Mathematical Analysis of a spatially coupled Reaction-Diffusion System for Signaling Networks in Biological Cells

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Contents

1 Introduction

This thesis is devoted to the mathematical analysis of a spatially coupled Reaction-Diffusion System for Signaling Networks in Biological Cells. Biological cells consist of a phospholipid bilayer membrane surrounding the cytoplasma and thereby seperating the cell from the ambient environment. Cells receive and process signals from the nearby environment and respond adequatly; these complex mechanism are entitled as Signaling Networks. As a key player in membrane trafficking, signal transduction or cytoskeleton organization, GTP-binding proteins (GTPase) have been identified. These protein families were detected on the cell membrane, on the membrane of inner compartments, the so-called endosomes, and in the cytoplasma. All have in common that they shuttle between an active and inactive state. In addition, these proteins diffuse in the cytosol and with lateral diffusion on the cell membrane. A mathematical model that describes the relative density of these proteins either in their active or inactive state, led to an evolution system of partial differential equations of this type in [RR12]. The goal of this thesis is the analysis of a generalized system in order to prove that this mathematical model is well-posed. This generalized model accounts for additional biological observations, such as generalized reaction rates and position dependent diffusion on the cell membrane but still covers the system from [RR12].

The main challenges and difficulties of this thesis are the interaction of processes in the three-dimensionsal bulk (the cytoplasma) and the two-dimensional boundary surface (cell membrane), the particular coupling in the form of an outflow boundary condition for the volume equation and a source term in the membrane equation and the structure of the nonlinearities. Some authors in the field of mathematical biology consider local membrane processes by assuming a locally flat boundary. Global three-dimensional approaches assume spherical symmetries and other simplifications without coupling effects. In contrast to that, we assume a global approach establishing a rather general domain in $\mathbb{R}^3$ as a model of a cell or an endosome. According to a survey of M. Pierre, see [Pie10], the choice of the nonlinear growth rates is crucial to prove global in time existence results in the field of Reaction-Diffusion Systems. Throughout this thesis we focus on sublinear growth rates, since it covers a major part of common model approaches in mathematical biology and allows to obtain the well-posedness of the system.

In this thesis we consider spatially coupled systems of the following type: A spatially coupled system of three species with specific regularity assumptions on the given bulk will be denoted by (FCRD). It includes surface operators as well as nonlinearities that allow for classical solutions. A generalized version with position dependent diffusion operators in divergence form is denoted by (GFCRD). Afterwards, we assume that the cytosolic diffusion constant of (GFCRD) tends to infinity. Then, the given system formally converges to a reduction, given by a two variable system on the membrane, but including a nonlocal contribution that is a remnant of the spatial coupling. Such a reduced sytem is similar to the so-called Shadow Systems in the analysis of a two variable system in a flat domain, see [Ni11; Kee78].

This thesis is structured as follows. We begin our deductions with an introduction of Signaling Networks and G-proteins. Afterwards, we specify constitutive assumptions for our spatially coupled Reaction-Diffusion Systems and introduce the formal reduction in form of a system similar to Shadow Systems to state the main results of this thesis. Next, we give a survey of the relevant literature and introduce some notation. In Chapter 2 we prove the existence of classical solutions of (FCRD) with a maximum principle and Schauder techniques. In Chapter 3 we show the well-posedness of (GFCRD). We apply these results in Chapter 4 to prove that (GFCRD) converges to a Shadow System. Chapter 5 will contain the summary of this thesis, in the Appendix we list auxiliary results and background material.

Signaling Networks and G-proteins One specific motivation for investigating biological cells and the behaviour of so-called Ras family G-proteins is the observation that in 15% of all human tumors a protein of the Ras G-protein family is mutated [BRW07]. Consisting of five principal families, i.e. Ras, Rho, Rab, Arf and Ran families [WRD05], these G-proteins control a huge variety of signaling, nutrient transport and formation processes of a cell. The receptor sites of these G-proteins can be manipulated by drugs to inhibit signal transduction processes and therefore influence propagation of deseases. The aim of all such experiments and their corresponding biological and mathematical models is to gain deeper insights in the complex interactions of proteins and trace in which mechanisms defects might occur to develop new defect prevention strategies.

The spatio-temporal evolution of G-proteins is determined by diffusion effects in correspondance to their local density and to reactions with other proteins and catalytic effectors. Throughout this thesis we call these G-proteins or *guanine-tri-phosphate-binding proteins* 'GTPases'. Besides the ability of a phosphate molecule to losely attach to proteins, there exists a tight binding of GTPase to *guanine-tri-phosphate (GTP)* on the cell membrane. We call the complex 'GTPase–GTP' the *active state* of GTPase. Active GTPase can for example interact with and activate so-called downstream targets inducing a signal which triggers a cellular response, see for example [SA05]. Moreover, activated GTPase proteins induce cell polarization effects which lead to pattern formation, see [RR14] and the references therein. In both cases, the activation-deactivation cycle is part of a signal transduction chain; Signaling Networks occur.

Reversely, if a GTPase–GTP complex replaces GTP with *guanine-di-phosphate (GDP)* it changes to its *inactive state* denoted by GTPase–GDP, see [Alb+08, p. 179]. Due to observations of [GRA05], there is a high affinity of inactive GTPase–GDP to bind to a *GDP-dissociatior-inhibitor (GDI)* inside the cell, inhibiting GTPase–GDP from binding to the cell membrane. In fact, inactive GTPase in complex with GDI plays a predominant role in the cytosol. How inactive GTPase attaches to the membrane is less clear. A possible mechanism is that a *GDI displacement factor (GDF)* decouples GDI and GTPase–GDP, mediating inactive GTPase to the cell membrane [Pfe03].

In general, the reaction speed on the cell membrane of the described activation and deactivation processes is slow. Observations yield that catalysis quickens these reactions. In particular, there exists a *guanine exchange factor (GEF)*, catalyzing the activation process and a *GTPase activating protein (GAP)* forcing hydrolysis, see [GRG05] and [BRW07]. In addition, as a self-sustaining feedback loop, there exists a cytosolic GEF-effector complex being recruited by GTPase–GTP, resulting in an amplified production of activated GTPase, see [GON06] and [Wed+03]. This mechanistic description was considered in [RR12] for a representant of the G-protein family, namely *Rab5–GTPase*. A schematic illustration of the GTPase cycle can be found in Figure 1.1, see also [RR12].

For the fully coupled models in the latter we assume that the diffusion inside the cytosol is constant at first. Due to observations in [Pos+04], the cytosolic diffusion is much larger than the diffusion on the membrane, therefore the formal assumption of infinite diffusion speed might be used in a model reduction approach. For lateral diffusion on the membrane we observe the following: According to [SK94], the cell membrane has a compartmentalized structure. In those compartments the diffusion speed differs from other compartments. The local diffusion in those compartments is constant and strictly positive, but jumps in diffusion coefficients may occur whenever the sharp border of a compartment will be passed.

This description is already a severe simplification of reality and shows how complex cell biological processes are and how many key players have to be integrated to obtain an adequate model of a signaling process in a cell.

Spatially coupled Reaction-Diffusion Systems We start our mathematical investigations by considering $I \stackrel{\text{def}}{=} (0,T)$ for $T > 0$ as a time-interval of observation and let $\Omega \subset \mathbb{R}^3$ be a bounded domain describing the cell. Throughout this thesis let $\Gamma \stackrel{\text{def}}{=} \partial\Omega$ be the smooth boundary of Ω modelling the cell membrane, and let $\nu : \Gamma \to S^2$ denote the outer normal vector of Ω. Let corresponding time-space cylinders be denoted by

$$\Omega_T \stackrel{\text{def}}{=} \Omega \times (0,T) \quad \text{and} \quad \Gamma_T \stackrel{\text{def}}{=} \Gamma \times (0,T)$$

and the closure of Ω_T be denoted by $\overline{\Omega}_T$. For an intermediate value $t \in (0,T)$ we set $\Omega_t \stackrel{\text{def}}{=} \Omega \times (0,t)$, resp. $\Gamma_t \stackrel{\text{def}}{=} \Gamma \times (0,t)$. The densities of the GTPase proteins are denoted by (V,u,v) with unknowns $V : \overline{\Omega} \times I \to \mathbb{R}$ and $u, v : \Gamma \times I \to \mathbb{R}$. Here, V describes the inactive GDP-bound state inside the cytosol, v denotes the density of the inactive GTPase–GDP state of molecules attached to the membrane and, finally, u describes the active GTPase–GTP state on the cell membrane. All densities depend on the position $x \in \Omega$ and $x \in \Gamma$, respectively and on time $t \in I$. We write $V = V(x,t)$, $u = u(x,t)$ and $v = v(x,t)$. In addition, we assume that nonnegative initial conditions V_0, u_0, and v_0 are given.

Deterministic descriptions of reaction and attachment / deattachment processes are based on ordinary differential equations where biochemical reactions are translated into

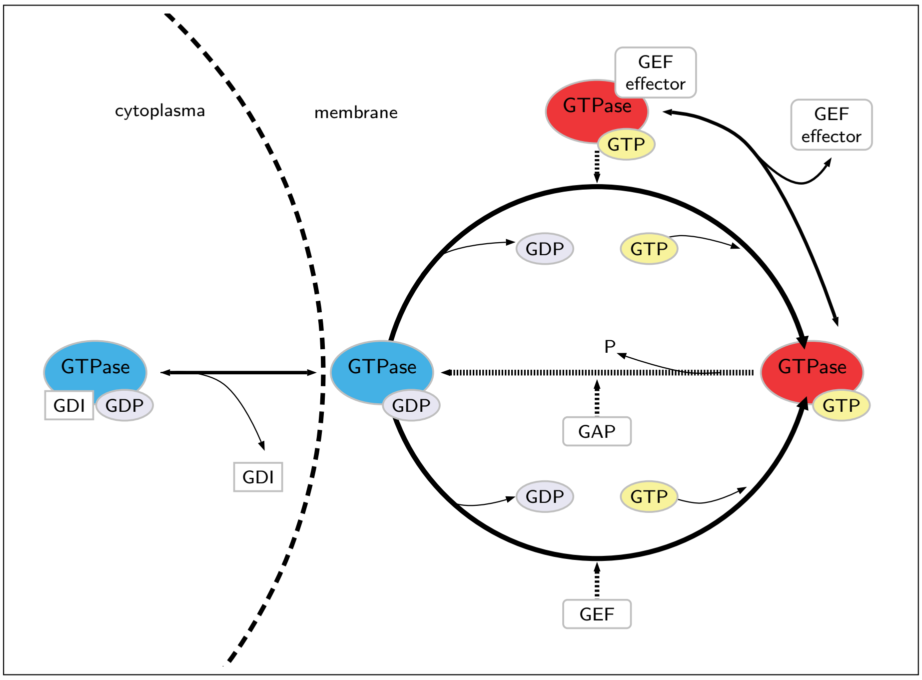

Figure 1.1: *This figure displays a schematic Rab5–GTPase cycle. Rab5–GTPase is inactive in the cytosol and attaches to the membrane loosing the attached GDI-complex and shuttles back to the inside. The GTPase–GDP complex on the cell membrane is also called inactive. On the membrane GTPase–GDP is replaced by GTPase–GTP and becomes active. An additional catalysed activation process is being displayed in form of a GEF–effector complex.*

linear and nonlinear rate laws, see [KS09]. Fundamental reaction laws are for example the *law of mass action*, catalysis reactions modeled by *Michaelis-Menten enzyme kinetics* or cut-off functions that model saturation phenomena called *Langmuir rate law*, see [KS09, p. 1ff.], [Nel08, p. 433ff.], [Kel09, p. 8f.]. We cover these reaction effects on the cell membrane Γ in the following way: For $f_1, f_2 : \mathbb{R}^2_+ \to \mathbb{R}$ and $q_1, q_2 : \mathbb{R}^2_+ \to \mathbb{R}$ we set $f : \mathbb{R}^2 \to \mathbb{R}$ and $q : \mathbb{R}^3 \to \mathbb{R}$, such that

$$f(u,v) \overset{\text{def}}{=} f_1(u,v)v - f_2(u,v)u,$$
$$q(u,v,V) \overset{\text{def}}{=} q_1(u,v)V - q_2(u,v)v$$

holds, where f_i, q_i are nonnegative and bounded. These describe activation / deactivation contributions in the case of f and attachment / deattachment contributions in the case of q, respectively. We will specify assumptions more precisely in Section 1.1.

We are facing diffusion processes on the cell boundary and inside the cell. Let $D > 0$ denote the diffusion constant in Ω. Lateral diffusion on the cell membrane will mathematically be described by the action of the Laplace-Beltrami operator Δ_Γ. With the help of lateral differential operators in divergence form we describe jumps in diffusion coefficients in dependence of the position on the membrane. Let

$$A_u, A_v : \Gamma \to \mathbb{R}^{3\times 3},\ A_u(p), A_v(p) : \mathrm{T}_p\Gamma \to \mathrm{T}_p\Gamma \text{ with } p \in \Gamma$$

be linear tangent operators on the cell boundary Γ and consider the associated differential operator $\nabla_\Gamma \cdot A_u \nabla_\Gamma$ and $\nabla_\Gamma \cdot A_v \nabla_\Gamma$. The Laplace-Beltrami operator then corresponds to the special case $A_u = A_v = \mathrm{Id}$. A detailed introduction to differential geometrical concepts is presented in the Appendix, see Subsection A.1.3.

The most general model we consider in this thesis is the *generalized fully coupled Reaction-Diffusion System* (GFCRD) given by the following equations:

$$\text{(GFCRD)}\begin{cases} \partial_t u = \nabla_\Gamma \cdot (A_u \nabla_\Gamma u) + f_1(u,v)v - f_2(u,v)u & \text{on } \Gamma \times I, \\ \partial_t v = \nabla_\Gamma \cdot (A_v \nabla_\Gamma v) - f_1(u,v)v + f_2(u,v)u & \\ \qquad\quad + q_1(u,v)V - q_2(u,v)v & \text{on } \Gamma \times I, \\ \partial_t V = D\Delta V & \text{in } \Omega \times I, \\ -D\nabla V \cdot \nu = q_1(u,v)V - q_2(u,v)v & \text{on } \Gamma \times I, \\ V(\cdot,0) = V_0 & \text{in } \Omega, \\ v(\cdot,0) = v_0 \text{ and } u(\cdot,0) = u_0 & \text{on } \Gamma. \end{cases}$$

This initial value problem (GFCRD) covers the biological effects discussed before, in particular, we allow for jumps in diffusion constants on the cell membrane. We notice that (GFCRD) is given in divergence form including a Robin-boundary condition. This sometimes called *third-type boundary condition* covers mass transport from the cytosol to the membrane and back to the inside. In particular, the system introduced in [RR12] is covered within (GFCRD). As a first consequence we find that the initial mass of the system given by

$$m_0 \overset{\text{def}}{=} \int_\Omega V_0(x)\mathrm{d}x + \int_\Gamma (u_0(x) + v_0(x))\,\mathrm{d}\sigma \tag{1.1}$$

is being preserved over time, i.e. the time-derivative of $m(t)$ defined by

$$m(t) \overset{\text{def}}{=} \int_\Omega V(x,t)\mathrm{d}x + \int_\Gamma (u(x,t) + v(x,t))\,\mathrm{d}\sigma$$

is zero. Well-posedness of (GFCRD) highly depends on the regularity of the given differential operators A_u and A_v and the growth conditions and regularity assumptions on the nonlinearities f and q. Since we assumed A_u and A_v to be nonsmooth to cover jump effects in diffusion constants, classical solutions for (GFCRD) will in general not exist. We collect sufficient conditions for the differential operators and nonlinearities later to find weak solutions and in particular well-posedness of (GFCRD) in Theorem 1.2 in Section 1.1.

We now consider the case of regular data and a less general formulation that still covers many situations of interest. In this model variant we assume $A_u = \mathrm{Id}$ and $A_v = d \cdot \mathrm{Id}$ for a diffusion constant $d > 1$. Then, as a modification of (GFCRD) we introduce the *fully coupled Reaction-Diffusion System* (FCRD) given by

$$
\text{(FCRD)} \begin{cases}
\partial_t u = \Delta_\Gamma u + f_1(u,v)v - f_2(u,v)u & \text{on } \Gamma \times I, \\
\partial_t v = d\Delta_\Gamma v - f_1(u,v)v + f_2(u,v)u & \\
\qquad + q_1(u,v)V - q_2(u,v)v & \text{on } \Gamma \times I, \\
\partial_t V = D\Delta V & \text{in } \Omega \times I, \\
-D\nabla V \cdot \nu = q_1(u,v)V - q_2(u,v)v & \text{on } \Gamma \times I, \\
V(\cdot,0) = V_0 & \text{in } \overline{\Omega}, \\
v(\cdot,0) = v_0 \ \text{ and } u(\cdot,0) = u_0 & \text{on } \Gamma.
\end{cases}
$$

Even in this case, where diffusion on Γ is simply given by the Laplace-Beltrami operator Δ_Γ on Γ, the different diffusion constants still might lead to blow up effects in finite time, see [Pie10]. Existence and uniqueness is again depending on the choice of the nonlinearities. In Section 1.1 we collect conditions such that (FCRD) admits classical solutions, see Theorem 1.1.

In [RR12] the authors suggest a model reduction by sending the cytosolic diffusion to infinity, i.e. $D \to \infty$. Then, formally, the cytosolic concentration is spatially constant. In this case the variable V only depends on time. The equations on the boundary are no longer spatially coupled to the equation in the cytosol. The prize for this simplification is the appearance of a nonlocal functional on the boundary which is induced by the mass conservation property,

$$
V[u+v](t) = \frac{1}{|\Omega|} m_0 - \frac{1}{|\Omega|} \int_\Gamma (u+v)(t)\mathrm{d}\sigma.
$$

The model reduction of (GFCRD) is then given by

$$
\begin{cases}
\partial_t u = \nabla_\Gamma \cdot (A_u \nabla_\Gamma u) + f_1(u,v)v - f_2(u,v)u \ \text{on } \Gamma \times (0,T), & \text{(1.2a)} \\
\partial_t v = \nabla_\Gamma \cdot (A_v \nabla_\Gamma v) - f_1(u,v)v + f_2(u,v)u & \\
\qquad + q_1(u,v)V[u+v] - q_2(u,v)v \quad \text{on } \Gamma \times (0,T), & \text{(1.2b)} \\
V[u+v](t) = \frac{1}{|\Omega|} m_0 - \frac{1}{|\Omega|} \int_\Gamma (u+v)(t)\mathrm{d}\sigma \quad \text{for } t \in [0,T), & \text{(1.2c)} \\
v(\cdot,0) = v_0 \text{ and } u(\cdot,0) = u_0 \quad \text{on } \Gamma. & \text{(1.2d)}
\end{cases}
$$

We show in Theorem 1.3 rigorously that for $D \to \infty$ the system (GFCRD) tends to (1.2a)–(1.2d). In particular, the case in [RR12] is included in this Theorem.

The introduced models contain several mathematical difficulties such as nonsmooth differential operators in divergence form on a smooth manifold Γ, Reaction-Diffusion Systems on Γ, mass transport via Robin-boundary condition, control of growth rates for nonlinearities and convergence including a nonlocal functional on the boundary Γ. In particular the coupling between bulk and surface partial differential systems keeps us away from using standard theory to study the solvability of the particular models.

1.1 Main results

In this section we state the main results of this thesis. To improve readability, background material considering the occuring function spaces, facts and concepts from differential geometry are collected in the Appendix.

We begin with the notion of classical solutions of (FCRD), adequate system assumptions and the main result for (FCRD).

Classical solutions of (FCRD) The triplet (V, u, v) is called *classical solution* for (FCRD) if $V \in C^{2,1}(\Omega_T) \cap C^1(\overline{\Omega} \times (0,T))$ and $u,\ v \in C^{2,1}(\Gamma_T) \cap C^0(\overline{\Gamma_T})$ holds while (V, u, v) suffices (FCRD) in a pointwise sense. Here, we denote by $C^{2,1}$ the space of functions which are twice continuously differentiable in space and continuously differentiable in time. Let $C^{2+\alpha}$ denote the space of functions such that the second derivative is α-Hölder continuous. We assume sufficiently regular initial data and a compatibility condition.

Assumption 1.1 (Initial conditions for classical solutions)**.** *The initial data is prescribed to be essentially bounded, nonnegative and of class $C^{2+\alpha}$ for some $0 < \alpha < 1$, i.e.*

$$u_0,\ v_0 \in C^{2+\alpha}(\Gamma),\ V_0 \in C^{2+\alpha}(\overline{\Omega}) \text{ and } u_0, v_0, V_0 \geq 0.$$

Moreover, let the compatibility condition

$$-D\nabla V_0(x) \cdot \nu(x) = q_1(u_0(x), v_0(x))V_0(x) - q_2(u_0(x), v_0(x))v_0(x),\ x \in \Gamma \tag{1.3}$$

hold, for $\nu \in S^2$ being the outer normal vector of Ω.

A fundamental assumption to achieve long-time existence results for classical solutions in Reaction-Diffusion Systems are growth rates for the nonlinearities. For counterexamples and positive results on existence see [Pie10]. Here, motivated by the protein reaction laws from [RR12] we assume a sublinear regime and a specific decomposition of the rate laws into activation / deactivation and attachment / deattachement contributions.

Assumption 1.2 (Assumption for nonlinearities in the classical solutions case)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $q : \mathbb{R}^3 \to \mathbb{R}$, such that*

$$f(u, v) = f_1(u, v)v - f_2(u, v)u, \tag{1.4}$$

$$q(u, v, V) = q_1(u, v)V - q_2(u, v)v \tag{1.5}$$

holds. Let $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ *and* $q_1, q_2 : \mathbb{R}^2 \to \mathbb{R}$ *be twice continuously differentiable and nonnegative, i.e.*

$$f_1,\ f_2\ , q_1\ , q_2 \in C^2(\mathbb{R}^2), \tag{1.6}$$

$$f_1,\ f_2,\ q_1,\ q_2 \geq 0 \text{ on } \mathbb{R}^2. \tag{1.7}$$

In addition, let $C_q, C_f \geq 0$ *be constants, such that* $0 \leq q_j \leq C_q$ *and* $0 \leq f_j \leq C_f$ *for* $j = 1, 2$ *holds.*

Let $H^{2+\alpha,(2+\alpha)/2}$ denote parabolic Hölder spaces of class $2+\alpha$. These spaces consist of functions such that the second derivative in space is still α-Hölder continuous and the first derivative in time is $\alpha/2$-Hölder continuous, see for example [LSU68]. A more detailed description of parabolic Hölder spaces is presented in the Appendix.

Well-posedness of a system of partial differential equations in the sense of Hadamard's definition is to show the existence of a solution, the uniqueness of solutions and the continuous dependency on data, see [Eva10, S. 7]. Chapter 2 is devoted to the following result for (FCRD) on well-posedness.

Theorem 1.1 (Classical solutions for (FCRD))**.** *For* $T > 0$ *and* $0 < \alpha < 1$*, let initial data* (V_0, u_0, v_0) *satisfy Assumption 1.1. Moreover, let the nonlinearities of system* (FCRD) *be given as in Assumption 1.2. Then, the fully coupled system* (FCRD) *has a unique, nonnegative, classical solution* (V, u, v) *of parabolic Hölder class* $H^{2+\alpha,(2+\alpha)/2}$*. Moreover, classical solutions of* (FCRD) *depend continuously on its initial data. Therefore, the system* (FCRD) *is well-posed.*

Well-posedness of (GFCRD) As the main result we find the well-posedness for weak solutions of (GFCRD). Here, we can allow for weaker assumptions on initial data and nonlinearities.

Assumption 1.3 (Initial conditions for weak solutions)**.** *Let initial conditions be essentially bounded, nonnegative and* L^2*-interable, i.e.*

$$V_0 \in L^2(\Omega) \quad \text{and} \quad u_0, v_0 \in L^2(\Gamma)$$

with $u_0, v_0, V_0 \geq 0$.

Assumption 1.4 (Nonlinearities in the case of weak solutions)**.** *Let the nonlinearities* $f : \mathbb{R}^2 \to \mathbb{R}$ *and* $q : \mathbb{R}^3 \to \mathbb{R}$ *still be decomposed as in* (1.4) *and* (1.5) *and let the nonlinearities* $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ *and* $q_1, q_2 : \mathbb{R}^2 \to \mathbb{R}$ *be bounded and nonnegative, there exist constants* $C_q, C_f \geq 0$*, such that* $0 \leq q_j \leq C_q$ *and* $0 \leq f_j \leq C_f$ *holds for* $j = 1, 2$*. We further assume that the nonlinearities* $f_1,\ f_2,\ q_1,\ q_2$ *are Lipschitz continuous on* $\mathbb{R}^2$*,*

$$f_1, f_2, q_1, q_2 \in C^{0,1}(\mathbb{R}^2).$$

Assumption 1.5. *Let the linear differential operators $\mathcal{A}_u$ and $\mathcal{A}_v$ of the form*

$$\mathcal{A}_u w = \nabla_\Gamma \cdot (A_u \nabla_\Gamma w) \quad \textit{and} \quad \mathcal{A}_v w = \nabla_\Gamma \cdot (A_v \nabla_\Gamma w)$$

be uniformly elliptic on Γ. This means that $A_u, A_v : \Gamma \to \mathbb{R}^{3\times 3}$ with $A_u(p), A_v(p) : T_p\Gamma \to T_p\Gamma$ for every $p \in \Gamma$ satisfy the following ellipticity condition: there exist constants $c_u, c_v > 0$, such that

$$\xi \cdot A_u(p)\xi \geq c_u|\xi|^2 \quad \textit{and} \quad \xi \cdot A_v(p)\xi \geq c_v|\xi|^2 \tag{1.8}$$

holds for every $\xi \in T_p\Gamma$ and every $p \in \Gamma$. Moreover, let A_u and A_v be measurable and essentially bounded, i.e. there exist constants $C_{A_u}, C_{A_v} > 0$, such that

$$\|A_u\|_{L^\infty(\Gamma,\mathbb{R}^{3\times 3})} < C_{A_u} \quad \textit{and} \quad \|A_v\|_{L^\infty(\Gamma,\mathbb{R}^{3\times 3})} < C_{A_v} \tag{1.9}$$

holds.

We cannot expect classical solution in this context according to the fact that A_u and A_v are not differentiable, i.e. no pointwise solution concept exists. The concept of weak solutions takes the usual form for parabolic divergence form operators. For a precise formulation see Section 3.1. The following result will be proven in Chapter 3. It guarantees the well-posedness of (GFCRD) for weak solutions.

Theorem 1.2 (Well-posedness for (GFCRD) in a weak sense)**.** *For $T > 0$ and given initial conditions (V_0, u_0, v_0) let* (GFCRD) *satisfy Assumptions 1.3–1.5. Then, there exists a unique solution of* (GFCRD)*, i.e. a solution triplet (V, u, v) in $L^2(0, T; H^1(\Omega)) \times (L^2(0, T; H^1(\Gamma)))^2$. Moreover, solutions are nonnegative, essentially bounded on $(0, T)$, depend continuously on the initial data and are L^2-continuous. Therefore, the generalized system* (GFCRD) *is well-posed.*

Convergence to a Shadow System We have introduced a so-called Shadow System reduction for (GFCRD) in (1.2a)–(1.2d). Such systems lie in between an ODE reduction and the fully coupled model of partial differential equations. We show in Chapter 4 rigorously that (GFCRD) tends to (1.2a)–(1.2d).

