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Abstract

In this paper we study the problem of statistical inference for a continuous-time moving average Lévy process of the form

$$Z_t = \int_{\mathbb{R}} \mathcal{K}(t-s)\, dL_s, \quad t \in \mathbb{R}$$

with a deterministic kernel $\mathcal{K}$ and a Lévy process L. Especially the estimation of the Lévy measure ν of L from low-frequency observations of the process Z is considered. We construct a consistent estimator, derive its convergence rates and illustrate its performance by a numerical example. On the technical level, the main challenge is to establish a kind of exponential mixing for continuous-time moving average Lévy processes.

Keywords: moving average, Mellin transform, low-frequency estimation

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1. Introduction

Continuous-time Lévy-driven moving average processes of the form:

$$Z_t = \int_{-\infty}^{\infty} \mathcal{K}(s,t)\, dL_s$$

with a deterministic kernel $\mathcal{K}$ and a Lévy process $(L_t)_{t\in\mathbb{R}}$ build a large class of stochastic processes including semimartingales and non-semimartingales, cf. Basse and Pedersen [1], Basse-O'Connor and Rosinsky [2], Bender, Lindner and Schicks [3], as well as long-memory processes. Starting point was the paper by Rajput and Rosinski [4] providing conditions on the interplay between $\mathcal{K}$ and L such that Z is well defined. Continuous-time Lévy-driven moving average processes provide a unifying approach to many popular stochastic models like Lévy driven Ornstein-Uhlenbeck processes, fractional Lévy processes and CARMA processes. Furthermore, they are the building blocks of more involved models such as Lévy semistationary processes and ambit processes, which are popular in turbulence and finance, cf. Barndorff-Nielsen, Benth and Veraart [5].

Statistical inference for Ornstein-Uhlenbeck processes and CARMA processes is already well-established due to the special structure of the processes, for an overview see Brockwell and Lindner [6], whereas for general continuous-time Lévy driven moving average processes so far only partial results are known in the literature mainly concerning parameters which enter the kernel function, cf. Cohen and Lindner [7] for an approach via empirical moments or Zhang, Lin and Zhang [8] for a least squares approach. Further results concern limit theorems for the power variation, cf. Glaser [9], Basse-O'Connor, Lachieze-Rey and Podolskij [10], which may be used for statistical inference based on high-frequency data.

In this paper we consider a special case of stationary continuous-time Lévy-driven moving average processes of the form $Z_t = \int_{-\infty}^{\infty} \mathcal{K}(s-t)\, dL_s$ and aim to infer the unknown parameters of the driving Lévy process from its low-frequency observations. Our setting especially includes the case of Gamma-kernels of the form $\mathcal{K}(t) = t^{\alpha} e^{-\lambda t} 1_{[0,\infty)}(t)$ with $\lambda > 0$ and $\alpha > -1/2$, which serve as a popular kernel for applications in finance and turbulence, cf. Barndorff-Nielsen and Schmiegel [11]. The special symmetric case of the well-balanced Ornstein-Uhlenbeck process has been discussed in Schnurr and Woerner [12].

In fact, the resulting statistical problem is rather challenging for several reasons. On the one hand, the set of parameters, i.e., the so-called Lévy-triplet of the driving Lévy process contains, in general, an infinite dimensional object, a Lévy measure making the statistical problem nonparametric. On the other hand, the relation between the parameters of the underlying Lévy process (L_t) and those of the resulting moving average process (Z_t) is rather nonlinear and implicit, pointing out to a nonlinear ill-posed statistical problem. It turns out that in Fourier domain this relation becomes exponentially linear and has a form of multiplicative convolution. This observation underlies our estimation procedure, which basically consists of three steps. First, we estimate the marginal characteristic function of the Lévy-driven moving average process (Z_t). Then we estimate the Mellin transform of the second derivative of the log-transform of the characteristic function. Finally, an inverse Mellin transform technique is used to reconstruct the Lévy density of the underlying Lévy process.

The paper is organized as follows. In the next session, we explain our setup and discuss the correctness of our model. In Section 3, we present the estimation procedure. Our main theoretical results related to the rates of convergence of the estimates are given in Section 4. Next, in Section 5, we provide a numerical example, which shows the performance of our procedure. All proofs are collected in the appendix.

2. Setup

In this paper we study a stationary continuous-time moving average (MA) Lévy process $(Z_t)_{t\in\mathbb{R}}$ of the form:

$$Z_t = \int_{-\infty}^{\infty} \mathcal{K}(t-s)\, dL_s, \quad t \in \mathbb{R}, \tag{1}$$

where $\mathcal{K} : \mathbb{R} \to \mathbb{R}_+$ is a measurable function and $(L_t)_{t\in\mathbb{R}}$ is a two-sided Lévy process with the triplet $\mathcal{T} = (\gamma, \sigma^2, \nu)$. As shown in [4], under the conditions

$$\int_{\mathbb{R}} \int_{\mathbb{R}\setminus\{0\}} \left(|\mathcal{K}(s)x|^2 \wedge 1\right) \nu(dx)\, ds < \infty, \tag{2}$$

$$\sigma^2 \int_{\mathbb{R}} \mathcal{K}^2(s)\, ds < \infty, \tag{3}$$

$$\int_{\mathbb{R}} \left| \mathcal{K}(s) \left(\gamma + \int_{\mathbb{R}} x \left(1_{\{|x\mathcal{K}(s)|\le 1\}} - 1_{\{|x|\le 1\}} \right) \nu(dx) \right) \right| ds < \infty \tag{4}$$

the stochastic integral in (1) exists. In what follows, we assume that $\mathcal{K} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and the Lévy measure ν satisfies

$$\int x^2 \nu(dx) < \infty, \tag{5}$$

that is, the Lévy process L has finite second moment. In fact, (3) is trivial in this case; condition (2) directly follows from the inequality

$$\begin{aligned}\int_{\mathbb{R}} \int_{\mathbb{R}\setminus\{0\}} \left(|\mathcal{K}(s)x|^2 \wedge 1\right) \nu(dx)\, ds &\leq \int_{\mathbb{R}} \int_{\mathbb{R}\setminus\{0\}} |\mathcal{K}(s)x|^2\, \nu(dx)\, ds \\ &= \int_{\mathbb{R}} (\mathcal{K}(s))^2\, ds \cdot \int_{\mathbb{R}\setminus\{0\}} x^2 \nu(dx)\, ds.\end{aligned}$$

As to the condition (4), we have

$$\begin{aligned}&\int_{\mathbb{R}} \left| \mathcal{K}(s) \left(\gamma - \int_{\mathbb{R}} x 1_{\{|x|\leq 1\}} \nu(dx) \right) + \int_{\mathbb{R}} x\mathcal{K}(s) 1_{\{|x\mathcal{K}(s)|\leq 1\}} \nu(dx) \right| ds \\ &\qquad = \int_{\mathbb{R}} \left| \mathcal{K}(s) \mathbb{E}\left[L_1\right] - \int_{\mathbb{R}} x\mathcal{K}(s) 1_{\{|x\mathcal{K}(s)|>1\}} \nu(dx) \right| ds \\ &\qquad\qquad \leq |\mathbb{E}[L_1]| \int_{\mathbb{R}} \mathcal{K}(s)\, ds + \int_{\mathbb{R}} \int_{\mathbb{R}} x^2 \left(\mathcal{K}(s)\right)^2 \nu(dx)\, ds.\end{aligned}$$

In the sequel we assume that $\mathcal{K} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and

$$\int x^2 \nu(dx) < \infty. \tag{6}$$

Moreover, under the above assumptions, the process $(Z_t)_{t\in\mathbb{R}}$ is strictly stationary with the characteristic function of the form

$$\Phi(u) := \mathbb{E}\left[e^{\mathrm{i}uZ_t}\right] = \exp\left(\psi(u)\right), \tag{7}$$

where

$$\Psi(u) := \int_{\mathbb{R}} \psi(u\,\mathcal{K}(s))\, ds$$

and

$$\psi(u) := \mathrm{i}u\gamma - \sigma^2 u^2/2 + \int_{\mathbb{R}} \left(e^{\mathrm{i}ux} - 1 - \mathrm{i}ux 1_{\{|x|\leq 1\}}\right) \nu(dx).$$

Our main goal is the estimation of the parameters of the Lévy process L from low-frequency observations of the process Z given that the function $\mathcal{K}$ is known.

3. Mellin transform approach

3.1. Main idea

Let L be a Lévy process with the Lévy triplet (μ, σ^2, ν), where ν is an absolutely continuous w.r.t. to the Lebesgue measure on $\mathbb{R}_+$ and satisfies (6). Denote by $\nu(x)$ the density of ν and set $\overline{\nu}(x) := x^2\nu(x)$. For the sake of clarity we first assume that σ is known and ν is supported on $\mathbb{R}_+$, i.e. L is a sum of a Brownian motion and subordinator. Set

$$\Psi_\sigma(u) := \Psi(u) + \frac{\sigma^2 u^2}{2}\int_{\mathbb{R}} \mathcal{K}^2(x)\, dx.$$

It follows then

$$\Psi''_\sigma(u) \;=\; \int_{\mathbb{R}} \psi''\left(u\mathcal{K}(x)\right)\cdot \mathcal{K}^2(x)\, dx = -\int_{\mathbb{R}} \mathcal{F}[\overline{\nu}]\left(u\mathcal{K}(x)\right)\cdot \mathcal{K}^2(x)\, dx,$$

where $\mathcal{F}[\overline{\nu}]$ stands for the Fourier transform of $\overline{\nu}$. Next, let us compute the Mellin transform of Ψ''_σ:

$$\begin{aligned}
\mathcal{M}\left[\Psi''_\sigma\right](z) &= -\int_{\mathbb{R}_+}\left[\int_{\mathbb{R}} \mathcal{F}[\overline{\nu}]\left(u\mathcal{K}(x)\right)\cdot \mathcal{K}^2(x)\, dx\right] u^{z-1} du \\
&= -\int_{\mathbb{R}}\left[\int_{\mathbb{R}_+} \mathcal{F}[\overline{\nu}]\left(u\mathcal{K}(x)\,\right)\cdot u^{z-1} du\right] \mathcal{K}^2(x)\, dx \\
&= -\mathcal{M}[\mathcal{F}[\overline{\nu}]](z)\cdot\left[\int_{\mathbb{R}} \left(\mathcal{K}(x)\right)^{2-z} dx\right], \qquad (8)
\end{aligned}$$

for all z such that $\int_{\mathbb{R}} (\mathcal{K}(x))^{2-\mathrm{Re}(z)}\, dx < \infty$ and $\int_{\mathbb{R}_+} |\mathcal{F}[\overline{\nu}]\,(v)|\cdot v^{\mathrm{Re}(z)-1} dv < \infty$. Since $\overline{\nu} \in L_1(\mathbb{R}_+)$, it holds

$$\begin{aligned}
\mathcal{M}[\mathcal{F}[\overline{\nu}]](z) &= \int_0^\infty v^{z-1}\left[\int_0^\infty e^{\mathrm{i}xv}\overline{\nu}(x) dx\right] dv \\
&= \mathcal{M}\left[e^{\mathrm{i}\cdot}\right](z)\cdot \mathcal{M}\left[\overline{\nu}\right](1-z).
\end{aligned}$$

