (1.12)) to the case of an x-dependent combustion type nonlinearity. That is, we consider the equation

$$u_t = \sum_{i,j=1}^n a_{ij}(x) u_{x_i x_j} + f(x,u). \tag{1.17}$$

Under appropriate assumptions, which include the case that f is a combustion nonlinearity as used in [21], we find the existence of a traveling wave solution in (t,x)-coordinates with $0 < u < 1$, $u_t > 0$ and $c < 0$. For additional regularity assumptions, the solution can be transformed into a solution in moving frame coordinates. That is, the function $U(s,y) = U(k\cdot x - ct, x) = u(t,x)$ satisfies the equation

$$\sum_{i,j=1}^n a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})U + cU_s + f(y,U) = 0.$$

Under a monotonicity assumption on $f = f(x,z)$ near $z = 1$, we obtain the monotonicity for any traveling wave solution in t and the uniqueness of the traveling wave solution (u,c) up to a constant shift of u in t. We remark, that the condition $c < 0$ will be part of our definition of a traveling wave solution as opposed to Xin's definition. In order to obtain monotonicity and uniqueness of the traveling wave solution to (1.12), Xin assumes $g'(1) < 0$. Our assumptions on f for the monotonicity and uniqueness are similar, but slightly weaker even in the case that f does not depend on x. This will be achieved by a more precise use of a two-sided maximum principle.

We obtain the existence of a traveling wave solution for a class of nonlinearities f, that may depend explicitly on x. The case of a combustion nonlinearity as used in [21] is included in this class. In [21], Xin claims the existence of a traveling wave solution of (1.12) in moving frame coordinates, which is a classical solution of

$$\sum_{i,j=1}^n a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})U + cU_s + g(U) = 0, \tag{1.18}$$

where g is a $C^1[0,1]$-combustion nonlinearity. In this form of the equation, a higher regularity is required for a classical solution. This is because the equation in moving frame coordinates includes second order s-derivatives. Since $u_t = -cU_s$ and $u_{tt} = c^2U_{ss}$, this involves second order time derivatives. However, we believe that the existence of second order time derivatives cannot be expected for $g \in C^1$. We think that slightly more regularity for f is needed in order to prove Xin's existence result. Unfortunately, Xin does not give a proof of the regularity of his solution. We will provide a rigorous proof of the regularity of our traveling wave solution in (t,x)-coordinates in the regularity case $f \in C^1(T^n \times [0,1])$. Under slightly better regularity assumptions on f we will also prove the required regularity for the solution in moving frame coordinates.

We provide precise regularity assumptions on the matrix field $(a_{ij})_{i,j=1}^n$. For the existence of a traveling wave solution, we require $(a_{ij})_{i,j=1}^n$ to be $C^1(T^n, \mathbb{R}^{n\times n})$. For the monotonicity and uniqueness, the regularity $(a_{ij})_{i,j=1}^n \in C^0(T^n, \mathbb{R}^{n\times n})$ is sufficient. In [21], Xin gives no precise regularity assumptions on the matrix field $(a_{ij})_{i,j=1}^n$. The argumentation in his existence proof for traveling wave solutions of (1.12) involves second order derivatives of a_{ij}. Therefore, more regularity for the matrix field is needed in Xin's argumentation than for our result.

In [21], Xin uses an elliptic regularization method to prove the existence of a traveling wave solution. The same method will also be used in this thesis. After solving the regularized problem, a priori estimates are needed to pass from the regularized equation to the original equation. Xin uses a priori estimates, which he did not prove and which we consider unlikely to hold. We were unable to prove the estimates that Xin uses. Instead, we prove weaker estimates which are still sufficient to pass to the limit. These estimates are proved with the help of a theorem for elliptic regularization of Berestycki and Hamel [4]. This theorem requires higher regularity assumptions than we have. We solve this problem with an approximation of the coefficients and the nonlinearity f.

In [21], there arises a situation, where for a solution U of (1.18) the inequality $U_s \geq 0$ is known and $U_s > 0$ has to be proved. The analogous situation also arises for the regularized problem. Xin differentiates (1.18) with respect to s and similarly for the regularized equation. He then applies the maximum principle to obtain $U_s > 0$. However, it is not completely clear which maximum principle is meant, since the differentiated equation is not necessarily solved in the classical sense. The same situations also arise in our work for the function u. We developed a new use of Harnack's inequality, which is applied to a sequence of difference quotients, in order to obtain $u_t > 0$.

In both [21] and this thesis, the Leray-Schauder degree is used to solve the regularized problem. Some of the relevant properties of the mappings that it is applied to are used without proof in [21]. In this thesis, we will rigorously prove the precise setting for the Leray-Schauder argument in order to obtain the regularized solution.

Perhaps the main idea of this thesis is the following: Consider the regularized equation

$$\varepsilon\alpha(y)U_{ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})U + cU_s + f(y,U) = 0.$$

The regularized problem for a pair (U, c) consists of this equation, boundary conditions at $\partial((-b, b) \times T^n)$ and further conditions. After solving this problem, one has to pass to the limit $\varepsilon \to 0$ and $b \to \infty$. This requires a priori estimates not only for U, but also for c. The estimates are of the form $c_1 < c = c^{\varepsilon,b} < c_2 < 0$ for small ε and large b. In order to obtain the estimates $c^{\varepsilon,b} < c_2$, it is not sufficient to solve the regularized problem for the nonlinearity f, but one has to solve also for a different nonlinearity g. Then the regularized solution for g can be used as a comparison. This helps to weaken the necessary assumptions on f for the existence of the traveling wave solution.

This thesis is organized as follows: In chapter 2, we give the basic definitions and present our main results. Furthermore, we will discuss the work [21] in more detail and continue to work out some of the differences of our work and [21]. In chapter 3, we describe the plan of the proof. Chapter 4 contains the treatment of the regularized problem. This is the first half of the existence proof for the traveling wave solution. In chapter 5, we continue the existence proof with the analysis of the limits $\varepsilon \to 0$, $b \to \infty$. Chapter 6 contains our proof of monotonicity and uniqueness.

2. Main results

2.1. Setting of the problem and basic definitions

General assumptions and notation

Unless stated otherwise, we will always use the definitions and impose the assumptions that are described in this section.

When we speak of a domain, we always mean an open and connected set. The n-dimensional torus will be denoted by T^n. The set R_b is the cylinder

$$R_b \coloneqq [-b, b] \times T^n.$$

We say that a function is 1-periodic in y, if it is 1-periodic in every component of y. Oftentimes we will just express the 1-periodicity of a function in all or parts of its variables by giving it a domain of definition involving T^n.

By $A = A(x) = (a_{ij}(x))_{i,j=1}^n$ we will denote a $C^1(T^n)$-matrix field. It is assumed to be symmetric and uniformly elliptic, i.e. there exists $\mu > 0$ such that

$$\mu||\xi||^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \mu^{-1}||\xi||^2 \text{ for all } \xi \in \mathbb{R}^n \text{ and } x \in T^n.$$

By $k \in \mathbb{R}^n$ we will always denote a given unit vector (indicating a direction) and by $r(y)$ we will denote the scalar function

$$r(y) \coloneqq \sum_{i,j=1}^n a_{ij}(y)k_ik_j = k^T A(y)k.$$

We remark that we neglect the dependence of $r(y)$ on k in the notation, since k will be given for the problem. Moreover, let

$$r_{\min} = \min_{y\in T^n} r(y), \qquad r_{\max} = \max_{y\in T^n} r(y).$$

The $W_p^{2,1}$-spaces: Let $1 < p < \infty$ and Ω be a domain in $\mathbb{R}^{n+1}$, $u : \Omega \to \mathbb{R}$ be a function depending on $(t, x) = (t, x_1, ..., x_n)$. We say $u \in W_p^{2,1}(\Omega)$, if $u \in L^p(\Omega)$, $u_t \in L^p(\Omega)$, $u_{x_i} \in L^p(\Omega)$ for $i = 1, ..., n$ and $u_{x_ix_j} \in L^p(\Omega)$ for $i, j = 1, ..., n$. Furthermore, we say $u \in W_{p,loc}^{2,1}(\Omega)$, if for any domain $\Omega' \Subset \Omega$ we have $u \in W_p^{2,1}(\Omega')$. We always use $Du \coloneqq (u_{x_1}, ..., u_{x_n})$ and $D^2u \coloneqq (u_{x_ix_j})_{i,j=1}^n$.

The space $W_p^{2,1}(\Omega)$ becomes a reflexive Banach space with the norm

$$||u||^p_{W_p^{2,1}(\Omega)} := ||u||^p_{L^p(\Omega)} + ||u_t||^p_{L^p(\Omega)} + ||Du||^p_{L^p(\Omega)} + \left|\left|D^2 u\right|\right|^p_{L^p(\Omega)}.$$

Anisotropic Hölder spaces: The definition of the following spaces and norms are equivalent (but not identical) to those used in [16]. We use different symbols to represent them, so as to distinguish them more clearly from the corresponding isotropic spaces and norms. For $(t,x),(\tilde{t},\tilde{x}) \in \mathbb{R} \times \mathbb{R}^n$, we write

$$d\left((t,x),(\tilde{t},\tilde{x})\right) := \max\left\{||x-\tilde{x}||, \left|t-\tilde{t}\right|^{\frac{1}{2}}\right\}$$

for the parabolic distance of $X=(t,x)$ and $\tilde{X}=(\tilde{t},\tilde{x})$. Let $\alpha \in (0,1)$, Ω be a domain in $\mathbb{R}^{n+1}$ and $f:\Omega \to \mathbb{R}$ a function. We write $f \in C^{\alpha}_{\alpha/2}(\overline{\Omega})$, if the following Hölder seminorm is finite:

$$[f]_{C^{\alpha}_{\alpha/2}(\overline{\Omega})} := \sup_{X,\tilde{X}\in\Omega, X\neq\tilde{X}} \frac{\left|f(X)-f(\tilde{X})\right|}{d(X,\tilde{X})^{\alpha}}.$$

Now we define a norm on $C^{\alpha}_{\alpha/2}(\overline{\Omega})$ by

$$||f||_{C^{\alpha}_{\alpha/2}(\overline{\Omega})} := [f]_{C^{\alpha}_{\alpha/2}(\overline{\Omega})} + ||f||_{C^0(\overline{\Omega})}.$$

Moreover, we define $C^2_1(\overline{\Omega})$ as the set of functions $f:\Omega\to\mathbb{R}$, such that f is uniformly continuous on Ω and the derivatives f_t, f_{x_i} and $f_{x_ix_j}$ exist for $i,j=1,...,n$ on Ω and are uniformly continuous in Ω. Furthermore, we define

$$C^{2,\alpha}_{1,\alpha/2}(\overline{\Omega}) := \left\{f \in C^2_1(\overline{\Omega}) : f, f_t, f_{x_i}, f_{x_ix_j} \in C^{\alpha}_{\alpha/2}(\overline{\Omega}) \text{ for } i,j=1,...,n\right\}.$$

A norm on $C^{2,\alpha}_{1,\alpha/2}(\overline{\Omega})$ is given by

$$||f||_{C^{2,\alpha}_{1,\alpha/2}(\overline{\Omega})} := ||f||_{C^{\alpha}_{\alpha/2}(\overline{\Omega})} + ||f_t||_{C^{\alpha}_{\alpha/2}(\overline{\Omega})} + \sum_{i=1}^{n} ||f_{x_i}||_{C^{\alpha}_{\alpha/2}(\overline{\Omega})} + \sum_{i,j=1}^{n} \left|\left|f_{x_ix_j}\right|\right|_{C^{\alpha}_{\alpha/2}(\overline{\Omega})}.$$

Isotropic Hölder spaces: Let $\alpha \in (0,1)$ be a real number, Ω be a domain in $\mathbb{R}^n$ and $f:\Omega\to\mathbb{R}$ be a continuous function. We say $f \in C^{\alpha}(\overline{\Omega})$, if the following Hölder semi-norm is finite:

$$[f]_{C^{\alpha}(\overline{\Omega})} := \sup_{x,\tilde{x}\in\Omega, x\neq\tilde{x}} \frac{|f(x)-f(\tilde{x})|}{|x-\tilde{x}|^{\alpha}}.$$

A norm on $C^{\alpha}(\overline{\Omega})$ is defined by

$$||f||_{C^{\alpha}(\overline{\Omega})} := [f]_{C^{\alpha}(\overline{\Omega})} + ||f||_{C^0(\overline{\Omega})}.$$

For $k \in \mathbb{N}_0$ we say $f \in C^{k,\alpha}(\overline{\Omega})$, if for all multiindices β with length $|\beta| \leq k$, we have $D^{\beta} f \in C^{\alpha}(\overline{\Omega})$. A norm on $C^{k,\alpha}(\overline{\Omega})$ is given by

$$||f||_{C^{k,\alpha}(\overline{\Omega})} := \sum_{|\beta|\leq k} \left|\left|D^{\beta} f\right|\right|_{C^{\alpha}(\overline{\Omega})}.$$

Definition of traveling wave solutions

Let $f : T^n \times [0,1] \to \mathbb{R}$ be an arbitrary function which satisfies $f(x,0) = f(x,1) = 0$ for every $x \in T^n$. We consider the equation

$$u_t(t,x) - \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j}(t,x) = f(x, u(t,x)) \text{ for } (t,x) \in \mathbb{R}^{n+1}. \tag{2.1}$$

It has stationary states 0 and 1 by the assumptions on f, i.e. $u \equiv 0$ and $u \equiv 1$ are solutions.

For a given unit vector $k \in \mathbb{R}^n$ and an unknown $c \in \mathbb{R}$, we introduce the moving frame coordinates $s = k \cdot x - ct$, $y = x$. The vector k is indicating a direction. The real number $c \in \mathbb{R}$ takes the role of an unknown constant, which we call the wavespeed. In these coordinates, we search for a solution $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ of (2.1) that can be written as $u(t,x) = U(s,y)$, where $U(s,\cdot)$ is 1-periodic and satisfies the assumptions $U(-\infty,\cdot) = 0$ and $U(+\infty,\cdot) = 1$ and $0 \le U \le 1$. If $c = 0$, then u is a stationary solution. Otherwise, the periodicity of U in y can also be expressed by $u(t,x) = u\left(t + \frac{k \cdot e_i}{c}, x + e_i\right)$ for the unit vectors e_i, $i = 1, ..., n$.

Suppose for a moment, a solution u of this form is twice continuously differentiable in all variables. Then U will be C^2 as well and satisfy the equation

$$\sum_{i,j=1}^{n} a_{ij}(y)(k_i \partial_s + \partial_{y_i})(k_j \partial_s + \partial_{y_j}) U + cU_s + f(y,U) = 0. \tag{2.2}$$

However, solutions of parabolic equations usually have anisotropic regularity properties. In particular, u does not need to have a second time-derivative to be a classical solution of (2.1), but U needs to have a second s-derivative to be a classical solution of (2.2). Since $U_s = u_t$, it requires more regularity for U to be a solution of (2.2) than for u to be a solution of (2.1). We arrive at two possible definitions for traveling wave solutions.

Definition 2.1 (Traveling wave solution in original coordinates, "type I") *A pair (u,c) with a function $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and a real number $c < 0$ is called a traveling wave solution with wavespeed c of the equation (2.1), if u is a classical solution of*

$$-u_t + \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + f(x,u) = 0 \tag{2.3}$$

$$\left.\begin{aligned} &\lim_{t \to -\infty} u(t,x) = 0 \ \textit{and} \ \lim_{t \to \infty} u(t,x) = 1 \ \textit{locally uniformly in } x \in \mathbb{R}, \\ &u(t,x) = u\left(t + \frac{k \cdot e_i}{c}, x + e_i\right) \ \textit{for the unit vectors } e_i,\ i = 1, ..., n \\ &\qquad\qquad \textit{and } 0 \le u \le 1. \end{aligned}\right\} \tag{2.4}$$

Definition 2.2 (Traveling wave solution in transformed variables, "type II") *A pair (U,c) with a function $U:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ and a number $c<0$ is called traveling wave solution with wavespeed c of the equation* (2.1)*, if U is a classical solution of*

$$\sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s+\partial_{y_i})(k_j\partial_s+\partial_{y_j})U+cU_s+f(y,U)=0, \tag{2.5}$$

$$\left.\begin{array}{c} \lim\limits_{s\to-\infty}U(s,y)=0 \text{ and } \lim\limits_{s\to\infty}U(s,y)=1 \text{ uniformly in } y\in\mathbb{R}, \\ U(s,\cdot) \text{ is 1-periodic} \qquad \text{and } 0\le U\le 1. \end{array}\right\} \tag{2.6}$$

Remark 2.3 *Because of the periodicity of a traveling wave solution U of type II in y, we can also regard U as a function $U:\mathbb{R}\times T^n\to[0,1]$. This has the advantage that T^n is compact and yet has no boundary. We shall use whatever interpretation seems fit at the appropriate moment without mentioning. Also note that every traveling wave solution of type II with a negative wavespeed c can be transformed to a traveling wave solution of type I by reversing the initial transformation, that is by putting $t=\frac{s-k\cdot y}{-c}$ and $x=y$. However a traveling wave solution of type I does not necessarily have enough regularity to be transformed in a traveling wave solution of type II.*

On our way to the final result, we will have to consider a different type of problem, a regularization of our original problem. We will call it the elliptic regularization problem or the (ε,b)-problem.

Definition 2.4 (Elliptic regularization problem or (ε,b)-problem) *Let $f:\mathbb{R}^n\times[0,1]\to\mathbb{R}$ be a function. We assume that $f(\cdot,z)$ is 1-periodic in every component for every $z\in[0,1]$ and that $f(\cdot,0)=f(\cdot,1)=0$. Moreover, let $\alpha\in C^1(T^n)$, $\alpha>0$ and $\theta\in(0,1)$. Let $b>0$ and $\varepsilon>0$. Consider the following problem for a pair (U,c) with a function $U:[-b,b]\times\mathbb{R}^n\to\mathbb{R}$ and a real number $c\in\mathbb{R}$:*

$$\varepsilon\alpha(y)U_{ss}+\sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s+\partial_{y_i})(k_j\partial_s+\partial_{y_j})U+cU_s+f(y,U)=0, \tag{2.7}$$

$$\begin{cases} 0\le U\le 1, \qquad U(s,\cdot) \text{ is } 1-\text{periodic}, \\ U(-b,y)=0, \qquad U(b,y)=1 \text{ for all } y\in T^n, \\ \max\limits_{y\in T^n}U(0,y)=\theta. \end{cases} \tag{2.8}$$

This problem is an elliptic regularization of the problem (2.5), (2.6)*. We refer to it as the (ε,b)-problem or regularized problem. A pair $(U^{\varepsilon,b},c^{\varepsilon,b})=(U,c)$, which is a classical solution of the (ε,b)-problem, is called the (ε,b)-solution or regularized solution. We call $c=c^{\varepsilon,b}$ the wavespeed of the (ε,b)-solution. The constant θ is called the normalization constant. A solution to the problem is said to be normalized at θ.*

Remark 2.5 (Ellipticity of the (ε, b)-problem) *As the name suggests, the elliptic regularization problem or (ε, b)-problem is indeed an elliptic problem. We will demonstrate this in what follows. Consider the second order part of the operator on the left hand side of (2.7). It can be written as*

$$LU := \varepsilon\alpha(y)U_{ss} + \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}(y)k_i U_{sy_j} + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(y)k_j U_{y_i s} + \sum_{i,j=1}^{n} a_{ij}(y)U_{y_i y_j}.$$

We define the $(n+1)\times(n+1)$-matrix $B = B(y) = (b_{ij})_{i,j=0}^n$ by $b_{00} = \varepsilon\alpha$ and $b_{ij} = a_{ij}$ for $i, j = 1, ..., n$, as well as $b_{0j} = \sum_{i=1}^n a_{ij}(y)k_i$ for $j = 1, ..., n$ and $b_{i0} = \sum_{j=1}^n a_{ij}(y)k_j$ for $i = 1, ..., n$. With $A = (a_{ij})_{i,j=1}^n$, this reads in matrix notation:

$$B = \left(\begin{array}{c|c} \varepsilon\alpha + k^T Ak & k^T A \\ \hline Ak & A \end{array}\right).$$

With $(z_0, z_1, ..., z_n) := (s, y_1, ..., y_n)$, the operator can be written as

$$LU = \sum_{i,j=0}^{n} b_{ij}(z)U_{z_i z_j}.$$

We show that the matrix B is elliptic: Let $\xi = (\xi_0, \tilde{\xi}) = (\xi_0, \xi_1, ..., \xi_n) \in \mathbb{R}^{n+1}$ and μ be the uniform ellipticity constant of $A = (a_{ij})_{i,j=1}^n$. Due to Young's inequality, there holds $(a-b)^2 = a^2 - 2ab + b^2 \geq a^2 - \frac{a^2}{2} - 4b^2 + b^2 = \frac{a^2}{2} - 3b^2$. Using this and $|k| = 1$, we calculate

$$\begin{aligned}
\xi^T B\xi &= \xi_0^2(\varepsilon\alpha + k^T Ak) + \xi_0 k^T A\tilde{\xi} + \xi_0\tilde{\xi}^T Ak + \tilde{\xi}^T A\tilde{\xi} \\
&= \varepsilon\alpha\xi_0^2 + (\xi_0 k + \tilde{\xi})^T A(\xi_0 k + \tilde{\xi}) \geq \varepsilon\alpha\xi_0^2 + \mu\left|\xi_0 k + \tilde{\xi}\right|^2 \\
&\geq \min\{\varepsilon\alpha, \mu\}\xi_0^2 + \frac{1}{6}\min\{\varepsilon\alpha, \mu\}\left|\left|\tilde{\xi}\right| - |\xi_0 k|\right|^2 \\
&\geq \min\{\varepsilon\alpha, \mu\}\xi_0^2 + \frac{1}{6}\min\{\varepsilon\alpha, \mu\}\left(\frac{1}{2}\left|\tilde{\xi}\right|^2 - 3|\xi_0|^2\right) \\
&\geq \frac{1}{12}\min\{\varepsilon\alpha, \mu\}|\xi|^2.
\end{aligned}$$

Since $\min\{\varepsilon\alpha, \mu\} = \min\{\varepsilon\alpha(y), \mu\}$ is bounded away from 0, the ellipticity of the operator L follows.

Nonlinearities

We want to generalize the results about Xin for existence and uniqueness of traveling wave solutions in the case of periodic coefficients and a combustion nonlinearity to a more general type of nonlinearities. To this end, we now introduce several types of nonlinearities. We will comment on why we introduced these different types of nonlinearities after our plan of proof.

Definition 2.6 (Nonlinearity of basic type) *A function $f \in C^{0,1}(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ is called a nonlinearity of basic type, if:*

(i) $f(x,z) = 0$ *for all* $z \notin (0,1)$, $x \in \mathbb{R}^n$

(ii) $f(\cdot, z)$ *is* 1*-periodic in every component of* x *for every* $z \in [0,1]$.

(iii) There is $\theta \in (0,1)$ *such that* $f(\cdot,z) \le 0$ *for* $z \le \theta$ *and* $f(\cdot,z) \ge 0$ *for* $z \ge \theta$.

We define

$$\theta(f) := \sup\{\theta \in (0,1) : f(\cdot,z) \le 0 \text{ for } z \le \theta \text{ and } f(\cdot,z) \ge 0 \text{ for } z \ge \theta\} \in (0,1].$$

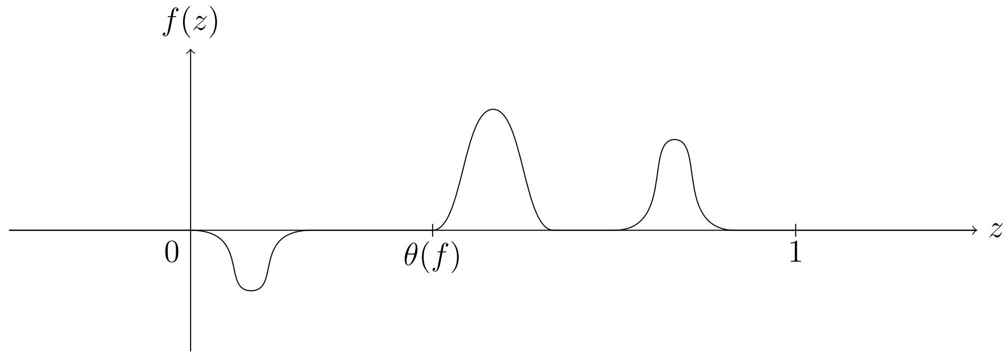


Figure 2.1.: An x-independent nonlinearity of basic type.

We will introduce several properties that nonlinearities of basic types can additionally have. Some of them refer to the (ε, b)-problem.

Definition 2.7 (Properties of nonlinearities) *Let f be a nonlinearity of basic type.*

Covering Property. *We say that f has the covering property, if*

(1) $\theta(f) \in (0,1)$,

(2) for every $z \in (\theta(f), 1)$, *there is* $x \in T^n$, *such that* $f(x,z) > 0$.

Strong Covering Property. *We say that f has the strong covering property if f has the covering property and there exists $z_0 \in (\theta(f), 1)$ with $f(x, z_0) > 0$ for all $x \in T^n$.*

Negative Wavespeed Property. *We say f has the negative wavespeed property (with respect to the matrix field $(a_{ij})_{i,j=1}^n$ and the unit vector k), if there exist constants $b_0 > 0$, $\varepsilon_0 > 0$ and $c_0 < 0$, such that for $b \ge b_0$, $0 < \varepsilon \le \varepsilon_0$, every solution $(U^{\varepsilon,b}, c^{\varepsilon,b})$ of* (2.7), (2.8) *satisfies $c \le c_0$.*

Combustion Type. *We say that f is of combustion type, if $f \ge 0$.*

In all the existence results, f is supposed to have the covering property. Nevertheless, it will be important to consider the (ε, b)-problem for nonlinearities without this property. The reason is that the (ε, b)-solutions in case of the nonlinearities without covering property are needed as comparison solutions. With these comparison solutions, we can then estimate the wavespeed in the case of nonlinearities with the strong covering property.

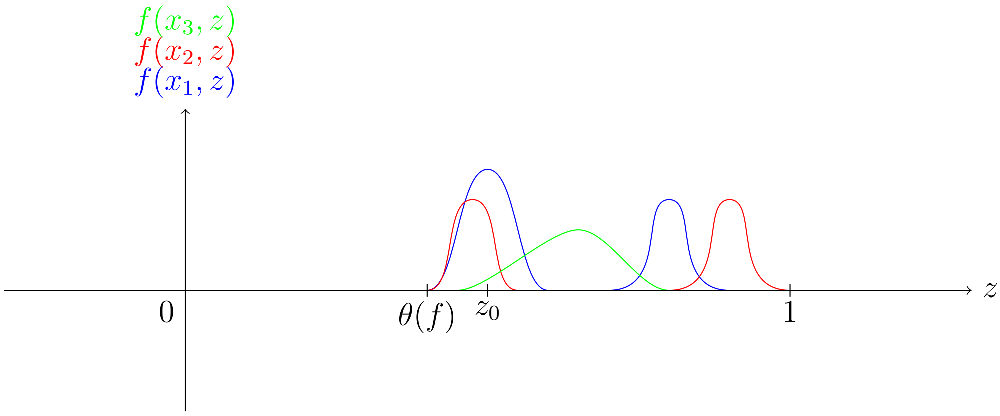


Figure 2.2.: A nonlinearity of combustion type with the strong covering property.

2.2. Main results

Our first result is an existence result under an abstract assumption on f.

Theorem 2.8 (An abstract existence result) *Let $(a_{ij})_{i,j=1}^n$ be a matrix field and k be a unit vector as in the general assumptions. Furthermore, let f be a nonlinearity of combustion type with $f|_{T^n\times[0,1]} \in C^1(T^n \times [0,1])$, which has the covering property and the negative wavespeed property (with respect to the matrix field and the vector k). Then there is a traveling wave solution (u, c) of type I as defined in definition 2.1 with $c < 0$. Moreover, the strict inequalities $0 < u < 1$ and $u_t > 0$ hold. Additionally, the regularity $u_t \in W^{2,1}_{p,loc}(\mathbb{R}^{n+1})$ holds for any $2 \le p < \infty$.*

As a corollary, we will get the following result. We conclude it by showing that nonlinearities of combustion type with the strong covering property have the negative wavespeed property.

Theorem 2.9 (Existence of traveling wave solutions of type I) *Let $(a_{ij})_{i,j=1}^n$ be a matrix field and k be a unit vector as in the general assumptions. Furthermore, let f be a nonlinearity of combustion type with $f|_{T^n\times[0,1]} \in C^1(T^n\times[0,1])$, which has the strong covering property. Then there is a traveling wave solution (u, c) of type I as defined in definition 2.1 with $c < 0$. Moreover, the strict inequalities $0 < u < 1$ and $u_t > 0$ hold. Additionally, we have $u_t \in W^{2,1}_{p,loc}(\mathbb{R}^{n+1})$ for any $2 \le p < \infty$.*

Under additional regularity assumptions on f, we can prove that the solution of type I has enough regularity to be transformed into a solution of type II. See remark 2.20.

Theorem 2.10 (Existence of traveling wave solutions of type II) *Let $(a_{ij})_{i,j=1}^n$ be a matrix field and k be a unit vector as in the general assumptions. Furthermore, let f be a nonlinearity of combustion type with $f|_{T^n\times[0,1]} \in C^1(T^n \times [0,1])$, which has the negative wavespeed property (for example because it has the strong covering property). If moreover $f_z \in C^{\alpha_0}(T^n \times [0,1])$ for some $\alpha_0 \in (0,1)$, then there is a traveling wave solution (U,c) of type II as defined in 2.2 with $c < 0$. Moreover, the strict inequalities $0 < U < 1$ and $U_s > 0$ hold.*

Theorems 2.8, 2.9 and 2.10 should be compared to theorem 0.1 of [21]. For the comparison, see the comments in remark 2.20 below.

For the monotonicity and uniqueness results, we require less regularity for the matrix field than in the general assumptions:

Theorem 2.11 (Monotonicity) *Let $(a_{ij})_{i,j=1}^n$ be a $C^0(T^n)$-matrix field, which is symmetric and uniformly elliptic. Furthermore, let f be a nonlinearity of combustion type with $f|_{T^n\times[0,1]} \in C^1(T^n \times [0,1])$. Suppose that there is $\varepsilon > 0$ such that*

$$f_z(x,z) \leq 0 \textit{ for all } (x,z) \in T^n \times [1-\varepsilon, 1].$$

Let (u,c) with $c < 0$ be a traveling wave solution of (2.3), (2.4) as in definition 2.1. Then u is increasing in t and the strict inequalities $0 < u < 1$ and $u_t > 0$ hold.

Theorem 2.12 (Uniqueness) *Let $(a_{ij})_{i,j=1}^n$ and f be as in theorem 2.11. Furthermore let (u,c) and (u',c') with $c,c' < 0$ be two traveling wave solutions of (2.3), (2.4) as in definition 2.1. Then $c = c'$ and there is some $t_0 \in \mathbb{R}$ such that $u(t+t_0,x) = u'(t,x)$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$.*

Theorems 2.11 and 2.12 should be compared to theorems 0.2 and 0.3 of [21]. For comments, see remark 2.23.

2.3. A critical discussion of the results of Xin

In this section we will describe the work of Xin [21]. Xin considers the reaction-diffusion equation in nondivergence form:

$$u_t = \sum_{i,j=1}^n a_{ij}(x)u_{x_ix_j} + g(u) \text{ on } \mathbb{R}^{n+1}. \tag{2.9}$$

He assumes that $(a_{ij}(x))_{i,j=1}^n$ is a positive definite matrix, smooth and 1-periodic in each component. Moreover, g is a combustion nonlinearity. By his definition, that means $g \in C^1([0,1],\mathbb{R})$ with $g \equiv 0$ in $[0,\theta]$ for some $\theta = \theta(g) \in (0,1)$ and $g > 0$ in $(\theta,1)$, as well as $g(1) = 0$. For a given unit vector k and an unknown constant c, Xin is looking for a solution of the form $U(s, y) = U(k \cdot x - ct, x)$, which satisfies $U(-\infty, y) = 0$, $U(+\infty, y) = 1$ and $U(s, \cdot)$ is 1-periodic in every component of y. In (s, y)-coordinates, the equation for U becomes

$$\sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})U + cU_s + g(U) = 0 \text{ on } \mathbb{R}^{n+1}, \tag{2.10}$$

$$0 \leq U \leq 1, \qquad U(-\infty, y) = 0, \qquad U(+\infty, y) = 1, \qquad U(s, \cdot) \text{ 1-periodic.} \tag{2.11}$$

2.3.1. Xin's main results

The main results of Xin can be collected in the following three theorems:

Theorem 2.13 (Existence) *The problem* (2.10), (2.11) *with a combustion nonlinearity g has a classical solution (U, c), which additionally satisfies*

$$\begin{aligned} 0 < U < 1 & \qquad \textit{for all } (s, y) \in \mathbb{R} \times T^n \\ U_s(s, y) > 0 & \qquad \textit{for all } (s, y) \in \mathbb{R} \times T^n \\ c < 0. & \end{aligned}$$

Theorem 2.14 (Monotonicity) *If the combustion nonlinearity g satisfies $g'(1) < 0$ and (U, c) is a classical solution of the problem* (2.10), (2.11), *then $U_s(s, y) > 0$ for all $(s, y) \in \mathbb{R} \times T^n$.*

Theorem 2.15 (Uniqueness) *If the combustion nonlinearity g satisfies $g'(1) < 0$ and (U, c), (U', c') are two classical solutions of the problem* (2.10), (2.11), *then $c = c'$ and there is $s_0 \in \mathbb{R}$, such that $U(s, y) = U'(s + s_0, y)$ for all $(s, y) \in \mathbb{R} \times T^n$.*

2.3.2. Xin's proof of existence

In order to prove theorem 2.13, Xin begins with an elliptic regularization. Consider the weight factor

$$r(y) := \sum_{i,j=1}^{n} a_{ij}(y)k_ik_j \tag{2.12}$$

and the linear elliptic operator

$$L_\varepsilon U := \varepsilon r(y)U_{ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})U.$$

Then the elliptic regularization problem for (U, c) (we like to call it the (ε, b)-problem) is the following problem:

$$L_\varepsilon U + cU_s + g(U) = 0 \text{ on } (-b, b) \times \mathbb{R}^n, \tag{2.13}$$

$$\left.\begin{array}{l} 0 \leq U \leq 1, \qquad U(s,\cdot) \text{ is } 1-\textit{periodic} \\ U(-b,y)=0, \qquad U(b,y)=1 \text{ for all } y \in T^n \\ \max\limits_{y\in T^n} U(0,y)=\theta(g). \end{array}\right\} \tag{2.14}$$

The condition $\max_{y\in T^n} U(0,y) = \theta(g)$ is called a normalization condition. The weight $r(y)$ in the ε-term makes some of the calculations easier.

Monotonicity for the (ε, b)-problem

Xin's treatment of the regularized problem begins with a monotonicity result: Any classical solution (U,c) of (2.13), (2.14) satisfies $U_s > 0$ on $(-b,b)\times T^n$.

Xin's proof can be summarized as follows:

(1) Xin's first step is to state that the strict inequality $0 < U < 1$ holds on $(-b,b)\times T^n$ by direct application of the maximum and minimum principle.

(2) He shows $U_s \geq 0$ with the help of the sliding domain method, which can be described as follows: For $\lambda > 0$, he considers the function $w(s,y,\lambda) = U(s+\lambda,y) - U(s-\lambda,y)$ for $(s,y)\in\overline{\Sigma_\lambda}$, where $\Sigma_\lambda = (-b+\lambda, b-\lambda)\times T^n$. It can be seen from (1), that $w(s,y,\lambda) > 0$ for $(s,y)\in\partial\Sigma_\lambda$. Then, the method proceeds in two steps.

(2a) In the first step, Xin shows that for λ close to b, there holds $w(s,y,\lambda) > 0$ for any $(s,y)\in\overline{\Sigma_\lambda}$. This can be shown using only the boundary conditions for U.

(2b) Xin defines

$$\lambda_0 = \inf\{\lambda > 0 | w(s,y,\lambda) > 0 \text{ for all } (s,y)\in\overline{\Sigma_\lambda}\}.$$

Then he proves $\lambda_0 = 0$ by a contradiction argument using the minimum principle and the fact that $w(s,y,\lambda) > 0$ on $\partial\Sigma_\lambda$ if $\lambda > 0$. From $\lambda_0 = 0$, the assertion $U_s \geq 0$ follows.

(3) The strict inequality $U_s > 0$ is then shown by differentiating (2.13) with respect to s and applying the minimum principle to U_s.

Remark 2.16 *In order to show $U < 1$ in step (1), one has to be a little bit more precise than in the proof of Xin. The reason is the following: If U assumes its maximum 1 at a point (s_0,y_0), then the term $g(U)$ in (2.13) does not have the correct sign to apply the maximum principle in a neighborhood of (s_0,y_0). But a simple linearization does the trick: $g(U) = g(U) - g(1) = \beta(s,y)(U-1)$ with a continuous and bounded β. The maximum principle can be applied to the function $U-1$ irrespectively of the sign of β, since the value of the attained maximum of $U-1$ is 0.*

Furthermore, in step (3) it cannot be expected, that U_s is still a classical solution of the differentiated equation. Therefore, the minimum principle for classical solutions cannot be applied here. One of multiple ways around this would be to apply a Harnack inequality to the sequence of difference quotients $\frac{U(s+h,y)-U(s,y)}{h}$, $h > 0$. See proposition 4.1.

Uniqueness of the (ε, b)-problem

Xin's work continues with a uniqueness result for classical solutions of (2.13), (2.14). Precisely, the result is the following: If there are two classical solution (U_1, c_1) and (U_2, c_2) of the problem, then $U_1 = U_2$ and $c_1 = c_2$.

In his proof, Xin uses a slight variant of the sliding domain method, which he used in step (2) of the proof of the monotonicity result for the (ε, b)-problem. He applies the sliding domain method to the function

$$w(s, y, \lambda) = U_1(s + \lambda, y) - U_2(s - \lambda, y).$$

Xin assumes that $c_1 > c_2$. By showing with the method that $U_1 = U_2$, he obtains a contradiction to $c_1 > c_2$. Consequently, there holds $c_1 \leq c_2$. The same argument shows $c_2 \leq c_1$. In the argumentation, the monotonicity $U_{1,s} > 0$ and $U_{2,s} > 0$ is used. After $c_1 = c_2$ has been established, the argumentation is repeated to show $U_1 = U_2$ directly.

A priori estimates for the existence of the (ε, b)-problem

In order to prove the existence of the (ε, b)-solution, Xin derives the following a priori estimates for the (ε, b)-problem.

Denote by R_b the cylinder $R_b = [-b, b] \times T^n$. Let

$$M := \sup_{y \in T^n, u \in [0,1]} \frac{g(u)}{r(y)} \qquad \text{and} \qquad R := \sup_{y \in T^n} r(y). \tag{2.15}$$

Then there is a constant $K = K(\varepsilon, b, M, R)$, such that any solution (U, c) of (2.13), (2.14) satisfies

$$|c| \leq K, ||U||_{C^1(R_b)} \leq K. \tag{2.16}$$

In the proof, Xin distinguishes two cases. In case $c > 0$, he compares U to the solution z of the problem

$$\begin{aligned} (1 + \varepsilon) z_{ss} + \frac{c}{R} z_s - M = 0, \\ z(-b) = 0, \qquad z(b) = 1. \end{aligned} \tag{2.17}$$

Using $U_s > 0$, Xin calculates $-L_\varepsilon (U - z)_{ss} - \frac{c}{R} r(y)(U - z)_s \geq 0$. From this, he deduces with the maximum principle $U \geq z$. In particular, there holds $\theta(f) = \max_{y \in T^n} U(0, y) \geq z(0)$. Xin calculates z explicitly. He obtains the formula

$$z(0) = \frac{MRb}{c} + (1 - e^{cb}(1 + \varepsilon)R) \frac{\frac{2MRb}{c} - 1}{e^{cb}(1 + \varepsilon)R - e^{-cb}(1 + \varepsilon)R}.$$

From this formula, he deduces $\lim_{c \to \infty} z(0) = 1$. Due to $u(0) \leq \theta(f) < 1$, there has to be an upper bound for c. The case $c < 0$ is treated analogously to find a lower bound for c.

Without proof, Xin claims that $0 \leq U \leq 1$ implies $||U||_{C^1(R_b)} \leq K$ by elliptic estimates.

Remark 2.17 *(i) The factor $r(y)$ in the operator L_ε in* (2.13) *is helpful in the calculation for $-L_\varepsilon(U-z)_{ss} - \frac{c}{R}r(y)(U-z)_s \geq 0$. The reason is that, when L_ε is applied to the y-independent function z, $r(y)$ factors out: $L_\varepsilon z = (\varepsilon+1)r(y)z_{ss}$. Otherwise the situation is slightly more complicated. See lemma 4.4*

(ii) Instead of (2.15),

$$M \geq \sup_{y\in T^n, u\in[0,1]} \frac{g(u)}{r(y)} \qquad \text{and} \qquad R \geq \sup_{y\in T^n} r(y)$$

is sufficient in order for the a priori estimates to hold, which follows from the proof of Xin. We think, it is actually necessary to have the a priori estimates in this form, because they are later applied to an entire family of problems with varying coefficients and nonlinearities. We follow up on this in remark 2.18 (*iii*).

(*iii*) *The constant K is used in the estimate $||U||_{C^1(R_b)} \leq K$. In our opinion, the constant K must therefore depend on the moduli of continuity of the second order coefficients and the ellipticity constant of L_ε. When applying the estimate to a family of problems, one then has to make sure that the family of second order coefficients admit a common modulus of continuity and ellipticity constant. See lemma 4.4. We follow up on this in remark 2.18* (*iii*).

Existence for the (ε, b)-problem

Xin's existence result for the (ε, b)-problem is the following: For every $\varepsilon > 0$ and $b > 0$, there exists a unique classical solution of (2.13), (2.14) on $R_b = [-b, b] \times T^n$.

Xin proves the existence with the Leray-Schauder degree: For $(v, c) \in C^1(R_b) \times \mathbb{R}$ and $\tau \in [0, 1]$, let $\phi_\tau(v, c)$ denote the unique solution U of the linear problem

$$\varepsilon[\tau r(y) + (1-\tau)]U_{ss} + \sum_{i,j=1}^{n} [\tau a_{ij}(y) + \delta_{ij}(1-\tau)](k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})U + cU_s = -\tau g(v), \tag{2.18}$$

$$U(-b, y) = 0, \qquad U(s, \cdot) \text{ 1-periodic}, \qquad U(b, y) = 1. \tag{2.19}$$

Xin defines

$$h_\tau(v, c) = \max_{s=0, y\in T^n} \phi_\tau(v, c)(s, y)$$

and the mapping

$$F_\tau(v, c) = (\phi_\tau(v, c), c - h_\tau(v, c) + \theta(g)).$$

A solution (u, c) of (2.13), (2.14) satisfies $u = \phi_1(u, c)$ and $h_1(u, c) = \theta(g)$. Vice versa, a pair (u, c) satisfying $u = \phi_1(u, c)$ and $h_1(u, c) = \theta(g)$ is a solution of (2.13), (2.14). Hence, for $(u, c) \in C^1(R_b) \times \mathbb{R}$, it is equivalent to be a solution of (2.13), (2.14) and to satisfy $F_1(u, c) = (u, c)$. Thus, the problem can be solved by finding a fixed point of F_1. The plan is now to find a domain D, such that for every $\tau \in [0, 1]$ the Leray-Schauder degree $d(id - F_\tau, D, 0)$ is well defined and $d(id - F_1, D, 0) = d(id - F_0, D, 0) = 1$ holds.

Without proof, Xin uses the continuity and compactness of the mapping

$$(\tau,(v,c)) \mapsto F_\tau(v,c), \qquad [0,1]\times C^1(R_b)\times\mathbb{R} \to C^1(R_b)\times\mathbb{R}.$$

He uses the a priori estimates (2.16) for the definition of the domain D:

$$D = \{(v,c) \in C^1(R_b)\times\mathbb{R} : ||v||_{C^1(R_b)} \le K, |c| \le K\}.$$

The constant K is chosen larger than in the a priori estimates (2.16). According to Xin, there holds $F_\tau(v,c) \ne (v,c)$ for all $(v,c) \in \partial D$ and $\tau \in [0,1]$. The reason he gives is that the a priori estimates only depend on

$$R_\tau = \sup_{y\in T^n}[\tau r(y) + (1-\tau)] \text{ and } M_\tau = \sup_{y\in T^n, z\in[0,1]} \frac{g(z)}{\tau r(y)+(1-\tau)}, \tag{2.20}$$

which have τ-independent bounds. Xin deduces that the Leray-Schauder degree is well-defined and by homotopy invariance

$$d(id - F_1, D, 0) = d(id - F_0, D, 0).$$

It therefore remains to calculate $d(id - F_0, D, 0)$.

Xin continues by demonstrating that for $\tau = 0$, the mapping $F_\tau(v,c) = F_0(v,c)$ takes a simpler form: Let $z_c = z_c(s)$ be the solution of

$$\begin{gathered}(1+\varepsilon)z_{c,ss} + cz_{c,s} = 0,\\ z_c(-b) = 0, \qquad z_c(b) = 1.\end{gathered}$$

If z_c is regarded as a function of (s,y), then z_c solves the linear problem (2.18), (2.19) for $\tau = 0$ and arbitrary $v \in C^1(R_b)$. Therefore, $z_c = \phi_0(v,c)$. Then $h_0(v,c) = z_c(0) =: h(c)$ for any $v \in C^1(R_b)$. Consequently, F_0 takes the simpler form

$$F_0(v,c) = (z_c, c - h(c) + \theta(g)).$$

The value $h(c) = z_c(0)$ can be explicitly calculated as

$$h(c) = \frac{1 - e^{ca/(1+\varepsilon)}}{e^{-ca/(1+\varepsilon)} - e^{ca/(1+\varepsilon)}}.$$

It can be seen that there is a unique $c_0 \in \mathbb{R}$ with $h(c_0) = \theta(g)$. Xin now argues that due to $h(K) > \theta(g)$ and $h(-K) < \theta(g)$, it follows that $d(h(\cdot) - \theta(g), (-K,K), 0) = 1$. (Actually Xin wrote $h(K) > 0$ and $h(-K) < 0$, but this must be a typing error.) The reason for $h(K) > \theta(g)$ and $h(-K) < \theta(g)$ is not given. However, it can be seen from the a priori estimates: In the situation $\tau = 0$, the constants in (2.15) in the proof of the a priori estimates are $M = 0$ and $R = 1$. Then the problem (2.17) is the same problem as the problem for z_c. Therefore $h(K) > \theta(g)$ and $h(-K) < \theta(g)$ follow from the proof of the a priori estimates.