Theorem 1.3 (Convergence to a Shadow System)**.** *Let Assumptions 1.3–1.5 hold and let $(D_k)_{k\in\mathbb{N}}$ be a sequence of diffusion coefficients and $(V_k, u_k, v_k)_{k\in\mathbb{N}}$ be a sequence of solutions of* (GFCRD) *with D replaced by D_k. Then, a subsequence of $(V_k, u_k, v_k)_{k\in\mathbb{N}}$ converges to a weak solution $(V_\infty, u_\infty, v_\infty)$ of* (1.2a)–(1.2d)*.*

1.2 Survey of literature

Biological observations on cell polarization, long-term behavior and diffusion effects on cell membranes led to a huge variety of mathematical models in the past decades. In particular Reaction-Diffusion Systems were set up to reasonably demonstrate and explain the observed effects, see [GP08; Wed+03] and the references therein. The biomathematical models evolved from basic two species models in $\mathbb{R}^2$ and basic reaction rates towards multispecies approaches in $\mathbb{R}^3$ with spatially coupled, nonsmooth diffusion operators on cell membranes, see for example [GP08; Gie+15; MCV15; MS15]. Besides biomathematical models, several publications present a linear stability analysis or numerical simulations to varify the occurance of Turing pattern formation or cell polarization properties, see for example [RR12; Gie+15; MCV15]. Rigorous existence results and well-posedness of solutions are more or less left as open questions. This thesis focusses on existence and well-posedness results for a specific spatially coupled Reaction–Diffusion System which is motivated by the above mentioned biological obeservations. In this survey we give a brief overview of mathematical difficulties in Reaction–Diffusion Systems and specific bulk-surface coupled systems and finally discuss model reductions in form of so-called Shadow Systems.

Mathematical difficulties Whereas existence of classical solutions for short-times is often rather easy, the development of adequate a priori estimates is the key step to find global in time solutions of Reaction-Diffusion Systems. This will mainly be achieved by maximum principles, invariant region approaches, abstract semigroup theory or uniform L^p-estimates, see for example [Smo83; Ali11]. We point out that these approaches are not providing a priori estimates in every case. According to [Pie10], even in rather simple systems blow-up effects may occur in finite time. Assumptions as nonnegativity and mass conservation are also not sufficient to ensure global existence. Besides different diffusion constants and growth rates of nonlinearities, in our particular case spatial coupling effects and nonflat metrics determine the behavior of Reaction–Diffusion Systems and makes it even more difficult to establish a priori estimates.

Bulk-surface coupling As a first basic example for the analysis of coupling effects between bulk and surface processes a sample system was stated in [ER13], where a stationary diffusion equation in the bulk was coupled to a linear stationary diffusion equation on the cell membrane with a linear Robin boundary condition. Such an elliptic problem can be seen as a toy problem for steady-states in parabolic initial-boundary problems. This system allows for unique solutions of class $H^2(\Omega) \times H^2(\Gamma)$ according to Lax-Milgram techniques and regularity theory.

In a recent preprint a general bulk surface Reaction–Diffusion System of similar type as (FCRD) is analyzed , cf. [MS15]. The respective Reaction–Diffusion System is defined on a bounded domain with smooth boundary. Several species inside the bulk may diffuse with different diffusion constants than the species attached to the boundary. The diffusion equations are coupled via a mass transport boundary condition of Robin-type. The authors show that if the given nonlinearities satisfy some particular polynomial growth condition and conservation of mass, then there exists a unique componentwise nonnegative, global classical solution, see [MS15, Theorem 3.3, p. 7]. In comparison to (FCRD), the considered Reaction–Diffusion System is generalized in the number of species, the dimension of the given domain and source/sink contributions for the bulk equation. The assumptions in [MS15] are different from ours. Super-linear growth is allowed in some of the variables but at the same time more restrictive assumptions on the structure of the nonlinearities are imposed. In particular, the assumptions in [MS15] are in general not satisfied for (FCRD). Moreover, only smooth coefficients and classical solutions are considered.

Cell polarization has motivated many further biomathematical models with bulk surface interactions in the past years, see for example [Mar+07]. Extending [GP08], the authors of [Gie+15] introduced a two-dimensional model where an annulus represents the two-dimensional section of a cell with the outer boundary part refering to the cell membrane and the inner boundary part is an inner cell compartment like an organelle. In a very similar manner compared to the nonsmooth diffusion operators in (GFCRD), the authors describe a single species reaction and diffusion equation on the outer membrane with mass transport to the cytosol. There is no flux to the inner boundary part, the diffusion operator inside the cell is also assumed to be nonsmooth. The focus of [Gie+15] lies more on numerical simulations for a model reduction and not on rigorous proofs on existence and uniqueness of this particular model but it shows the recent relevance and necessity for analyzing systems with nonsmooth differential operators.

Shadow Systems A complexity reduction for Reaction–Diffusion Systems in flat space called Shadow Systems were first introduced in [Kee78]. These reductions are well-studied for the Gierer–Meinhardt system, see for example [NL09]. A Shadow System reduction of a Reaction–Diffusion System is a formal limit of a multiple variable Reaction–Diffusion System where one of the diffusion coefficients is considered to be infinitely large. To figure out the different analytical behaviour of system reductions and the appearance of Shadow System we return to $\mathbb{R}^n$ models. The authors in [Mar+16] consider a system of two coupled Reaction-Diffusion equations in a given domain with zero flux boundary condition. This system has a unique nonnegative global-in-time solution under reasonable initial conditions. A vanishing diffusion coefficient then implies that the solution blows-up in finite time, so-called *diffusion-driven blow-up* occurs. This is an example for the occurance of a Shadow System revealing completely different analytical properties compared to the initial system.

We already mentioned the model proposed in [GP08] to study cell polarization properties. The framework we consider in our fully coupled Reaction-Diffusion System (FCRD) was motivated by the model proposed in [RR14]. Very similar, [GP08] and [RR14] introduce an asymptotic model reduction with a nonlocal term based on mass conservation. The results suggests that in both cases a diffusion-driven pattern formation occurs due to the fact that bulk diffusion is much larger than lateral diffusivity. With the results in Theorem 1.3 we find a rigorous justification of the asymptotic model reduction in [RR14].

1.3 Notation

For the sake of improved readability, background material and auxiliary results are collected in the Appendix. Still, it is necessary to introduce conventions on notation in this section. Let $w : \Omega \to \mathbb{R}$. If w is integrable over Ω, then we make use of the notation

$$\int_\Omega w \, \mathrm{d}\mathcal{L}^3 = \int_\Omega w(x) \, \mathrm{d}x = \int_\Omega w.$$

For Γ as above we denote by σ the surface area measure on Γ. If $\hat{w} : \Gamma \to \mathbb{R}$ is integrable with respect to σ, then

$$\int_\Gamma \hat{w}(x) \, \mathrm{d}\sigma(x) = \int_\Gamma \hat{w} \, \mathrm{d}\sigma = \int_\Gamma \hat{w}$$

denotes integration over Γ. Let $\tilde{w} : \Gamma \times I \to \mathbb{R}$, then integration over $\Gamma_T = \Gamma \times (0, T)$ for integrable $\tilde{w}$ will be achieved by Tonelli's Theorem for measure spaces, i.e.

$$\int_{\Gamma_T} \tilde{w}(x,t) \, \mathrm{d}(\sigma \otimes \mathcal{L}^1) = \int_0^T \int_\Gamma \tilde{w}(x,t) \, \mathrm{d}\sigma(x) \, \mathrm{d}\mathcal{L}^1(t) = \int_\Gamma \int_0^T \tilde{w} \, \mathrm{d}\mathcal{L}^1 \mathrm{d}\sigma = \int_{\Gamma_T} \tilde{w}.$$

2 Existence Theory for classical solutions

In this chapter we investigate the existence of classical solutions (V, u, v) for the fully coupled Reaction-Diffusion system given by

$$
\text{(FCRD)}\begin{cases}
\partial_t u = \Delta_\Gamma u + f_1(u,v)v - f_2(u,v)u & \text{on } \Gamma \times (0,T),\\
\partial_t v = d\Delta_\Gamma v - f_1(u,v)v + f_2(u,v)u & \\
\qquad + q_1(u,v)V - q_2(u,v)v & \text{on } \Gamma \times (0,T),\\
\partial_t V = D\Delta V & \text{in } \Omega \times (0,T),\\
-D\nabla V \cdot \nu = q_1(u,v)V - q_2(u,v)v & \text{on } \Gamma \times (0,T),\\
V(\cdot,0) = V_0 & \text{in } \overline{\Omega},\\
v(\cdot,0) = v_0 \ \text{ and } u(\cdot,0) = u_0 & \text{on } \Gamma
\end{cases}
$$

for given $T > 0$. Let Ω and Γ be given as in Section 1.1 and let Assumptions 1.1 and 1.2 hold as it was described in Section 1.1. The main result of this Chapter is to prove Theorem 1.1. The proof is divided into four major steps, see Section 2.1–Section 2.4. First, we give a detailed outline of the proof, introduce auxiliary problems and find a priori estimates. Then, we deduce the existence and uniqueness of solutions of a nonlinear parabolic system on the boundary Γ without spatial coupling to the bulk in suitable spaces with estimates in parabolic Hölder norms. In Section 2.3 we prove that there exist unique solutions of an initial-boundary value problem with mass-transport boundary condition in Ω for given input data on the boundary and obtain suitable estimates. We finish the proof of the existence result from Theorem 1.1 in Section 2.4 with the help of Schauder's Fixed-Point Theorem. Complementary, we deduce a result on uniform L^∞-bounds and continuous dependance on initial data.

2.1 Auxiliary problems and a priori estimates

First, we introduce auxiliary problems and a priori estimates. The idea is to rescale the system (FCRD) by multiplying the variables (V, u, v) with a time-dependent exponential factor $e^{-\lambda t}$, for fixed $\lambda > 0$ which will be specified later. In this rescaled framework we denote variables by $(\tilde{V}, \tilde{u}, \tilde{v})$. We specify auxiliary problems and deduce a priori estimates in this section. We begin with a detailed outline of the proof of Theorem 1.1.

2.1.1 Outline of the proof

We want to show that (FCRD) has a unique classical solution (V, u, v) on a time-interval $(0, T)$ for a given finite $T > 0$. Therefore, we multiply the variables (V, u, v) with a time-dependent exponential factor $e^{-\lambda t}$ for fixed $\lambda > 0$ that will be specified later. We consider $\tilde{V} : \overline{\Omega} \times [0, T] \to \mathbb{R}$, $\tilde{u}, \tilde{v} : \Gamma \times [0, T] \to \mathbb{R}$ with:

$$\tilde{V}(x,t) \stackrel{\text{def}}{=} e^{-\lambda t} V(x,t) \qquad \text{for } x \in \overline{\Omega},\ t \in [0,T], \tag{2.1}$$

$$\tilde{u}(x,t) \stackrel{\text{def}}{=} e^{-\lambda t} u(x,t) \qquad \text{for } x \in \Gamma,\ t \in [0,T], \tag{2.2}$$

$$\tilde{v}(x,t) \stackrel{\text{def}}{=} e^{-\lambda t} v(x,t) \qquad \text{for } x \in \Gamma,\ t \in [0,T]. \tag{2.3}$$

We justify in Subsection 2.1.2 that (FCRD) is then equivalent to

$$\begin{cases} \partial_t \tilde{u} = \Delta_\Gamma \tilde{u} + \tilde{f}_1(t,\tilde{u},\tilde{v})\tilde{v} - \tilde{f}_2(t,\tilde{u},\tilde{v})\tilde{u} - \lambda \tilde{u} & \text{on } \Gamma \times (0,T), \quad (2.4a) \\ \partial_t \tilde{v} = d\Delta_\Gamma \tilde{v} - \tilde{f}_1(t,\tilde{u},\tilde{v})\tilde{v} + \tilde{f}_2(t,\tilde{u},\tilde{v})\tilde{u} & \\ \qquad + \tilde{q}_1(t,\tilde{u},\tilde{v})\tilde{V} - \tilde{q}_2(t,\tilde{u},\tilde{v})\tilde{v} - \lambda \tilde{v} & \text{on } \Gamma \times (0,T), \quad (2.4b) \\ \partial_t \tilde{V} = D\Delta \tilde{V} - \lambda \tilde{V} & \text{in } \Omega \times (0,T), \quad (2.4c) \\ -D\nabla \tilde{V} \cdot \nu = \tilde{q}_1(t,\tilde{u},\tilde{v})\tilde{V} - \tilde{q}_2(t,\tilde{u},\tilde{v})\tilde{v} & \text{on } \Gamma \times (0,T), \quad (2.4d) \\ \tilde{V}(\cdot,0) = V_0 & \text{in } \overline{\Omega}, \quad (2.4e) \\ \tilde{u}(\cdot,0) = u_0 \text{ and } \tilde{v}(\cdot,0) = v_0 & \text{on } \Gamma. \quad (2.4f) \end{cases}$$

where the nonlinearities are now explicitly time-dependent, see a corresponding definition in (2.9)–(2.10).

We establish an updating procedure to prove that (2.4a)–(2.4f) has a unique solution $(\tilde{V}, \tilde{u}, \tilde{v})$ of parabolic Hölder class $H^{2+\alpha,(2+\alpha)/2}$. Therefore, we start with a given function $\tilde{V} : \overline{\Omega} \times [0, T] \to \mathbb{R}$ with $\tilde{V} \in H^{\beta,\beta/2}(\overline{\Omega_T})$ for a Hölder coefficient $0 < \beta \le \alpha$ to be chosen later. Here, according to Assumption 1.1 we assume that the initial data $\tilde{V}(\cdot, 0) = V_0$ is of class $C^{2+\alpha}(\overline{\Omega})$. A possible choice for $\tilde{V}$ would be to continue the initial profile V_0 to a function which is constant in time on $(0, T)$ and multiply it afterwards with $e^{-\lambda t}$.

We consider $\tilde{V}$ as input data for the following system of nonlinear partial differential equations on the boundary

$$\begin{cases} \partial_t \tilde{u} = \Delta_\Gamma \tilde{u} + \tilde{f}_1(t,\tilde{u},\tilde{v})\tilde{v} - \tilde{f}_2(t,\tilde{u},\tilde{v})\tilde{u} - \lambda\tilde{u} & \text{on } \Gamma \times (0,T), \quad (2.5a) \\ \partial_t \tilde{v} = d\Delta_\Gamma \tilde{v} - \tilde{f}_1(t,\tilde{u},\tilde{v})\tilde{v} + \tilde{f}_2(t,\tilde{u},\tilde{v})\tilde{u} & \\ \qquad + \tilde{q}_1(t,\tilde{u},\tilde{v})\tilde{V} - \tilde{q}_2(t,\tilde{u},\tilde{v})\tilde{v} - \lambda\tilde{v} & \text{on } \Gamma \times (0,T), \quad (2.5b) \\ \tilde{u}(\cdot,0) = u_0 \text{ and } \tilde{v}(\cdot,0) = v_0 & \text{on } \Gamma. \quad (2.5c) \end{cases}$$

We show the existence of unique solutions $(\tilde{u},\tilde{v})$ of (2.5a)–(2.5c) on $\Gamma \times [0,T]$ of parabolic Hölder class $(2+\beta)$. In addition, we control the solution pair $(\tilde{u},\tilde{v})$ uniformly by $C^0([0,T];C^{1+\sigma}(\Gamma))$-norms for any $0 \le \sigma < 1$ and by adequate parabolic Hölder norms in dependence of the given data, see Section 2.2. One central ingredient is the development L^∞-estimates. If $\tilde{V}$ is essentially bounded on $\Omega \times (0,T)$ by a constant $\tilde{\Lambda}_1 > 0$, then $\tilde{u}$, $\tilde{v}$ are essentially bounded on $\Gamma \times (0,T)$ by a constant $\tilde{\Lambda}_2 > 0$, see Lemma 2.3 (i).

The next step of this updating procedure is to find unique solutions and estimates of an updated function V_{new}, see Section 2.3. In the rescaled framework we multiply the updated variable V_{new} with a time-dependent exponential factor $e^{-\lambda t}$ for fixed $\lambda > 0$ and consider $\tilde{V}_{\text{new}} : \overline{\Omega} \times [0,T] \to \mathbb{R}$ with

$$\tilde{V}_{\text{new}}(x,t) \overset{\text{def}}{=} e^{-\lambda t} V_{\text{new}}(x,t) \quad \text{for} \quad x \in \overline{\Omega},\ t \in [0,T]. \tag{2.6}$$

Consider $\tilde{v}$ and $\tilde{u}$ as input data for a Robin-boundary problem given by

$$\begin{cases} \partial_t \tilde{V}_{\text{new}} = D\Delta \tilde{V}_{\text{new}} - \lambda \tilde{V}_{\text{new}} & \text{in } \Omega \times (0,T), \quad (2.7a) \\ -D\nabla \tilde{V}_{\text{new}} \cdot \nu = \tilde{q}_1(t,\tilde{u},\tilde{v})\tilde{V}_{\text{new}} - \tilde{q}_2(t,\tilde{u},\tilde{v})\tilde{v} & \text{on } \Gamma \times (0,T), \quad (2.7b) \\ \tilde{V}_{\text{new}}(\cdot,0) = V_0 & \text{on } \overline{\Omega}. \quad (2.7c) \end{cases}$$

This will be done with the help of nonlinear Schauder theory described in [LSU68, Theorem 5.3, p. 320f.] and [LSU68, Theorem 7.1, p. 478]. Besides obtaining a unique classical solution $\tilde{V}_{\text{new}}$, these Theorems provide two crucial estimates: first, the parabolic Hölder norm of $\tilde{V}_{\text{new}}$ of order $(2+\beta)$ is estimated by the initial data and parabolic Hölder norms of $\tilde{u}$ and $\tilde{v}$ of order $(1+\beta)$. Second, there exists $0 < \kappa < 1$ depending on $C^0([0,T];C^1(\Gamma))$-norms of $\tilde{u}$ and $\tilde{v}$ such that the parabolic Hölder norm of $\tilde{V}_{\text{new}}$ of order κ will be estimated in terms of system constants and $C^0([0,T];C^1(\Gamma))$-norms of $\tilde{u}$ and $\tilde{v}$. For a suitable choice of β depending on κ and α the estimates the parabolic β-Hölder norm of $\tilde{V}_{\text{new}}$ is only depending on given data, see Proposition 2.6 for details. This estimate in the parabolic Hölder norm of order β will be the basis to perform a fixed-point argument for the updating procedure in the latter. The aforementioned strategy of considering a given function $\tilde{V}$ and finding an updated function $\tilde{V}_{\text{new}}$ is a decoupling procedure of the given problem (2.4a)–(2.4f).

In Section 2.4 we use the updating strategy to show that there is a fixed-point, such that $\tilde{V} = \tilde{V}_{\text{new}}$ holds. According to compactness arguments in terms of Hölder norms and Schauder's Fixed-Point Theorem, there exists a limit object $(\tilde{V}, \tilde{u}, \tilde{v})$ of parabolic Hölder class β. By repeating the updating structure we find that $(\tilde{V}, \tilde{u}, \tilde{v})$ solves (2.5a)–(2.5c) and (2.7a)–(2.7c), and therefore is also a solution of (2.4a)–(2.4f) of parabolic Hölder class $2+\beta$. In particular, we apply a bootstrapping strategy to find that the solutions are of parabolic Hölder class $2+\alpha$ as the initial data indicated. In addition, we find that this system depends continuously on the given initial data. Consequently, by rescaling, (V, u, v) given by (2.1)–(2.3) is then a classical solution of (FCRD). We deduce uniqueness with the independent results of Chapter 3.

The tools and spaces we use in this Chapter are collected in Section A.1 of the Appendix. We introduce global constants that depend on the system constants, i.e. $\tilde{\Lambda}_1 > 0$ will be specified later, we set

$$\tilde{\Lambda}_2 \overset{\text{def}}{=} \max\{\|u_0\|_{L^\infty(\Gamma)}, \|v_0\|_{L^\infty(\Gamma)}\} \quad \text{and} \quad C_{fq} \overset{\text{def}}{=} \max\{C_f, C_q\}. \tag{2.8}$$

2.1.2 Auxiliary problems

First, we show that with the definition of the rescaled framework in (2.1)–(2.3) we find (FCRD) to be equivalent to (2.4a)–(2.4f).

Assume that (V, u, v) is a classical solution of (FCRD) and $\lambda > 0$ fixed, then the regularity of (V, u, v) carries over to $(\tilde{V}, \tilde{u}, \tilde{v})$ by (2.1)–(2.3) and for example the time-derivative of $\tilde{u}$ is derived by

$$\begin{aligned}\partial_t \tilde{u} &= e^{-\lambda t}\partial_t u - \lambda e^{-\lambda t} u \\ &= e^{-\lambda t}\left(\Delta_\Gamma u + f_1(u,v)v - f_2(u,v)u\right) - \lambda e^{-\lambda t} u \\ &= \Delta_\Gamma \tilde{u} + f_1(e^{\lambda t}\tilde{v}, e^{\lambda t}\tilde{v})\tilde{v} - f_2(e^{\lambda t}\tilde{u}, e^{\lambda t}\tilde{v})\tilde{u} - \lambda\tilde{u},\end{aligned}$$

where we used the first equation of (FCRD). Note that now f_i and q_i are explicitly depending on t for $i = 1, 2$. Therefore, we define

$$\tilde{f}_i : \mathbb{R} \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}, \quad \tilde{f}_i(t, \tilde{u}_k, \tilde{v}_k) \overset{\text{def}}{=} f_i(e^{\lambda t}\tilde{u}_k, e^{\lambda t}\tilde{v}_k), \tag{2.9}$$

$$\tilde{q}_i : \mathbb{R} \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}, \quad \tilde{q}_i(t, \tilde{u}_k, \tilde{v}_k) \overset{\text{def}}{=} q_i(e^{\lambda t}\tilde{u}_k, e^{\lambda t}\tilde{v}_k), \tag{2.10}$$

for $i = 1, 2$. The corresponding equations for $\tilde{v}$ and $\tilde{V}$ follow in the same way. Since there is no time-derivative included in the Robin-boundary condition, we simply multiply with $e^{-\lambda t}$ and find equation (2.4d). In addition, this rescaling operation leaves the initial conditions untouched having $\tilde{u}(\cdot, 0) = u_0$, $\tilde{v}(\cdot, 0) = v_0$ and $\tilde{V}(\cdot, 0) = V_0$, respectively, on the corresponding domains. Therefore, classical solutions of (FCRD) are classical solutions of (2.4a)–(2.4f).

Vice versa, if $(\tilde{V}, \tilde{u}, \tilde{v})$ is a classical solution of (2.4a)–(2.4f) for fixed $\lambda > 0$, then there exists a uniquely defined triple (V, u, v) given by (2.1)–(2.3), such that

$$\begin{aligned}\partial_t v = \partial_t(e^{\lambda t}\tilde{v}) &= \lambda e^{\lambda t}\tilde{v} + e^{\lambda t}\partial_t\tilde{v} \\ &= \lambda v + e^{\lambda t}\left(\Delta_\Gamma\tilde{v} - \tilde{f}_1(t,\tilde{u},\tilde{v})\tilde{v} + \tilde{f}_2(t,\tilde{u},\tilde{v})\tilde{u} + \tilde{q}_1(t,\tilde{u},\tilde{v})\tilde{V} - \tilde{q}_2(t,\tilde{u},\tilde{v})\tilde{v} - \lambda\tilde{v}\right) \\ &= \Delta_\Gamma v - f_1(u,v)v + f_2(u,v)u + q_1(u,v)V + q_2(u,v)v\end{aligned}$$

holds on $\Gamma \times (0,T)$. Here, we used the definition of the nonlinearities and that (2.4b) holds true. With a similar calculation the other equations follow. The initial conditions remain the same and the boundary condition follows by multiplying with $e^{\lambda t}$. In sum we find that (2.4a)–(2.4f) is a rescaled system of (FCRD) and the solution concepts coincide for fixed $\lambda > 0$. In the same fashion we find a justification for the subsystems (2.5a)–(2.5c) and (2.7a)–(2.7c) from (2.4a)–(2.4f).

We need to introduce another auxiliary system that yields a suitable comparison function for the solution of the bulk system. The crucial point about maximum bounds and a priori estimates is to control the boundary flux condition of the inner variable $\tilde{V}_{\text{new}}$. Therefore, we introduce $\tilde{\mu} > 0$ to be a fixed constant to be chosen later. Since we only work in the rescaled framework we introduce the problem of a heat equation inside Ω, with constant boundary flux of amount $\tilde{\mu}$ observed on a time-interval $[0,T]$ with nonnegative initial data $\tilde{\Psi}_0 \in C^{2+\alpha}(\overline{\Omega})$ given by

$$\begin{cases} \partial_t \tilde{\Psi} = D\Delta\tilde{\Psi} - \lambda\tilde{\Psi} & \text{in } \Omega \times (0,T), \quad (2.11a)\\ -D\nabla\tilde{\Psi}\cdot\nu = -\tilde{\mu} & \text{on } \Gamma \times (0,T), \quad (2.11b)\\ \tilde{\Psi}(\cdot,0) = \tilde{\Psi}_0 & \text{on } \overline{\Omega}, \quad (2.11c) \end{cases}$$

where $\tilde{\Psi} : \overline{\Omega} \times [0,T] \to \mathbb{R}$ with $\tilde{\Psi} = \tilde{\Psi}(x,t)$. We assume that the compatibility condition regarding initial data is satisfied, i.e. $-D\nabla\tilde{\Psi}_0 \cdot \nu = -\tilde{\mu}$ holds. Then, according to [LSU68, IV. Theorem 5.3, p. 320f.], (2.11a)–(2.11c) has a unique classical solution $\tilde{\Psi} \in H^{2+\beta,(2+\beta)/2}(\overline{\Omega_T})$ for $0 < \beta \le \alpha$. In the latter it is necessary to obtain an upper bound for $\tilde{\Psi}$ on $\overline{\Omega} \times [0,T]$ and an explicit dependency on the diffusion constant $D > 0$. Therefore, we formulate the following statement.

Lemma 2.1 (Uniform maximum bound for the auxiliary problem)**.** *Let $\tilde{\Psi}$ be the unique solution of (2.11a)–(2.11c) of class $H^{2+\beta,(2+\beta)/2}(\overline{\Omega_T})$. Then, there exists a constant $c_0 = c_0(\Omega, D) > 0$ being uniformly bounded for diffusion constants $D \ge 1$, such that*

$$\tilde{\Psi}(x,t) \le \sup_{\overline{\Omega}} \tilde{\Psi}_0 + \tilde{\mu} e^{c_0 T} \tag{2.12}$$

for any $(x,t) \in \overline{\Omega} \times [0,T]$.

Proof. We define two subproblems to split boundary and initial conditions from the original problems. First, let

$$\begin{cases} \partial_t \vartheta_1 = D\Delta\vartheta_1 - \lambda\vartheta_1 & \text{in } \Omega \times (0,T), \quad (2.13a)\\ -D\nabla\vartheta_1\cdot\nu = -\tilde{\mu} & \text{on } \Gamma \times (0,T), \quad (2.13b)\\ \vartheta_1(\cdot,0) = \tilde{\Psi}_0 - \sup_{\overline{\Omega}} \tilde{\Psi}_0 & \text{on } \overline{\Omega} \quad (2.13c) \end{cases}$$

hold. Second, we investigate the behavior of

$$\begin{cases} \partial_t \vartheta_2 = D\Delta\vartheta_2 - \lambda\vartheta_2 & \text{in } \Omega \times (0,T), \quad (2.14a) \\ -D\nabla\vartheta_2 \cdot \nu = 0 & \text{on } \Gamma \times (0,T), \quad (2.14b) \\ \vartheta_2(\cdot,0) = \sup_{\Omega} \tilde{\Psi}_0 & \text{on } \overline{\Omega}. \quad (2.14c) \end{cases}$$

We notice that the initial data is in both cases compatible to the Neumann-boundary condition. Both problems, (2.13a)–(2.13c) and (2.14a)–(2.14c) have unique classical solutions.