Note that the Mellin transform $\mathcal{M}\left[\overline{\nu}\right](1-z)$ is defined for all z with $\mathrm{Re}(z) \in (0,1)$, provided $\overline{\nu}$ is bounded at 0. Next, using the fact that

$$\mathcal{M}[e^{\mathrm{i}\cdot}](z) = \Gamma(z)\left[\cos(\pi z/2) + \mathrm{i}\sin(\pi z/2)\right] = \Gamma(z)e^{\mathrm{i}\pi z/2}$$

for all z with $\mathrm{Re}(z) \in (0,1)$ (see [13], 5.1-5.2), we get

$$\mathcal{M}[\Psi''_\sigma](z) = Q(z)\cdot \mathcal{M}[\overline{\nu}](1-z), \quad \mathrm{Re}(z) \in (0,1),$$

where

$$Q(z) := -\Gamma(z)e^{\mathrm{i}\pi z/2} \int_{\mathbb{R}} (\mathcal{K}(x))^{2-z}\, dx. \tag{9}$$

Finally, we apply the inverse Mellin transform to get

$$\begin{aligned} \overline{\nu}(x) &= \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathcal{M}[\overline{\nu}](z)x^{-z}dz \\ &= \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathcal{M}[\Psi''_\sigma](1-z)}{Q(1-z)} x^{-z}dz \end{aligned} \tag{10}$$

for $c \in (0,1)$. The formula (10) connects the weighted Levy density $\overline{\nu}$ to the characteristic exponent Ψ_σ of the process Z and forms the basis for our estimation procedure.

Remark 1. *If σ^2 is supposed to be unknown, one can estimate it by noting that for a properly chosen bounded kernel w with $\operatorname{supp}(w) \subseteq [1,2]$ and $\int_0^\infty w(u)\, du = 1$,*

$$\begin{aligned} \int_{\mathbb{R}_+} w_n(u)\Psi''(u)\, du &= -\sigma^2 \int_{\mathbb{R}} \mathcal{K}^2(x)\, dx \\ &\quad - \int_{\mathbb{R}} \int_{\mathbb{R}_+} w_n(u)\mathcal{F}[\overline{\nu}]\left(u\mathcal{K}(x)\right)\mathcal{K}^2(x)\, du\, dx \\ &= -\sigma^2 \int_{\mathbb{R}} \mathcal{K}^2(x)\, dx \\ &\quad - \int_{\mathbb{R}} \int_{\mathbb{R}_+} w(u)\mathcal{F}[\overline{\nu}]\left(uU_n\mathcal{K}(x)\right)\mathcal{K}^2(x)\, du\, dx \end{aligned}$$

with $w_n(u) := U_n^{-1}w(u/U_n)$ and some sequence $U_n \to \infty$. Suppose that $|\mathcal{F}[\overline{\nu}](u)| \leq C(1+u)^{-\alpha}$ for all $u \geq 0$ and some constants $\alpha > 0$, $C > 0$, then

$$\left| \int_{\mathbb{R}} \int_{\mathbb{R}_+} w(u)\mathcal{F}[\overline{\nu}]\left(uU_n\mathcal{K}(x)\right)\mathcal{K}^2(x)\, du\, dx \right| \leq \|w\|_\infty \int_{\mathbb{R}} \frac{\mathcal{K}^2(x)}{(1+U_n\mathcal{K}(x))^\alpha}\, dx \to 0$$

as $n \to \infty$. For example, in the case of a one-sided exponential kernel $\mathcal{K}(x) = e^{-x}\mathbb{I}(x \geq 0)$, we derive

$$\int_{\mathbb{R}} \frac{\mathcal{K}^2(x)}{(1+U_n\mathcal{K}(x))^\alpha}\, dx = \frac{1}{U_n^{-2}} \int_0^{U_n} \frac{z}{(1+z)^\alpha}\, dz \lesssim \begin{cases} U_n^{-\alpha}, & \alpha < 2, \\ U_n^{-2}\log(U_n), & \alpha = 2, \\ U_n^{-2}, & \alpha > 2, \end{cases}$$

as $n \to \infty$.

Remark 2. *Let us remark on the general case where the jump part of L is not necessary a subordinator. In this case one can show that*

$$\frac{\mathcal{M}\left[\Psi''_\sigma(-\cdot)\right](u)+\mathcal{M}\left[\Psi''_\sigma(\cdot)\right](u)}{2} = \\ -\left\{\mathcal{M}\left[\overline{\nu}_+\right](1-z)+\mathcal{M}\left[\overline{\nu}_-\right](1-z)\right\} \\ \cdot\cos\left(\frac{\pi z}{2}\right)\Gamma(z)\cdot\int_{\mathbb{R}}\left(\mathcal{K}(x)\right)^{2-z}dx$$

and

$$\frac{\mathcal{M}\left[\Psi''_\sigma(\cdot)\right](u)-\mathcal{M}\left[\Psi''_\sigma(-\cdot)\right](u)}{2i} = \\ -\left\{\mathcal{M}\left[\overline{\nu}_+\right](1-z)-\mathcal{M}\left[\overline{\nu}_-\right](1-z)\right\} \\ \cdot\sin\left(\frac{\pi z}{2}\right)\Gamma(z)\cdot\int_{\mathbb{R}}\left(\mathcal{K}(x)\right)^{2-z}dx,$$

where $\overline{\nu}_+(x)=\nu(x)\cdot 1(x\geq 0)$ and $\overline{\nu}_-(x)=\nu(-x)\cdot 1(x\geq 0)$. Using the above formulas, one can express $\mathcal{M}\left[\overline{\nu}_-\right]$, $\mathcal{M}\left[\overline{\nu}_+\right]$ in terms of $\mathcal{M}\left[\Psi''_\sigma(-\cdot)\right]$, $\mathcal{M}\left[\overline{\nu}_-\right]$ and apply the Mellin inversion formula to reconstruct $\overline{\nu}_-$ and $\overline{\nu}_+$.

3.2. Estimation procedure

Assume that the process Z is observed on the equidistant time grid $\{\Delta, 2\Delta, \ldots, n\Delta\}$. Our aim is to estimate the Lévy density ν of the process L. First we approximate the Mellin transform of the function

$$\Psi''_\sigma(u)=\frac{\Phi''(u)}{\Phi(u)}-\left(\frac{\Phi'(u)}{\Phi(u)}\right)^2+\sigma^2\|\mathcal{K}\|^2_{L^2}$$

via

$$\mathcal{M}_n[\Psi''_\sigma](1-z):=\int_0^{U_n}\left[\frac{\Phi''_n(u)}{\Phi_n(u)}-\left(\frac{\Phi'{}_n(u)}{\Phi_n(u)}\right)^2+\sigma^2\|\mathcal{K}\|^2_{L^2}\right]u^{-z}\,du, \qquad (11)$$

where

$$\Phi_n(u):=\frac{1}{n}\sum_{k=1}^{n}\exp\{\mathrm{i}Z_{k\Delta}u\}$$

and a sequence $U_n \to \infty$ as $n \to \infty$. Second, by regularising the inverse Mellin transform, we define

$$\overline{\nu}_n(x) := \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}V_n}^{c+\mathrm{i}V_n} \frac{\mathcal{M}_n[\Psi''_\sigma](1-z)}{Q(1-z)} x^{-z} dz \tag{12}$$

for some $c \in (0,1)$ and some sequence $V_n \to \infty$, which will be specified later. In the next section we study the properties of the estimate $\overline{\nu}_n(x)$. In particular, we show that $\overline{\nu}_n(x)$ converges to $\overline{\nu}(x)$ and derive the corresponding convergence rates.

4. Convergence

Assume that the following conditions hold.

(AN) For some $A > 0$ and $\alpha \in (0,1), \gamma > 0, c \in (0,1)$ the Lévy density ν fulfills

$$\int_{\mathbb{R}} (1+|y|)^\alpha \left| \mathcal{F}[\overline{\nu}](y) \right| dy \le A, \tag{13}$$

$$\int_{\mathbb{R}} e^{\gamma |u|} \left| \mathcal{M}[\overline{\nu}](c+\mathrm{i}u) \right| du \le A, \tag{14}$$

$$\int_{\mathbb{R}_+} (x \vee x^2)\, \nu(x)\, dx \le A. \tag{15}$$

Theorem 1. *Suppose that (AN) holds, $\mathcal{K}$ is a nonnegative kernel with $\mathcal{K} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Denote $D_j(u) := (\Phi_n^{(j)}(u) - \Phi^{(j)}(u))/\Phi(u)$, $j = 0, 1, 2, \ldots$ Let for any real valued function f on $\mathbb{R}$, $\|f\|_{U_n} := \sup_{u \in [-U_n, U_n]} |f(u)|$. Fix some $K > 0$ and denote*

$$\mathcal{A}_K := \left\{ \max_{j=0,1,2} \|D_j\|_{U_n} \ge K\varepsilon_n \right\}, \qquad K \ge 0.$$