Xin concludes that by the product property of the degree

$$d(id - F_0, D, 0) = d(id - z_{c_0}, ||u||_{C^1(R_b)} \le K, 0)\cdot d(h(\cdot) - \theta(g), (-K,K), 0) = 1 \cdot 1 = 1.$$

This finishes Xin's proof of existence of the (ε, b)-solution.

Remark 2.18 *(i) Xin is not specific as to how he wants to apply the product property of the degree at the end of the proof or which one of the product properties. There is a product property for cartesian products and one for the composition of functions, but we do not see how to apply either of them to get Xin's result directly. At least the mapping F_0 does not have the necessary structure for the product property for cartesian products, namely F_0 is not of the form $F_0(v,c) = (F_1(v), F_2(c))$. In our situation, we will prove $d(id - F_0, D, 0) = 1$ differently, see proposition 4.8.*

(ii) We prove the compactness of the mapping

$$(\tau, (v,c)) \mapsto F_\tau(v,c), \qquad [0,1] \times C^1(R_b) \times \mathbb{R} \to C^1(R_b) \times \mathbb{R}$$

in lemma 4.7.

(iii) We follow up on remark 2.17 (ii) and (iii): As stated in remark 2.17 (iii), we believe that the constant K in the a priori estimates (2.16) *has to depend on the moduli of continuity of the second order coefficients and the ellipticity constant of the operator. In the situation of the existence proof for the (ε, b)-problem, the operator on the left hand side of 2.18 has coefficients, that depend on τ. The a priori estimates are applied to the entire family of operators with $\tau \in [0,1]$. Therefore, one has to make sure, that the family of operators admits a τ-independent ellipticity constant and τ-independent moduli of continuity for the second order coefficients. This is possible, therefore there is no major problem in Xin's argumentation, only an inaccuracy.*

In the existence proof, Xin also reasoned that the estimates (2.16) *as applied in the proof, only depend on R_τ and M_τ as given in* (2.20) *and R_τ and M_τ can be bounded independently of τ. This argument is inaccurate as well, since it is unknown if the constant K in the a priori estimates depends continuously on R_τ and M_τ. The precise argument is to give the a priori estimates in the sense of remark 2.17 (ii). Then it can be seen that in the situation of the existence proof for the (ε, b)-problem, the a priori estimates do not depend on the individual R_τ and M_τ, but only on the τ-independent bounds for R_τ and M_τ.*

A priori estimates for $(U^{\varepsilon,b}, c^{\varepsilon,b})$

Before passing to the limit, Xin derives a priori estimates for the wavespeed of the (ε, b)-solution.

The exact result is as follows: Let $(U,c) = (U^{\varepsilon,b}, c^{\varepsilon,b})$ be the solution of (2.13), (2.14). There exist constants $c_1, c_2 < 0$ and $\varepsilon_0, b_0 > 0$, such that

$$c_1 < c^{\varepsilon,b} < c_2 < 0 \quad \text{for} \quad 0 < \varepsilon < \varepsilon_0 \text{ and } b \geq b_0. \tag{2.21}$$

Xin also gives estimates for $U^{\varepsilon,b}$. The result is:

$$||U^{\varepsilon,b}||_{C^1_{loc}} \leq C \tag{2.22}$$

for $0 < \varepsilon \leq \varepsilon_0$ and $b \geq b_0$. According to Xin, the result (2.22) is obtained from parabolic estimates, but he does not give a proof.

Xin's proof of the estimates (2.21) for the wavespeed is as follows: Xin uses a comparison principle for wavespeeds. He proves this principle with the sliding domain method similarly as in the proof of the uniqueness of the (ε, b)-solution. The comparison principle is applied as follows: With $r(y)$ as in (2.12), let $r_{\max} = \max_{y\in T^n} r(y)$ and $r_{\min} = \min_{y\in T^n} r(y)$. Furthermore, let (u_r, c_r) with $u_r : \mathbb{R} \to \mathbb{R}$ be the solution of

$$(1+\varepsilon)u_{r,ss} + \frac{c_r}{r_{\min}} u_{r,s} + \frac{1}{r_{\max}} g(u_r) = 0,$$

$$u_r(-b) = 0, \qquad u_r(0) = \theta(g), \qquad u_r(b) = 1.$$

Since u_r is y-independent, it also solves

$$L_\varepsilon u_r + \frac{c_r r(y)}{r_{\min}} u_{r,s} + \frac{r(y)}{r_{\max}} g(u_r) = 0.$$

There holds $\frac{r(y)}{r_{\min}} \geq 1$ and $\frac{r(y)}{r_{\max}} g(u) \leq g(u)$. For comparison, we recall that U satisfies the equation

$$L_\varepsilon U + cU_s + g(U) = 0.$$

The weight $\frac{r(y)}{r_{\min}}$ in the $c_r u_{r,s}$ term is therefore larger then the weight 1 in the cU_s term. With the nonlinearities $\frac{r(y)}{r_{\max}} g$ and g, the inequality is the opposite. In this situation, Xin's comparison principle implies $c^{\varepsilon,b} < c_r$ under the condition, that there holds $c_r < 0$.

According to Xin, it is known by a 1-D traveling wave result, that there exist constants $\varepsilon_0 > 0$, $b_0 > 0$ and $c_2 < 0$, such that $c_r < c_2 < 0$ holds for $0 < \varepsilon \leq \varepsilon_0$ and $b \geq b_0$. However, he does not give a source for that. Since c_r is negative, the above comparison holds. The other bound is proved analogously.

Remark 2.19 *We believe that none of the standard parabolic or elliptic estimates gives the result* (2.22)*, when ε does not stay away from* 0*. We are not sure how Xin obtains these estimates. In our situation, we will derive weaker a priori estimates, which are still sufficient to pass to the limit, see lemma 5.4. For these estimates, we will use theorem A.2, which is taken from [4]. This makes it necessary to replace $r(y)$ in L_ε from* (2.13) *by a constant. Unfortunately, this results in slightly more complicated proofs of lemma 4.4, lemma 4.10 and lemma 4.13.*

Passing to the Limit

Using (2.21) and (2.22), Xin chooses a sequence $\varepsilon_n \to 0$, $b_n \to \infty$, such that

$$c_n := c^{\varepsilon_n, b_n} \to c < 0 \text{ for } n \to \infty \text{ and } U_n := U^{\varepsilon_n, b_n} \to U \text{ locally uniformly in } \mathbb{R} \times T^n. \tag{2.23}$$

From the estimate (2.22), Xin concludes that U is locally Lipschitz. Without proof, he claims that parabolic regularity implies that U is locally a classical solution of

$$\sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})U + cU_s + g(U) = 0, \tag{2.24}$$

$$\max_{y \in T^n} U(0,y) = \theta(f) \text{ and } U(s,\cdot) \text{ 1-periodic.} \tag{2.25}$$

The inequalities $U_s \geq 0$ and $0 \leq U \leq 1$ for U follow from the respective inequalities $U_{n,s} > 0$ and $0 \leq U_n \leq 1$ for U_n.

Possibilities for the limit behavior of U for $s \to \pm\infty$

Let U be the limit function from (2.23). From the monotonicity and boundedness of U, Xin deduces that the limits $U^-(y) := \lim_{s \to -\infty} U(s,y)$ and $U^+(y) := \lim_{s \to +\infty} U(s,y)$ exist.

According to Xin, it is obvious that there is a sequence $s_m \to \infty$, such that

$$\lim_{m \to \infty} U_s(s_m, y) = 0, \qquad \lim_{m \to \infty} U_{ss}(s_m, y) = 0 \text{ for all } y \in T^n. \tag{2.26}$$

By multiplying (2.24) with a smooth test function $\psi(y)$ and integrating over $y \in T^n$ and integrating by parts, one obtains

$$\sum_{i,j=1}^{n} \left(\int_y (\psi a_{ij}) k_i k_j U_{ss} - 2(\psi a_{ij})_{y_j} k_i U_s + (\psi a_{ij})_{y_i y_j} U \right) + c \int_y \psi U_s + \int_y \psi g(U) = 0. \tag{2.27}$$

Inserting $s = s_m$ into (2.27), one concludes from (2.26) and $m \to \infty$:

$$\sum_{i,j=1}^{n} \int_y (\psi a_{ij})_{y_i y_j} U^+ + \int_y \psi g(U^+) = 0. \tag{2.28}$$

According to Xin, $U^+(y)$ is therefore a weak solution of

$$\sum_{i,j=1}^{n} a_{ij} U^+_{y_i y_j} + g(U^+) = 0. \tag{2.29}$$

In (2.28) Xin then uses as ψ a function $m(y)$ with the properties

$$\left. \begin{array}{c} \displaystyle\sum_{i,j=1}^{n} (m(y) a_{ij}(y))_{y_i y_j} = 0, \\ m(y) > 0, \qquad \displaystyle\int_{T^n} m(y) = 1, \qquad m(y) \text{ 1-periodic.} \end{array} \right\} \tag{2.30}$$

The existence of such a function is proved in [2]. This yields $\int_{T^n} m(y) g(U^+) = 0$. Furthermore, by $g(U^+) \geq 0$ and $m > 0$ it follows that $g(U^+) = 0$. Using this in (2.29), one finds

$$\sum_{i,j=1}^{n} a_{ij}(y) U^+_{y_i y_j} = 0.$$

According to Xin, elliptic regularity implies that U^+ is a classical solution of this equation. If this is shown, it follows from the maximum principle that U^+ is constant. In the same way it follows that U^- is constant.

By the normalization in (2.25) and $U_s \geq 0$, the inequalities $U^- \leq \theta(f)$ and $U^+ \geq \theta(f)$ follow. Since $g(U^+) = 0$ and $g > 0$ in $(\theta(f), 1)$, there only remain the two possibilities $U^+ = 1$ and $U^+ = \theta(f)$. For U^-, there still remain all possible values in $[0, \theta(g)]$.

Remark 2.20 *(i) Under the assumptions of Xin, we could not verify that U is a classical solution of* (2.24)*, since this includes the existence of* U_{ss} *in the classical sense. We could only verify* $U_{ss} \in L^p_{loc}$*. We think that slightly more regularity assumptions on g are needed for the existence of* U_{ss} *in the classical sense. See theorems 2.8, 2.9 and 2.10, as well as lemma 5.6.*

(ii) The derivative U_{ss} *is not only used in Xin's main result, but also in the further proof, for example in the part that we described in* (2.26)*. It is not clear why* (2.26) *holds. Furthermore, it is left unclear by Xin, in what sense* (2.26) *is supposed to hold and if this is sufficient to deduce* (2.28)*.*

(iii) Xin does not specify how smooth the matrix field a_{ij} *has to be in his existence theorem. In his formula, which we repeated in* (2.27)*, he uses second derivatives of* a_{ij}*. Therefore, at least the existence of the second derivatives of* a_{ij} *is needed for Xin's argumentation. In our situation, after passing to the limit, we proceed differently to show that the limits* $u^{\pm}$ *are constant (see lemma 5.5, proposition 5.7, lemma 5.8). That way, we will only need* $a_{ij} \in C^1(T^n)$ *to prove our results.*

Reducing the possibilities for the limit behavior of U for $s \to \pm\infty$

The next step in the existence proof of Xin is the following: If $U^+ = \theta(g)$, then $U \equiv \theta(g)$.

A consequence of this result is that either $U^+ = 1$ or $U \equiv \theta(g)$, since there are only the possibilities $U^+ = 1$ and $U^+ = \theta(g)$ as described above. Since $U \equiv \theta(g)$ would furthermore imply $U^- = \theta(g)$, it would be sufficient to prove $U^- \neq \theta(g)$, in order to obtain $U^+ = 1$ automatically.

The result is proved by Xin with a two-sided strong parabolic maximum principle. He derives this maximum principle from the usual one sided strong parabolic maximum principle and the periodicity of U.

An exponential solution

In order to finish the investigation of the limit behavior of U for $t \to \pm\infty$, Xin needs a certain solution of the homogeneous equation to compare it with U. Xin's result is the following: Consider the equation

$$\varepsilon r(y)\Phi^{\varepsilon}_{ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})\Phi^{\varepsilon} + c^{\varepsilon}\Phi^{\varepsilon}_s = 0,$$

where $c_1 < c^\varepsilon < c_2 < 0$ for $\varepsilon \in [0, \varepsilon_0)$ with the constants c_1, c_2 and ε_0 from the a priori estimates (2.21). Then there is a solution of the form $\Phi^\varepsilon = e^{\lambda_\varepsilon s}\Psi(y,\varepsilon)$ with $\lambda_\varepsilon > 0$ and a function $\Psi > 0$, which is 1-periodic in y. Moreover, there holds

$$\lim_{\varepsilon\to 0} \lambda_\varepsilon = \lambda_0 > 0 \ \text{ and } \ \lim_{\varepsilon\to 0} \Psi(y,\varepsilon) = \Psi(y,0) > 0. \tag{2.31}$$

Xin shows this by plugging in the ansatz for Φ^ε into the equation. This leads to the equation

$$\sum_{i,j=1}^{n} \frac{a_{ij}}{r(y)}\Psi_{y_i y_j} + 2\lambda_\varepsilon \sum_{i,j=1}^{n} \frac{a_{ij}k_i}{r(y)}\Psi_{y_j} + \frac{c^\varepsilon \lambda_\varepsilon}{r(y)}\Psi = -(1+\varepsilon)\lambda_\varepsilon^2 \Psi.$$

Finding λ_ε and $\Phi(y,\varepsilon)$ then comes down to finding $\lambda = \lambda_\varepsilon$ such that $\rho(\lambda) = -(1+\varepsilon)\lambda_\varepsilon^2$, where $\rho(\lambda)$ is the periodic principal eigenvalue of the operator given by

$$L\Psi = \sum_{i,j=1}^{n} \frac{a_{ij}}{r(y)}\Psi_{y_i y_j} + 2\lambda_\varepsilon \sum_{i,j=1}^{n} \frac{a_{ij}k_i}{r(y)}\Psi_{y_j} + \frac{c^\varepsilon \lambda_\varepsilon}{r(y)}\Psi.$$

(For the properties of the periodic principal eigenvalue, see lemma 5.13.) Using the monotonicity of $\rho(\lambda)$ in the zeroth order coefficient, one can see

$$\frac{c^\varepsilon \lambda}{r_{min}} \le \rho(\lambda) \le \frac{c^\varepsilon \lambda}{r_{max}}.$$

Using this inequality and making use of the continuous dependence of $\rho(\lambda)$ on λ, one can find $\lambda = \lambda_\varepsilon$ with $\rho(\lambda) = -(1+\varepsilon)\lambda^2$ with the intermediate value theorem. Without further proof, Xin states that λ_ε depends continuously on ε and deduces (2.31).

Remark 2.21 *Xin seems to neglect entirely in both the result for the exponential solution and the proof, that the wavespeed c from the solution $(U,c) = (U^{\varepsilon,b}, c^{\varepsilon,b})$ of* (2.13), (2.14) *also depends on b. We are not sure why he denotes it only by c^ε. Moreover, it is not completely clear to us from Xin's proof why λ_ε and $\Phi(y,\varepsilon)$ are continuous in ε (and in b). Especially since he has not proved, that $c^{\varepsilon,b}$ depends continuously on ε and b. It is also not clear, what he means by c^ε (or rather $c^{\varepsilon,b}$) in the case $\varepsilon = 0$ (and $b = \infty$). A priori, $c^{\varepsilon,b}$ might have more than one limit point for $\varepsilon \to 0$, $b \to \infty$, when f is such that Xin's uniqueness result theorem 2.15 does not apply. It is essential that one can assure both $\Psi(y,\varepsilon,b) \ge \theta(f)$ and $\Phi(y,\varepsilon,b) \le C$. In lemma 5.14, we use the Harnack inequality to show this instead of Xin's continuity argument.*

Conclusion of the existence proof

Xin continues by showing $\Phi^\varepsilon \ge U^{\varepsilon,b}$ on $[-b,0]\times T^n$ with the maximum principle. Letting $\varepsilon \to 0$ and $b \to \infty$, this yields the following inequality for the limit function U:

$$U(s,y) \le e^{\lambda_0 s}\Psi(y,0) \text{ for } (s,y) \in (-\infty,0)\times T^n.$$

This allows to conclude $U^- = \lim_{s\to-\infty} U(s,y) = 0$. As discussed earlier, this also implies $U^+ = 1$. This finishes the investigation of the limit behavior of U for $s \to \pm\infty$.

At this point, only the strict inequalities $0 < U < 1$ and $U_s > 0$ are left to show, in order to establish that U is the solution from theorem 2.13. Xin does not discuss the inequality $0 < U < 1$ at all. He shows the other inequality $U_s > 0$ by differentiating (2.24) with respect to s and applying the maximum principle.

Remark 2.22 *We are not sure if there is enough regularity to differentiate* (2.24) *with respect to s in the classical sense and to apply the classical maximum principle, since we think, that U_{ss} does not have to exist in the classical sense. In lemma 5.17 we will instead use the Harnack inequality to show $u_t > 0$.*

2.3.3. Xin's proof of monotonicity and uniqueness

A minimum principle and the sign of the wavespeed

The first step in order to show the monotonicity and uniqueness of traveling wave solutions under the assumptions of the theorems 2.14 and 2.15, is to prove a two-sided minimum principle. By that we mean the following. If a function u is the solution of a parabolic equation and achieves its minimum at a point (t_0, x_0), then under the right assumptions, the minimum principle implies that u is constant in $t \leq t_0$. We call this a one-sided minimum principle, since u only has to be constant in the past, but not in the future. In some situations however, a two-sided minimum principle is required, which implies that u has to be constant in the past and the future.

Xin's two-sided minimum principle is as follows: Consider the operator L, which is given by

$$Lv := (\nabla_y + k\partial_s)(a(y)(\nabla_y + k\partial_s)v) + b(y)^T \cdot (\nabla_y + k\partial_s)v + cv_s + \beta(s, y)v.$$

Assume that the coefficients are smooth and 1-periodic in y and that $c < 0$. Let $v = v(s, y)$ be 1-periodic in y and a classical solution of $Lv \leq 0$ in $\mathbb{R} \times T^n$. Suppose further that v assumes its minimum. If now either $\beta \leq 0$ and the attained minimum is nonpositive or the attained minimum has the value 0 (then the sign of β is not relevant), it follows that v is constant.

The next step is to consider the sign of the wavespeed c of the traveling wave solution (U, c). Xin finds that any solution (U, c) of (2.10), (2.11) satisfies $c < 0$. He proves this by testing (2.10) with the function from (2.30).

Monotonicity and Uniqueness

Xin's proof of theorem 2.14 can be summarized as follows:

(1) Xin's first step is to state that the strict inequality $0 < U < 1$ holds by direct application of the maximum and minimum principle.

(2) He shows $U_s \geq 0$ with a variant of the sliding domain method: For $\lambda > 0$, he considers the function $w(s, y, \lambda) = U(s + \lambda, y) - U(s, y)$ for $(s, y) \in \mathbb{R} \times T^n$. Then the method proceeds in two steps.

(2a) Xin shows that for sufficiently large λ, there holds $w(s, y, \lambda) > 0$ for all $(s, y) \in \mathbb{R} \times T^n$ as follows: Given $N > 0$, there is $\lambda_0 = \lambda_0(N)$, such that $w(s, y, \lambda) > 0$ for $\lambda > \lambda_0$ and $(s, y) \in [-N, N] \times T^n$. This is a consequence of the boundary conditions at infinity for U. The constant N can be chosen sufficiently large, such that outside of $[-N, N] \times T^n$, w cannot have a nonpositive minimum. Xin shows this by using the two sided minimum principle and the fact that $w(s, y, \lambda) \to 0$ for $s \to \pm\infty$. Hence, $w(s, y, \lambda) > 0$ for $\lambda > \lambda_0$ and $(s, y) \in \mathbb{R} \times T^n$.

(2b) Xin defines

$$\mu := \{\lambda > 0 : w(s, y, \lambda) \geq 0 \text{ for } (s, y) \in \mathbb{R} \times T^n\}.$$

The set in the definition of μ is not empty due to $(2a)$. Xin shows $\mu = 0$ with the help of the two-sided minimum principle. This implies $U_s \geq 0$.

(3) The strict inequality $U_s > 0$ is shown by differentiating (2.10) with respect to s and applying the minimum principle to U_s.

Remark 2.23 *(i) As in the proof of the monotonicity for the (ε, b)-problem, a linearization is necessary to prove $U < 1$. Compare remark 2.16.*

(ii) In the proof of the monotonicity, Xin uses the two-sided minimum principle, to show that w cannot have a negative minimum in $(-\infty, N) \times T^n$. However, in order to show that w cannot have a negative minimum in $(N, \infty) \times T^n$, he writes the equation for w as

$$\sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})w + cw_s + \beta(s, y)w = 0.$$

As a consequence of $g'(1) < 0$ and the choice of N, the strict inequality $\beta < 0$ holds in $(N, \infty) \times T^n$. If w had a negative minimum in $(N, \infty) \times T^n$, Xin would obtain an immediate contradiction to the equation. The condition $g'(1) < 0$ in his monotonicity theorem and in the uniqueness theorem is only necessary to make sure that $\beta < 0$ holds here. However, by better usage of the two sided minimum principle, $\beta \leq 0$ would be sufficient. Therefore the condition $g'(1) < 0$ can be relaxed a bit. See theorems 2.11 and 2.12.

(iii) In step (3) it cannot be expected that U_s is still a classical solution of the differentiated equation. Therefore, the minimum principle for classical solutions cannot be applied here. One possible way around this would be to apply a Harnack inequality to the sequence of difference quotients $\frac{U(s+h,y)-U(s,y)}{h}$, $h > 0$. Compare our proof of lemma 5.17.

Theorem 2.15 is proved with a similar variant of the sliding domain method.

3. Plan of proof

3.1. Existence

In the existence part we always assume the matrix field $(a_{ij})_{i,j=1}^n$ to be as in the general assumptions, i.e. $(a_{ij})_{i,j=1}^n$ is a $C^1(T^n)$-matrix field, which is symmetric and uniformly elliptic. The assumptions for the nonlinearity f may vary and will therefore be given locally.

The general plan of the proof of existence is mostly as in [21]. Very roughly, it can be described as follows: We consider the (ε, b)-problem (2.7), (2.8) from definition 2.4. It serves as an elliptic regularization of the original problem in moving frame coordinates. For $\varepsilon > 0$ and $b > 0$, we obtain a unique solution $(U^{\varepsilon,b}, c^{\varepsilon,b})$ of the problem. We proceed by proving estimates for the wavespeed $c^{\varepsilon,b}$ of the form $c_1 \leq c^{\varepsilon,b} \leq c_2 < 0$ for $0 < \varepsilon \leq \varepsilon_0$ and $b \geq b_0$. By a transformation $(s, y) = T^{\varepsilon,b}(t, x) := (k \cdot x - c^{\varepsilon,b}t, x)$, we obtain a function $u^{\varepsilon,b} = u^{\varepsilon,b}(t, x)$ in the original coordinates. We derive local estimates of the form $\left|\left|\nabla_x u^{\varepsilon,b}\right|\right|_{L^\infty_{loc}} \leq C$ and $\left|\left|u_t^{\varepsilon,b}\right|\right|_{L^2_{loc}} \leq C$. Then, we pass to the limit $\varepsilon \to 0$, $b \to \infty$ (for a subsequence) and show that the limit function has all the properties of the desired traveling wave solution.

The (ε, b)-problem

The results concerning monotonicity, uniqueness and existence for the (ε, b)-problem hold for any nonlinearity f of basic type as defined in definition 2.6.

Monotonicity and Uniqueness

The investigation of the (ε, b)-problem begins with a monotonicity result. This monotonicity result holds for a slightly more general class of problems, which includes the (ε, b)-problem. It is only more general in so far as the term εU_s can be replaced by the more general term $c\beta(y)U_s$ with a weight β. The monotonicity result for the more general problem is later used in the proof of a comparison principle for wavespeeds.

Let $U = U(s, y)$, defined in $[-b, b] \times T^n$, be a solution of the problem. The proof of the monotonicity result starts with the strict inequality $0 < U < 1$ in $(-b, b) \times T^n$. This is proved by applying minimum and maximum principles. From there, we will continue by proving $U_s \geq 0$ with the sliding domain method. The method can be briefly described as follows: For $\lambda \in (0, b)$, the function $W_\lambda(s, y) := U(s+\lambda, y) - U(s-\lambda, y)$ is considered in $[-b+\lambda, b-\lambda] \times T^n$. For λ close to b, the function W_λ is shown to be positive. Based on this, we will see with the help of minimum principle that $W_\lambda \geq 0$ is still positive for arbitrarily small $\lambda > 0$. This implies $U_s \geq 0$. The strict inequality $U_s > 0$ is then obtained by application of the Harnack inequality to the sequence of difference quotients.

Subsequently, the uniqueness of (U, c) is proved. (Uniqueness only holds with the normalization.) The method of the proof is a different variation of the sliding domain method. Assume the existence of two solutions (U_1, c_1) and (U_2, c_2). Then, the function $W_\lambda(s, y) := U_1(s + \lambda, y) - U_2(s - \lambda, y)$ is investigated with the sliding domain method. It is applied three times consecutively. With the first application, the assumption that $c_1 > c_2$ is led to a contradiction, then this is repeated with the assumption $c_2 > c_1$. Once it is known that $c_1 = c_2$, the method is repeated to show $U_1 = U_2$.

Existence

In order to show the existence of the (ε, b)-solutions, we start by deriving a priori estimates for a family of problems. For fixed (ε, b), but varying coefficients and f, we derive estimates for $|c|$ and $||U||_{C^1(R_b)}$ (where R_b denotes the cylinder $[-b, b] \times T^n$). The a priori estimates for c are obtained by finding comparison solutions z that lie strictly above U or strictly below in R_b. (This depends on whether we are in the case $c > 0$ or in the case $c < 0$.) Using the normalization condition $\max_{y \in T^n} U(0, y) = \theta$, we obtain either $z(0) \geq \theta$ or $z(0) \leq \theta$. The comparison is achieved with the help of the maximum and minimum principle and the monotonicity result for the (ε, b)-solution. The comparison functions z will be given as solutions of ordinary differential equations. Hence, we will be able to calculate them explicitly in terms of c etc. From this, we will obtain inequalities for c and from that the estimate for $|c|$.

Based on the estimate for $|c|$, the norm $||U||_{C^1(R_b)}$ will then be estimated using L^p-estimates. These have the advantage that the L^p-norm of $f(y, U)$ can be estimated independently of U, since f is always bounded.

Using these estimates, the existence of the (ε, b)-solutions can be proved with the help of the Leray-Schauder degree as follows. For given $v \in C^1([-b, b] \times T^n)$ and $c \in \mathbb{R}$, consider the linear problem

$$\varepsilon\alpha(y)u_{ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})u + cu_s = -f(y, v),$$

$$u(-b, \cdot) = 0, \qquad u(s, \cdot) \text{ 1-periodic}, \qquad u(+b, \cdot) = 1.$$

We will formulate the (ε, b)-problem in terms of a fixed point problem involving this linear problem. This fixed point problem can be investigated with the Leray-Schauder degree.

First, however, we have to solve the linear problem. We will do that by solving the equation on bounded domains Ω_m, where Ω_m is an exhaustion of $[-b, b] \times \mathbb{R}^n$. After finding suitable estimates, we can pass to the limit $m \to \infty$. We arrive at a bounded solution of the equation. Then we use the maximum principle lemma A.1 on unbounded domains from Berestycki, Hamel and Rossi (see [7]). It implies the uniqueness of the bounded solutions. From this, we can derive the periodicity.

Afterwards, we consider the linear problem with a homotopy of coefficients α_τ and $(a_{ij}^\tau)_{i,j=1}^n$ and nonlinearities f_τ. Denote by $\phi_\tau(v, c)$ the solution of the linear problem. We prove continuity and compactness properties of the mapping $(\tau, v, c) \mapsto \phi_\tau(v, c)$ in appropriate spaces. Using this and the estimates for the (ε, b)-problem, we can create an adequate setting for the Leray-Schauder degree. We will choose the homotopy of coefficients such that we have our original coefficients for $\tau = 1$ and an easy situation for $\tau = 0$. Then we obtain the existence result for the (ε, b)-problem from the Leray-Schauder degree.

Estimates for $c^{\varepsilon,b}$

After the (ε, b)-problem is solved, we have to derive bounds for the wavespeed $c^{\varepsilon,b}$ of the form $c_1 \leq c^{\varepsilon,b} \leq c_2 < 0$ for $0 < \varepsilon \leq \varepsilon_0$, $b \geq b_0$. The lower bound c_1 can be found for all nonlinearities of basic type. This can be done by finding suitable comparison solutions $z = z(s)$ and $w = w(s)$ on the left half $[-b, 0] \times T^n$ and right half $[0, b] \times T^n$ of R_b, which both lie above U and satisfy $z(0) = w(0) = \theta$. That means that they touch U at a point $(0, y_0)$ with $U(0, y_0) = \theta$. This implies the inequality $z_s(0) \leq U_s(0, y_0) \leq w_s(0)$. From this, we obtain an inequality for c, because z and w can be calculated explicitly in terms of c.

The upper bound c_2 can be found for a nonlinearity of basic type f under an additional assumption: The nonlinearity f lies above an y-independent nonlinearity $g = g(z)$ of basic type, which satisfies a certain integral condition. This integral condition depends on the oscillations of $r(y) = \sum_{i,j=1}^n a_{ij}(y) k_i k_j$. In particular, the assumption is satisfied for f, if f is a nonlinearity of combustion type with the strong covering property. Under the above assumptions, the bound is obtained with the help of a comparison principle for wavespeeds. This comparison principle will be proved with the help of the sliding domain method.

Estimates for $u^{\varepsilon,b}$

From here on we will assume the following definition for $u^{\varepsilon,b}$. Consider the (ε, b)-problem with the weight $\alpha = 1$ in (2.7). Let f be a nonlinearity of combustion type with the covering property and the negative wavespeed property (with respect to the matrix field $(a_{ij})_{i,j=1}^n$ and the unit vector k). Furthermore, let $\theta := \theta(f)$ be the normalization constant. We emphasize that it possible to normalize the (ε, b)-solution at any $\theta \in (0, 1)$. However, when passing to the limit, we really want the (ε, b)-solution to be normalized at $\theta(f)$. In this situation, let $(U^{\varepsilon,b}, c^{\varepsilon,b})$ be the (ε, b)-solution.

Before passing to the limit, we return to (t, x)-coordinates. We consider the transformation $s = k \cdot x - c^{\varepsilon,b} t$, $y = x$ and define $u^{\varepsilon,b}(t, x) := U^{\varepsilon,b}(s, y)$.

We will then derive a priori estimates for $u^{\varepsilon,b}(t, x)$. We obtain local L^∞-estimates for $\nabla_x u^{\varepsilon,b}$ by using the theorem A.2 from Berestycki and Hamel (see [4]). It is designed for elliptic regularization, this theorem was also used for example in [6]). The requirements in the theorem involve the C^3-regularity of $u^{\varepsilon,b}$, which we do not have. Therefore, we have to solve this problem with an approximation of the coefficients. We furthermore derive local L^2-bounds for $u_t^{\varepsilon,b}$ by testing. The uniform estimates for the x-gradient enable us to carry over information from the normalization condition of the (ε, b)-solution to the limit function.

Passing to the limit

The estimates allow us to choose a subsequence, such that $c^{\varepsilon,b}$ converges to a real number $c < 0$ and the function $u^{\varepsilon,b}$ converges weakly in $H^1_{loc}(\mathbb{R}^{n+1})$, strongly in $L^2_{loc}(\mathbb{R}^{n+1})$ and almost everywhere in $\mathbb{R}^{n+1}$. After passing to the limit, some properties for the limit function u follow naturally from corresponding properties of $u^{\varepsilon,b}$. These are $0 \le u \le 1$ and $u_t \ge 0$, as well as the periodicity condition $u(t, x) = u(t+\frac{k \cdot e_i}{c}, x+e_i)$ for the standard unit vectors e_i, $i = 1, ..., n$. With the help of the estimates for the x-gradient of $u^{\varepsilon,b}$ and the normalization $\max_{y \in T^n} U^{\varepsilon,b}(0, y) = \theta(f)$ and possibly choosing another subsequence, we obtain a weak form of the normalization condition for u: There exists $d > 0$ with

$$\max_{x \in [0,1]^n} u(-d, x) \le \theta(f), \qquad \theta(f) \le \max_{x \in [0,1]^n} u(d, x).$$

It also follows easily that u is a weak solution of (2.1), rewritten in divergence form.

The regularity for the limit function is more difficult. First, we use an interior regularity result for weak solutions to see that $D^2u \in L^2_{loc}$ and that u is a strong solution of (2.1). From there, $C^{2,\alpha}_{1,\alpha/2,loc}$-regularity is derived with an approximation and interior estimates (both L^p-estimates and Schauder estimates). More regularity for u_t, including the additional regularity which is needed for a solution of type II, is derived by the technique of difference quotients.

Due to $u_t \ge 0$ and $0 \le u \le 1$, we obtain the existence of the limits $\lim_{t \to \pm\infty} u(t, x) =: u^{\pm}(x)$. From the periodicity condition for u, it is easily seen that $u^{\pm}$ are periodic functions. After deriving suitable estimates for the derivatives of u, we can show that $u^{\pm}$ are stationary states of (2.1). Therefore, $u^{\pm}$ satisfy the corresponding elliptic equation (2.1) with $u_t = 0$ to (2.1) classically. Together with the minimum principle this implies, that $u^{\pm}$ have to be constants. In case of u^+, we have to make use of the fact, that f is of combustion type. The reason is, if f had both signs, then there might be non-constant stationary states. This is the only point, where we cannot avoid using the combustion property of f.

We want to further reduce the amount of possible values for u^+ and u^-. Since u^+ is constant, it has to be in $[\theta(f), 1]$ due to the normalization and monotonicity of u. By the covering property, the only possibilities for u^+ are therefore $u^+ = \theta$ and $u^+ = 1$. With the maximum principle, one can further show that either $u^+ = 1$ or $u \equiv \theta(f)$, which would imply $u^- = \theta(f)$. Consequently, it only remains to prove u^-. Then $u^+ = 1$ will follow automatically.

The proof of $u^- = 0$ uses a certain exponential solution of the homogeneous equation, which can be compared to $U^{\varepsilon,b}$. From this comparison, we obtain $u^- = 0$. The exponential solution can be found with the help of the periodic principle eigenvalue.

It remains to prove the strict inequalities $0 < u < 1$ and $u_t > 0$. The inequality $0 < u < 1$ follows similarly as the corresponding inequality in the proof of the monotonicity of the (ε, b)-problem. However, there is a slight difference. For the inequality $0 < u$, we have to make use of a two-sided minimum principle, because the usual parabolic left-sided minimum principle is not sufficient. Afterwards, the inequality $u_t > 0$ is proved as in the case of the (ε, b)-problem by applying the Harnack inequality to a sequence of difference quotients. The existence result is then completely proved.

3.2. Monotonicity and uniqueness

In this section, we always consider a matrix field $(a_{ij})_{i,j=1}^n$ and a nonlinearity f with the assumptions of the theorems 2.11 and 2.12.

For the proofs of the monotonicity and uniqueness, the underlying method is the same as in [21]: the sliding domain method.

A minimum principle

Before starting with the monotonicity, we have to derive a two-sided minimum principle. The reason is that in some situations the usual left-sided parabolic minimum principle is not sufficient. The two-sided minimum principle can be obtained from the usual parabolic minimum principle and the periodicity condition $u(t, x) = u(t + \frac{k \cdot e_i}{c}, x + e_i)$ for the standard unit vectors e_i, $i = 1, ..., n$.

Monotonicity

In the proof of the monotonicity result, we start with the inequality $0 < u < 1$. It can be proved exactly in the same way as the corresponding inequality in the existence result.

Now $u_t \geq 0$ can be proved by a variation of the sliding domain method. It is best described in moving frame coordinates, given by $(s, y) := (k \cdot x - ct, y)$. Consider the function $W_\lambda(s, y) = U(s+\lambda, y) - U(s, y)$. We will show that for large $\lambda > 0$, the function W_λ is positive. This can be proved by making use of the strict inequality $0 < U < 1$, the boundary conditions at infinity and our two-sided minimum principle. Subsequently, we will prove, that $W_\lambda \geq 0$ even holds for arbitrarily small $\lambda > 0$. This can be seen using the two-sided minimum principle. From this, the inequality $U_s \geq 0$ and therefore $u_t \geq 0$ follows.

The strict inequality $u_t > 0$ follows as the respective property in the existence proof.

Uniqueness

The uniqueness is proved with a slight variation of the sliding domain method, which was used for the monotonicity. Let (u_1, c_1) and (u_2, c_2) be two solutions. We will shortly describe the method in moving frame coordinates. (These are given by $(s, y) := (k \cdot x - c_1 t, y)$ and $(s, y) := (k \cdot x - c_2 t, y)$, respectively). Let U_1 and U_2 be the functions u_1 and u_2 expressed in moving frame coordinates.

The method will be applied three times consecutively. In the first application, assume $c_2 < c_1$. For $\lambda \in \mathbb{R}$, consider the function $W_\lambda(s, y) = U_1(s + \lambda, y) - U_2(s, y)$. We will show, that for large $\lambda > 0$, the function W_λ is positive. This can be proved by making use of the strict inequality $0 < U < 1$, the boundary conditions at infinity and our two-sided minimum principle. Let μ denote the infimum of all $\lambda \in \mathbb{R}$, such that $W_\lambda \geq 0$. With the two-sided minimum principle, we will derive $W_\mu = 0$. This means, that U_1 is a shift of U_2 in s by μ. From this, we will obtain a contradiction to $c_2 < c_1$.

The second application is performed with c_1 and c_2 interchanged. This yields $c_1 = c_2$. From the third application of the method we will see $W_\mu = 0$ directly. Hence, that U_1 is only a shift of U_2 and likewise for u_1 and u_2. This finishes the proof.

4. Existence of traveling wave solutions: the (ε, b)-problem

4.1. Elliptic regularization or the (ε, b)-problem

We begin with the elliptic regularization of the problem.

4.1.1. Monotonicity and uniqueness of the (ε, b)-solution

We examine the monotonicity of the regularized solution and receive the following result. We show a more general result than Xin, since we need more than the monotonicity of the regularized solution in the proof of a comparison principle later on.

Proposition 4.1 *Let f be a nonlinearity of basic type and $\alpha, \beta \in C^1(T^n)$ with $\alpha > 0$. Moreover, let (U, c) with $U : [-b, b] \times \mathbb{R}^n \to \mathbb{R}$ be a classical solution, that is*

$$U \in C^2((-b, b) \times \mathbb{R}^n) \cap C^0([-b, b] \times \mathbb{R}^n),$$

of the following problem:

$$\varepsilon\alpha(y)U_{ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})U + c\beta(y)U_s + f(y, U) = 0, \tag{4.1}$$

$$\left.\begin{array}{ll} 0 \le U \le 1, & U(s, \cdot) \textit{ is } 1-\textit{periodic}, \\ U(-b, y) = 0, & U(b, y) = 1 \textit{ for all } y \in T^n, \\ \max_{y \in T^n} U(0, y) = \theta \in (0, 1). & \end{array}\right\} \tag{4.2}$$

Then the strict inequalities $0 < U(s, y) < 1$ and $U_s(s, y) > 0$ hold for all $(s, y) \in (-b, b) \times T^n$.

Proof *Step 1: The strict inequality* $0 < U < 1$. The first step is to show that we have the strict inequality $0 < U < 1$ in $(-b, b) \times T^n$. We want to be very precise in the first step, due to our observation in remark 2.16. By assumption there holds $0 \le U \le 1$. Suppose that U achieves the value 1 at a point $(s_0, y_0) \in (-b, b) \times T^n$. We have

$$L_\varepsilon U := \varepsilon\alpha(y)U_{ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})U + c\beta(y)U_s = -f(y, U),$$

and, at this point, no knowledge about the sign of f. In a neighborhood of (s_0, y_0), there holds $-f(y,U) \le 0$. But this is the wrong sign to apply the maximum principle (and $f(y_0, U(s_0, y_0)) = 0$ is not sufficient to apply it either). To solve this problem, we linearize the term $f(y,U)$. We put

$$d(s,y) := \begin{cases} \dfrac{f(y,U(s,y)) - f(y,1)}{U-1} & \text{if } U(s,y) \neq 1 \\ 0 & \text{otherwise.} \end{cases}$$

For the second case of the definition, we note that f is not necessarily differentiable. Because of $L_\varepsilon U = L_\varepsilon(U-1)$ and $f(y,1) = 0$, we have $L_\varepsilon(U-1) + d(s,y)(U-1) = 0$. Clearly, we have $|d| \le L$, where L is the Lipschitz constant of f. Hence, $|d|$ is bounded, and this allows us to apply the maximum principle. Note that in this case the sign of d does not matter, because the attained maximum of $U-1$ has the value 0. We see that $U - 1 \equiv 0$, which is a contradiction to $U(-b, \cdot) = 0$. This contradiction implies $U(s,y) < 1$ for $(s,y) \in (-b,b) \times T^n$.

Showing that $U(s,y) > 0$ for $(s,y) \in (-b,b) \times T^n$ can be done in the same fashion. In conclusion

$$0 < U(s,y) < 1 \text{ for } (s,y) \in (-b,b) \times T^n.$$

Step 2: The inequality $U_s \ge 0$. For $\lambda \in (0,b)$, we consider the function

$$w_\lambda(s,y) = U(s+\lambda, y) - U(s-\lambda, y)$$

with $(s,y) \in \overline{\Sigma_\lambda}$, where $\Sigma_\lambda = (-b+\lambda, b-\lambda) \times T^n$. Equation (4.1) for U implies that w_λ satisfies

$$\begin{aligned} &\varepsilon\alpha(y) w_{\lambda,ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})w_\lambda \\ &\qquad + c\beta(y)w_{\lambda,s} + f(y, U(s+\lambda,y)) - f(y, U(s-\lambda,y)) = 0 \text{ on } \Sigma_\lambda. \end{aligned} \tag{4.3}$$

At the boundary of Σ_λ, the function w_λ is always positive: Setting $s = -b+\lambda$ we find

$$w_\lambda(-b+\lambda, y) = U(2\lambda - b, y) - U(-b,y) = U(2\lambda - b, y) > 0,$$

and setting $s = +b - \lambda$ we find

$$w_\lambda(b-\lambda, y) = U(b,y) - U(b-2\lambda, y) = 1 - U(b - 2\lambda, y) > 0,$$

since we obtained $0 < U(s,y) < 1$ for $(s,y) \in (-b,b) \times T^n$ in step 1.

Step 2a: Large λ. Loosely speaking, the idea of the subsequent argument is as follows: If λ is close to b, because of $s \in (-b+\lambda, b-\lambda)$, we find s is close to 0 and therefore $s+\lambda$ is close to b and $s-\lambda$ is close to $-b$. Then $U(s+\lambda,y) - U(s-\lambda,y)$ is close to $1 - 0 = 1$. In particular w_λ is positive.

Let us now start with the rigorous proof. By uniform continuity of U up to the boundary, for every $\frac{1}{2} > \varepsilon > 0$ there is a $\delta > 0$, such that for $s_1 \in (b-\delta, b)$ and $s_2 \in (-b, -b+\delta)$ we have

$$|U(s_1,y) - 1| < \varepsilon \text{ and } |U(s_2,y)| < \varepsilon.$$

Consequently $0 < b - \lambda < \frac{\delta}{2}$ yields for $(s, y) \in \Sigma_\lambda$:

$$s + \lambda \in (-b + 2\lambda, b) \subset (b - \delta, b) \text{ and } s - \lambda \in (-b, b - 2\lambda) \subset (-b, -b + \delta)$$

and thus

$$|U(s + \lambda, y) - 1| < \varepsilon \text{ and } |U(s - \lambda, y)| < \varepsilon \text{ for } (s, y) \in \overline{\Sigma_\lambda}.$$

Therefore, by $0 < \varepsilon < \frac{1}{2}$,

$$w_\lambda(s, y) = U(s + \lambda, y) - U(s - \lambda, y) > 0 \text{ for all } (s, y) \in \overline{\Sigma_\lambda}.$$

This finishes the calculation for λ close to b.