1st step: For $0 < \delta \leq 1/4$ let $h_\delta \in C^\infty(\mathbb{R})$ be a function satisfying

$$h_\delta(r) = \begin{cases} r & \text{for } |r| \leq \delta, \\ 2\delta & \text{for } r \geq 3\delta, \\ -2\delta & \text{for } r \leq -3\delta, \end{cases}$$

with a smooth monotone transition on $(-3\delta, -\delta)$ and $(\delta, 3\delta)$, respectively. Moreover, $|h'_\delta| \leq 1$ and $|h''_\delta| \leq \frac{1}{\delta}$ holds. For $\Gamma = \partial\Omega$ we introduce the *signed distance function* $\operatorname{sdist} : \mathbb{R}^3 \to \mathbb{R}$ given by

$$\operatorname{sdist}(x,\Gamma) \stackrel{\text{def}}{=} \operatorname{dist}(x,\Omega) - \operatorname{dist}(x,\Omega^{c}) \text{ for } x \in \mathbb{R}^3.$$

For $\gamma > 0$, let $\phi : \mathbb{R}^3 \to [1/2, 3/2]$ be defined as

$$\phi(x) \stackrel{\text{def}}{=} 1 - h_\delta\left(\frac{\operatorname{sdist}(x,\Gamma)}{\gamma}\right).$$

According to [GT01, Lemma 14.16, p. 355], sdist is in particular of class C^2 sufficiently close to the boundary of Ω since Γ was assumed to be smooth. Then, by the definition of h_δ and the assumptions on Γ, $\phi \in C^2(\mathbb{R}^3)$ for $\gamma\delta$ sufficiently small. In particular, $\phi(x) = 1$ for $x \in \Gamma$, $1 \leq \phi \leq 3/2$ on $\overline{\Omega}$. For the gradient of ϕ we find

$$\begin{aligned} \nabla\phi(x) &= -h'_\delta\left(\frac{\operatorname{sdist}(x,\Gamma)}{\gamma}\right)\frac{1}{\gamma}\nabla\operatorname{sdist}(x,\Gamma) \\ &= -\frac{1}{\gamma}\nu(x) \quad \text{for } x \in \Gamma, \end{aligned}$$

where we used that on Γ the gradient of the signed distance function is given by the outer normal vector ν for Ω. Moreover, we set $c_* \stackrel{\text{def}}{=} \gamma\delta$ and find that there exists a constant $c_1 > 0$ depending on the geometry of Γ and δ and $c_2 > 0$ depending on the geometry of Γ, such that

$$|\nabla\phi| \leq \frac{c_1}{\gamma} \quad \text{and} \quad |\Delta\phi| \leq \frac{c_2}{\gamma^2}|h''_\delta| \leq \frac{c_2}{\gamma c_*}$$

holds.

2nd step: Let $\overline{\lambda} > 0$ be a constant to be chosen later. Let $\tilde{\vartheta}_1 : \overline{\Omega} \times [0,T] \to \mathbb{R}$ be defined as

$$\tilde{\vartheta}_1(x,t) \stackrel{\text{def}}{=} e^{-\overline{\lambda}t}\vartheta_1(x,t)\phi(x) \text{ for } (x,t) \in \overline{\Omega} \times [0,T].$$

Hence, we obtain

$$\begin{aligned}
\partial_t\tilde{\vartheta}_1 - &D\Delta\tilde{\vartheta}_1 + \lambda\tilde{\vartheta}_1 + 2D\nabla\tilde{\vartheta}_1 \cdot \frac{\nabla\phi}{\phi} \\
&= -\overline{\lambda}\tilde{\vartheta}_1 + e^{-\overline{\lambda}t}\phi\partial_t\vartheta_1 - De^{-\overline{\lambda}t}\phi\Delta\vartheta_1 - 2De^{-\overline{\lambda}t}\nabla\vartheta_1 \cdot \nabla\phi \\
&\quad - De^{-\overline{\lambda}t}\vartheta_1\Delta\phi + \lambda\tilde{\vartheta}_1 + 2De^{-\overline{\lambda}t}\nabla\vartheta_1 \cdot \nabla\phi + 2De^{-\overline{\lambda}t}\frac{\vartheta_1}{\phi}|\nabla\phi|^2 \\
&= -\overline{\lambda}\tilde{\vartheta}_1 + e^{-\overline{\lambda}t}\phi\left(D\Delta\vartheta_1 - \lambda\vartheta_1\right) - De^{-\overline{\lambda}t}\phi\Delta\vartheta_1 \\
&\quad - De^{-\overline{\lambda}t}\vartheta_1\Delta\phi + \lambda\tilde{\vartheta}_1 + 2D\frac{\tilde{\vartheta}_1}{\phi^2}|\nabla\phi|^2 \\
&= \tilde{\vartheta}_1\left(2D\frac{|\nabla\phi|^2}{\phi^2} - D\frac{\Delta\phi}{\phi} - \overline{\lambda}\right),
\end{aligned} \tag{2.15}$$

where we used (2.13a)–(2.13c). With $\overline{\lambda} = {}^{Dc_1}\!/\!_{\gamma^2} + {}^{Dc_2}\!/\!_{\gamma c_*}$ we find that the factor in the brackets on the right-hand side of (2.15) is strictly negative. For the boundary condition of $\tilde{\vartheta}_1$ we compute

$$\begin{aligned}
-D\nabla\tilde{\vartheta}_1 \cdot \nu &= -De^{-\overline{\lambda}t}\phi\nabla\vartheta_1 \cdot \nu - De^{-\overline{\lambda}t}\vartheta_1\nabla\phi \cdot \nu \\
&= -e^{-\overline{\lambda}t}\tilde{\mu} + D\tilde{\vartheta}_1\frac{1}{\gamma},
\end{aligned} \tag{2.16}$$

where we used the properties of ϕ from the first step.

3rd step: We set $M \stackrel{\text{def}}{=} \sup_{\overline{\Omega}_T} \tilde{\vartheta}_1 = \tilde{\vartheta}_1(x_0,t_0)$ and find that $M \leq 0$ holds for $t_0 = 0$ and for all $x_0 \in \overline{\Omega}$ according to the given initial condition. If $x_0 \in \Omega$ and $t_0 > 0$, then

$$0 \leq \left(\partial_t\tilde{\vartheta}_1 - D\Delta\tilde{\vartheta}_1 + 2D\nabla\tilde{\vartheta}_1 \cdot \frac{\nabla\phi}{\phi}\right)(x_0,t_0) < 0$$

if $M > 0$, which is a contradiction. For $x_0 \in \Gamma$ and $t_0 > 0$ we obtain with (2.16)

$$0 \geq -D\nabla\tilde{\vartheta}_1 \cdot \nu(x_0,t_0) = -e^{-\overline{\lambda}t_0}\tilde{\mu} + \frac{D}{\gamma}\tilde{\vartheta}_1(x_0,t_0).$$

Since $\tilde{\vartheta}_1(x_0,t_0) = M$, we find

$$M \leq \frac{\gamma\tilde{\mu}e^{-\overline{\lambda}t_0}}{D} \leq \frac{\gamma\tilde{\mu}}{D}.$$

Consequently, $\tilde{\vartheta}_1 \leq \frac{\gamma\tilde{\mu}}{D}$. We set $\gamma \stackrel{\text{def}}{=} D$ to find hat

$$\vartheta_1(x,t) \leq e^{\overline{\lambda}t}\tilde{\mu} \leq e^{\left(\frac{c_1}{D}+\frac{c_2}{c_*}\right)t}\tilde{\mu} \leq e^{c_0 t}\tilde{\mu}$$

holds, for a constant $c_0 > 0$ depending on the geometry of Γ and D remaining bounded for positive diffusion constant $D > 0$.

4th step: We consider (2.14a)–(2.14c). We use [Jos07, Satz 4.1.1, p. 80] to obtain

$$\vartheta_2 \leq \sup_{(\overline{\Omega}\times\{0\})\cup\Gamma_T} \vartheta_2 = \sup_{\overline{\Omega}\times\{0\}} \vartheta_2 = \sup_{\overline{\Omega}} \tilde{\Psi}_0.$$

By setting $\tilde{\Psi} \stackrel{\text{def}}{=} \vartheta_1 + \vartheta_2$, we reassamble the auxiliary problem (2.11a)–(2.11c) with maximum bound

$$\tilde{\Psi}(x,t) \leq \sup_{\overline{\Omega}} \tilde{\Psi}_0 + \tilde{\mu}e^{c_0 T} \quad \text{for all} \quad (x,t) \in \overline{\Omega}\times[0,T]$$

to finish the proof. □

2.1.3 Nonnegativity and a priori estimates

In the following we work in the rescaled framework considering variables $\tilde{V}, \tilde{V}_{\text{new}}, \tilde{u}, \tilde{v}$ introduced in Subsection 2.1.2. In this case the purpose of rescaling is the following: we want to find a finite but large parameter $\lambda > 0$, such that the values of all members of the updating procedure remain in a nonnegative rectangular, namely $[0, \tilde{\Lambda}_1] \times [0, \tilde{\Lambda}_2]^2$ for $\tilde{\Lambda}_1 > 0$, $\tilde{\Lambda}_2 > 0$. In Lemma 2.2 we prove that the updating procedure accounts for nonnegative solutions, where in Lemma 2.3 we deduce maximum bounds for the updating procedure.

Lemma 2.2. *Let Assumptions 1.1–1.2 hold. Assume $\tilde{V}$ is nonnegative. For a constant $C = C(\Omega) > 0$ set*

$$\lambda \geq \max\left\{\frac{C(\Omega)C_q^2}{D},\, 2(C_f + C_q)\right\}. \tag{2.17}$$

Then, the updating procedure yields that $\tilde{V}_{new}, \tilde{u}, \tilde{v}$ satisfying (2.5a)–(2.5c) *and* (2.7a)–(2.7c) *are all nonnegative for all times $t \in (0,T)$.*

Proof. We assumed that $(\tilde{V}_{\text{new}}, \tilde{u}, \tilde{v})$ satisfy the given rescaled systems. Therefore, we find that the negative parts given by

$$\tilde{V}_{\text{new}}^- \stackrel{\text{def}}{=} \max\{-\tilde{V}_{\text{new}}, 0\},\ \tilde{u}^- \stackrel{\text{def}}{=} \max\{-\tilde{u}, 0\} \text{ and } \tilde{v}^- \stackrel{\text{def}}{=} \max\{-\tilde{v}, 0\},$$

are admissible test-functions for (2.5a)–(2.5c) and (2.7a)–(2.7c). We use $-\tilde{V}_{\text{new}}^-$, $-\tilde{u}^-$ and $-\tilde{v}^-$ as testfunctions and the fact that $\tilde{v} = \tilde{v}^+ - \tilde{v}^- \geq -\tilde{v}^-$ holds, $\tilde{u} \geq -\tilde{u}^-$ respectively.

With the nonnegativity of $\tilde{V}$ we obtain for the sum of (2.5a)–(2.5c) and (2.7a)–(2.7c) that

$$
\begin{aligned}
&-\frac{1}{2}\int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}|\tilde{V}^-_{\mathrm{new}}|^2 - \frac{1}{2}\int_\Gamma \left(\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{u}^-|^2 + \frac{\mathrm{d}}{\mathrm{d}t}|\tilde{v}^-|^2\right) \\
&\quad= \int_\Omega (\partial_t \tilde{V}_{\mathrm{new}})(-\tilde{V}^-_{\mathrm{new}}) + \int_\Gamma ((\partial_t\tilde{u})(-\tilde{u}^-) + (\partial_t\tilde{v})(-\tilde{v}^-)) \\
&\quad= \int_\Omega \left(\lambda(\tilde{V}^-_{\mathrm{new}})^2 + D|\nabla\tilde{V}^-_{\mathrm{new}}|^2\right) + \int_\Gamma \left(\lambda((\tilde{u}^-)^2 + (\tilde{v}^-)^2) + |\nabla_\Gamma\tilde{u}^-|^2 + d|\nabla_\Gamma\tilde{v}^-|^2\right) \\
&\qquad+ \int_\Gamma \left(-\tilde{q}_1(t,\tilde{u},\tilde{v})(\tilde{V}^-_{\mathrm{new}})^2 + \tilde{q}_2(t,\tilde{u},\tilde{v})\tilde{v}\tilde{V}^-_{\mathrm{new}} + \tilde{q}_1(t,\tilde{u},\tilde{v})\tilde{V}\tilde{v}^- - \tilde{q}_2(t,\tilde{u},\tilde{v})(\tilde{v}^-)^2\right) \\
&\qquad+ \int_\Gamma \left(-\tilde{f}_1(t,\tilde{u},\tilde{v})(\tilde{v}^-)^2 + \tilde{f}_2(t,\tilde{u},\tilde{v})\tilde{u}\tilde{v}^-\right) \\
&\qquad+ \int_\Gamma \left(\tilde{f}_1(t,\tilde{u},\tilde{v})\tilde{v}\tilde{u}^- - \tilde{f}_2(t,\tilde{u},\tilde{v})(\tilde{u}^-)^2\right) \qquad (2.18)
\end{aligned}
$$

holds. We estimate (2.18) from below by

$$
\begin{aligned}
&\int_\Omega \left(\lambda(\tilde{V}^-_{\mathrm{new}})^2 + D|\nabla\tilde{V}^-_{\mathrm{new}}|^2\right) + \int_\Gamma \left(\lambda((\tilde{u}^-)^2 + (\tilde{v}^-)^2) + |\nabla_\Gamma\tilde{u}^-|^2 + d|\nabla_\Gamma\tilde{v}^-|^2\right) \\
&\qquad+ \int_\Gamma \left(-C_q(\tilde{V}^-_{\mathrm{new}})^2 - C_q\tilde{v}^-\tilde{V}^-_{\mathrm{new}} - C_q(\tilde{v}^-)^2\right) \\
&\qquad+ \int_\Gamma \left(-C_f(\tilde{v}^-)^2 - C_f\tilde{u}^-\tilde{v}^- - C_f\tilde{v}^-\tilde{u}^- - C_f(\tilde{u}^-)^2\right) \\
&\quad\geq \int_\Omega \left(\lambda(\tilde{V}^-_{\mathrm{new}})^2 + D|\nabla\tilde{V}^-_{\mathrm{new}}|^2\right) + \int_\Gamma \left(\lambda(\tilde{u}^-)^2 + (\tilde{v}^-)^2) + |\nabla_\Gamma\tilde{u}^-|^2 + d|\nabla_\Gamma\tilde{v}^-|^2\right) \\
&\qquad- \int_\Gamma \left(\frac{3}{2}C_q(\tilde{V}^-_{\mathrm{new}})^2 + \left(\frac{3}{2}C_q + 2C_f\right)(\tilde{v}^-)^2 + 2C_f(\tilde{u}^-)^2\right), \qquad (2.19)
\end{aligned}
$$

where we applied Young's Inequality, estimates for the negative parts and Assumption 1.2 which remains valid for $\tilde{f}_i$ and $\tilde{q}_i$ for $i = 1, 2$. We apply the Trace Theorem, see Lemma A.5 from the Appendix with $\varepsilon = D/3C_q$ to find that the right-hand side of (2.19) is estimated from below by

$$
\begin{aligned}
&\int_\Omega \left(\lambda - \frac{C(\Omega)C_q^2}{D}\right)(\tilde{V}^-_{\mathrm{new}})^2 + \frac{D}{2}\int_\Omega |\nabla\tilde{V}^-_{\mathrm{new}}|^2 + \int_\Gamma (\lambda - 2C_f)(\tilde{u}^-)^2 + \int_\Gamma |\nabla_\Gamma\tilde{u}^-|^2 \\
&\qquad+ \int_\Gamma \left(\lambda - \left(\frac{3}{2}C_q + 2C_f\right)\right)(\tilde{v}^-)^2 + \int_\Gamma d|\nabla_\Gamma\tilde{v}^-|^2 \qquad (2.20)
\end{aligned}
$$

with a constant $C = C(\Omega) > 0$. We obtain from (2.19) and (2.20) with $\lambda > 0$ from (2.17) that

$$
\frac{1}{2}\int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}|\tilde{V}^-_{\mathrm{new}}|^2 + \frac{1}{2}\int_\Gamma \left(\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{u}^-|^2 + \frac{\mathrm{d}}{\mathrm{d}t}|\tilde{v}^-|^2\right) \leq 0
$$

holds, therefore the solutions remain nonnegative whenever the initial values are nonnegative. □

Lemma 2.3 (A priori estimates)**.** *Let Assumptions 1.1–1.2 hold, $\tilde{\Lambda}_2$ and C_{fq} be given as in* (2.8)*. Let*

$$\lambda \geq \max\left\{\frac{C_{fq}^2}{D}C(\Omega), 4C_{fq}, C_{fq}\left(1 + C_1 C_{fq} + \frac{2\|V_0\|_{L^\infty(\Omega)} + 1}{\tilde{\Lambda}_2}\right)\right\}, \tag{2.21}$$

such that Condition (2.17) *from Lemma 2.2 is satisfied with $C_1 = C_1(T, \Omega, D) > 0$ and $C(\Omega) > 0$. There exists a constant $\tilde{\Lambda}_1 > 0$ depending on $T, D, C_f, C_q, \Omega, V_0, u_0$ and v_0 with the following properties:*

(i) *If $\|\tilde{V}\|_{L^\infty(\Omega_T)} \leq \tilde{\Lambda}_1$ and $\tilde{u}$, $\tilde{v}$ solve* (2.5a)–(2.5c)*, then $\|\tilde{u}\|_{L^\infty(\Gamma_T)}, \|\tilde{v}\|_{L^\infty(\Gamma_T)} \leq \tilde{\Lambda}_2$ and,*

(ii) *if $\|\tilde{u}\|_{L^\infty(\Gamma_T)}, \|\tilde{v}\|_{L^\infty(\Gamma_T)} \leq \tilde{\Lambda}_2$ and $\tilde{V}_{new}$ is the solution of* (2.7a)–(2.7c)*, then the updated function $\tilde{V}_{new}$ remains essentially bounded by $\tilde{\Lambda}_1$, i.e.*

$$\|\tilde{V}_{new}\|_{L^\infty(\Omega_T)} \leq \tilde{\Lambda}_1$$

uniformly in every update step.

Proof. The function $(\tilde{V}_{\text{new}} - \tilde{\Psi})_+$ is an admissible testfunction for the auxiliary system (2.11a)–(2.11c), with partial integration we find that

$$\begin{aligned} 0 = &\int_\Omega (\partial_t \tilde{\Psi})(\tilde{V}_{\text{new}} - \tilde{\Psi})_+ + \int_\Omega D\nabla\tilde{\Psi} \cdot \nabla(\tilde{V}_{\text{new}} - \tilde{\Psi})_+ \\ &+ \lambda \int_\Omega \tilde{\Psi}(\tilde{V}_{\text{new}} - \tilde{\Psi})_+ - \int_\Gamma \tilde{\mu}(\tilde{V}_{\text{new}} - \tilde{\Psi})_+ \end{aligned} \tag{2.22}$$

holds. By testing (2.7a)–(2.7c) with $(\tilde{V}_{\text{new}} - \tilde{\Psi})_+$ we obtain that

$$\begin{aligned} 0 = &\int_\Omega (\partial_t \tilde{V}_{\text{new}})(\tilde{V}_{\text{new}} - \tilde{\Psi})_+ + D\int_\Omega \nabla\tilde{V}_{\text{new}} \cdot \nabla(\tilde{V}_{\text{new}} - \tilde{\Psi})_+ \\ &+ \int_\Gamma (\tilde{q}_1(t,\tilde{u},\tilde{v})\tilde{V}_{\text{new}} - \tilde{q}_2(t,\tilde{u},\tilde{v})\tilde{v})(\tilde{V}_{\text{new}} - \tilde{\Psi})_+ + \lambda\int_\Omega \tilde{V}_{\text{new}}(\tilde{V}_{\text{new}} - \tilde{\Psi})_+ \end{aligned} \tag{2.23}$$

holds. Analogously, we find for (2.5a)–(2.5c) tested by $(\tilde{u} - \tilde{\Lambda}_2)_+$ and $(\tilde{v} - \tilde{\Lambda}_2)_+$ that

$$\begin{aligned} 0 = &\int_\Gamma (\partial_t \tilde{u})(\tilde{u} - \tilde{\Lambda}_2)_+ + \int_\Gamma \nabla_\Gamma\tilde{u} \cdot \nabla_\Gamma(\tilde{u} - \tilde{\Lambda}_2)_+ + \lambda\int_\Gamma \tilde{u}(\tilde{u} - \tilde{\Lambda}_2)_+ \\ &+ \int_\Gamma (-\tilde{f}_1(t,\tilde{u},\tilde{v})\tilde{v} + \tilde{f}_2(t,\tilde{u},\tilde{v})\tilde{u})(\tilde{u} - \tilde{\Lambda}_2)_+, \end{aligned} \tag{2.24}$$

$$\begin{aligned} 0 = &\int_\Gamma (\partial_t \tilde{v})(\tilde{v} - \tilde{\Lambda}_2)_+ + d\int_\Gamma \nabla_\Gamma\tilde{v} \cdot \nabla_\Gamma(\tilde{v} - \tilde{\Lambda}_2)_+ + \lambda\int_\Gamma \tilde{v}(\tilde{v} - \tilde{\Lambda}_2)_+ \\ &+ \int_\Gamma \left(\tilde{f}_1(t,\tilde{u},\tilde{v})\tilde{v} - \tilde{f}_2(t,\tilde{u},\tilde{v})\tilde{u} - \tilde{q}_1(t,\tilde{u},\tilde{v})\tilde{V} + \tilde{q}_2(t,\tilde{u},\tilde{v})\tilde{v}\right)(\tilde{v} - \tilde{\Lambda}_2)_+. \end{aligned} \tag{2.25}$$

We drop the arguments of $\tilde{f}_i$ and $\tilde{q}_i$ for $i = 1, 2$, substract (2.22) from (2.23) and add this to the sum of (2.24) and (2.25) to find

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega (\tilde{V}_{\text{new}} - \tilde{\Psi})_+^2 + \int_\Gamma (\tilde{v} - \tilde{\Lambda}_2)_+^2 + \int_\Gamma (\tilde{u} - \tilde{\Lambda}_2)_+^2\right) \\
&\quad = \int_\Omega (\partial_t(\tilde{V}_{\text{new}} - \tilde{\Psi}))(\tilde{V}_{\text{new}} - \tilde{\Psi})_+ + \int_\Gamma (\partial_t \tilde{v})(\tilde{v} - \tilde{\Lambda}_2)_+ + \int_\Gamma (\partial_t \tilde{u})(\tilde{u} - \tilde{\Lambda}_2)_+ \\
&\quad = -D\int_\Omega |\nabla(\tilde{V}_{\text{new}} - \tilde{\Psi})_+|^2 - \int_\Gamma |\nabla_\Gamma(\tilde{u} - \tilde{\Lambda}_2)_+|^2 - d\int_\Gamma |\nabla_\Gamma(\tilde{v}_k - \tilde{\Lambda}_2)_+|^2 \\
&\qquad - \lambda\int_\Omega (\tilde{V}_{\text{new}} - \tilde{\Psi})_+^2 - \lambda\int_\Gamma \tilde{u}(\tilde{u} - \tilde{\Lambda}_2)_+ - \lambda\int_\Gamma \tilde{v}(\tilde{v} - \tilde{\Lambda}_2)_+ \\
&\qquad + \int_\Gamma \left(\tilde{f}_1\tilde{v} - \tilde{f}_2\tilde{u}\right)(\tilde{u} - \tilde{\Lambda}_2)_+ + \int_\Gamma \left(-\tilde{f}_1\tilde{v} + \tilde{f}_2\tilde{u} + \tilde{q}_1\tilde{V} - \tilde{q}_2\tilde{v}\right)(\tilde{v} - \tilde{\Lambda}_2)_+ \\
&\qquad + \int_\Gamma \left(-\tilde{q}_1\tilde{V}_{\text{new}} + \tilde{q}_2\tilde{v}\right)(\tilde{V}_{\text{new}} - \tilde{\Psi})_+ - \tilde{\mu}\int_\Gamma (\tilde{V}_{\text{new}} - \tilde{\Psi})_+. \qquad (2.26)
\end{aligned}$$

We have to control the right-hand side of (2.26) appropriately. To obtain a higher clarity we demonstrate the arising computations separately. With $C_{fq} > 0$ we find for the following nonlinear expressions that

$$\begin{aligned}
&\int_\Gamma \left(\tilde{f}_1\tilde{v} - \tilde{f}_2\tilde{u}\right)(\tilde{u} - \tilde{\Lambda}_2)_+ \\
&\quad \le \int_\Gamma C_{fq}\tilde{v}(\tilde{u} - \tilde{\Lambda}_2)_+ = \int_\Gamma C_{fq}(\tilde{v} - \tilde{\Lambda}_2 + \tilde{\Lambda}_2)(\tilde{u} - \tilde{\Lambda}_2)_+ \\
&\quad = \int_\Gamma C_{fq}(\tilde{v} - \tilde{\Lambda}_2)(\tilde{u} - \tilde{\Lambda}_2)_+ + \int_\Gamma C_{fq}\tilde{\Lambda}_2(\tilde{u} - \tilde{\Lambda}_2)_+ \\
&\quad \le \int_\Gamma \frac{C_{fq}}{2}(\tilde{v} - \tilde{\Lambda}_2)_+^2 + \int_\Gamma \frac{C_{fq}}{2}(\tilde{u} - \tilde{\Lambda}_2)_+^2 + \int_\Gamma C_{fq}\tilde{\Lambda}_2(\tilde{u} - \tilde{\Lambda}_2)_+, \qquad (2.27)
\end{aligned}$$

holds, where we used Young's Inequality. With $\|\tilde{V}\|_{L^\infty(\Omega_T)} \le \tilde{\Lambda}_1$ we obtain

$$\begin{aligned}
&\int_\Gamma \left(-\tilde{f}_1\tilde{v} + \tilde{f}_2\tilde{u} + \tilde{q}_1\tilde{V} - \tilde{q}_2\tilde{v}\right)(\tilde{v} - \tilde{\Lambda}_2)_+ \\
&\quad \le \int_\Gamma \frac{C_{fq}}{2}\left((\tilde{u} - \tilde{\Lambda}_2)_+^2 + (\tilde{v} - \tilde{\Lambda}_2)_+^2\right) + \int_\Gamma C_{fq}\left(\tilde{\Lambda}_2 + \tilde{\Lambda}_1\right)(\tilde{v} - \tilde{\Lambda}_2)_+. \qquad (2.28)
\end{aligned}$$

The remaining nonlinear expression is defined on Γ but has to be controlled by expressions defined on Ω. Therefore, the Trace Theorem according to Lemma A.5 with $\varepsilon = {}^D/_{C_{fq}}$ and a constant $C = C(\Omega) > 0$ yields

$$\begin{aligned}
&\int_\Gamma \left(-\tilde{q}_1 \tilde{V}_{\text{new}} + \tilde{q}_2 \tilde{v}\right) (\tilde{V}_{\text{new}} - \tilde{\Psi})_+ \\
&\quad \le \int_\Gamma C_{fq} \tilde{v} (\tilde{V}_{\text{new}} - \tilde{\Psi})_+ \\
&\quad = \int_\Gamma C_{fq} (\tilde{v} - \tilde{\Lambda}_2)(\tilde{V}_{\text{new}} - \tilde{\Psi})_+ + \int_\Gamma C_{fq} \tilde{\Lambda}_2 (\tilde{V}_{\text{new}} - \tilde{\Psi})_+ \\
&\quad \le \int_\Gamma \frac{C_{fq}}{2} (\tilde{v} - \tilde{\Lambda}_2)_+^2 + \int_\Gamma \frac{C_{fq}}{2} (\tilde{V}_{\text{new}} - \tilde{\Psi})_+^2 + \int_\Gamma C_{fq} \tilde{\Lambda}_2 (\tilde{V}_{\text{new}} - \tilde{\Psi})_+ \\
&\quad \le \int_\Gamma \frac{C_{fq}}{2} (\tilde{v} - \tilde{\Lambda}_2)_+^2 + \frac{D}{2} \int_\Omega |\nabla (\tilde{V}_{\text{new}} - \tilde{\Psi})_+|^2 \\
&\qquad + \int_\Omega \frac{C_{fq}^2}{2D} C(\Omega) (\tilde{V}_{\text{new}} - \tilde{\Psi})_+^2 + \int_\Gamma C_{fq} \tilde{\Lambda}_2 (\tilde{V}_{\text{new}} - \tilde{\Psi})_+ .
\end{aligned} \tag{2.29}$$