Let ε_n, U_n be two sequences of positive numbers such that $\varepsilon_n \to 0, U_n \to \infty$ as $n \to \infty$, and moreover

$$K\varepsilon_n \left(1 + \|\Psi'_\sigma\|_{U_n}\right) \le 1/2.$$

Choosing ε_n and U_n in such a way is always possible, since $\Psi'_\sigma(0) = \psi'(0)\int \mathcal{K}(s)\,ds$ is finite. Then on the set $\overline{\mathcal{A}}_K$ the estimate $\overline{\nu}_n(x)$ given by (12) with the same $c \in (0,1)$ as in (14) satisfies

$$\sup_{x\in\mathbb{R}_+} \{x^c \,|\overline{\nu}_n(x) - \overline{\nu}(x)|\} \;\leq\; \frac{1}{2\pi}\int_{\{|v|\leq V_n\}} \frac{\Omega_n}{|Q(1-c-iv)|}\,dv + \frac{A}{2\pi}e^{-\gamma V_n},$$

where Q as in (9), V_n is a sequence of positive numbers and

$$\Omega_n = 2K\varepsilon_n U_n^{1-c}\Big(2 + \|\Psi''_\sigma\|_{U_n} + \|\Psi'_\sigma\|^2_{U_n} + 3\,\|\Psi'_\sigma\|_{U_n}\Big) + \left(A + \frac{2^\alpha A}{1-c}\right)\int_{\mathbb{R}}\big[\mathcal{K}(x)\big]^{c+1}\big[1+U_n\mathcal{K}(x)\big]^{-\alpha}\,dx.$$

Remark 3. *Note that in case of $\operatorname{supp}(\nu) \subseteq \mathbb{R}_+$, the sum $2 + \|\Psi''_\sigma\|_{U_n} + \|\Psi'_\sigma\|^2_{U_n} + 3\,\|\Psi'_\sigma\|_{U_n}$ can be uniformly bounded. Indeed,*

$$\big|\psi'(u) - \sigma^2 u\big| \;=\; \big|i\mu + \int_{\mathbb{R}_+} ixe^{iux}\nu(x)dx\big| \leq \mu + \int_{\mathbb{R}_+} x\nu(x)dx \leq \mu + A,$$

by (15). Analogously,

$$\big|\psi''(u) - \sigma^2\big| \;=\; \left|\int_{\mathbb{R}_+} x^2 e^{iux}\nu(x)dx\right| \leq \int_{\mathbb{R}_+} x^2\nu(x)dx \leq A.$$

Therefore

$$\begin{aligned}
\big\|\Psi'_\sigma\big\|_{U_n} &= \left\|\int_{\mathbb{R}}(\psi'(u\mathcal{K}(x)) - \sigma^2 u)\mathcal{K}(x)dx\right\|_{U_n} \leq (\mu + A)\,\|\mathcal{K}\|_{L^1},\\
\big\|\Psi''_\sigma\big\|_{U_n} &= \left\|\int_{\mathbb{R}}(\psi''(u\mathcal{K}(x)) - \sigma^2)\mathcal{K}^2(x)dx\right\|_{U_n} \leq A\|\mathcal{K}\|^2_{L^2},
\end{aligned}$$

where the integrals in the right-hand sides are bounded due to the assumption $\mathcal{K} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Example 1. *Consider a tempered stable Lévy process (L_t) with*

$$\nu(x) = x^{-\eta-1}\cdot e^{-\lambda x}, \quad x \geq 0, \quad \eta \in (0,1), \quad \lambda > 0. \tag{16}$$

Since

$$\mathcal{M}[\overline{\nu}](z) = \lambda^{\eta-z-1}\Gamma(z-\eta+1), \quad \mathsf{Re}(z) > \eta - 1,$$

we derive that (14) *holds for all* $0 < \gamma < \pi/2$ *and* $\alpha > 0$ *due to the asymptotic properties of the Gamma function. Furthermore,*

$$\mathcal{F}[\overline{\nu}](u) = (iu - \lambda)^{-(2-\eta)}\Gamma(2-\eta)$$

and hence (13) *holds for any* $0 < \alpha < 2 - \eta$*. Moreover,* ν *satisfies* (15)*.*

Let us now estimate the probability of the event $\mathcal{A}_K$. The following result holds.

Theorem 2. *Suppose that the following assumptions are fulfilled.*

1. *The kernel* $\mathcal{K}$ *satisfies*

$$\sum_{j=-\infty}^{\infty} \left| \mathcal{F}[\mathcal{K}] \left(2\pi \frac{j}{\Delta} \right) \right| \leq K^* \tag{17}$$

and

$$(\mathcal{K} \star \mathcal{K})(\Delta j) \leq \kappa_0 \, |j|^{\kappa_1} e^{-\kappa_2 |j|}, \quad \forall\, j \in \mathbb{Z} \tag{18}$$

for some positive constants K^*,κ_0, κ_1 *and* κ_2*, such that the all eigenvalues of the matrix* $((\mathcal{K} \star \mathcal{K})(\Delta(j-k))_{k,j\in\mathbb{Z}}$ *are bounded from below and above by two finite positive constants.*

2. *The Lévy measure* ν *satisfies*

$$\int_{|x|>1} e^{Rx} \nu(dx) \leq A_R$$

for some $R > 0$ *and* $A_R > 0$*.*

Then under the choice

$$\varepsilon_n = \sqrt{\frac{\log(n)}{n}} \cdot \exp\left(C_1 \sigma^2 U_n^2 \int (\mathcal{K}(x))^2 \, dx \right)$$

with $C_1 = A/2$*, it holds for any* $K > 0$

$$\mathbb{P}\Big(\mathcal{A}_K\Big) \leq \frac{C_2}{\sqrt{K}} \frac{\sqrt{U_n} n^{(1/4)-C_3 K^2}}{\log^{1/4}(n)},$$

where the positive constants C_1, C_2 *may depend on* K^**,* A_R *and* κ_i*,* $i = 1, 2$*. Hence by an appropriate choice of* K *we can ensure that* $\mathbb{P}(\mathcal{A}_K) \to 0$ *as* $n \to \infty$*.*

Example 2. *Consider the class of symmetric kernels of the form*

$$\mathcal{K}(x) = |x|^r e^{-\rho|x|}, \tag{19}$$

where r is a nonnegative integer and $\rho > 0$. Let us check the assumptions of Theorem 2. We have

$$\mathcal{F}[\mathcal{K}](u) = \Gamma(r+1)\left[\frac{1}{(iu-\rho)^{r+1}} + \frac{1}{(-iu-\rho)^{r+1}}\right]$$

and (17) *holds. Assumption* (18) *is proved in Lemma 2.*

Corollary 1. *Consider again a class of kernels of the form*

$$\mathcal{K}(x) = |x|^r e^{-\rho|x|},$$

where r is a nonnegative integer and $\rho > 0$, and assume that the Lévy measure ν satisfies the set of assumptions (AN). Then

$$\Omega_n \lesssim K\varepsilon_n U_n^{1-c} + U_n^{-\alpha}, \quad n \to \infty$$

and

$$\int_{\{|v|\le V_n\}} \frac{1}{|Q(1-c-iv)|}\,dv \lesssim \begin{cases} V_n^{c+3/2}, & r = 0, \\ V_n^{c+1}, & r \ge 1. \end{cases}$$

As a result we have on $\overline{\mathcal{A}}_K$

$$\sup_{x\in\mathbb{R}_+} \{x^c\,|\overline{\nu}_n(x) - \overline{\nu}(x)|\} \lesssim V_n^{\zeta}\left(\varepsilon_n U_n^{(1-c)} + U_n^{-\alpha}\right) + e^{-\gamma V_n}$$

with $\zeta = c + 1 + I\!I\{r=0\}/2$. By taking $V_n = \varkappa \log(U_n)$ with $\varkappa > \alpha/\gamma$ and $U_n = \theta \log^{1/2}(n)$ for any $\theta < \left(A\int(\mathcal{K}(x))^2\,dx\right)^{-1/2}$,

$$\sup_{x\in\mathbb{R}_+} \{x^c\,|\overline{\nu}_n(x) - \overline{\nu}(x)|\} \lesssim \log^{-\alpha/2}(n), \quad n \to \infty.$$

4.1. Discussion

The proof of Theorem 2 is based on some kind of exponential mixing for the general Lévy-driven moving average processes of the form (1). In fact, such mixing properties were established in the literature only for the processes Z corresponding to the exponential kernel function $\mathcal{K}$, see, e.g. [14]. The assumption of Theorem 2 may seem to be strong, but as shown above, are fulfilled for the family of kernels (19).

5. Numerical example

5.1. Simulation.

Consider the integral $Z_t := \int_{\mathbb{R}} \mathcal{K}(t-s)\, dL_s$ with the kernel $\mathcal{K}(x) = e^{-|x|}$ and the Lévy process

$$L_t = L_t^{(1)} I\!I\{t>0\} - L_{-t}^{(2)} I\!I\{t<0\},$$

constructed from the independent compound Poisson processes

$$L_t^{(1)} \stackrel{d}{=} L_t^{(2)} \stackrel{d}{=} \sum_{k=1}^{N_t} \xi_k,$$

where N_t is a Poisson process with intensity λ, and $\xi_1, \xi_2, ...$ are independent r.v.'s with standard exponential distribution. Note that the Lévy density of the process $L_t^{(1)}$ is $\nu(x) = \lambda e^{-x}$.

For $k = 1, 2$, denote the jump times of $L_t^{(k)}$ by $s_1^{(k)}, s_2^{(k)}, ...$ and the corresponding jump sizes by $\xi_1^{(k)}, \xi_2^{(k)}, ...$ Then

$$Z_t = \sum_{j=0}^{\infty} \mathcal{K}(t - s_j^{(1)})\xi_j^{(1)} - \sum_{j=0}^{\infty} \mathcal{K}(t + s_j^{(2)})\xi_j^{(2)}.$$

In practice, we truncate both series in the last representation by finding a value $x_{max} := \max_{x\in\mathbb{R}_+}\{\mathcal{K}(x) > \alpha\}$ for a given level α. Let

$$\tilde{Z}_t \;=\; \sum_{k\in K^{(1)}} \mathcal{K}(t - s_j^{(1)})\xi_j^{(1)} - \sum_{k\in K^{(2)}} \mathcal{K}(t + s_j^{(2)})\xi_j^{(2)},$$

where

$$\begin{aligned} K^{(1)} &:= \left\{k : \max(0, t - x_{max}) < s_k^{(1)} < t + x_{max}\right\}, \\ K^{(2)} &:= \left\{k : 0 < s_k^{(2)} < \max(0, -t + x_{max})\right\}. \end{aligned}$$

For simulation study, we take $\lambda = 1$, $\alpha = 0.01$ (and therefore $x_{\max} = 6.908$). Typical trajectory of the process $\tilde{Z}_t$ is presented on Figure 1.