Step 2b: Arbitrary λ. We set

$$\lambda_0 := \inf\{\lambda \in (0, b) : w_\lambda(s, y) > 0 \text{ for } (s, y) \in \overline{\Sigma_\lambda}\}.$$

We have just proved in step 2a that the set on the right hand side is not empty. If we can show $\lambda_0 = 0$, this implies $U_s \geq 0$ in $(-b, b) \times T^n$. Suppose for a contradiction $\lambda_0 \neq 0$. We therefore have $\lambda_0 \in (0, b)$. By definition of λ_0 and continuity, there holds $w_{\lambda_0} \geq 0$. Since w_{λ_0} is strictly positive on $\partial\Sigma_{\lambda_0}$ (see start of step 2), there are two possibilities:

(*i*) There is (s_0, y_0) with $w_{\lambda_0}(s_0, y_0) = 0$ for some $(s_0, y_0) \in \Sigma_{\lambda_0}$. We define

$$d_\lambda(s, y) := \begin{cases} \dfrac{f(y, U(s + \lambda, y)) - f(y, U(s - \lambda, y))}{U(s + \lambda, y) - U(s - \lambda, y)} & \text{if } U(s + \lambda, y) \neq U(s - \lambda, y) \\ 0 & \text{otherwise.} \end{cases}$$

For the second case of the definition, we note that f is not necessarily differentiable everywhere. With this definition of d_λ, equation (4.3) for w_{λ_0} reads

$$\varepsilon\alpha(y) w_{\lambda_0, ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j}) w_{\lambda_0} + c\beta(y) w_{\lambda_0, s} + d_{\lambda_0}(s, y) w_{\lambda_0} = 0.$$

Similarly as in step 1, $|d_{\lambda_0}| \leq L$, where L is the Lipschitz constant of f. Since the value of the minimum is 0, the strong elliptic minimum principle can be applied regardless of the sign of d_{λ_0}. Therefore, we can infer that $w_{\lambda_0} \equiv 0$, which contradicts $w_{\lambda_0}(b - \lambda_0, y) > 0$.

(*ii*) There holds $w_{\lambda_0}(s, y) > 0$ for all $(s, y) \in \overline{\Sigma_{\lambda_0}}$. The minimality of λ_0 implies the existence of sequences $0 < \lambda_n \nearrow \lambda_0$ and $(s_n, y_n) \in \Sigma_{\lambda_n}$, such that $w_{\lambda_n}(s_n, y_n) \leq 0$. If we are able to find a convergent subsequence $(s_n, y_n) \to (s_0, y_0) \in \overline{\Sigma_{\lambda_0}}$, we obtain $w_{\lambda_0}(s_0, y_0) \leq 0$ by continuity of the mapping $(\lambda, s, y) \mapsto w_\lambda(s, y)$, which is a contradiction to the setting of (*ii*). Finding a convergent subsequence is possibly due to the 1-periodicity of w in y (y_n can be chosen bounded) and the fact that $s_n \in [-b, b]$ is bounded.

In both cases (i) and (ii), we obtain a contradiction. This yields $\lambda_0 = 0$ and hence $U_s \geq 0$.

Step 3: The strict inequality $U_s > 0$. Let us now show the strict inequality $U_s > 0$. We suppose for a contradiction that there is a point $(s_0, y_0) \in (-b, b) \times T^n$ with $U_s(s_0, y_0) = 0$.

Let us once more start with a formal argument: Differentiating (2.7) with respect to s, we obtain the following equation for U_s:

$$\varepsilon\alpha(y)(U_s)_{ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})U_s + c\beta(y)(U_s)_s + f_z(y, U)U_s = 0.$$

The strong maximum principle yields for the solution U_s the strict inequality $U_s > 0$ in $(-b, b) \times T^n$. The argument is not applicable directly, since we cannot differentiate in a classical sense, we do not have enough regularity for that (see remark 2.16).

We provide now the rigorous proof using the Harnack inequality, applied to the difference quotients of U is s. We choose $R > 0$ such that $B_{3R}(s_0, y_0) \subset (-b, b) \times T^n$. For $0 < h < R$ we know that $v^h := \frac{U(s+h,y)-U(s,y)}{h} \geq 0$ in $B_{2R}(s_0, y_0)$ by what we proved in step 2. Moreover, in $B_{2R}(s_0, y_0)$ we have for v^h the equation

$$\varepsilon\alpha(y)v^h_{ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})v^h + c\beta(y)v^h_s + d^h v^h = 0,$$

where

$$d^h(s, y) = \begin{cases} \dfrac{f(y, U(s + h, y)) - f(y, U(s, y))}{U(s + h, y) - U(s, y)} & \text{if } U(s + h, y) \neq U(s, y) \\ 0 & \text{otherwise.} \end{cases}$$

(As above, we note for the second case of the definition, that f needs not be differentiable everywhere.) The Harnack inequality (see for example Corollary 9.25 in [12]) yields

$$\sup_{B_R(s_0,y_0)} v^h \leq C \inf_{B_R(s_0,y_0)} v^h. \tag{4.4}$$

The constant $C > 0$ in the Harnack inequality depends on certain bounds on the coefficients, but not on the individual coefficients. Since $\left|d^h\right| \leq L$, where L is the Lipschitz constant of f, the constant C depends on L and on the other coefficients (which do not depend on h). Therefore, C does not depend on h. For $h \to 0$, the right hand side of (4.4) tends to 0 due to the assumption $U_s(s_0, y_0) = 0$. Consequently, the left hand side tends to 0 as well, which implies $U_s = 0$ on $B_R(s_0, y_0)$. We thus have proved, that the set of zeros of U_s is open in $(-b, b) \times T^n$. Since it is also closed in $(-b, b) \times T^n$, we have $U_s \equiv 0$ in $(-b, b) \times T^n$. By the boundary conditions on U, this is impossible. This contradiction implies $U_s > 0$ in $(-b, b) \times T^n$. ∎

As a corollary we obtain the monotonicity for the regularized solution.

Corollary 4.2 (Monotonicity of the regularized solution) *Let f be a nonlinearity of basic type and (U, c) be a classical solution of the (ε, b)-problem (2.7), (2.8). Then $0 < U(s, y) < 1$ and $U_s(s, y) > 0$ for all $(s, y) \in (-b, b) \times T^n$.*

Proof Apply proposition 4.1 with $\beta \equiv 1$. ■

Based on corollary 4.2 we can prove the uniqueness of the regularized solution. The method of the proof will be very similar to step 2 of the proof of proposition 4.1, the sliding domain method. The previously proved monotonicity will also be used in the proof.

Proposition 4.3 (Uniqueness of the regularized solution) *Let f be a nonlinearity of basic type and let (U_i, c_i), $i = 1, 2$, be two classical solutions of the (ε, b)-problem (2.7), (2.8). Then $c_1 = c_2$ and $U_1 = U_2$.*

Proof *Step 1: $c_1 \leq c_2$.* The first step is to prove $c_1 \leq c_2$, and we suppose for a contradiction that $c_1 > c_2$. We proceed similarly as in step 2 of the proof of proposition 4.1: For $\lambda \in (0, b)$, we consider the function

$$w_\lambda(s, y) = U_1(s + \lambda, y) - U_2(s - \lambda, y)$$

with $(s, y) \in \overline{\Sigma_\lambda}$, where $\Sigma_\lambda = (-b + \lambda, b - \lambda) \times T^n$.

Since equation (2.7) is shift invariant in s, the functions $U_1(s + \lambda, y)$ and $U_2(s - \lambda, y)$ satisfy on Σ_λ the equations

$$\varepsilon\alpha(y) U_{1,ss}(s + \lambda, y) + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j}) U_1(s + \lambda, y) + c_1 U_{1,s}(s + \lambda, y)$$
$$+ f(y, U_1(s + \lambda, y)) = 0$$

and

$$\varepsilon\alpha(y) U_{2,ss}(s - \lambda, y) + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j}) U_2(s - \lambda, y) + c_2 U_{2,s}(s - \lambda, y)$$
$$+ f(y, U_2(s - \lambda, y)) = 0.$$

Subtracting these two equations, we obtain for $(s, y) \in \Sigma_\lambda$ the equation

$$\varepsilon\alpha(y) w_{\lambda,ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j}) w_\lambda$$
$$+ c_1 U_{1,s} - c_2 U_{2,s} + f(y, U_1) - f(y, U_2) = 0,$$

where $w_\lambda = w_\lambda(s, y)$, $U_1 = U_1(s + \lambda, y)$ and $U_2 = U_2(s - \lambda, y)$. We define

$$\gamma_\lambda(s, y) := \begin{cases} \dfrac{f(y, U_1(s + \lambda, y)) - f(y, U_2(s - \lambda, y))}{U_1(s + \lambda, y) - U_2(s - \lambda, y)} & \text{if } U_1(s + \lambda, y) \neq U_2(s - \lambda, y) \\ 0 & \text{otherwise.} \end{cases}$$

For the second case of the definition, we note that f is not necessarily differentiable everywhere. With this definition of γ_λ, the equation can be rewritten as:

$$\varepsilon\alpha(y)w_{\lambda,ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})w_\lambda + c_1 w_{\lambda,s} + \gamma_\lambda(s,y)w_\lambda = (c_2 - c_1)U_{2,s} < 0. \tag{4.5}$$

The inequality holds due to the assumption $c_1 > c_2$ and the inequality $U_{2,s} > 0$ from lemma 4.2. Note that γ_λ is bounded by the Lipschitz constant of f, which we need in order to apply the maximum principle later on.

The strict inequalities $0 < U_1 < 1$ and $0 < U_2 < 1$ hold in $(-b, b) \times T^n$ by corollary 4.2. Using this, we can see as in the proof of proposition 4.1, that $w_\lambda > 0$ on $\partial\Sigma_\lambda$: Setting $s = -b + \lambda$ we see

$$w_\lambda(-b + \lambda, y) = U_1(2\lambda - b, y) - U_2(-b, y) = U_1(2\lambda - b, y) > 0$$

and setting $s = +b - \lambda$ we see

$$w_\lambda(b - \lambda, y) = U_1(b, y) - U_2(b - 2\lambda, y) = 1 - U_2(b - 2\lambda, y) > 0.$$

Step 1a: Large λ. In the same fashion as in step 2a of the proof of proposition 4.1, we can see $w_\lambda(s, y) > 0$ on $\overline{\Sigma_\lambda}$ for λ close enough to b: By uniform continuity of U_1 and U_2 up to the boundary, for every $\frac{1}{2} > \varepsilon > 0$ there is a $\delta > 0$, such that for $s_1 \in (b - \delta, b)$ and $s_2 \in (-b, -b + \delta)$ we have

$$|U_1(s_1, y) - 1| < \varepsilon \text{ and } |U_2(s_2, y)| < \varepsilon.$$

Consequently $0 < b - \lambda < \frac{\delta}{2}$ yields for $(s, y) \in \Sigma_\lambda$:

$$s + \lambda \in (-b + 2\lambda, b) \subset (b - \delta, b) \text{ and } s - \lambda \in (-b, b - 2\lambda) \subset (-b, -b + \delta)$$

and thus

$$|U_1(s + \lambda, y) - 1| < \varepsilon \text{ and } |U_2(s - \lambda, y)| < \varepsilon \text{ for } (s, y) \in \overline{\Sigma_\lambda}.$$

Therefore, by $0 < \varepsilon < \frac{1}{2}$,

$$w_\lambda(s, y) = U_1(s + \lambda, y) - U_2(s - \lambda, y) > 0 \text{ for all } (s, y) \in \overline{\Sigma_\lambda}.$$

This finishes the calculation for λ close to b.

Step 1b: Arbitrary λ. We define

$$\lambda_0 := \inf\{\lambda \in (0, b) : w_\lambda(s, y) > 0 \text{ for all } (s, y) \in \overline{\Sigma_\lambda}\}.$$

By step 1a, the set on the right hand side of the definition is nonempty. We want to show $\lambda_0 = 0$. We suppose that on the contrary $\lambda_0 > 0$. The definition of λ_0 implies $w_{\lambda_0}(s, y) \geq 0$ due to continuity reasons. Since $w_{\lambda_0} > 0$ on $\partial\Sigma_{\lambda_0}$ (see right before step 1a), two cases are possible:

(i) There is a point $(s_0, y_0) \in \Sigma_{\lambda_0}$ with $w_{\lambda_0}(s_0, y_0) = 0$. Because of equation (4.5), the elliptic minimum principle implies $w_{\lambda_0} \equiv 0$. This holds irrespectively of the sign of γ_{λ_0}, because the value of the minimum is 0. However, this is a contradiction to $w(b - \lambda_0, y, \lambda_0) > 0$.

(ii) There holds $w_{\lambda_0} > 0$ on $\overline{\Sigma_{\lambda_0}}$. Because of the minimality of λ_0, there are sequences $\lambda_n \nearrow \lambda_0$ and $(s_n, y_n) \in \Sigma_{\lambda_n}$ such that $w_{\lambda_n}(s_n, y_n) \leq 0$. The sequence $s_n \in [-b, b]$ is bounded. Moreover, the sequence y_n can be chosen so that it is bounded due to the 1-periodicity of w in y. This allows us to choose a convergent subsequence $(s_n, y_n, \lambda_n) \to (s_0, y_0, \lambda_0)$. Hence, we obtain $w_{\lambda_0}(s_0, y_0) \leq 0$, which is a contradiction to the setting of (ii).

In both cases (i) and (ii), we obtain a contradiction. This yields $\lambda_0 = 0$.

Step 1c: The equality $U_1 = U_2$. From $\lambda_0 = 0$, which was proved in step 1b, we obtain $w_0 \geq 0$. This is equivalent to $U_1 \geq U_2$. For $s = \pm b$, we have $w_0(s, y) = 0$ due to the boundary conditions for U_1 and U_2. Consequently, the strong minimum principle implies that either $w_0 > 0$ in Σ_0 or $w_0 \equiv 0$ (by the 1-periodicity in y, w_0 has to take its minimum in $\overline{\Sigma_0}$). The normalization of the regularized solutions makes $w_0 > 0$ in Σ_0 impossible. We can see this as follows: Let us suppose $w_0(s, y) > 0$ for all $(s, y) \in \Sigma_0$. This implies in particular $U_1(0, y) > U_2(0, y)$ for all $y \in T^n$. Thus

$$\theta = \max_{y \in T^n} U_2(0, y) =: U_2(0, y_{max}) < U_1(0, y_{max}) \leq \max_{y \in T^n} U_1(0, y) = \theta.$$

Since this is a contradiction, $w_0 \equiv 0$ must hold, and hence also $U_1 = U_2$.

Step 1d: Conclusion of $c_1 \leq c_2$. By the previous step, we have $w_0 \equiv 0$. This is contradiction to the strict inequality in (4.5). Therefore, the original assumption $c_1 > c_2$ at the beginning of step 1 must have been false. Hence, $c_1 \leq c_2$.

Step 2a: $c_1 = c_2$. The argument from step 1 can be repeated with c_1 and c_2 interchanged to show $c_2 \leq c_1$ as well. Consequently $c_1 = c_2$ holds.

Step 2b: Proof of $U_1 = U_2$. The arguments of step 1 to step 1c) can be repeated with a minor modification to prove $U_1 = U_2$ directly. The modification is the following: Since now $c_1 = c_2$ is known, the strict inequality (4.5) is replaced by

$$\varepsilon\alpha(y) w_{\lambda,ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j}) w_\lambda + c_1 w_{\lambda,s} + \gamma_\lambda(s, y) w_\lambda = 0.$$

We note, that the strict inequality in (4.5) has only been used in step 1d, but not in the steps 1 to 1c). The repetition of steps 1 to 1c with the mentioned modification concludes the proof of $U_1 = U_2$. ■

4.1.2. Existence of the (ε, b)-solution

A priori estimates for $(U^{\varepsilon,b}, c^{\varepsilon,b})$

To show the existence of a solution $(U^{\varepsilon,b}, c^{\varepsilon,b})$ of the regularized problems (2.7), (2.8), we require a priori estimates for $(U^{\varepsilon,b}, c^{\varepsilon,b})$. The estimates in the following lemma are a slight variation of Xin's a priori estimates (lemma 2.1 in [21], see remarks 2.17 and 2.19):

Lemma 4.4 (A priori estimates) *Let f be a nonlinearity of basic type and (U, c) be a classical solution of (2.7), (2.8). Moreover, we suppose $U \in C^2((-b, b) \times T^n) \cap C^1([-b, b] \times T^n)$. Let $\tilde{\mu}$ be an ellipticity constant of the occurring operator L given by*

$$LU := \varepsilon\alpha(y)U_{ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})U + cU_s$$

and let $M, R, \bar{r} > 0$ be such that

$$M \geq \sup_{y\in T^n, z\in[0,1]} \frac{|f(y,z)|}{r(y)}, \qquad R \geq \sup_{y\in T^n} r(y), \qquad \bar{r} \geq \max_{y\in T^n} \frac{\alpha(y)}{r(y)},$$

where $r(y) = \sum_{i,j=1}^{n} a_{ij}(y)k_ik_j$. Furthermore let $\Lambda > 0$ be such that

$$||a_{ij}||_\infty \leq \Lambda \text{ for } i, j = 1, ..., n \text{ and } ||\varepsilon\alpha(y)||_\infty \leq \Lambda$$

and let ω_{ij}, ω be moduli of continuity for the coefficients a_{ij} and $\varepsilon\alpha(y)$. Then there is a constant $K = K(\theta, \varepsilon, b, \bar{r}, R, M, \tilde{\mu}, \Lambda, \{\omega_{ij} | i, j = 1, ..., n\}, \omega)$, such that every solution (U, c) of the above problem (2.7), (2.8) satisfies the estimates $|c| \leq K$ and $||U||_{C^1(R_b)} \leq K$, where $R_b = [-b, b] \times T^n$.

Proof *Step 1: The estimates for the wavespeed c.* We want to compare U to a solution of an ordinary differential equation, which can be calculated explicitly. Doing so, we have to distinguish between the two possible signs of c.

Case 1: Consider first the case $c > 0$. Let $z = z(s)$ be the solution of

$$(1 + \bar{r}\varepsilon)z_{ss} + \frac{c}{R}z_s - M = 0,$$
$$z(-b) = 0, \qquad z(b) = 1.$$

Before we compare it to the function U, we have to calculate z explicitly, because we need information about the sign of z_{ss}. The general solution of the ODE is given by

$$z(s) = \frac{MRs}{c} + D + C\exp\left(\frac{-cs}{R(1+\bar{r}\varepsilon)}\right).$$

The boundary conditions give

$$0 = z(-b) = -\frac{MRb}{c} + D + C\exp\left(\frac{cb}{R(1+\bar{r}\varepsilon)}\right)$$

and

$$1 = z(b) = \frac{MRb}{c} + D + C\exp\left(\frac{-cb}{R(1+\bar{r}\varepsilon)}\right).$$

From this we can calculate

$$C = \frac{\frac{2MRb}{c} - 1}{\exp\left(\frac{cb}{R(1+\bar{r}\varepsilon)}\right) - \exp\left(\frac{-cb}{R(1+\bar{r}\varepsilon)}\right)}$$

and

$$D = \frac{MRb}{c} - \frac{\frac{2MRb}{c} - 1}{\exp\left(\frac{cb}{R(1+\bar{r}\varepsilon)}\right) - \exp\left(\frac{-cb}{R(1+\bar{r}\varepsilon)}\right)} \exp\left(\frac{cb}{R(1+\bar{r}\varepsilon)}\right).$$

We observe $sign(z_{ss}) = sign(C) = sign(\frac{2MRb}{c} - 1)$ and therefore $z_{ss} < 0$ for $c > 2MRb$. Since we are looking for an upper bound for c, it is sufficient to consider the case $c > 2MRb$ only. Since we want to apply the minimum principle, we calculate

$$\begin{aligned}
&-\varepsilon\alpha(y)(U-z)_{ss} - \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})(U-z) - \frac{c}{R}r(y)(U-z)_s \\
&= f(y,U) + (r(y) + \varepsilon\alpha(y))z_{ss} + \frac{cr(y)}{R}z_s + \left(c - \frac{cr(y)}{R}\right)U_s \\
&\geq r(y)\left[(1 + \varepsilon\frac{\alpha(y)}{r(y)})z_{ss} + \frac{c}{R}z_s + \frac{f(y,U)}{r(y)}\right] \\
&\geq r(y)\left[(1+\bar{r}\varepsilon)z_{ss} + \frac{c}{R}z_s - M\right] = 0.
\end{aligned}$$

In the calculation we have used that $U_s \geq 0$ of corollary 4.2 and $c > 0$ at the first inequality sign. At the second inequality sign we used $z_{ss} < 0$.

Due to the 1-periodicity in y, the function $U - z$ achieves its minimum in $[-b, b] \times T^n$. Therefore and because of the preceding calculation, we can apply the strong minimum principle to the function $U - z$. We obtain that it achieves its minimum at the boundary. Because of

$$U - z|_{s=-b} = U - z|_{s=b} = 0,$$

it follows that we have $U \geq z$ in $[-b, b] \times T^n$. In particular

$$\theta = \max_{y \in T^n} U(0, y) \geq z(0).$$

From the formula for z we can see

$$z(0) = D + C = \frac{MRb}{c} + \left(1 - \exp\left(\frac{cb}{R(1+\bar{r}\varepsilon)}\right)\right) \frac{\frac{2MRb}{c} - 1}{\exp\left(\frac{cb}{R(1+\bar{r}\varepsilon)}\right) - \exp\left(\frac{-cb}{R(1+\bar{r}\varepsilon)}\right)}.$$

One can see $z(0) \to 1$ for $c \to \infty$. On the other hand we obtained $z(0) \leq \theta < 1$ by comparison with U. Therefore there is a $K_{c>0} = K_{c>0}(\theta, \varepsilon, b, \bar{r}, R, M)$ with $0 \leq c \leq K_{c>0}$.

Case 2: We now consider the case $c < 0$. Let $z = z(s)$ be the solution of

$$(1 + \bar{r}\varepsilon) z_{ss} + \frac{c}{R} z_s + M = 0,$$
$$z(-b) = 0, \qquad z(b) = 1.$$

The general solution of the ODE is given by

$$z(s) = -\frac{MRs}{c} + D + C \exp\left(\frac{-cs}{R(1 + \bar{r}\varepsilon)} \right).$$

The boundary conditions read

$$0 = z(-b) = \frac{MRb}{c} + D + C \exp\left(\frac{cb}{R(1 + \bar{r}\varepsilon)} \right)$$

and

$$1 = z(b) = -\frac{MRb}{c} + D + C \exp\left(\frac{-cb}{R(1 + \bar{r}\varepsilon)} \right).$$

This leads to

$$C = \frac{-\frac{2MRb}{c} - 1}{\exp\left(\frac{cb}{R(1+\bar{r}\varepsilon)}\right) - \exp\left(\frac{-cb}{R(1+\bar{r}\varepsilon)}\right)}$$

and

$$D = -\frac{MRb}{c} + \frac{\frac{2MRb}{c} + 1}{\exp\left(\frac{cb}{R(1+\bar{r}\varepsilon)}\right) - \exp\left(\frac{-cb}{R(1+\bar{r}\varepsilon)}\right)} \exp\left(\frac{cb}{R(1 + \bar{r}\varepsilon)} \right).$$

Similarly as in the first case, we have $sign(z_{ss}) = sign(C)$. It is sufficient to consider the case $c < -2MRb$, for which we have $C > 0$ (both numerator and denominator are negative) and hence $z_{ss} > 0$. In order to use the maximum principle, we calculate analogously to the first case:

$$\begin{aligned}
& - \varepsilon\alpha(y)(U - z)_{ss} - \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})(U - z) - \frac{c}{R} r(y)(U - z)_s \\
& = f(y, U) + (r(y) + \varepsilon\alpha(y)) z_{ss} + \frac{cr(y)}{R} z_s + \left(c - \frac{cr(y)}{R} \right) U_s \\
& \leq r(y) \left[(1 + \varepsilon\frac{\alpha(y)}{r(y)}) z_{ss} + \frac{c}{R} z_s + \frac{f(y, U)}{r(y)} \right] \\
& \leq r(y) \left[(1 + \bar{r}\varepsilon) z_{ss} + \frac{c}{R} z_s + M \right] = 0.
\end{aligned}$$

In the calculation, we have used $U_s \geq 0$ and $c < 0$ at the first inequality sign. At the second inequality sign we used $z_{ss} > 0$ for $c < -2MRb$.

We apply the strong maximum principle to the function $U - z$. Due to

$$U - z|_{s=-b} = U - z|_{s=b} = 0,$$

we obtain $U \leq z$. In particular

$$0 < \theta = \max_{y \in [0,1]} U(0, y) \leq z(0),$$

where

$$z(0) = D + C = -\frac{MRb}{c} + \left(1 - \exp\left(\frac{cb}{R(1+\bar{r}\varepsilon)}\right)\right) \frac{-\frac{2MRb}{c} - 1}{\exp\left(\frac{cb}{R(1+\bar{r}\varepsilon)}\right) - \exp\left(\frac{-cb}{R(1+\bar{r}\varepsilon)}\right)}.$$

We read off that for $c \to -\infty$, there holds $z(0) \to 0$. From this and $z(0) \geq \theta$ we deduce that there is a constant $K_{c<0} = K_{c<0}(\theta, \varepsilon, b, \bar{r}, R, M)$, such that $0 > c > -K_{c<0}$.

Both cases $c > 0$ and $c < 0$ combined yield $|c| \leq K(\theta, \varepsilon, b, \bar{r}, R, M)$ with $K = K(\theta, \varepsilon, b, \bar{r}, R, M) = \max\{K_{c<0}, K_{c>0}\}$.

Step 2: Estimates for U. It remains to show the estimates for $||U||_{C^1(R_b)}$. To obtain these estimates, we will use L^p-estimates.

Clearly we have $||U||_{C^1(R_b)} = ||U||_{C^1([-b,b]\times[0,1]^n)}$ by periodicity. For some $\delta > 0$, we choose the bounded Lipschitz-domain

$$\Omega := (-b, b) \times (-\delta, 1 + \delta)^n.$$

Let furthermore

$$\Omega_1 := \left(-b, \frac{b}{2}\right) \times (0,1)^n \text{ and } \Omega_2 := \left(-\frac{b}{2}, b\right) \times (0,1)^n,$$

as well as

$$P_1 := \{-b\} \times [-\delta, 1+\delta]^n \text{ and } P_2 := \{b\} \times [-\delta, 1+\delta]^n.$$

Then P_1, P_2 are $C^{1,1}$ boundary portions of Ω with $U = 0$ on P_1 and $U - 1 = 0$ on P_2. Furthermore $\Omega_1 \Subset \Omega \cup P_1$ and $\Omega_2 \Subset \Omega \cup P_2$.

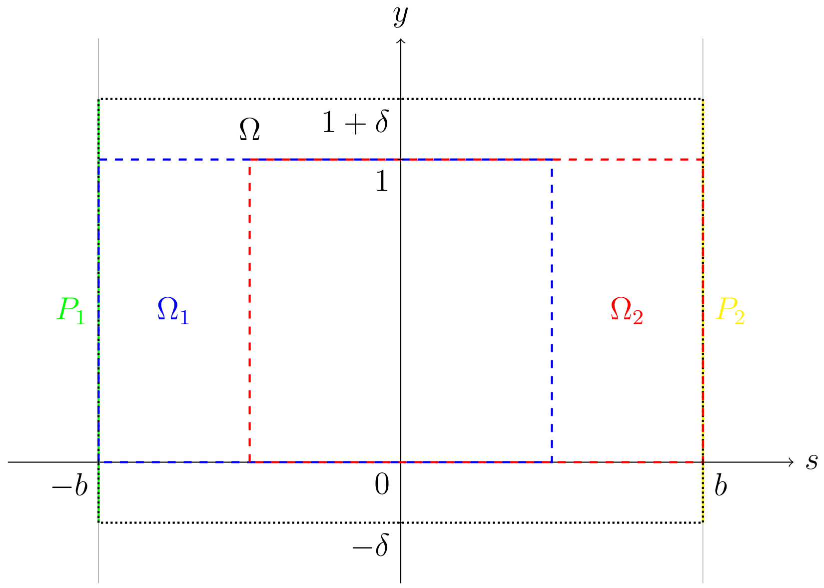
In this situation, L^p-estimates for some arbitrary $1 < p < \infty$ yield (see for example [12], Theorem 9.13)

$$||U||_{W^{2,p}(\Omega_1)} \leq C_1 \left(||U||_{L^p(\Omega)} + ||f(y,U)||_{L^p(\Omega)}\right) \leq C_1 \left(|\Omega|^{\frac{1}{p}} + MR\,|\Omega|^{\frac{1}{p}}\right)$$

and

$$||U-1||_{W^{2,p}(\Omega_2)} \leq C_2 \left(||U-1||_{L^p(\Omega)} + ||f(y,U)||_{L^p(\Omega)}\right) \leq C_2 \left(|\Omega|^{\frac{1}{p}} + MR\,|\Omega|^{\frac{1}{p}}\right).$$

We have used here that $||f||_{L^\infty} \leq MR$. The two constants C_i, $i = 1, 2$ depend on c, but c is already bounded by $K(\theta, \varepsilon, b, \bar{r}, R, M)$. Altogether, the two constants C_i,



$i = 1, 2$ depend on $\tilde{\mu}, \max\{\Lambda, K(\theta, \varepsilon, b, \bar{r}, R, M)\}, P_i, \Omega_i, \Omega$, the moduli of continuity of the coefficients a_{ij} and $\varepsilon\alpha(y)$ and the already established bounds for c from step 1, namely $K(\theta, \varepsilon, b, \bar{r}, R, M)$. Since $||U||_{L^p((-b,b)\times(0,1)^n)} \leq (2b)^{\frac{1}{p}}$ and

$$||U||_{W^{2,p}((-b,b)\times(0,1)^n)} \leq ||U||_{W^{2,p}(\Omega_1)} + ||U-1||_{W^{2,p}(\Omega_2)} + ||U||_{L^p((-b,b)\times(0,1)^n)},$$

we obtain

$$||U||_{W^{2,p}((-b,b)\times(0,1)^n)} \leq \tilde{K}(\theta, \varepsilon, b, \bar{r}, R, M, \tilde{\mu}, \Lambda, \{\omega_{ij}|i, j = 1, ..., n\}, \omega).$$

We have omitted the non-relevant dependencies on Ω and so forth. We choose p such that $1 < 2 - \frac{n}{p}$ and Sobolev-Imbeddings give us the desired estimate

$$||U||_{C^1(R_b)} \leq K(\theta, \varepsilon, b, \bar{r}, R, M, \tilde{\mu}, \Lambda, \{\omega_{ij}|i, j = 1, ..., n\}, \omega).$$

Choosing the larger one of the two bounds (for $|c|$ of step 1 and for $||U||_{C^1(R_b)}$ of step 2) finishes the proof. ∎

The linear problem

The regularized solution we are looking for will be constructed as a fixed point of a certain linear elliptical boundary value problem with periodicity conditions in y. We treat the solvability of this linear problem in the next lemma.

Lemma 4.5 (Solvability of the linear problem) *Let f be a nonlinearity of basic type as in definition 2.6. Furthermore, let $\beta \in C^1(T^n)$, $\beta > 0$ and $(a_{ij}(x))_{i,j=1}^n$ as* *in the general assumptions. Moreover, let $v \in C^1(R_b)$, where $R_b = [-b, b] \times T^n$. Then the problem*

$$LU := \beta(y)U_{ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})U + cU_s = -f(y, v) \text{ on } [-b, b] \times \mathbb{R}^n,$$
$$U(-b, \cdot) = 0, \qquad U(s, \cdot) \text{ 1-periodic}, \qquad U(b, \cdot) = 1,$$

has a unique classical solution U. Moreover, we even have $U \in C^{2,\alpha}(R_b)$ for any $\alpha \in (0, 1)$.

Proof Let us initially informally discuss the idea of the proof. We will begin by showing that there is a unique bounded solution of $LU = -f(y, v)$ on $[-b, b] \times \mathbb{R}^n$. This can be done by solving the equation on a sequence of increasing domains Ω_m with appropriate boundary values. With the help of the maximum principle and a supersolution one can then show that such a sequence of solutions stays bounded. We derive local $C^{2,\alpha}$-estimates. This allows us to choose a subsequence, which converges locally in C^2. Hence, we obtain a bounded solution on the entire domain. With a maximum principle for unbounded domains one can show the uniqueness of this bounded solution and deduce from that the periodicity of this solution.

Step 1: Construction of a sequence of solutions on bounded domains. For $m \in \mathbb{N}$ we choose Ω_m to be a bounded domain with $C^{2,\alpha}$ boundary, such that $\Omega_m \subset \Omega_{m+1}$ and $(-b, b) \times (-m, m)^n \subset \Omega_m$ for all $m \in \mathbb{N}$. Furthermore, we choose boundary values: Let

$$\varphi_m(s, y) = \varphi(s, y) = \frac{s + b}{2b} \text{ for all } m \in \mathbb{N}. \tag{4.6}$$

Then $\varphi_m \in C^{2,\alpha}(\overline{\Omega_m})$.

In this situation, we know from Schauder-theory (see for example [12] theorem 6.14), that there is a solution $u_m \in C^{2,\alpha}(\overline{\Omega_m})$ of

$$Lu_m = -f(y, v) \text{ on } \Omega_m, \qquad u_m = \varphi_m \text{ on } \partial\Omega_m.$$

Step 2: Uniform bounds for u_m. We show now that the sequence of solutions stays bounded. By this, we mean that there exists $C_0 > 0$, which is independent of m, such that $||u_m||_{C^0(\overline{\Omega_m})} \leq C_0$. For this purpose we define

$$w(s, y) = -\exp(l \cdot s).$$

Here, l is a constant, which we choose sufficiently large, such that for some $\gamma > 0$ we have (with $r(y) = \sum_{i,j=1}^{n} a_{ij}(y)k_ik_j$)

$$-Lw = ((\beta(y) + r(y))l^2 + cl)w \geq \gamma > 0 \text{ on } [-b, b] \times \mathbb{R}^n.$$

Hence, for $K = \frac{||f||_{L^\infty}}{\gamma}$ and all $m \in \mathbb{N}$ we have

$$L(u_m - Kw) \geq -f(y, v) + \gamma K \geq 0,$$
$$L(u_m + Kw) \leq -f(y, v) - \gamma K \leq 0.$$

Therefore (and because w is negative and bounded on $[-b, b] \times \mathbb{R}^n$) we get from the maximum principle

$$\sup_{\Omega_m} u_m \leq \sup_{\Omega_m}(u_m - Kw) \leq \sup_{\partial\Omega_m}(u_m - Kw) \leq 1 + K\,||w||_{L^\infty([-b,b]\times\mathbb{R}^n)} =: C_0.$$

Similarly, we get from the minimum principle

$$\inf_{\Omega_m} u_m \geq \inf_{\Omega_m}(u_m + Kw) \geq \inf_{\partial\Omega_m}(u_m + Kw) \geq 0 - K\,||w||_{L^\infty([-b,b]\times\mathbb{R}^n)} \geq -C_0.$$

Together, this implies

$$||u_m||_{C^0(\overline{\Omega_m})} \leq C_0.$$

This concludes the second step.

Step 3: $C^{2,\alpha}$-estimates. Now we derive local $C^{2,\alpha}([-b, b] \times \mathbb{R}^n)$ estimates in order to pass to the limit $m \to \infty$ in the step 4.

Step 3a: Domains, that do not touch the boundary. First, we consider domains that do not touch the boundary of $[-b, b] \times \mathbb{R}^n$. Let Ω' and Ω be domains that satisfy $\Omega' \Subset \Omega \Subset (-b, b) \times \mathbb{R}^n$. There exists $m_0 = m_0(\Omega) \in \mathbb{N}$, such that for $m \geq m_0$, there holds $\Omega \subset \Omega_m$. Interior Schauder estimates for u_m give (see for example [12] corollary 6.3) for $m \geq m_0$:

$$||u_m||_{C^{2,\alpha}(\overline{\Omega'})} \leq C\left(||u_m||_{C^0(\overline{\Omega})} + ||f(y,v)||_{C^\alpha(\overline{\Omega})}\right) \leq C\left(C_0 + ||f(y,v)||_{C^\alpha(\overline{\Omega})}\right) =: \tilde{C}.$$

Therefore $||u_m||_{C^{2,\alpha}(\overline{\Omega'})} \leq \tilde{C}$. The constant $\tilde{C}$ depends on Ω' and Ω, but not on m. That is, we have $C^{2,\alpha}$-estimates on bounded domains that do not touch the boundary of $(-b, b) \times \mathbb{R}^n$.

Step 3b: Domains that touch the boundary. Let x_0 be a boundary point of $[-b, b] \times \mathbb{R}^n$, i.e. either $x_0 = (-b, \overline{x}_0)$ or $x_0 = (b, \overline{x}_0)$ with $\overline{x}_0 \in \mathbb{R}^n$. For $\rho > 0$, let Ω' be a domain of the form

$$\Omega' = \big((-b, b) \times \mathbb{R}^n\big) \cap B_\rho(x_0).$$

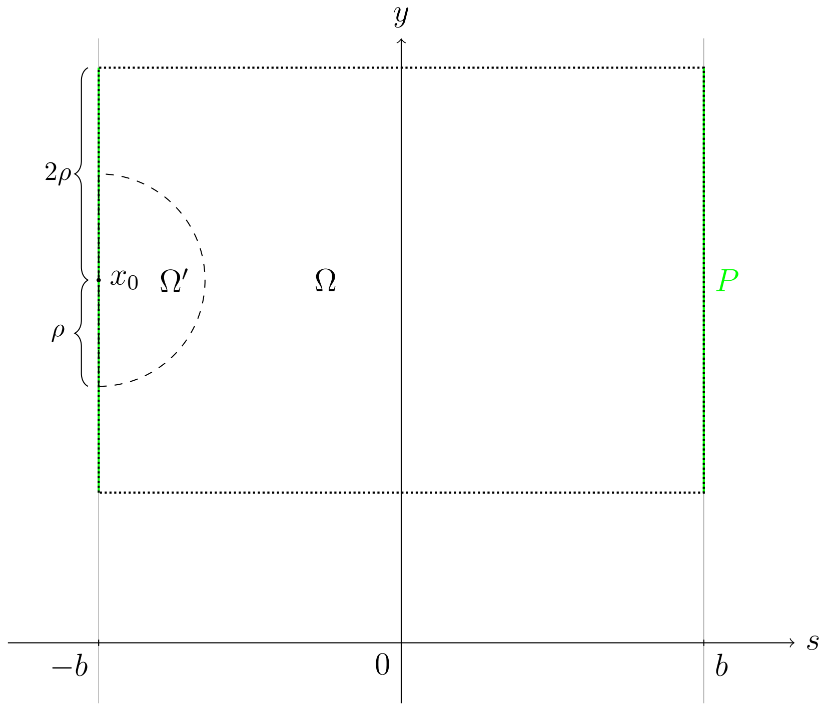
Additionally, we take a larger domain

$$\Omega = (-b, b) \times \big(\overline{x}_0 + (-2\rho, 2\rho)^n\big).$$

Moreover, we take a $C^{2,\alpha}$ boundary portion P of Ω:

$$P = \Big[\{-b\} \times \big(\overline{x}_0 + (-2\rho, 2\rho)^n\big)\Big] \cup \Big[\{b\} \times \big(\overline{x}_0 + (-2\rho, 2\rho)^n\big)\Big].$$

Clearly, there holds $\rho < dist(x_0, \partial\Omega \setminus P)$ (which is necessary to apply the following Schauder estimates). There exists $m_0 = m_0(\Omega)$, such that for $m \geq m_0$, we have $\Omega \subset \Omega_m$.



With the affine function $\varphi = \frac{s+b}{2b}$ as in (4.6) we have $\varphi \in C^{2+\alpha}(\Omega)$ and $u_m = \varphi$ on P for $m \geq m_0$. For this situation Schauder estimates yield (see for example [12] corollary 6.7)

$$
\begin{aligned}
||u_m||_{C^{2,\alpha}(\overline{\Omega'})} &\leq C\left(||u_m||_{C^0(\overline{\Omega})} + ||\varphi||_{C^{2,\alpha}(\overline{\Omega})} + ||f(y,v)||_{C^\alpha(\overline{\Omega})}\right) \\
&\leq C\left(C_0 + ||\varphi||_{C^{2,\alpha}(\overline{\Omega})} + ||f(y,v)||_{C^\alpha(\overline{\Omega})}\right).
\end{aligned}
$$

The right hand side of this inequality is independent of m, which concludes step 3b.

Step 4: $m \to \infty$. From step 3 we now have estimates $||u_m||_{C^{2,\alpha}_{loc}([-b,b]\times\mathbb{R}^n)} \leq C$ and $||u_m||_{C^0(\overline{\Omega_m})} \leq C_0$. By the Arzelà-Ascoli theorem and a diagonal argument we can therefore choose a subsequence (which will again be named u_m) such that $u_m \to U$ in $C^2_{loc}([-b,b]\times\mathbb{R}^n)$, where

$$U \in C^2_{loc}([-b,b]\times\mathbb{R}^n) \cap L^\infty([-b,b]\times\mathbb{R}^n).$$

This is sufficient to pass to the limit in the equation and we get $LU = -f(y,v)$. We also obtain the boundary conditions $U(-b,\cdot) = 0$ and $U(b,\cdot) = 1$. Elliptic regularity says in this situation that (see for example [12] lemma 6.18)

$$U \in C^{2,\alpha}_{loc}([-b,b]\times\mathbb{R}^n). \tag{4.7}$$

Step 5: Periodicity and $U \in C^{2,\alpha}(R_b)$. In this step we show that $U(s,\cdot)$ is 1-periodic. To this end let $1 \leq m \leq n$ be arbitrary and $\tilde{U}(s,y) := U(s, y+e_m)$ where e_m is the m-th unit vector. Let $\Omega = (-b,b) \times \mathbb{R}^n$. We calculate

$$\begin{aligned}
&L\tilde{U}(s,y)\\
&= \beta(y)\tilde{U}_{ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})\tilde{U} + c\tilde{U}_s\\
&= \beta(y+e_m)U_{ss}(s,y+e_m) + \sum_{i,j=1}^{n} a_{ij}(y+e_m)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})U(s,y+e_m)\\
&\qquad\qquad + cU_s(s,y+e_m)\\
&= (LU)(s,y+e_m) = f(y+e_m, v(s,y+e_m)) = f(y, v(s,y)) = LU(s,y).
\end{aligned}$$

Therefore $L(U - \tilde{U}) = 0$ and $U - \tilde{U} = 0$ on $\partial\Omega$. Since Ω is unbounded we need a maximum principle for unbounded domains. We use lemma A.1 from Berestycki, Hamel and Rossi. Let $w(s,y) = -\exp(l \cdot s)$ as in step 2, such that $-Lw \geq \gamma > 0$. Then $\tilde{w}(s,y) := w(s,y) + 2\exp(l \cdot b)$ still satisfies $L\tilde{w} \geq \gamma > 0$ and also $\tilde{w} \geq \tilde{\gamma} > 0$ for some $\tilde{\gamma} > 0$. Since $U - \tilde{U}$ is bounded, we can apply Lemma A.1 (ii) to the function $U - \tilde{U}$ and get $U - \tilde{U} \leq 0$. The same can be done for $\tilde{U} - U$. Consequently $\tilde{U} = U$. Since $m \in \{1, ..., n\}$ was arbitrary, we conclude that U is 1-periodic. This, together with (4.7), implies $U \in C^{2,\alpha}(R_b)$.

Step 6: Uniqueness. In the argumentation of step 5, we have already seen that there exists at most one classical solution of the problem (without the periodicity condition) which is bounded (Lemma A.1 is only applicable to bounded functions). However, a solution of the problem (with the periodicity condition) is necessarily bounded and therefore unique. ∎

In order to apply the Leray-Schauder degree, we have to study how the solution of the linear problem behaves under homotopy of the coefficients and f. In doing so, we will use the following Schauder type estimates in a half-periodic setting. The estimates will also be applied in another proof.

Lemma 4.6 (Schauder type estimates) *Denote by R_b the cylinder $R_b = [-b,b] \times T^n$ and by $x = (x_0, x_1, ..., x_n)$ its points with $x_0 \in [-b,b]$, $(x_1, ..., x_n) \in T^n$. We consider the operator L given by*

$$Lu = \sum_{i,j=0}^{n} a_{ij}(x)u_{x_i,x_j} + \sum_{i=0}^{n} b_i(x)u_{x_i} + c(x)u \text{ on } R_b.$$

For the coefficients of the operator we assume $a_{ij}, b_i, c \in C^\alpha(R_b)$ for $i,j = 1, ..., n$ and the matrix field $(a_{ij})_{i,j=0}^n$ is supposed to satisfy

$$\sum_{i,j=0}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda \, ||\xi||^2 \text{ for all } x \in R_b, \xi \in \mathbb{R}^{n+1}$$

for some $\lambda > 0$. Let $\Lambda > 0$ be such that

$$||a_{ij}||_{C^\alpha(R_b)} \leq \Lambda, \qquad ||b_i||_{C^\alpha(R_b)} \leq \Lambda, \qquad ||c||_{C^\alpha(R_b)} \leq \Lambda.$$

Let $f \in C^\alpha(R_b)$ and $\varphi \in C^{2,\alpha}(R_b)$. Moreover, let $u \in C^{2,\alpha}(R_b)$ be a solution of

$$Lu = f \text{ in } R_b, \qquad u = \varphi \text{ on } \partial R_b.$$

Then there is a constant $C = C(n, \alpha, \lambda, \Lambda, b)$, such that

$$||u||_{C^{2,\alpha}(R_b)} \leq C \left(||u||_{C^0(R_b)} + ||\varphi||_{C^{2,\alpha}(R_b)} + ||f||_{C^\alpha(R_b)} \right). \tag{4.8}$$

Proof First of all we note that for functions on R_b, due to the 1-periodicity in y, the following holds. If $\Omega \subset \mathbb{R}^{n+1}$ is a domain with with $[-b, b] \times [0,1]^n \subset \bar{\Omega}$ and if $g \in C^{k,\alpha}(R_b)$ with $k \in \mathbb{N}$, then $||g||_{C^{k,\alpha}(R_b)} = ||g||_{C^{k,\alpha}(\bar{\Omega})}$.