Moreover, we notice that

$$-\lambda \int_\Gamma \tilde{u} (\tilde{u} - \tilde{\Lambda}_2)_+ = -\lambda \int_\Gamma (\tilde{u} - \tilde{\Lambda}_2)_+^2 - \lambda \int_\Gamma \tilde{\Lambda}_2 (\tilde{u} - \tilde{\Lambda}_2)_+ , \tag{2.30}$$

$$-\lambda \int_\Gamma \tilde{v} (\tilde{v} - \tilde{\Lambda}_2)_+ = -\lambda \int_\Gamma (\tilde{v} - \tilde{\Lambda}_2)_+^2 - \lambda \int_\Gamma \tilde{\Lambda}_2 (\tilde{v} - \tilde{\Lambda}_2)_+ \tag{2.31}$$

holds. Then, we estimate the right-hand side of (2.26) from above by using (2.27)–(2.31) by

$$\begin{aligned}
&-\frac{D}{2} \int_\Omega |\nabla (\tilde{V}_{\text{new}} - \tilde{\Psi})_+|^2 + \left(\frac{C_{fq}^2}{2D} C(\Omega) - \lambda \right) \int_\Omega (\tilde{V}_{\text{new}} - \tilde{\Psi})_+^2 \\
&\qquad + (C_{fq} - \lambda) \int_\Gamma (\tilde{u} - \tilde{\Lambda}_2)_+^2 + \left(\frac{3}{2} C_{fq} - \lambda \right) \int_\Gamma (\tilde{v} - \tilde{\Lambda}_2)_+^2 \\
&\qquad + (C_{fq} - \lambda) \tilde{\Lambda}_2 \int_\Gamma (\tilde{u} - \tilde{\Lambda}_2)_+ + (C_{fq} (\tilde{\Lambda}_1 + \tilde{\Lambda}_2) - \lambda \tilde{\Lambda}_2) \int_\Gamma (\tilde{v} - \tilde{\Lambda}_2)_+ \\
&\qquad + (C_{fq} \tilde{\Lambda}_2 - \tilde{\mu}) \int_\Gamma (\tilde{V}_{\text{new}} - \tilde{\Psi})_+ ,
\end{aligned} \tag{2.32}$$

where we dropped gradient expressions on Γ. Here, we choose $\tilde{\mu} = C_{fq} \tilde{\Lambda}_2$ and choose $\tilde{\Psi}_0$, such that $\tilde{\Psi}_0$ satisfies the compatibility condition

$$-D \nabla \tilde{\Psi}_0 \cdot \nu = -C_{fq} \tilde{\Lambda}_2 ,$$

$V_0 \le \Psi_0$ and $\|\tilde{\Psi}_0\|_{L^\infty(\Omega)} \le 2 \|V_0\|_{L^\infty(\Omega)} + 1$, which accounts for vanishing initial of V_0. According to this choice we apply Lemma 2.1 and (2.12) to find that $\tilde{\Psi}$ is essentially bounded on $\overline{\Omega}_T$ with a constant $C_1 = C_1(D, \Omega, T) > 0$, i.e.

$$\|\tilde{\Psi}\|_{L^\infty(\Omega_T)} \le \|\tilde{\Psi}_0\|_{L^\infty(\Omega)} + C_1 C_{fq} \tilde{\Lambda}_2 .$$

We extend the condition for λ from (2.17) and choose

$$\lambda \geq \max\left\{\frac{C_{fq}^2}{D}C(\Omega), 4C_{fq}, C_{fq}\left(1 + C_1 C_{fq} + \frac{2\|V_0\|_{L^\infty(\Omega)+1}}{\tilde{\Lambda}_2}\right)\right\},$$

which is solely depending on system constants. Considering

$$\tilde{\Lambda}_1 \stackrel{\text{def}}{=} 2\|V_0\|_{L^\infty(\Omega)} + 1 + C_1 C_{fq}\tilde{\Lambda}_2,$$

then all coefficients of (2.32) are less or equal zero. The settings above and the estimates from (2.26)–(2.32) imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|(\tilde{V}_{\text{new}} - \tilde{\Psi})_+(t)\|^2_{L^2(\Omega)} + \|(\tilde{u}_k - \tilde{\Lambda}_2)_+(t)\|^2_{L^2(\Gamma)} + \|(\tilde{v}_k - \tilde{\Lambda}_2)_+(t)\|^2_{L^2(\Gamma)}\right) \leq 0$$

holds for all $t \in (0, T)$. We obtain with the assumptions of this lemma that

$$\begin{aligned}\|(\tilde{V}_{\text{new}} - \tilde{\Psi})_+(t)\|^2_{L^2(\Omega)} &+ \|(\tilde{u}_k - \tilde{\Lambda}_2)_+(t)\|^2_{L^2(\Gamma)} + \|(\tilde{v}_k - \tilde{\Lambda}_2)_+(t)\|^2_{L^2(\Gamma)} \\ &\leq \|V_0 - \Psi_0\|^2_{L^2(\Omega)} + \|u_0 - \tilde{\Lambda}_2\|^2_{L^2(\Gamma)} + \|v_0 - \tilde{\Lambda}_2\|^2_{L^2(\Gamma)} = 0 \end{aligned} \tag{2.33}$$

for every $t \in (0, T)$. In particular, assertion (i) follows, since the boundedness for $\tilde{u}$ and $\tilde{v}$ just relies on the boundedness assumption for $\tilde{V}$ and is independent from calculations for $\tilde{V}_{\text{new}}$. For the rescaled functions we conclude that there exists a finite $\lambda > 0$, such that

$$\begin{aligned}\|\tilde{u}\|_{L^\infty(\Gamma_T)},\ \|\tilde{v}\|_{L^\infty(\Gamma_T)} &\leq \tilde{\Lambda}_2, \\ \|\tilde{V}_{\text{new}}\|_{L^\infty(\Omega_T)} \leq \|\tilde{\Psi}\|_{L^\infty(\Omega_T)} &\leq 2\|V_0\| + 1 + C_1 C_{fq}\tilde{\Lambda}_2\end{aligned}$$

where we used Lemma 2.1. These maximum bounds are only depending on the initial values, T and system constants. Therefore, the second assertion holds. □

2.2 Solutions on the boundary

In this Subsection we find that there exist solutions $\tilde{u}, \tilde{v}$ of (2.5a)–(2.5c) in the parabolic Hölder space $H^{2+\beta,(2+\beta)/2}(\overline{\Gamma_T})$, where $0 < \beta \le \alpha$ if only u_0, v_0 are of class $C^{2+\alpha}(\Gamma)$ and $\tilde{V} \in H^{\beta,\beta/2}(\overline{\Omega_T})$ holds. We denote the corresponding Hölder norms by $|\cdot|^{(2+\beta)}_{\Gamma_T}$ and $|\cdot|^{(2+\beta)}_{\Omega_T}$, respectively. A detailed description of Hölder spaces is presented in the Appendix, see Section A.1. For given $q \in (2,\infty)$ we use parabolic Sobolev spaces denoted by $W^{2,1}_q(\Gamma_T)$, see [LSU68, p. 5] for further details. Here, let $\|\cdot\|^{2,1}_{q,\Gamma_T}$ denote the corresponding norm. We prove the following statement:

Proposition 2.4 (Solutions on the boundary)**.** *Let Assumptions 1.1–1.2 hold, let $T > 0$ and assume $0 < \beta \le \alpha$. Let $\|\tilde{V}\|_{L^\infty(\Omega_T)} < \tilde{\Lambda}_1$ for $\tilde{\Lambda}_1 > 0$ from Lemma 2.3, let $\lambda > 0$ satisfy Conditions* (2.17), (2.21) *and $\tilde{V}$ be of class $H^{\beta,\beta/2}(\overline{\Omega_T})$. Then, the following assertions hold:*

(i) *there exists a unique solution $(\tilde{u}, \tilde{v}) \in \left(H^{2+\beta,(2+\beta)/2}(\overline{\Gamma_T})\right)^2$ of* (2.5a)–(2.5c)*,*

(ii) *for a constant $C^{uv}_1 = C^{uv}_1(T,\Omega,\beta) > 0$ the solution $(\tilde{u}, \tilde{v})$ satisfies the estimates*

$$|\tilde{u}|^{(2+\beta)}_{\Gamma_T}, |\tilde{v}|^{(2+\beta)}_{\Gamma_T} \le C^{uv}_1 \left(|\tilde{V}|^{(\beta)}_{\Omega_T} + \|u_0\|_{C^{2+\alpha}(\Gamma)} + \|v_0\|_{C^{2+\alpha}(\Gamma)}\right),$$

(iii) *for all $0 \le \sigma < 1$ there exists a constant $C^{uv}_2 = C^{uv}_2(T,\Omega,\sigma) > 0$, such that*

$$|\tilde{u}|^{(\sigma)}_{\Gamma_T},\ |\tilde{v}|^{(\sigma)}_{\Gamma_T} \le C^{uv}_2 \left(\tilde{\Lambda}_1 + \tilde{\Lambda}_2 + \|u_0\|_{C^{2+\alpha}(\Gamma)} + \|v_0\|_{C^{2+\alpha}(\Gamma)}\right),$$

(iv) *for all $0 \le \sigma < 1$ there exists a constant $C^{uv}_3 > 0$ depending on T, $\tilde{\Lambda}_1$, $\tilde{\Lambda}_2$, σ, $\|u_0\|_{C^{2+\alpha}(\Gamma)}$ and $\|v_0\|_{C^{2+\alpha}(\Gamma)}$, such that*

$$\|\tilde{u}\|_{C^0([0,T];C^{1+\sigma}(\Gamma))}, \|\tilde{v}\|_{C^0([0,T];C^{1+\sigma}(\Gamma))} \le C^{uv}_3.$$

Before we start with the proof of Proposition 2.4, we have to prove Lemma 2.5. In the proof we are referring to a family of local charts $\{\varphi_\alpha\}_\alpha$ to cover the smooth manifold Γ. Since Γ is compact we find that there exists a finite number $N \in \mathbb{N}$ and local charts $\varphi_i : U_i \subset \mathbb{R}^2 \to W_i \subset \Gamma$ for $i = 1, \dots, N$, such that Γ is covered. These local charts imply a pull back metric denoted by g. This atlas will be denoted by $\mathcal{A} = (W_i, \varphi_i)_{i=1,\dots,N}$. A more detailed description is presented in Subsection A.1.3 in the Appendix.

Lemma 2.5. *Let $d \ge 1$ and $q \in (2,\infty)$ be arbitrary. We assume that $F \in C^0(\overline{\Gamma_\delta})$ for given $\delta > 0$ and $w_0 \in C^2(\Gamma)$. Let*

$$w \in C([0,\delta), C^2(\Gamma)) \cap C^{2,1}(\Gamma \times (0,\delta))$$

be a solution of

$$\begin{cases} \partial_t w - d\Delta_\Gamma w = F \text{ on } \Gamma \times (0,\delta), & (2.34a) \\ \qquad w(\cdot,0) = w_0 \text{ on } \Gamma. & (2.34b) \end{cases}$$

Then, for any $q \geq 2$ there exists a constant $C_q^w > 0$ depending on δ, q and Γ, such that

$$\|w\|_{q,\Gamma_\delta}^{2,1} \leq C_q^w \left(\|F\|_{C^0(\overline{\Gamma_\delta})} + \|w_0\|_{C^2(\Gamma)} \right)$$

holds. Moreover, for any $0 \leq \sigma < 1$ there exists a constant $C_\sigma^w > 0$ depending on δ, q, σ and Γ, such that

$$\|w\|_{C^0([0,\delta];C^{1+\sigma}(\Gamma))} + |w|_{\Gamma_\delta}^{(\sigma)} \leq C_\sigma^w \left(\|F\|_{C^0(\overline{\Gamma_\delta})} + \|w_0\|_{C^{2+\alpha}(\Gamma)} \right).$$

Proof. The proof consists of three steps. First, we localize the given solution and pull back to a family of flat domains in the plane to find an auxiliary problem. For the auxiliary problem we establish estimates in parabolic Sobolev spaces which will be applied to the original problem resulting in the first claim. In addition, we deduce suitable estimates in Hölder spaces of order σ in the third step.

1st step: In the following we choose a suitable atlas $\mathcal{A}$. Let $\widetilde{W}_i \subset\subset W_i \subset \Gamma$ for $i = 1, \ldots, N$ be compactly included subsets of Γ, such that

$$\bigcup_{i=1}^{N} \widetilde{W}_i \supset \Gamma$$

holds and such that local parametrizations $\varphi_i : B^2(0,2) \to W_i$ exist, such that

$$\varphi_i \left(B^2(0,2) \right) = W_i \quad \text{and} \quad \varphi_i \left(B^2(0,1) \right) = \widetilde{W}_i$$

holds for all $i = 1, \ldots, N$. Let $B^2(0,r)$ denote the ball of radius r with center zero in $\mathbb{R}^2$. By restricting ourselves to $\widetilde{W}_i$ on Γ or $B^2(0,2)$ in $\mathbb{R}^2$, respectively, and setting $\widetilde{w}_i \stackrel{\text{def}}{=} w \circ \varphi_i$ we obtain that $\widetilde{w}_i$ satisfies

$$\begin{cases} \partial_t \widetilde{w}_i - d\,\widetilde{A} : D^2 \widetilde{w}_i + \widetilde{B} \cdot \nabla \widetilde{w}_i = \widetilde{F}_i & \text{in} \quad B^2(0,2) \times (0,\delta), & (2.35a) \\ \widetilde{w}_i(\cdot,0) = \widetilde{w}_{i,0} \stackrel{\text{def}}{=} w_0 \circ \varphi_i & \text{in} \quad B^2(0,2), & (2.35b) \end{cases}$$

where $\widetilde{A}, \widetilde{B} : B^2(0,2) \to \mathbb{R}$ are uniformly bounded by a constant determined only by the atlas and where $\widetilde{F}_i = F \circ \varphi_i$.

2nd step: We will next establish $\|\cdot\|_{q,\Gamma_\delta}^{2,1}$-estimates for $\widetilde{w}_i$ for an arbitrary fixed $q \geq 2$. Therefore, we first assume that $\widetilde{w}_{i,0} = 0$ and extend $\widetilde{w}_i$ by setting $\widetilde{w}_i = 0$ on $B^2(0,2) \times (-\infty, 0)$. Moreover, we set

$$\widetilde{F}(\cdot,t) = 0 \quad \text{for} \quad t \in (-\infty, 0),$$

to find that $\widetilde{w}_i$ is a $W^{2,1}$-solution of

$$\partial_t \widetilde{w}_i - d\,\widetilde{A} : D^2 \widetilde{w}_i + \widetilde{B} \cdot \nabla \widetilde{w}_i = \widetilde{F}_i \quad \text{in} \quad B^2(0,2) \times (-\infty,\delta). \tag{2.35a'}$$

It follows from the interior L^p-estimates [Lie96, Theorem 7.22, p. 175] that

$$\|D^2\widetilde{w}_i\|_{L^q(B^2(0,1)\times(0,\delta))} + \|\partial_t\widetilde{w}_i\|_{L^q(B^2(0,1)\times(0,\delta))} \\ \leq c_1\left(\|\widetilde{F}_i\|_{L^q(B^2(0,2)\times(0,\delta))} + \|\widetilde{w}_i\|_{L^q(B^2(0,2)\times(0,\delta))}\right) \tag{2.36}$$

holds, where $c_1 > 0$ only depends on the diffusion constant d and the atlas $\mathcal{A}$. Now

$$\|\widetilde{F}_i\|_{L^q(B^2(0,2)\times(0,\delta))} \leq c_2\|F\|_{L^q(\Gamma\times(0,\delta))} \quad \text{and} \quad \|\widetilde{w}_i\|_{L^q(B^2(0,2)\times(0,\delta))} \leq c_3\|w\|_{L^q(\Gamma\times(0,\delta))}$$

for constants $c_2, c_3 > 0$ only depending on $\mathcal{A}$. Using an Ehrling-type inequality [Rou13, p. 207] we deduce that

$$\|\widetilde{w}_i\|^{2,1}_{q,(B^2(0,1)\times(0,\delta))} \leq c_4\left(\|F\|_{L^q(\Gamma\times(0,\delta))} + \|w\|_{L^q(\Gamma\times(0,\delta))}\right) \tag{2.37}$$

for $c_4 > 0$. In the case of general initial datum $\widetilde{w}_{i,0}$ we can apply this result to $\widetilde{w}_i - \widetilde{w}_{i,0}$ and deduce that

$$\|\widetilde{w}_i\|^{2,1}_{q,(B^2(0,1)\times(0,\delta))} \leq c_5\left(\|F\|_{L^q(\Gamma\times(0,\delta))} + \|w\|_{L^q(\Gamma\times(0,\delta))} + \|w_0\|_{C^2(\Gamma)}\right)$$

for $c_5 > 0$ depending on d, $\mathcal{A}$ and δ. Using that $\widetilde{w}_i = w \circ \varphi_i$ we find

$$\|w\|^{2,1}_{q,\widetilde{W}_i\times(0,\delta)} \leq c_6\left(\|F\|_{L^q(\Gamma\times(0,\delta))} + \|w\|_{L^q(\Gamma\times(0,\delta))} + \|w_0\|_{C^2(\Gamma)}\right)$$

and, since $\Gamma \subset \cup_{i=1}^N \widetilde{W}_i$ we obtain

$$\|w\|^{2,1}_{q,\Gamma\times(0,\delta)} \leq c_7\left(\|F\|_{L^q(\Gamma\times(0,\delta))} + \|w\|_{L^q(\Gamma\times(0,\delta))} + \|w_0\|_{C^2(\Gamma)}\right), \tag{2.38}$$

where $c_7 > 0$ only depends on d, $\mathcal{A}$ and δ. Next, we observe with (2.38) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|w(\cdot,t)\|^q_{L^q(\Gamma)} = \int_\Gamma q\partial_t w(\cdot,t)|w(\cdot,t)|^{q-1} \\
&\quad\leq q\|\partial_t w(\cdot,t)\|_{L^q(\Gamma)}\|w(\cdot,t)\|^{q-1}_{L^q(\Gamma)} \\
&\quad\leq c_8\left(\|F(\cdot,t)\|_{L^q(\Gamma)} + \|w(\cdot,t)\|_{L^q(\Gamma\times(0,\delta))} + \|w_0\|_{C^2(\Gamma)}\right)\|w(\cdot,t)\|^{q-1}_{L^q(\Gamma)} \\
&\quad\leq c_8\left(\|w(\cdot,t)\|^q_{L^q(\Gamma)} + \left(\|F(\cdot,t)\|_{L^q(\Gamma)} + \|w_0\|_{C^2(\Gamma)}\right)\|w(\cdot,t)\|^{q-1}_{L^q(\Gamma)}\right) \\
&\quad\leq c_8\|w(\cdot,t)\|^q_{L^q(\Gamma)} + c_8\left(\|F(\cdot,t)\|^q_{L^q(\Gamma)} + \|w_0\|^q_{C^2(\Gamma)}\right),
\end{aligned} \tag{2.39}$$

where $c_8 = c_8(q) > 0$ changes from line to line and we used Hölder's and Young's Inequality. We apply Gronwall's Lemma A.9 to (2.39) to find a bound for the L^q-norm of $\widetilde{w}_i$ and in particular

$$\|w\|_{L^q(\Gamma\times(0,\delta))} \leq c_9\left(\|F\|_{L^q(\Gamma\times(0,\delta))} + \|w_0\|_{C^2(\Gamma)}\right) \tag{2.40}$$

with a constant $c_9 > 0$. We finally obtain from (2.40) and (2.38) that

$$\|w\|_{q,\Gamma\times(0,\delta)}^{2,1} \le C_q^w \left(\|F\|_{L^q(\Gamma\times(0,\delta))} + \|w_0\|_{C^2(\Gamma_\delta)}\right), \tag{2.41}$$

where $C_q^w > 0$ depends on δ, d, q and $\mathcal{A}$ and therefore on the geometry of Γ.

3rd step: It remains to prove the estimate for the parabolic Hölder norm for $|w|_{\Gamma_\delta}^{(\sigma)}$ and its first spatial derivative. In the flat case, such that $\Gamma \subset \mathbb{R}^2$ would be an open domain with sufficiently smooth boundary, [LSU68, Lemma 3.3, p. 80] implies the following. For any $0 \le \sigma < 1$, we choose $q \in (4, \infty)$, such that $q > 4/(1-\sigma)$ holds. Then, there exist constants $\tilde{c}_1, \tilde{c}_2 > 0$ depending on q, σ and Γ, such that

$$|\tilde{w}_i|_{\Gamma_\delta}^{(\sigma)} \le \tilde{c}_1 \|\tilde{w}_i\|_{\Gamma\times(0,\delta)}^{2,1} \quad \text{and} \quad |D\tilde{w}_i|_{\Gamma_\delta}^{(\sigma)} \le \tilde{c}_2 \|\tilde{w}_i\|_{\Gamma\times(0,\delta)}^{2,1}$$

holds. Using a suitable atlas $\mathcal{A}$ as above one therefore deduces the estimates claimed in the Lemma. □

Proof of Proposition 2.4. We divide the proof into three steps. We begin with an existence and uniqueness result for solutions $(\tilde{u}, \tilde{v})$ of (2.5a)–(2.5c) for a short-time $\delta > 0$ according to the results of Lamm. Afterwards, we treat the given equations separately and find suitable estimates in parabolic Sobolev spaces and parabolic Hölder spaces with the help of Lemma 2.5. According to uniform estimates of parabolic Hölder norms of order $(2+\alpha)$ we deduce that the short-time solution can be continued to $[0, T]$ such that assertions (i)–(iv) hold.

1st step: For convenience, we restate system (2.5a)–(2.5c), i.e.

$$\begin{cases} \partial_t \tilde{u} = \Delta_\Gamma \tilde{u} + \tilde{f}_1(t,\tilde{u},\tilde{v})\tilde{v} - \tilde{f}_2(t,\tilde{u},\tilde{v})\tilde{u} - \lambda\tilde{u} & \text{on } \Gamma\times(0,T), \quad (2.5a) \\ \partial_t \tilde{v} = d\Delta_\Gamma \tilde{v} - \tilde{f}_1(t,\tilde{u},\tilde{v})\tilde{v} + \tilde{f}_2(t,\tilde{u},\tilde{v})\tilde{u} & \\ \qquad\qquad + \tilde{q}_1(t,\tilde{u},\tilde{v})\tilde{V} - \tilde{q}_2(t,\tilde{u},\tilde{v})\tilde{v} - \lambda\tilde{v} & \text{on } \Gamma\times(0,T), \quad (2.5b) \\ \tilde{u}(\cdot,0) = u_0 \text{ and } \tilde{v}(\cdot,0) = v_0 & \text{on } \Gamma, \quad (2.5c) \end{cases}$$

where $\tilde{V}$ is assumed to be essentially bounded by $\tilde{\Lambda}_1 > 0$ from Lemma 2.3 and being in the space $H^{\beta,\beta/2}(\overline{\Omega_T})$. Then, for the system (2.5a)–(2.5c), we find with Assumption 1.2 that the coefficients are β-Hölder continuous. Then, [Lam02, Satz 2.4.5, p. 57] in combination with [Lam02, Bemerkung 2.4.4, p. 56] and the observation that this Theorem holds also true for systems of strongly parabolic equations on a compact manifold Γ yields that there exists a $\delta > 0$ and a unique solution $(\tilde{u}, \tilde{v})$ of (2.5a)–(2.5c) of class $H^{2+\beta,(2+\beta)/2}(\overline{\Gamma_\delta})$. We notice that $\delta > 0$ depends on the Lipschitz-constants of the nonlinearities, d, $\|u_0\|_{C^{2+\alpha}(\Gamma)}$, $\|v_0\|_{C^{2+\alpha}(\Gamma)}$, $|\tilde{V}|_{\Omega_T}^{(\beta)}$.

2nd step: We treat (2.5a) and (2.5b) on $(0, \delta)$ separately to work with scalar parabolic equations on Γ. According to the regularity assumptions for the initial conditions u_0 and v_0 we deduce parabolic regularity results in Sobolev spaces and Hölder spaces to find the desired estimates afterwards. We begin with the observation that since $\|\tilde{V}\|_{L^\infty(\Omega_T)} \le \tilde{\Lambda}_1$ Lemma 2.3 is applicable. We find that

$$\|\tilde{u}\|_{L^\infty(\Gamma_\delta)}, \|\tilde{v}\|_{L^\infty(\Gamma_\delta)} \le \tilde{\Lambda}_2$$

for $\tilde{\Lambda}_2 > 0$ from Condition (2.8). Let functions $\tilde{F}, \tilde{G} : \Gamma \times (0, \delta) \to \mathbb{R}$ be given by

$$\begin{aligned}\tilde{F}(x,t) &\stackrel{\text{def}}{=} \tilde{f}_1(t, \tilde{u}(x,t), \tilde{v}(x,t))\tilde{v}(x,t) - \tilde{f}_2(t, \tilde{u}(x,t), \tilde{v}(x,t))\tilde{u}(x,t) - \lambda\tilde{u}(x,t),\\ \tilde{G}(x,t) &\stackrel{\text{def}}{=} -\tilde{f}_1(t, \tilde{u}(x,t), \tilde{v}(x,t))\tilde{v}(x,t) + \tilde{f}_2(t, \tilde{u}(x,t), \tilde{v}(x,t))\tilde{u}(x,t)\\ &\qquad + \tilde{q}_1(t, \tilde{u}(x,t), \tilde{v}(x,t))\tilde{V}(x,t) - \tilde{q}_2(t, \tilde{u}(x,t), \tilde{v}(x,t))\tilde{v}(x,t) - \lambda\tilde{v}(x,t),\end{aligned}$$

for $(x,t) \in \Gamma \times (0,\delta)$. Since $\tilde{u}$, $\tilde{v}$ and $\tilde{V}$ are known objects we obtain a pair of scalar parabolic equations given by

$$\begin{cases} \partial_t \tilde{u} = \Delta_\Gamma \tilde{u} + \tilde{F} & \text{on } \Gamma \times (0,\delta), \quad (2.43a)\\ \partial_t \tilde{v} = d\Delta_\Gamma \tilde{v} + \tilde{G} & \text{on } \Gamma \times (0,\delta), \quad (2.43b)\\ \tilde{u}(\cdot,0) = u_0 \text{ and } \tilde{v}(\cdot,0) = v_0 & \text{on } \Gamma. \quad (2.43c)\end{cases}$$

With the deductions above we have

$$\|\tilde{F}\|_{L^\infty(\Gamma_\delta)} \le c_1\tilde{\Lambda}_2 \quad \text{and} \quad \|\tilde{G}\|_{L^\infty(\Gamma_\delta)} \le c_1(\tilde{\Lambda}_1 + \tilde{\Lambda}_2) \tag{2.44}$$

where $c_1 > 0$ is a constant depending on the system constants C_f, C_q and on λ. Note that $\tilde{F}$ and $\tilde{G}$ are continuous on $\Gamma \times [0,\delta]$ and $\tilde{u}$ is a unique solution of (2.43a) and $\tilde{v}$ is a unique solution of (2.43b) with the corresponding initial data.