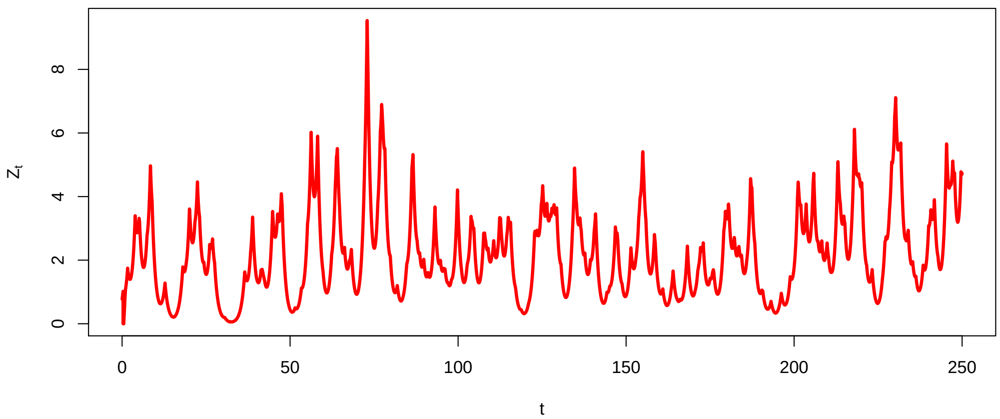

Figure 1: Typical trajectory of the process Z_t constructed from the compound Poisson process with positive jumps.

5.2. General idea of the estimation procedure.

In practice the estimation procedure described in Section 3.1 can be slightly simplified under the assumption that the Lèvy process L has no drift. In this case, one can consider the first derivative of the function $\Psi_\sigma(u)$ instead of the second, and get that

$$\mathcal{M}[\Psi'_\sigma](z) = \widetilde{Q}(z) \cdot \mathcal{M}[\overline{\nu}](1-z), \quad \mathrm{Re}(z) \in (0,1),$$

where

$$\widetilde{Q}(z) = \mathrm{i}\Gamma(z) \exp\{\mathrm{i}\pi z/2\} \int_{\mathbb{R}} (K(x))^{1-z}\, dx.$$

The estimation scheme mainly follows the original idea: we first estimate the Mellin transform of the function Ψ'_σ, and then infer on the Lévy measure ν by applying the Mellin transform techniques. Below we describe these steps in more details.

Estimation of the Mellin transform of $\Psi'(\cdot)$. The most natural estimate is

$$\mathcal{M}_n[\Psi'](1-z) := \mathrm{i} \int_0^{U_n} \frac{\mathrm{mean}(Z_{k\Delta} e^{\mathrm{i}u Z_{k\Delta}})}{\mathrm{mean}(e^{\mathrm{i}u Z_{k\Delta}})} u^{-z} du. \tag{20}$$

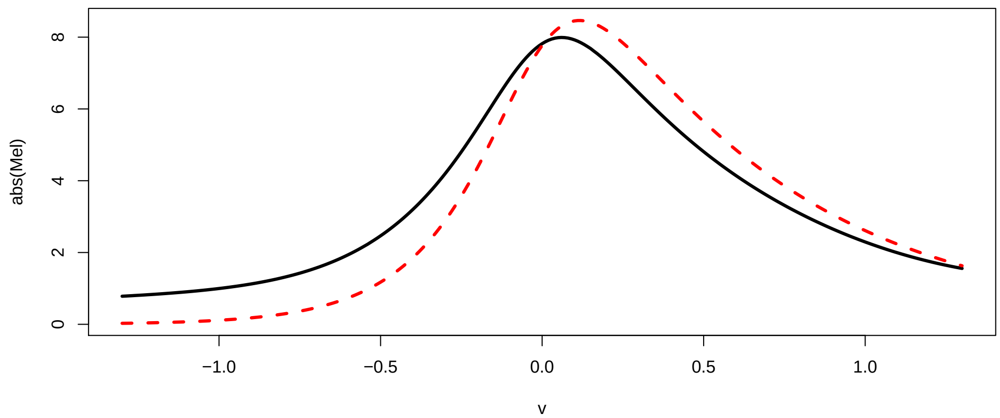

Figure 2: Absolute values of the empirical (black solid) and theoretical (red dashed) Mellin transforms of the function $\Psi'(\cdot)$ depending on the imaginary part of the argument.

In order to improve the numerical rates of convergence of the integral involved in (20), we slightly modify this estimate:

$$\mathcal{M}_n[\Psi'](1-z) := \mathrm{i} \int_0^{U_n} \left[\frac{\operatorname{mean}(Z_{k\Delta} e^{\mathrm{i}uZ_{k\Delta}})}{\operatorname{mean}(e^{\mathrm{i}uZ_{k\Delta}})} - \operatorname{mean}(Z) e^{\mathrm{i}u} \right] u^{-z} du \\ + 2\mathrm{i}\lambda\Gamma(1-z)\exp\{\mathrm{i}\pi(1-z)/2\}.$$

Note that $\mathcal{M}_n[\Psi'](1-z)$ is also a consistent estimate of $\mathcal{M}[\Psi'](1-z)$ (since $\operatorname{mean}(Z) \to 2\lambda$), but involves the integral with better convergence properties. In our case $\mathcal{M}[\overline{\nu}](z) = \lambda\Gamma(1+z)$, and therefore the Mellin transform of the function Ψ' is equal to

$$\mathcal{M}[\Psi'](1-z) = \widetilde{Q}(1-z) \cdot \mathcal{M}[\overline{\nu}](z) = 2\mathrm{i}\lambda \frac{\Gamma(1-z)\Gamma(1+z)}{z} e^{\mathrm{i}\pi(1-z)/2}.$$

We estimate $\mathcal{M}[\Psi'](1-z)$ for $z = c + \mathrm{i}v_k$, where c is fixed and $v_k,\ k = 1, \ldots, K$, are taken on the equidistant grid from $(-V_n)$ to V_n with step $\delta = 2V_n/K$. Typical behavior of the the Mellin transform $\mathcal{M}[\Psi'](1-z)$ and its estimate $\mathcal{M}_n[\Psi'](1-z)$ is illustrated by Figure 2.

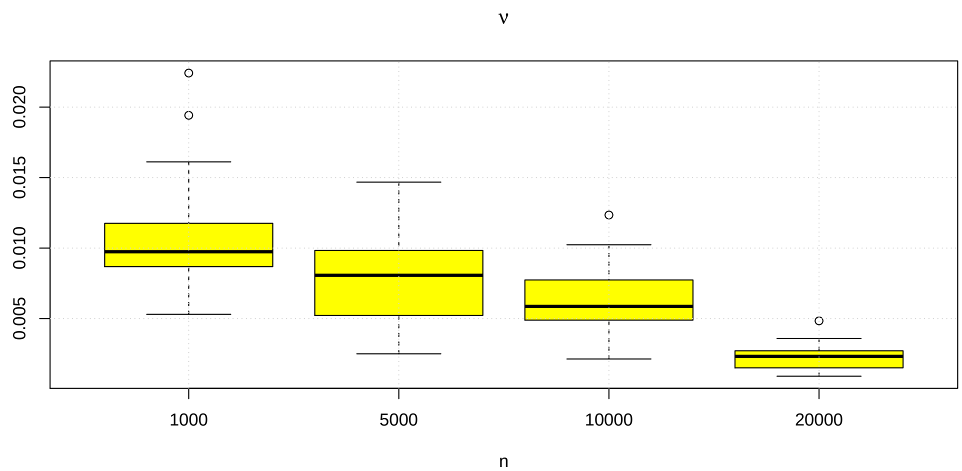

Figure 3: Boxplot of the estimate $\mathcal{R}(\tilde{\nu}_n^\star)$ based on 20 simulation runs.

Estimation of $\nu(x)$. Finally, we estimate the Lévy density $\nu(x)$ by

$$\tilde{\nu}_n(x) := \frac{\delta}{2\pi x} \sum_{k=1}^{K} \mathrm{Re} \left\{ \frac{\mathcal{M}_n[\Psi'](1-c-\mathrm{i}v_k)}{\widetilde{Q}(1-c-\mathrm{i}v)} \cdot x^{-(c+\mathrm{i}v_k)} \right\}$$

and measure the quality of this estimate by the L^2-norm on the interval $[1,3]$:

$$\mathcal{R}(\tilde{\nu}_n) = \int_1^3 \left(\tilde{\nu}_n(x) - \nu(x)\right)^2 dx.$$

To show the convergence of this estimate, we made simulations with different values of n. The parameters U_n and V_n are chosen by numerical optimization of $\mathcal{R}(\tilde{\nu}_n)$. The results of this optimization, for different values of n, as well as the means and variances of the estimate $\tilde{\nu}_n$ based on 20 simulation runs, are given in the next table.

n	U_n	V_n	mean $(\mathcal{R}(\tilde{\nu}_n))$	Var $(\mathcal{R}(\tilde{\nu}_n))$
1000	0.4	1.1	0.0109	$1.62 * 10^{-5}$
5000	0.4	1.2	0.0079	$9.07 * 10^{-6}$
10000	0.5	1.3	0.0063	$6.56 * 10^{-6}$
20000	0.3	1.3	0.0023	$9.15 * 10^{-7}$

The boxplots of this estimate based on 20 simulation runs are presented on Figure 3.