We take the boundary point $\bar{x} = (-b, 0_{T^n}) \in R_b$ and a sufficiently large $\rho > 0$, such that $[-b, b] \times [0,1]^n \subset B_\rho(\bar{x})$. Furthermore, we consider the larger domain $\Omega = (-b, b) \times (-2\rho, 2\rho)^n$ and a boundary portion $P = \{-b\} \times [-2\rho, 2\rho]^n \cup \{b\} \times [-2\rho, 2\rho]^n$. Clearly, P is a $C^{2,\alpha}$ boundary portion of Ω. There holds $\rho < dist(\bar{x}, \partial\Omega \setminus P)$. In this situation, Schauder estimates yield (see for example [12] Corollary 6.7)

$$\begin{aligned} ||u||_{C^{2,\alpha}(R_b)} = ||u||_{C^{2,\alpha}(\overline{B_\rho(\bar{x}) \cap \Omega})} &\leq C \left(||u||_{C^0(\bar{\Omega})} + ||\varphi||_{C^{2,\alpha}(\bar{\Omega})} + ||f||_{C^\alpha(\bar{\Omega})} \right) \\ &= C \left(||u||_{C^0(R_b)} + ||\varphi||_{C^{2,\alpha}(R_b)} + ||f||_{C^\alpha(R_b)} \right). \end{aligned}$$

At the last equality sign, we have used the inclusion $[-b, b] \times [0,1]^n \subset \overline{B_\rho(\bar{x}) \cap \Omega}$. The constant C depends on n, α, λ, Λ and $B_\rho(\bar{x}) \cap \Omega$. The choice of ρ in the definition of $B_\rho(\bar{x})$ and Ω can be given in terms of b. This finishes the proof. ■

With these estimates we can study the behavior of the linear problem under a change of the coefficients and f:

Lemma 4.7 (Behavior of the linear problem under a homotopy) *Let f be a nonlinearity of basic type. Furthermore, let $R_b = [-b, b] \times T^n$, $E_1 = C^1(R_b)$ and $E_2 = E_1 \times \mathbb{R}$. For $(v, c) \in E_2$ and $\tau \in [0,1]$, let $u = \varphi_\tau(v, c)$ be the unique solution from lemma 4.5 of the problem*

$$\begin{aligned} &L_\tau U + cU_s \\ &:= \varepsilon[\tau\alpha(y) + (1-\tau)]U_{ss} + \sum_{i,j=1}^n [\tau a_{ij}(y) + \delta_{ij}(1-\tau)](k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})U + cU_s \\ &= -\tau f(y, v), \\ &U(-b, \cdot) = 0, \qquad U(s, \cdot) \text{ 1-periodic}, \qquad U(b, \cdot) = 1. \end{aligned}$$

Then the mapping

$$G : [0,1] \times E_2 \to E_1, \qquad (\tau, (v, c)) \mapsto \varphi_\tau(v, c)$$

is continuous and compact.

PROOF We will show the continuity of the mapping at a point $(\tau,(v,c)) \in [0,1] \times E_2$. So let $(\tau,(v,c)),(\tilde{\tau},(\tilde{v},\tilde{c})) \in [0,1] \times E_2$ and $U = \varphi_\tau(v,c)$, $\tilde{U} = \varphi_{\tilde{\tau}}(\tilde{v},\tilde{c})$.

Step 1: The first step is to show a weaker result. We claim that the mapping

$$\tilde{G} : [0,1] \times E_2 \to C^0(R_b), \qquad (\tau,(v,c)) \mapsto \varphi_\tau(v,c)$$

is continuous ($\tilde{G}$ coincides with G, but the image E_1 is replaced by $C^0(R_b)$). This will be shown with the maximum principle and the minimum principle. These can be applied due to the periodicity of U and $\tilde{U}$ in y. We calculate

$$\begin{aligned} &L_{\tilde{\tau}}(\tilde{U}-U) + \tilde{c}(\tilde{U}-U)_s \\ &= (L_\tau - L_{\tilde{\tau}})U + (c-\tilde{c})U_s - \tilde{\tau} f(y,\tilde{v}) + \tau f(y,v) \\ &= (L_\tau - L_{\tilde{\tau}})U + (c-\tilde{c})U_s - (\tilde{\tau}-\tau) f(y,v) - \tilde{\tau}(f(y,\tilde{v}) - f(y,v)). \end{aligned} \tag{4.9}$$

Using $U \in C^2(R_b)$ by lemma 4.5, we see that all of the four terms on the right hand side become small in the $L^\infty(R_b)$ norm, when $(\tilde{\tau},(\tilde{v},\tilde{c}))$ is close to $(\tau,(v,c))$. In the last term, one has to use the Lipschitz-continuity of f to see this. Hence, for every $\eta > 0$, there is $1 > \delta > 0$, such that

$$\left|\left| L_{\tilde{\tau}}(\tilde{U}-U) + \tilde{c}(\tilde{U}-U)_s \right|\right|_{L^\infty(R_b)} \leq \eta \text{ for } (\tilde{\tau},(\tilde{v},\tilde{c})) \in B_\delta(\tau,(v,c)). \tag{4.10}$$

Step 1a): Finding a comparison function. Let $w(s,y) = -\exp(l \cdot s)$, where $l \in \mathbb{N}$ is still to be chosen. We define

$$r_\tau(y) := \sum_{i,j=1}^{n} [\tau a_{ij}(y) + \delta_{ij}(1-\tau)] k_i k_j \text{ and } \alpha_\tau(y) := \tau\alpha(y) + (1-\tau).$$

With this notation we have

$$\begin{aligned} -L_{\tilde{\tau}} w - \tilde{c} w_s &= [\varepsilon \alpha_{\tilde{\tau}}(y) + r_{\tilde{\tau}}(y)] l^2 \exp(l \cdot s) + \tilde{c}\, l \exp(l \cdot s) \\ &\geq ([\varepsilon \min\{\alpha(y),1\} + \min\{r_1(y),1\}] l^2 + \tilde{c}\, l) \exp(l \cdot s). \end{aligned}$$

We can see that there is some possibly small $\gamma > 0$, such that for some large enough l, there holds

$$-L_{\tilde{\tau}} w - \tilde{c} w_s \geq \gamma > 0 \text{ for all } \tilde{\tau} \in [0,1] \text{ and } \tilde{c} \text{ with } |\tilde{c} - c| \leq 1. \tag{4.11}$$

Step 1b): Carrying out the comparison. Let η and δ be as in inequality (4.10) and w and γ be as in step 1a). Moreover, let $K = \frac{\eta}{\gamma}$ and $(\tilde{\tau},(\tilde{v},\tilde{c})) \in B_\delta(\tau,(v,c))$. Then we have by (4.11) and (4.10):

$$L_{\tilde{\tau}}(\tilde{U} - U - Kw) + \tilde{c}(\tilde{U} - U - Kw)_s \geq -\eta + K\gamma = 0.$$

By the maximum principle and the negativity of w, we obtain

$$\sup_{R_b}\left(\tilde{U} - U\right) \leq \sup_{R_b}\left(\tilde{U} - U - Kw\right) = \sup_{\partial R_b}\left(\tilde{U} - U - Kw\right) = \sup_{\partial R_b}(-Kw) \leq \eta\frac{||w||_\infty}{\gamma}.$$

Analogously, we have

$$L_{\tilde{\tau}}(\tilde{U} - U + Kw) + \tilde{c}(\tilde{U} - U + Kw)_s \leq \eta - K\gamma = 0$$

and by the minimum principle

$$\inf_{R_b} \left(\tilde{U} - U\right) \geq \inf_{R_b} \left(\tilde{U} - U + Kw\right) = \inf_{\partial R_b} \left(\tilde{U} - U + Kw\right) = \inf_{\partial R_b} Kw \geq -\eta \frac{||w||_\infty}{\gamma}.$$

In conclusion, we have the following: For every $\eta > 0$, there exists $\delta > 0$, such that

$$\left|\left|\tilde{U} - U\right|\right|_\infty \leq \eta \frac{||w||_\infty}{\gamma} \text{ for } (\tilde{\tau}, (\tilde{v}, \tilde{c})) \in B_\delta(\tau, (v, c)).$$

This is the continuity of the mapping $\tilde{G}$.

Step 2: The second step is to show the continuity of the mapping

$$\tilde{\tilde{G}} : [0,1] \times E_2 \to C^{2,\alpha}(R_b), \qquad (\tau, (v, c)) \mapsto \varphi_\tau(v, c).$$

This is a stronger result than the continuity of the mapping G. To prove it, we want to apply the Schauder type estimates from lemma 4.6 to the family of elliptic operators ($\tilde{\tau}$ and $\tilde{c}$ are not fixed) $L_{\tilde{\tau}} - \tilde{c}\partial_s$ on the left hand side of equation (4.9). However, these estimates involve a constant that depends on the ellipticity constants and some bounds on the coefficients. It is, however, easy to see that the family operators admits a common ellipticity constant $\tilde{\mu}$ and also a common upper bound Λ on the $C^\alpha(\overline{\Omega})$-Norm of the second-order coefficients. Furthermore we only have to consider $\tilde{c}$ in the vicinity of c, therefore by further increasing Λ, it is sufficient to consider $\tilde{c}$ with $||\tilde{c}||_{C^\alpha(\overline{\Omega})} \leq \Lambda$ (of course $\tilde{c}$ is constant). On the boundary we have $\tilde{U} - U = 0 =: \varphi$. The Schauder type estimates (4.8) from lemma 4.6 yield

$$\left|\left|\tilde{U} - U\right|\right|_{C^{2,\alpha}(R_b)} \leq C \left(\left|\left|\tilde{U} - U\right|\right|_{C^0(R_b)} + \left|\left|L_{\tilde{\tau}}(\tilde{U} - U) + \tilde{c}(\tilde{U} - U)_s\right|\right|_{C^\alpha(R_b)} \right). \quad (4.12)$$

Here, $C = C(n, \alpha, \tilde{\mu}, \Lambda, b)$. In particular, C does not depend on $\tilde{\tau}, \tilde{c}$ or $\tilde{v}$.

We have already proven the continuity of the mapping $\tilde{G}$ in step 1. Therefore, we have

$$\left|\left|\tilde{U} - U\right|\right|_{C^0(R_b)} \to 0 \text{ as } (\tilde{\tau}, (\tilde{v}, \tilde{c})) \to (\tau, (v, c)) \text{ in } [0,1] \times E_2$$

on the right hand side of (4.12). To estimate the second term on the right hand side of (4.12), we take a look at our calculation (4.9) again. From it, we can even deduce more, namely

$$\left|\left|L_{\tilde{\tau}}(\tilde{U} - U) + \tilde{c}(\tilde{U} - U)_s\right|\right|_{C^\alpha(R_b)} \to 0 \text{ as } (\tilde{\tau}, (\tilde{v}, \tilde{c})) \to (\tau, (v, c)) \text{ in } [0,1] \times E_2.$$

Therefore, it follows from (4.12) that

$$\left|\left|\tilde{U} - U\right|\right|_{C^{2,\alpha}(R_b)} \to 0 \text{ as } (\tilde{\tau}, (\tilde{v}, \tilde{c})) \to (\tau, (v, c)) \text{ in } [0,1] \times E_2.$$

This is the continuity of the mapping $\tilde{\tilde{G}}$.

Step 3: The continuity and compactness of the mapping G. The continuity of the mapping $\tilde{\tilde{G}}$ (as shown in step 2) directly implies the continuity of the mapping G. Furthermore, it yields the compactness of the mapping G. This follows from the compactness of inclusion mapping

$$\iota : C^{2,\alpha}(R_b) \to E_1 = C^1(R_b), \qquad u \mapsto u$$

and $G = \iota \circ \tilde{\tilde{G}}$. ∎

Proof of existence for the (ε, b)-Problem

Now that we studied the linear problem, we are ready to show the existence of the (ε, b)-solution. For the following proposition, compare with proposition 2.1 in [21]:

Proposition 4.8 *We denote by R_b the cylinder $[-b, b] \times T^n$. Let f be a nonlinearity of basic type. Then for every $\varepsilon > 0$, $b > 0$ and $\theta \in (0, 1)$, there is a unique classical solution (U, c) of the (ε, b)-problem (2.7), (2.8) on R_b. This solution has the regularity $U \in C^{2,\alpha}(R_b)$ for any $\alpha \in (0, 1)$.*

Proof Let us briefly discuss the structure of the proof. The first step will be to define a family of mappings F_τ with $\tau \in [0, 1]$ with a common domain of definition E. The mappings will be defined using a corresponding elliptic boundary value problem involving the parameter τ. Moreover, it is shown that $(id - F_1)(u, c) = 0$ if and only if (u, c) is the desired solution (U, c) of the (ε, b)-problem.

The second step will be to find a suitable subdomain D of E, such that the following holds: Firstly, for all $\tau \in [0, 1]$ the Leray-Schauder degree $d(id - F_\tau, D, 0)$ is well-defined. Secondly, for all $\tau \in [0, 1]$ we have $id - F_\tau \neq 0$ in $E \setminus D$. We then deduce from the homotopy invariance, that $d(id - F_1, D, 0) = d(id - F_0, D, 0)$. The last step is to show $d(id - F_0, D, 0) \neq 0$.

Step 0: Notation. We introduce some notation. We consider the spaces $E_1 = C^1(R_b)$ and $E_2 = E_1 \times \mathbb{R}$. For a parameter $\tau \in [0, 1]$, we define the coefficients $\alpha_\tau = \tau\alpha + (1-\tau) \in C^1(R_b)$ and the $C^1(R_b)$ matrix fields $(a^\tau_{ij})^n_{i,j=1} = (\tau a_{ij} + \delta_{ij}(1-\tau))^n_{i,j=1}$. Furthermore, we consider the nonlinearities $f_\tau := \tau f$. Since f is a nonlinearity of basic type, all f_τ are of basic type as well. Using this notation, we define the linear operators L_τ, given by

$$L_\tau u := \varepsilon\alpha_\tau(y)u_{ss} + \sum_{i,j=1}^{n} a^\tau_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})u.$$

We will also use the notation $r_\tau(y) = \sum^n_{i,j=1} a^\tau_{ij}(y)k_ik_j$.

Step 1a: The mappings F_τ. With $(v, c) \in E_2$ and $\tau \in [0, 1]$, we consider the elliptic boundary value problem

$$\begin{gathered} L_\tau u + cu_s = -f_\tau(y, v), \\ u(-b, \cdot) = 0, \qquad u(s, \cdot) \text{ 1-periodic}, \qquad u(+b, \cdot) = 1. \end{gathered} \tag{4.13}$$

As proved in lemma 4.5, this problem has a unique classical solution in $C^{2,\alpha}(R_b) \subset E_1$. We denote it by $\phi_\tau(v,c)$. Furthermore, we set

$$h_\tau(v,c) := \max_{y \in T^n, s=0} \phi_\tau(v,c)(s,y).$$

Now, we are ready to define the mappings F_τ:

$$\begin{aligned} F_\tau : E_2 &\to E_2 \\ (u,c) &\mapsto (\phi_\tau(u,c), c - h_\tau(u,c) + \theta). \end{aligned}$$

Step 1b: Let us make a simple observation: For given $(v,c) \in E_2$ and $\tau \in [0,1]$, it is possible that the function $\phi_\tau(v,c)$ has values that are not contained in $[0,1]$. If, however, $v = \phi_\tau(v,c)$, it follows easily from the maximum and minimum principles, that $0 \leq \phi_\tau(v,c) \leq 1$. This is because of the fact that the nonlinearities f_τ all vanish outside of $[0,1]$.

Due to this observation, for every $(u_\tau, c_\tau) \in E_2$ with $(id - F_\tau)(u_\tau, c_\tau) = 0$, there holds $0 \leq u_\tau \leq 1$. We note, that any $(u,c) \in E_2$ with $(id - F_1)(u,c) = 0$ also satisfies all other properties of the (ε, b)-problem (2.7), (2.8) and therefore is the (ε, b)-solution. The converse is also true: If (u,c) is the (ε, b)-solution, then $(id - F_1)(u,c) = 0$. Therefore (u,c) is the desired solution of the (ε, b)-problem if and only if $(id - F_1)(u,c) = 0$.

If we have $(u_\tau, c_\tau) \in E_2$ with $(id - F_\tau)(u_\tau, c_\tau) = 0$, then $0 \leq u_\tau \leq 1$, as we have just observed. Therefore, such (u_τ, c_τ), is an (ε, b)-solution in its own right: If in (2.7), (2.8), the coefficients α_τ and the matrix field $(a^\tau_{ij})^n_{i,j=1}$ take the place of α and $(a_{ij})^n_{i,j=1}$ and f_τ takes the place of f, then (u_τ, c_τ) solves (2.7), (2.8). Consequently, we can say, that there is at most one such (u_τ, c_τ) for every $\tau \in [0,1]$ by proposition 4.3. More importantly, it follows that the a priori estimates from lemma 4.4 can be applied.

Step 2: Applying the Leray-Schauder degree. We claim that the mapping $(\tau, (v,c)) \to F_\tau(v,c)$ from $[0,1] \times E_2$ to E_2 is continuous and compact. This can be seen as follows: The mapping has two components. We note that it was proved in lemma 4.7, that the mapping $(\tau, (v,c)) \to \phi_\tau(v,c)$ is a continuous and compact mapping from $[0,1] \times E_2$ to E_1. This also implies the continuity and compactness of the mapping $(\tau, (v,c)) \to h_\tau(v,c)$ from $[0,1] \times E_2$ to $\mathbb{R}$. (See the definition of h_τ in step 1a.) Consequently, the continuity and compactness for the second component is proved. Therefore the mapping $(\tau, (v,c)) \to F_\tau(v,c)$ from $[0,1] \times E_2$ to E_2 is continuous and compact, as we claimed.

Step 2a: Finding a suitable domain of definition D. To use the Leray-Schauder degree for the family of functions $id - F_\tau$, we have to restrict the functions F_τ to a bounded set $D \subset E_2$, such that $F_\tau(v,c) \neq (v,c)$ for all $(v,c) \in \partial D$. This is necessary for the Leray-Schauder degree $d(id - F_\tau, D, 0)$ to be well defined. More strongly, we want a possible zero of $id - F_\tau$ to be contained in D. That means $F_\tau(v,c) \neq (v,c)$ for all $(v,c) \in E_2 \setminus D$.

To find such a set D we will use the a priori estimates from lemma 4.4. We already observed in step 1b that this lemma can be applied to all pairs $(u_\tau, c_\tau) \in E_2$ with $(id - F_\tau)(u_\tau, c_\tau) = 0$. The family of operators $\{L_\tau : \tau \in [0,1]\}$ admits a common ellipticity constant $\tilde{\mu}$. Also there is a $\Lambda > 0$, such that

$$\left|\left|a^\tau_{ij}\right|\right|_\infty \leq \Lambda \text{ and } ||\varepsilon \alpha_\tau||_\infty \leq \Lambda \text{ for all } \tau \in [0,1].$$

There are $\bar{r} > 0$, $R > 0$ and $M > 0$ such that

$$\left|\left|\frac{\alpha_\tau}{r_\tau}\right|\right|_\infty \leq \bar{r}, ||r_\tau||_\infty \leq R \text{ and } M \geq \sup_{y \in T^n, u \in [0,1]} \left|\frac{f_\tau(y,u)}{r_\tau(y)}\right| \text{ for all } \tau \in [0,1].$$

Moreover the families of coefficients $\{a^\tau_{ij} : \tau \in [0,1]\}$ and $\{\varepsilon\alpha_\tau : \tau \in [0,1]\}$ are clearly equicontinuous and therefore admit common moduli of continuity ω_{ij} and ω independently of τ. Now lemma 4.4 yields the existence of a constant

$$K = K(\theta, \varepsilon, b, \bar{r}, R, M, \tilde{\mu}, \Lambda, \{\omega_{ij} | i,j = 1, ..., n\}, \omega),$$

which is independent of τ, such that every fixed point (u_τ, c_τ) of F_τ satisfies $|c_\tau| \leq K$ and $||u_\tau||_{C^1(R_b)} \leq K$. Hence we choose

$$D = \{(u,c) \in E_2 : ||u||_{C_1(R_b)} \leq K+1, |c| \leq K+1\}.$$

Step 2b: Since the mapping $(\tau, (v,c)) \to F_\tau(v,c)$ from $[0,1] \times \overline{D}$ to E_2 is also continuous and compact, for every $\tau \in [0,1]$ the mapping $F_\tau : \overline{D} \to E_2$ is continuous and compact. By our choice of D we have $(id - F_\tau)(v,c) \neq 0$ for all $(v,c) \in \partial D$. Hence the Leray-Schauder degree $d(id - F_\tau, D, 0)$ is well defined. Because of the homotopy invariance, it is independent of τ. We have, in particular,

$$d(id - F_1, D, 0) = d(id - F_0, D, 0). \tag{4.14}$$

Step 3: Computing $d(id - F_0, D, 0)$. It remains to show $d(id - F_0, D, 0) \neq 0$. We will do so slightly differently as Xin in [21], see remark 2.18 for the reason. First, we will show that the mapping F_0 takes a simpler form than the other F_τ. To this end, let $z_c(s)$ be the unique solution of

$$(1+\varepsilon) z_{c,ss} + c z_{c,s} = 0,$$
$$z_c(-b) = 0, \qquad z_c(+b) = 1.$$

If we regard z_c as a function of (s,y), then z_c also solves (4.13) with $\tau = 0$. By uniqueness $z_c = \phi_0(v,c)$. Actually, $\phi_0(v,c)$ is independent of v because of $f_0 = 0$. Therefore $h_0(v,c)$ and $F_0(v,c)$ are also independent of v. We set $h(c) := h_0(v,c)$. As we have seen, F_0 is then given by

$$F_0(v,c) = (z_c, c - h(c) + \theta) =: (F_0^{(1)}(v,c), F_0^{(2)}(v,c)).$$

We are therefore in a position to calculate F_0 explicitly. An easy calculation yields

$$z_c(s,y) = \frac{s+b}{2b} \text{ if } c = 0 \qquad \text{and} \qquad z_c(s,y) = C \cdot \exp\left(-\frac{c}{1+\varepsilon} s\right) + D \text{ if } c \neq 0,$$

where

$$C = \frac{-\exp\left(\frac{cb}{1+\varepsilon}\right)}{\exp\left(\frac{2cb}{1+\varepsilon}\right) - 1} \qquad \text{and} \qquad D = \frac{\exp\left(\frac{2cb}{1+\varepsilon}\right)}{\exp\left(\frac{2cb}{1+\varepsilon}\right) - 1}.$$

Because z_c is independent of y, we see

$$h(c) = z_c(0) = C + D = \frac{\exp\left(\frac{2cb}{1+\varepsilon}\right) - \exp\left(\frac{cb}{1+\varepsilon}\right)}{\exp\left(\frac{2cb}{1+\varepsilon}\right) - 1} = \frac{\exp\left(\frac{cb}{1+\varepsilon}\right)}{\exp\left(\frac{cb}{1+\varepsilon}\right) + 1},$$

where the last form is also valid for $c = 0$.

Step 3a: Finding a zero (u_0, c_0) of $id - F_0$. The function h is strictly increasing. Moreover, $h(c) \to 1$ for $c \to \infty$, as well as $h(c) \to 0$ for $c \to -\infty$. Consequently, there is a unique $c_0 \in \mathbb{R}$ such that $h(c_0) = \theta$. Therefore, there is a unique zero (u_0, c_0) of $(id - F_0)$ given by $u_0 = z_{c_0}$ and $h(c_0) = \theta$. We know from step 2 that $|c_0| \leq K$ and $||u_0||_{C^1(R_b)} \leq K$.

Step 3b: Using the index of the zero (u_0, c_0). Since (u_0, c_0) is unique, the excision property of the Leray Schauder degree yields $d(id - F_0, D, 0) = d(id - F_0, B_\delta(u_0, c_0), 0)$ for some small $\delta > 0$. By the subsequent lemma 4.9 we know $F_0 \in C^1(E_2, E_2)$. The differential can be written in matrix-like form:

$$DF_0(v_0, c_0)(v, c) = \begin{pmatrix} 0 & D_c F_0^{(1)}(v_0, c_0) \\ 0 & 1 + h'(c_0) \end{pmatrix} \begin{pmatrix} v \\ c \end{pmatrix}.$$

Therefore

$$D(id - F_0)(v_0, c_0)(v, c) = \begin{pmatrix} id_{E_1} & - D_c F_0^{(1)}(v_0, c_0) \\ 0 & - h'(c_0) \end{pmatrix} \begin{pmatrix} v \\ c \end{pmatrix}.$$

Since $h' > 0$ we see from the upper diagonal structure that $D(id - F_0)$ is invertible regardless of $D_c F_0^{(1)}(v_0, c_0)$. Consequently

$$d(id - F_0, D, 0) = d(id - F_0, B_\delta(u_0, c_0), 0) = \text{Index}(id - F_0, (u_0, c_0)) \neq 0.$$

By homotopy invariance we conclude $d(id - F_1, D, 0) \neq 0$, which yields the existence of the desired solution. Its $C^{2,\alpha}(R_b)$-regularity follows from the regularity of the solution of the linear problem in lemma 4.5. ∎

Lemma 4.9 *The mapping F_0 from the proof of proposition 4.8 satisfies $F_0 \in C^1(E_2, E_2)$.*

Proof Because $F_0(v, c)$ is constant in v it is enough to prove that the mapping is continuously differentiable in c. The second component $F_0^{(2)}$ was given by $c - h(c) + \theta$, where

$$h(c) = z_c(0) = \frac{\exp\left(\frac{cb}{1+\varepsilon}\right)}{\exp\left(\frac{cb}{1+\varepsilon}\right) + 1}.$$

Clearly, $F_0^{(2)}$ is continuously differentiable. The first component $F_0^{(1)}$ was given by z_c, where z_c is the solution of

$$\begin{aligned} (1 + \varepsilon) z_{c,ss} + c z_{c,s} &= 0, \\ z_c(-b) = 0, \qquad z_c(+b) &= 1. \end{aligned}$$

We will use the implicit function theorem in Banach spaces to show that

$$(c \mapsto z_c) \in C^1(\mathbb{R}, C^2([-b,b])) \subset C^1(\mathbb{R}, C^1([-b,b])).$$

Let $w(s) := \frac{s+b}{2b}$, then $\tilde{z}_c(s) := z_c(s) - w(s)$ satisfies

$$\begin{aligned} (1+\varepsilon)\tilde{z}_{c,ss} + c\tilde{z}_{c,s} + \frac{c}{2b} &= 0, \\ \tilde{z}_c(-b) = 0, \qquad \tilde{z}_c(+b) &= 0. \end{aligned}$$

Consider the space $A := \{z \in C^2([-b,b]) : z(-b) = z(b) = 0\}$ and the mapping

$$G : \mathbb{R} \times A \to C^0([-b,b]), \qquad (c,z) \mapsto (1+\varepsilon)z_{ss} + cz_s + \frac{c}{2b}.$$

For arbitrary $c \in \mathbb{R}$, we know $G(c, \tilde{z}_c) = 0$. Hence, $c \mapsto \tilde{z}_c$ is the implicit function that will be delivered by the implicit function theorem. We only need it to verify the smoothness of this mapping.

Clearly $G \in C^1(\mathbb{R} \times A, C^0([-b,b]))$ with the differential

$$DG(\bar{c}, \bar{z}) : \mathbb{R} \times A \to C^0([-b,b]), \qquad (c,z) \mapsto (1+\varepsilon)z_{ss} + \bar{c}z_s + c\bar{z}_s.$$

Moreover, the partial differential

$$D_zG(\bar{c}, \bar{z}) : A \to C^0([-b,b]), \qquad z \mapsto (1+\varepsilon)z_{ss} + \bar{c}z_s$$

is an Isomorphism. This is true since for every fixed $\bar{c}$ and $f \in C^0([-b,b])$, the equation

$$\begin{aligned} (1+\varepsilon)z_{ss} + \bar{c}z_s &= f, \\ z_c(-b) = 0, \qquad z_c(+b) &= 0 \end{aligned}$$

has a unique solution $z \in A$.

Now, the implicit function theorem in Banach spaces yields the local existence of a function $g = g(c)$, which (as we already mentioned) must coincide with $c \mapsto \tilde{z}_c$. Furthermore, the implicit function theorem implies $g \in C^1(\mathbb{R}, A)$ (since we know that g exists globally). This implies the weaker statement

$$(c \mapsto z_c = \tilde{z}_c + w) \in C^1(\mathbb{R}, C^1([-b,b])).$$

When we interpret z_c as a function of s and y, which is constant in y, we also get $(c \mapsto z_c) \in C^1(\mathbb{R}, C^1(R_b))$. ■

4.2. Bounds on the wavespeed $c^{\varepsilon,b}$

4.2.1. Lower bounds on the wavespeed

Finding bounds for the wavespeed $c^{\varepsilon,b}$ away from $-\infty$ is possible for any nonlinearity of basic type, as long as the (ε, b)-solution is normalized by a constant θ that satisfies property (iii) of definition 2.6. It is furthermore possible to find these bounds directly, without comparing the wavespeed to the wavespeed of an (ε, b)-solution for another nonlinearity, as opposed to the situation for the upper bounds of the wavespeed.

Lemma 4.10 (Lower bounds for the wavespeed) *Let f be a nonlinearity of basic type and θ be as in property (iii) of definition 2.6. Then there are $\varepsilon_0 > 0$ and $b_0 > 0$ and a constant $c_1 < 0$, such that for every $0 < \varepsilon \leq \varepsilon_0$ and $b \geq b_0$ the unique classical solution $(U, c) = (U^{b,\varepsilon}, c^{b,\varepsilon})$ of (2.7), (2.8) satisfies $c_1 \leq c^{b,\varepsilon}$.*

PROOF The basic idea of the proof is to find two comparison functions z and w on $[-b, 0] \times T^n$ and $[0, b] \times T^n$ for U that are actually independent of y and that touch U at a point $(0, y_0)$. This will result in an inequality $z_s(0) \leq U_s(0, y_0) \leq w_s(0)$, which gives an inequality for c from which the desired estimate can be deduced. These comparison functions will be found as solutions of ordinary differential boundary value problems.

Of course, it is sufficient to assume that $c < 0$, since we are interested in bounds for c away from $-\infty$.

Step 1: A comparison function on $[-b, 0] \times T^n$. Let $r(y) = \sum_{i,j=1}^n a_{ij}(y) k_i k_j$, $\bar{r} = \left|\left|\frac{\alpha}{r}\right|\right|_\infty$ and $R = ||r||_\infty$. Consider the problem

$$(1 + \bar{r}\varepsilon) z_{ss} + \frac{c}{R} z_s = 0 \text{ in } [-b, 0],$$
$$z(-b) = 0, \qquad z(0) = \theta.$$

The general solution of the ODE is given by

$$z(s) = D + C \exp\left(\frac{-cs}{R(1 + \bar{r}\varepsilon)}\right).$$

The boundary conditions give

$$0 = z(-b) = D + C \exp\left(\frac{cb}{R(1 + \bar{r}\varepsilon)}\right) \qquad \text{and } \theta = z(0) = D + C.$$

Therefore

$$C = \frac{\theta}{1 - \exp\left(\frac{cb}{R(1+\bar{r}\varepsilon)}\right)} \qquad \text{and } D = \theta - C = -\theta \frac{\exp\left(\frac{cb}{R(1+\bar{r}\varepsilon)}\right)}{1 - \exp\left(\frac{cb}{R(1+\bar{r}\varepsilon)}\right)}.$$

From $c < 0$ we see $C > 0$ and consequently $z_{ss} > 0$.

This helps us to compare U and z:

$$\begin{aligned}
& -\varepsilon\alpha(y)(U - z)_{ss} - \sum_{i,j=1}^n a_{ij}(y)(k_i \partial_s + \partial_{y_i})(k_j \partial_s + \partial_{y_j})(U - z) - \frac{c}{R} r(y)(U - z)_s \\
&= f(y, U) + (r(y) + \varepsilon\alpha(y)) z_{ss} + \frac{cr(y)}{R} z_s + (c - \frac{cr(y)}{R}) U_s \\
&\leq r(y)[(1 + \varepsilon\frac{\alpha(y)}{r(y)}) z_{ss} + \frac{c}{R} z_s] \\
&\leq r(y)[(1 + \bar{r}\varepsilon) z_{ss} + \frac{c}{R} z_s] = 0.
\end{aligned}$$

In the calculation, we have used $f(y, U) \leq 0$ for $s \leq 0$ (by monotonicity of U in s) and $U_s \geq 0$ and $c < 0$ at the first inequality sign and $z_{ss} > 0$ at the second one. At the boundary, we have

$$U - z|_{s=-b} = 0 \qquad \text{and } U - z|_{s=0} \leq 0.$$

Consequently, we have by the maximum principle $U \leq z$ in $[-b, 0] \times T^n$.

Step 2: A first inequality for $U_s(0, y_0)$. With the help of the comparison function z from step 1, we derive an inequality for $U_s(0, y_0)$ for a specific y_0: There exists $y_0 \in T^n$ such that $U(0, y_0) = \theta = z(0)$. We obtain $U_s(0, y_0) \geq z_s(0)$. Therefore for $0 < \varepsilon \leq \varepsilon_0$, $b \geq b_0 > 0$ for some ε_0, b_0

$$\begin{aligned} U_s(0, y_0) \geq z_s(0) &= \frac{-c}{R(1+\bar{r}\varepsilon)} \cdot \frac{\theta}{1 - \exp\left(\frac{cb}{R(1+\bar{r}\varepsilon)}\right)} \\ &\geq \frac{-c}{R(1+\bar{r}\varepsilon_0)} \cdot \theta =: C_1 \cdot (-c). \end{aligned} \tag{4.15}$$

Step 3: A comparison function on $[0, b] \times T^n$. On the interval $[0, b]$ we will have to distinguish two cases to find a comparison function for U. Let $r_1 = \bar{r}$ and $r_2 = \inf_{y \in T^n} \frac{\alpha(y)}{r(y)}$, R be as above and $M = \sup_{y \in T^n, z \in [0,1]} \left|\frac{f(y,z)}{r(y)}\right|$. For $l = 1, 2$ consider the problem

$$\begin{gathered} (1 + r_l\varepsilon) w_{l,ss} + \frac{c}{R} w_{l,s} + M = 0 \text{ on } [0, b], \\ w_l(0) = \theta, \qquad w_l(b) = 1. \end{gathered}$$

Then the general solution of the ODE is

$$w_l(s) = -\frac{MRs}{c} + D + C \exp\left(\frac{-cs}{R(1 + r_l\varepsilon)}\right)$$

and the boundary conditions yield

$$\theta = D + C, \qquad 1 = -\frac{MRb}{c} + D + C \exp\left(\frac{-cb}{R(1 + r_l\varepsilon)}\right).$$

Therefore

$$C = \frac{1 - \theta + \frac{MRb}{c}}{\exp\left(\frac{-cb}{R(1+r_l\varepsilon)}\right) - 1} \qquad \text{and} \qquad D = \theta - \frac{1 - \theta + \frac{MRb}{c}}{\exp\left(\frac{-cb}{R(1+r_l\varepsilon)}\right) - 1}.$$

For $w_{l,ss}$, we obtain

$$w_{l,ss} = \left(\frac{-c}{R(1 + r_l\varepsilon)}\right)^2 \frac{1 - \theta + \frac{MRb}{c}}{\exp\left(\frac{-cb}{R(1+r_l\varepsilon)}\right) - 1} \exp\left(\frac{-cs}{R(1 + r_l\varepsilon)}\right).$$

Hence $w_{l,ss} \geq 0$ for $c < -\frac{MRb}{1-\theta}$ and $w_{l,ss} \leq 0$ for $c \geq -\frac{MRb}{1-\theta}$ (for both $l = 1$ and $l = 2$).

In the proof of lemma 4.4 we had a similar situation, however, there we only wanted an estimate for fixed b with a b-dependent bound and could therefore assume the case $c < -2MRb$ in the corresponding situation. Here, b is not fixed and we want a b-independent bound, so we have to take both cases into consideration.

For the first case that $l = 1$ and $c < -\frac{MRb}{1-\theta}$ or for the second case that $l = 2$ and $c \geq -\frac{MRb}{1-\theta}$, we have

$$\begin{aligned}
&-\varepsilon\alpha(y)(U-w_l)_{ss} - \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})(U-w_l) - \frac{c}{R}r(y)(U-w_l)_s \\
&= f(y,U) + (r(y)+\varepsilon\alpha(y))w_{l,ss} + \frac{cr(y)}{R}w_{l,s} + (c - \frac{cr(y)}{R})U_s \\
&\leq r(y)\left[(1+\varepsilon\frac{\alpha(y)}{r(y)})w_{l,ss} + \frac{c}{R}w_{l,s} + \frac{f(y,U)}{r(y)}\right] \\
&\leq r(y)\left[(1+r_l\varepsilon)w_{l,ss} + \frac{c}{R}w_{l,s} + M\right] = 0.
\end{aligned}$$

In the calculation, we have used at the last inequality sign, that in the first case $w_{1,ss} \geq 0$ and $\frac{\alpha(y)}{r(y)} \leq r_1$ and in the second case $w_{2,ss} \leq 0$ and $\frac{\alpha(y)}{r(y)} \geq r_2$. Furthermore we have used $U_s \geq 0$ and $c < 0$ at the first inequality sign. Since

$$U - w_l|_{s=0} \leq 0 \qquad \text{and } U - w_l|_{s=b} = 0,$$

we have $U \leq w_1$ for $c < -\frac{MRb}{1-\theta}$ and $U \leq w_2$ for $c \geq -\frac{MRb}{1-\theta}$ due to the maximum principle.

Step 4: A second inequality for $U_s(0,y_0)$. With y_0 from step 2 and w_l from step 3, we have $U(0,y_0) = \theta = w_l(0)$ for $l = 1,2$. We obtain $U_s(0,y_0) \leq w_{l,s}(0)$ for $l = 1$ and $c < -\frac{MRb}{1-\theta(f)}$ or $l = 2$ and $c \geq -\frac{MRb}{1-\theta(f)}$. In both cases and for $0 < \varepsilon \leq \varepsilon_0$ and $b \geq b_0$, there holds

$$\begin{aligned}
U_s(0,y_0) &\leq \max\{w_{1,s}(0), w_{2,s}(0)\} \\
&= \max\left\{\frac{-c}{R(1+r_l\varepsilon)}\frac{1-\theta+\frac{MRb}{c}}{\exp\left(\frac{-cb}{R(1+r_l\varepsilon)}\right)-1} : l = 1,2\right\} - \frac{MR}{c} \\
&\leq \max\left\{\frac{-c}{R(1+r_l\varepsilon)}\frac{1-\theta}{\exp\left(\frac{-cb}{R(1+r_l\varepsilon)}\right)-1} : l = 1,2\right\} - \frac{MR}{c} \\
&\leq \frac{-c}{R}\frac{1-\theta}{\exp\left(\frac{-cb}{R(1+r_1\varepsilon)}\right)-1} - \frac{MR}{c} \leq \frac{-c}{R}\frac{1-\theta}{\frac{-cb}{R(1+r_1\varepsilon)}} - \frac{MR}{c} \\
&= \frac{(1+r_1\varepsilon)(1-\theta)}{b} - \frac{MR}{c} \leq \frac{(1+r_1\varepsilon_0)(1-\theta)}{b_0} - \frac{MR}{c} =: C_2 - \frac{MR}{c}.
\end{aligned}$$

Step 3: Combining the inequalities for c. Combining the inequality from step 4 with (4.15), we obtain

$$C_1 \cdot (-c) \leq C_2 - \frac{MR}{c}.$$

Clearly, this inequality can only hold for $c \geq -K$ with some constant $K > 0$. This concludes the proof. Note that the distinction between the two cases would not have been necessary in the case $\alpha(y) = r(y)$ (see remark 2.19). However, in order to obtain a priori estimates later, we need α to be constant (see lemma 5.4). ■

4.2.2. Upper bounds on the wavespeed

To find upper bounds for the wavespeed, or more precisely bounds for $c^{\varepsilon,b}$ away from 0, we need further properties of the involved nonlinearity. We can find these upper bounds for nonlinearities of basic type which lie above an x-independent nonlinearity g with a certain integral property. This includes nonlinearities of combustion type with the strong covering property.

As opposed to the lower bounds, we cannot find the upper bounds for the wavespeed directly. Instead, we have to compare the wavespeed to the wavespeed of a more simple one dimensional problem. To do so, we need a comparison principle for the wavespeeds. The following proposition is a modification of proposition 1.3 from [21], where we basically merged two parts into one.

Proposition 4.11 (Comparison principle) *We consider functions $0 < \beta_1 \leq \beta_2$ and $\alpha > 0$ with $\beta_1, \beta_2, \alpha \in C^1(T^n)$ and nonlinearities of basic type $f_2 \leq f_1$. Moreover, let (U_l, c_l) $(l = 1, 2)$ be classical solutions of*

$$\begin{aligned} \varepsilon\alpha(y)U_{l,ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})U_l& \\ + c_l\beta_l(y)U_{l,s} + f_l(y, U_l) = 0,& \end{aligned} \tag{4.16}$$

$$\left.\begin{aligned} 0 \leq U_l \leq 1, &\qquad U_l(s,\cdot) \text{ 1-periodic },\\ U_l(-b,y) = 0, &\qquad U_l(b,y) = 1 \text{ for all } y \in T^n,\\ \max_{y\in T^n} U_l(0,y) = \theta. & \end{aligned}\right\} \tag{4.17}$$

If now either $c_2 < 0$ or $\beta_1 = \beta_2$, then $c_1 \leq c_2$ follows.

Proof As in step 2 of the proof of proposition 4.1, we use the sliding domain method. Assume for a contradiction $c_2 < c_1$. The goal is to prove $U_1 = U_2$ and derive a contradiction from that.

Step 1: For $\lambda \in (0, b)$, we define the function

$$w_\lambda(s, y) = U_1(s + \lambda, y) - U_2(s - \lambda, y)$$

on $\Sigma_\lambda = (-b+\lambda, b-\lambda) \times T^n$. It follows from equation (4.16) that w_λ satisfies the following equation (where $U_1 = U_1(s+\lambda, y)$ and $U_2 = U_2(s-\lambda, y)$):

$$\begin{aligned}\varepsilon\alpha(y) w_{\lambda,ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j}) w_\lambda \\ + c_1\beta_1(y)U_{1,s} - c_2\beta_2(y)U_{2,s} + f_1(y, U_1) - f_2(y, U_2) = 0.\end{aligned}$$

This implies

$$\begin{aligned}\varepsilon\alpha(y) w_{\lambda,ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j}) w_\lambda + c_2\beta_2(y) w_{\lambda,s} + f_1(y, U_1) - f_1(y, U_2) \\ = -c_1\beta_1(y)U_{1,s} + c_2\beta_2(y)U_{1,s} + f_2(y, U_2) - f_1(y, U_2) \\ \leq (c_2 - c_1)\beta_1(y)U_{1,s} + c_2(\beta_2(y) - \beta_1(y))U_{1,s} < 0.\end{aligned} \tag{4.18}$$

In this calculation, the first inequality holds due to $f_2 \leq f_1$. The second inequality holds because of $c_2 - c_1 < 0$ (by assumption), $\beta_1 > 0$ and $U_{1,s} > 0$ (because of proposition 4.1), as well as either $c_2 < 0$ and $\beta_1 \leq \beta_2$ or $\beta_1 = \beta_2$. We define

$$\gamma_\lambda(s, y) = \begin{cases} \dfrac{f_1(y, U_1(s+\lambda, y)) - f_1(y, U_2(s-\lambda, y))}{U_1(s+\lambda, y) - U_2(s-\lambda, y)} & \text{if } U_1(s+\lambda, y) \neq U_2(s-\lambda, y) \\ 0 & \text{otherwise.} \end{cases}$$

We note that γ_λ is bounded by the Lipschitz constant of f_1. Using the definition of γ_λ in (4.18), we obtain

$$\varepsilon\alpha(y) w_{\lambda,ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j}) w_\lambda + c_2\beta_2(y) w_{\lambda,s} + \gamma_\lambda(s, y) w_\lambda \leq 0. \tag{4.19}$$

Step 2: The function w_λ at the boundary. At the boundary of Σ_λ, the function w_λ is always positive. This can be seen as follows: We recall that $0 < U_1 < 1$ and $0 < U_2 < 1$ in $(-b, b) \times T^n$ by proposition 4.1. Therefore, setting $s = -b + \lambda$ we find

$$w_\lambda(-b+\lambda, y) = U_1(2\lambda - b, y) - U_2(-b, y) = U_1(2\lambda - b, y) > 0,$$

and setting $s = +b - \lambda$ we find

$$w_\lambda(b-\lambda, y) = U_1(b, y) - U_2(b - 2\lambda, y) = 1 - U_2(b - 2\lambda, y) > 0.$$

Step 3: Large λ. By uniform continuity of U_1 and U_2 up to the boundary, for every $\frac{1}{2} > \varepsilon > 0$ there is a $\delta > 0$, such that for $s_1 \in (b - \delta, b)$ and $s_2 \in (-b, -b + \delta)$ we have

$$|U_1(s_1, y) - 1| < \varepsilon \text{ and } |U_2(s_2, y)| < \varepsilon.$$

Consequently $0 < b - \lambda < \frac{\delta}{2}$ yields for $(s, y) \in \Sigma_\lambda$:

$$s + \lambda \in (-b + 2\lambda, b) \subset (b - \delta, b) \text{ and } s - \lambda \in (-b, b - 2\lambda) \subset (-b, -b + \delta)$$

and thus

$$|U_1(s+\lambda, y) - 1| < \varepsilon \text{ and } |U_2(s-\lambda, y)| < \varepsilon \text{ for } (s,y) \in \overline{\Sigma_\lambda}.$$

Therefore, by $0 < \varepsilon < \frac{1}{2}$,

$$w_\lambda(s,y) = U_1(s+\lambda, y) - U_2(s-\lambda, y) > 0 \text{ for all } (s,y) \in \overline{\Sigma_\lambda}.$$

This finishes the calculation for λ close to b.

Step 4: Arbitrary λ. We set

$$\lambda_0 = \inf\{\lambda \in (0,b) : w_\lambda(s,y) > 0 \text{ for } (s,y) \in \overline{\Sigma_\lambda}\}.$$

In step 3 we have seen that the set on the right hand side of the definition is not empty. We want to show $\lambda_0 = 0$. Suppose for a contradiction $\lambda_0 \in (0,b)$. The definition of λ_0 implies $w_{\lambda_0}(s,y) \geq 0$ due to continuity reasons. Since $w_{\lambda_0} > 0$ on $\partial\Sigma_{\lambda_0}$, there are two possibilities:

(1) There is $(s_0, y_0) \in \Sigma_{\lambda_0}$ such that $w_{\lambda_0}(s_0, y_0) = 0$. The function w_{λ_0} satisfies the inequality (4.19) with $\lambda = \lambda_0$. Therefore, the strong elliptic minimum principle implies $w_{\lambda_0}(s,y) \equiv 0$. This holds irrespectively of the sign of γ, since the minimum has the value 0. However, that is a contradiction to $w_{\lambda_0}(b - \lambda_0, y) > 0$.