Then, Lemma 2.5 implies that for any $q \ge 2$ there exists constants $C_q^{\tilde{u}}, C_q^{\tilde{v}} > 0$ depending on δ, q and Γ, such that the following estimates for parabolic Sobolev norms hold

$$\begin{aligned}\|\tilde{u}\|_{q,\Gamma_\delta}^{2,1} &\le C_q^{\tilde{u}}\left(\|\tilde{F}\|_{C^0(\overline{\Gamma_\delta})} + \|u_0\|_{C^2(\Gamma)}\right),\\ \|\tilde{v}\|_{q,\Gamma_\delta}^{2,1} &\le C_q^{\tilde{v}}\left(\|\tilde{G}\|_{C^0(\overline{\Gamma_\delta})} + \|v_0\|_{C^2(\Gamma)}\right),\end{aligned}$$

and for any $0 \le \sigma < 1$ there exists $q \ge 4$ satisfying $q > 1/(1-\sigma)$ and constants $C_\sigma^{\tilde{u}}$, $C_\sigma^{\tilde{v}} > 0$ depending on δ, q, σ and Γ such that the parabolic σ-Hölder norms are estimated by

$$|\tilde{u}|_{\Gamma_\delta}^{(\sigma)} \le C_\sigma^{\tilde{u}}\left(\|\tilde{F}\|_{C^0(\overline{\Gamma_\delta})} + \|u_0\|_{C^2(\Gamma)}\right), \tag{2.45}$$

$$|\tilde{v}|_{\Gamma_\delta}^{(\sigma)} \le C_\sigma^{\tilde{v}}\left(\|\tilde{G}\|_{C^0(\overline{\Gamma_\delta})} + \|v_0\|_{C^2(\Gamma)}\right). \tag{2.46}$$

In particular, this holds true for $\sigma = \beta$. Moreover, we obtain that with Lemma 2.5 that

$$\begin{aligned}\|\tilde{u}\|_{C^0([0,\delta];C^{1+\sigma}(\Gamma))} &\le C_\sigma^{\tilde{u}}\left(\|\tilde{F}\|_{C^0(\overline{\Gamma_\delta})} + \|u_0\|_{C^2(\Gamma)}\right),\\ \|\tilde{v}\|_{C^0([0,\delta];C^{1+\sigma}(\Gamma))} &\le C_\sigma^{\tilde{v}}\left(\|\tilde{G}\|_{C^0(\overline{\Gamma_\delta})} + \|v_0\|_{C^2(\Gamma)}\right).\end{aligned}$$

With this in hand we go back to the definition of $\tilde{F}$ and $\tilde{G}$ and find with the regularity of $\tilde{f}_1$, $\tilde{f}_2$, $\tilde{q}_1$ and $\tilde{q}_2$ and, additionally with $\tilde{V} \in H^{\beta,\beta/2}(\overline{\Omega_T})$ that there exists a constant $C_{FG} = C_{FG}(\delta,\Gamma,\beta) > 0$ such that

$$|\tilde{F}|_{\Gamma_\delta}^{(\beta)}, |\tilde{G}|_{\Gamma_\delta}^{(\beta)} \le C_{FG}\left(\tilde{\Lambda}_1 + \tilde{\Lambda}_2 + \|u_0\|_{C^{2+\alpha}(\Gamma)} + \|v_0\|_{C^{2+\alpha}(\Gamma)} + |\tilde{V}|_{\Omega_T}^{(\beta)}\right).$$

This regularity property of the right-hand sides of (2.43a)–(2.43b) gives with [Lam02, Satz 2.4.3, p. 55] in combination with [Lam02, Bemerkung 2.4.4., p. 56] an estimate on the Hölder norms of class $(2+\beta)$, i.e. there exists a constant $C_{uv} = C_{uv}(\delta, \Gamma, \beta) > 0$ such that

$$|\tilde{u}|^{(2+\beta)}_{\Gamma_\delta}, |\tilde{v}|^{(2+\beta)}_{\Gamma_\delta} \leq C_{uv}\left(\tilde{\Lambda}_1 + \tilde{\Lambda}_2 + \|u_0\|_{C^{2+\alpha}(\Gamma)} + \|v_0\|_{C^{2+\alpha}(\Gamma)} + |\tilde{V}|^{(\beta)}_{\Omega_T}\right)$$

holds on $(0,\delta)$. Moreover, we deduce from (2.45) and (2.46) that

$$\|\tilde{u}\|_{C^0([0,\delta];C^{1+\sigma}(\Gamma))}, \|\tilde{v}\|_{C^0([0,\delta];C^{1+\sigma}(\Gamma))} \leq \widetilde{C}^{uv}_3,$$

where $\widetilde{C}^{uv}_3$ depends on δ, $\tilde{\Lambda}_1$, $\tilde{\Lambda}_2$, σ and the initial data of $\tilde{u}$ and $\tilde{v}$ of class $2+\alpha$.

3rd step: If $\delta = T$ holds, then assertions (i)–(iii) follows. If $\delta < T$, then we assume that $(0,\delta)$ is the maximal interval of existence for $\tilde{u}$ and $\tilde{v}$. Pick any $\hat{t} \in (\delta - \varepsilon, \delta)$, $\varepsilon > 0$ and let $\delta_2 > 0$. We introduce $\hat{u}, \hat{v} : \Gamma \times (0, \delta_2) \to \mathbb{R}$ satisfying (2.5a)-(2.5b) with initial condition $\hat{u}(\cdot, 0) = \tilde{u}(\cdot, \hat{t})$ and $\hat{v}(\cdot, 0) = \tilde{v}(\cdot, \hat{t})$. The second step implies that the initial conditions have enough regularity to apply the first step. Then, $(\hat{u}, \hat{v})$ is a solution on $(\hat{t}, \hat{t} + \delta_2)$ of class $H^{2+\beta,(2+\beta)/2}(\Gamma \times [\hat{t}, \hat{t} + \delta_2])$. We notice that according to the first step, the existence time $\delta_2 > 0$ is independent of $\hat{t}$ and only depending on $\|\tilde{u}(\hat{t})\|_{C^{2+\beta}(\overline{\Omega})}$, $\|\tilde{v}(\hat{t})\|_{C^{2+\beta}(\overline{\Omega})}$, the nonlinearities and $|\tilde{V}|^{(\beta)}_{\Omega_T}$. Therefore, $\hat{t} + \delta_2 > \delta$ for $(\delta - \hat{t}) \ll 1$ sufficiently small. This yields that $(0,\delta)$ was not maximal, a contradiction to our assumption. Step two implies also that the claimed estimates hold on $(0,T)$. In sum, we find that assertions (i)–(iv) hold. □

2.3 Solutions in the bulk

In this section we obtain that there exists a unique solution $\tilde{V}_{\text{new}}$ of (2.7a)–(2.7c) for given $\tilde{u}$, $\tilde{v}$ in suitable parabolic Hölder spaces of order $2+\beta$. The following Proposition is based on [LSU68, Theorem 5.3, p. 320] and [LSU68, Theorem 7.1, p. 478].

Proposition 2.6 (Solutions for Robin-boundary conditions)**.** *Let* $0 < \beta \leq \alpha$, $T > 0$ *and Assumptions 1.1–1.2 hold. Let* $\lambda > 0$ *be given as in Conditions* (2.17) *and* (2.21)*. Assume* $\tilde{u}, \tilde{v} \in H^{2+\beta,(2+\beta)/2}(\overline{\Gamma_T})$ *with* $\|\tilde{u}\|_{L^\infty(\Gamma_T)}, \|\tilde{v}\|_{L^\infty(\Gamma_T)} \leq \tilde{\Lambda}_2$ *with constants* $\tilde{\Lambda}_1, \tilde{\Lambda}_2 > 0$ *as introduced in Lemma 2.3. Then*

(i) *there exists a unique solution* $\tilde{V}_{new} \in H^{2+\beta,(2+\beta)/2}(\overline{\Omega_T})$ *of* (2.7a)–(2.7c)*,*

(ii) *for a constant* $C_1^V > 0$ *depending on* Ω *and* $|\tilde{u}|_{\Gamma_T}^{(1+\beta)} + |\tilde{v}|_{\Gamma_T}^{(1+\beta)}$*, the solution* $\tilde{V}_{new}$ *satisfies the estimate*

$$|\tilde{V}_{new}|_{\Omega_T}^{(2+\beta)} \leq C_1^V \left(1 + \|V_0\|_{C^{2+\alpha}(\overline{\Omega})}\right)$$

(iii) *there exists* $0 < \kappa < 1$ *depending on* $\|\tilde{u}\|_{C^0([0,T];C^1(\Gamma_T))}$, $\|\tilde{v}\|_{C^0([0,T];C^1(\Gamma_T))}$, T, $\tilde{\Lambda}_1$ *and* $\tilde{\Lambda}_2$ *and a constant* $C_2^V > 0$ *depending on* $\|\tilde{u}\|_{C^0([0,T];C^1(\Gamma_T))}$, $\|\tilde{v}\|_{C^0([0,T];C^1(\Gamma_T))}$, T, $\tilde{\Lambda}_1$, $\tilde{\Lambda}_2$ *and* $\|V_0\|_{C^\beta(\overline{\Omega})}$*, such that*

$$|\tilde{V}_{new}|_{\Omega_T}^{(\kappa)} \leq C_2^V .$$

Proof. The proof is based on the deductions in [LSU68, p. 318ff.] obtaining Schauder estimates of a parabolic Robin-boundary problem. We state [LSU68, problem (5.4), p. 318ff.] and switch to the variable $\tilde{V}_{\text{new}}$ to avoid misconceptions. Let $T > 0$, then we consider for linear uniformly parabolic operators $\mathcal{L}$ with smooth coefficients

$$\begin{cases} \mathcal{L}\tilde{V}_{\text{new}}(x,t) \overset{\text{def}}{=} (\partial_t - D\Delta)\tilde{V}_{\text{new}}(x,t) = \psi(x,t) \quad \text{in } \Omega_T, & (2.47) \\ \tilde{V}_{\text{new}}\Big|_{t=0} = \phi(x) \quad \text{in } \overline{\Omega}, & (2.48) \\ \displaystyle\sum_{i=1}^{3} \mathfrak{b}_i(x,t)\frac{\partial \tilde{V}_{\text{new}}}{\partial x_i} + b(x,t)\tilde{V}_{\text{new}}\Bigg|_{\Gamma_T} = \Phi(x,t) \text{ on } \Gamma_T & (2.49) \end{cases}$$

with additional compatibility conditions of order zero for $x \in \Gamma$ in $t = 0$, i.e.

$$\left(\sum_{i=1}^{3} \mathfrak{b}_i(x,t)\frac{\partial \tilde{V}_{\text{new}}}{\partial x_i} + b(x,t)\tilde{V}_{\text{new}}\right)\Bigg|_{t=0} = \Phi(x,0) \text{ for all } x \in \Gamma, \tag{2.50}$$

see Ladyzhenskaja et. al. [LSU68, p. 320f.] for further details. We specify $\mathfrak{b}_i$, ψ, φ, b and Φ in the latter. We restate the following Theorem.

Theorem 2.7 ([LSU68, Theorem 5.3, p. 320f.], Existence and uniqueness in parabolic Hölder-spaces of order $2+\beta$). *Let $\beta \in (0,1)$ and Γ be $C^{2+\beta}$-regular, the coefficients of the operator $\mathcal{L}$ shall be in the class $H^{\beta,\beta/2}(\overline{\Omega_T})$ and $\mathfrak{b}_i$, $b \in H^{1+\beta,(1+\beta)/2}(\overline{\Gamma_T})$. Then for any $\psi \in H^{\beta,\beta/2}(\overline{\Omega_T})$, $\phi \in C^{2+\beta}(\overline{\Omega})$ and $\Phi \in H^{1+\beta,(1+\beta)/2}(\overline{\Gamma_T})$ satisfying compatibility condition of order zero, problem (2.47)–(2.49) has a unique solution $\tilde{V}_{new}$ of class $H^{2+\beta,(2+\beta)/2}(\overline{\Omega_T})$ and there exists a constant $c = c(\Gamma,\Omega,\mathcal{L},\mathfrak{b},b) > 0$ being independent of ϕ and Φ with*

$$|\tilde{V}_{new}|^{(2+\beta)}_{\Omega_T} \le c\left(|\psi|^{(\beta)}_{\Omega_T} + \|\phi\|_{C^{2+\beta}(\overline{\Omega})} + |\Phi|^{(1+\beta)}_{\Gamma_T}\right).$$

Note that we stated the result of Theorem 2.7 for the case of $(2+\beta)$-regularity. In particular it yields much higher regularity, if all other regularity assumptions are higher.

To prove Proposition 2.6, we show that Theorem 2.7 can be applied, set $\psi \equiv 0$ and let $\phi = V_0$, $\mathfrak{b} = -D\nu$, $b = \tilde{q}_1(t,\tilde{u},\tilde{v})$ and we set $\Phi = \tilde{q}_2(t,\tilde{u},\tilde{v})\tilde{v}$. Then, equations (2.47)–(2.49) are just the same as equations (2.7a)–(2.7c). We assumed that Γ is a smooth manifold, in particular it is $C^{2+\beta}$-regular. The coefficients of the operator $\mathcal{L}$ are of class $H^{\beta,\beta/2}(\overline{\Omega_T})$, conditions on $\mathfrak{b}$ are fulfilled as well as for ψ. With Assumption 1.1, initial conditions are of class $C^{2+\alpha}$, $\alpha \ge \beta$, moreover the compatibility conditions of order $\lfloor(1+\beta)/2\rfloor = 0$ hold, see equation (2.50) compared to equation (1.3). Assumption 1.2 yields that the nonlinearities f_i, q_i, $i = 1,2$ are twice continuously differentiable, in particular this holds for $\tilde{f}_i$ and $\tilde{q}_i$. This implies that for $\tilde{u},\tilde{v} \in H^{2+\beta,(2+\beta)/2}(\overline{\Gamma_T})$ the nonlinearities $b = \tilde{q}_1(t,\tilde{u},\tilde{v}) \in H^{1+\beta,(1+\beta)/2}(\overline{\Gamma_T})$ as well as $\Phi \in H^{1+\beta,(1+\beta)/2}(\overline{\Gamma_T})$. Theorem 2.7 yields a unique solution $\tilde{V}_{\text{new}} \in H^{2+\beta,(2+\beta)/2}(\overline{\Omega_T})$ with the estimate

$$\begin{aligned}
|\tilde{V}_{\text{new}}|^{(2+\beta)}_{\Omega_T} &\le c\left(\|\phi\|_{C^{2+\beta}(\overline{\Omega})} + |\Phi|^{(1+\beta)}_{\Gamma_T}\right)\\
&\le c\left(\|V_0\|_{C^{2+\beta}(\overline{\Omega})} + |q_2(\tilde{u},\tilde{v})\tilde{v}|^{(1+\beta)}_{\Gamma_T}\right)\\
&\le c\left(\|V_0\|_{C^{2+\beta}(\overline{\Omega})} + c_1\left(|\tilde{u}|^{(1+\beta)}_{\Gamma_T} + |\tilde{v}|^{(1+\beta)}_{\Gamma_T}\right)\right)\\
&\le c\left(\|V_0\|_{C^{2+\beta}(\overline{\Omega})} + |\tilde{u}|^{(1+\beta)}_{\Gamma_T} + |\tilde{v}|^{(1+\beta)}_{\Gamma_T}\right),
\end{aligned}$$

where $c_1 > 0$ depends on the system constant C_q and bounds on the derivative of q_2. This proves Proposition 2.6 (i)–(ii). We remark that Theorem 2.7 assures existence of the solution on the a priori given time-interval $[0,T)$.

To prove assertion (iii) we want to apply [LSU68, Theorem 7.1, p. 478]. There are two crucial conditions to check. First, the function $\theta(x,t,\tilde{V}_{\text{new}}) \stackrel{\text{def}}{=} -\tilde{q}_1(t,\tilde{u},\tilde{v})\tilde{V}_{\text{new}} + \tilde{q}_2(t,\tilde{u},\tilde{v})\tilde{v}$ representing the boundary condition has to satisfy the following requirements: there exists a constant $\mu > 0$ such that

$$\left|\theta, \frac{\partial\theta}{\partial\tilde{V}_{\text{new}}}, \frac{\partial\theta}{\partial x}\right| \le \mu$$

holds, see [LSU68, condition (7.5), p. 476]. With Lemma 2.3 and Assumption 1.2 we find that

$$|\theta| \le C_q(\tilde{\Lambda}_1 + \tilde{\Lambda}_2) \text{ and } \left|\frac{\partial\theta}{\partial\tilde{V}_{\text{new}}}\right| = |-\tilde{q}_1| \le C_q.$$

Moreover, we compute

$$\begin{aligned}\left|\frac{\partial\theta}{\partial x}\right| &= \left|\left(-\frac{\partial\tilde{q}_1}{\partial\tilde{u}}(t,\tilde{u},\tilde{v})\tilde{u}_x - \frac{\partial\tilde{q}_1}{\partial\tilde{v}}(t,\tilde{u},\tilde{v})\tilde{v}_x\right)\tilde{V}\right.\\ &\qquad\left. + \left(\frac{\partial\tilde{q}_2}{\partial\tilde{u}}(t,\tilde{u},\tilde{v})\tilde{u}_x + \frac{\partial\tilde{q}_2}{\partial\tilde{v}}(t,\tilde{u},\tilde{v})\tilde{v}_x\right)\tilde{v} + \tilde{q}_2(t,\tilde{u},\tilde{v})\tilde{v}_x\right|\\ &\le 2L_q\tilde{\Lambda}_1\left(|\tilde{u}_x| + |\tilde{v}_x|\right) + 2L_q\tilde{\Lambda}_2\left(|\tilde{u}_x| + |\tilde{v}_x|\right) + C_q|\tilde{v}_x|,\end{aligned}$$

where $L_q > 0$ denotes a local Lipschitz-constant of the nonlinearity. The appearing spatial derivatives of $\tilde{u}$ and $\tilde{v}$ are controlled by $\|\tilde{u}\|_{C^0([0,T];C^1(\Gamma_T))}$ and $\|\tilde{v}\|_{C^0([0,T];C^1(\Gamma_T))}$. Therefore, [LSU68, Condition (7.5), p. 476] is satisfied with μ depending on $\tilde{\Lambda}_1$, $\tilde{\Lambda}_2$, $\|\tilde{u}\|_{C^0([0,T];C^1(\Gamma_T))}$, $\|\tilde{v}\|_{C^0([0,T];C^1(\Gamma_T))}$ and given system constants.

Second, the estimate in [LSU68, Theorem 7.1, p. 478] first only controls oscillations locally in suitable subsets of the time-space cylinder Ω_T (adjacent to a portion of Γ that is contained in a chart of Γ). For $\varepsilon > 0$ we define

$$\Omega_\varepsilon \stackrel{\text{def}}{=} \{x \in \Omega,\ \operatorname{dist}(x,\Gamma) > \varepsilon\}.$$

By $B(x_i, 2\varepsilon)$ we denote balls of radius 2ε with center $x_i \in \Gamma$. According to the compactness of Γ there exists a finite set of points $\{x_i\}_{i=1\ldots N}$ for $N \in \mathbb{N}$, such that

$$\overline{\Omega} \subset \bigcup_{i=1}^{N} B(x_i, 2\varepsilon) \cup \Omega_\varepsilon.$$

To localize solutions on Γ we introduce a family of compactly supported functions $\{\eta_i\}_{i=1,\ldots,N}$ such that $\eta_i \in C_c^\infty(W_i)$ with $\sum_{i=1}^N \eta_i(x) \equiv 1$ for every $x \in \Gamma$ and $W_i = \Gamma \cap B(x_i, 4\varepsilon)$. We find that $\mathcal{T} \stackrel{\text{def}}{=} (W_i, \varphi_i, \eta_i)$ is a local trivialization triplet on Γ. Then,

$$\operatorname{dist}(B(x_i, 2\varepsilon), \Gamma \setminus B(x_i, 4\varepsilon)) \ge \varepsilon.$$

In this setting [LSU68, Theorem 7.1, p. 478] is applicable to

$$\Omega_i \stackrel{\text{def}}{=} \Omega_\varepsilon \cup B(x_i, 2\varepsilon).$$

To be more precise, for $\varrho \in (0, \varepsilon)$ we set

$$Q_\varrho \stackrel{\text{def}}{=} B(x,\varrho) \times (t_0, t_0 + \varrho^2) \quad \text{for} \quad x \in \overline{\Omega} \text{ and } t_0 \in (0,T).$$

Then, for all $Q_\varrho \subset (\Omega_i \times (0,\delta))$ not intersecting $(\Gamma \setminus W_i) \times (0,\delta)$ there exists $0 < \kappa < 1$ determined by $\tilde{\Lambda}_1$ and μ and constants $c_i > 0$ depending on $\tilde{\Lambda}_1$, μ and $\|V_0\|_{C^\beta(\overline{\Omega})}$ and the distances of x to $\Gamma \setminus W_i$ which are at least ε, such that the oscillations of $\tilde{V}_{\text{new}}$ are bounded, i.e.

$$\operatorname{osc}\{\tilde{V}_{\text{new}}, Q_\varrho \cap \Omega_T\} \le c_i \varrho^\kappa.$$

We point out that μ depends on $\|\tilde{u}\|_{C^0([0,T];C^1(\Gamma_T))}$ and $\|\tilde{v}\|_{C^0([0,T];C^1(\Gamma_T))}$ for arbitrary small $\sigma > 0$. With $\overline{\Omega} \subset \cup_i \Omega_i$ and [LSU68, Definition (1.7), p. 7] the bounds on oscillations are sufficient to control the κ-Hölder norm of $\tilde{V}_{\text{new}}$. This finishes the proof. □

2.4 Classical solutions to the fully coupled model

The aim of this Subsection is to find that the updating procedure introduced in Subsection 2.1.1 has a fixed-point $\tilde{V} = \tilde{V}_{\text{new}}$. We use Schauder's Fixed-Point Theorem to find this particular fixed-point and conclude that $(\tilde{V}, \tilde{u}, \tilde{v})$ is a classical solution of the rescaled system (2.4a)–(2.4f). We use the independent results of Chapter 3 to find that this solution is unique. Moreover, we deduce in Proposition 2.11 that solutions depend continuously on their initial data, also using an L^2-continuity property from Chapter 3.

Proposition 2.8 (Existence and uniqueness result for classical solutions of (FCRD))**.** *Let Assumptions 1.1, 1.2 hold, let $T > 0$ be arbitrary. Then,* (FCRD) *has a unique solution (V, u, v) of class $H^{2+\alpha,(2+\alpha)/2}$ on the time-interval $[0, T]$.*

Before we start with the proof of Proposition 2.8, we state Schauder's Fixed-Point Theorem for convenience.

Theorem 2.9 ([Sch13, p. 309], Schauder's Fixed-Point Theorem)**.** *Let X be a Banach space. Suppose $M \subset X$ is non-empty, bounded and convex. Then, every continuous and compact operator $F : M \to M$ has a fixed-point in M.*

Proof of Proposition 2.8. In this proof we follow the updating procedure introduced in Subsection 2.1.1. For arbitrary $T > 0$ we set

$$M \overset{\text{def}}{=} \left\{ \tilde{V} \in H^{\beta,\beta/2}(\overline{\Omega_T}) : \ \|\tilde{V}\|_{L^\infty(\Omega_T)} \le \tilde{\Lambda}_1, \ |\tilde{V}|^{(\beta)}_{\Omega_T} \le \Lambda_1^* \right\},$$

for $\tilde{\Lambda}_1 > 0$ as introduced in Lemma 2.3 and $\Lambda_1^* > 0$ to be specified later. Let $F : M \to M$ describe the updating proceduce by $F : \tilde{V} \mapsto \tilde{V}_{\text{new}}$. We verify that the conditions of Theorem 2.9 are satisfied in this situation.

Let $\beta = \min\{\alpha, \kappa\} \in (0, 1)$ for given $\alpha, \kappa \in (0, 1)$, where κ was introduced in Proposition 2.6. We show below that κ only depends on the data, cf. (2.53). A possible choice for an initial $\tilde{V}$ would be to continue $V_0 \in C^{2+\alpha}(\overline{\Omega})$ as a constant function onto the time-interval $[0, T)$ an multiply it with $e^{-\lambda t}$, where $\lambda > 0$ is already chosen, see Condition (2.21). Then, with an apropriate choice of $\Lambda_1^* > 0$ we obtain that M is non-empty.

Consider $\tilde{V} \in M$, then $\|V_0\|_{L^\infty(\Omega)} \le \tilde{\Lambda}_1$ holds and we obtain with Lemma 2.3 (i) that

$$\|\tilde{u}\|_{L^\infty(\Gamma_T)}, \|\tilde{v}\|_{L^\infty(\Gamma_T)} \le \tilde{\Lambda}_2.$$

Moreover, the nonnegativity result from Lemma 2.2 holds. Then, with Proposition 2.4 we find a unique solution $(\tilde{u}, \tilde{v}) \in \left(H^{2+\beta,(2+\beta)/2}(\overline{\Gamma_T})\right)^2$ of (2.5a)–(2.5c), where the following estimates hold for any $0 \le \sigma < 1$:

$$|\tilde{u}|^{(2+\beta)}_{\Gamma_T}, \ |\tilde{v}|^{(2+\beta)}_{\Gamma_T} \le C_1^{uv} \left(|\tilde{V}|^{(\beta)}_{\Omega_T} + \|u_0\|_{C^{2+\alpha}(\Gamma)} + \|v_0\|_{C^{2+\alpha}(\Gamma)} \right), \tag{2.51}$$

$$|\tilde{u}|^{(\sigma)}_{\Gamma_T}, \ |\tilde{v}|^{(\sigma)}_{\Gamma_T} \le C_2^{uv} \left(\tilde{\Lambda}_1 + \tilde{\Lambda}_2 + \|u_0\|_{C^{2+\alpha}(\Gamma)} + \|v_0\|_{C^{2+\alpha}(\Gamma)} \right), \tag{2.52}$$

$$\|\tilde{u}\|_{C^0([0,T];C^{1+\sigma}(\Gamma))}, \|\tilde{v}\|_{C^0([0,T];C^{1+\sigma}(\Gamma))} \le C_3^{uv}, \tag{2.53}$$

for constants $C_1^{uv} = C_1^{uv}(T, \Omega, \beta) > 0$, $C_2^{uv} = C_2^{uv}(T, \Omega, \sigma) > 0$ and $C_3^{uv} > 0$ depending on $T, \tilde{\Lambda}_1, \tilde{\Lambda}_2, \sigma$, $\|u_0\|_{C^{2+\alpha}(\Gamma)}$ and $\|v_0\|_{C^{2+\alpha}(\Gamma)}$. Proposition 2.6 yields that for given $(\tilde{u}, \tilde{v}) \in H^{2+\beta,(2+\beta)/2}(\overline{\Gamma_T})$ there exists a unique solution $\tilde{V}_{\text{new}} \in H^{2+\beta,(2+\beta)/2}(\overline{\Omega_T})$ of (2.7a)–(2.7c), such that the estimate

$$|\tilde{V}_{\text{new}}|_{\Omega_T}^{(2+\beta)} \leq C_1^V \left(\|V_0\|_{C^{2+\alpha}(\overline{\Omega})} + |\tilde{u}|_{\Gamma_T}^{(1+\beta)} + |\tilde{v}|_{\Gamma_T}^{(1+\beta)} \right) \tag{2.54}$$

hold for a constant $C_1^V = C_1^V(\Omega) > 0$ and such that

$$|\tilde{V}_{\text{new}}|_{\Omega_T}^{(\kappa)} \leq C_2^V, \tag{2.55}$$

holds, where C_2^V depends on $\|\tilde{u}\|_{C^0([0,T];C^1(\Gamma))}$, $\|\tilde{v}\|_{C^0([0,T];C^1(\Gamma))}$, T, $\tilde{\Lambda}_1$, $\tilde{\Lambda}_2$ and $\|V_0\|_{C^\beta(\overline{\Omega})}$ and κ depends on $\|\tilde{u}\|_{C^0([0,T];C^1(\Gamma))}$, $\|\tilde{v}\|_{C^0([0,T];C^1(\Gamma))}$, T, $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$. Combining this with (2.53) we obtain that κ depends only T, $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ and that there exists a constant $\Lambda_1^* > 0$, such that

$$|\tilde{V}_{\text{new}}|_{\Omega_T}^{(\beta)} \leq \Lambda_1^*$$

holds, where Λ_1^* depends on T, $\tilde{\Lambda}_1$, $\tilde{\Lambda}_2$ and β. In addition, Lemma 2.3 (ii) implies that $\|\tilde{V}_{\text{new}}\|_{L^\infty(\Omega_T)} \leq \tilde{\Lambda}_1$ holds. This finishes the updating procedure with a new function $\tilde{V}_{\text{new}}$.