Appendix A. Proof of Theorem 1

Denote $G_j(u) = \Psi^{(j)}_{\sigma,n}(u) - \Psi^{(j)}_{\sigma}(u)$, $j = 1, 2$, where

$$\Psi_{\sigma,n}(u) = \log \Phi_n(u) + \frac{\sigma^2 u^2}{2} \int_{\mathbb{R}} \mathcal{K}^2(x)\, dx.$$

Then

$$\begin{aligned} G_1(u) &= \frac{D_1(u) - D_0(u)\Psi'_\sigma(u)}{1 + D_0(u)}, && \text{(A.1)} \\ G_2(u) &= \frac{\left(\Psi''_\sigma(u) + (\Psi'_\sigma(u))^2 + \Psi'_\sigma(u) G_1(u)\right) D_0(u)}{1 + D_0(u)} \\ &\quad - \frac{(2\Psi'_\sigma(u) + G_1(u))\, D_1(u)}{1 + D_0(u)} + \frac{D_2(u)}{1 + D_0(u)}. && \text{(A.2)} \end{aligned}$$

We have

$$\begin{aligned} \overline{\nu}_n(x) - \overline{\nu}(x) &= \frac{1}{2\pi \mathrm{i}} \int_{c - \mathrm{i}V_n}^{c + \mathrm{i}V_n} \left[\frac{\mathcal{M}_n[\Psi''_\sigma](1 - z) - \mathcal{M}[\Psi''_\sigma](1 - z)}{Q(1 - z)} \right] x^{-z}\, dz \\ &\quad - \frac{1}{2\pi x} \int_{\{|v| \geq V_n\}} \mathcal{M}[\overline{\nu}](c + \mathrm{i}v)\, x^{-(c + \mathrm{i}v)}\, dv \end{aligned}$$

and

$$\begin{aligned} x^c \left(\overline{\nu}_n(x) - \overline{\nu}(x)\right) &= \frac{1}{2\pi} \int_{\{|v| \leq V_n\}} \frac{R_1(v) + R_2(v)}{Q(1 - c - \mathrm{i}v)} x^{-\mathrm{i}v}\, dv \\ &\quad - \frac{1}{2\pi} \int_{\{|v| \geq V_n\}} \mathcal{M}[\overline{\nu}](c + \mathrm{i}v)\, x^{-\mathrm{i}v}\, dv, \qquad \text{(A.3)} \end{aligned}$$

where

$$R_1(v) := \int_0^{U_n} G_2(u) u^{-c - \mathrm{i}v}\, du$$

and

$$R_2(v) := -\int_{U_n}^{\infty} \Psi''_\sigma(u)\, u^{-c - \mathrm{i}v}\, du.$$

We have on $\overline{\mathcal{A}_K}$, under the assumption $K\varepsilon_n(1+\|\Psi'_\sigma\|_{U_n}) \leq 1/2$, that the denominator of the fractions in G_1 and G_2 can be lower bounded as follows:

$$\min_{u\in[-U_n,U_n]} |1+D_0(u)| \geq 1 - \max_{u\in[-U_n,U_n]} |D_0(u)| \geq 1 - K\varepsilon_n \geq 1/2.$$

Therefore,

$$\begin{aligned}
\|G_1\|_{U_n} &\leq 2K\varepsilon_n\left(1+\|\Psi'_\sigma\|_{U_n}\right) \leq 1 \\
\|G_2\|_{U_n} &\leq 2K\varepsilon_n\Big(1+\|\Psi''_\sigma\|_{U_n} + \left\|(\Psi'_\sigma)^2\right\|_{U_n} \\
&\quad + (1+\|\Psi'_\sigma\|_{U_n})\,\|G_1\|_{U_n} + 2\,\|\Psi'_\sigma\|_{U_n}\Big),
\end{aligned}$$

Thus

$$|R_1(v)| \leq 2KU_n^{1-c}\varepsilon_n\Big(2+\|\Psi''_\sigma\|_{U_n} + \|\Psi'_\sigma\|^2_{U_n} + 3\,\|\Psi'_\sigma\|_{U_n}\Big).$$

Since

$$\Psi''_\sigma(u) = -\int_\infty^\infty \mathcal{K}^2(x)\cdot\mathcal{F}[\overline{\nu}](u\mathcal{K}(x))\,dx,$$

it holds for any $z\in\mathbb{C}$

$$\begin{aligned}
\int_{U_n}^\infty \Psi''_\sigma(u)u^{-z}\,dy &= -\int_{-\infty}^\infty \mathcal{K}^2(x)\left[\int_{U_n}^\infty \mathcal{F}[\overline{\nu}](u\mathcal{K}(x))u^{-z}\,du\right]dx \\
&= -\int_{-\infty}^\infty [\mathcal{K}(x)]^{z+1}\left[\int_{U_n\mathcal{K}(x)}^\infty \mathcal{F}[\overline{\nu}](v)v^{-z}\,dv\right]dx.
\end{aligned}$$

Next, for any fixed $x\in\mathbb{R}$, we can upper bound the inner integral in the right-hand side of the last formula:

$$\begin{aligned}
&\left|\int_{U_n\mathcal{K}(x)}^\infty \mathcal{F}[\overline{\nu}](v)v^{-z}\,dv\right| \\
&\qquad\leq (1+U_n\mathcal{K}(x))^{-\alpha}\cdot\int_0^\infty v^{-\mathrm{Re}(z)}(1+v)^\alpha\,|\mathcal{F}[\overline{\nu}](v)|\,dv.
\end{aligned}$$

Due to (13) we get that for any z with $\mathrm{Re}(z)\in(0,1)$ it holds

$$\int_0^\infty v^{-\mathrm{Re}(z)}(1+v)^\alpha\,|\mathcal{F}[\overline{\nu}](v)|\,dv < \frac{\bar{\delta}}{1-\mathrm{Re}(z)} + A$$

with $\bar{\delta} = 2^\alpha \int_{\mathbb{R}_+} x^2 \nu(x) dx \le 2^\alpha A$ due to (15). Finally, we conclude that

$$|R_2(v)| := \left| \int_{U_n}^{\infty} \Psi''_\sigma(y) y^{-c-\mathrm{i}v} \, dy \right| \le \left(\frac{\bar{\delta}}{1-c} + A \right) \times \int_{\mathbb{R}} [\mathcal{K}(x)]^{c+1} \left(1 + U_n \mathcal{K}(x)\right)^{-\alpha} dx.$$

Now an upper bound for the last term in (A.3) follows from the assumption on the Mellin transform of the function $\bar{\nu}$. Indeed, since (14) is assumed, it holds

$$\left| \int_{\{|u| \ge V_n\}} \mathcal{M}[\overline{\nu}](c + \mathrm{i}u) \, x^{-\mathrm{i}u} \, du \right| \le e^{-\gamma V_n} \int_{\{|u| \ge V_n\}} e^{\gamma V_n} \left| \mathcal{M}[\overline{\nu}](c + \mathrm{i}u) \right| \, du \le A e^{-\gamma V_n}.$$

This observation completes the proof.

Appendix B. Proof of Corollary 1

For the sake of simplicity we consider the case $\rho = 1$. We divide the proof into several steps. For the sake of simplicity we assume that either the kernel $\mathcal{K}$ is symmetric or is supported on $\mathbb{R}_+$, so that it suffices to study the integral over $\mathbb{R}_+$.

1. Upper bound for $\Lambda_n := \int_{\mathbb{R}_+} \left[\mathcal{K}(x)\right]^{c+1} \left[1 + U_n \mathcal{K}(x)\right]^{-\alpha} dx.$ Note that the function $\mathcal{K}(x) = x^r e^{-x}$ has two intervals of monotonicity on $\mathbb{R}_+$: $[0, r]$ and $[r, \infty)$. Denote the corresponding inverse functions by $g_1 : [0, r^r e^{-r}] \to [0, r]$

and $g_2 : [0, r^r e^{-r}] \to [r, \infty)$. Then

$$\begin{aligned}
\Lambda_n &= \left(\int_0^r + \int_r^\infty\right) [\mathcal{K}(x)]^{c+1} [1 + U_n \mathcal{K}(x)]^{-\alpha}\, dx \\
&= \int_0^{r^r e^{-r}} w^{c+1} (1 + U_n w)^{-\alpha} g_1'(w) dw \\
&\qquad\qquad + \int_{r^r e^{-r}}^0 w^{c+1} (1 + U_n w)^{-\alpha} g_2'(w) dw \\
&= \int_0^{r^r e^{-r}} w^{c+1} (1 + U_n w)^{-\alpha} G(w) dw \\
&= U_n^{-c-2} \left(\int_0^1 + \int_1^{r^r e^{-r} U_n}\right) y^{c+1} (1 + y)^{-\alpha} \cdot G(y/U_n)\, dy \\
&\qquad\qquad\qquad\qquad =: J_1 + J_2,
\end{aligned}$$

where $G(\cdot) = g_1'(\cdot) - g_2'(\cdot)$. In what follows, we separately analyze the summands J_1 and J_2.

1a. Upper bound for J_1**.**. Clearly, the behavior of the function $G(\cdot)$ at zero is crucial for the analysis of J_1. Since $\mathcal{K}(g_1(y)) = y$ for any $y \in [0, r^r e^{-r}]$, we get $g_1(0) = 0$ and moreover as $y \to 0$,

$$g_1'(y) = \frac{1}{\mathcal{K}'(g_1(y))} = \frac{1}{[g_1(y)]^{r-1} e^{-g_1(y)} (r - g_1(y))} \asymp \frac{1}{r[g_1(y)]^{r-1}}.$$

Analogously, due to $\mathcal{K}(g_2(y)) = y$ for any $y \in [0, r^r e^{-r}]$, we conclude that $\lim_{y\to 0} g_2(y) = +\infty$, and as $y \to 0$

$$\begin{aligned}
g_2'(y) = &\frac{1}{[g_2(y)]^{r-1} e^{-g_2(y)} (r - g_2(y))} \\
&\qquad\qquad \asymp \frac{-1}{[g_2(y)]^r e^{-g_2(y)}} = \frac{-1}{\mathcal{K}(g_2(y))} = \frac{-1}{y}.
\end{aligned}$$

For further analysis of the asymptotic behaviour of $g_1(\cdot)$ we apply the asymptotic iteration method. We are interested in the behaviour of the solution $g_1(y)$ of the equation

$$f(x) := x^r e^{-x} - y = 0$$

as $y \to 0$. Note that the distinction between the solutions is in the asymptotic behaviour as $y \to 0$: $g_1(y) \to 0$, $g_2(y) \to \infty$. Let us iteratively apply the recursion

$$\varphi_{n+1} = \varphi_n - \frac{f(\varphi_n)}{f'(\varphi_n)} = \varphi_n - \frac{\varphi_n^r e^{-\varphi_n} - y}{\varphi_n^{r-1} e^{-\varphi_n} (r - \varphi_n)}, \qquad n = 1, 2, ...$$

Motivated by the power series expansion of the function e^{-x} at zero,

$$x^r e^{-x} = x^r - x^{r+1} + \frac{1}{2} x^{r+2} + o(x^{r+2}),$$

we take for the initial approximation of $g_1(y)$, the function $\varphi_0 = y^{1/r}$. Then

$$\begin{aligned} \varphi_1(y) &= y^{1/r} - \frac{y e^{-y^{1/r}} - y}{y^{(r-1)/r} e^{-y^{1/r}} (r - y^{1/r})} \\ &= y^{1/r} \left(1 - \frac{e^{-y^{1/r}} - 1}{e^{-y^{1/r}} (r - y^{1/r})} \right) \\ &= y^{1/r} + O(y^{2/r}). \end{aligned}$$

Finally, we conclude that as $y \to 0$,

$$G(y) = \frac{1}{r y^{(r-1)/r}} (1 + o(1)) + \frac{1}{y} (1 + o(1)) = \frac{1}{y} (1 + o(1)).$$

Therefore J_1 can be upper bounded as follows:

$$J_1 \leq C_3 U_n^{-c-1} \int_0^1 y^c (1+y)^{-\alpha} (1 + o(1)) \, dy.$$

The integral in the right-hand side converges iff $\int_0^1 y^c dy < \infty$. Since $c \in (0, 1)$, we get $J_1 \lesssim U_n^{-c-1}$.