(2) There holds $w_{\lambda_0} > 0$ in $\overline{\Sigma_{\lambda_0}}$. The minimality of λ_0 implies the existence of a sequence $0 < \lambda_n \nearrow \lambda_0$, along with a sequence $(s_n, y_n) \in \Sigma_{\lambda_n}$, such that $w_{\lambda_n}(s_n, y_n) \leq 0$. After chosing a convergent subsequence $(s_n, y_n) \to (s_0, y_0)$, we find $w_{\lambda_0}(s_0, y_0) \leq 0$, a contradiction to the assumption of (2). Chosing this convergent subsequence is possible, because y_n can be chosen so that it is bounded (because of the 1-periodicity of w in y) and the boundedness of $s_n \in [-b, b]$.

We have therefore obtained $\lambda_0 = 0$.

Step 5: Deducing $U_1 = U_2$ and deriving a contradiction. From $\lambda_0 = 0$, it follows that $w_0 \geq 0$. This is equivalent to $U_1(s,y) \geq U_2(s,y)$ for $(s,y) \in [-b,b] \times T^n$. By the normalization assumption, there holds $\max_{y \in T^n} U_l(0,y) = \theta$ for $l = 1,2$. Consequently, there exists y_0 with $U_2(0, y_0) = \theta$. From this, we obtain $w_0(0, y_0) \leq 0$ and thus $w_0(0, y_0) = 0$. The minimum principle yields $w_0 \equiv 0$ and hence $U_1 = U_2$.

Subtracting the respective equations (4.16) for U_1 and U_2 from one another, we obtain from $U_1 = U_2$:

$$c_1\beta_1(y)U_{1,s} + f_1(y, U_1) = c_2\beta_2(y)U_{1,s} + f_2(y, U_1).$$

This implies $c_1\beta_1(y) \leq c_2\beta_2(y)$. In the case that $c_2 < 0$, it follows that $c_1 < 0$ and $c_1 \leq c_2$. This contradicts the assumption $c_2 < c_1$. In the case that $\beta_1 = \beta_2$, it follows that $c_1 \leq c_2$ as well. Again, this contradicts the assumption. This finishes the proof. ■

The existence of the comparison solution in the one dimensional situation is given in the next lemma. It is a consequence of the more general proposition 4.8. Moreover we derive estimates for the wavespeed of the comparison solution.

Lemma 4.12 (Existence and wavespeed of the comparison solution) *(i) Let ε, b, $\bar{r}$, $R > 0$ and let $g = g(z)$ be an x-independent nonlinearity of basic type and $0 < \theta < 1$. Then the problem*

$$\begin{aligned} &(1+\bar{r}\varepsilon)u_{ss} + \frac{c}{R}u_s + g(u) = 0 \text{ on } [-b,b], \\ &u(-b) = 0, \qquad u(0) = \theta, \qquad u(b) = 1 \end{aligned} \tag{4.20}$$

has a unique classical solution (u, c).

(ii) In addition to the assumptions of (i), let $\theta \in (0,1)$ be such that $g(z) \le 0$ for $z \le \theta$ (such θ does exist by property (iii) of definition 2.6) and

$$\int_0^1 g(z)\,dz > 0.$$

Then there exist $\varepsilon_0 > 0$, $b_0 > 0$ and $c_0 < 0$ such that $c < c_0$ holds for $0 < \varepsilon \le \varepsilon_0$, $b \ge b_0$.

Proof (i) We choose coefficients $\alpha(y) = \bar{r}$ and $a_{ij}(y) = \delta_{ij}$ for $i, j = 1, ..., n$. (Note that $\sum_{i,j=1}^n a_{ij}(y)k_ik_j = 1$.) With these coefficients α and a_{ij}, we apply proposition 4.8 to obtain a unique solution $(U, \tilde{c})$ of problem (2.7), (2.8). The coefficients are independent of y. Hence, for arbitrary $L \in \mathbb{R}$ and the $i-th$ unit vector e_i, $i = 1, ..., n$, the function given by $\tilde{U}(s, y) = U(s, y + Le_i)$ solves (2.7) and (2.8). By uniqueness, which was proved in proposition 4.3, $\tilde{U} = U$ follows. Since L was arbitrary, it follows that $u := U$ is independent of y and solves

$$\begin{aligned} &(1+\bar{r}\varepsilon)u_{ss} + \tilde{c}u_s + g(u) = 0 \text{ on } [-b,b], \\ &u(-b) = 0, \qquad u(0) = \theta, \qquad u(b) = 1. \end{aligned}$$

By putting $c = R\tilde{c}$, we see that (u, c) is the desired solution.

(ii) *Step 1: Obtaining an inequality involving c.* We multiply (4.20) by u_s and integrate from some $t_0 \le 0$ to b. We obtain

$$\begin{aligned} \frac{1}{2}u_s^2(b) - \frac{1}{2}u_s^2(t_0) &= \frac{-c}{R(1+\bar{r}\varepsilon)}\int_{t_0}^{b} u_s^2\,ds - \frac{1}{(1+\bar{r}\varepsilon)}\int_{t_0}^{b} g(u)u_s\,ds \\ &= \frac{-c}{R(1+\bar{r}\varepsilon)}\int_{t_0}^{b} u_s^2\,ds - \frac{1}{(1+\bar{r}\varepsilon)}\int_{u(t_0)}^{1} g(z)\,dz \\ &\le \left(\frac{-c}{R(1+\bar{r}\varepsilon)}\right)_+ \int_{-b}^{b} u_s^2\,ds - \frac{1}{(1+\bar{r}\varepsilon)}\int_0^1 g(z)\,dz. \end{aligned}$$

At the inequality, we have used $t_0 \leq 0$, which implies $u(t_0) \leq \theta$ and therefore $g(z) \leq 0$ for $z \leq u(t_0) \leq \theta$ by the assumption on θ.

Step 1a): Estimating the term $-\frac{1}{2}u_s^2(t_0)$. For arbitrary $t_0 \leq 0$, it does not seem possible to obtain good enough estimates for the term $-\frac{1}{2}u_s^2(t_0)$ on the left hand side of the calculation. However, it is sufficient to find a single $t_0 \leq 0$, for which we can get a good estimate of the term. The existence of such t_0 can be seen with an abstract argument: We know $u(-b) = 0$ and $u(0) = \theta$. We claim that there exists $t_0 \in [-b, 0]$, such that

$$u_s(t_0) = \min\{u_s(t) : -b \leq t \leq 0\} \leq \frac{\theta}{b}.$$

Indeed, from the contrary statement we could easily conclude $u(0) > \theta$, which is not true.

By the existence of such t_0, we further have $-\frac{1}{2}u_s^2(t_0) \geq -\frac{1}{2}\frac{\theta^2}{b^2} \geq -\frac{\theta^2}{b^2}$. (Keep in mind $u_s \geq 0$ by corally 4.2.) Using this and $\frac{1}{2}u_s^2(b) \geq 0$ in the above inequality, we obtain

$$-\frac{\theta^2}{b^2} \leq \left(\frac{-c}{R(1+\bar{r}\varepsilon)}\right)_+ \int_{-b}^{b} u_s^2\, ds - \frac{1}{(1+\bar{r}\varepsilon)} \int_0^1 g(z)\, dz. \tag{4.21}$$

Step 2: Ruling out $c \geq 0$ for large b and small ε. We consider first the case $c \geq 0$. Let $\varepsilon_0 > 0$ be arbitrary. Then, for $0 < \varepsilon \leq \varepsilon_0$, the inequality (4.21) simplifies to

$$-\frac{\theta^2}{b^2} \leq -\frac{1}{(1+\bar{r}\varepsilon)} \int_0^1 g(z)\, dz \leq -\frac{1}{(1+\bar{r}\varepsilon_0)} \int_0^1 g(z)\, dz.$$

We observe that the right hand side of this inequality does not depend on b and is negative. However, the left hand side tends to 0 as $b \to \infty$. Consequently, for sufficiently large $b_1 > 0$, the inequality fails to hold if $b \geq b_1$. This implies $c < 0$ for $0 < \varepsilon \leq \varepsilon_0$ and $b \geq b_1$.

Step 3: Bounds away from 0. Now we turn to the case $c \leq 0$. We choose some fixed $K < 0$ and distinguish the cases $c < K$ and $c \geq K$. If $c < K$, because as we wanted, c is away from 0.

Step 3a): A priori estimates for the case $c \in [K, 0]$. For the case $c \in [K, 0]$ we will need an estimate for $||u_s||_{C^0([-b,b])}$. Without loss of generality let $b \geq 1$. We derive estimates for $||u_s||_{C^0(I)}$ for intervals I of length $|I| = 1$, which do not depend on I. Then we achieve the estimate for $||u_s||_{C^0([-b,b])}$ by covering $[-b, b]$ with intervals of length of 1.

Let $I \subset [-b, b]$ be an arbitrary interval with length $|I| = 1$. We already know $u_s \geq 0$ and $\int_{-b}^{b} u_s\, ds = u(b) - u(-b) = 1$. Therefore, we have $||u_s||_{L^1([-b,b])} = 1$. In particular, this implies $||u_s||_{L^1(I)} \leq 1$. From (4.20), we obtain

$$|u_{ss}| \leq \frac{|K|}{R(1+\bar{r}\varepsilon)} u_s + \frac{1}{1+\bar{r}\varepsilon} \, ||g||_\infty .$$

Hence, we have

$$||u_{ss}||_{L^1(I)} \leq \frac{|K|}{R(1+\bar{r}\varepsilon)}\, ||u_s||_{L^1(I)} + \frac{1}{1+\bar{r}\varepsilon}\, ||g||_\infty \, |I| \leq \frac{|K|}{R} + ||g||_\infty \, .$$

The right hand side of this inequality does not depend on b or ε. Therefore we have

$$||u_s||_{C^0(I)} \leq C\, ||u_s||_{W^{1,1}(I)} \leq \tilde{C}.$$

The constant C only depends on $|I| = 1$, but not on I itself. (Note that for intervals of smaller length, the constant C does become larger, therefore we chose to only use intervals I of the exact length 1.) Consequently, $\tilde{C}$ is independent of b, ε and I. By covering $[-b, b]$ with intervals of length 1, we obtain

$$||u_s||_{C^0([-b,b])} \leq \tilde{C}. \tag{4.22}$$

Step 3b): Applying the estimates for $||u_s||_{C^0([-b,b])}$. Let ε_0 be as in step 2 and $0 < \varepsilon \leq \varepsilon_0$. Moreover, let $b_2 > 0$ and $b \geq b_2$. Using the estimates (4.22) in (4.21), we obtain

$$\begin{aligned}
-\frac{\theta^2}{b^2} &\leq \left(\frac{-c}{R(1+\bar{r}\varepsilon)}\right)_+ \int_{-b}^{b} u_s \, ds \cdot ||u_s||_{C^0([-b,b])} - \frac{1}{(1+\bar{r}\varepsilon)} \int_0^1 g(z)\, dz \\
&= \frac{-c}{R(1+\bar{r}\varepsilon)}\, ||u_s||_{C^0([-b,b])} - \frac{1}{(1+\bar{r}\varepsilon)} \int_0^1 g(z)\, dz \\
&\leq \frac{-c\tilde{C}}{R} - \frac{1}{(1+\bar{r}\varepsilon_0)} \int_0^1 g(z)\, dz.
\end{aligned}$$

For $b \geq b_2$ and $0 < \varepsilon \leq \varepsilon_0$, this implies

$$c \leq \frac{R}{\tilde{C}} \left(-\frac{1}{(1+\bar{r}\varepsilon_0)} \int_0^1 g(z)\, dz + \frac{\theta^2}{b_2^2} \right) =: \bar{c}.$$

The right hand side of the inequality is independent of b and ε. Moreover, it is negative for b_2 sufficiently large. Putting $c_0 := \max\{K, \bar{c}\}$ and $b_0 := \max\{b_1, b_2\}$ with b_1 from step 2 finishes the proof. ∎

In the next lemma, we show that nonlinearities lying above appropriate x-independent nonlinearities with certain integral properties have the negative wavespeed property.

Lemma 4.13 (Negative wavespeed property in the general case) *Let f be a nonlinearity of basic type. Moreover, let the matrix field $(a_{ij})_{i,j=1}^n$, the vector k and the (k-dependent) numbers $r_{\max}$, $r_{\min}$ be as in the general assumptions. Furthermore,* *let there exist an x-independent nonlinearity $g = g(z)$ of basic type with* $\int_0^1 g_+(z)\,dz >$ $\frac{r_{\max}}{r_{\min}} \int_0^1 g_-(z)\,dz$ *and* $f(x,z) \geq g(z)$ *for all* $(x,z) \in T^n \times [0,1]$*. Then* f *has the negative wavespeed property with respect to the matrix field* $(a_{ij})_{i,j=1}^n$ *and direction* k*.*

Proof *Step 1: A modification of the nonlinearity* g*.* We claim that there is a factor $q > 1$ and an x-independent nonlinearity $\bar{g} = \bar{g}(z)$ of basic type with $\int_0^1 \bar{g}(z)\,dz > 0$ and $f(x,z) \geq \tilde{q}\bar{g}(z)$ for all $\tilde{q} \in [1, q\frac{r_{\max}}{r_{\min}}]$ and $(x,z) \in T^n \times [0,1]$. This can be seen as follows: Define $\bar{g}$ by

$$\bar{g}(z) = \begin{cases} g(z), & \text{if } g(z) \leq 0 \\ \dfrac{r_{\min}}{q r_{\max}} g(z), & \text{if } g(z) > 0. \end{cases}$$

Clearly, we can choose $q > 1$, but close enough to 1 such that still $\int_0^1 \bar{g}(z)\,dz > 0$. It is easy to see that $\bar{g}$ is still a nonlinearity of basic type. Moreover, for any $\tilde{q} \in [1, q\frac{r_{\max}}{r_{\min}}]$, we have $\tilde{q}\bar{g} \leq g \leq f$.

Step 2: The negative wavespeed property of f*.* We show that f has the negative wavespeed property with the help of $\bar{g}$ from step 1. Let $(U,c) = (U^{\varepsilon,b}, c^{\varepsilon,b})$ be the solution of problem (2.7), (2.8) (with f as nonlinearity). Moreover, let $\bar{r} = \min_{y \in T^n} \frac{\alpha(y)}{r(y)}$. By lemma 4.12 (i), there is a unique solution $(\tilde{u}, \tilde{c}) = (\tilde{u}^{\varepsilon,b}, \tilde{c}^{\varepsilon,b})$ of the problem

$$\begin{aligned} &(1 + \bar{r}\varepsilon)\tilde{u}_{ss} + \frac{\tilde{c}^{\varepsilon,b}}{r_{\min}}\tilde{u}_s + \frac{\bar{g}(\tilde{u})}{r_{\min}} = 0 \text{ on } [-b,b], \\ &\tilde{u}(-b) = 0, \qquad \tilde{u}(0) = \theta(f), \qquad \tilde{u}(b) = 1. \end{aligned}$$

First, we have to estimate $\tilde{c}^{\varepsilon,b}$. We note, that $\theta(f) \in (0,1)$. Otherwise, $\theta(f) = 1$ would imply $f \leq 0$, which clearly cannot be the case here. This is important, since it is an assumption of part (ii) of lemma 4.12. Secondly, we note that $\bar{g}(z) \leq 0$ for $z \leq \theta(f)$ is satisfied because of $\bar{g} \leq f$. Therefore, the assumptions lemma 4.12 (ii) are met. The lemma yields the existence of constants $b_0 > 0$, $\tilde{\varepsilon} > 0$ and $c_0 < 0$, such that for $b \geq b_0$ and $0 < \varepsilon \leq \tilde{\varepsilon}$ there holds $\tilde{c}^{\varepsilon,b} \leq c_0$.

We want to compare $c^{\varepsilon,b}$ to $\tilde{c}^{\varepsilon,b}$ with the help of the comparison principle proposition 4.11. This requires $\tilde{c}^{\varepsilon,b} < 0$. Therefore we assume $0 < \varepsilon \leq \varepsilon_0$ and $b \geq b_0$. We define

$$\beta^\varepsilon(y) = \frac{r(y)(1 + \varepsilon\frac{\alpha(y)}{r(y)})}{r_{\min}(1 + \bar{r}\varepsilon)} \text{ and } g^\varepsilon(y,z) = \beta^\varepsilon(y)\bar{g}(z).$$

Since $\bar{g}$ is a nonlinearity of basic type, the same holds for g^ε for every $\varepsilon > 0$. Because $\tilde{u}$ is independent of y, we see that $\tilde{u}$ satisfies

$$\begin{aligned}
&\varepsilon\alpha(y)\tilde{u}_{ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})\tilde{u} + \tilde{c}\beta^\varepsilon(y)\tilde{u}_s + g^\varepsilon(y,\tilde{u})\\
&= (r(y) + \varepsilon\alpha(y))\tilde{u}_{ss} + \tilde{c}\beta^\varepsilon(y)\tilde{u}_s + g^\varepsilon(y,\tilde{u})\\
&= \frac{r(y)(1+\varepsilon\frac{\alpha(y)}{r(y)})}{(1+\bar{r}\varepsilon)}\left((1+\bar{r}\varepsilon)\tilde{u}_{ss} + \frac{\tilde{c}}{r_{\min}}\tilde{u}_s + \frac{\bar{g}(\tilde{u})}{r_{\min}}\right) = 0.
\end{aligned}$$

For some small $\varepsilon_0 \le \tilde{\varepsilon}$ and $0 < \varepsilon \le \varepsilon_0$, the inequality $1 \le \beta^\varepsilon \le q\frac{r_{\max}}{r_{\min}}$ holds. Therefore $g^\varepsilon(z) = \beta^\varepsilon\bar{g}(z) \le f(x,z)$ by step 1. This and $\tilde{c} = \tilde{c}^{\varepsilon,b} < 0$ let us apply proposition 4.11, which yields $c^{\varepsilon,b} = c \le \tilde{c} = \tilde{c}^{\varepsilon,b} \le c_0$ for $0 < \varepsilon \le \varepsilon_0$. ∎

Remark 4.14 *If the modification process in step 1 of the previous proof is applied to a C^1-nonlinearity g, its modification $\bar{g}$ will in general only be Lipschitz continuous. This is actually the reason, why we solved the (ε, b)-problem for nonlinearities that are only Lipschitz continuous.*

As a corollary, we obtain the negative wavespeed property for nonlinearities of combustion type and the strong covering property. This is actually the most important case for us.

Lemma 4.15 (Negative wavespeed property in the combustion case) *Let f be a nonlinearity of combustion type that satisfies the strong covering property. Then f has the negative wavespeed property for all matrix fields that satisfy the general assumptions and for all directions k.*

PROOF By the strong covering property of f, there exists $u_0 > \theta(f)$ such that $f(x,u_0) > 0$ for all $x \in T^n$. By continuity reasons, it is easy to conclude that there is an x-independent nonlinearity g of combustion type, such that $0 \le g(z) \le f(x,z)$ for all $(x,z) \in T^n \times [0,1]$ and which is nontrivial. Since $g_- = 0$, the assumptions of lemma 4.13 hold for any matrix field $(a_{ij})_{i,j=1}^n$ that satisfies the general assumptions and any direction k. Applying the lemma concludes the proof. ∎

The following proposition is a synopsis of the results on upper bounds and lower bounds for the wavespeed. It combines the results as we need them.

Proposition 4.16 (Upper bounds and lower bounds) *Let f be a nonlinearity of basic type which has the negative wavespeed property with respect to the matrix field $(a_{ij})_{i,j=1}^n$ and direction k. (For example f as in lemma 4.13 or lemma 4.15.) Let $(U,c) = (U^{\varepsilon,b}, c^{\varepsilon,b})$ be the solutions of problem (2.7), (2.8). Then there are constants $c_1, c_2 < 0$ and $b_0, \varepsilon_0 > 0$, so that $c_1 \le c^{\varepsilon,b} \le c_2 < 0$ for $b \ge b_0$, $0 < \varepsilon \le \varepsilon_0$.*

PROOF By lemma 4.10 there are constants $\varepsilon_1 > 0$, $b_1 > 0$ and $c_1 < 0$, such that $c^{\varepsilon,b}$ satisfies $c_1 \le c^{\varepsilon,b}$ for $0 < \varepsilon \le \varepsilon_1$ and $b \ge b_1$. Moreover, by definition of the negative wavespeed property, there are constants $\varepsilon_2 > 0$, $b_2 > 0$ and $c_2 < 0$, such that $c^{\varepsilon,b}$ satisfies $c^{\varepsilon,b} \le c_2 < 0$ for $0 < \varepsilon \le \varepsilon_2$ and $b \ge b_2$. With $b_0 := \max\{b_1, b_2\}$ and $\varepsilon_0 := \min\{\varepsilon_1, \varepsilon_2\}$ the assertion follows. ∎

5. Analysis of limits $\varepsilon \to 0$, $b \to \infty$

5.1. Preparations for passing to the limit

Consider the weight factor α in the ε-term of (2.7). In order to solve the (ε, b)-problem and to estimate the wavespeed $c^{\varepsilon,b}$ of the (ε, b)-problem, choosing an arbitrary $\alpha \in C^1(T^n)$ with $\alpha > 0$ was sufficient. However, for the task of deriving a priori estimates, we will use theorem A.2 from Berestycki and Hamel. This theorem requires that α does not depend on y. Therefore, we will use $\alpha \equiv 1$ as a weight factor in the following.

5.1.1. Returning to (t, x) coordinates

The (ε, b)-problem is the regularization of a transformed problem in (s, y)-coordinates. Before we pass to the limit $\varepsilon \to 0$, $b \to \infty$, we want to return to (t, x)-coordinates (which is possible for $c^{\varepsilon,b} < 0$). There are multiple reasons: First of all, we want to apply the aforementioned theorem A.2 of Berestycki and Hamel to derive suitable a priori estimates. Furthermore, by returning to (t, x)-coordinates, we will have the usual parabolic theory at our disposal after passing to the limit. Moreover, we do so for regularity reasons, see remark 2.20.

Definition 5.1 (The (ε, b)-solution in (t, x)-coordinates) *Assume that the nonlinearity f is of combustion type and has the covering property and the negative wavespeed property with respect to the matrix field $(a_{ij})_{i,j=1}^n$. (For example, due to lemma 4.15, this is the case if f has the strong covering property.) Moreover, let $f|_{T^n \times [0,1]} \in C^1(T^n \times [0, 1])$. Let $(U^{\varepsilon,b}, c^{\varepsilon,b})$ be the solution of* (2.7), (2.8) *with the weight factor $\alpha = 1$ in* (2.7) *and normalization constant $\theta = \theta(f)$ in* (2.8). *This solution exists by proposition 4.8. Let ε_0, b_0 and c_1, c_2 be the constants from proposition 4.16. Assuming $b \geq b_0$ and $0 < \varepsilon \leq \varepsilon_0$, we have the estimates $c_1 \leq c^{\varepsilon,b} \leq c_2 < 0$ for the wavespeed $c^{\varepsilon,b}$. We consider the transformation*

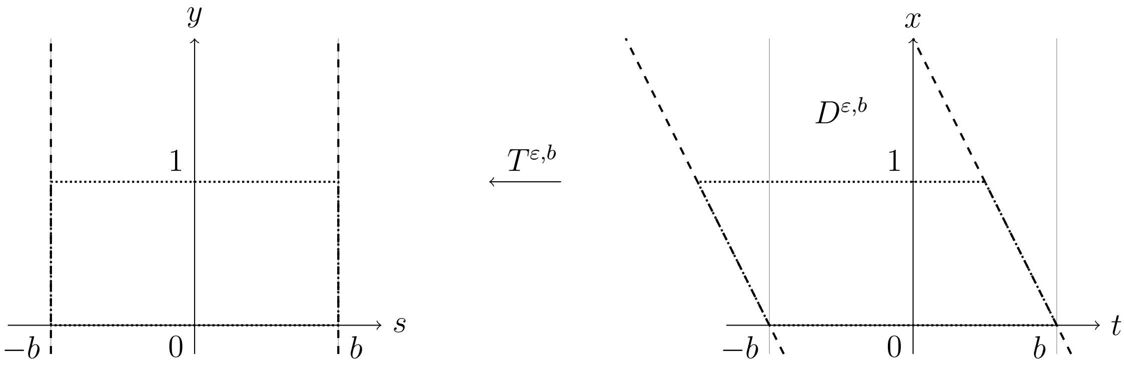
$$(s, y) = (k \cdot x - c^{\varepsilon,b} t, x) =: T^{\varepsilon,b}(t, x).$$

With the transformation, we define the function

$$u^{\varepsilon,b}(t, x) := U^{\varepsilon,b}(T^{\varepsilon,b}(t, x)) = U^{\varepsilon,b}(s, y). \tag{5.1}$$

Since the inverse transformation is given by $x = y$ and $t = \frac{s - k \cdot y}{-c^{\varepsilon,b}}$, the function $u^{\varepsilon,b}$ is defined on $\overline{D^{\varepsilon,b}}$, where

$$D^{\varepsilon,b} := \left\{ (t, x) \in \mathbb{R}^{n+1} : \frac{-b - k \cdot x}{-c^{\varepsilon,b}} < t < \frac{b - k \cdot x}{-c^{\varepsilon,b}} \right\}.$$



According to the chain rule, there hold

$$u_t(t,x) = -c^{\varepsilon,b} U_s(s,y), \qquad u_{tt}(t,x) = (c^{\varepsilon,b})^2 U_{ss}(s,y)$$

and

$$u_{x_i}(t,x) = (k_i\partial_s + \partial_{y_i})U(s,y), \qquad u_{x_ix_j}(t,x) = (k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})U(s,y).$$

By (2.7), the function $u^{\varepsilon,b}$ satisfies

$$-u_t^{\varepsilon,b} + \frac{\varepsilon u_{tt}^{\varepsilon,b}}{(c^{\varepsilon,b})^2} + \sum_{i,j=1}^n a_{ij}(x) u_{x_ix_j}^{\varepsilon,b} = -f(x, u^{\varepsilon,b}). \tag{5.2}$$

Furthermore, by (2.8) the function $u^{\varepsilon,b}$ satisfies

$$u^{\varepsilon,b}\left(t + \frac{k\cdot e_i}{c^{\varepsilon,b}}, x + e_i\right) = u^{\varepsilon,b}(t,x) \tag{5.3}$$

for the unit vectors e_i, $i = 1, ..., n$. Due to corollary 4.2, there holds $U_s^{\varepsilon,b} > 0$ and thus

$$u_t^{\varepsilon,b} > 0 \tag{5.4}$$

(since $c^{\varepsilon,b} < 0$).

5.1.2. A priori estimates for $u^{\varepsilon,b}$

In order to obtain estimates for the x-gradient of the function $u^{\varepsilon,b}$ from definition 5.1, we want to use theorem A.2 from Berestycki and Hamel. However this theorem requires more regularity on $u^{\varepsilon,b}$ than we have, namely $u^{\varepsilon,b} \in C^3$. Therefore, the theorem will not be applied to $u^{\varepsilon,b}$ directly, but to an approximation. In the following lemma, we will study the behavior (in (s,y)-coordinates) of the (ε,b)-problem under an approximation of the coefficients and the nonlinearity.

Lemma 5.2 (Behavior of the (ε, b)-problem under approximation) *We denote by R_b the cylinder $[-b, b] \times T^n$. Let $\alpha = 1$, f be a nonlinearity of basic type with $f|_{T^n\times[0,1]} \in C^1(T^n \times [0,1])$ and $(a_{ij})_{i,j=1}^n$ be as in the general assumptions. Let $(U, c) = (U^{\varepsilon,b}, c^{\varepsilon,b})$ be the corresponding solution of problem (2.7), (2.8) with normalization constant $\theta = \theta(f)$, whose existence is guaranteed by proposition 4.8. That means the following: Given the operator*

$$LU := \varepsilon U_{ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})U, \tag{5.5}$$

the pair (U, c) satisfies

$$LU + cU_s = -f(y, U) \tag{5.6}$$

and

$$\left.\begin{array}{lll} 0 \le U \le 1, & U(s,\cdot) \text{ is } 1-\text{periodic} & \\ U(-b,y) = 0, & U(b,y) = 1 & \text{for all } y \in T^n, \\ \max_{y\in T^n} U(0,y) = \theta(f). & & \end{array}\right\} \tag{5.7}$$

For small $\delta > 0$, let $(a_{ij})^\delta$ be a $C^1(T^n)$-matrix field which is symmetric and uniformly elliptic. For $i, j = 1, ..., n$, we assume

$$\left|\left|a_{ij}^\delta - a_{ij}\right|\right|_{C^1(T^n)} \to 0 \text{ for } \delta \to 0. \tag{5.8}$$

Moreover, let f^δ be a nonlinearity of basic type with $f^\delta|_{T^n\times[0,1]} \in C^1(T^n \times [0,1])$ such that

$$\left|\left|f^\delta - f\right|\right|_{C^1(T^n\times[0,1])} \to 0 \text{ for } \delta \to 0. \tag{5.9}$$

Furthermore, let $(U^\delta, c^\delta) = (U^{\delta,\varepsilon,b}, c^{\delta,\varepsilon,b})$ be the corresponding solutions of the problem (2.7), (2.8), also with normalization constant $\theta(f)$ (not $\theta(f^\delta)$). This means the following: Given the operator

$$L^\delta U^\delta := \varepsilon U_{ss}^\delta + \sum_{i,j=1}^{n} a_{ij}^\delta(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})U^\delta, \tag{5.10}$$

the pair (U^δ, c^δ) satisfies

$$L^\delta U^\delta + c^\delta U_s^\delta = -f^\delta(y, U^\delta) \tag{5.11}$$

and

$$\left.\begin{array}{lll} 0 \le U^\delta \le 1, & U^\delta(s,\cdot) \text{ is } 1-\text{periodic} & \\ U^\delta(-b,y) = 0, & U^\delta(b,y) = 1 & \text{for all } y \in T^n, \\ \max_{y\in T^n} U^\delta(0,y) = \theta(f). & & \end{array}\right\} \tag{5.12}$$

Then $\left|c^\delta - c\right| \to 0$ and $\left|\left|U^\delta - U\right|\right|_{C^2(R_b)} \to 0$ for $\delta \to 0$.

Proof We will show that any subsequence of $\delta \to 0$ has another subsequence, again denoted by δ, such that $|c^\delta - c| \to 0$ and $||U^\delta - U||_{C^2(R_b)} \to 0$ for $\delta \to 0$. Then the assertion $|c^\delta - c| \to 0$ and $||U^\delta - U||_{C^2(R_b)} \to 0$ for $\delta \to 0$ follows. To this end, consider an arbitrary subsequence of $\delta \to 0$, again denoted by δ.

Step 1: Estimates for $|c^\delta|$ *and* $||U^\delta||_{C^1(R_b)}$. In the first step, we will acquire estimates for c^δ with the help of the a priori estimates in lemma 4.4, which we have used to prove the existence of the (ε, b)-solution. By r^δ, we denote the function $r^\delta(y) := \sum_{i,j=1}^n a_{ij}^\delta(y)k_ik_j$. Let δ_0, M, R, $\bar{r}$ be such that

$$M \geq \sup_{y\in T^n, z\in[0,1]} \frac{|f^\delta(y,z)|}{r^\delta(y)}, \qquad R \geq \sup_{y\in T^n} r^\delta(y), \qquad \bar{r} \geq \max_{y\in T^n} \frac{1}{r^\delta(y)}$$

for all $0 < \delta \leq \delta_0$. These constants exist due to (5.8) and (5.9). Furthermore, let $\Lambda > 0$ such that $||a_{ij}^\delta||_\infty \leq \Lambda$, $||\varepsilon||_\infty \leq \Lambda$. The assumption (5.8) implies the equicontinuity of the family $\{a_{ij}^\delta : 0 < \delta \leq \delta_0\}$ for every $i, j = 1, ..., n$. Therefore, the family admits a common modulus of continuity ω_{ij} for every $i, j = 1, ..., n$. Clearly, the constant ε has a modulus of continuity ω. Also the family of operators L^δ admits a common ellipticity constant $\tilde{\mu}$ for its second order part. Hence, by lemma 4.4 there is a constant $K = K(\theta(f), \varepsilon, b, \bar{r}, R, M, \tilde{\mu}, \Lambda, \{\omega_{ij}|i,j=1,...,n\}, \omega)$, such that

$$|c^\delta| \leq K \text{ and } ||U^\delta||_{C^1(R_b)} \leq K. \tag{5.13}$$

Step 2: Estimates for $||U^\delta||_{C^{2,\alpha}(R_b)}$. The $C^1(R_b)$-estimates from step 1 will help us to obtain better estimates, which we need to pass to the limit in the equation. Let $\alpha \in (0,1)$ be an arbitrary Hölder exponent and $0 < \delta \leq \delta_0$ with δ_0 from step 1. Because of the estimates (5.13) and because $||f^\delta||_{C^1(T^n\times[0,1])}$ is bounded by (5.9), we obtain

$$||f^\delta(y, U^\delta)||_{C^\alpha(R_b)} \leq C. \tag{5.14}$$

Since a_{ij}^δ is bounded in $C^1(R_b)$ due to (5.8), it is also bounded in $C^\alpha(R_b)$. Hence, there is $\tilde{\Lambda} \geq K$ with K from step 1, such that $||a_{ij}^\delta||_{C^\alpha(R_b)} \leq \tilde{\Lambda}$ and $|c^\delta| \leq \tilde{\Lambda}$. Let $\varphi(s,y) = \frac{b+s}{2b}$. Then $U^\delta = \varphi$ on ∂R_b. By lemma 4.6, there is a constant $C = C(n, \alpha, \tilde{\mu}, \tilde{\Lambda}, b)$ such that

$$||U^\delta||_{C^{2,\alpha}(R_b)} \leq C(||U^\delta||_{C^0(R_b)} + ||\varphi||_{C^{2,\alpha}(R_b)} + ||f^\delta(y,U^\delta)||_{C^\alpha(R_b)}) \leq \tilde{C}. \tag{5.15}$$

The second inequality holds due to (5.14) and $0 \leq U^\delta \leq 1$ independently of δ. Therefore, we have proved that the sequence U^δ is bounded in $C^{2,\alpha}(R_b)$.

Step 3: Passing to a subsequence. By the estimates (5.13) and (5.15), there is a subsequence of (U^δ, c^δ), a $\tilde{c} \in \mathbb{R}$ and some $\tilde{U} \in C^2(R_b)$, such that

$$|c^\delta - \tilde{c}| \to 0 \text{ and } \left|\left|U^\delta - \tilde{U}\right|\right|_{C^2(R_b)} \to 0 \text{ for } \delta \to 0.$$

This convergence is sufficient to pass to the limit in the equation (5.11) and its conditions (5.12). We obtain

$$\varepsilon \tilde{U}_{ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})\tilde{U} + \tilde{c}\tilde{U}_s = -f(y,\tilde{U}),$$

$$\begin{cases} 0 \le \tilde{U} \le 1, & \tilde{U}(s,\cdot) \text{ is } 1-\textit{periodic}, \\ \tilde{U}(-b,y) = 0, & \tilde{U}(b,y) = 1 \qquad \text{for all } y \in T^n, \\ \max\limits_{y\in T^n} \tilde{U}(0,y) = \theta(f). \end{cases}$$

However, by uniqueness of the (ε, b)-solution (see proposition 4.3), we have $\tilde{c} = c$ and $\tilde{U} = U$. Hence, the lemma is proved. ∎

Remark 5.3 (Modification of the nonlinearity f) *As we already mentioned, we want to apply theorem A.2 to obtain a certain a priori estimate for the function $u^{\varepsilon,b}$ from definition 5.1. Let f be a nonlinearity with properties as in the definition of $u^{\varepsilon,b}$. Formally, the regularity $f|_{T^n\times[0,1]} \in C^1(T^n \times [0,1])$ is not sufficient to apply the theorem. However, the function $u^{\varepsilon,b}$ only takes values in $[0,1]$. We can therefore modify f outside of $T^n \times [0,1]$ to a function $\tilde{f} \in C_b^1(T^n \times \mathbb{R})$. Then, the function $u^{\varepsilon,b}$ remains a solution of (5.2) with $\tilde{f}$ instead of f. We will define $\tilde{f}$ by a point reflection of f at $z = 1$: For $x \in T^n$ and $z \in \mathbb{R}$, we define*

$$\tilde{f}(x,z) := \begin{cases} f(x,z), & \textit{if } z \le 1, \\ -f(x, 2-z), & \textit{if } z > 1. \end{cases}$$

Since $f \ge 0$ by the combustion property and $f(x,z) \le 0$ for $z \in [0,\theta]$ with $\theta \in (0,1)$ by property (iii) from definition 2.6, we actually have $f(x,z) = 0$ for $z \in (0,\theta)$. Because also $f(x,z) = 0$ for $z \notin [0,1]$, the nonlinearity f actually has the regularity $f \in C_b^1(T^n \times (-\infty,1])$. Then for $\tilde{f}$, the regularity $\tilde{f} \in C_b^1(T^n \times \mathbb{R})$ follows from construction. Furthermore, by this construction, $\tilde{f} = \tilde{f}(x,z)$ is a function that is still periodic in x and compactly supported in z. Therefore, the modified f and all its first order derivatives are even uniformly continuous on $T^n \times \mathbb{R}$.

Lemma 5.4 (A priori estimates for $u^{\varepsilon,b}$) *Consider $u^{\varepsilon,b} : D^{\varepsilon,b} \to [0,1]$ and $c^{\varepsilon,b}$ as defined in definition 5.1 for $0 < \varepsilon \le \varepsilon_0$, $b \ge b_0$. For every bounded Lipschitz domain $\Omega \subset \mathbb{R}^{n+1}$, there are constants $C = C(\Omega)$, $\tilde{b} = \tilde{b}(\Omega) \ge b_0 > 0$ and $0 < \tilde{\varepsilon} = \tilde{\varepsilon}(\Omega) \le \varepsilon_0$, such that the following holds: For $0 < \varepsilon \le \tilde{\varepsilon}$ and $b \ge \tilde{b}$, there holds $\Omega \Subset D^{\varepsilon,b}$ and furthermore*

$$\left|\left|\nabla_x u^{\varepsilon,b}\right|\right|_{L^\infty(\Omega)} \le C(\Omega), \tag{5.16}$$

$$\left|\left|u_t^{\varepsilon,b}\right|\right|_{L^1(\Omega)} \le C(\Omega), \tag{5.17}$$

and

$$\left|\left|u_t^{\varepsilon,b}\right|\right|_{L^2(\Omega)} \le C(\Omega). \tag{5.18}$$

PROOF *Step 0: The inclusion for* Ω. Let Ω be a bounded Lipschitz domain. We recall

$$D^{\varepsilon,b} = \{(t,x) \in \mathbb{R}^{n+1} : \frac{-b - k\cdot x}{-c^{\varepsilon,b}} < t < \frac{b - k\cdot x}{-c^{\varepsilon,b}}\}$$

and the estimates $c_1 \le c^{\varepsilon,b} \le c_2 < 0$ for $b \ge b_0$, $0 < \varepsilon \le \varepsilon_0$ from proposition 4.16. Hence, there exists $\tilde{b} = \tilde{b}(\Omega) \ge b_0$ such that the following holds: For $b \ge \tilde{b}$ and $0 < \varepsilon \le \varepsilon_0$, we have $\Omega \Subset D^{\varepsilon,b}$. Moreover, for some constant K_0, there holds

$$dist(\overline{\Omega}, \partial D^{\varepsilon,b}) \ge K_0 > 0. \tag{5.19}$$

Step 1: Estimating the x-gradient. In order to obtain the estimates (5.16), we want to apply theorem A.2. By remark 5.3, we can assume $f \in C^1_b(T^n \times \mathbb{R})$ and that f and all of its first order derivatives are uniformly continuous on $T^n \times \mathbb{R}$.

However, as already mentioned, directly applying the theorem to $u^{\varepsilon,b}$ would require that $u^{\varepsilon,b} \in C^3$. We usually do not have such a regularity, since $(a_{ij})^n_{i,j=1}$ is only a $C^1(T^n)$ matrix field and we only have $f \in C^1_b(T^n \times \mathbb{R})$ as well. In order to be sure that $u \in C^3$, we would at least require $(a_{ij})^n_{i,j=1}$ to be a $C^{1,\alpha}(T^n)$ matrix field and f to be such that the function $f(y, u^{\varepsilon,b})$ is $C^{1,\alpha}$ for some $\alpha \in (0,1)$. Since the estimates in theorem A.2 only involve the C^1-Norms of the matrix field and the nonlinearity, we can solve this problem by an approximation.

Step 1a: Defining the approximative coefficients. We define $\eta \in C^\infty(\mathbb{R}^n)$ by

$$\eta(x) := \begin{cases} C \exp\left(\dfrac{1}{|x|^2 - 1}\right) & for\ |x| \le 1, \\ 0 & \text{otherwise} \end{cases}$$

and choose C such that $\int_{\mathbb{R}^n} \eta = 1$. Then, we take the standard mollifier on $\mathbb{R}^n$ given by $\eta_\delta(x) := \frac{1}{\delta^n}\eta(\frac{x}{\delta})$ and define $a^\delta_{ij} := \eta_\delta * a_{ij}$. Consequently, we have $a^\delta_{ij} \in C^\infty(T^n)$. Since a_{ij} and its first order derivatives are uniformly continuous, there holds

$$\left|\left|a^\delta_{ij} - a_{ij}\right|\right|_{C^1(T^n)} \to 0 \text{ for } \delta \to 0. \tag{5.20}$$

Analogously, let $\tilde{\eta}_\delta$ be the standard mollifier on $\mathbb{R}^{n+1}$ and define $f^\delta = \tilde{\eta}_\delta * f$. Then $f^\delta \in C^\infty(T^n \times \mathbb{R})$. Since f and its first order derivatives are uniformly continuous (see the assumption at the beginning of step 1), there holds

$$\left|\left|f^\delta - f\right|\right|_{C^1_b(T^n \times \mathbb{R})} \to 0 \text{ for } \delta \to 0. \tag{5.21}$$

In particular, this implies

$$\left|\left|f^\delta - f\right|\right|_{C^1(T^n \times [0,1])} \to 0 \text{ for } \delta \to 0. \tag{5.22}$$

Let us show that for $\delta < \theta(f)$, the relevant part of f^δ, namely $f^\delta|_{T^n \times [0,1]}$ coincides with the relevant part of a nonlinearity of basic type as in definition 2.6. This means that the trivial continuation of $f^\delta|_{T^n \times [0,1]}$ is a nonlinearity of basic type. This is necessary in order to apply lemma 5.2. We can see it as follows: Due to the way the modified f has been constructed in remark 5.3 and because $\tilde{\eta}_\delta$ is supported in $\overline{B_\delta(0)}$, there holds $f^\delta(x,z) = 0$ for $z \in (-\infty, \theta(f) - \delta)$. By symmetry of $\tilde{\eta}$ and the way the modification was constructed by mirroring, there holds $f^\delta(x,1) = 0$. Lastly, we have to show $f^\delta(x,z) \geq 0$ for $z < 1$: For any (x_0, z_0) with $z_0 < 1$, we have

$$\begin{aligned} f(x_0,z_0) &= \int\limits_{\mathbb{R}^{n+1}} f(x,z)\tilde{\eta}_\delta((x,z)-(x_0,z_0))\,d(x,z) \\ &= \int\limits_{\mathbb{R}^n\times(-\infty,1)} f(x,z)\{\tilde{\eta}_\delta((x_0,z_0)-(x,z)) - \tilde{\eta}_\delta((x_0,z_0)-(x,2-z))\}\,d(x,z). \end{aligned}$$

The expression in the curly bracket is nonnegative, since $|z - z_0| \leq |2 - z|$ for $z \leq 1$ and also $f(x,z) \geq 0$ for $z \leq 1$. Therefore $f(x_0, z_0) \geq 0$.

Step 1b: Carrying out the approximation. We now have that $f^\delta|T^n \times [0,1]$ coincides the relevant part of a nonlinearity of basic type. Moreover, we have the necessary convergences (5.20) and (5.22) of a^δ_{ij} and f^δ. Altogether, the assumptions of lemma 5.2 hold. We take $(U^\delta, c^\delta) = (U^{\delta,\varepsilon,b}, c^{\delta,\varepsilon,b})$ from lemma 5.2. With consider the transformation

$$(s,y) = (k \cdot x - c^{\varepsilon,b}t, x) =: T^{\varepsilon,b}(t,x).$$

(We emphasize that the transformation is not $s = k \cdot x - c^{\delta,\varepsilon,b}t$.) Then we define

$$u^{\delta,\varepsilon,b}(t,x) := U^{\delta,\varepsilon,b}(s,y) = U^{\delta,\varepsilon,b}(T^{\varepsilon,b}(t,x)).$$

The function $u^{\delta,\varepsilon,b}$ is defined on $\overline{D^{\varepsilon,b}}$. In definition 5.1, the function $u^{\varepsilon,b}$ was defined using the same transformation $T^{\varepsilon,b}$ by $u^{\varepsilon,b}(t,x) = U^{\varepsilon,b}(T^{\varepsilon,b}(t,x))$. Consequently, we have $u^{\delta,\varepsilon,b} = U^{\delta,\varepsilon,b} \circ T^{\varepsilon,b}$ and $u^{\varepsilon,b} = U^{\varepsilon,b} \circ T^{\varepsilon,b}$. Furthermore, by lemma 5.2 we also have $\left|\left|U^{\delta,\varepsilon,b} - U^{\varepsilon,b}\right|\right|_{C^2(R_b)} \to 0$ for $\delta \to 0$. Therefore

$$\left|\left|u^{\delta,\varepsilon,b} - u^{\varepsilon,b}\right|\right|_{C^2(\overline{D^{\varepsilon,b}})} \to 0 \text{ for } \delta \to 0 \tag{5.23}$$

as well.