According to the aforementioned estimates, see Lemma 2.3, Proposition 2.4 and Proposition 2.6 we obtain that F maps from M back to M. We study this function F in more detail: From the above estimates we find that the image of F satisfies $F(\tilde{V}) \in H^{2+\beta,(2+\beta)/2}(\overline{\Omega_T})$, in particular we find with (2.54), (2.51) and (2.55) and a constant $c_1 > 0$ that

$$\begin{aligned} |F(\tilde{V})|_{\Omega_T}^{(2+\beta)} &\leq C_1^V(\Omega) \left(\|V_0\|_{C^{2+\alpha}(\overline{\Omega})} + |\tilde{u}|_{\Gamma_T}^{(1+\beta)} + |\tilde{v}|_{\Gamma_T}^{(1+\beta)} \right) \\ &\leq c_1(T, \Omega, \beta) \left(\|V_0\|_{C^{2+\alpha}(\overline{\Omega})} + \|u_0\|_{C^{2+\alpha}(\Gamma)} + \|v_0\|_{C^{2+\alpha}(\Gamma)} + |\tilde{V}|_{\Omega_T}^{(\beta)} \right) \\ &\leq c_1(T, \Omega, \beta) \left(\|V_0\|_{C^{2+\alpha}(\overline{\Omega})} + \|u_0\|_{C^{2+\alpha}(\Gamma)} + \|v_0\|_{C^{2+\alpha}(\Gamma)} + \Lambda_1^* \right). \end{aligned} \tag{2.56}$$

Therefore, $F(\tilde{V})$ is contained in a bounded subset of $H^{2+\beta,(2+\beta)/2}(\overline{\Omega_T})$ and there exists a compact embedding into $H^{\beta,\beta/2}(\overline{\Omega_T})$, see for example [Alt12, Theorem 8.6, p. 321] for the elliptic setting, for parabolic spaces this is true since Ω_T has compact closure and Arzelà-Ascoli's Theorem finds application. Therefore, F is a compact operator.

To find that F is continuous and compact we assume a second solution $(\tilde{V}_{\text{new}}^{(2)}, \tilde{u}^{(2)}, \tilde{v}^{(2)})$ with same initial data generated by a different initiating function $\tilde{V}^{(2)} \in H^{\beta,\beta/2}(\overline{\Omega_T})$. Then, these are solutions of (2.5a)–(2.5c) and (2.7a)–(2.7c). By substracting the corresponding equations we find a new difference system. The boundary equations are given by

$$\begin{aligned}\partial_t(\tilde{u}-\tilde{u}^{(2)}) &= \Delta_\Gamma(\tilde{u}-\tilde{u}^{(2)}) + \tilde{f}_1(t,\tilde{u},\tilde{v})\tilde{v} - \tilde{f}_1(t,\tilde{u}^{(2)},\tilde{v}^{(2)})\tilde{v}^{(2)} \\ &\quad - \tilde{f}_2(t,\tilde{u},\tilde{v})\tilde{u} + \tilde{f}_2(t,\tilde{u}^{(2)},\tilde{v}^{(2)})\tilde{u}^{(2)}, \\ \partial_t(\tilde{v}-\tilde{v}^{(2)}) &= d\Delta_\Gamma(\tilde{v}-\tilde{v}^{(2)}) - \tilde{f}_1(t,\tilde{u},\tilde{v})\tilde{v} + \tilde{f}_1(t,\tilde{u}^{(2)},\tilde{v}^{(2)})\tilde{v}^{(2)} \\ &\quad + \tilde{f}_2(t,\tilde{u},\tilde{v})\tilde{u} - \tilde{f}_2(t,\tilde{u}^{(2)},\tilde{v}^{(2)})\tilde{u}^{(2)} + \tilde{q}_1(t,\tilde{u},\tilde{v})\tilde{V} - \tilde{q}_1(t,\tilde{u}^{(2)},\tilde{v}^{(2)})\tilde{V}^{(2)} \\ &\quad - \tilde{q}_2(t,\tilde{u},\tilde{v})\tilde{v} + \tilde{q}_2(t,\tilde{u}^{(2)},\tilde{v}^{(2)})\tilde{v}^{(2)}\end{aligned}$$

on $\Gamma \times (0,T)$. The bulk equation and the corresponding boundary flux equation are given by

$$\begin{aligned}\partial_t(\tilde{V}_{\text{new}}-\tilde{V}^{(2)}_{\text{new}}) &= D\Delta(\tilde{V}_{\text{new}}-\tilde{V}^{(2)}_{\text{new}}) && \text{on } \Omega\times(0,T), \\ -D\nabla(\tilde{V}_{\text{new}}-\tilde{V}^{(2)}_{\text{new}})\cdot\nu &= \tilde{q}_1(t,\tilde{u},\tilde{v})\tilde{V}_{\text{new}} - \tilde{q}_1(t,\tilde{u}^{(2)},\tilde{v}^{(2)})\tilde{V}^{(2)}_{\text{new}} \\ &\quad - \tilde{q}_2(t,\tilde{u},\tilde{v})\tilde{v} + \tilde{q}_2(t,\tilde{u}^{(2)},\tilde{v}^{(2)})\tilde{v}^{(2)} && \text{on } \Gamma\times(0,T),\end{aligned}$$

with complementary initial conditions

$$\begin{aligned}(\tilde{u}-\tilde{u}^{(2)})(\cdot,0) = 0,\ (\tilde{v}-\tilde{v}^{(2)})(\cdot,0) = 0 \text{ on } \Gamma, \\ (\tilde{V}_{\text{new}}-\tilde{V}^{(2)}_{\text{new}})(\cdot,0) = 0 \text{ on } \overline{\Omega}.\end{aligned}$$

We apply the Mean Value Theorem for the given nonlinearities in the boundary flux condition

$$\begin{aligned}\tilde{q}_1(t,\tilde{u},\tilde{v})\tilde{V}_{\text{new}} - &\tilde{q}_1(t,\tilde{u}^{(2)},\tilde{v}^{(2)})\tilde{V}^{(2)}_{\text{new}} - \tilde{q}_2(t,\tilde{u},\tilde{v})\tilde{v} + \tilde{q}_2(t,\tilde{u}^{(2)},\tilde{v}^{(2)})\tilde{v}^{(2)} \\ &= \tilde{q}_1(t,\tilde{u},\tilde{v})(\tilde{V}_{\text{new}}-\tilde{V}^{(2)}_{\text{new}}) + (\tilde{q}_1(t,\tilde{u},\tilde{v}) - \tilde{q}_1(t,\tilde{u}^{(2)},\tilde{v}^{(2)}))\tilde{V}^{(2)}_{\text{new}} \\ &\quad - \tilde{q}_2(t,\tilde{u},\tilde{v})(\tilde{v}-\tilde{v}^{(2)}) + (-\tilde{q}_1(t,\tilde{u},\tilde{v}) + \tilde{q}_2(t,\tilde{u}^{(2)},\tilde{v}^{(2)}))\tilde{v}^{(2)} \\ &= \tilde{q}_1(t,\tilde{u},\tilde{v})(\tilde{V}_{\text{new}}-\tilde{V}^{(2)}_{\text{new}}) + \tilde{V}^{(2)}_{\text{new}}\ \nabla\tilde{q}_1|_{\xi_1}\cdot\Big((\tilde{u}-\tilde{u}^{(2)}),(\tilde{v}-\tilde{v}^{(2)})\Big)^T \\ &\quad - \tilde{q}_2(t,\tilde{u},\tilde{v})(\tilde{v}-\tilde{v}^{(2)}) + \tilde{v}^{(2)}\ \nabla\tilde{q}_2|_{\xi_2}\cdot\Big((\tilde{u}-\tilde{u}^{(2)}),(\tilde{v}-\tilde{v}^{(2)})\Big)^T,\end{aligned}$$

for $\xi_1, \xi_2 \in \mathbb{R}^2$. In this case we apply [LSU68, Theorem 5.3, p. 320] to this structure to find that

$$|\tilde{V}_{\text{new}}-\tilde{V}^{(2)}_{\text{new}}|^{(2+\beta)}_{\Omega_T} \leq c_2(\Omega)\left(|\tilde{u}-\tilde{u}^{(2)}|^{(1+\beta)}_{\Gamma_T} + |\tilde{v}-\tilde{v}^{(2)}|^{(1+\beta)}_{\Gamma_T}\right), \tag{2.57}$$

with $c_2 = c_2(\Omega) > 0$. We exemplarily examine the second equation on the surface, in particular the nonlinearities are of interest. We obtain

$$\begin{aligned}\Phi_2 \overset{\text{def}}{=} & -\tilde{f}_1(t,\tilde{u},\tilde{v})\tilde{v} + \tilde{f}_1(t,\tilde{u}^{(2)},\tilde{v}^{(2)})\tilde{v}^{(2)} + \tilde{f}_2(t,\tilde{u},\tilde{v})\tilde{u} - \tilde{f}_2(t,\tilde{u}^{(2)},\tilde{v}^{(2)})\tilde{u}^{(2)} \\ & + \tilde{q}_1(t,\tilde{u},\tilde{v})\tilde{V} - \tilde{q}_1(t,\tilde{u}^{(2)},\tilde{v}^{(2)})\tilde{V}^{(2)} - \tilde{q}_2(t,\tilde{u},\tilde{v})\tilde{v} + \tilde{q}_2(t,\tilde{u}^{(2)},\tilde{v}^{(2)})\tilde{v}^{(2)} \\ = & (\tilde{v} - \tilde{v}^{(2)})(-\tilde{f}_1(t,\tilde{u},\tilde{v})) + \tilde{v}^{(2)}(-\tilde{f}_1(t,\tilde{u},\tilde{v}) + \tilde{f}_1(t,\tilde{u}^{(2)},\tilde{v}^{(2)})) \\ & + \tilde{f}_2(t,\tilde{u},\tilde{v})(\tilde{u} - \tilde{u}^{(2)}) + (\tilde{f}_2(t,\tilde{u},\tilde{v}) - \tilde{f}_2(t,\tilde{u}^{(2)},\tilde{v}^{(2)}))\tilde{u}^{(2)} \\ & + \tilde{q}_1(t,\tilde{u},\tilde{v})(\tilde{V} - \tilde{V}^{(2)}) + (\tilde{q}_1(t,\tilde{u},\tilde{v}) - \tilde{q}_1(t,\tilde{u}^{(2)},\tilde{v}^{(2)}))\tilde{V}^{(2)} \\ & + (\tilde{v} - \tilde{v}^{(2)})(-\tilde{q}_2(t,\tilde{u},\tilde{v})) + (\tilde{q}_2(t,\tilde{u}^{(2)},\tilde{v}^{(2)}) - \tilde{q}_2(t,\tilde{u},\tilde{v}))\tilde{v}^{(2)}. \end{aligned} \tag{2.58}$$

Since the nonlinearities are in particular differentiable we obtain with the Mean Value Theorem from (2.58)

$$\begin{aligned} & (\tilde{v} - \tilde{v}^{(2)})(-\tilde{f}_1(t,\tilde{u},\tilde{v}) - \tilde{q}_2(t,\tilde{u},\tilde{v})) \\ & \quad + \tilde{f}_2(t,\tilde{u},\tilde{v})(\tilde{u} - \tilde{u}^{(2)}) + \tilde{q}_1(t,\tilde{u},\tilde{v})(\tilde{V} - \tilde{V}^{(2)}) \\ & \quad + \nabla\tilde{f}_1\Big|_{(\xi_1^1,\xi_2^1)} \cdot \Big((\tilde{u}^{(2)} - \tilde{u}), (\tilde{v}^{(2)} - \tilde{v})\Big)^T \tilde{v}^{(2)} \\ & \quad + \nabla\tilde{f}_2\Big|_{(\xi_1^2,\xi_2^2)} \cdot \Big((\tilde{u} - \tilde{u}^{(2)}), (\tilde{v} - \tilde{v}^{(2)})\Big)^T \tilde{u}^{(2)} \\ & \quad + \nabla\tilde{q}_1|_{(\xi_1^3,\xi_2^3)} \cdot \Big((\tilde{u} - \tilde{u}^{(2)}), (\tilde{v} - \tilde{v}^{(2)})\Big)^T \tilde{V}^{(2)} \\ & \quad + \nabla\tilde{q}_2|_{(\xi_1^4,\xi_2^4)} \cdot \Big((\tilde{u}^{(2)} - \tilde{u}), (\tilde{v}^{(2)} - \tilde{v})\Big)^T \tilde{v}^{(2)}, \end{aligned}$$

where $(\xi_1^i, \xi_2^i) \in \mathbb{R}^2$, $i = 1, \dots, 4$ are intermediate points and the gradient expression shall not be effecting the time-variable of the nonlinearities. The coefficient expressions with $(\tilde{u} - \tilde{u}^{(2)})$, $(\tilde{v} - \tilde{v}^{(2)})$ and $(\tilde{V} - \tilde{V}^{(2)})$ are uniformly bounded with a uniform bound controlled by $|\tilde{V}|_{\Omega_T}^{(2+\beta)}$, $|\tilde{V}^{(2)}|_{\Omega_T}^{(2+\beta)}$ and (2.56). Analogously we define Φ_1 and find also expressions depending on $(\tilde{u} - \tilde{u}^{(2)})$ and $(\tilde{v} - \tilde{v}^{(2)})$. If we consider these two equations as a system on the surface, then Proposition 2.4 implies that there exists a constant $c_3 = c_3(T, \Omega, \beta) > 0$, such that

$$|\tilde{u} - \tilde{u}^{(2)}|_{\Gamma_T}^{(2+\beta)},\ |\tilde{v} - \tilde{v}^{(2)}|_{\Gamma_T}^{(2+\beta)} \leq c_3 \left(|\tilde{V} - \tilde{V}^{(2)}|_{\Omega_T}^{(\beta)}\right), \tag{2.59}$$

holds. In sum we find from (2.57) in combination (2.59) that

$$\begin{aligned} |\tilde{V}_{\text{new}} - \tilde{V}_{\text{new}}^{(2)}|_{\Omega_T}^{(2+\beta)} & \leq c_2(\Omega) \left(|\tilde{u} - \tilde{u}^{(2)}|_{\Gamma_T}^{(1+\beta)} + |\tilde{v} - \tilde{v}^{(2)}|_{\Gamma_T}^{(1+\beta)}\right) \\ & \leq c_4 \left(|\tilde{V} - \tilde{V}^{(2)}|_{\Omega_T}^{(\beta)}\right), \end{aligned} \tag{2.60}$$

for a constant $c_4 > 0$. Therefore, F is a continuous and compact operator.

In this situation Schauder's Fixed-Point Theorem 2.9 can be applied to F. There exists a fixed-point $\tilde{V} = \tilde{V}_{\text{new}}$ in $H^{\beta,\beta/2}(\overline{\Omega_T})$. We notice that this fixed-point does not have to be unique at first sight. With Proposition 2.4 we find $(\tilde{u}, \tilde{v}) \in H^{2+\beta,(2+\beta)/2}(\overline{\Gamma_T})$ and in turn with Proposition 2.6 that $\tilde{V}$ is in $H^{2+\beta,(2+\beta)/2}(\overline{\Omega_T})$. Then, $(\tilde{V}, \tilde{u}, \tilde{v})$ is a triplet of class $H^{2+\beta,(2+\beta)/2}$ and we find that equations (2.5a)–(2.5c) and (2.7a)–(2.7c) are solved. This implies that they are no longer decoupled. Since $\tilde{V}$ is of class $H^{2+\beta,(2+\beta)/2}(\overline{\Omega_T})$, it is of class $H^{\alpha,\alpha/2}(\overline{\Omega_T})$. Then, according to Proposition 2.4 we find that $\tilde{u}$ and $\tilde{v}$ are of parabolic Hölder class $(2+\alpha)$ and therefore in particular $\tilde{V}$ is parabolic $(2+\alpha)$-Hölder regular.

Since we deduced these equations by a rescaling argument from (FCRD), the rescaled functions (V, u, v) are a solution triplet of (FCRD) of parabolic Hölder class $2+\alpha$ since the scaling factor $e^{-\lambda t}$ is sufficiently smooth, see also Subsection 2.1.2.

Moreover, we find that (V, u, v) is also a weak solution of (GFCRD). Chapter 3 implies that weak solutions of (GFCRD) are unique, therefore (V, u, v) as a solution of (FCRD) is unique. This completes the proof. □

Proposition 2.10 (Uniform estimates for classical solutions)**.** *Let* $\tilde{\Lambda}_1$, $\tilde{\Lambda}_2$, C_{fq}, C_1, C_2 *be given as in Lemma 2.3 and*

$$\lambda \geq \max\left\{\frac{C_{fq}^2}{D}C(\Omega), 4C_{fq}, C_{fq}\left(1 + C_1 C_{fq} + \frac{2\|V_0\|_{L^\infty(\Omega)} + 1}{\tilde{\Lambda}_2}\right)\right\}.$$

Then, the unique classical solution (V, u, v) *of* (FCRD) *is nonnegative and uniformly bounded,*

$$\|u\|_{L^\infty(\Gamma_T)}, \|v\|_{L^\infty(\Gamma_T)} \leq e^{\lambda T}\tilde{\Lambda}_2,$$
$$\|V\|_{L^\infty(\Omega_T)} \leq e^{\lambda T}\tilde{\Lambda}_1.$$

Proof. We consider the rescaled framework for variables $(\tilde{V}, \tilde{u}, \tilde{v})$ being the unique solution of the rescaled fully coupled system (FCRD) on $[0, T]$. According to the proof of Proposition 2.8, the set M contains the unique fixed-point $\tilde{V}$ being essentially bounded by $\tilde{\Lambda}_1$ on Ω_T. With Lemma 2.3 (i) we obtain that $\tilde{u}$ and $\tilde{v}$ are essentially bounded by $\tilde{\Lambda}_2$. With the definitions posed in (2.1)–(2.3) we find the desired estimates for V, u and v. □

Proposition 2.11 (Continuous dependence of classical solutions on initial data)**.** *Let the assumptions of Proposition 2.8 hold. Then, solutions* (V, u, v) *of* (FCRD) *depend continuously on their initial data. For* $(V^{(1)}, u^{(1)}, v^{(1)})$ *with initial data* $(V_0^{(1)}, u_0^{(1)}, v_0^{(1)})$ *and* $(V^{(2)}, u^{(2)}, v^{(2)})$ *with initial data* $(V_0^{(2)}, u_0^{(2)}, v_0^{(2)})$ *being solutions of* (FCRD) *on* $[0, T]$, *then there exists a constant* $\hat{C} > 0$, *such that*

$$\begin{aligned} &|V^{(1)} - V^{(2)}|_{\overline{\Omega}_T}^{(2+\alpha)} + |u^{(1)} - u^{(2)}|_{\overline{\Gamma}_T}^{(2+\alpha)} + |v^{(1)} - v^{(2)}|_{\overline{\Gamma}_T}^{(2+\alpha)} \\ &\quad \leq \hat{C}(T, \Omega, \alpha)\left(\|V_0^{(1)} - V_0^{(2)}\|_{C^{2+\alpha}(\overline{\Omega})} + \|u_0^{(1)} - u_0^{(2)}\|_{C^{2+\alpha}(\Gamma)} + \|v_0^{(1)} - v_0^{(2)}\|_{C^{2+\alpha}(\Gamma)}\right). \end{aligned}$$

Proof. We work in the rescaled framework and assume that two different solutions $(\tilde{V}^{(1)}, \tilde{u}^{(1)}, \tilde{v}^{(1)})$ and $(\tilde{V}^{(2)}, \tilde{u}^{(2)}, \tilde{v}^{(2)})$ of the rescaled system related to (FCRD) are given with initial data $(V_0^{(1)}, u_0^{(1)}, v_0^{(1)})$ in the case of $(\tilde{V}^{(1)}, \tilde{u}^{(1)}, \tilde{v}^{(1)})$ and $(V_0^{(2)}, u_0^{(2)}, v_0^{(2)})$ for $(\tilde{V}^{(2)}, \tilde{u}^{(2)}, \tilde{v}^{(2)})$. We remark that the inital data in this framework coincides with the given initial data without rescaling. Then, we find for $\tilde{u}^{(1)} - \tilde{u}^{(2)}$ that

$$\begin{aligned}\partial_t(\tilde{u}^{(1)} - \tilde{u}^{(2)}) = \Delta_\Gamma(\tilde{u}^{(1)} - \tilde{u}^{(2)}) &+ \tilde{f}_1(t, \tilde{u}^{(1)}, \tilde{v}^{(1)})\tilde{v}^{(1)} - \tilde{f}_1(t, \tilde{u}^{(2)}, \tilde{v}^{(2)})\tilde{v}^{(2)} \\ &- \tilde{f}_2(t, \tilde{u}^{(1)}, \tilde{v}^{(1)})\tilde{u}^{(1)} + \tilde{f}_2(t, \tilde{u}^{(2)}, \tilde{v}^{(2)})\tilde{u}^{(2)} \qquad \text{on } \Gamma \times (0,T)\end{aligned}$$

holds, where $\tilde{v}^{(1)} - \tilde{v}^{(2)}$ satisfies

$$\begin{aligned}\partial_t(\tilde{v}^{(1)} - \tilde{v}^{(2)}) = d\Delta_\Gamma(\tilde{v}^{(1)} - \tilde{v}^{(2)}) &- \tilde{f}_1(t, \tilde{u}^{(1)}, \tilde{v}^{(1)})\tilde{v}^{(1)} + \tilde{f}_1(t, \tilde{u}^{(2)}, \tilde{v}^{(2)})\tilde{v}^{(2)} \\ &+ \tilde{f}_2(t, \tilde{u}^{(1)}, \tilde{v}^{(1)})\tilde{u}^{(1)} - \tilde{f}_2(t, \tilde{u}^{(2)}, \tilde{v}^{(2)})\tilde{u}^{(2)} \\ &+ \tilde{q}_1(t, \tilde{u}^{(1)}, \tilde{v}^{(1)})\tilde{V}^{(1)} - \tilde{q}_1(t, \tilde{u}^{(2)}, \tilde{v}^{(2)})\tilde{V}^{(2)} \\ &- \tilde{q}_2(t, \tilde{u}^{(1)}, \tilde{v}^{(1)})\tilde{v}^{(1)} + \tilde{q}_2(t, \tilde{u}^{(2)}, \tilde{v}^{(2)})\tilde{v}^{(2)} \qquad \text{on } \Gamma \times (0,T).\end{aligned}$$

For the bulk equations we find

$$\begin{aligned}\partial_t(\tilde{V}^{(1)} - \tilde{V}^{(2)}) &= D\Delta(\tilde{V}^{(1)} - \tilde{V}^{(2)}) && \text{on } \Omega \times (0,T), \\ -D\nabla(\tilde{V}^{(1)} - \tilde{V}^{(2)}) \cdot \nu &= \tilde{q}_1(t, \tilde{u}^{(1)}, \tilde{v}^{(1)})\tilde{V}^{(1)} - \tilde{q}_1(t, \tilde{u}^{(2)}, \tilde{v}^{(2)})\tilde{V}^{(2)} \\ &\quad - \tilde{q}_2(t, \tilde{u}^{(1)}, \tilde{v}^{(1)})\tilde{v}^{(1)} + \tilde{q}_2(t, \tilde{u}^{(2)}, \tilde{v}^{(2)})\tilde{v}^{(2)} && \text{on } \Gamma \times (0,T).\end{aligned}$$

The initial data is given by

$$(\tilde{u}^{(1)} - \tilde{u}^{(2)})(\cdot, 0) = u_0^{(1)} - u_0^{(2)} \quad \text{and} \quad (\tilde{v}^{(1)} - \tilde{v}^{(2)})(\cdot, 0) = v_0^{(1)} - v_0^{(2)}$$

on Γ and

$$(\tilde{V}^{(1)} - \tilde{V}^{(2)})(\cdot, 0) = V_0^{(1)} - V_0^{(2)} \qquad \text{on } \overline{\Omega}.$$

With the same calculations as in the proof of Proposition 2.8 we rewrite the right-hand sides of this difference system to separate the differences $\tilde{u}^{(1)} - \tilde{u}^{(2)}$, $\tilde{v}^{(1)} - \tilde{v}^{(2)}$ and $\tilde{V}^{(1)} - \tilde{V}^{(2)}$, respectively. This linearization procedure now reveals the same structure as (2.5a)–(2.5c) and (2.7a)–(2.7c). We successively apply Proposition 2.4 and Proposition 2.6 to the differences of the given functions in their corresponding parabolic Hölder norms of order $2 + \alpha$ and find with regard to the dependencies in the given constants that

$$\begin{aligned}&|\tilde{u}^{(1)} - \tilde{u}^{(2)}|_{\Gamma_T}^{(2+\alpha)} + |\tilde{v}^{(1)} - \tilde{v}^{(2)}|_{\Gamma_T}^{(2+\alpha)} + |\tilde{V}^{(1)} - \tilde{V}^{(2)}|_{\Omega_T}^{(2+\alpha)} \\ &\quad \le c_1(T, \Omega, \alpha) \Big(|\tilde{V}^{(1)} - \tilde{V}^{(2)}|_{\Omega_T}^{(\alpha)} + \|u_0^{(1)} - u_0^{(2)}\|_{C^{2+\alpha}(\Gamma)} + \|v_0^{(1)} - v_0^{(2)}\|_{C^{2+\alpha}(\Gamma)} \\ &\qquad + \|V_0^{(1)} - V_0^{(2)}\|_{C^{2+\alpha}(\overline{\Omega})} + |\tilde{u}^{(1)} - \tilde{u}^{(2)}|_{\Gamma_T}^{(1+\alpha)} + |\tilde{v}^{(1)} - \tilde{v}^{(2)}|_{\Gamma_T}^{(1+\alpha)} \Big) \\ &\quad \le c_2(T, \Omega, \alpha) \Big(|\tilde{V}^{(1)} - \tilde{V}^{(2)}|_{\Omega_T}^{(\alpha)} + \|V_0^{(1)} - V_0^{(2)}\|_{C^{2+\alpha}(\overline{\Omega})} \\ &\qquad + \|u_0^{(1)} - u_0^{(2)}\|_{C^{2+\alpha}(\Gamma)} + \|v_0^{(1)} - v_0^{(2)}\|_{C^{2+\alpha}(\Gamma)} \Big) \end{aligned} \tag{2.61}$$

for constants $c_1, c_2 > 0$. We use an Ehrling-type estimate, see for example [Rou13, p. 207], to deduce an estimate of the remaining Hölder-norm of order α for $\varepsilon > 0$ and a constant $c_3 > 0$ that

$$|\tilde{V}^{(1)} - \tilde{V}^{(2)}|_{\Omega_T}^{(\alpha)} \le \varepsilon |\tilde{V}^{(1)} - \tilde{V}^{(2)}|_{\Omega_T}^{(2+\alpha)} + c_3(\varepsilon) \|\tilde{V}^{(1)} - \tilde{V}^{(2)}\|_{L^2(\Omega_T)}$$

holds. We absorb $|\tilde{V}^{(1)} - \tilde{V}^{(2)}|_{\Omega_T}^{(2+\alpha)}$ to the left-hand side of (2.61) and find with the continuous dependence on initial data from Chapter 3 that

$$\begin{aligned}
&|\tilde{u}^{(1)} - \tilde{u}^{(2)}|_{\Gamma_T}^{(2+\alpha)} + |\tilde{v}^{(1)} - \tilde{v}^{(2)}|_{\Gamma_T}^{(2+\alpha)} + |\tilde{V}^{(1)} - \tilde{V}^{(2)}|_{\Omega_T}^{(2+\alpha)} \\
&\quad \le c_4(T, D, \Omega, \alpha) \Big(\|V_0^{(1)} - V_0^{(2)}\|_{L^2(\Omega)} + \|u_0^{(1)} - u_0^{(2)}\|_{L^2(\Gamma)} + \|v_0^{(1)} - v_0^{(2)}\|_{L^2(\Gamma)} \\
&\qquad + \|V_0^{(1)} - V_0^{(2)}\|_{C^{2+\alpha}(\overline{\Omega})} + \|u_0^{(1)} - u_0^{(2)}\|_{C^{2+\alpha}(\Gamma)} + \|v_0^{(1)} - v_0^{(2)}\|_{C^{2+\alpha}(\Gamma)} \Big) \\
&\quad \le \hat{C}(T, D, \Omega, \alpha) \Big(\|V_0^{(1)} - V_0^{(2)}\|_{C^{2+\alpha}(\overline{\Omega})} + \|u_0^{(1)} - u_0^{(2)}\|_{C^{2+\alpha}(\Gamma)} + \|v_0^{(1)} - v_0^{(2)}\|_{C^{2+\alpha}(\Gamma)} \Big)
\end{aligned}$$

holds, what was asserted. □

3 Weak Existence Theory

In this Chapter we present a proof of Theorem 1.2. We show existence, uniqueness, uniform boundedness, nonnegativity and continuous dependency on initial data for a weak solution $(V,u,v) \in L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Gamma)) \times L^2(0,T;H^1(\Gamma))$ for the generalized fully coupled Reaction-Diffusion system (GFCRD) given by

$$\text{(GFCRD)} \begin{cases} \partial_t u = \nabla_\Gamma \cdot (A_u \nabla_\Gamma u) + f_1(u,v)v - f_2(u,v)u & \text{on } \Gamma \times I, \\ \partial_t v = \nabla_\Gamma \cdot (A_v \nabla_\Gamma v) - f_1(u,v)v + f_2(u,v)u & \\ \qquad + q_1(u,v)V - q_2(u,v)v & \text{on } \Gamma \times I, \\ \partial_t V = D\Delta V & \text{in } \Omega \times I, \\ -D\nabla V \cdot \nu = q_1(u,v)V - q_2(u,v)v & \text{on } \Gamma \times I, \\ V(\cdot,0) = V_0 & \text{in } \Omega, \\ v(\cdot,0) = v_0 \text{ and } u(\cdot,0) = u_0 & \text{on } \Gamma. \end{cases}$$

Here A_u, A_v are given as in Assumption 1.5, Assumption 1.3 provides conditions on the initial data and nonlinearities are specified in Assumption 1.4. In Section 3.1 we present a weak formulation of (GFCRD) and deduce a time-discrete recursive approximation scheme in Section 3.2. We solve this recursive approximation scheme using the theory of compactly perturbed monotone operators. Afterwards, we prove the nonnegativity of these time-discrete solutions. In Section 3.3 we show that there exists a limit triplet (V,u,v) of the time-discrete approximation and that (V,u,v) solves (GFCRD). Section 3.4 is devoted to the continuous dependency of solutions on the initial data which implies uniqueness. Additionally, we find that the unique weak solution is nonnegative and satisfies a maximum principle. On the one hand, these bounds are of interest regarding numerical stability and provide on the other hand precise dependencies of maximum bounds on system constants, which will be applied in Chapter 4.