1b. Asymptotic behaviour of J_2. Analogously, the asymptotic behavior of J_2 crucially depends on the behavior of $G(y)$ at the point $y = r^r e^{-r}$. Note that as $y \to r^r e^{-r}$,

$$g_k'(y) = \frac{1}{\mathcal{K}'(g_k(y))} = \frac{1}{[g_k(y)]^{r-1} e^{-g_k(y)} (r - g_k(y))} \asymp \frac{C}{r - g_k(y)}$$

for $k=1,2$. Taking logarithms of both parts of the equation $x^r e^{-x}=y$ and changing the variables $u=x-r$ and $\delta=r^r e^{-r}-y$, we arrive at the equality

$$u = r\log\left(1+\frac{u}{r}\right) - \log\left(1-\frac{\delta}{r^r e^{-r}}\right).$$

Consider this equality as $u\to 0$ and $\delta\to 0+$, we get

$$u = r\left(\frac{u}{r}-\frac{1}{2}\frac{u^2}{r^2}\right)+\frac{\delta}{r^r e^{-r}}+O\left(\delta^2\right)+O(u^3),$$

and therefore

$$u = \pm\sqrt{2r^{1-r}e^r}\cdot\sqrt{\delta}+O\left(\delta\right)+O(u^{3/2})$$

corresponding to the functions g_1 and g_2. Finally, we conclude

$$|G(y)| \asymp \frac{C\sqrt{2}}{\sqrt{r^{1-r}e^r}}\frac{1}{\sqrt{r^r e^{-r}-y}}, \qquad y\to r^r e^{-r},$$

and therefore

$$J_2 \quad\sim\quad U_n^{-c-3/2}\int_1^{r^r e^{-r}U_n} y^{c+1}\left(1+y\right)^{-\alpha}\cdot\frac{1}{\sqrt{r^r e^{-r}U_n-y}}\,dy.$$

We change the variable in the last integral:

$$z=\sqrt{\frac{r^r e^{-r}U_n-1}{r^r e^{-r}U_n-y}}, \qquad y=r^r e^{-r}U_n+\frac{1-r^r e^{-r}U_n}{z^2},$$

and get with $\widetilde{U}_n=r^r e^{-r}U_n$

$$J_2 \asymp U_n^{-c-3/2}\int_1^{\infty}\left(\widetilde{U}_n+\frac{1-\widetilde{U}_n}{z^2}\right)^{c+1} \cdot\left(1+\widetilde{U}_n+\frac{1-\widetilde{U}_n}{z^2}\right)^{-\alpha}\cdot\frac{z}{\sqrt{\widetilde{U}_n-1}}\frac{2(\widetilde{U}_n-1)}{z^3}dz.$$

Therefore,

$$J_2 \asymp C_4 U_n^{-c-3/2}\widetilde{U}_n^{c+1}\left(\widetilde{U}_n+1\right)^{-\alpha}\sqrt{\widetilde{U}_n-1}, \qquad n\to\infty,$$

with some constant $C_4>0$ and we conclude that $J_2\asymp C_5 U_n^{-\alpha}$ as $n\to\infty$. To sum up, $\Lambda_n \lesssim U_n^{-\min(\alpha,c+1)} = U_n^{-\alpha}$ as $n\to\infty$.

2. ***Upper bound for*** $H_n := \int_{\{|v|\le V_n\}} |Q(1-c-\boldsymbol{i}v)|^{-1}\, dv$. Recall that

$$H_n = \int_{\{|v|\le V_n\}} \frac{e^{-\pi v/2}}{\left|\Gamma(1-c-\mathrm{i}v)\right| \cdot \left|\int_{\mathbb{R}} (\mathcal{K}(x))^{c+1+\mathrm{i}v}\, dx\right|}\, dv$$

Note that for our choice of the function $\mathcal{K}(\cdot)$, it holds for any $z \in \mathbb{C}$

$$\int_{\mathbb{R}} (K(x))^{z}\, dx = 2\int_{\mathbb{R}_+} (x^r e^{-x})^z dx = 2\left[\lim_{R\to+\infty} \int_{\gamma_R(z)} u^{rz} e^{-u} du\right] \cdot z^{-(rz+1)},$$

where $\gamma_R(z)$ is the part of the complex line $\{(x\,\mathrm{Re}(z), x\,\mathrm{Im}(z)),\ x \in [0,R]\}$. Note that due to the Cauchy theorem, for any z with positive real part

$$\int_{\mathbb{R}_+} u^{rz} e^{-\rho u} du = \lim_{R\to+\infty} \int_{\gamma_R(z)} u^{rz} e^{-u} du + \lim_{R\to+\infty} \int_{c_R} u^{rz} e^{-u} du \tag{B.1}$$

with $c_R := \{(R\cos(\theta), R\sin(\theta))\,, \theta \in (0, \arctan(\mathrm{Im}(z)/\,\mathrm{Re}(z))\}$. Since the last limit in (B.1) is equal to 0, we conclude that

$$\int_{\mathbb{R}} (K(x))^{c+1+\mathrm{i}v}\, dx = 2\,\Gamma\Big(r(c+1)+1+\mathrm{i}vr\Big) \cdot e^{-(r(c+1)+1+\mathrm{i}vr)\cdot\log(c+1+\mathrm{i}v)}.$$

Next, using the fact that there exists a constant $\bar{C} > 0$ such that $|\Gamma(\alpha+\mathrm{i}\beta)| \ge \bar{C}|\beta|^{\alpha-1/2} e^{-|\beta|\pi/2}$ for any $\alpha \ge -2, |\beta| \ge 2$ (see Corollary 7.3 from [15]), we get that

$$\frac{e^{-\pi v/2}}{|\Gamma(1-c-\mathrm{i}v)|} \le v^{c-1/2},$$

and moreover

$$\left|\int_{\mathbb{R}} (K(x))^{c+1+\mathrm{i}v}\, dx\right| = 2\frac{|\Gamma(r(c+1)+1+\mathrm{i}vr)|}{((c+1)^2+v^2)^{(r(c+1)+1)/2}\, e^{-vr\arctan(v/(c+1))}}.$$

The asymptotic behavior of the last expression depends on the value r. More precisely,

$$\left|\int_{\mathbb{R}} (K(x))^{c+1+\mathrm{i}v}\, dx\right| \sim \begin{cases} 2\frac{c(vr)^{r(c+1)+1/2} e^{-vr\pi/2}}{((c+1)^2+v^2)^{(r(c+1)+1)/2} e^{-vr\arctan(v/(c+1))}} \sim v^{-1/2}, \\ \qquad\qquad \text{if } r = 1, 2, ..., \\ v^{-1}, \qquad\qquad \text{if } r = 0. \end{cases}$$

as $v \to +\infty$. Finally, we conclude that $H_n \lesssim V_n^{c+1}$, if $r = 1, 2, ...,$ and $H_n \lesssim V_n^{c+3/2}$ if $r = 0$.

Appendix C. Mixing properties of the Lévy-based MA processes

Theorem 3. *Let (L_t) be a Lévy process with Lévy triplet (μ, σ^2, ν), where $\sigma > 0$ and $\operatorname{supp}(\nu) \subseteq \mathbb{R}_+$. Consider a Lévy-based moving average process of the form*

$$Z_s = \int \mathcal{K}(s-t)\, dL_t, \quad s \geq 0$$

with a nonnegative kernel $\mathcal{K}$. Fix some $\Delta > 0$ and denote

$$Z_S := (Z_{j\Delta})_{j \in S}$$

for any subset S of $\{1, \ldots, n\}$. Fix two natural numbers m and p such that $m + p \leq n$. For any subsets $S \subseteq \{1, \ldots, m\}$ and $S' \subseteq \{p+m, \ldots, n\}$, let g and g' be two real valued functions on $\mathbb{R}^{|S|}$ and $\mathbb{R}^{|S'|}$ satisfying

$$\max\left\{ \left\| e^{-R_S^\top \cdot} g \right\|_{L^1}, \left\| e^{-R_{S'}^\top \cdot} g' \right\|_{L^1} \right\} < \infty$$

for some $R_S \in \mathbb{R}_+^{|S|}$ and $R_{S'} \in \mathbb{R}_+^{|S'|}$, and denote $C_\circ := \left\| e^{-R_S^\top \cdot} g \right\|_{L^1} \cdot \left\| e^{-R_{S'}^\top \cdot} g' \right\|_{L^1}$. Suppose that the Fourier transform $\widehat{\mathcal{K}}$ of $\mathcal{K}$ fulfils

$$K^* := \sum_{j=-\infty}^{\infty} \left| \widehat{\mathcal{K}}\left(2\pi \frac{j}{\Delta} \right) \right| < \infty$$

and

$$\int_{|x|>1} e^{R^* x} x^2 \nu(dx) \leq A_{R^*}$$

for $R^* = \frac{\|R_{S \cup S'}\|_\infty K^*}{\Delta}$. Then

$$\begin{aligned} \left| \operatorname{Cov}\left(g(Z_S), g'(Z_{S'}) \right) \right| \;&\leq\; C_R C_\circ \max_{|l|>p} \left(\mathcal{K} \star \mathcal{K} \right)(l\Delta) \\ &\qquad \times \int \|u_{S \cup S'} - i R_{S \cup S'}\|^2 \exp\left(-\sigma^2 \lambda_{S \cup S'}(u) \right) du_{S \cup S'}, \end{aligned} \tag{C.1}$$

where $\lambda_S(u) := \sum_{k,j \in S} u_k u_j (\mathcal{K} \star \mathcal{K})(\Delta(k-j))$ for any $u \in \mathbb{R}^n$ and $C_R =$ $\exp(\sigma^2 \lambda_{S \cup S'}(R_{S \cup S'}))$.

