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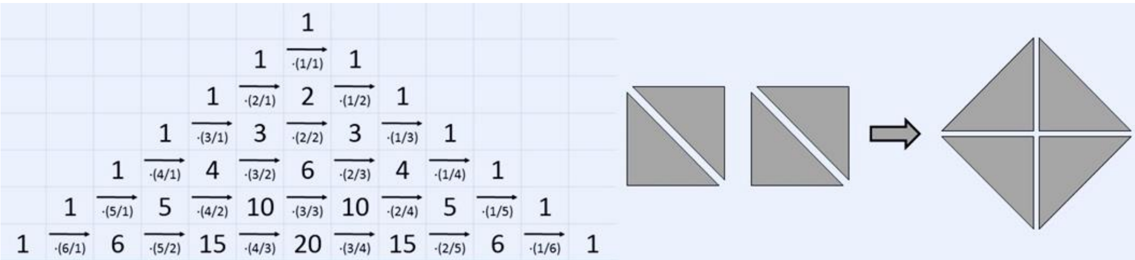
Examples of Elementary Mathematical Discoveries

The working mathematician knows a great variety of strategies in his search for new results. In modern mathematics education, however, the art of mathematical discovery is often reduced to just one of those strategies: induction (see the first example). To oppose this monoculture I try to find and promote convincing alternatives.

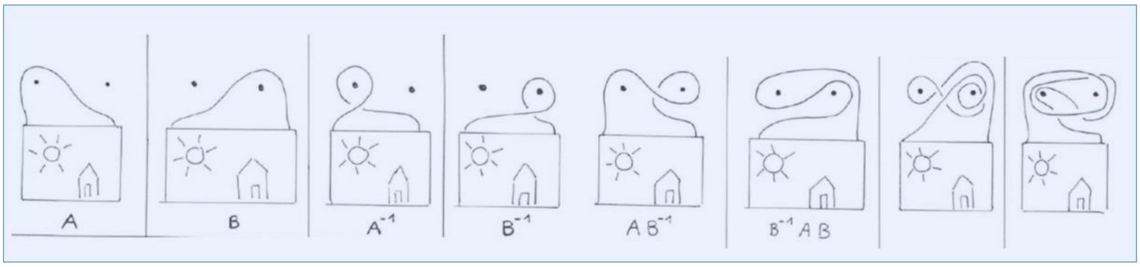
Recognizing a Pattern (Induction) - Binomial Coefficients: With which number do we have to multiply the *k*-th number in the *n*-th row of the left figure below to get the *k+1*-th number in the *n*-th row? Express the *k*-th number in the *n*-th row as a product of fractions.



Learning from a Special Case - Pythagorean Theorem: The right figure above shows how one can put two congruent squares together to form one bigger square. Can you adjust this approach such that it also works for two squares of different size?

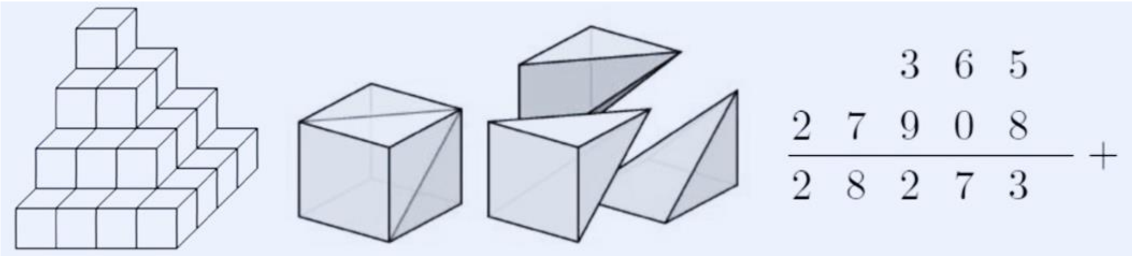
Learning from an Analogous Case - Number of Regular Polytopes: A convex polyhedron built from equilateral triangles can only have *3*, *4* or *5* triangles around each vertex. Hence there are only *3* Platonic solids consisting of triangles. How many tetrahedra fit around an edge of a convex polytope? What does this tell us about the number of regular polytopes consisting of tetrahedra?

Algebraizing a Geometrical Context - Foolish Hanging of a Picture: The following figure shows different hangings of a picture using two nails. Each hanging is encoded by a word in the letters *A* and *B*. Which words belong to the last two hangings? Find a hanging using *n* nails such that the picture falls down if you remove any of the *n* nails.



In Institut für Mathematik und Informatik Heidelberg (Hrsg.), *Beiträge zum Mathematikunterricht 2016* (S. x–y). Münster: WTM-Verlag

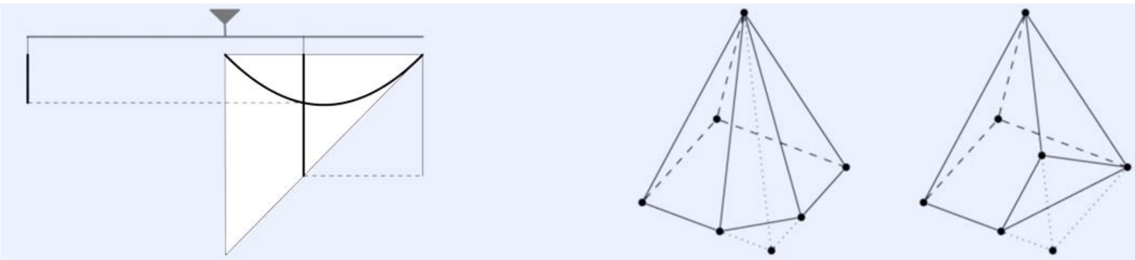
Geometrizing an Arithmetical Context - Formula for the Sum of Squares: The number $1^2 + 2^2 + 3^2 + 4^2$ can be visualized by the pyramid in the left figure below. Use the fact that three such pyramids fit into a cube to derive a formula for the sum of the first *n* square numbers.



Alienating a Familiar Object - *10*-adic Numbers: Two natural numbers given in decimal notation are usually added as indicated in the right figure above. Instead of adding finite sequences of digits we could add infinite sequences of digits by the same method. What properties does this addition have? What can we say about periodic sequences? Do those infinite sequences behave differently in comparison to the natural numbers?

Analysing an Existing Proof - Law of Cosines: What happens if we apply Euclid's well-known proof of the Pythagorean Theorem (found in the first book of the Elements) to an arbitrary (not right-angled) triangle?

Following a Global Recipe - Volume of a Sphere: The left figure below indicates how one can determine the area of a parabolic segment using the law of moments. Try to determine the volume of a sphere in a similar way.



Discovering En Passant (Serendipity) - Euler's Polyhedron Formula: If we wanted to give each polyhedron a name in a systematic way, we could simply call them according their number of faces. Then, however, the triangular prism and the square pyramid would get the same name, as they both have *5* faces. We can easily resolve this by introducing the number of vertices as a distinctive feature. This leads to compound names: *6*-angled pentahedron for the triangular prism and *5*-angled pentahedron for the square pyramid. The right figure above shows two *6*-angled hexahedra, which are combinatorially different, since the left one has a pentagonal face and the right one doesn't. It seems natural to consider the number of edges as an additional distinctive feature. Would this work?