3.1 Weak formulation of (GFCRD)

In this Section we introduce a weak formulation of (GFCRD). The triplet (V, u, v) is called *weak solution* of (GFCRD), if the functions $V \in L^2(0,T;H^1(\Omega))$ and $u, v \in L^2(0,T;H^1(\Gamma))$ satisfy the equations of the *fully coupled weak system* (WS) given by

$$\text{(WS)}\quad \begin{cases} \displaystyle\int_{\Omega_T} \partial_t \eta_1 (V - V_0) = D \int_{\Omega_T} \nabla V \cdot \nabla \eta_1 + \int_{\Gamma_T} q(V,u,v)\eta_1, \\ \displaystyle\int_{\Gamma_T} \partial_t \eta_2 (u - u_0) = \int_{\Gamma_T} A_u \nabla_\Gamma u \cdot \nabla_\Gamma \eta_2 - \int_{\Gamma_T} f(u,v)\eta_2, \\ \displaystyle\int_{\Gamma_T} \partial_t \eta_3 (v - v_0) = \int_{\Gamma_T} A_v \nabla_\Gamma v \cdot \nabla_\Gamma \eta_3 + \int_{\Gamma_T} f(u,v)\eta_3 - \int_{\Gamma_T} q(V,u,v)\eta_3, \end{cases}$$

for all

$$\begin{aligned} \eta_1 &\in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \\ \eta_2, \eta_3 &\in H^1(0,T;L^2(\Gamma)) \cap L^2(0,T;H^1(\Gamma)). \end{aligned}$$

Here we assume vanishing final data of the testfunctions, i.e. $\eta_1(\cdot,T) \equiv 0$, $\eta_2(\cdot,T) \equiv 0$ and $\eta_3(\cdot,T) \equiv 0$ in the trace sense.

We remark the following: If (V,u,v) is a strong solution triplet of (GFCRD), then (V,u,v) is also a solution triplet of (WS). We exemplarily varify this claim for the bulk equation. Let $\eta_1 \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ with $\eta_1(T) \equiv 0$. For V being a strong solution an integration by parts in time and space yields that

$$\begin{aligned} &\int_{\Omega_T} \partial_t \eta_1 (V - V_0) - D \int_{\Omega_T} \nabla V \cdot \nabla \eta_1 - \int_{\Gamma_T} q(V,u,v)\eta_1 \\ &\quad = -\int_{\Omega_T} \eta_1 \partial_t (V - V_0) + \int_{\Omega} \eta_1 (V - V_0)\Big|_0^T + D \int_{\Omega_T} (\Delta V)\eta_1 \\ &\qquad - D \int_{\Gamma_T} \nabla V \cdot \nu\, \eta_1 - \int_{\Gamma_T} q(V,u,v)\eta_1 \\ &\quad = -\int_{\Omega_T} \eta_1 \partial_t V - \int_{\Omega} \eta_1(\cdot,0)(V - V_0)(\cdot,0) + D \int_{\Omega_T} (\Delta V)\eta_1 \\ &\qquad - \int_{\Gamma_T} \eta_1 \left(D\nabla V \cdot \nu - q(V,u,v) \right) \\ &\quad = \int_{\Omega_T} \eta_1 \left(-\partial_t V + D\Delta V \right) + \int_{\Gamma_T} \eta_1 \left(-D\nabla V \cdot \nu + q(V,u,v) \right) = 0 \end{aligned}$$

is satisfied for $V(\cdot,0) = V_0$ in Ω. In the particular case of strong solutions we find the V lies in $H^1(0,T;(H^1(\Omega))^*)$, i.e. there exists a time-continuous representant V according to the density of smooth functions in $H^1(0,T;(H^1(\Omega))^*)$. Therefore, the initial data is attained. The remaining equations of system (WS) are obtained in the same way.

3.2 Time-discrete approximation

In this section we introduce the notion of time-discretization and compactly perturbed monotone operators in order to show that there exists a solution to (GFCRD). At the end of the section we show that the solutions are nonnegative if the initial data was nonnegative.

3.2.1 A time-discretization scheme

Here we formulate an implicit Euler scheme for (WS). Let $0 < h < h_0$ be a time-step for $0 < h_0 < 1$ fixed. For fixed $i \in \{1, \dots, \lfloor \frac{T}{h} \rfloor + 1\}$, and already determined

$$(V_{i-1}, u_{i-1}, v_{i-1}) \quad \text{in} \quad L^2(\Omega) \times L^2(\Gamma) \times L^2(\Gamma)$$

the aim is to find solutions $(V_i, u_i, v_i) : \Omega \times \Gamma \times \Gamma \to \mathbb{R}^3$, with $V_i \in H^1(\Omega)$, $u_i, v_i \in H^1(\Gamma)$, such that the recursive scheme $(\text{WS})_i^h$ given by

$$(\text{WS})_i^h \quad \begin{cases} \dfrac{1}{h} \displaystyle\int_\Omega (V_i - V_{i-1})\eta_1 + D \int_\Omega \nabla V_i \cdot \nabla \eta_1 = \int_\Gamma -q(V_i, u_i, v_i)\eta_1, & (3.1) \\ \dfrac{1}{h} \displaystyle\int_\Gamma (u_i - u_{i-1})\eta_2 + \int_\Gamma A_u \nabla u_i \cdot \nabla \eta_2 = \int_\Gamma f(u_i, v_i)\eta_2, & (3.2) \\ \dfrac{1}{h} \displaystyle\int_\Gamma (v_i - v_{i-1})\eta_3 + \int_\Gamma A_v \nabla v_i \cdot \nabla \eta_3 = \int_\Gamma (q(V_i, u_i, v_i) - f(u_i, v_i))\eta_3 & (3.3) \end{cases}$$

is satisfied for all $\eta_1 \in H^1(\Omega)$, $\eta_2, \eta_3 \in H^1(\Gamma)$. For given initial data V_0, u_0 and v_0 according to Assumption 1.3, this recursive scheme is a time-discrete approximation of (WS) for a given time-step h.

We obtain the recursive scheme $(\text{WS})_i^h$ via integration by parts in time in (WS) and discretizing the time-derivatives. Then, the system reduces from a parabolic system to a recursive elliptic system.

3.2.2 Reformulation as a variational inequality

Let $M^1 \stackrel{\text{def}}{=} H^1(\Omega) \times H^1(\Gamma) \times H^1(\Gamma)$ with the product norm on M^1 given by

$$\|(V, u, v)\|_{M^1} \stackrel{\text{def}}{=} \|V\|_{H^1(\Omega)} + \|u\|_{H^1(\Gamma)} + \|v\|_{H^1(\Gamma)}$$

and let $M^0 \stackrel{\text{def}}{=} L^2(\Omega) \times L^2(\Gamma) \times L^2(\Gamma)$. Let $(M^1)^*$ be the dual space of M^1. We define the following operators

$$F : M^1 \longrightarrow (M^1)^*, \quad F_1 : M^1 \longrightarrow H^1(\Omega)^*, \quad F_2, F_3 : M^1 \longrightarrow H^1(\Gamma)^*,$$

where

$$\begin{aligned}\langle\eta_1, F_1(V_i,u_i,v_i)\rangle &\overset{\text{def}}{=} \int_\Omega \frac{1}{h}(V_i - V_{i-1})\eta_1 + \int_\Omega D\nabla V_i\cdot\nabla\eta_1 + \int_\Gamma q(V_i,u_i,v_i)\eta_1,\\ \langle\eta_2, F_2(u_i,v_i)\rangle &\overset{\text{def}}{=} \int_\Gamma \frac{1}{h}(u_i - u_{i-1})\eta_2 + \int_\Gamma A_u\nabla_\Gamma u_i\cdot\nabla_\Gamma\eta_2 - \int_\Gamma f(u_i,v_i)\eta_2,\\ \langle\eta_3, F_3(V_i,u_i,v_i)\rangle &\overset{\text{def}}{=} \int_\Gamma \frac{1}{h}(v_i - v_{i-1})\eta_3 + \int_\Gamma A_v\nabla_\Gamma v_i\cdot\nabla_\Gamma\eta_3\\ &\qquad + \int_\Gamma f(u_i,v_i)\eta_3 - \int_\Gamma q(V_i,u_i,v_i)\eta_3,\\ \langle(\eta_1,\eta_2,\eta_3), F(V_i,u_i,v_i)\rangle &\overset{\text{def}}{=} \langle\eta_1, F_1(V_i,u_i,v_i)\rangle + \langle\eta_2, F_2(u_i,v_i)\rangle + \langle\eta_3, F_3(V_i,u_i,v_i)\rangle\end{aligned}$$

for all $(\eta_1,\eta_2,\eta_3)\in M^1$ and a given previous solutions triplet $(V_{i-1},u_{i-1},v_{i-1})\in M^0$. Exemplarily we varify the well-posedness of F_1 using Assumption 1.4, i.e.

$$\begin{aligned}&\int_\Omega \frac{1}{h}(V_i - V_{i-1})\eta_1 + \int_\Omega D\nabla V_i\cdot\nabla\eta_1 + \int_\Gamma q(V_i,u_i,v_i)\eta_1\\ &\qquad\le \frac{1}{h}(\|V_i\|_{L^2(\Omega)} + \|V_{i-1}\|_{L^2(\Omega)})\|\eta_1\|_{L^2(\Omega)} + D\|\nabla V_i\|_{L^2(\Omega)}\|\nabla\eta_1\|_{L^2(\Omega)}\\ &\qquad\quad + \|q_1(u_i,v_i)V_i - q_2(u_i,v_i)v_i\|_{L^2(\Gamma)}\|\eta_1\|_{L^2(\Gamma)}\\ &\qquad\le \frac{1}{h}\Big((1+D)\|V_i\|_{H^1(\Omega)} + \|V_{i-1}\|_{L^2(\Omega)}\Big)\|\eta_1\|_{H^1(\Omega)}\\ &\qquad\quad + C_q(\|V_i\|_{L^2(\Gamma)} + \|v_i\|_{L^2(\Gamma)})C_{\text{tr}}\|\eta_1\|_{H^1(\Omega)}\\ &\qquad\le C(\Omega)\|\eta_1\|_{H^1(\Omega)}\left(\left(\frac{1}{h}(1+D) + C_qC(\Omega)\right)\|V_i\|_{H^1(\Omega)}\right)\\ &\qquad\quad + C(\Omega)\|\eta_1\|_{H^1(\Omega)}\Big(C_q\|v_i\|_{L^2(\Gamma)} + \|V_{i-1}\|_{L^2(\Omega)}\Big)\end{aligned}$$

Here, we applied the Trace Theorem with $C(\Omega) > 0$, see Corollary A.6, to estimate the boundary values of η_1 and V_i. This yields that the operator norm of F_1 is bounded for L^2-bounded v_i and $V_i\in H^1(\Omega)$. According to Assumption 1.4 the nonlinearity $f(u_i,v_i)$ can be estimated by a constant $C_f > 0$, u_i and v_i. This yields that the operator F_2, resp. F_3, is well-posed with bounded $\|v_i\|_{L^2(\Gamma)}$ and $\|u_i\|_{H^1(\Gamma)}$ norms, resp. bounded $\|u_i\|_{L^2(\Gamma)}$, $\|v_i\|_{H^1(\Gamma)}$ and $\|V_i\|_{H^1(\Omega)}$ norms.

Lemma 3.1 (A variational formulation)**.** *Fix $i\in\{1,\dots,\lfloor\frac{T}{h}\rfloor+1\}$ and assume that $(V_{i-1},u_{i-1},v_{i-1})\in M^0$ holds. If $(V_i,u_i,v_i)\in M^1$ solves the variational inequality*

$$\langle(V_i,u_i,v_i) - (\eta_1,\eta_2,\eta_3), F(V_i,u_i,v_i)\rangle \le 0, \quad \textit{for all } (\eta_1,\eta_2,\eta_3)\in M^1, \tag{3.4}$$

then (V_i,u_i,v_i) solves the time-discrete system $(WS)_i^h$.

Proof. Since (V_i,u_i,v_i) is a solution of (3.4) the inequality

$$\begin{aligned}0 &\ge \langle(V_i,u_i,v_i) - (\eta_1,\eta_2,\eta_3), F(V_i,u_i,v_i)\rangle\\ &= \langle V_i - \eta_1, F_1(V_i,u_i,v_i)\rangle + \langle u_i - \eta_2, F_2(u_i,v_i)\rangle + \langle v_i - \eta_3, F_3(V_i,u_i,v_i)\rangle,\end{aligned} \tag{3.5}$$

holds for every $(\eta_1, \eta_2, \eta_3) \in M^1$. We choose an arbitrary $\zeta \in H^1(\Omega)$. Testing (3.5) with $(\eta_1, \eta_2, \eta_3) = (V_i + \zeta, u_i, v_i)$ and $(V_i - \zeta, u_i, v_i)$ yields

$$\begin{aligned} 0 &= \langle \zeta, F_1(V_i, u_i, v_i) \rangle \\ &= \int_\Omega \frac{1}{h}(V_i - V_{i-1})\zeta + D \int_\Omega \nabla V_i \cdot \nabla \zeta + \int_\Gamma q(V_i, u_i, v_i)\zeta. \end{aligned}$$

Since $\zeta \in H^1(\Omega)$ was arbitrary, this implies that equation (3.1) holds. Testing with $(\eta_1, \eta_2, \eta_3) = (V_i, u_i \pm \zeta, v_i)$ and $(\eta_1, \eta_2, \eta_3) = (V_i, u_i, v_i \pm \zeta)$, where $\zeta \in H^1(\Gamma)$, we achieve that equations (3.2) and (3.3) hold and Lemma 3.1 follows. □

3.2.3 Weak solutions to the time-discretization

In order to show that there exists a weak solution to the time-discrete system $(\mathrm{WS})_i^h$ for given $h > 0$ and a fixed time-step i, we represent F as a sum of two operators. The first part of the operator will be linear and monotone, where the second part is a compact pertubation. We begin with the following definitions.

Let M^1 be given as above being a separable, reflexive Banach space. Let $(M^1)^*$ be the dual space of M^1 and $\langle \cdot, \cdot \rangle : M^1 \times (M^1)^* \to \mathbb{R}$ be the duality product. We say that the map $F : M^1 \to (M^1)^*$ is *bounded* if F maps bounded subsets of M^1 on bounded subsets of $(M^1)^*$. Additionally, F is *coercive* if

$$\frac{\langle \eta, F(\eta) \rangle}{\|\eta\|_{M^1}} \to \infty \text{ for } \eta \in M^1 \tag{3.6}$$

holds for $\|\eta\|_{M^1} \to \infty$. We say that F is a *compactly perturbed monotone operator* if for $L, K : M^1 \to (M^1)^*$, we have that $F(\eta) = L(\eta) + K(\eta)$ for $\eta \in M^1$ satisfying the following properties:

- L is a monotone operator, i.e. for all $\eta,\ \zeta \in M^1$ the inequality
$$\langle \eta - \zeta, L(\eta) - L(\zeta) \rangle \geq 0 \tag{3.7}$$
holds with equality iff $\eta = \zeta$ (strictness),
- L is continuous on finite dimensional subspaces,
- $K : (M^1, \text{weak}) \to ((M^1)^*, \text{weak-}*)$ is continuous and
- $\langle \cdot, K(\cdot) \rangle : (M^1, \text{weak}) \to \mathbb{R}$ is lower semi-continuous.

Moreover, we say that K is a *completely continuous operator* if for an arbitrary sequence $\{\zeta_l\} \subset M^1$ and $\zeta \in M^1$ the weak convergence $\zeta_l \rightharpoonup \zeta$ in M^1 for $l \to \infty$ implies that $K(\zeta_l) \to K(\zeta)$ strong in $(M^1)^*$ for $l \to \infty$, i.e. $K : (M^1, \text{weak}) \to ((M^1)^*, \text{strong})$.

According to these definitions, we are able to formulate the following abstract result about compactly perturbed monotone operators, see [Sch13, Corollary 17.20, p. 342]. It yields existence of solutions to a variational identity if the respective operator satisfies appropriate structure assumptions.

Proposition 3.2. *Let M^1 be a separable, reflexive Banach space. Let $F : M^1 \to (M^1)^*$ be a compactly perturbed monotone operator. Moreover, let F be bounded and coercive. Then there exists $\zeta \in M^1$ such that the variational inequality*

$$\langle \zeta - \eta, F(\zeta) \rangle \leq 0$$

is satisfied for all $\eta \in M^1$.

We now define the operators L and K and prove the necessary conditions of Proposition 3.2 to apply it afterwards. Let $L, K : M^1 \to (M^1)^*$ for $\eta \stackrel{\text{def}}{=} (\eta_1, \eta_2, \eta_3) \in M^1$ be defined by

$$\begin{aligned}
\langle \eta, L(V_i, u_i, v_i) \rangle \stackrel{\text{def}}{=} & D \int_\Omega \nabla V_i \cdot \nabla \eta_1 + \int_\Gamma A_u \nabla_\Gamma u_i \cdot \nabla_\Gamma \eta_2 + \int_\Gamma A_v \nabla_\Gamma v_i \cdot \nabla_\Gamma \eta_3 \\
& + \frac{1}{h} \int_\Omega V_i \eta_1 + \frac{1}{h} \int_\Gamma (u_i \eta_2 + v_i \eta_3), \qquad (3.8) \\
\langle \eta, K(V_i, u_i, v_i) \rangle \stackrel{\text{def}}{=} & \int_\Gamma (q(V_i, u_i, v_i)\eta_1 - f(u_i, v_i)\eta_2 + (f(u_i, v_i) - q(V_i, u_i, v_i))\eta_3) \\
& - \frac{1}{h} \left(\int_\Omega V_{i-1}\eta_1 + \int_\Gamma u_{i-1}\eta_2 + \int_\Gamma v_{i-1}\eta_3 \right). \qquad (3.9)
\end{aligned}$$

We remark that $K = K^{(i)}$ is depending on the previous time-step. First, we show that the operator L has the following properties:

Lemma 3.3. *We have*

(i) $F = (L + K)$,

(ii) $L : M^1 \to (M^1)^*$ *is linear,*

(iii) L *is bounded,*

(iv) L *is coercive and*

(v) L *is monotone.*

Proof.

(i) This property follows immediately.

(ii) Since the differential operators $A_u \nabla_\Gamma, A_v \nabla_\Gamma$ and ∇ are linear the claim follows.

(iii) Consider $(\zeta_1, \zeta_2, \zeta_3) \in M^1$. Applying Cauchy-Schwarz' Inequality, we estimate that

$$\begin{aligned}
\langle \eta, L(\zeta_1, \zeta_2, \zeta_3)\rangle &\le D\|\nabla\zeta_1\|_{L^2(\Omega)} \|\nabla\eta_1\|_{L^2(\Omega)} + \|A_u\nabla_\Gamma\zeta_2\|_{L^2(\Gamma)} \|\eta_2\|_{L^2(\Gamma)} \\
&\quad + \|A_v\nabla_\Gamma\zeta_3\|_{L^2(\Gamma)} \|\eta_3\|_{L^2(\Gamma)} + \frac{1}{h}\|\zeta_1\|_{L^2(\Omega)} \|\eta_1\|_{L^2(\Omega)} \\
&\quad + \frac{1}{h}\|\zeta_2\|_{L^2(\Gamma)} \|\eta_2\|_{L^2(\Gamma)} + \frac{1}{h}\|\zeta_2\|_{L^2(\Gamma)} \|\eta_3\|_{L^2(\Gamma)} \\
&\le \max\left\{D, \frac{1}{h}\right\} \|\zeta_1\|_{H^1(\Omega)}\|\eta_1\|_{H^1(\Omega)} \\
&\quad + \max\left\{C_{A_u}, \frac{1}{h}\right\} \|\zeta_2\|_{H^1(\Gamma)}\|\eta_2\|_{H^1(\Gamma)} \\
&\quad + \max\left\{C_{A_v}, \frac{1}{h}\right\} \|\zeta_3\|_{H^1(\Gamma)}\|\eta_3\|_{H^1(\Gamma)} \\
&\le c_1\|\eta\|_{M^1}\|(\zeta_1, \zeta_2, \zeta_3)\|_{M^1}
\end{aligned}$$

holds, where we used uniform boundedness of A_u and A_v, see Assumption 1.5. Here, $c_1 > 0$ depends on h, D, C_{A_u} and C_{A_v}. This shows that

$$\|L(\zeta_1, \zeta_2, \zeta_3)\|_{(M^1)^*} \le c_1\|(\zeta_1, \zeta_2, \zeta_3)\|_{M^1}$$

holds, hence $L : M^1 \to (M^1)^*$ is bounded and thereby continuous.

(iv) The coercivity of L follows by testing with $(\zeta_1, \zeta_2, \zeta_3) \in M^1$, i.e.

$$\begin{aligned}
\langle(\zeta_1, \zeta_2, \zeta_3), L(\zeta_1, \zeta_2, \zeta_3)\rangle &= D\int_\Omega |\nabla\zeta_1|^2 + \int_\Gamma A_u\nabla_\Gamma\zeta_2 \cdot \nabla_\Gamma\zeta_2 + \int_\Gamma A_v\nabla_\Gamma\zeta_3 \cdot \nabla_\Gamma\zeta_3 \\
&\quad + \frac{1}{h}\int_\Omega \zeta_1^2 + \frac{1}{h}\int_\Gamma \left(\zeta_2^2 + \zeta_3^2\right) \\
&\ge D\|\nabla\zeta_1\|^2_{L^2(\Omega)} + \frac{1}{h}\|\zeta_1\|^2_{L^2(\Omega)} + c_u\|\nabla_\Gamma\zeta_2\|^2_{L^2(\Gamma)} \\
&\quad + \frac{1}{h}\|\zeta_2\|^2_{L^2(\Gamma)} + \frac{1}{h}\|\zeta_3\|^2_{L^2(\Gamma)} + c_v\|\nabla_\Gamma\zeta_3\|^2_{L^2(\Gamma)},
\end{aligned} \tag{3.10}$$

where we used the uniform ellipticity condition (1.8), see Assumption 1.5. Then inequality (3.10) yields

$$\begin{aligned}
\langle(\zeta_1, \zeta_2, \zeta_3), L(\zeta_1, \zeta_2, \zeta_3)\rangle &\ge \min\left\{\frac{1}{h}, D\right\} \|\zeta_1\|^2_{H^1(\Omega)} \\
&\quad + \min\left\{\frac{1}{h}, c_u\right\} \|\zeta_2\|^2_{H^1(\Gamma)} + \min\left\{\frac{1}{h}, c_v\right\} \|\zeta_3\|^2_{H^1(\Gamma)} \\
&\ge \min\left\{\frac{1}{h}, D, c_u, c_v\right\} \left(\|\zeta_1\|^2_{H^1(\Omega)} + \|\zeta_2\|^2_{H^1(\Gamma)} + \|\zeta_3\|^2_{H^1(\Gamma)}\right) \\
&\ge c_2\|(\zeta_1, \zeta_2, \zeta_3)\|^2_{M^1},
\end{aligned}$$

where we used that $0 < h < h_0 < 1$. Here, $c_2 > 0$ depends on D, c_u and c_v. Therefore, L is coercive.

(v) For two elements $\eta, \zeta \in M^1$ we compute for L

$$\langle \eta - \zeta, L(\eta) - L(\zeta) \rangle = \langle \eta - \zeta, L(\eta - \zeta) \rangle \geq c_2 \|\eta - \zeta\|_{M^1}^2 > 0,$$

for all $\eta \neq \zeta$. Here, we used the linerarity and coercivity properties of L. Hence, L is a strictly monotone operator.

□

We remark that the continuity of L implies the continuity on finite dimensional subspaces. Now we let the operator K come into play, which covers the nonlinearities of the system.

Lemma 3.4. *We have*

(i) *K is bounded,*

(ii) *$F = L + K$ is bounded,*

(iii) *K is a completely continuous operator,*

(iv) *$K : (M^1, weak) \to ((M^1)^*, weak\text{-}*)$ is continuous,*

(v) *$\langle \cdot, K(\cdot) \rangle$ is lower semi-continuous and*

(vi) *$F = L + K$ is coercive.*

Proof.