Proof. We have for any $S \subseteq \{1, \dots, n\}$

$$\begin{aligned}\Phi_S(u_S - \mathrm{i}R_S) &:= \mathbb{E}\left[\exp\left(i\sum_{j\in S} u_j Z_{j\Delta} + \sum_{j\in S} R_j Z_{j\Delta}\right)\right] \\ &= \exp\left(\int \psi\left(\sum_{j\in S}(u_j - \mathrm{i}R_j)\,\mathcal{K}(t - j\Delta)\right)\, dt\right),\end{aligned}$$

where $u_S := (u_j \in \mathbb{R},\ j \in S)$ and $R_S := (R_j \in \mathbb{R}_+,\ j \in S)$, provided

$$\mathbb{E}\left[\exp\left(\sum_{j\in S} R_j Z_{j\Delta}\right)\right] < \infty.$$

Denote for any subsets $S \subseteq \{1, \dots, m\}$ and $S' \subseteq \{p+m, \dots, n\}$,

$$\begin{aligned}&D(u_S - \mathrm{i}R_S, u_{S'} - \mathrm{i}R_{S'}) \\ &\qquad := \Phi_{S,S'}(u_S - \mathrm{i}R_S, u_{S'} - \mathrm{i}R_{S'}) - \Phi_S(u_S - \mathrm{i}R_S)\Phi_{S'}(u_{S'} - \mathrm{i}R_{S'}),\end{aligned}$$

where it is assumed that

$$\mathbb{E}\left[\exp\left(\sum_{j\in S\cup S'} R_j Z_{j\Delta}\right)\right] < \infty$$

Then using the elementary inequality $|e^z - e^y| \le (|e^z| \vee |e^y|)\,|y - z|$, $y, z \in \mathbb{C}$, we derive

$$\begin{aligned}&|D(u_S - \mathrm{i}R_S, u_{S'} - \mathrm{i}R_{S'})| \\ &\quad \le \{|\Phi_{S,S'}(u_S - \mathrm{i}R_S, u_{S'} - \mathrm{i}R_{S'})| \vee |\Phi_S(u_S - \mathrm{i}R_S)\Phi_{S'}(u_{S'} - \mathrm{i}R_{S'})|\} \times \\ &\left|\int \left\{\psi\left(\sum_{j\in S\cup S'}(u_j - \mathrm{i}R_j)\,\mathcal{K}(x - j\Delta)\right) - \psi\left(\sum_{j\in S}(u_j - \mathrm{i}R_j)\,\mathcal{K}(x - j\Delta)\right)\right.\right. \\ &\qquad\qquad \left.\left. - \psi\left(\sum_{j\in S'}(u_j - \mathrm{i}R_j)\,\mathcal{K}(x - j\Delta)\right)\right\} dx\right|.\end{aligned}$$

Due to Lemma 1 and the Poisson summation formula, we derive

$$
\begin{aligned}
&|D(u_S - \mathrm{i}R_S, u_{S'} - \mathrm{i}R_{S'})| \\
&\quad \le \{|\Phi_{S,S'}(u_S - \mathrm{i}R_S, u_{S'} - \mathrm{i}R_{S'})| \vee |\Phi_S(u_S - \mathrm{i}R_S)\Phi_{S'}(u_{S'} - \mathrm{i}R_{S'})|\} \times \\
&\qquad\qquad \left[\sum_{j\in S}\sum_{l\in S'} |(u_l - \mathrm{i}R_l)\,(u_j - \mathrm{i}R_j)|\,(\mathcal{K}\star\mathcal{K})\,((j-l)\Delta)\right] \\
&\qquad\qquad\qquad\qquad \times \int y^2 e^{\frac{y\|R\|_\infty K^*}{\Delta}}\,\nu(dy).
\end{aligned}
$$

We have

$$
\begin{aligned}
&\mathrm{Cov}\,(g(Z_S), g'(Z_{S'})) \\
&\quad = \int_{\mathbb{R}_+^{|S|}}\int_{\mathbb{R}_+^{|S'|}} g(x_S)g'(x_{S'})\,(p_{S,S'}(x_S,x_{S'}) - p_S(x_S)p_{S'}(x_{S'}))\;dx_S\,dx_{S'}.
\end{aligned}
$$

and the Parseval's identity implies

$$
\begin{aligned}
\mathrm{Cov}\,(g(Z_S), g(Z_{S'})) \;=\; & \frac{1}{(2\pi)^{|S|+|S'|}}\int_{\mathbb{R}^{|S|}}\int_{\mathbb{R}^{|S'|}} \widehat{g}(\mathrm{i}R_S - u_S)\widehat{g}(\mathrm{i}R_{S'} - u_{S'}). \\
& \times D(u_S - \mathrm{i}R_S, u_{S'} - \mathrm{i}R_{S'})\,du_S\,du_{S'},
\end{aligned}
$$

$\widehat{g}$ stands for the Fourier transform of g. Hence

$$
\begin{aligned}
&|\mathrm{Cov}\,(g(Z_S), g'(Z_{S'}))| \\
&\qquad \le \frac{C_{\mathfrak{i}Rc}}{(2\pi)^{|S|+|S'|}}\int_{\mathbb{R}^{|S|}}\int_{\mathbb{R}^{|S'|}} |D(u_S - \mathrm{i}R_S, u_{S'} - \mathrm{i}R_{S'})|\;du_S\,du_{S'}.
\end{aligned}
$$

Furthermore, for any set $S \in \{1,\dots,n\}$, we have

$$
\int \psi\left(\sum_{j\in S}(u_j - \mathrm{i}R_j)\,\mathcal{K}(s - j\Delta)\right)\,ds \;\;\le\;\; -\sigma^2\lambda_S(u) + \sigma^2\lambda_S(R).
$$

As a result

$$
|\Phi_S(u_S - \mathrm{i}R_S)| \le C_R \exp\left(-\sigma^2\lambda_S(u)\right)
$$

and

$$
\begin{aligned}
|D(u_S - \mathrm{i}R_S, u_{S'} - \mathrm{i}R_{S'})| \;\;\le\;\; & \max_{|l|>p}\,(\mathcal{K}\star\mathcal{K})\,(l\Delta)\sum_{j\in S}\sum_{l\in S'}|(u_l - \mathrm{i}R_l)\,(u_j - \mathrm{i}R_j)| \\
& C_R\,\exp\left(-\sigma^2\lambda_{S\cup S'}(u)\right).
\end{aligned}
$$

□

Lemma 1. *Set*

$$\psi(z) = \int_0^\infty (\exp(zx) - 1)\nu(dx)$$

for any $z \in \mathbb{C}$, *such that the integral* $\int_{|x|>1} \exp(\mathrm{Re}(z)x)\nu(dx)$ *is finite. Then*

$$|\psi(z_1 + z_2) - \psi(z_1) - \psi(z_2)| \leq 2\,|z_1|\,|z_2| \int x^2 e^{x(\mathrm{Re}(z_1)+\mathrm{Re}(z_2))}\nu(dx),$$

provided the integral $\int x^2 e^{x(\mathrm{Re}(z_1)+\mathrm{Re}(z_2))}\nu(dx)$ *is finite.*

Proof. We have

$$\begin{aligned}
&\psi(z_1 + z_2) - \psi(z_1) - \psi(z_2) \\
&\qquad = \int_0^\infty (\exp((z_1 + z_2)x) - \exp(z_1 x) - \exp(z_2 x) + 1)\nu(dx) \\
&\qquad\qquad = \int_0^\infty (\exp(z_1 x) - 1)(\exp(z_2 x) - 1)\nu(dx).
\end{aligned}$$

Since

$$\begin{aligned}
|\exp(z) - 1| &= \left|e^{\mathrm{Re}(z)} e^{i\mathrm{Im}(z)} - 1\right| \\
&= \left|e^{\mathrm{Re}(z)}\left(e^{i\mathrm{Im}(z)} - 1\right) + e^{\mathrm{Re}(z)} - 1\right| \\
&\leq |\mathrm{Im}(z)|\, e^{\mathrm{Re}(z)} + \left|e^{\mathrm{Re}(z)} - 1\right| \\
&\leq (|\mathrm{Re}(z)| + |\mathrm{Im}(z)|)\, e^{\mathrm{Re}(z)} \\
&\leq \sqrt{2}\,|z|\, e^{\mathrm{Re}(z)},
\end{aligned}$$

we get

$$\begin{aligned}
|\psi(z_1 + z_2) - \psi(z_1) - \psi(z_2)| &\leq \int_0^\infty |\exp(z_1 x) - 1|\,|\exp(z_2 x) - 1|\,\nu(dx) \\
&\leq 2\,|z_1|\,|z_2| \int x^2 e^{x(\mathrm{Re}(z_1)+\mathrm{Re}(z_2))}\nu(dx).
\end{aligned}$$

□

Lemma 2. *Let* $\mathcal{K}(x) = |x|^r e^{-\rho|x|}$ *with some* $r \in \mathbb{N} \cup \{0\}$ *and* $\rho > 0$. *Then*

$$\frac{(\mathcal{K} \star \mathcal{K})(\Delta(k - j))}{(\mathcal{K} \star \mathcal{K})(0)} \leq \kappa_0\,(j - k)^{\kappa_1} e^{-\kappa_2(j-k)} \tag{C.2}$$

for all $j > k$ with $\kappa_2 = \Delta\rho,\ \kappa_1 = 2r+1,$ and

$$\kappa_0 \;=\; \frac{(2r+3)}{2}\max\left\{\frac{\Delta^{2r+1}}{2^{2r}}, \max_{m=0,\dots,r}\left\{C_r^m\frac{(r+m)!}{(2r)!}(2\rho\Delta)^{r-m}\right\}\right\}$$

with $C_r^m = \binom{r}{m}$. Moreover, all eigenvalues of the matrix $((\mathcal{K}\star\mathcal{K})(\Delta(k-j)))_{k,j\in\mathbb{Z}}$ are bounded from below and above by two finite positive numbers, provided κ_2 (equivalently ρ) is large enough.