Showing the regularity $u^{\delta,\varepsilon,b} \in C^\infty(\overline{D^{\varepsilon,b}})$ is not difficult (and much more than we actually need). Let R_b denote as usual the cylinder $[-b,b] \times T^n$. The function $U^{\delta,\varepsilon,b}$ has the regularity $U^{\delta,\varepsilon,b} \in C^{2,\alpha}(R_b)$ by proposition 4.8. We will apply theorem 6.19 of [12]. By what is said right after the proof of this theorem in [12], it can be applied locally in the following way: Let $\tilde{\Omega} = (-b,b) \times (-1,2)^n$. There holds $L^\delta U^{\delta,\varepsilon,b} = -f^\delta(y, U^{\delta,\varepsilon,b})$ on $\tilde{\Omega}$ with L^δ from (5.11). The domain $\tilde{\Omega}$ has a $C^{3,\alpha}$ boundary portion $P = \{-b\} \times (-1,2)^n \cup \{b\} \times (-1,2)^n$. Boundary values of $U^{\delta,\varepsilon,b}$ on this boundary portion are given by the function $\varphi(s,y) = \frac{b+s}{2b}$. We have $\varphi \in C^{3,\alpha}(\overline{\tilde{\Omega}})$ and $\varphi = U^{\delta,\varepsilon,b}$ on P. Since $f^\delta \in C^\infty(T^n \times \mathbb{R})$, we have $f^\delta(y, U^\delta) \in C^{1,\alpha}(\overline{\tilde{\Omega}})$. Since $a^\delta_{ij} \in C^\infty(T^n)$, the coefficients of L^δ are in $C^{1,\alpha}(\overline{\tilde{\Omega}})$. The local version of the theorem 6.19 in [12] yields $U^{\delta,\varepsilon,b} \in C^{3,\alpha}(\tilde{\Omega} \cup P)$. By periodicity $U^{\delta,\varepsilon,b} \in C^{3,\alpha}(R_b)$ follows. Because f^δ, a^δ_{ij} and φ are actually arbitrarily smooth, the argument can be iterated to obtain $U^{\delta,\varepsilon,b} \in C^\infty(R_b)$. Therefore, $u^{\delta,\varepsilon,b} = U^{\delta,\varepsilon,b} \circ T^{\varepsilon,b} \in C^\infty(\overline{D^{\varepsilon,b}})$.

Step 1c: Estimating the x-gradient of the approximative (ε, b)-solution. From equation (5.11), we can see that the approximative (ε, b)-solution $u^{\delta,\varepsilon,b}$ satisfies the equation

$$-\frac{c^{\delta,\varepsilon,b}}{c^{\varepsilon,b}} u_t^{\delta,\varepsilon,b} + \frac{\varepsilon u_{tt}^{\delta,\varepsilon,b}}{(c^{\varepsilon,b})^2} + \sum_{i,j=1}^n a_{ij}^\delta(x) u_{x_i x_j}^{\delta,\varepsilon,b} = -f^\delta(x, u^{\delta,\varepsilon,b}).$$

If $\delta > 0$ is sufficiently small, such that $c^{\delta,\varepsilon,b} < 0$, this is equivalent to

$$-u_t^{\delta,\varepsilon,b} + \frac{\varepsilon u_{tt}^{\delta,\varepsilon,b}}{c^{\varepsilon,b} c^{\delta,\varepsilon,b}} + \sum_{i,j=1}^n \frac{c^{\varepsilon,b}}{c^{\delta,\varepsilon,b}} a_{ij}^\delta(x) u_{x_i x_j}^{\delta,\varepsilon,b} = -\frac{c^{\varepsilon,b}}{c^{\delta,\varepsilon,b}} f^\delta(x, u^{\delta,\varepsilon,b}). \tag{5.24}$$

Because of the estimates for $c^{\varepsilon,b}$, we can (if necessary) decrease $\tilde{\varepsilon} > 0$ and increase $\tilde{b} > 0$, such that for $0 < \varepsilon \le \tilde{\varepsilon}$ and $b \ge \tilde{b}$ we have $0 < \frac{\varepsilon}{(c^{\varepsilon,b})^2} < 1$. Now let $0 < \varepsilon \le \tilde{\varepsilon}$, $b \ge \tilde{b}$ with ε and b fixed. There is $\delta_0 = \delta_0(b, \varepsilon) > 0$ such that for $0 < \delta \le \delta_0$ we have $0 < \frac{\varepsilon}{c^{\varepsilon,b} c^{\delta,\varepsilon,b}} < 1$. Using the convergences (5.20) and (5.21), by possibly decreasing $\delta_0 > 0$ further, we can achieve

$$\left|\left| \frac{c^{\varepsilon,b}}{c^{\delta,\varepsilon,b}} a_{ij}^\delta \right|\right|_{C^1(\overline{D^{\varepsilon,b}})} + \left|\left| \frac{c^{\varepsilon,b}}{c^{\delta,\varepsilon,b}} \partial_u f^\delta \right|\right|_{L^\infty(D^{\varepsilon,b} \times \mathbb{R})} \le 2\,||a_{ij}||_{C^1(T^n)} + 2\,||\partial_u f||_{L^\infty(T^n \times \mathbb{R})} =: K_1$$

for $0 < \delta \le \delta_0$. (We have regarded $a_{ij}^\delta = a_{ij}^\delta(x)$ as a function that depends on t in the expression with the $C^1(\overline{D^{\varepsilon,b}})$-norm.) Using the convergence (5.21) again, we can furthermore achieve

$$\left|\left| \frac{c^{\varepsilon,b}}{c^{\delta,\varepsilon,b}} f^\delta \right|\right|_{L^\infty(T^n \times \mathbb{R})} \le 2\,||f||_{L^\infty(T^n \times \mathbb{R})} =: K_2,$$

$$\left|\left| \frac{c^{\varepsilon,b}}{c^{\delta,\varepsilon,b}} \nabla_x f^\delta \right|\right|^2_{L^\infty(T^n \times \mathbb{R})} \le 2\,||\nabla_x f||^2_{L^\infty(T^n \times \mathbb{R})} =: K_3$$

for $0 < \delta \le \delta_0$ with a possibly even smaller δ_0. The factor $\frac{\varepsilon}{c^{\varepsilon,b} c^{\delta,\varepsilon,b}} \in (0,1)$ in front of $u_{tt}^{\delta,\varepsilon,b}$ in equation (5.24) now takes the role of the factor ε in front of u_{tt} in equation (A.1), and we can apply theorem A.2 to $u^{\delta,\varepsilon,b}$. We obtain for $(t,x) \in \overline{\Omega}$

$$\left|\nabla_x u^{\delta,\varepsilon,b}(t,x)\right|^2 \le C\left(1 + \frac{1}{d((t,x), \partial D^{\varepsilon,b})^2}\right) \le C\left(1 + \frac{1}{K_0^2}\right),$$

with K_0 from (5.19). The constant C in the theorem is given by an expression that can be estimated by

$$\begin{aligned} C &\le C_1\left[||u^{\delta,\varepsilon,b}||_{L^\infty(D^{\varepsilon,b})} \left(\underset{D^{\varepsilon,b}}{osc}(u^{\delta,\varepsilon,b}) + \left|\left| \frac{c^{\varepsilon,b}}{c^{\delta,\varepsilon,b}} f^\delta \right|\right|_{L^\infty(T^n \times \mathbb{R})} \right) + \left|\left| \frac{c^{\varepsilon,b}}{c^{\delta,\varepsilon,b}} \nabla_x f^\delta \right|\right|^2_{L^\infty(T^n \times \mathbb{R})} \right] \\ &\le C_1[1(1 + K_2) + K_3] \end{aligned}$$

for $0 < \delta \leq \delta_0(\varepsilon, b)$. The constant C_1 depends only on n, $\tilde{\mu}$, K_1, where $\tilde{\mu}$ is a common ellipticity constant for the family of matrix fields $\{(\frac{c^{\varepsilon,b}}{c^{\delta,\varepsilon,b}} a_{ij}^{\delta})_{i,j=1}^n : 0 < \delta \leq \delta_0\}$. Clearly, this family admits a common ellipticity constant for a possible even smaller δ_0 because of $\left|\left|a_{ij}^{\delta} - a_{ij}\right|\right|_{C^1(T^n)} \to 0$ for $\delta \to 0$ and $\frac{c^{\varepsilon,b}}{c^{\delta,\varepsilon,b}} \to 1$ for $\delta \to 0$. Therefore

$$\left|\left|\nabla_x u^{\delta,\varepsilon,b}\right|\right|^2_{L^\infty(\Omega)} \leq C_1[1(1+K_2)+K_3]\left(1+\frac{1}{K_0^2}\right) =: \tilde{C}^2.$$

The constant $\tilde{C}$ is independent of ε and b. By (5.23) it follows that

$$\left|\left|\nabla_x u^{\varepsilon,b}\right|\right|_{L^\infty(\Omega)} \leq \tilde{C} \text{ for } 0 < \varepsilon \leq \tilde{\varepsilon}, b \geq \tilde{b}.$$

This is (5.16).

Step 2: Deriving the L^1 and L^2-bounds. For the rest of the proof we will more shortly write u for $u^{\varepsilon,b}$. To see the L^1-bounds of u_t, we remember $u_t \geq 0$ and $0 \leq u \leq 1$. Therefore $\int_I u_t\,dt \leq 1$ for any interval I. Now the L^1-bounds (5.17) follow from the fubini theorem.

To obtain the L^2-bounds for u_t, we choose a function $\eta \in C_c^\infty(D^{\varepsilon,b})$ with $0 \leq \eta \leq 1$ and $\eta = 1$ on $\bar{\Omega}$. First we test equation (5.2) with $u\eta$ and obtain

$$\int\limits_{supp(\eta)} -u_t u\eta + \frac{\varepsilon u_{tt} u\eta}{(c^{\varepsilon,b})^2} + \sum_{i,j=1}^n a_{ij}(x) u_{x_i x_j} u\eta + f(x,u)u\eta\, d\mathcal{L} = 0.$$

This can be rewritten by integration by parts as

$$\begin{aligned}\int\limits_{supp(\eta)} \frac{\varepsilon u_t^2 \eta}{(c^{\varepsilon,b})^2}\, d\mathcal{L} = \int\limits_{supp(\eta)} & -u_t u\eta - \frac{\varepsilon u_t u\eta_t}{(c^{\varepsilon,b})^2} - \sum_{i,j=1}^n (\partial_{x_j} a_{ij}(x)) u_{x_i} u\eta \\ & - \sum_{i,j=1}^n a_{ij}(x) u_{x_i} u_{x_j} \eta - \sum_{i,j=1}^n a_{ij}(x) u_{x_i} u\eta_{x_j} + f(x,u)u\eta\, d\mathcal{L}.\end{aligned}$$

The integrals of all of the six terms on the right hand side are bounded independently of ε and b. For the first term, this is due to the L^1-bounds $||u_t||_{L^1(supp(\eta))} \leq C$ by (5.17), $0 \leq u \leq 1$ and $\eta \in L^\infty$. For the second term, the reasons are $0 < \varepsilon \leq \tilde{\varepsilon}$, (5.17), $0 \leq u \leq 1$, $\eta_t \in L^\infty$ and because of $c^{\varepsilon,b} < c_2 < 0$. For the third, fourth and fifth term, we use the already derived L^∞-bounds for the x-gradient of u, $0 \leq u \leq 1$ and $a_{ij}, \partial_{x_j} a_{ij}, \eta \in L^\infty$. For the sixth term, we use $f \in L^\infty$, $0 \leq u \leq 1$ and $\eta \in L^\infty$. Altogether we obtain

$$\int\limits_{supp(\eta)} \frac{\varepsilon u_t^2 \eta}{(c^{\varepsilon,b})^2}\, d\mathcal{L} \leq C. \tag{5.25}$$

Now we test (5.2) with $u_t\eta^2$. We have

$$\begin{aligned}
\int_\Omega u_t^2\, d\mathcal{L} &\le \int_{supp(\eta)} u_t^2\eta^2\, d\mathcal{L}\\
&= \int_{supp(\eta)} \frac{\varepsilon u_{tt}u_t\eta^2}{(c^{\varepsilon,b})^2}\, d\mathcal{L} + \int_{supp(\eta)} \sum_{i,j=1}^n a_{ij}(x)u_{x_ix_j}u_t\eta^2\, d\mathcal{L} + \int_{supp(\eta)} f(x,u)u_t\eta^2\, d\mathcal{L}\\
&=: T_1 + T_2 + T_3.
\end{aligned}$$

Clearly, $|T_3| \le ||f\eta^2||_\infty\, ||u_t||_{L^1(supp(\eta))} \le C$. Moreover

$$|T_1| = \left| \int_{supp(\eta)} \frac{\varepsilon\frac{1}{2}\partial_t(u_t^2)\eta^2}{(c^{\varepsilon,b})^2}\, d\mathcal{L} \right| = \left| \int_{supp(\eta)} \frac{\varepsilon u_t^2\eta\eta_t}{(c^{\varepsilon,b})^2}\, d\mathcal{L} \right| \le ||\eta_t||_\infty \left| \int_{supp(\eta)} \frac{\varepsilon u_t^2\eta}{(c^{\varepsilon,b})^2}\, d\mathcal{L} \right| \le C$$

by (5.25). It remains to consider the term T_2:

$$\begin{aligned}
|T_2| &\le \left| \int_{supp(\eta)} \sum_{i,j=1}^n \partial_{x_j}(a_{ij}(x)u_{x_i})u_t\eta^2\, d\mathcal{L} \right| + \left| \int_{supp(\eta)} \sum_{i,j=1}^n (\partial_{x_j}a_{ij}(x))u_{x_i}u_t\eta^2\, d\mathcal{L} \right|\\
&=: |T_4| + |T_5|\,.
\end{aligned}$$

Clearly, $|T_5| \le C$, because of $\eta^2\partial_{x_j}a_{ij} \in L^\infty$, (5.16) and $||u_t||_{L^1(supp(\eta))} \le C$. Let us consider the term $T4$ now. After an integration by parts in x_j, T_4 can by symmetry of a_{ij} and a subsequent integration by parts in t be rewritten as

$$\begin{aligned}
T_4 &= -\int_{supp(\eta)} \sum_{i,j=1}^n a_{ij}(x)u_{x_i}u_{tx_j}\eta^2\, d\mathcal{L} - \int_{supp(\eta)} \sum_{i,j=1}^n a_{ij}(x)u_{x_i}u_t 2\eta\eta_{x_j}\, d\mathcal{L}\\
&= -\int_{supp(\eta)} \frac{1}{2}\partial_t(\sum_{i,j=1}^n a_{ij}(x)u_{x_i}u_{x_j})\eta^2\, d\mathcal{L} - \int_{supp(\eta)} \sum_{i,j=1}^n a_{ij}(x)u_{x_i}u_t 2\eta\eta_{x_j}\, d\mathcal{L}\\
&= +\int_{supp(\eta)} \frac{1}{2}(\sum_{i,j=1}^n a_{ij}(x)u_{x_i}u_{x_j})\eta\eta_t\, d\mathcal{L} - \int_{supp(\eta)} \sum_{i,j=1}^n a_{ij}(x)u_{x_i}u_t 2\eta\eta_{x_j}\, d\mathcal{L}.
\end{aligned}$$

On the right hand side the first integral can be estimated using (5.16) for u_{x_i} and u_{x_j}, because all the other factors are bounded and the integrand has support in $supp(\eta)$. For the second integral we can use (5.16) for u_{x_i} and the $L^1(supp(\eta))$-bounds for u_t. All the other factors are bounded. Therefore $|T_4| \le C$. Together we have proved

$$\int_\Omega u_t^2\, d\mathcal{L} \le C$$

for $0 < \varepsilon \le \tilde{\varepsilon}$ and $b \ge \tilde{b}$. Here it might be necessary to increase $\tilde{b}$ and decrease $\tilde{\varepsilon}$, since we used the estimates (5.16) and (5.17) not on Ω, but on the larger set $supp(\eta)$. On the larger set, the estimates hold for a possible smaller $\tilde{\varepsilon}$ and $\tilde{b}$. ∎

5.2. Passage to the limit

In this section, we start from $u^{\varepsilon,b}$ as defined in definition 5.1. We prove the existence of the limit function for $\varepsilon \to 0$ and $b \to \infty$. Afterwards, we show that the limit function u is a traveling wave solution of type I and furthermore has the properties claimed in theorems 2.8 and 2.9.

Under the additional regularity assumptions on f in theorem 2.10, we will show in lemma 5.6, that u_{tt} exists in the classical sense and that consequently u can be transformed into a solution of type II.

5.2.1. Existence of the limit function u and its regularity

Under the assumptions of either of the theorems 2.8, 2.9 or 2.10, the assumptions of the following lemma are met.

Lemma 5.5 *Let the assumptions of definition 5.1 hold. Consider the function* $u^{\varepsilon,b} : \overline{D^{\varepsilon,b}} \to [0,1]$ *and the corresponding wavespeed* $c^{\varepsilon,b}$*, which where given in definition 5.1. Then there are sequences* $b_m \to \infty$ *and* $\varepsilon_m \to 0$ *such that* $c^{(m)} := c^{\varepsilon_m,b_m} \to c < 0$*, and* $u_m := u^{\varepsilon_m,b_m}$ *converges weakly in* $H^1_{loc}(\mathbb{R}^{n+1})$ *to a function* $u : \mathbb{R}^{n+1} \to \mathbb{R}$*. The function* u *satisfies* $0 \le u \le 1$*, it is nondecreasing in* t *and a classical solution of:*

$$-u_t + \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + f(x,u) = 0 \text{ on } \mathbb{R}^{n+1}. \tag{5.26}$$

Moreover, it has the regularity $u \in C^{2,\alpha}_{1,\alpha/2,loc}(\mathbb{R}^{n+1})$ *for any* $\alpha \in (0,1)$*. It satisfies the "pulsating wave" condition*

$$u\left(t + \frac{k \cdot e_i}{c}, x + e_i\right) = u(t,x) \tag{5.27}$$

for the unit vectors e_i*,* $i = 1, ..., n$*. Additionally, there is some time-like number* $d > 0$ *such that*

$$\max_{x\in[0,1]^n} u(-d,x) \le \theta(f), \qquad \theta(f) \le \max_{x\in[0,1]^n} u(d,x). \tag{5.28}$$

Proof *Step 1: Existence of the limit function and some properties.* Let ε_0, b_0 and c_1, c_2 be the constants from definition 5.1. We choose sequences ε_m and b_m, such that $0 < \varepsilon_m \le \varepsilon_0$ and $b_m \ge b_0$, as well as $\varepsilon_m \to 0$ and $b_m \to \infty$ for $m \to \infty$. Then $c_1 \le c^{(m)} \le c_2 < 0$. Therefore there is a subsequence (which will be denoted with the same name) such that $c^{(m)} \to c$ for some $c < 0$. The bounds for $u^{\varepsilon,b}$ given in lemma 5.4 together with $0 \le u^{\varepsilon,b} \le 1$ clearly imply H^1_{loc}-bounds. Consequently, there is another subsequence and a function $u \in H^1_{loc}(\mathbb{R}^{n+1})$, such that $u_m \to u$ weakly in $H^1_{loc}(\mathbb{R}^{n+1})$, strongly in $L^2_{loc}(\mathbb{R}^{n+1})$ and pointwise almost everywhere. From $0 \le u_m \le 1$, we obtain $0 \le u \le 1$. The monotonicity of u in t follows from the monotonicity of u_m in t, which holds by (5.4).

Step 2: Weak solution. In the second step, we show that u is a weak solution of the equation in divergence form. We test the equation (5.2) for u_m with some test function $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$. For m large enough we have $supp(\varphi) \Subset D^{\varepsilon_m, b_m}$ by lemma 5.4 and

$$0 = \int\limits_{\mathbb{R}^{n+1}} -u_{m,t}\varphi + \frac{\varepsilon_m u_{m,tt}}{(c^{(m)})^2}\varphi + \sum_{i,j=1}^n a_{ij}(x) u_{m,x_ix_j}\varphi + f(x,u_m)\varphi \, d\mathcal{L}$$

$$= \int\limits_{\mathbb{R}^{n+1}} -u_{m,t}\varphi - \frac{\varepsilon_m u_{m,t}}{(c^{(m)})^2}\varphi_t - \sum_{i,j=1}^n a_{ij}(x) u_{m,x_i}\varphi_{x_j} - \sum_{i,j=1}^n (\partial_{x_j} a_{ij}(x)) u_{m,x_i}\varphi + f(x,u_m)\varphi \, d\mathcal{L}.$$

In the last term we will use $u_m \to u$ almost everywhere and therefore $f(x,u_m) \to f(x,u)$ almost everywhere, which together with $f \in L^\infty$ implies $f(x,u_m) \to f(x,u)$ in L^2_{loc}. In all the other term we will use $u_m \to u$ weakly in H^1_{loc}. Moreover, we will use $\frac{\varepsilon_m}{(c^{(m)})^2} \to 0$ by the bounds on $c^{(m)}$. Therefore, letting $m \to \infty$ yields for all $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$:

$$\int\limits_{\mathbb{R}^{n+1}} -u_t\varphi - \sum_{i,j=1}^n a_{ij}(x) u_{x_i}\varphi_{x_j} - \sum_{i,j=1}^n (\partial_{x_j} a_{ij}(x)) u_{x_i}\varphi + f(x,u)\varphi \, d\mathcal{L} = 0.$$

Hence, $u \in H^1_{loc}(\mathbb{R}^{n+1})$ is a weak solution of

$$\tilde{L}u := -u_t + \sum_{i,j=1}^n \partial_{x_i}(a_{ij}(x) u_{x_j}) - \sum_{j=1}^n (\sum_{i=1}^n \partial_{x_i} a_{ij}(x)) u_{x_j} = -f(x,u) \text{ on } \mathbb{R}^{n+1}. \tag{5.29}$$

Step 3: The $W^{2,1}_{2,loc}(\mathbb{R}^{n+1})$-regularity of u. In this step, we prove $u_{x_ix_j} \in L^2_{loc}(\mathbb{R}^{n+1})$ for $i,j = 1,...,n$. We will use interior regularity of weak solutions. To this end, let $Q, Q' \subset \mathbb{R}^{n+1}$ be bounded domains with $Q' \Subset Q$. We begin to check the necessary assumptions for the interior regularity: We have $u \in H^1(Q)$ and (5.29) holds on Q. The matrix field $(a_{ij})_{i,j=1}^n$ is a $C^1(\overline{Q})$-matrix field, which is symmetric and uniformly elliptic. The coefficients $\sum_{j=1}^n (\partial_{x_j} a_{ij}(x))$ are in $L^\infty(Q)$ for $i = 1,..,n$. Under these assumptions, interior regularity of weak solutions (see for example theorem 6.6 of [16]) implies $u_{x_ix_j} \in L^2(Q')$ for every $i,j = 1,...,n$. Since Q' was arbitrary, it follows that $u \in W^{2,1}_{2,loc}(\mathbb{R}^{n+1})$ and u is locally a strong solution of

$$Lu := -u_t + \sum_{i,j=1}^n a_{ij}(x) u_{x_ix_j} = -f(x,u) \text{ on } \mathbb{R}^{n+1}.$$

Step 4: Cutting off in time. For arbitrary real numbers $t_1 < t_2$ consider a function $\eta \in C_c^\infty(\mathbb{R})$ with $\eta = 1$ on $[t_1, t_2]$ and $0 \le \eta \le 1$. The function $\tilde{u}$ given by $\tilde{u}(t,x) = \eta(t)u(t,x)$ satisfies $\tilde{u} \in W^{2,1}_{2,loc}(\mathbb{R}^{n+1})$ and is a strong solution of the equation

$$L\tilde{u} = L(\eta u) = \eta L u - \eta_t u = -\eta f(x,u) - \eta_t u =: \tilde{f} \text{ on } \mathbb{R}^{n+1}.$$

Step 5: Obtaining a strong and classical approximative solution. In this step, we prove the existence of an approximative solution v^ε for the truncated $\tilde{u}$ that is both strong and classical. Afterwards, we deduce $u \in W^{2,1}_{p,loc}(\mathbb{R}^{n+1})$ for every $2 \leq p < \infty$. For $\varepsilon > 0$, let $\varphi_\varepsilon \in C_c^\infty(B_\varepsilon(0))$ (where $B_\varepsilon(0)$ denotes the ball in the euclidean metric) be the standard mollifier on $\mathbb{R}^{n+1}$ (for its definition check the proof of lemma 5.4). With $\tilde{u}$, $\tilde{f}$ and η from step 4, we define $\psi_\varepsilon := \tilde{u} * \varphi_\varepsilon$ (ψ_ε will take the role of boundary conditions) and $f_\varepsilon := \tilde{f} * \varphi_\varepsilon$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary (say $C^{2,\alpha}$). Consider the bounded cylinder $Q = I \times \Omega$ with an open interval $I = (T_1, T_2)$, such that $supp(\eta) \Subset I$. Let $\mathcal{P}Q$ denote its parabolic boundary and $CQ = \{T_1\} \times \partial\Omega$ its corners. For ε sufficiently small, there holds $\psi_\varepsilon = 0$ and $f_\varepsilon = 0$ in the vicinity of CQ. Therefore, the following compatibility condition holds:

$$L\psi_\varepsilon = 0 = f_\varepsilon \text{ on } CQ. \tag{5.30}$$

With that in mind, we consider the parabolic problem

$$\begin{aligned} -v_{\varepsilon,t} + \sum_{i,j=1}^{n} a_{ij}(x) v_{\varepsilon,x_i x_j} &= f_\varepsilon \text{ on } Q, \\ v_\varepsilon &= \psi_\varepsilon \text{ on } \mathcal{P}Q. \end{aligned} \tag{5.31}$$

We check the assumptions which guarantee the existence of a smooth solution v_ε to the problem: The boundary conditions ψ_ε and the right hand side f_ε are smooth (even $\psi_\varepsilon \in C^\infty(\overline{Q})$ and $f_\varepsilon \in C^\infty(\overline{Q})$, which is more than needed). The matrix field $(a_{ij})_{i,j=1}^n$ is symmetric, uniformly elliptic and $a_{ij} \in C^1(T^n)$ (which is more than needed). Furthermore, the compatibility conditions (5.30) for the boundary conditions are satisfied. Lastly, the domain Ω in the definition of Q has a sufficiently smooth boundary. Therefore, there exists a unique solution $v_\varepsilon \in C^{2,\alpha}_{1,\alpha/2}(\overline{Q})$ of (5.31) (see theorem 5.14 in [16]).

Furthermore, for arbitrary $2 \leq p < \infty$ we have $\psi_\varepsilon \in W^{2,1}_p(Q)$ and $f_\varepsilon \in L^p(Q)$. Therefore, there also exists a unique strong solution in $W^{2,1}_p(Q)$ of (5.31) by theorem 7.32 in [16]. But since $v_\varepsilon \in C^{2,\alpha}_{1,\alpha/2}(\overline{Q}) \subset W^{2,1}_p(Q)$ for any $p \geq 2$, the strong solution for any $p \geq 2$ has to coincide with v_ε. Define $w_\varepsilon := \tilde{u} - v_\varepsilon$. Due to steps 3 and 4, we have $w_\varepsilon \in W^{2,1}_p(Q)$ for $p = 2$. By the estimate given in the mentioned theorem (used with the boundary values $w_\varepsilon = \tilde{u} - \psi_\varepsilon$ on $\mathcal{P}Q$ and $p = 2$), we obtain

$$\begin{aligned} &||w_\varepsilon||_{L^2(Q)} + ||Dw_\varepsilon||_{L^2(Q)} + \left|\left|D^2 w_\varepsilon\right|\right|_{L^2(Q)} + ||\partial_t w_\varepsilon||_{L^2(Q)} \\ \leq &C\bigg(\left|\left|\tilde{f} - f_\varepsilon\right|\right|_{L^2(Q)} + ||\tilde{u} - \psi_\varepsilon||_{L^2(Q)} + ||D(\tilde{u} - \psi_\varepsilon)||_{L^2(Q)} \\ &\qquad + \left|\left|D^2(\tilde{u} - \psi_\varepsilon)\right|\right|_{L^2(Q)} + ||\partial_t(\tilde{u} - \psi_\varepsilon)||_{L^2(Q)} \bigg). \end{aligned}$$

The right hand side tends to 0 as $\varepsilon \to 0$, which implies

$$||\tilde{u} - v_\varepsilon||_{W^{2,1}_2(Q)} \to 0 \text{ as } \varepsilon \to 0. \tag{5.32}$$

Step 6: Deriving $W^{2,1}_p(Q')$-estimates for v_ε and deducing $u \in W^{2,1}_{p,loc}(\mathbb{R}^{n+1})$. Let $2 \leq p < \infty$. In order to derive $W^{2,1}_p(Q')$-estimates for v_ε, we first want to estimate $\sup_Q |v_\varepsilon|$.

To do so, we will use the following well known maximum estimates. We have $v_\varepsilon \in C^{2,\alpha}_{1,\alpha/2}(\overline{Q})$ (where $v_\varepsilon \in C^2_{1,loc}(Q) \cap C^0(\overline{Q})$ would be sufficient). The operator in (5.31) has no zeroth order term, therefore we have $j = 0$ as an upper bound for the coefficient of the zeroth order term. Then we have (see for example theorem 2.10 of [16])

$$\sup_Q |v_\varepsilon| \le e^{(j+1)(T_2-T_1)} \left(\sup_{\mathcal{P}Q} |\psi_\varepsilon| + \sup_Q |f_\varepsilon| \right) \le e^{T_2-T_1} \left(1 + \sup_{\mathbb{R}^{n+1}} \left| \tilde{f} \right| \right) =: K_1. \tag{5.33}$$

Here, we have used: $\sup |\psi_\varepsilon| = \sup |\tilde{u} * \varphi_\varepsilon| \le \sup |\tilde{u}| \le 1$ and likewise for f_ε.

Now let $Q' = [T_3, T_4] \times \Omega' \subset Q$ be a subcylinder of Q, such that $d(Q', \mathcal{P}Q) > 0$ and Ω' has smooth boundary and furthermore $[t_1, t_2] \subset [T_3, T_4]$. Interior L^p-estimates (see for example theorem 7.22 in [16]) imply

$$\begin{aligned} &\left|\left|D^2 v_\varepsilon\right|\right|_{L^p(Q')} + ||\partial_t v_\varepsilon||_{L^p(Q')} \le C \left(||Lv_\varepsilon||_{L^p(Q)} + ||v_\varepsilon||_{L^p(Q)} \right) \\ \le &\, C \left(||f_\varepsilon||_{L^p(Q)} + K_1 |Q|^{\frac{1}{p}} \right) \le C \left(\sup_{\mathbb{R}^{n+1}} \left| \tilde{f} \right| + K_1 \right) |Q|^{\frac{1}{p}} =: K_2. \end{aligned}$$

From this, we estimate $||Dv_\varepsilon||_{L^p(Q')}$ by interpolation. We have $v_\varepsilon(t, \cdot) \in C^2(\overline{\Omega'})$ for any fixed $t \in [T_3, T_4]$ and by an interpolation inequality (see lemma 7.20 in [16])

$$||Dv_\varepsilon(t,\cdot)||_{L^p(\Omega')} \le \left|\left|D^2 v_\varepsilon(t,\cdot)\right|\right|_{L^p(\Omega')} + C(n,\Omega) \, ||v_\varepsilon(t,\cdot)||_{L^p(\Omega')} .$$

Taking the $L^p([T_3, T_4])$-norm on both sides and using Minkowski's inequality, we obtain from this

$$||Dv_\varepsilon||_{L^p(Q')} \le \left|\left|D^2 v_\varepsilon\right|\right|_{L^p(Q')} + C(n,\Omega) \, ||v_\varepsilon||_{L^p(Q')} \le K_2 + C(n,\Omega) K_1 \, |Q'|^{\frac{1}{p}} .$$

Altogether, we have $||v_\varepsilon||_{W^{2,1}_p(Q')} \le C$.

Hence, for a subsequence and some $\hat{u} \in W^{2,1}_p(Q')$, we have $v_\varepsilon \to \hat{u}$ weakly in $W^{2,1}_p(Q')$ for $\varepsilon \to 0$. By $p \ge 2$, this implies $v_\varepsilon \to \hat{u}$ strongly in $L^2(Q')$. Since (5.32) also implies $v_\varepsilon \to \tilde{u}$ strongly in $L^2(Q')$, we can deduce $\tilde{u} = \hat{u}$ on Q' almost everywhere. Therefore, $\tilde{u} \in W^{2,1}_p(Q')$. Due to $\eta = 1$ on $[t_1, t_2] \subset [T_3, T_4]$, we have $u \in W^{2,1}_p((t_1, t_2) \times \Omega')$ for every $2 \le p < \infty$. Since t_1, t_2 and Ω', Ω were essentially arbitrary, we deduce $u \in W^{2,1}_{p,loc}(\mathbb{R}^{n+1})$ for every $2 \le p < \infty$.

Step 7: Using interior Schauder estimates for v_ε. Knowing $u \in W^{2,1}_{p,loc}(\mathbb{R}^{n+1})$ for every $2 \le p < \infty$, we take another look at the sequence v_ε by using interior Schauder estimates. To do so, we have to find an estimate for $||f_\varepsilon||_{C^\alpha_{\alpha/2}(\overline{Q})} = \left|\left| \tilde{f} * \varphi_\varepsilon \right|\right|_{C^\alpha_{\alpha/2}(\overline{Q})}$. We can do that, since we now know $u \in W^{2,1}_{p,loc}(\mathbb{R}^{n+1})$ for every $p \ge 2$. Indeed, for sufficiently large p, we have by Sobolev imbedding

$$u \in W^{2,1}_{p,loc}(\mathbb{R}^{n+1}) \subset W^{1,p}_{loc}(\mathbb{R}^{n+1}) \subset C^{0,\alpha}_{loc}(\mathbb{R}^{n+1}) \subset C^\alpha_{\alpha/2,loc}(\mathbb{R}^{n+1}).$$

Therefore, $f(x,u) \in C^\alpha_{\alpha/2,loc}(\mathbb{R}^{n+1})$ and thus $\tilde{f} = -\eta f(x,u) - \eta_t u \in C^\alpha_{\alpha/2,loc}(\mathbb{R}^{n+1})$. Let $Q_\varepsilon = \{z \in \mathbb{R}^{n+1} : dist(z, Q) < \varepsilon\}$ (where we mean the usual euclidean distance).

Furthermore, we denote by $d((t,x),(\tilde{t},\tilde{x}))$ the parabolic distance of two points as defined in the general assumptions. For $(t,x),(\tilde{t},\tilde{x}) \in Q$, we calculate

$$\begin{aligned}
&\left|f_\varepsilon(t,x) - f_\varepsilon(\tilde{t},\tilde{x})\right| \\
&\leq \int\limits_{supp(\varphi_\varepsilon)} \varphi_\varepsilon(s,y) \left|\tilde{f}(t-s,x-y) - \tilde{f}(\tilde{t}-s,\tilde{x}-y)\right| \, d(s,y) \\
&\leq \int\limits_{supp(\varphi_\varepsilon)} \varphi_\varepsilon(s,y) \left|\left|\tilde{f}\right|\right|_{C^\alpha_{\alpha/2}(\overline{Q_\varepsilon})} d((t-s,x-y),(\tilde{t}-s,\tilde{x}-y))^\alpha \, d(s,y) \\
&= \left|\left|\tilde{f}\right|\right|_{C^\alpha_{\alpha/2}(\overline{Q_\varepsilon})} d((t,x),(\tilde{t},\tilde{x}))^\alpha.
\end{aligned}$$

Similarly,

$$\sup_Q |f_\varepsilon| \leq \sup_{Q_1} \left|\tilde{f}\right| \text{ for } 0 < \varepsilon \leq 1.$$

Together, this implies

$$||f_\varepsilon||_{C^\alpha_{\alpha/2}(\overline{Q})} \leq \left|\left|\tilde{f}\right|\right|_{C^\alpha_{\alpha/2}(\overline{Q_1})} \text{ for } 0 < \varepsilon \leq 1. \tag{5.34}$$

Now interior Schauder estimates yield (in this form, the estimates follow easily from the estimates in theorem 4.9 [16]):

$$||v_\varepsilon||_{C^{2,\alpha}_{1,\alpha/2}(\overline{Q'})} \leq C\left(||v_\varepsilon||_{C^0(\overline{Q})} + ||f_\varepsilon||_{C^\alpha_{\alpha/2}(\overline{Q})}\right) \leq C\left(K_1 + \left|\left|\tilde{f}\right|\right|_{C^\alpha_{\alpha/2}(\overline{Q_1})}\right) \tag{5.35}$$

by (5.33) and (5.34). Consequently, for a subsequence and some $\tilde{v} \in C^2_1(\overline{Q'})$, we have $||v_\varepsilon - \tilde{v}||_{C^2_1(\overline{Q'})} \to 0$ for $\varepsilon \to 0$. By the estimate on v_ε, we even have $\tilde{v} \in C^{2,\alpha}_{1,\alpha/2}(\overline{Q'})$. But by (5.32), $\tilde{v} = \tilde{u}$ follows. Therefore, we obtain $\tilde{u} \in C^{2,\alpha}_{1,\alpha/2}(\overline{Q'})$. Due to $\eta = 1$ on $[t_1,t_2] \subset [T_3,T_4]$, we have $u \in C^{2,\alpha}_{1,\alpha/2}(\overline{(t_1,t_2) \times \Omega'})$. Since t_1, t_2 and Ω', Ω were essentially arbitrary, we deduce $u \in C^{2,\alpha}_{1,\alpha/2,loc}(\mathbb{R}^{n+1})$. This finishes the proof of the regularity.

Step 8: Further properties of u. From (5.3) we immediately see (5.27). From (5.16) we see that for some $d > 0$ by taking another subsequence we can achieve $u_m(d,\cdot) \to \tilde{u}$ in $C^0([0,1]^n)$ and likewise for $-d$ instead of d. Since $u_m \to u$ almost everywhere in $\mathbb{R}^{n+1}$, it follows from the Fubini theorem, that for $t \in \mathbb{R} \setminus N$, where N is a set of measure 0, there holds $u_m(t,\cdot) \to u(t,\cdot)$ almost everywhere in $[0,1]^n$. If we choose $d \in \mathbb{R} \setminus N$, we obtain $\tilde{u} = u(d,\cdot)$ and likewise for $-d$. Therefore we have:

$$\max_{x \in [0,1]^n} u_m(-d,x) \to \max_{x \in [0,1]^n} u(-d,x) \qquad \text{and} \quad \max_{x \in [0,1]^n} u_m(d,x) \to \max_{x \in [0,1]^n} u(d,x) \tag{5.36}$$

for $m \to \infty$.

We now discuss the appropriate choice of d. There holds

$$u_m(-d,x) = U^{\varepsilon_m,b_m}(k \cdot x + c^{(m)}d, x) \text{ and } u_m(d,x) = U^{\varepsilon_m,b_m}(k \cdot x - c^{(m)}d, x).$$

Since $c^{(m)} \le c_2 < 0$, we can choose d large enough, such that $k \cdot x + c^{(m)} d \le 0$ and $k \cdot x - c^{(m)} d \ge 0$ for $x \in [0,1]^n$ and all m. By the monotonicity of U^{ε_m, b_m} in s, we thus have

$$u_m(-d, x) \le U^{\varepsilon_m, b_m}(0, x) \le u_m(d, x).$$

Taking the maximum over $[0,1]^n$, we obtain

$$\max_{x \in [0,1]^n} u_m(-d, x) \le \theta(f) \le \max_{x \in [0,1]^n} u_m(d, x).$$

From this, (5.28) follows with the help of (5.36). ■

In the next lemma, we derive further regularity properties of u. These include $u_{tt} \in L^p_{loc}(\mathbb{R}^{n+1})$ for any $2 \le p < \infty$. Under the regularity assumptions of theorem 2.10, we prove a stronger regularity result. This implies that u_{tt} exists classically.

Lemma 5.6 (Additional regularity) (i) *In the situation of lemma 5.5, the function u has the additional regularity $u_t \in W^{2,1}_{p,loc}(\mathbb{R}^{n+1})$ for any $2 \le p < \infty$. Moreover, $v := u_t$ is locally a strong solution of*

$$-v_t + \sum_{i,j=1}^{n} a_{ij}(x) v_{x_i x_j} + f_z(x, u) v = 0. \tag{5.37}$$

(ii) *In addition to the assumptions of (i), we suppose that f has the additional regularity $f_z \in C^{0,\alpha_0}_{loc}(\mathbb{R}^{n+1})$ for some $\alpha_0 \in (0,1)$. Then $u_t \in C^{2,\alpha}_{1,\alpha/2,loc}(\mathbb{R}^{n+1})$ for any $\alpha \in (0, \alpha_0)$. In particular, in this case, u_{tt} exists classically. With the transformation $s = k \cdot x - ct$, $y = x$ and $U(s, y) := u(t, x)$ satisfies (2.5) classically.*

Proof For both regularity statements (i) and (ii), we use the technique of difference quotients. (i) We prove first that $u_t \in W^{2,1}_{p,loc}(\mathbb{R}^{n+1})$ for any $2 \le p < \infty$. Consider the difference quotients of u in t:

$$v^h(t, x) := D^h_t u(t, x) := \frac{u^h(t, x) - u(t, x)}{h} := \frac{u(t + h, x) - u(t, x)}{h}.$$

Let $\Omega, \tilde{\Omega} \subset \mathbb{R}^n$ be domains with smooth boundaries and $\Omega \Subset \tilde{\Omega}$. By Q, we denote the cylinder $Q := (t_1, t_2) \times \Omega$ in $\mathbb{R}^{n+1}$. For $\varepsilon > 0$, let $Q_\varepsilon := (t_1 - \varepsilon, t_2 + \varepsilon) \times \tilde{\Omega}$. Then we have $Q \Subset Q_\varepsilon$. By the mean value theorem, we obtain

$$\left|\left|v^h\right|\right|_{C^0(\overline{Q_\varepsilon})} \le ||u_t||_{C^0(\overline{Q_{2\varepsilon}})} =: C_1 \text{ for } 0 < |h| < \varepsilon. \tag{5.38}$$

The difference quotient v^h satisfies the equation

$$-v^h_t + \sum_{i,j=1}^{n} a_{ij}(x) v^h_{x_i x_j} = -\frac{f(x, u^h) - f(x, u)}{h} =: f^h(t, x). \tag{5.39}$$

Applying the mean value theorem twice yields

$$\left|\left|f^h\right|\right|_{C^0(\overline{Q_\varepsilon})} \le ||f||_{C^1(T^n \times [0,1])} \, ||u_t||_{C^0(\overline{Q_{2\varepsilon}})} =: C_2 \text{ for } 0 < |h| < \varepsilon. \tag{5.40}$$

By (5.38), (5.40), interior L^p-estimates yield (see theorem 7.22 in [16]):

$$\begin{aligned}\left|\left|D^2 v^h\right|\right|_{L^p(Q)} + \left|\left|v_t^h\right|\right|_{L^p(Q)} &\le C\left(\left|\left|f^h\right|\right|_{L^p(Q_\varepsilon)} + \left|\left|v^h\right|\right|_{L^p(Q_\varepsilon)}\right)\\ &\le C(C_2 + C_1)\left|Q_\varepsilon\right|^{\frac{1}{p}}\end{aligned}$$

for $0 < |h| < \varepsilon$. Here, we have used $Q \Subset Q_\varepsilon$, which implies $d(Q, \mathcal{P}Q_\varepsilon) > 0$, where $\mathcal{P}Q_\varepsilon$ denotes the parabolic boundary of Q_ε. The constant C does depend on ε, but not on h. As in the proof of 5.5, we conclude from this by interpolation:

$$\left|\left|v^h\right|\right|_{W_p^{2,1}(Q)} \le \tilde{C} \text{ for } 0 < |h| < \varepsilon$$

and some constant $\tilde{C}$, which does not depend on h.

Hence, for a subsequence and some $\tilde{v} \in W_p^{2,1}(Q)$, we have $v^h \to \tilde{v}$ weakly in $W_p^{2,1}(Q)$. By Sobolev embedding also $v^h \to \tilde{v}$ strongly in $L^p(Q)$. But we have $v^h \to u_t$ pointwise as well and therefore $u_t = \tilde{v}$ almost everywhere in Q. Consequently, $u_t \in W_p^{2,1}(Q)$. Since Q was essentially arbitrary, we conclude $u_t \in W_{p,loc}^{2,1}(\mathbb{R}^{n+1})$. Differentiating (5.26) with respect to t yields (5.37).

(ii) Now we prove $u_t \in C_{1,\alpha/2,loc}^{2,\alpha}(\mathbb{R}^{n+1})$ for any $\alpha \in (0, \alpha_0)$ under the additional regularity assumption on f. Let $\alpha \in (0, \alpha_0)$ be arbitrary and some $\varepsilon > 0$ be fixed. We consider the difference quotients v^h and f^h and the domains Q and Q_ε as in the proof of (i). Some of the estimates from (i) will also be used. We can show, that there is a constant $C_3 > 0$, such that

$$\left|\left|f^h\right|\right|_{C^\alpha_{\alpha/2}(\overline{Q_\varepsilon})} \le C_3 \text{ for } 0 < |h| < \varepsilon. \tag{5.41}$$

However, for the sake of clarity, we postpone this estimate to the end of the proof. With the help of interior Schauder estimates (in this form, they follow easily from theorem 4.9 in [16]), we conclude from (5.39), (5.38) and (5.41):

$$\left|\left|v^h\right|\right|_{C^{2,\alpha}_{1,\alpha/2}(\overline{Q})} \le C\left(\left|\left|v^h\right|\right|_{C^0(\overline{Q_\varepsilon})} + \left|\left|f^h\right|\right|_{C^\alpha_{\alpha/2}(\overline{Q_\varepsilon})}\right) \le C(C_1 + C_3) \text{ for } 0 < |h| < \varepsilon.$$

The constant C depends on ε, but not on h.