Proof. We have

$$(\mathcal{K}\star\mathcal{K})(0) = 2\int_0^\infty x^{2r}e^{-2\rho x}\,dx = 2(2\rho)^{-2r-1}\Gamma(2r+1)$$

and

$$\int_{\mathbb{R}}\mathcal{K}_{\Delta j}(v)\mathcal{K}_{\Delta k}(v)\,dv = \left(\int_{-\infty}^{\Delta k}+\int_{\Delta k}^{\Delta j}+\int_{\Delta j}^{\infty}\right)\mathcal{K}_{\Delta j}(v)\mathcal{K}_{\Delta k}(v)\,dv$$
$$=: I_1+I_2+I_3,$$

where $\mathcal{K}_t(s) := \mathcal{K}(s-t),\ \forall s,t\in\mathbb{R}_+$. In the sequel we separately consider integrals I_1, I_2, I_3. We have

$$\begin{aligned}
I_1 &= \int_{\Delta j}^{\infty}(v-\Delta j)^r(v-\Delta k)^r e^{-2\rho v+\Delta\rho(j+k)}dv \\
&= \int_{\mathbb{R}_+}u^r(u+\Delta(j-k))^r e^{-2\rho u-\rho\Delta(j-k)}du \\
&= e^{-\rho\Delta(j-k)}\int_{\mathbb{R}_+}u^r\left(\sum_{m=0}^{r}C_r^m u^m(\Delta(j-k))^{r-m}\right)e^{-2\rho u}du \\
&= \left[\sum_{m=0}^{r}C_r^m(r+m)!\frac{\Delta^{r-m}}{(2\rho)^{r+m+1}}(j-k)^{r-m}\right]e^{-\rho\Delta(j-k)},
\end{aligned}$$

because $\int_{\mathbb{R}_+}u^{r+m}e^{-2\rho u}du = 2^{-(r+m+1)}\Gamma(r+m+1) = (2\rho)^{-(r+m+1)}(r+m)!$.

$$I_2 = \int_{\Delta k}^{\Delta j}\left[-(v-\Delta j)(v-\Delta k)\right]^r e^{-\rho\Delta(j-k)}dv$$
$$\le \frac{\Delta^{2r+1}}{2^{2r}}(j-k)^{2r+1}e^{-\rho\Delta(j-k)},$$

because maximum of the quadratic function $f(v) := -(v-\Delta j)(v-\Delta k)$ is attained at the point $v = \Delta(k+j)/2$ and is equal to $(\Delta^2/4)(j-k)^2$.

$$\begin{aligned} I_3 &= \int_{-\infty}^{\Delta k} (\Delta j - v)^r (\Delta k - v)^r e^{2\rho v - \rho\Delta(j+k)} dv = \\ &= \int_{\mathbb{R}_+} (u + \Delta(j-k))^r u^r e^{-2\rho u - \rho\Delta(j-k)} du = I_1. \end{aligned}$$

Next, the well-known Gershgorin circle theorem implies that the minimal eigenvalue of the matrix $((\mathcal{K}\star\mathcal{K})(\Delta(k-j)))_{k,j\in\mathbb{Z}}$ is bounded from below by

$$(\mathcal{K}\star\mathcal{K})(0) - 2\sum_{l>0}(\mathcal{K}\star\mathcal{K})(l) = (\mathcal{K}\star\mathcal{K})(0)\left[1 - 2\kappa_0 \sum_{l>0} l^{\kappa_1} e^{-\kappa_2 l}\right].$$

Note that for any natural number $\kappa_1 > 0$

$$\sum_{l\geq 1} l^{\kappa_1} e^{-\kappa_2 l} = (-1)^{\kappa_1} \frac{d^{\kappa_1}}{dx^{\kappa_1}} \left(\frac{e^{-x}}{1-e^{-x}}\right)\Bigg|_{x=\kappa_2}.$$

Hence the minimal eigenvalue of the matrix $((\mathcal{K}\star\mathcal{K})(\Delta(k-j)))_{k,j\in\mathbb{Z}}$ is bounded from below by a positive number, if κ_2 is large enough. Analogously the maximal eigenvalue of the matrix $((\mathcal{K}\star\mathcal{K})(\Delta(k-j)))_{k,j\in\mathbb{Z}}$ is bounded from above by

$$(\mathcal{K}\star\mathcal{K})(0) + 2\sum_{l>0}(\mathcal{K}\star\mathcal{K})(l) = (\mathcal{K}\star\mathcal{K})(0)\left[1 + 2\kappa_0 \sum_{l>0} l^{\kappa_1} e^{-\kappa_2 l}\right]$$

which is finite. □

Appendix D. Proof of Theorem 2

The rest of the proof of Theorem 2 basically follows the same lines as the proof of Proposition 3.3 from [16]. First note that

$$\max_{|u|\leq U_n} \frac{|\Phi_n(u) - \Phi(u)|}{|\Phi(u)|} \leq \exp\left\{C_1 \sigma^2 U_n^2 \int_{\mathbb{R}} (\mathcal{K}(x))^2 dx\right\} \cdot \max_{|u|\leq U_n} \Big|\Phi_n(u) - \Phi(u)\Big|$$

for n large enough. Next, we separately consider the real and imaginary parts of the difference between $\Phi_n(u)$ and $\Phi(u)$. Denote

$$S_n(u) := n\,\mathrm{Re}\,(\Phi_n(u) - \Phi(u)) = \sum_{k=1}^{n} \left[\cos\left(uZ_{k\Delta}\right) - \mathbb{E}\left[\cos\left(uZ_{k\Delta}\right)\right]\right]$$

Since $S_n(u)$ is a sum of centred real-valued random variables, bounded by 2 and satisfying (C.1) with (C.2), there exist a positive constant c_1 such that

$$\mathbb{P}\left\{|S_n(u)| \geq x\right\} \leq \exp\left\{\frac{-c_1 x^2}{2n + x\log(n)\log\log(n)}\right\}, \quad \forall\, x \geq 0, \tag{D.1}$$

see Theorem 1 from [17]. In order to apply now the classical chaining argument, we divide the interval $[-U_n, U_n]$ by $2J$ equidistant points $(u_j) =: \mathcal{G}$, where $u_j = U_n(-J + j)/J, \quad j = 1, \ldots, 2J$. Applying (D.1), we get for any $x \geq 0$,

$$\mathbb{P}\left\{\max_{u_j \in \mathcal{G}} |S_n(u_j)| \geq x/2\right\} \leq 2J\exp\left\{\frac{-c_1 x^2}{8n + 2x\log(n)\log\log(n)}\right\}. \tag{D.2}$$

Note that for any $u \in [-U_n, U_n]$ there exists a point $u^\star \in \mathcal{G}$ such that $|u - u^\star| \leq U_n/J$ and therefore for all $k \in 1, \ldots, n$,

$$|\cos(uZ_{k\Delta}) - \cos(u^\star Z_{k\Delta})| \leq |Z_{k\Delta}| \cdot |u - u^\star| \leq |Z_{k\Delta}| \cdot U_n/J.$$

Next, we get

$$\begin{aligned} &\mathbb{P}\left\{\max_{|u|\leq U_n} |S_n(u)| \geq x\right\} \\ &\quad \leq \mathbb{P}\left\{\max_{u_j\in\mathcal{G}} |S_n(u_j)| \geq x/2\right\} + \mathbb{P}\left\{\sum_{k=1}^{n} \left(|Z_{k\Delta}| + \mathbb{E}\left[|Z_{k\Delta}|\right]\right) U_n/J \geq x/2\right\}. \end{aligned}$$

Applying (D.2) and the Markov inequality, we arrive at

$$\begin{aligned} &\mathbb{P}\left\{\max_{|u|\leq U_n} |S_n(u)| \geq x\right\} \\ &\qquad\qquad \leq 2J\exp\left\{\frac{-c_1 x^2}{8n + 2x\log(n)\log\log(n)}\right\} + \frac{4U_n}{xJ} n\mathbb{E}\left|Z_\Delta\right|, \end{aligned}$$

where $\mathbb{E}\left[|Z_\Delta|\right] \leq \left(\mathbb{E}\left[|Z_\Delta|^2\right]\right)^{1/2}$ is finite due to (6). The choice

$$J \;=\; \operatorname{floor}\left(\sqrt{\frac{U_n n}{x}\cdot\exp\left\{\frac{c_1x^2}{8n+2x\log(n)\log\log(n)}\right\}}\right),$$

where floor$(\cdot)$ stands for the largest integer smaller than the argument, leads to the estimate

$$\begin{aligned}\mathbb{P}\left\{\max_{|u|\leq U_n}|S_n(u)|\geq x\right\} \;&\leq\; c_2\sqrt{\frac{U_n n}{x}}\exp\left\{\frac{-c_1x^2}{16n+4x\log(n)\log\log(n)}\right\}\\ &\leq\; c_2\sqrt{\frac{U_n n}{x}}\exp\left\{\frac{-c_3x^2}{n}\right\},\end{aligned}$$

which holds for n large enough with $c_2 = 2\left(1+\mathbb{E}\left[|Z_\Delta|\right]\right), c_3 = c_1/17$, provided $x \lesssim n^{1-\varepsilon}$ with some $\varepsilon > 0$. Finally,

$$\begin{aligned}&\mathbb{P}\left\{\max_{|u|\leq U_n}|S_n(u)|\geq x\right\}\\ &\quad\geq \mathbb{P}\left\{\max_{|u|\leq U_n}\left|\operatorname{Re}\left(\frac{\Phi_n(u)-\Phi(u)}{\Phi(u)}\right)\right|\geq\frac{x}{n}\exp\left\{C_1\sigma^2U_n^2\int_{\mathbb{R}}(\mathcal{K}(x))^2dx\right\}\right\}.\end{aligned}$$

Therefore, the choice

$$x = Kn\exp\left\{-C_1\sigma^2U_n^2\int_{\mathbb{R}}(\mathcal{K}(x))^2dx\right\}\varepsilon_n/2 = K\sqrt{n\log(n)}/2$$

with any positive K leads to

$$\mathbb{P}\left\{\max_{|u|\leq U_n}\left|\operatorname{Re}\left(\frac{\Phi_n(u)-\Phi(u)}{\Phi(u)}\right)\right|\geq\frac{K\varepsilon_n}{2}\right\}\leq\frac{\sqrt{2}c_2}{\sqrt{K}}\frac{\sqrt{U_n}n^{(1/4)-c_3(K^2/4)}}{\log^{1/4}(n)}.$$

Since the same statement holds for the imaginary bound of $\left(\Phi_n(u)-\Phi(u)\right)/\Phi(u)$, we arrive at the desired result.

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