Hence, for a subsequence and some $\tilde{v} \in C_1^2(\overline{Q})$, we have $\left|\left|v^h - \tilde{v}\right|\right|_{C_1^2(\overline{Q})} \to 0$. Moreover, the estimates on v^h imply $\tilde{v} \in C^{2,\alpha}_{1,\alpha/2}(\overline{Q})$. But since $v^h \to u_t$ pointwise, we have $u_t = \tilde{v}$. Consequently, $u_t \in C^{2,\alpha}_{1,\alpha/2}(\overline{Q})$.

It remains to verify the estimate (5.41) to finish the proof. Suppose $\tilde{\alpha} \in (0,1)$. We calculate for $(t,x), (\tilde{t},\tilde{x}) \in \overline{Q_{2\varepsilon}}$ (where the constant C may change from line to line):

$$\begin{aligned}&\left|f_z(x, u(t,x)) - f_z(\tilde{x}, u(\tilde{t},\tilde{x}))\right|\\ &\le C\left(\left(\left|\left|x - \tilde{x}\right|\right|^2 + \left|u(t,x) - u(\tilde{t},\tilde{x})\right|^2\right)^{\frac{1}{2}}\right)^{\alpha_0}\\ &\le C\left(\left|\left|x - \tilde{x}\right|\right| + d((t,x),(\tilde{t},\tilde{x}))^{\tilde{\alpha}}\right)^{\alpha_0}\\ &\le Cd\left((t,x),(\tilde{t},\tilde{x})\right)^{\tilde{\alpha}\alpha_0}.\end{aligned}$$

For the last inequality, we used that $||x-\tilde{x}||^{1-\tilde{\alpha}}$ is bounded for $(t,x),(\tilde{t},\tilde{x}) \in \overline{Q_{2\varepsilon}}$. Therefore, the function $(t,x) \mapsto f_z(x,u(t,x))$ is a $C^{\alpha}_{\alpha/2}(\overline{Q_{2\varepsilon}})$-function for any $\alpha \in (0,\alpha_0)$.

Now let $g(t,x) := f(x,u(t,x))$. We obtain $g_t(t,x) = f_z(x,u(t,x))u_t(t,x)$. Consequently, $g_t \in C^{\alpha}_{\alpha/2}(\overline{Q_{2\varepsilon}})$, since the product of two $C^{\alpha}_{\alpha/2}(\overline{Q_{2\varepsilon}})$ functions is again a $C^{\alpha}_{\alpha/2}(\overline{Q_{2\varepsilon}})$ function.

Assume $0<|h|<\varepsilon$ and $(t,x),(\tilde{t},\tilde{x}) \in \overline{Q_{\varepsilon}}$. We calculate for $\alpha \in (0,\alpha_0)$:

$$
\begin{aligned}
&\left|f^h(t,x) - f^h(\tilde{t},\tilde{x})\right| \\
&= \left| \frac{1}{h}\int_t^{t+h} g_t(s,x)\,ds - \frac{1}{h}\int_{\tilde{t}}^{\tilde{t}+h} g_t(s,\tilde{x})\,ds \right| \\
&= \left| \frac{1}{h}\int_t^{t+h} g_t(s,x)\,ds - \frac{1}{h}\int_{\tilde{t}}^{\tilde{t}+h} g_t(s,x)\,ds + \frac{1}{h}\int_{\tilde{t}}^{\tilde{t}+h} g_t(s,x) - g_t(s,\tilde{x})\,ds \right| \\
&\le \frac{1}{h}\int_t^{t+h} \left|g_t(s,x) - g_t(s+\tilde{t}-t,x)\right|\,ds + \frac{1}{h}\int_{\tilde{t}}^{\tilde{t}+h} \left|g_t(s,x) - g_t(s,\tilde{x})\right|\,ds \\
&\le \frac{1}{h}\int_t^{t+h} ||g_t||_{C^{\alpha}_{\alpha/2}(\overline{Q_{2\varepsilon}})} \left|t-\tilde{t}\right|^{\alpha/2}\,ds + \frac{1}{h}\int_{\tilde{t}}^{\tilde{t}+h} ||g_t||_{C^{\alpha}_{\alpha/2}(\overline{Q_{2\varepsilon}})} \left|x-\tilde{x}\right|^{\alpha}\,ds \\
&\le Cd\left((t,x),(\tilde{t},\tilde{x})\right)^{\alpha}.
\end{aligned}
$$

Together with $\left|\left|f^h\right|\right|_{C^0(\overline{Q_\varepsilon})} \le ||f_z||_{C^0(T^n\times[0,1])}\,||u_t||_{C^0(\overline{Q_{2\varepsilon}})}$ by mean value theorem, we have proved (5.41). ∎

Note that lemma 5.6 resolves the problem that was mentioned in remark 2.20 (i).

5.2.2. Possibilities for the values at infinity

In this subsection, we prove that for $t \to \pm\infty$, the limit function $u(t,x)$ from lemma 5.5 converges locally uniformly in x to functions $u^+(x)$ and $u^-(x)$. We will prove that these functions have to be constant. We will show that there are only two possibilities for u^+, namely $u^+=1$ and $u^+=\theta(f)$. Furthermore, we will show that the possibility $u^+=\theta(f)$ implies that u is constant and therefore $u^-=\theta(f)$. Hence, by proving $u^-=0$, one automatically obtains $u^+=1$.

In order to prove that u^+ and u^- are constant, we have to derive a differential equation for these functions. For this task, we have to obtain estimates for the derivatives of u. The first step to these estimates will be the following lemma.

Proposition 5.7 (Global estimates) *In the situation of lemma 5.5, the function u is globally Lipschitz continuous. Furthermore, the derivatives u_t, Du and D^2u are uniformly bounded on $\mathbb{R}\times\mathbb{R}^n$.*

PROOF Let $Q_r = Q_r(z_0) = Q_r(t_0, x_0) \subset \mathbb{R} \times \mathbb{R}^n$ be the cube with edge length r centered around $z_0 = (t_0, x_0)$. We want to have an estimate on $||u_t||_{C^0(\overline{Q_r})}$, $||Du||_{C^0(\overline{Q_r})}$ and $||D^2u||_{C^0(\overline{Q_r})}$, which does not depend on z_0. Once we have such a local estimate, we can conclude that these derivatives are globally bounded and therefore u is globally Lipschitz as well. This can be done by estimating the $C^{2,\alpha}_{1,\alpha/2}(\overline{Q_r})$ norm:

$$||u_t||_{C^0(\overline{Q_r})} + ||Du||_{C^0(\overline{Q_r})} + \left|\left|D^2u\right|\right|_{C^0(\overline{Q_r})} \leq C \, ||u||_{C^{2,\alpha}_{1,\alpha/2}(\overline{Q_r})} . \tag{5.42}$$

Step 1: An estimate for $||u||_{C^{2,\alpha}_{1,\alpha/2}(\overline{Q_r})}$. By interior Schauder estimates, $||u||_{C^{2,\alpha}_{1,\alpha/2}(\overline{Q_r})}$ can be estimated by an expressing not involving values of u on the parabolic boundary of Q_r, only depending on $||u||_{C^0(\overline{Q_{2r}})}$ and $||f(x,u)||_{C^{\alpha}_{\alpha/2}(\overline{Q_{2r}})}$. The following estimates follow easily for example from [16] theorem 4.9:

$$\begin{aligned} ||u||_{C^{2,\alpha}_{1,\alpha/2}(\overline{Q_r})} &\leq K \left(||u||_{C^0(\overline{Q_{2r}})} + ||f(x,u)||_{C^{\alpha}_{\alpha/2}(\overline{Q_{2r}})} \right) \\ &\leq K \left(1 + ||f(x,u)||_{C^{\alpha}_{\alpha/2}(\overline{Q_{2r}})} \right) . \end{aligned} \tag{5.43}$$

The constant K only depends on n, α, $||a_{ij}||_{C^\alpha(T^n)}$, the ellipticity constants of matrix field $(a_{ij})_{i,j=1}^n$ and the parabolic distance of $Q_r(z_0)$ and $Q_{2r}(z_0)$ (which does not depend on z_0). In particular, the estimate does not depend on z_0.

Step 2: An estimate for $||f(x,u)||_{C^{\alpha}_{\alpha/2}(\overline{Q_{2r}})}$. It remains to estimate $||f(x,u)||_{C^{\alpha}_{\alpha/2}(\overline{Q_{2r}})}$. This involves estimating the Hölder seminorm of u on $\overline{Q_{2r}}$, for which we introduced the notation $[u]_{C^{\alpha}_{\alpha/2}(\overline{Q_{2r}})}$. With the help of a theorem from Krylov and Safonov, we can estimate $[u]_{C^{\alpha}_{\alpha/2}(\overline{Q_{2r}})}$ for a specific Hölder-exponent $\alpha = \sigma$, which depends on the dimension n and the ellipticity constant μ of $(a_{ij})_{i,j=1}^n$. The Krylov-Safonov estimates (see [15] theorem 4.3) state for $(t_1, x_1), (t_2, x_2) \in Q_{2r}$

$$\begin{aligned} |u(t_1,x_1) - u(t_2,x_2)| &\leq N \left(||u||_{C^0(\overline{Q_{3r}})} + ||f(x,u)||_{L^{n+1}(Q_{3r})} \right) d\left((t_1,x_1),(t_2,x_2)\right)^\sigma \\ &\leq N \left(1 + (3r)^{\frac{1}{n+1}} \, ||f||_{C^0(T^n \times [0,1])} \right) d\left((t_1,x_1),(t_2,x_2)\right)^\sigma \end{aligned}$$

Consequently, we have $[u]_{C^{\sigma}_{\sigma/2}(\overline{Q_{2r}})} \leq N \left(1 + (3r)^{\frac{1}{n+1}} \, ||f||_{C^0(T^n \times [0,1])} \right) =: C$. Here, the constant N depends on the parabolic distance between ∂Q_{3r} and Q_{2r} (which does not depend on the center z_0), the ellipticity constants of $(a_{ij})_{i,j=1}^n$ and the dimension n. Consequently, C does not depend on z_0.

Using $[u]_{C^{\sigma}_{\sigma/2}(\overline{Q_{2r}})} \leq C$, it is easy to see that $||f(x,u)||_{C^{\sigma}_{\sigma/2}(\overline{Q_{2r}})}$ is bounded independently of z_0. We want to mention that, alternatively, the estimates for $[u]_{C^{\alpha}_{\alpha/2}(\overline{Q_{2r}})}$ can be obtained with interior L^p-estimates (see theorem 7.22 in [16]). We have already used these estimates in the proofs of the lemmas 5.5 and 5.6. In this case, the estimates for

$[u]_{C^{\alpha}_{\alpha/2}(\overline{Q_{2r}})}$ can be obtained for any $\alpha \in (0,1)$ by choosing appropriate p.

Step 3: Combining the estimates. Combining the estimates for $||f(x,u)||_{C^{\sigma}_{\sigma/2}(\overline{Q_{2r}})}$ with the estimates (5.43), we obtain that $||u||_{C^{2,\sigma}_{1,\sigma/2}(\overline{Q_{2r}(z_0)})}$ is bounded independently of z_0. Therefore, we see from (5.42) that $||u_t||_{C^0(\overline{Q_r(z_0)})} + ||Du||_{C^0(\overline{Q_r(z_0)})} + ||D^2u||_{C^0(\overline{Q_r(z_0)})}$ is bounded independently of z_0. Consequently all these derivatives are bounded globally and u is globally Lipschitz-continuous. ■

In the following lemma we show the existence of the limit functions u^+ and u^-. The estimates of the previous proposition will be used to derive equations for u^+ and u^-. With the help of these equations, it is proved that u^+ and u^- are constant.

Lemma 5.8 (Boundary values at infinity are constant) *In the situation of lemma 5.5, the limits*

$$u^+(x) = \lim_{t\to+\infty} u(t,x) \text{ and}$$
$$u^-(x) = \lim_{t\to-\infty} u(t,x)$$

exist locally uniformly in $x \in \mathbb{R}^n$. The functions u^+ and u^- are constant. Furthermore, there are only two possibilities for u^+. We either have $u^+ \equiv 1$ or $u^+ \equiv \theta(f)$. For u^-, we have the inequality $0 \le u^- \le \theta(f)$.

PROOF *Step 1: Existence and periodicity of the limits.* $u(t,x)$ is monotone in t and bounded, so clearly the pointwise limits exist. From (5.27), it follows immediately that $u^+(x) = u^+(x+e_i)$ and $u^-(x) = u^-(x+e_i)$ for the unit vectors e_i, $i = 1,...,n$. In other words, u^+ and u^- are 1-periodic functions.

Step 2: Deriving a differential equation for u^+ and u^-. We will now derive an equation for u^+ and u^- using the Lipschitz estimates from proposition 5.7. To this end, let Ω and Ω' be bounded domains in $\mathbb{R}^n$ with $[0,1]^n \subset \Omega \Subset \Omega'$. Furthermore, let D and D' be the cylindrical domains $D = (0,T)\times\Omega$ and $D' = (-1,T)\times\Omega'$. Consider the operator

$$L = \partial_t - \sum_{i,j=1}^{n} a_{ij}(x)\partial_{x_i x_j}$$

and the sequence of functions $u_m(t,x) := u(t+m,x)$, as well as $f_m(t,x) := f(x, u_m(t,x))$. Then $Lu_m = f_m$ in D'. By proposition 5.7 we have $||f_m||_{C^{\alpha}_{\alpha/2}(\overline{D'})} \le C$ independently of m. Interior Schauder estimates yield (as mentioned above, in this form they follow easily from thereom 4.9 in [16])

$$||u_m||_{C^{2,\alpha}_{1,\alpha/2}(\overline{D})} \le K\left(||u_m||_{C^0(\overline{D'})} + ||f_m||_{C^{\alpha}_{\alpha/2}(\overline{D'})}\right) \le K(1+C) = \tilde{C}. \tag{5.44}$$

On the one hand, $u_m(t,x) \to u^+(x)$ uniformly on $\overline{D}$ for $m \to +\infty$, on the other hand the estimate (5.44) implies by the Arzelà-Ascoli theorem the existence of a subsequence, such that $\partial_t u_m$, Du_m and $D^2 u_m$ converge uniformly on $\overline{D}$. Due to (5.44), the limit function is even a $C^{2,\alpha}_{1,\alpha/2}(\overline{D})$-function, even though the convergence does not take place in $C^{2,\alpha}_{1,\alpha/2}(\overline{D})$. Therefore, we have $u^+ \in C^{2,\alpha}_{1,\alpha/2}(\overline{D})$ (if considered as a function of t and x) and $Lu_m \to Lu^+$ uniformly on $\overline{D}$ for $m \to \infty$. Moreover, $f_m(t,x) \to f(x, u^+(x))$ uniformly on $\overline{D}$ for $m \to \infty$. Consequently, we can pass to the limit in the equation. We obtain that u^+ is a classical solution of $Lu^+ = f(x,u^+)$ on Ω. The t-derivative of u^+ vanishes, because u^+ does not depend on t. Since Ω was arbitrary, it follows that u^+ is a classical solution of

$$\sum_{i,j=1}^{n} a_{ij}(x) u^+_{x_i x_j} = -f(x,u^+) \leq 0 \text{ on } \mathbb{R}^n.$$

By the same reasoning, u^- is a classical solution of

$$\sum_{i,j=1}^{n} a_{ij}(x) u^-_{x_i x_j} = -f(x,u^-) \leq 0 \text{ on } \mathbb{R}^n.$$

Step 3: Conclusion. Since u^+ is 1-periodic, it must assume its minimum. Hence, the minimum principle implies that u^+ is constant. That in turn means $Lu^+ = 0$ and therefore $f(x,u^+) = 0$ for all $x \in T^n$. Due to the same reasoning, u^- must be constant (and satisfy $f(x,u^-) = 0$ for all $x \in T^n$).

By (5.28), there is $x_0 \in [0,1]^n$, such that $u(d,x_0) \geq \theta(f)$. Since $u(t,x)$ is nondecreasing in t, there holds $u(t,x_0) \geq \theta(f)$ for $t \geq d$. Since $u(t,x) \to u^+(x)$ for $t \to \infty$ and u^+ is constant, we obtain $u^+ \geq \theta(f)$. Moreover, by (5.28), there holds $u(-d,x) \leq \theta(f)$ for all $x \in [0,1]^n$. Therefore, we can deduce $u^- \leq \theta(f)$ in the same fashion.

If now $u^+ \in (\theta(f), 1)$, there exists $x \in T^n$ with $f(x,u^+) > 0$ by the covering property of f. This is a contradiction to $f(x,u^+) = 0$ for all $x \in T^n$. Therefore, only the possibilities $u^+ = \theta(f)$ or $u^+ = 1$ remain.

Lastly, we have to prove that the convergence for $t \to \pm\infty$ takes place locally uniformly. This can be deduced by Dini's theorem, since $u(t,x)$ is monotone in t and we already know that the limit functions u^+ and u^- are continuous. ∎

Remark 5.9 *The possibilities for u^- and u^+ are slightly asymmetrical at this point. Due to lemma 5.8, both u^- and u^+ are constant. However, for u^+ there remain only two possibilities $u^+ = \theta(f)$ and $u^+ = 1$, while there remains a continuum of possibilities for u^-, namely $u^- \in [0, \theta(f)]$. The reason is that the covering property only concerns $f(\cdot, z)$ for $z \in (\theta(f), 1)$. Therefore, the information $f(x,u^-) = 0$ for all $x \in T^n$ does not help to say anything about u^- here. Thus, for u^- any value in $[0, \theta(f)]$ is still a possibility at this point.*

The possibilities for the combination of possible values of u^+ and u^- will be reduced further in the following lemma, which is the analogue of lemma 3.2 of [21].

Lemma 5.10 *In the situation of lemma 5.5, consider the constants u^+ and u^- from lemma 5.8. If u^+ satisfies $u^+ = \theta(f)$, then $u \equiv \theta(f)$ on $\mathbb{R}^{n+1}$ and consequently $u^- = \theta(f)$.*

Proof In this proof, we will use the maximum principle in a slightly different way than Xin does in the proof of lemma 3.2 in [21]. We do so, to avoid the use of lemma 6.1, which corresponds to proposition 4.1 in [21]. We will use the fact that u is nondecreasing to turn the usual one-side parabolic maximum principle into a two-sided one.

Let us begin with the proof. We know $u(t,x) \to u^+$ as $t \to \infty$ for all $x \in \mathbb{R}^n$. Therefore, if $u^+ = \theta(f)$, the monotonicity of u in t (lemma 5.5) implies $u(t,x) \leq \theta(f)$ for all $(t,x) \in \mathbb{R}^{n+1}$. This has two consequences. Firstly, u achieves its maximum $\theta(f)$ due to (5.28). Secondly, due to the combustion property of f, we have $f(x,u(t,x)) = 0$ for all $(t,x) \in \mathbb{R}^{n+1}$. Therefore, u is a classical solution of

$$-u_t + \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} = 0.$$

Therefore, the strong parabolic maximum principle is available. Denote by (t_0, x_0) the point, where u achieves its maximum $u(t_0,x_0) = \theta(f)$. Due to the monotonicity of u in t and $u \leq \theta(f)$, the maximum is also achieved in any point $(\tilde{t}, x_0)$ with $\tilde{t} \geq t_0$. By the maximum principle, u has to be equal to $\theta(f)$ in $\{(t,x) \in \mathbb{R}^{n+1} : t \leq \tilde{t}\}$ for any $\tilde{t} \geq t_0$. Hence, we obtain $u \equiv \theta(f)$ on $\mathbb{R}^{n+1}$. ∎

We want to remark that instead of using the combustion property of f in the previous proof to infer $f(x,u(t,x)) = 0$ for all $(t,x) \in \mathbb{R}^{n+1}$, it would have sufficed to use property (iii) from definition 2.6 and obtain $f(x,u(t,x)) \leq 0$ for all $(t,x) \in \mathbb{R}^{n+1}$.

Remark 5.11 *Assume that in the situation of lemmas 5.5, 5.8, we can show $u^- = 0$. Then it follows from lemma 5.10 together with lemma 5.8 that $u^+ = 1$. In order to show $u^- = 0$, we need a special exponential solution of the equation that we can compare with the solution of the (ε, b)-problem as we pass to the limit.*

5.2.3. The periodic principle eigenvalue and an exponential solution

It is known that for elliptic operators L in nondivergence form with smooth and periodic coefficients, there exists a unique real number $\lambda_p = \lambda_p(L)$ and a periodic positive eigenfunction ϕ such that $L\phi = \lambda_p \phi$. This is mentioned, for example, in [7]. In [21], Xin also makes use of it. However, we were unable to find a proof in literature for the existence and uniqueness of $\lambda_p(L)$ and a source for its further properties. In particular, we need to know how $\lambda_p(L)$ depends on the coefficients of L. Therefore we want to give a proof for the results that we need. First, however, we have to solve the linear problem in the periodic setting. After that we show the existence of λ_p and its properties. We then apply this result to obtain a special exponential solution, which we can use as a comparision function to show $u^- = 0$.

Lemma 5.12 (Solvability of the linear periodic problem) *Consider the elliptic operator L given by*

$$Lu := \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} b_i(x) u_{x_i} + c(x) u.$$

We assume that L has $C^{0,\alpha}(T^n)$ coefficients and that $c(x) \le c_0 < 0$ for all $x \in T^n$. For some $\mu > 0$, the matrix field is assumed to satisfy

$$\mu \left|\left|\xi\right|\right|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \mu^{-1} \left|\left|\xi\right|\right|^2 \text{ for all } \xi \in \mathbb{R}^n \text{ and } x \in T^n.$$

Then for every $f \in C^{0,\alpha}(T^n)$, there is a unique solution $u \in C^2(T^n)$ of $Lu = f$. Moreover, u has the regularity $u \in C^{2,\alpha}(T^n)$ and satisfies the estimate

$$||u||_{C^0(T^n)} \le \frac{||f||_{C^0(T^n)}}{|c_0|}. \tag{5.45}$$

PROOF *Step 1: Choosing an approximating sequence for the solution.* We choose a sequence of $C^{2,\alpha}$-domains $\Omega_m \subset \mathbb{R}^n$, $m \in \mathbb{N}$, such that

$$\Omega_m \Subset \Omega_{m+1} \text{ and } \bigcup_{m=1}^{\infty} \Omega_m = \mathbb{R}^n.$$

Clearly, under the given assumptions, there is a unique solution $u_m \in C^{2,\alpha}(\overline{\Omega_m})$ of the problem

$$Lu_m = f \text{ in } \Omega_m, \qquad u_m = 0 \text{ on } \partial\Omega_m.$$

Step 2: A maximum estimate for u_m. There holds

$$L\left(u_m - \frac{||f||_{C^0(T^n)}}{|c_0|}\right) = f(x) - \frac{||f||_{C^0(T^n)}}{|c_0|}c(x) \ge 0.$$

It follows from the weak maximum principle that

$$\sup_{\Omega_m}\left(u_m - \frac{||f||_{C^0(T^n)}}{|c_0|}\right) \le \sup_{\partial\Omega_m}\left(u_m - \frac{||f||_{C^0(T^n)}}{|c_0|}\right)_+ = 0. \tag{5.46}$$

for every $m \in \mathbb{N}$. Similarly,

$$L\left(u_m + \frac{||f||_{C^0(T^n)}}{|c_0|}\right) = f(x) + \frac{||f||_{C^0(T^n)}}{|c_0|}c(x) \le 0$$

and by the weak minimum principle

$$\inf_{\Omega_m}\left(u_m + \frac{||f||_{C^0(T^n)}}{|c_0|}\right) \ge \inf_{\partial\Omega_m}\left(u_m + \frac{||f||_{C^0(T^n)}}{|c_0|}\right)_- = 0. \tag{5.47}$$

Together, (5.46) and (5.47) imply

$$||u_m||_{C^0(\overline{\Omega_m})} \le \frac{||f||_{C^0(T^n)}}{|c_0|} \text{ for every } m \in \mathbb{N}. \tag{5.48}$$

Step 3: Local $C^{2,\alpha}$-estimates for u_m. Let Ω and Ω' be bounded domains in $\mathbb{R}^n$ with $\Omega \Subset \Omega'$. There exists $m_0 = m_0(\Omega') \in \mathbb{N}$, such that $\Omega' \Subset \Omega_m$ holds for $m \geq m_0$. For $m \geq m_0$, interior Schauder estimates yield together with (5.48):

$$\begin{aligned} ||u_m||_{C^{2,\alpha}(\overline{\Omega})} &\leq C \left(||u_m||_{C^0(\overline{\Omega'})} + ||f||_{C^{0,\alpha}(\overline{\Omega'})} \right) \\ &\leq C \left(\frac{||f||_{C^0(T^n)}}{|c_0|} + ||f||_{C^{0,\alpha}(\overline{\Omega'})} \right). \end{aligned} \tag{5.49}$$

The constant C depends on n, α, the coefficients of L, on Ω and Ω', but not on m. Therefore, the right hand side of (5.49) does not depend on m. Hence, we obtained $C^{2,\alpha}_{loc}$-estimates for u_m.

Step 4: Passing to the limit. By the Arzelà-Ascoli theorem and a diagonal argument, we can therefore choose a subsequence, such that $u_m \to u$ in $C^2_{loc}(\mathbb{R}^n)$. From the estimate (5.49), we even obtain $u \in C^{2,\alpha}_{loc}(\mathbb{R}^n)$, even though the convergence only takes place in $C^2_{loc}(\mathbb{R}^n)$. By the $C^2_{loc}(\mathbb{R}^n)$ convergence of u_m, we can pass to the limit in the equation and obtain $Lu = f$ on $\mathbb{R}^n$. Moreover, by (5.48) we deduce

$$||u||_{L^\infty(\mathbb{R}^n)} \leq \frac{||f||_{C^0(T^n)}}{|c_0|}. \tag{5.50}$$

In particular, u is bounded.

Step 5: Periodicity and Uniqueness of the solution. With the help of theorem A.1, we can conclude that u is periodic: The assumptions of the theorem are satisfied because for the constant function $v = 1$ we have $-Lv = -c(x) \geq -c_0 > 0$ and v is clearly nonnegative. Let e_i be the $i-th$ unit vector and define $\tilde{u}(x) := u(x + e_i)$. By the periodicity of the coefficients of L and the periodicity of f, we obtain $L\tilde{u} = f$. Therefore, there holds $L(u - \tilde{u}) = 0$ and $u - \tilde{u}$ is bounded. Theorem A.1 (ii) implies (note $\partial\mathbb{R}^n = \emptyset$, therefore no boundary values have to be checked) that $u - \tilde{u} = 0$. Since $i = 1, ..., n$ was arbitrary, we conclude the periodicity of u. Due to the same reasoning, we also obtain the uniqueness of the solution. From $u \in C^{2,\alpha}_{loc}(\mathbb{R}^n)$ and the periodicity, we deduce $u \in C^{2,\alpha}(T^n)$. The claimed estimate for $||u||_{C^0(T^n)}$ follows from (5.50). ■

The solution of the linear problem will now be used to obtain the existence of the periodic principle eigenvalue. The proof uses the Krein-Rutman theorem.

Lemma 5.13 (The periodic principle eigenvalue) *Consider the elliptic operators L and L_0, which are given by*

$$Lu := L_0 u + c(x)u := \sum_{i,j=1}^n a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^n b_i(x) u_{x_i} + c(x)u.$$

We assume, that they have $C^{0,\alpha}(T^n)$ coefficients and for some $\mu > 0$, the matrix fields satisfies

$$\mu \, ||\xi||^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq \mu^{-1} \, ||\xi||^2 \ \textit{for all } \xi \in \mathbb{R}^n \textit{ and } x \in T^n.$$

(a) *There is a unique* $\lambda_p = \lambda_p(L) \in \mathbb{R}$, *such that there exists a 1-periodic function* $\phi \in C^2(T^n)$ *with* $\phi > 0$ *and* $L\phi = \lambda_p \phi$. *Moreover,* $\phi \in C^{2,\alpha}(T^n)$ *and* ϕ *is unique up to multiplication with a positive constant.* λ_p *is called the periodic principal eigenvalue and* ϕ *the periodic principal eigenfunction of* L.

(b) *Assume* $c_1, c_2 \in C^{0,\alpha}(T^n)$ *with* $c_1(x) \le c_2(x)$ *for all* $x \in \mathbb{R}^n$ *and define the operators* $L_j u := L_0 u + c_j(x)u$ *for* $j = 1, 2$. *Then there holds* $\lambda_p(L_1) \le \lambda_p(L_2)$. *Moreover, for the operator* L, *there holds*

$$\inf_{x \in T^n} c(x) \le \lambda_p(L) \le \sup_{x \in T^n} c(x). \tag{5.51}$$

(c) *For* $m \in \mathbb{N}$, *consider the elliptic operators* $L^{(m)}$, *which are given by*

$$L^{(m)}u = \sum_{i,j=1}^{n} a_{ij}^{(m)}(x) u_{x_i x_j} + \sum_{i=1}^{n} b_i^{(m)}(x) u_{x_i} + c^{(m)}(x)u.$$

We assume that each $L^{(m)}$ *satisfies the same assumptions as* L. *Furthermore, for* $1 \le i, j \le n$, *we assume the following convergences of the coefficients:*

$$\begin{aligned} \left|\left|a_{ij}^{(m)} - a_{ij}\right|\right|_{C^{0,\alpha}(T^n)} &\to 0 \text{ for } m \to \infty, \\ \left|\left|b_i^{(m)} - b_i\right|\right|_{C^{0,\alpha}(T^n)} &\to 0 \text{ for } m \to \infty \text{ and} \\ \left|\left|c^{(m)} - c\right|\right|_{C^{0,\alpha}(T^n)} &\to 0 \text{ for } m \to \infty. \end{aligned}$$

Then $\lambda_p\left(L^{(m)}\right) \to \lambda_p(L)$ *for* $m \to \infty$.

PROOF *(a) Step 1: Existence.* We show the assertion with the Krein-Rutman theorem as it is given in [8] theorem 1.2. For $\xi \in \mathbb{R}$, we define the operator $L_\xi u := Lu - \xi u$. We choose ξ sufficiently large, such that $c(x) - \xi \le c_0 < 0$. Then by lemma 5.12, for every $f \in C^{0,\alpha}(T^n)$ and in particular for every $f \in C^1(T^n)$, there is a unique solution $u \in C^{2,\alpha}(T^n)$ of the problem $-L_\xi u = f$. Therefore with $X := C^1(T^n)$, the operator $T : X \to X$, $f \mapsto (-L_\xi)^{-1}(f)$ is well defined.

We show that the operator T is compact: Set $u := Tf$, which is equivalent to $-L_\xi u = f$. By (5.45) we have

$$||u||_{C^0(T^n)} \le \frac{||f||_{C^0(T^n)}}{|c_0|}.$$

We choose a bounded domain Ω with $[0,1]^n \subset \Omega$. Then interior Schauder estimates yield

$$\begin{aligned} ||u||_{C^{2,\alpha}(T^n)} = ||u||_{C^{2,\alpha}([0,1]^n)} &\le C\left(||u||_{C^0(\overline{\Omega})} + ||f||_{C^{0,\alpha}(\overline{\Omega})}\right) \\ &\le C\left(\frac{||f||_{C^0(T^n)}}{|c_0|} + ||f||_{C^{0,\alpha}(\overline{\Omega})}\right) \le \tilde{C}\, ||f||_{C^1(T^n)}. \end{aligned}$$

Hence, we have $||Tf||_{C^{2,\alpha}(T^n)} \leq \tilde{C}\,||f||_{C^1(T^n)}$. Consequently, T is a compact linear operator from X to X.

Let $K = \{u \in C^1(T^n) : u \geq 0\}$ be the cone of nonnegative functions. Clearly, this cone has a nonempty interior. In other words, it is a solid cone. The interior is given by $K^\circ = \{u \in C^1(T^n) : u > 0\}$.

We have to show that T is a strongly positive operator. (Every cone induces a partial ordering on a Banach space. See [8] chapter 1.) By definition, this means that we have to show $T(K \setminus \{0\}) \subset K^\circ$. To this end, let $f \in K \setminus \{0\}$, that is $f \geq 0$, but $f \not\equiv 0$. As above, set $u := Tf$, which is equivalent to $-L_\xi u = f$. The function u does achieve its minimum due to periodicity. There are two possibilities. Either u is strictly positive or u assumes a nonpositive minimum. Assume the latter is the case. Then, by the minimum principle, u has to be constant and $u \leq 0$ holds. This implies $f = -L_\xi u = -(c(x) - \xi)u \leq 0$, in contradiction to $f \geq 0$, $f \not\equiv 0$. There remains the only possibility $u > 0$, that is $u \in K^\circ$. Hence, T is a strongly positive operator.

Altogether we know that $T : X \to X$ is a compact linear operator and strongly positive. We can therefore apply the Krein-Rutman theorem (theorem 1.2 in [8]). We obtain that the spectral radius $r(T)$ is positive and $r(T)$ is a simple eigenvalue of T with an eigenvector $\phi \in K^\circ$. Moreover, there is no other eigenvalue with an eigenvector in K°.

Hence, we have $(-L_\xi)^{-1}(\phi) = T\phi = r(T)\phi$. This implies $\phi \in C^{2,\alpha}(T^n)$ and due to $r(T) > 0$ also $\frac{1}{r(T)}\phi = -L_\xi\phi$. We can write this as $L\phi = (\xi - \frac{1}{r(T)})\phi$. Hence, $\lambda := (\xi - \frac{1}{r(T)})$ is an eigenvalue of L with positive eigenfunction $\phi \in C^{2,\alpha}(T^n)$.

Step 2: Uniqueness. Suppose there is a function $\tilde{\phi} \in C^2(T^n)$ with $\tilde{\phi} > 0$ and a real number $\tilde{\lambda}$, such that $L\tilde{\phi} = \tilde{\lambda}\tilde{\phi}$. With L_ξ as in step 1, we can rewrite this as $-L_\xi\tilde{\phi} = (\xi - \tilde{\lambda})\tilde{\phi}$. This implies $\xi \neq \tilde{\lambda}$, because otherwise $-L_\xi\tilde{\phi} = 0$ and thus $\tilde{\phi} = 0$. However, we assumed $\tilde{\phi} > 0$. Therefore $\frac{1}{(\xi - \tilde{\lambda})}\tilde{\phi} = T\tilde{\phi}$ with the operator T from step 1. Since $\tilde{\phi} \in K^\circ$, the Krein Rutman theorem implies $\frac{1}{(\xi - \tilde{\lambda})} = r(T)$. Hence, $\lambda = \tilde{\lambda}$ with the λ from the end of step 1. This finishes the proof of (a).

(b) Denote by ϕ_1 and ϕ_2 the periodic principle eigenfunctions of L_1 and L_2. Since they can be chosen up to multiplication with positive constants, it is possible to choose them in such a way that

$$\max_{x \in T^n} \phi_1 - \phi_2 = \phi_1(x_0) - \phi_2(x_0) = 0. \tag{5.52}$$

We have

$$L_1\phi_1 - L_2\phi_2 = L_0(\phi_1 - \phi_2) + c_1(x)\phi_1 - c_2(x)\phi_2 = \lambda_p(L_1)\phi_1 - \lambda_p(L_2)\phi_2. \tag{5.53}$$

Due to (5.52), we have $L_0(\phi_1 - \phi_2)(x_0) \leq 0$ and $\phi_1(x_0) = \phi_2(x_0)$. Using this in (5.53), we obtain

$$(c_1(x_0) - c_2(x_0))\phi_1(x_0) \geq (\lambda_p(L_1) - \lambda_p(L_2))\phi_1(x_0).$$

Since $c_1 \leq c_2$ and $\phi_1 > 0$, we deduce $0 \geq (c_1(x_0) - c_2(x_0))\phi_1(x_0)$ and consequently $\lambda_p(L_1) \leq \lambda_p(L_2)$.

In order to prove (5.51), we observe that for a constant $\bar{c}$, the operator given by $\bar{L}\phi := L_0\phi + \bar{c}\phi$ satisfies $\lambda_p(\bar{L}) = \bar{c}$ with a positive constant as eigenfunction. Therefore, (5.51) follows from the just proved monotonicity of λ_p in the zeroth order coefficient.

(c) Step 1: Estimates for the periodic principal eigenvalue and eigenfunction. Let $\phi^{(m)}$ denote the periodic principal eigenfunction of $L^{(m)}$. This means

$$L^{(m)}\phi^{(m)} = \sum_{i,j=1}^{n} a_{ij}^{(m)}(x)\phi_{x_i x_j}^{(m)} + \sum_{i=1}^{n} b_i^{(m)}(x)\phi_{x_i}^{(m)} + c^{(m)}(x)\phi^{(m)} = \lambda_p\left(L^{(m)}\right)\phi^{(m)}.$$

Equivalently, by defining a new operator $M^{(m)}$, this can be written as

$$M^{(m)}\phi^{(m)} := \sum_{i,j=1}^{n} a_{ij}^{(m)}(x)\phi_{x_i x_j}^{(m)} + \sum_{i=1}^{n} b_i^{(m)}(x)\phi_{x_i}^{(m)} + \left(c^{(m)}(x) - \lambda_p\left(L^{(m)}\right)\right)\phi^{(m)} = 0.$$

Since $\phi^{(m)}$ is only unique up to multiplication with a positive constant, we can choose $\phi^{(m)}$ such that $\left|\left|\phi^{(m)}\right|\right|_{C^0(T^n)} = 1$. Due to the inequality from (b), we have $\left|\lambda_p\left(L^{(m)}\right)\right| \leq \left|\left|c^{(m)}\right|\right|_{C^0(T^n)}$. From this inequality and the assumed convergences for the coefficients, we deduce that there exists $m_0 \in \mathbb{N}$, such that for $m \geq m_0$ the following estimates for the coefficients hold:

$$\left|\left|c^{(m)} - \lambda_p\left(L^{(m)}\right)\right|\right|_{C^{0,\alpha}(T^n)} \leq 2\left|\left|c^{(m)}\right|\right|_{C^{0,\alpha}(T^n)} \leq 2\left|\left|c\right|\right|_{C^{0,\alpha}(T^n)} + 1, \tag{5.54}$$

$$\left|\left|b_i^{(m)}\right|\right|_{C^{0,\alpha}(T^n)} \leq \left|\left|b_i\right|\right|_{C^{0,\alpha}(T^n)} + 1, \qquad \left|\left|a_{ij}^{(m)}\right|\right|_{C^{0,\alpha}(T^n)} \leq \left|\left|a_{ij}\right|\right|_{C^{0,\alpha}(T^n)} + 1, \tag{5.55}$$

as well as

$$\frac{\mu}{2}\left|\left|\xi\right|\right|^2 \leq \sum_{i,j=1}^{n} a_{ij}^{(m)}(x)\xi_i\xi_j \leq \left(\frac{\mu}{2}\right)^{-1}\left|\left|\xi\right|\right|^2 \text{ for all } \xi \in \mathbb{R}^n \text{ and } x \in T^n. \tag{5.56}$$

By $M^{(m)}\phi^{(m)} = 0$, (5.54), (5.55) and (5.55), one can see from interior Schauder estimates and the periodicity of $\phi^{(m)}$ that

$$\left|\left|\phi^{(m)}\right|\right|_{C^{2,\alpha}(T^n)} \leq C\left|\left|\phi^{(m)}\right|\right|_{C^0(T^n)} = C. \tag{5.57}$$

The constant C depends on the bounds obtained in (5.54), (5.55) and (5.55), on α and on n, but not on m.

Step 2: Passing to the limit. As we have seen above, $\lambda_p\left(L^{(m)}\right)$ is a bounded sequence due to $\left|\lambda_p\left(L^{(m)}\right)\right| \leq \left|\left|c^{(m)}\right|\right|_{C^0(T^n)}$ and (5.54). Moreover, due to (5.57), $\left|\left|\phi^{(m)}\right|\right|_{C^{2,\alpha}(T^n)} \leq C$. Therefore, for any subsequence, we can choose another subsequence, such that the following holds: There is a function $\tilde{\phi} \in C^2(T^n)$ with $\tilde{\phi} \geq 0$ and $\left|\left|\tilde{\phi}\right|\right|_{C^0(T^n)} = 1$, as well as real number $\tilde{\lambda}$, such that $\phi^{(m)} \to \tilde{\phi}$ in $C^2(T^n)$ and $\lambda_p\left(L^{(m)}\right) \to \tilde{\lambda}$ for $m \to \infty$. Passing to the limit, we see $L\tilde{\phi} = \tilde{\lambda}\tilde{\phi}$.

We want to use the uniqueness of the principle eigenvalue to conclude $\lambda = \tilde{\lambda}$. In order to do so, we still have to show the strict inequality $\tilde{\phi} > 0$. Only then it will be clear, that $\tilde{\phi}$ is a periodic principle eigenfunction of L. The weaker inequality $\tilde{\phi} \geq 0$ is already known. Assume that there exists x_0 with $\tilde{\phi}(x_0) = 0$. We write $L\tilde{\phi} = \tilde{\lambda}\tilde{\phi}$ as

$$\sum_{i,j=1}^{n} a_{ij}(x)\tilde{\phi}_{x_i x_j} + \sum_{i=1}^{n} b_i(x)\tilde{\phi}_{x_i} + (c(x) - \tilde{\lambda})\tilde{\phi} = 0.$$

Hence, the strong minimum principle implies $\tilde{\phi} \equiv 0$ irrespectively of the sign of $c(x) - \tilde{\lambda}$. But this is a contradiction to $\left\|\tilde{\phi}\right\|_{C^0(T^n)} = 1$. Consequently, the strict inequality $\tilde{\phi} > 0$ holds. Therefore, we know that $\tilde{\phi}$ is a periodic principle eigenfunction of L. By the uniqueness of the periodic principle eigenvalue, this implies $\tilde{\lambda} = \lambda_p(L)$. We have shown that any subsequence has another subsequence with $\lambda_p\left(L^{(m)}\right) \to \lambda_p(L)$. But this means the convergence of the original sequence as claimed. ∎

Lemma 5.14 (Exponential solution) *Let $c^{\varepsilon,b}$ be the wavespeed from definition 5.1. We recall the estimates $c_1 \leq c^{\varepsilon,b} \leq c_2 < 0$ for $0 < \varepsilon \leq \varepsilon_0$ and $b \geq b_0$. Consider the equation*

$$L^{\varepsilon,b}U := \varepsilon U_{ss} + \sum_{i,j=1}^{n} a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})U + c^{\varepsilon,b}U_s = 0. \tag{5.58}$$

Then there are a real number $\omega_{\varepsilon,b} > 0$ and a function $\Psi^{\varepsilon,b} \in C^2(T^n)$ with $\Psi^{\varepsilon,b}(y) > 0$, such that (5.58) *has the solution*

$$\Phi^{\varepsilon,b}(s,y) = e^{\omega_{\varepsilon,b}s}\Psi^{\varepsilon,b}(y).$$

For $\omega_{\varepsilon,b}$ and $\Psi^{\varepsilon,b}$, the following estimates hold:

$$\inf_{0<\varepsilon\leq\varepsilon_0, b\geq b_0} \omega_{\varepsilon,b} \geq \omega_0 > 0, \inf_{0<\varepsilon\leq\varepsilon_0, b\geq b_0, y\in T^n} \Psi^{\varepsilon,b}(y) = 1, \sup_{0<\varepsilon\leq\varepsilon_0, b\geq b_0, y\in T^n} \Psi^{\varepsilon,b}(y) \leq C < \infty. \tag{5.59}$$

Proof The proof is a variation of the proof of lemma 3.3 in [21], see remark 2.21.

Step 1: Reformulating the problem with the periodic principle eigenvalue. Consider the ansatz

$$\Phi^{\varepsilon,b}(s,y) = e^{\omega_{\varepsilon,b}s}\Psi^{\varepsilon,b}(y).$$

Writing more shortly ω for $\omega_{\varepsilon,b}$ and Ψ for $\Psi^{\varepsilon,b}$, the ansatz yields the equation

$$\begin{aligned} 0 &= L^{\varepsilon,b}\Phi^{\varepsilon,b} \\ &= (\varepsilon + r(y))\omega^2 e^{\omega s}\Psi + e^{\omega s}\sum_{i,j=1}^{n} a_{ij}(y)\Psi_{y_i y_j} + 2\omega e^{\omega s}\sum_{i,j=1}^{n} a_{ij}(y)k_i\Psi_{y_j} + \omega c^{\varepsilon,b}e^{\omega s}\Psi. \end{aligned}$$

This is equivalent to

$$-\omega^2\Psi = \sum_{i,j=1}^{n} \frac{1}{\varepsilon + r(y)} a_{ij}(y)\Psi_{y_i y_j} + 2\omega \sum_{i,j=1}^{n} \frac{1}{\varepsilon + r(y)} a_{ij}(y) k_i \Psi_{y_j} + \omega \frac{1}{\varepsilon + r(y)} c^{\varepsilon,b}\Psi.$$

We define the coefficients

$$a_{ij}^{\varepsilon,b,\omega}(y) := \frac{1}{\varepsilon + r(y)} a_{ij}(y), \qquad b_j^{\varepsilon,b,\omega}(y) := 2\omega \sum_{i=1}^{n} \frac{1}{\varepsilon + r(y)} a_{ij}(y) k_i$$

and

$$c^{\varepsilon,b,\omega}(y) := \omega \frac{1}{\varepsilon + r(y)} c^{\varepsilon,b}.$$

With these coefficients we define the elliptic operator $M_\omega = M_\omega^{\varepsilon,b}$ by

$$M_\omega^{\varepsilon,b}\Psi := \sum_{i,j=1}^{n} a_{ij}^{\varepsilon,b,\omega}(y)\Psi_{y_i y_j} + \sum_{j=1}^{n} b_j^{\varepsilon,b,\omega}(y)\Psi_{y_j} + c^{\varepsilon,b,\omega}(y)\Psi.$$

Then the equation can be written more shortly as $M_\omega \Psi = -\omega^2\Psi$.

Step 2: The existence of $\omega^{\varepsilon,b}$ and its estimate. By step 1, we have to find $\omega = \omega_{\varepsilon,b}$ such that the periodic principal eigenvalue of M_ω equals $-\omega^2$. Due to lemma 5.13 (*c*), $\lambda_p(M_\omega)$ depends continuously on ω. Hence, we can use the intermediate value theorem to find ω. From the inequality (5.51) and the negativity of $c^{\varepsilon,b}$, we obtain

$$\frac{c^{\varepsilon,b}\omega}{\varepsilon + r_{\min}} \le \lambda_p(M_\omega) \le \frac{c^{\varepsilon,b}\omega}{\varepsilon + r_{\max}} < 0. \tag{5.60}$$

This implies $-\omega^2 > \lambda_p(M_\omega)$ if $0 < \omega < -\frac{c^{\varepsilon,b}}{\varepsilon + r_{\max}}$ and $-\omega^2 < \lambda_p(M_\omega)$ if $\omega > -\frac{c^{\varepsilon,b}}{\varepsilon + r_{\min}}$. By the intermediate value theorem, there is $\omega = \omega_{\varepsilon,b}$ with $\lambda_p(M_{\omega_{\varepsilon,b}}^{\varepsilon,b}) = -\omega_{\varepsilon,b}^2$. From (5.60), we obtain furthermore

$$-\frac{c_2}{\varepsilon_0 + r_{\max}} \le -\frac{c^{\varepsilon,b}}{\varepsilon + r_{\max}} \le \omega_{\varepsilon,b} \le -\frac{c^{\varepsilon,b}}{\varepsilon + r_{\min}} \le -\frac{c_1}{r_{\min}}. \tag{5.61}$$

Therefore, the existence of the exponential solution is proved, as well as the estimate

$$\inf_{0<\varepsilon\le\varepsilon_0, b\ge b_0} \omega_{\varepsilon,b} \ge -\frac{c_2}{\varepsilon_0 + r_{\max}} =: \omega_0 > 0.$$

Step 3: The estimates for $\Psi^{\varepsilon,b}$. The corresponding eigenfunction $\Psi^{\varepsilon,b}$ to $\lambda_p(M_{\omega_{\varepsilon,b}}^{\varepsilon,b})$ is only unique up to multiplication with a positive constant. Hence, the function $\Psi^{\varepsilon,b}$ can be chosen such that

$$\inf_{y\in T^n} \Psi^{\varepsilon,b}(y) = 1 \text{ for every } 0 < \varepsilon \le \varepsilon_0, b \ge b_0.$$

Then, the second property of (5.59) is satisfied by this normalization.

The last of the properties will follow from the Harnack inequality. Let us check the assumptions for the inequality. We have $M^{\varepsilon,b}_{\omega_{\varepsilon,b}}\Psi^{\varepsilon,b} = -\omega^2_{\varepsilon,b}\Psi^{\varepsilon,b}$. By putting $\tilde{M}^{\varepsilon,b}_{\omega_{\varepsilon,b}} := M^{\varepsilon,b}_{\omega_{\varepsilon,b}} + \omega^2_{\varepsilon,b}$ we can write this as $\tilde{M}^{\varepsilon,b}_{\omega_{\varepsilon,b}}\Psi^{\varepsilon,b} = 0$. With the help of (5.61), it is clear that the coefficients of $\tilde{M}^{\varepsilon,b}_{\omega_{\varepsilon,b}}$ have L^∞-bounds that do not depend on ε or b. The matrix field $(a^{\varepsilon,b,\omega_{\varepsilon,b}}_{ij})^n_{i,j=1}$ actually does not depend on $\omega_{\varepsilon,b}$ and not on b either. The only dependence is on ε. One can see that the family of matrix fields $\{(a^{\varepsilon,b,\omega_{\varepsilon,b}}_{ij})^n_{i,j=1} : 0 < \varepsilon \leq \varepsilon_0, b \geq b_0\}$ admits an ellipticity constant that only depends on ε_0. We choose a ball $B_R(0)$ with $[0,1]^n \subset B_R(0)$. Then we apply the Harnack inequality (see for example Corollary 9.25 in [12]), which by $\tilde{M}^{\varepsilon,b}_{\omega_{\varepsilon,b}}\Psi^{\varepsilon,b} = 0$ and $\Psi^{\varepsilon,b} > 0$ gives us

$$\sup_{y\in T^n} \Psi^{\varepsilon,b}(y) = \sup_{y\in B_R(0)} \Psi^{\varepsilon,b}(y) \leq C \inf_{y\in B_R(0)} \Psi^{\varepsilon,b}(y) = C.$$

The constant C depends on n and the bounds for the coefficients and the ellipticity constant. Therefore, C is independent of ε and b. The third property in (5.59) is proved. ∎

5.2.4. Behavior of u at infinity

We are now ready to show that our limit function u from lemma 5.5 shows the correct behavior at $s = \pm\infty$.

Lemma 5.15 *Consider the situation of lemma 5.5. Then the constants u^+ and u^- from lemma 5.8 satisfy $u^- = 0$ and $u^+ = 1$.*

Proof The proof of the lemma is essentially as in the proof of theorem 0.1 in [21]. It only remains to show that $u^- = 0$. Then $u^+ = 1$ follows automatically as explained in remark 5.11.

We recall that $u(t,x)$ is the limit function of a sequence u^{ε_n,b_n} from lemma 5.5, where $u^{\varepsilon,b}$ was defined in definition 5.1 via the coordinate transformation $(s,y) = (k\cdot x - c^{\varepsilon,b}t, x)$ of the function $U^{\varepsilon,b}(s,y)$. In order to show $u^- = 0$, we compare $U^{\varepsilon,b}$ with the exponential solution $\Phi^{\varepsilon,b}(s,y) = e^{\omega_{\varepsilon,b}s}\Psi^{\varepsilon,b}(y)$ from lemma 5.14. Due to the estimate (5.59) we have $\Psi^{\varepsilon,b} \geq \theta(f)$. Moreover, $\theta(f) \geq U^{\varepsilon,b}(s,y)$ for $s \leq 0$ holds by normalization of $U^{\varepsilon,b}$ and the monotonicity of $U^{\varepsilon,b}$ in s. Consequently, we have $f(y, U^{\varepsilon,b}(s,y)) = 0$ for $(s,y) \in [-b,0] \times T^n$ due to the combustion property of f. Hence, $U^{\varepsilon,b}$ satisfies

$$\varepsilon U^{\varepsilon,b}_{ss} + \sum_{i,j=1}^n a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})U^{\varepsilon,b} + c^{\varepsilon,b}U^{\varepsilon,b}_s = 0 \text{ in } [-b,0]\times T^n.$$

Therefore, $W^{\varepsilon,b} := U^{\varepsilon,b} - \Phi^{\varepsilon,b}$ satisfies

$$\varepsilon W^{\varepsilon,b}_{ss} + \sum_{i,j=1}^n a_{ij}(y)(k_i\partial_s + \partial_{y_i})(k_j\partial_s + \partial_{y_j})W^{\varepsilon,b} + c^{\varepsilon,b}W^{\varepsilon,b}_s = 0 \text{ in } [-b,0]\times T^n.$$

At the boundary, we have

$$\Phi^{\varepsilon,b}(0,y) \geq \theta(f) \geq U^{\varepsilon,b}(0,y) \text{ for all } y \in T^n$$

and

$$\Phi^{\varepsilon,b}(-b,y) > 0 = U^{\varepsilon,b}(-b,y) \text{ for all } y \in T^n.$$

We thus have $W^{\varepsilon,b} \leq 0$ on $\partial((-b,0) \times T^n)$. Due to the 1-periodicity in y we can apply the strong maximum principle to $W^{\varepsilon,b}$. We obtain $W^{\varepsilon,b} \leq 0$ on $[-b,0] \times T^n$. Applying the estimates (5.59), we obtain

$$U^{\varepsilon,b}(s,y) \leq \Phi^{\varepsilon,b}(s,y) = e^{\omega_{\varepsilon,b}s}\Psi^{\varepsilon,b}(y) \leq Ce^{\omega_0 s} \text{ for all } (s,y) \in [-b,0] \times T^n.$$

In (t,x) coordinates, this means

$$\begin{aligned} u^{\varepsilon,b}(t,x) &\leq C\exp(\omega_0(k \cdot x - c^{\varepsilon,b}t)) \\ \text{on } &\left\{(t,x) : \frac{-b - k \cdot x}{-c^{\varepsilon,b}} \leq t \leq \frac{0 - k \cdot x}{-c^{\varepsilon,b}}\right\}. \end{aligned}$$

If we now take $\varepsilon = \varepsilon_n$ and $b = b_n$ from lemma 5.5 and let $n \to \infty$, we obtain

$$\begin{aligned} u(t,x) &\leq C\exp(\omega_0(k \cdot x - ct)) \\ \text{on } &\left\{(t,x) : t < \frac{0 - k \cdot x}{-c}\right\}. \end{aligned}$$

This implies $u(t,x) \to 0$ for $t \to -\infty$ locally uniformly (we recall $c < 0$) in x and therefore $u^- = 0$. This implies that both boundary values u^- and u^+ are as claimed. ∎

We want to remark that, as in the proof of lemma 5.10, instead of using the combustion property of f to infer $f(x,u(t,x)) = 0$ for all $(t,x) \in \mathbb{R}^{n+1}$, it would have sufficed to use property *(iii)* from definition 2.6 and obtain $f(x,u(t,x)) \leq 0$ for all $(t,x) \in \mathbb{R}^{n+1}$.

5.2.5. The strict inequalities $0 < u < 1$ and $u_t > 0$

Lemma 5.16 *In the situation of lemma 5.5, the function u satisfies the strict inequality $0 < u < 1$.*

Proof *Step 1: The inequality $u < 1$.* Suppose there is (t_0,x_0) such that $u(t_0,x_0) = 1$. From $f(x,1) = 0$, we obtain:

$$\begin{aligned} 0 &= -u_t + \sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j} + f(x,u) \\ &= -(u-1)_t + \sum_{i,j=1}^{n} a_{ij}(x)(u-1)_{x_ix_j} + f(x,u) - f(x,1). \end{aligned}$$

We define

$$d(t,x) := \begin{cases} \dfrac{f(x,u) - f(x,1)}{u-1} & \text{if } u \neq 1 \\ f_z(x,1) & \text{if } u = 1. \end{cases}$$

The function d is continuous and bounded. We obtain the equation

$$0 = -(u-1)_t + \sum_{i,j=1}^{n} a_{ij}(x)(u-1)_{x_i x_j} + d(t,x)(u-1).$$

Since $(u-1)(t_0,x_0) = 0$, the function $u-1$ obtains its maximum with the value 0. Due to the parabolic maximum principle we obtain $u \equiv 1$ for $t \leq t_0$ irrespectively of the sign of $d(t,x)$. However, this contradicts $u(t,x) \to 0$ for $t \to -\infty$. The strict inequality $u < 1$ is proved.

Step 2: The inequality $0 < u$. To show $0 < u$, we have to work differently. The reason is that we could use the usual left-sided parabolic maximum principle to show $u < 1$, but we need a right-sided minimum principle (or a two-sided one) to show $0 < u$. We have

$$-u_t + \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} = -f(x,u) \leq 0.$$

The inequality for f holds due to the combustion property. If $u(t_0,x_0) = 0$, the parabolic minimum principle implies $u \equiv 0$ for $t \leq t_0$. However, as opposed to the situation in the proof of lemma 5.10, this does not help. Moreover, the monotonicity does not help to turn the one-sided parabolic minimum principle into a two-sided one. Instead, the condition (5.27) helps us to do so. We refer forward to lemma 6.1 (i), which implies $u = 0$ on $\mathbb{R}^{n+1}$. This contradicts $u(t,x) \to 1$ for $t \to \infty$. Hence, $0 < u$.

Again, we remark that it is possible to avoid to use the combustion property of f in step 2 by using a linearization as in step 1. ∎

Lemma 5.17 *In the situation of lemma 5.5, the following holds for the function u: The time derivative u_t satisfies the strict inequality $u_t > 0$.*

Proof From lemma 5.5 we already know $u_t \geq 0$. We could use the additional regularity from lemma 5.6, reformulate (5.37) into a weak form and use a strong maximum principle for weak solutions. Instead we are going to give an argument with a Harnack inequality, that does not involve the additional regularity. See remark 2.22.

Suppose there is (t_0,x_0) such that $u_t(t_0,x_0) = 0$. For $h > 0$, we consider the difference quotient of u in t:

$$v^h(t,x) := D_t^h u(t,x) := \frac{u^h(t,x) - u(t,x)}{h} := \frac{u(t+h,x) - u(t,x)}{h}.$$

Due to the equation (5.26), the difference quotient v^h classically satisfies the equation

$$-v_t^h + \sum_{i,j=1}^{n} a_{ij}(x)v_{x_i x_j}^h + \frac{f(x,u^h) - f(x,u)}{h} = 0.$$

We define the coefficient d^h by

$$d^h(t,x) := \begin{cases} \dfrac{f(x,u^h(t,x)) - f(x,u(t,x))}{u^h(t,x) - u(t,x)} & \text{if } u^h(t,x) \neq u(t,x) \\ f_z(x,u(t,x)) & \text{if } u^h(t,x) = u(t,x). \end{cases}$$

Clearly, d^h is continuous for every $h > 0$ and bounded by $||f_z||_{C^0(T^n \times [0,1])}$ independently of h. By the regularity of the matrix field $(a_{ij})_{i,j=1}^n$, we see that v^h is also a weak solution of

$$-v_t^h + \sum_{i,j=1}^n \partial_{x_j}(a_{ij}(x)v_{x_i}^h) - \sum_{i=1}^n (\sum_{j=1}^n \partial_{x_j}(a_{ij}(x))v_{x_i}^h) + d^h(t,x)v^h. \tag{5.62}$$

Since v^h is nonnegative on $\mathbb{R}^{n+1}$ due to the monotonicity of u in t, we can apply the Harnack inequality from theorem 6.27 in [16]. With slight change of the notation in [16] we set

$$Q((t,x),R) := \{(\tilde{t},\tilde{x}) \in \mathbb{R}^{n+1} : d((\tilde{t},\tilde{x}),(t,x)) < R, \tilde{t} < t\}$$

and $\Theta((t,x),R) := Q((t-4R^2,x),R)$. Then the Harnack inequality reads:

$$\sup_{Q((t_0-R^2,x_0),R/2)} v^h = \sup_{\Theta((t_0,x_0),R/2)} v^h \leq C \inf_{Q((t_0,x_0),R)} v^h. \tag{5.63}$$

The constant C depends on R, an ellipticity constant for the matrix field $(a_{ij})_{i,j=1}^n$, on L^∞-bounds for the coefficients in (5.62), but not on the specific coefficients. This is important, since d^h depends on h. As stated above, d^h is bounded by $||f_z||_{C^0(T^n \times [0,1])}$ independently of h. The other coefficients are independent of h altogether. Hence, C does not depend on h.

For every fixed R, the right hand side of (5.63) converges to 0 as $h \to 0$. This is due to the assumption that $u_t(t_0,x_0) = 0$ and because of $v^h(t_0,x_0) \to u_t(t_0,x_0)$. It follows that $v^h \to 0$ in $Q((t_0-R^2,x_0),R/2)$. Consequently, we have $u_t = 0$ in $Q((t_0-R^2,x_0),R/2)$. Since $R > 0$ was arbitrary, we can deduce from this that $u_t(t,x_0) = 0$ for every $t < t_0$. From lemma 5.8 and lemma 5.15, we know $u(t,x_0) \to 0$ for $t \to -\infty$. Therefore, it follows that $u(t_0,x_0) = 0$. But by lemma 5.16, there holds the strict inequality $0 < u < 1$, which is a contradiction. Hence, $u_t > 0 \in \mathbb{R}^{n+1}$ is proved. ∎

5.2.6. Conclusion of the existence proofs

Under the assumptions of either of the theorems 2.8, 2.9 or 2.10, the assumptions of lemma 5.5 are met. The limit function u from this lemma is a classical solution of (5.26), that is of (2.3). Moreover, u satisfies the periodicity condition from (2.4) and $0 \leq u \leq 1$. The functions u has the desired behavior at infinity due to the lemmas 5.8 and 5.15. Therefore (2.4) is satisfied and consequently u is a traveling wave solution of type I. Moreover, by the lemmas 5.16 and 5.17, the strict inequalities $0 < u < 1$ and $u_t > 0$ hold. This finishes the proofs of theorems 2.8 and 2.9. To finish the proof of theorem 2.10, it remains to apply lemma 5.6 (*ii*).

6. Monotonicity and uniqueness of traveling wave solutions

6.1. Minimum principles

In some situations, the usual left-sided parabolic minimum principle is not sufficient. By the periodicity condition of traveling wave solutions, it is possible to deduce two-sided minimum principles from the usual one-sided one. As opposed to proposition 4.1 in [21], we state the following minimum principle in nondivergence form, which makes it possible to lower the regularity assumptions for the monotonicity and uniqueness. Also, we think that our proof of the minimum principles is simpler than the proof of proposition 4.1 in [21].

Lemma 6.1 (Two-sided minimum principle) *Consider the domain D, where either $D = \mathbb{R} \times \mathbb{R}^n$ or for some $q \in \mathbb{R}$ either*

$$D = \left\{ (t,x) \in \mathbb{R} \times \mathbb{R}^n : t < \frac{q - k \cdot x}{-c} \right\}$$

or

$$D = \left\{ (t,x) \in \mathbb{R} \times \mathbb{R}^n : t > \frac{q - k \cdot x}{-c} \right\}.$$

On D, let the operator L be given by

$$Lu := -u_t + \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + \beta(t,x) u \text{ on } D.$$

For the coefficients of L, we assume: The matrix field $(a_{ij})_{i,j=1}^n$ is uniformly elliptic, continuous and 1-periodic. The coefficient β is continuous. Moreover, for a real number $c < 0$ and a unit vector k, there holds $\beta(t,x) = \beta\left(t + \frac{k \cdot e_i}{c}, x + e_i\right)$ for the standard unit vectors e_i, $i = 1, ..., n$. Let u be a classical solution of the differential inequality $Lu \leq 0$ on D with $u(t,x) = u\left(t + \frac{k \cdot e_i}{c}, x + e_i\right)$ for e_i, $i = 1, ..., n$. We suppose that u achieves its minimum at a point $(t_0, x_0) \in D$. Assume that one of the following three conditions holds:

(i) $\beta \equiv 0$ *in* D,

(ii) $\beta \leq 0$ *in* D *and* $u(t_0, x_0) < 0$,

(iii) $u \geq 0$ *in* D *and* $u(t_0, x_0) = 0$.

Then u is constant on D.

Proof Let $m = u(t_0, x_0)$. If one of the conditions (i), (ii) or (iii) holds, we can apply the usual strong parabolic minimum principle (see for example [11] chapter 2 section 2 theorems 4' and 5). We obtain $u \equiv m$ on $D \cap \{t \leq t_0\}$. Since k is a unit vector, there is a j such that $k_j \neq 0$. In every case, the domain D has the property that $\left(t_0 + l\frac{k\cdot e_j}{c}, x_0 + le_j\right) \in D$ if and only if $(t_0, x_0) \in D$. Because for any $l \in \mathbb{Z}$, u also achieves its minimum m at $\left(t_0 + l\frac{k\cdot e_j}{c}, x_0 + le_j\right)$, we can conclude that $u \equiv m$ on $D \cap \left\{t \leq t_0 + l\frac{k\cdot e_j}{c}\right\}$ for any $l \in \mathbb{Z}$ and therefore $u \equiv m$ on D. ■

6.2. Proof of the monotonicity theorem 2.11

We recall that in theorem 2.11, $(a_{ij})_{i,j=1}^n$ only needs to be a $C^0(T^n)$-matrix field, which is symmetric and uniformly elliptic.

Proof (of theorem 2.11) *Step 1: The strict inequality* $0 < u < 1$. The first step is to show $0 < u < 1$ in $\mathbb{R} \times \mathbb{R}^n$. However, this can be deduced from (2.3) exactly in the way as in the proof of lemma 5.16.

Step 2: The inequality $u_t \geq 0$. Some things are easier described in (s, y)-coordinates than in (t, x)-coordinates and vice versa. Therefore, we will use both coordinate systems in the proof. We define the coordinate transformation $(s, y) := (k \cdot x - ct, y)$. The inverse transformation is given by

$$(t, x) = T(s, y) := \left(\frac{s - k \cdot y}{-c}, y\right).$$

With the help of the transformation T, we define the function U by

$$U(s, y) := u(T(s, y)) = u(t, x).$$

Let us remark that the function U is not necessarily a classical solution of (2.5), because this might require additional regularity. We refer to lemma 5.6. However, the function U does satisfy (2.6). The uniformity of the boundary conditions comes from the fact that u assumes its boundary values locally uniformly in x and the transformed function U is 1-periodic in y.

We define the shifts $U_\lambda(s, y) := U(s+\lambda, y)$ in (s, y)-coordinates and the corresponding shifts $u_\lambda(t, x) := u\left(t + \frac{\lambda}{-c}, x\right)$ in (t, x)-coordinates. Then we have $U_\lambda = u_\lambda \circ T$. For $\lambda > 0$, we consider a function W_λ in (s, y)-coordinates, which is given by

$$W_\lambda(s, y) := U(s + \lambda, y) - U(s, y) = U_\lambda(s, y) - U(s, y).$$

Furthermore, we consider the corresponding function w_λ in (t, x)-coordinates, which is given by

$$w_\lambda(t, x) := u\left(t + \frac{\lambda}{-c}, x\right) - u(t, x) = u_\lambda(t, x) - u(t, x).$$

Again, we have $W_\lambda = w_\lambda \circ T$.

Let θ be as in property (iii) of definition 2.6. There exists $\varepsilon < \theta$, such that

$$f_z(y,z) \le 0 \text{ for all } (y,z) \in T^n \times [1-\varepsilon, 1], \tag{6.1}$$

as assumed in the theorem. Due to the combustion property of f and $0 < \varepsilon < \theta$, there also holds

$$f(y,z) = 0 \text{ for } (y,z) \in T^n \times [0,\varepsilon]. \tag{6.2}$$

Since U satisfies the uniform boundary conditions in (2.6), there exists some $N = N(\varepsilon) > 0$, such that

$$\begin{aligned} |U(s,y) - 1| &< \varepsilon \text{ if } s \ge N, \\ |U(s,y)| &< \varepsilon \text{ if } s \le -N. \end{aligned} \tag{6.3}$$

Step 2a): Three assertions concerning minima. We formulate three assertions:

Assertion 1: There holds either $w_\lambda > 0$ on $\mathbb{R} \times \mathbb{R}^n$ or w_λ has a global minimum.

Assertion 2: If w_λ has a global minimum and $\min w_\lambda = 0$, then $w_\lambda \equiv 0$.

Assertion 3: If w_λ has a global minimum and $w_\lambda(t_0,x_0) = w_\lambda(T(s_0,y_0)) = \min w_\lambda < 0$, then $s_0 \in [-N,N]$.

Assertion 1 follows from the fact that $w_\lambda(T(s,y)) \to 0$ for $s \to \pm\infty$ uniformly in y and that $w_\lambda(T(s,y))$ is 1-periodic in y, as well as the continuity of w_λ. Assertion 2 can be seen as follows: By shift-invariance of (2.3), we obtain for w_λ the equation

$$-w_{\lambda,t} + \sum_{i,j=1}^{n} a_{ij}(x) w_{\lambda,x_i x_j} + f(x,u_\lambda) - f(x,u) = 0. \tag{6.4}$$

We define the coefficient

$$\begin{aligned} \beta_\lambda(t,x) &= \int_0^1 f_z \left(x, \xi u \left(t + \frac{\lambda}{-c}, x \right) + (1-\xi) u(t,x) \right) d\xi \\ &= \int_0^1 f_z \left(x, \xi U(s+\lambda, y) + (1-\xi) U(s,y) \right) d\xi. \end{aligned} \tag{6.5}$$

Using this coefficient, (6.4) can be written as

$$-w_{\lambda,t} + \sum_{i,j=1}^{n} a_{ij}(x) w_{\lambda,x_i x_j} + \beta_\lambda(t,x) w_\lambda = 0. \tag{6.6}$$

Now Assertion 2 follows from (6.6) and lemma 6.1 (iii).

To prove Assertion 3 suppose first that $s_0 \in (-\infty, -N)$. We have $U(s,y) < \varepsilon$ for $s < -N$, which is equivalent to $u(t,x) < \varepsilon < \theta$ for $t < \frac{N - k\cdot x}{-c}$. Therefore, by (6.2), there holds $f(x,u(t,x)) = 0$ on the domain

$$D = \left\{ (t,x) \in \mathbb{R} \times \mathbb{R}^n : t < \frac{N - k \cdot x}{-c} \right\}.$$

Hence, equation (6.4) implies

$$-w_{\lambda,t} + \sum_{i,j=1}^{n} a_{ij}(x) w_{\lambda,x_i x_j} = -f(x, u_\lambda) \leq 0 \text{ on } D.$$

The usual parabolic minimum principle implies (we do not need lemma 6.1 here) that w_λ is constant on $D \cap \{t < t_0\}$. However, this is a contradiction to $w_\lambda(t, x_0) \to 0$ for $t \to -\infty$.

Now suppose $s_0 \in (N, \infty)$. For $s > N$ or equivalently for $t > \frac{N-k\cdot x}{-c}$, we see from (6.5) and (6.1), that $\beta_\lambda(t,x) \leq 0$. Therefore, by (6.6) and lemma 6.1 *(ii)*, we obtain $w_\lambda \equiv w_\lambda(t_0, x_0) < 0$ on $\left\{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t > \frac{N-k\cdot x}{-c}\right\}$. This is a contradiction to $w_\lambda(t, x_0) \to 0$ for $t \to \infty$. The only remaining possibility is $s_0 \in [-N, N]$ as claimed. Hence, Assertion 3 is proved.

Step 2b): Large λ. In this step, we prove that there exists $\lambda_0 > 0$, such that $w_\lambda > 0$ for $\lambda > \lambda_0$. (Actually $w_\lambda \geq 0$ would be sufficient, but the proof is practically the same.) Due to the inequality $0 < u < 1$ (and likewise for U) from step 1, we have

$$\max_{s \in [-N,N], y \in T^n} U(s,y) =: M < 1.$$

Since U satisfies (2.6), we can choose λ_0 sufficiently large, such that $\lambda > \lambda_0$ implies $U(s+\lambda, y) > M$ for $s \in [-N, N], y \in \mathbb{R}^n$. Then we have

$$w_\lambda(T(s,y)) > 0 \text{ for } (s,y) \in [-N,N] \times T^n \text{ and } \lambda > \lambda_0.$$

We continue by showing that we even have $w_\lambda > 0$ on $\mathbb{R} \times \mathbb{R}^n$ for $\lambda > \lambda_0$. For fixed $\lambda > \lambda_0$ either $w_\lambda > 0$ on $\mathbb{R} \times \mathbb{R}^n$ or w_λ has a global nonpositive minimum by assertion 1. In the latter case, we either have $w_\lambda \equiv 0$ by assertion 2, if the minimum has the value 0. But this is a contradiction to $w_\lambda > 0$ in $T([-N,N] \times T^n)$. Or, when we have a negative minimum, we obtain a contradiction from assertion 3. Therefore, the only remaining possibility is $w_\lambda > 0$ on $\mathbb{R} \times \mathbb{R}^n$.

Step 2c): Arbitrary λ. We define

$$\mu := \inf\{\lambda > 0 : w_\lambda \geq 0\}.$$

The set on the right hand side is not empty due to step 2b). If $\mu = 0$, it follows that $u_t \geq 0$, since $\frac{w_\lambda}{\lambda} \to u_t$ pointwise for $\lambda \to 0$. In order to show $\mu = 0$, we assume for a contradiction that $\mu > 0$. The inequality $w_\mu \geq 0$ holds by definition of μ and continuity of $w_\lambda(t,x)$ in λ. By assertion 2, there either holds $w_\mu > 0$ or $w_\mu \equiv 0$. Suppose for a contradiction, that $w_\mu > 0$. Due to the minimality of μ and Assertion 1, there is a sequence $0 < \mu_n \nearrow \mu$ and (s_n, y_n) with

$$w_{\mu_n}(T(s_n, y_n)) = \min_{\mathbb{R} \times T^n} w_{\mu_n}(T(s,y)) < 0.$$

Then $s_n \in [-N, N]$ by Assertion 3. Also, by 1-periodicity in y, the sequence y_n can be chosen such that $y_n \in [0,1]^n$. Consequently, we can find a convergent subsequence of (s_n, y_n, μ_n) and obtain a point (s, y) with $w_\mu(T(s, y)) \leq 0$. This contradicts our assumption $w_\mu > 0$. Therefore, we find $w_\mu \equiv 0$.

That means $u_\mu - u \equiv 0$ or, in the (s, y)-coordinates,

$$U(s + \mu, y) - U(s, y) \equiv 0 \text{ for } (s, y) \in \mathbb{R} \times T^n.$$

In other words, U is periodic in s with μ as period. However, $U(s, y) \to 0$ for $s \to -\infty$ and $U(s, y) \to 1$ for $s \to \infty$. This can only hold if $\mu = 0$. Accordingly, we have $\mu = 0$. This is a contradiction to our assumption $\mu > 0$. Therefore $\mu = 0$ and thus $u_t \geq 0$.

Step 3: The inequality $u_t > 0$. Now that we know that $u_t \geq 0$, we can see the strict inequality $u_t > 0$ by exactly the same argumentation as in lemma 5.17. ∎

6.3. Proof of uniqueness theorem 2.12

Proof (of theorem 2.12) *Step 1: Rescaling.* Suppose for a contradiction $c' < c < 0$. We recall the situation for u and u'. For u, we have

$$-u_t + \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + f(x, u) = 0,$$

$$\lim_{t \to -\infty} u(t, x) = 0, \ \lim_{t \to \infty} u(t, x) = 1 \text{ locally uniformly in } x \in \mathbb{R},$$
$$u(t, x) = u\left(t + \frac{k \cdot e_i}{c}, x + e_i\right) \text{ for the unit vectors } e_i,\ i = 1, ..., n$$
$$\text{and } 0 \leq u \leq 1.$$

For u', we have

$$-u'_t + \sum_{i,j=1}^{n} a_{ij}(x) u'_{x_i x_j} + f(x, u') = 0,$$

$$\lim_{t \to -\infty} u'(t, x) = 0, \ \lim_{t \to \infty} u'(t, x) = 1 \text{ locally uniformly in } x \in \mathbb{R},$$
$$u'(t, x) = u'\left(t + \frac{k \cdot e_i}{c'}, x + e_i\right) \text{ for the unit vectors } e_i,\ i = 1, ..., n$$
$$\text{and } 0 \leq u' \leq 1.$$

We want to rescale u', so that it matches the periodicity condition of u. We define

$$\tilde{u}(t, x) := u'\left(\frac{c}{c'} t, x\right). \tag{6.7}$$

Then

$$\tilde{u}\left(t + \frac{k \cdot e_i}{c}, x + e_i\right) = u'\left(\frac{c}{c'}\left(t + \frac{k \cdot e_i}{c}\right), x + e_i\right) = u'\left(\frac{c}{c'} t + \frac{k \cdot e_i}{c'}, x + e_i\right)$$
$$= u'\left(\frac{c}{c'} t, x\right) = \tilde{u}(t, x).$$

Hence, the periodicity conditions for u and $\tilde{u}$ match. We have $\tilde{u}_t(t,x) = \frac{c}{c'}u'_t\left(\frac{c}{c'}t,x\right)$. Due to $u'_t > 0$ by theorem 2.11 and $c' < c < 0$ by assumption, this leads to

$$-\tilde{u}_t + \sum_{i,j=1}^{n} a_{ij}(x)\tilde{u}_{x_i x_j} + f(x,\tilde{u}) = \left(1 - \frac{c}{c'}\right) u'_t > 0. \tag{6.8}$$

The arguments in the equation are $\tilde{u} = \tilde{u}(t,x)$ and $u'_t = u'_t\left(\frac{c}{c'}t,x\right)$.

Step 2: Applying the sliding domain method. In analogy to the proof of theorem 2.11 we define

$$u_\lambda(t,x) := u\left(t + \frac{\lambda}{-c}, x\right) \text{ and } w_\lambda(t,x) := u_\lambda - \tilde{u} \text{ for } \lambda \in \mathbb{R}.$$

In contrast to the proof of theorem 2.11, we allow here any $\lambda \in \mathbb{R}$ instead of only $\lambda > 0$. By shift invariance of the equation for u in t, we have

$$-u_{\lambda,t} + \sum_{i,j=1}^{n} a_{ij}(x)u_{\lambda,x_i x_j} + f(x,u_\lambda) = 0.$$

Subtracting (6.8) from this equation, we obtain an inequality for w_λ:

$$-w_{\lambda,t} + \sum_{i,j=1}^{n} a_{ij}(x)w_{\lambda,x_i x_j} + f(x,u_\lambda) - f(x,\tilde{u}) = -\left(1 - \frac{c}{c'}\right) u'_t < 0. \tag{6.9}$$

As in the proof of theorem 2.11, we will use both (t,x)-coordinates and (s,y)-coordinates: Define $(s,y) := (k \cdot x - ct, x)$ and the inverse transformation

$$(t,x) = T(s,y) := \left(\frac{s - k \cdot y}{-c}, y\right).$$

With the help of the transformation T, we define

$$U = u \circ T \text{ and } \tilde{U} = \tilde{u} \circ T. \tag{6.10}$$

Moreover, we define the shift U_λ by $U_\lambda(s,y) := U(s+\lambda, y)$. Then also $U_\lambda = u_\lambda \circ T$ holds. As in the proof of theorem 2.11, we note that U is not necessarily a classical solution of (2.5) and neither has $\tilde{U}$ to be a classical solution of the corresponding differential inequality, because this requires additional regularity. However, U and $\tilde{U}$ do satisfy (2.6), in particular the uniform boundary conditions.

Let θ be as in property *(iii)* of definition 2.6. By assumption on f in theorem 2.11, there exists $0 < \varepsilon < \theta$, such that

$$f_z(y,z) \le 0 \text{ for } (y,z) \in T^n \times [1-\varepsilon, 1]. \tag{6.11}$$

Due to the combustion property of f and $0 < \varepsilon < \theta$, there also holds

$$f(y,z) = 0 \text{ for } (y,z) \in T^n \times [0, \varepsilon]. \tag{6.12}$$

Because of the uniform boundary conditions for U and $\tilde{U}$, we can choose $N = N(\varepsilon)$, such that

$$\begin{aligned} |U(s,y)-1| &< \varepsilon \quad \text{and} \quad |U'(s,y)-1| < \varepsilon \text{ for } s > N, \\ |U(s,y)| &< \varepsilon \quad \text{and} \quad |U'(s,y)| < \varepsilon \text{ for } s < -N. \end{aligned} \tag{6.13}$$

Step 2a): Three assertions concerning minima. As in the proof of 2.11, we will formulate three assertions:
Assertion 1: There holds either $w_\lambda > 0$ or w_λ has a global minimum.
Assertion 2: If w_λ has a global minimum and $\min w_\lambda = 0$, then $w_\lambda \equiv 0$.
Assertion 3: If w_λ has a global minimum and $w_\lambda(t_0, x_0) = w_\lambda(T(s_0, y_0)) = \min w_\lambda < 0$, then $s_0 \in [-N, N + \lambda_-]$, where $\lambda_- = \max\{-\lambda, 0\}$.
Assertion 1 follows from the fact that $w_\lambda(T(s,y)) \to 0$ uniformly in y and that $w_\lambda(T(s,y))$ is 1-periodic in y, as well as the continuity of w_λ. Assertion 2 can be seen as follows: We define the coefficient

$$\begin{aligned} \beta_\lambda(t,x) &= \int_0^1 f_z\left(x, \xi u\left(t + \frac{\lambda}{-c}, x\right) + (1-\xi)\tilde{u}(t,x)\right) d\xi \\ &= \int_0^1 f_z\left(x, \xi U(s+\lambda, y) + (1-\xi)\tilde{U}(s,y)\right) d\xi. \end{aligned} \tag{6.14}$$

Then, inequality (6.9) can be written as

$$-w_{\lambda,t} + \sum_{i,j=1}^n a_{ij}(x) w_{\lambda,x_i x_j} + \beta_\lambda(t,x) w_\lambda < 0. \tag{6.15}$$

Now Assertion 2 follows from (6.15) and lemma 6.1 (*iii*). We remark that we did not use the strict inequality in (6.15).

To prove Assertion 3 suppose first that $s_0 \in (-\infty, -N)$. We have $\tilde{U}(s,y) < \varepsilon$ for $s < -N$ or equivalently $\tilde{u}(t,x) < \varepsilon$ for $t < \frac{N-k\cdot x}{-c}$. Therefore, by (6.12), there holds $f(x, u(t,x)) = 0$ on the domain

$$D = \left\{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t < \frac{N - k\cdot x}{-c}\right\}.$$

Then the equation (6.9) implies

$$-w_{\lambda,t} + \sum_{i,j=1}^n a_{ij}(x) w_{\lambda,x_i x_j} = -f(x, u_\lambda) \leq 0 \text{ on } D.$$

The usual parabolic minimum principle implies (we do not need lemma 6.1 here) that w_λ is constant on $D \cap \{t < t_0\}$ with $(t_0, x_0) = T(s_0, y_0)$. This is a contradiction to $w_\lambda(t, x_0) \to 0$ for $t \to -\infty$.

Now suppose $s_0 \in (N + \lambda_-, \infty)$. For $s > N + \lambda_-$, which is equivalent to $t > \frac{(N+\lambda_-)-k\cdot x}{-c}$, we see from (6.11), (6.13) and (6.14), that $\beta_\lambda(t,x) \leq 0$. Therefore, we obtain from (6.15) and lemma 6.1 (*ii*) that $w_\lambda \equiv w_\lambda(t_0, x_0) < 0$ on $\left\{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t > \frac{(N+\lambda_-)-k\cdot x}{-c}\right\}$. This is a contradiction to $w_\lambda(t, x_0) \to 0$ for $t \to \infty$. The only remaining possibility is $s_0 \in [-N, N+\lambda_-]$ as claimed. Hence, Assertion 3 is proved.

Step 2b): Large λ. In this step, we show that there exists $\lambda_0 > 0$, such that $w_\lambda > 0$ for $\lambda > \lambda_0$. (Actually $w_\lambda \geq 0$ would be sufficient.) We have the inequality $0 < u' < 1$ from theorem 2.11. The rescaled function $\tilde{u}$ from (6.7) and the transformed $\tilde{U}$ from (6.10) inherit this inequality. Hence, we have

$$\max_{s\in[-N,N],y\in T^n} \tilde{U}(s,y) =: M < 1.$$

Since U satisfies (2.6), we can choose $\lambda_0 > 0$ sufficiently large, such that $\lambda > \lambda_0$ implies $U(s+\lambda, y) > M$ for all $(s,y) \in [-N,N] \times T^n$. Then there holds

$$w_\lambda(T(s,y)) > 0 \text{ for } (s,y) \in [-N,N] \times T^n \text{ and } \lambda > \lambda_0. \tag{6.16}$$

Now for fixed $\lambda > \lambda_0$ either $w_\lambda > 0$ on $\mathbb{R}\times\mathbb{R}^n$ or w_λ has a global nonpositive minimum by assertion 1. In the latter case, we either have $w_\lambda \equiv 0$ by assertion 2, if the minimum has the value 0. But this is a contradiction to $w_\lambda > 0$ in $T([-N,N]\times T^n)$. Or, when w_λ has a negative minimum, we obtain from assertion 3 that is is taken in $T([-N, N+\lambda_-]\times T^n) = T([-N,N] \times T^n)$ (since $\lambda > \lambda_0 > 0$). This contradicts (6.16). Therefore, the only remaining possibility is $w_\lambda > 0$ on $\mathbb{R} \times \mathbb{R}^n$.

Step 2c): Arbitrary λ. We define

$$\mu = \inf\{\lambda \in \mathbb{R} : w_\lambda \geq 0 \text{ on } \mathbb{R} \times \mathbb{R}^n\}.$$

The set on the right hand side is not empty due to step 2b). There holds $-\infty < \mu \leq \lambda_0$ ($\mu = -\infty$ is not possible due to the boundary conditions for u and $\tilde{u}$). From the definition of μ and the continuity of $w_\lambda(t,x)$ in λ, we obtain $w_\mu \geq 0$. By assertion 2 there either holds $w_\mu > 0$ or $w_\mu \equiv 0$. Suppose for a contradiction that $w_\mu > 0$ holds. By minimality of μ and assertion 1, there is a sequence $\mu - 1 < \mu_n \nearrow \mu$ and (s_n, y_n) with

$$w_{\mu_n}(T(s_n, y_n)) = \min_{\mathbb{R}\times T^n} w_{\mu_n}(T(s,y)) < 0.$$

Then $s_n \in [-N, N+\mu_-+1]$ by Assertion 3. Due to the 1-periodicity in y, the sequence y_n can be chosen such that $y_n \in [0,1]^n$. Consequently, we can find a convergent subsequence of (s_n, y_n, μ_n) and obtain a point (s,y) with $w_\mu(T(s,y)) \leq 0$. This contradicts the assumption $w_\mu > 0$. Therefore, we obtain $w_\mu \equiv 0$.

Step 2d): Obtaining a contradiction. From $w_\mu \equiv 0$, we obtain a contradiction to the strict inequality in (6.15). Therefore, the assumption $c' < c$ must have been false and we obtain $c' \geq c$.

Step 3: Ruling out $c < c'$. By repeating steps 1 - 2d) with c and c' interchanged, we obtain $c \geq c'$ and together $c = c'$.

Step 4: Conclusion of the proof. In step 3 we have found $c = c'$. The entire procedure can be repeated with minor modifications to show $u_\mu = u'$ for some μ directly. The rescaling in step 1 is now unnecessary. Therefore, we can skip step 1. Then steps 2-2c) can be performed with a slight modification. The modification is that the strict inequality sign in (6.15) is replaced with an equality sign. Note that the strict inequality was never used in steps 2-2c), it was only used in step 2d). Therefore steps 2-2c) can be repeated to obtain $w_\mu \equiv 0$ for some $\mu \in \mathbb{R}$. With $t_0 := \mu$, the t_0 of the theorem is found and the proof is finished. ∎

A. Appendix

Lemma A.1 (Berestycki, Hamel, Rossi [7]) *Consider the operator given by*

$$-\mathcal{L}u(x) := -\sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}u(x) - \sum_{i=1}^{n} b_i(x)\partial_i u(x) - c(x)u(x)$$

with $a_{ij}, b_i, c \in L^\infty(\Omega)$ *and* $(a_{ij})_{i,j=1}^n$ *uniformly elliptic. Let* Ω *be a general unbounded domain. Assume that there exist a positive constant* ε *and a nonnegative function* $v \in C^2(\Omega) \cap C(\overline{\Omega})$ *such that:* $-\mathcal{L}v \geq \varepsilon > 0$ *in* Ω *and, if* $\Omega \neq \mathbb{R}^n$, $\inf_{\partial\Omega} v > 0$. *Then we have the following:*

(*i*) $\inf_\Omega v > 0$,

(*ii*) *if* $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *is such that* $\sup_\Omega u < \infty$, $-\mathcal{L}u \leq 0$ *and, in case* $\Omega \neq \mathbb{R}^n$, $u \leq 0$ *on* $\partial\Omega$, *then* $u \leq 0$ *in* Ω.

Proof See [7] lemma 2.1. ∎

Theorem A.2 (Berestycki and Hamel [4]) *Let* Ω *be an open subset of* $\mathbb{R} \times \mathbb{R}^N$. *Let* $(\alpha^{ij})_{1\leq i,j\leq N}$ *be a* $C^1(\overline{\Omega})$ *symmetric matrix field such that there exists* $\sigma > 0$ *with*

$$\forall (X,\xi) \in \overline{\Omega} \times \mathbb{R}^N, \qquad \sum_{1\leq i,j\leq N} \alpha^{ij}(X)\xi_i\xi_j \geq \sigma\,|\xi|^2.$$

Let $(\beta^i)_{1\leq i\leq N}$ *be a* $C^1(\overline{\Omega})$ *vector field and* $f = f(X,u)$ *be a* $C^1_{loc}(\Omega \times \mathbb{R})$ *function such that* $\partial_u f$ *is bounded in* $\Omega \times \mathbb{R}$. *Let* $b \geq 0$ *be such that, for all* $1 \leq i,j \leq N$,

$$\left|\left|\alpha^{ij}\right|\right|_{C^1(\overline{\Omega})} + \left|\left|b^i\right|\right|_{C^1(\overline{\Omega})} + ||\partial_u f||_{L^\infty(\Omega\times\mathbb{R})} \leq b.$$

Let $0 \leq \varepsilon \leq 1$ *and let* u *be a solution of class* $C^3(\Omega) \cap L^\infty_{loc}(\Omega)$ *of the equation*

$$\varepsilon u_{tt} - u_t + \alpha^{ij}(X)u_{ij} + \beta^i(X)u_i + f(X,u) = 0 \text{ in } \Omega. \tag{A.1}$$

Then, for all $X \in \Omega$,

$$|\nabla_x u(X)|^2 \leq C\left(1 + \frac{1}{d(X,\partial\Omega)^2}\right),$$

where

$$C = C_1[||u||_{L^\infty(B_X)}\,(osc_{B_X}(u) + ||f||_{L^\infty(B_X\times[m_x,M_X])}) + ||\nabla_x f||^2_{L^\infty(B_x\times[m_X,M_X])}],$$

$B_X = B_{d(X,\partial\Omega)/2}(X)$, $m_X = \inf_{B_X} u$, *and* $M_X = \sup_{B_X} u$. *The constant* $C_1 = C_1(N,\sigma,b)$ *depends only on* N, σ, *and* b.

Proof See [4] theorem 1.2. ∎

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