(i) Assume that $(V_{i-1}, u_{i-1}, v_{i-1}) \in M^0$. Let $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in B \subset M^1$, where B is a bounded subset of M^1. We estimate with Cauchy-Schwarz' Inequality to find

$$\begin{aligned}
|\langle \eta, K(\zeta) \rangle| = \Bigg| &\int_\Gamma q(\zeta_1, \zeta_2, \zeta_3)(\eta_1 - \eta_3) + \int_\Gamma f(\zeta_2, \zeta_3)(\eta_3 - \eta_2) \\
&- \frac{1}{h} \int_\Omega V_{i-1}\eta_1 - \frac{1}{h} \left(\int_\Gamma u_{i-1}\eta_2 + \int_\Gamma v_{i-1}\eta_3 \right) \Bigg| \\
\leq \int_\Gamma &\left(|q_1(\zeta_2, \zeta_3)| \, |\zeta_1| \, |\eta_1 - \eta_3| + |q_2(\zeta_2, \zeta_3)| \, |\zeta_3| \, |\eta_1 - \eta_3| \right) \\
&+ \int_\Gamma \left(|f_1(\zeta_2, \zeta_3)| \, |\zeta_3| \, |\eta_3 - \eta_2| + |f_2(\zeta_2, \zeta_3)| \, |\zeta_2| \, |\eta_3 - \eta_2| \right) \\
&+ \frac{1}{h} \Big(\|V_{i-1}\|_{L^2(\Omega)} \|\eta_1\|_{L^2(\Omega)} + \|u_{i-1}\|_{L^2(\Gamma)} \|\eta_2\|_{L^2(\Gamma)} \\
&\quad + \|v_{i-1}\|_{L^2(\Gamma)} \|\eta_3\|_{L^2(\Gamma)} \Big) .
\end{aligned} \tag{3.11}$$

We apply Assumption 1.4 for f and q and see that with respect to the right-hand side of (3.11) we obtain

$$
\begin{aligned}
|\langle \eta, K(\zeta_1,\zeta_2,\zeta_3)\rangle| \le \int_\Gamma C_q|\zeta_1|\,|\eta_1-\eta_3| + \int_\Gamma C_q|\zeta_3|\,|\eta_1-\eta_3| \qquad (3.12)\\
+ \int_\Gamma C_f|\zeta_3|\,|\eta_3-\eta_2| + \int_\Gamma C_f|\zeta_2|\,|\eta_3-\eta_2| \qquad (3.13)\\
+ \frac{1}{h}\Big(\|V_{i-1}\|_{L^2(\Omega)}\|\eta_1\|_{L^2(\Omega)} + \|u_{i-1}\|_{L^2(\Gamma)}\|\eta_2\|_{L^2(\Gamma)}\\
+\|v_{i-1}\|_{L^2(\Gamma)}\|\eta_3\|_{L^2(\Gamma)}\Big)\,. \qquad (3.14)
\end{aligned}
$$

We estimate (3.14) from above with Hölder's Inequality and the Trace Theorem to obtain

$$
\begin{aligned}
&C_q(\|\zeta_1\|_{L^2(\Gamma)} + \|\zeta_3\|_{L^2(\Gamma)})(\|\eta_1\|_{L^2(\Gamma)} + \|\eta_3\|_{L^2(\Gamma)})\\
&\quad + C_f(\|\zeta_3\|_{L^2(\Gamma)} + \|\zeta_2\|_{L^2(\Gamma)})(\|\eta_3\|_{L^2(\Gamma)} + \|\eta_2\|_{L^2(\Gamma)})\\
&\quad + \frac{1}{h}\|\eta\|_{M^1}\Big(\|V_{i-1}\|_{L^2(\Omega)} + \|u_{i-1}\|_{L^2(\Gamma)} + \|v_{i-1}\|_{L^2(\Gamma)}\Big)\\
&\le C_q(C(\Omega)\|\zeta_1\|_{H^1(\Omega)} + \|\zeta_3\|_{H^1(\Gamma)})\cdot(C(\Omega)\|\eta_1\|_{H^1(\Omega)} + \|\eta_3\|_{H^1(\Gamma)})\\
&\quad + C_f(\|\zeta_3\|_{H^1(\Gamma)} + \|\zeta_2\|_{H^1(\Gamma)})(\|\eta_3\|_{H^1(\Gamma)} + \|\eta_2\|_{H^1(\Gamma)})\\
&\quad + \frac{1}{h}\|\eta\|_{M^1}\Big(\|V_{i-1}\|_{L^2(\Omega)} + \|u_{i-1}\|_{L^2(\Gamma)} + \|v_{i-1}\|_{L^2(\Gamma)}\Big)\\
&\le 2C_qC(\Omega)\|(\zeta_1,\zeta_2,\zeta_3)\|_{M^1}(\|\eta_1\|_{H^1(\Omega)} + \|\eta_3\|_{H^1(\Gamma)})\\
&\quad + 2C_f\|(\zeta_1,\zeta_2,\zeta_3)\|_{M^1}(\|\eta_2\|_{H^1(\Gamma)} + \|\eta_3\|_{H^1(\Gamma)})\\
&\quad + \frac{1}{h}\|\eta\|_{M^1}\Big(\|V_{i-1}\|_{L^2(\Omega)} + \|u_{i-1}\|_{L^2(\Gamma)} + \|v_{i-1}\|_{L^2(\Gamma)}\Big)\\
&\le \|\eta\|_{M^1}\frac{1}{h}\Big(\|V_{i-1}\|_{L^2(\Omega)} + \|u_{i-1}\|_{L^2(\Gamma)} + \|v_{i-1}\|_{L^2(\Gamma)}\Big)\\
&\quad + 4\|\eta\|_{M^1}(C_qC(\Omega) + C_f)\|(\zeta_1,\zeta_2,\zeta_3)\|_{M^1}
\end{aligned}
$$

holds, where $C(\Omega) > 0$ may change from line to line. Hence, K is bounded on bounded subsets $B \subset M^1$.

(ii) Since L and K are bounded, $F = L + K$ has the same property.

(iii) K is completely continuous if for $\{\zeta^l\} \subset M^1$, $\zeta \in M^1$, $\zeta^l \rightharpoonup \zeta$ implies strong convergence $K(\zeta^l) \to K(\zeta)$ in $(M^1)^*$ for $l \to \infty$. Therefore, we have to control the operator norm of $K(\zeta^l) - K(\zeta)$. The expressions from the previous time-step in K cancel, since they are not depending on ζ or ζ^l. We define the sequence $(\zeta^l)_l$ in M^1 as $\zeta^l \overset{\text{def}}{=} (\zeta_1^l, \zeta_2^l, \zeta_3^l)$ and write

$$
q^l \overset{\text{def}}{=} q(\zeta_1^l,\zeta_2^l,\zeta_3^l) = q_1^l\zeta_1^l + q_2^l\zeta_3^l = q_1(\zeta_2^l,\zeta_3^l)\zeta_3^l + q_2(\zeta_2^l,\zeta_3^l)\zeta_3^l, \qquad (3.15)
$$

$$
f^l \overset{\text{def}}{=} f(\zeta_2^l,\zeta_3^l) = f_1^l\zeta_2^l + f_2^l\zeta_3^l = f_1(\zeta_2^l,\zeta_3^l)\zeta_2^l + f_2(\zeta_2^l,\zeta_3^l)\zeta_3^l. \qquad (3.16)
$$

If $\|K(\zeta^l) - K(\zeta)\|_{(M^1)^*}$ should vanish for $l \to \infty$, it is sufficient to show that the right-hand side of the expression

$$\sup_{\|\eta\|_{M^1}\leq 1} |\langle \eta, K(\zeta_l) - K(\zeta)\rangle| \leq \sup_{\|\eta\|_{M^1}\leq 1} \Bigg(\left|\int_\Gamma \eta_1(q^l - q)\right| + \left|\int_\Gamma \eta_2(f - f^l)\right| + \left|\int_\Gamma \eta_3(f^l - f)\right| + \left|\int_\Gamma \eta_3(q - q^l)\right| \Bigg)$$

vanishes for $l \to \infty$. Here, let q and f denote the nonlinearities in dependence of the limit objectst $(\zeta_1, \zeta_2, \zeta_3)$ and (ζ_2, ζ_3), respectively. Let $s \in (\frac{4}{3}, 2)$ and r be the Hölder conjugate, i.e. $\frac{1}{r} + \frac{1}{s} = 1$, then $r \in (2, 4)$. With this in hand, one computes with (3.15) and (3.16) and Hölder's Inequality that

$$\begin{aligned} |\langle \eta, K(\zeta^l) - K(\zeta)\rangle| \leq{}& \|\eta_1\|_{L^r(\Gamma)} \Big(\|q_1^l\zeta_1^l - q_1\zeta_1\|_{L^s(\Gamma)} + \|q_2\zeta_3 - q_2^l\zeta_3^l\|_{L^s(\Gamma)} \Big) \\ &+ \|\eta_2\|_{L^r(\Gamma)} \Big(\|f_1^l\zeta_2^l - f_1\zeta_2\|_{L^s(\Gamma)} + \|f_2\zeta_3 - f_2^l\zeta_3^l\|_{L^s(\Gamma)} \Big) \\ &+ \|\eta_3\|_{L^r(\Gamma)} \Big(\|f_1^l\zeta_2^l - f_1\zeta_2\|_{L^s(\Gamma)} + \|f_2\zeta_3 - f_2^l\zeta_3^l\|_{L^s(\Gamma)} \Big) \\ &+ \|\eta_3\|_{L^r(\Gamma)} \Big(\|q_1\zeta_1 - q_1^l\zeta_1^l\|_{L^s(\Gamma)} + \|q_2^l\zeta_3^l - q_2\zeta_3\|_{L^s(\Gamma)} \Big) \end{aligned} \tag{3.17}$$

holds. The aim is now to estimate the right-hand side of (3.17) by the norm on M^1, therefore we want to apply Sobolev's Embedding Theorem A.7. With the setting of Hölder conjugates above we find that the embedding $H^1(\Gamma) \hookrightarrow L^r(\Gamma)$ is compact, since $0 > -\frac{2}{r}$ is always satisfied. We have to treat $\eta_1 \in L^r(\Gamma)$ slightly different, since we need the embedding $H^1(\Omega) \hookrightarrow L^r(\Gamma)$ to be compact. The condition for compact embeddings here is $1 - \frac{3}{2} > -\frac{2}{r}$, which is satisfied since $r < 4$. In the latter we use constants $c_1, c_2 > 0$ from Sobolev's Embedding Theorem A.7. Additionally, we introduce a constant $c_0 > 0$ being the maximum of the aforementioned constants. Thus, the right-hand side of inequality (3.17) is estimated from above by

$$\begin{aligned} & c_1\|\eta_1\|_{H^1(\Omega)} \Big(\|q_1^l\zeta_1^l - q_1\zeta_1\|_{L^s(\Gamma)} + \|q_2\zeta_3 - q_2^l\zeta_3^l\|_{L^s(\Gamma)} \Big) \\ &\quad + c_2\|\eta_2\|_{H^1(\Gamma)} \Big(\|f_1^l\zeta_2^l - f_1\zeta_2\|_{L^s(\Gamma)} + \|f_2\zeta_3 - f_2^l\zeta_3^l\|_{L^s(\Gamma)} \Big) \\ &\quad + c_2\|\eta_3\|_{H^1(\Gamma)} \Big(\|f_1^l\zeta_2^l - f_1\zeta_2\|_{L^s(\Gamma)} + \|f_2\zeta_3 - f_2^l\zeta_3^l\|_{L^s(\Gamma)} \Big) \\ &\quad + c_2\|\eta_3\|_{H^1(\Gamma)} \Big(\|q_1\zeta_1 - q_1^l\zeta_1^l\|_{L^s(\Gamma)} + \|q_2^l\zeta_3^l - q_2\zeta_3\|_{L^s(\Gamma)} \Big) \\ &\leq c_0\|\eta\|_{M^1} \Big(\|q_1^l\zeta_1^l - q_1\zeta_1\|_{L^s(\Gamma)} + \|q_2\zeta_3 - q_2^l\zeta_3^l\|_{L^s(\Gamma)} + \|f_1^l\zeta_2^l - f_1\zeta_2\|_{L^s(\Gamma)} \\ &\qquad + \|f_2\zeta_3 - f_2^l\zeta_3^l\|_{L^s(\Gamma)} + \|f_1^l\zeta_2^l - f_1\zeta_2\|_{L^s(\Gamma)} + \|f_2\zeta_3 - f_2^l\zeta_3^l\|_{L^s(\Gamma)} \\ &\qquad + \|q_1\zeta_1 - q_1^l\zeta_1^l\|_{L^s(\Gamma)} + \|q_2^l\zeta_3^l - q_2\zeta_3\|_{L^s(\Gamma)} \Big) \\ &\leq 2c_0\|\eta\|_{M^1} \Big(\|q_1^l\zeta_1^l - q_1\zeta_1\|_{L^s(\Gamma)} + \|q_2\zeta_3 - q_2^l\zeta_3^l\|_{L^s(\Gamma)} \\ &\qquad + \|f_1^l\zeta_2^l - f_1\zeta_2\|_{L^s(\Gamma)} + \|f_2\zeta_3 - f_2^l\zeta_3^l\|_{L^s(\Gamma)} \Big). \end{aligned} \tag{3.18}$$

Now, we focus on the expression in between the brackets of the right-hand side of (3.18), since this is the expression that has to vanish for $l \to \infty$. We estimate using Triangle Inequality to find

$$\begin{aligned}
&\|q_1^l\zeta_1^l - q_1\zeta_1\|_{L^s(\Gamma)} + \|q_2\zeta_3 - q_2^l\zeta_3^l\|_{L^s(\Gamma)} \\
&\qquad + \|f_1^l\zeta_2^l - f_1\zeta_2\|_{L^s(\Gamma)} + \|f_2\zeta_3 - f_2^l\zeta_3^l\|_{L^s(\Gamma)} \\
&\qquad \le \|q_1^l(\zeta_1^l - \zeta_1)\|_{L^s(\Gamma)} + \|\zeta_1(q_1^l - q_1)\|_{L^s(\Gamma)} + \|\zeta_3(q_2 - q_2^l)\|_{L^s(\Gamma)} \\
&\qquad + \|q_2^l(\zeta_3 - \zeta_3^l)\|_{L^s(\Gamma)} + \|f_1^l(\zeta_2^l - \zeta_2)\|_{L^s(\Gamma)} + \|\zeta_2(f_1^l - f_1)\|_{L^s(\Gamma)} \\
&\qquad + \|\zeta_3(f_2 - f_2^l)\|_{L^s(\Gamma)} + \|f_2^l(\zeta_3 - \zeta_3^l)\|_{L^s(\Gamma)}. \qquad (3.19)
\end{aligned}$$

Then, we use Assumption 1.4 on q and f to have boundedness of q_i^l and f_i^l, $i = 1, 2$ such that the right-hand side of (3.19) is estimated from above by

$$\begin{aligned}
&C_q\|\zeta_1^l - \zeta_1\|_{L^s(\Gamma)} + \|\zeta_1(q_1^l - q_1)\|_{L^s(\Gamma)} + \|\zeta_3(q_2 - q_2^l)\|_{L^s(\Gamma)} \\
&\quad + C_q\|\zeta_3 - \zeta_3^l\|_{L^s(\Gamma)} + C_f\|\zeta_2^l - \zeta_2\|_{L^s(\Gamma)} + \|\zeta_2(f_1^l - f_1)\|_{L^s(\Gamma)} \\
&\quad + \|\zeta_3(f_2 - f_2^l)\|_{L^s(\Gamma)} + C_f\|\zeta_3 - \zeta_3^l\|_{L^s(\Gamma)}. \qquad (3.20)
\end{aligned}$$

In (3.20) there are two cases occuring: first, terms that are linear in ζ^l, ζ and second, the convergence of nonlinearity sequences. We prove the following claims.

Claim. *If* $(\zeta_1^l, \zeta_2^l, \zeta_3^l) \rightharpoonup (\zeta_1, \zeta_2, \zeta_3)$ *in* M^1, *then*

$$\|\zeta_1^l - \zeta_1\|_{L^s(\Gamma)}, \|\zeta_2^l - \zeta_2\|_{L^s(\Gamma)}, \|\zeta_3^l - \zeta_3\|_{L^s(\Gamma)} \longrightarrow 0, \text{ for } l \to \infty. \qquad (3.21)$$

Proof. The compact version of Sobolev's Embedding Theorem A.7, yields that the embedding $H^1(\Omega) \hookrightarrow L^s(\Gamma)$ is compact for $s < 4$ and that $H^1(\Gamma) \hookrightarrow L^s(\Gamma)$ is compact for any $s \in [1, \infty)$. The choice of $s \in (\frac{4}{3}, 2)$ guarantees that both compact embeddings hold. After choosing a subsequence of ζ^l, which we are denoting with the same index (l), the sequence ζ^l converges strongly in $(L^s(\Gamma))^3$. That proves the convergence in (3.21). □

Claim. *If* $(\zeta_1^l, \zeta_2^l, \zeta_3^l) \rightharpoonup (\zeta_1, \zeta_2, \zeta_3)$ *in* M^1, *then*

$$\begin{aligned}
&\|\zeta_1(q_1^l - q_1)\|_{L^s(\Gamma)}, \|\zeta_3(q_2 - q_2^l)\|_{L^s(\Gamma)}, \\
&\quad \|\zeta_2(f_1^l - f_1)\|_{L^s(\Gamma)}, \|\zeta_3(f_2^l - f_2)\|_{L^s(\Gamma)} \to 0 \qquad (3.22)
\end{aligned}$$

for $l \to \infty$.

Proof. We use Hölder's Inequality and constants $c_3, c_4 > 0$ to obtain

$$\begin{aligned}
\|\zeta_1(q_1^l - q_1)\|_{L^s(\Gamma)} &\le \|\zeta_1\|_{L^{2s}(\Gamma)}\|q_1^l - q_1\|_{L^{2s}(\Gamma)} \le c_3\|\zeta_1\|_{H^1(\Omega)}\|q_1^l - q_1\|_{L^{2s}(\Gamma)}, \\
\|\zeta_3(q_2 - q_2^l)\|_{L^s(\Gamma)} &\le \|\zeta_3\|_{L^{2s}(\Gamma)}\|q_2 - q_2^l\|_{L^{2s}(\Gamma)} \le c_4\|\zeta_3\|_{H^1(\Gamma)}\|q_2 - q_2^l\|_{L^{2s}(\Gamma)}, \\
\|\zeta_2(f_1^l - f_1)\|_{L^s(\Gamma)} &\le \|\zeta_2\|_{L^{2s}(\Gamma)}\|f_1^l - f_1\|_{L^{2s}(\Gamma)} \le c_4\|\zeta_2\|_{H^1(\Gamma)}\|f_1^l - f_1\|_{L^{2s}(\Gamma)}, \\
\|\zeta_3(f_2 - f_2^l)\|_{L^s(\Gamma)} &\le \|\zeta_3\|_{L^{2s}(\Gamma)}\|f_2 - f_2^l\|_{L^{2s}(\Gamma)} \le c_4\|\zeta_3\|_{H^1(\Gamma)}\|f_1 - f_1^l\|_{L^{2s}(\Gamma)},
\end{aligned}$$

where we used Sobolev's Embedding Theorem A.7 with $s < 2$. The previous claim implies that $(\zeta_1^l, \zeta_2^l, \zeta_3^l) \to (\zeta_1, \zeta_2, \zeta_3)$ is strongly converging in $\left(L^{2s}(\Gamma)\right)^3$ for $s < 2$. Strong convergence in L^p-spaces implies pointwise, almost everywhere convergence for a suitable subsequence. Here, we denote this subsequence by (l) again. The nonlinearities f and q are supposed to be continuous functions, see Assumption 1.4, wherefore the pointwise almost everywhere convergence remains valid. We have that

$$q_1(\zeta_2^l, \zeta_3^l) \to q_1(\zeta_2, \zeta_3),\ q_2(\zeta_2^l, \zeta_3^l) \to q_2(\zeta_2, \zeta_3) \text{ pointwise a.e.,}$$
$$f_1(\zeta_2^l, \zeta_3^l) \to f_1(\zeta_2, \zeta_3),\ f_2(\zeta_2^l, \zeta_3^l) \to f_2(\zeta_2, \zeta_3) \text{ pointwise a.e.}$$

holds. With Assumption 1.4 we find that

$$|q_1^l|, |q_2^l| \le C_q \quad \text{and} \quad |f_1^l|, |f_2^l| \le C_f$$

holds, hence $C_q, C_f > 0$ are integrable L^{2s}-majorants. Then, by Generalized Lebesgue Convergence Theorem A.5, we find

$$q_1(\zeta_2^l, \zeta_3^l) \to q_1(\zeta_2, \zeta_3), \quad q_2(\zeta_2^l, \zeta_3^l) \to q_2(\zeta_2, \zeta_3) \text{ in } L^{2s}(\Gamma),$$
$$f_1(\zeta_2^l, \zeta_3^l) \to f_1(\zeta_2, \zeta_3), \quad f_2(\zeta_2^l, \zeta_3^l) \to f_2(\zeta_2, \zeta_3) \text{ in } L^{2s}(\Gamma).$$

Hence, the claim and the convergence in (3.22) is proved. □

According to (3.17)–(3.20) and the claims above we find that

$$\begin{aligned}\sup_{\|\eta\|_{M^1}\le 1} |\langle \eta, K(\zeta^l) - K(\zeta)\rangle| \le \sup_{\|\eta\|_{M^1}\le 1} & 2c_5\|\eta\|_{M^1} \Big(C_q\|\zeta_1^l - \zeta_1\|_{L^s(\Gamma)} \\ & + (C_f + C_q)\|\zeta_3 - \zeta_3^l\|_{L^s(\Gamma)} + C_f\|\zeta_2^l - \zeta_2\|_{L^s(\Gamma)} \\ & + \|\zeta_1(q_1^l - q_1)\|_{L^s(\Gamma)} + \|\zeta_3(q_2 - q_2^l)\|_{L^s(\Gamma)} \\ & + \|\zeta_2(f_1^l - f_1)\|_{L^s(\Gamma)} + \|\zeta_3(f_2 - f_2^l)\|_{L^s(\Gamma)}\Big) \to 0\end{aligned}$$

for $l \to \infty$ and a constant $c_5 > 0$. Therefore, K is completely continuous.

(iv) Complete continuity of K implies (weak, weak-$*$)-continuity of K, hence the property follows.

(v) For $\zeta^l \to \zeta$ the complete continuity of K implies $K(\zeta^l) \to K(\zeta)$ in $(M^1)^*$, hence

$$\langle \zeta, K(\zeta)\rangle = \lim_{l\to\infty} \langle \zeta^l, K(\zeta^l)\rangle = \liminf_{l\to\infty} \langle \zeta^l, K(\zeta^l)\rangle$$

holds, in particular $\langle \cdot, K(\cdot)\rangle$ is lower semi-continuous.

(vi) To prove coercivity of F we calculate

$$\begin{aligned}\langle \zeta, F(\zeta)\rangle &= \langle \zeta, L(\zeta)\rangle + \langle \zeta, K(\zeta)\rangle \\ &= D\|\nabla\zeta_1\|^2_{L^2(\Omega)} + \int_\Gamma A_u \nabla_\Gamma \zeta_2 \cdot \nabla_\Gamma \zeta_2 + \int_\Gamma A_v \nabla_\Gamma \zeta_3 \cdot \nabla_\Gamma \zeta_3 \\ &\quad + \frac{1}{h}\left(\|\zeta_1\|^2_{L^2(\Omega)} + \|\zeta_2\|^2_{L^2(\Gamma)} + \|\zeta_3\|^2_{L^2(\Gamma)}\right) \\ &\quad + \int_\Gamma (q(\zeta)\zeta_1 - f(\zeta_2,\zeta_3)\zeta_2 + f(\zeta_2,\zeta_3)\zeta_3 - q(\zeta)\zeta_3) \\ &\quad - \frac{1}{h}\left(\int_\Omega V_{i-1}\zeta_1 + \int_\Gamma u_{i-1}\zeta_2 + \int_\Gamma v_{i-1}\zeta_3\right). \end{aligned} \tag{3.23}$$

Since we are interested in coercivity we are focussing on finding a lower bound with respect to norms tending quadratically to infinity for $\|\zeta\|_{M^1} \to \infty$. For the nonlinearities we calculate

$$\begin{aligned}\int_\Gamma q(\zeta)\zeta_1 &= \int_\Gamma (q_1(\zeta_2,\zeta_3)\zeta_1 - q_2(\zeta_2,\zeta_3)\zeta_3)\zeta_1 \\ &\geq \int_\Gamma q_1(\zeta_2,\zeta_3)\zeta_1^2 - \int_\Gamma C_q|\zeta_1|\,|\zeta_3| \\ &\geq \int_\Gamma q_1(\zeta_2,\zeta_3)\zeta_1^2 - \frac{1}{2}C_q\int_\Gamma \zeta_1^2 - \frac{1}{2}C_q\int_\Gamma \zeta_3^2. \end{aligned} \tag{3.24}$$

In the same way follows

$$\begin{aligned} -\int_\Gamma f(\zeta_2,\zeta_3)\zeta_2 &= \int_\Gamma (-f_1(\zeta_2,\zeta_3)\zeta_3 + f_2(\zeta_2,\zeta_3)\zeta_2)\zeta_2 \\ &\geq \int_\Gamma -C_f|\zeta_3|\,|\zeta_2| + \int_\Gamma f_2(\zeta_2,\zeta_3)\zeta_2^2 \\ &\geq -\frac{1}{2}C_f\int_\Gamma \zeta_3^2 - \frac{1}{2}C_f\int_\Gamma \zeta_2^2 + \int_\Gamma f_2(\zeta_2,\zeta_3)\zeta_2^2. \end{aligned} \tag{3.25}$$

We then compute

$$\begin{aligned}\int_\Gamma (f(\zeta_2,\zeta_3) - q(\zeta))\zeta_3 &= \int_\Gamma (f_1(\zeta_2,\zeta_3)\zeta_3 - f_2(\zeta_2,\zeta_3)\zeta_2)\zeta_3 \\ &\quad + \int_\Gamma (-q_1(\zeta_2,\zeta_3)\zeta_1 + q_2(\zeta_2,\zeta_3)\zeta_3)\zeta_3 \\ &\geq \int_\Gamma (f_1(\zeta_2,\zeta_3) + q_2(\zeta_2,\zeta_3))\zeta_3^2 - C_f\int_\Gamma |\zeta_2|\,|\zeta_3| - \int_\Gamma C_q|\zeta_1|\,|\zeta_3| \\ &\geq \int_\Gamma (f_1(\zeta_2,\zeta_3) + q_2(\zeta_2,\zeta_3))\zeta_3^2 \\ &\quad - \int_\Gamma \left(\frac{1}{2}C_f\zeta_2^2 + \frac{1}{2}C_f\zeta_3^2 + \frac{1}{2}C_q\zeta_1^2 + \frac{1}{2}C_q\zeta_3^2\right). \end{aligned} \tag{3.26}$$

Inequalities (3.24)–(3.26) imply that

$$\begin{aligned}
&\int_\Gamma (q(\zeta)\zeta_1 - f(\zeta_2,\zeta_3)\zeta_2 + (f(\zeta_2,\zeta_3) - q(\zeta))\zeta_3) \\
&\quad \geq \int_\Gamma q_1(\zeta_2,\zeta_3)\zeta_1^2 + \int_\Gamma f_2(\zeta_2,\zeta_3)\zeta_2^2 + \int_\Gamma (f_1(\zeta_2,\zeta_3) + q_2(\zeta_2,\zeta_3))\zeta_3^2 \\
&\qquad - \int_\Gamma (C_q\zeta_1^2 + C_q\zeta_3^2 + C_f\zeta_3^2 + C_f\zeta_2^2) \\
&\quad \geq \int_\Gamma -C_q(\zeta_1^2 + \zeta_3^2) - \int_\Gamma C_f(\zeta_2^2 + \zeta_3^2)
\end{aligned} \tag{3.27}$$

holds. These estimates hold on the boundary Γ. To control ζ_1 by an appropriate term we use

$$\|\zeta_1\|^2_{L^2(\Gamma)} \leq C(\Omega)\frac{1}{\varepsilon}\|\zeta_1\|^2_{L^2(\Omega)} + \varepsilon\|\nabla\zeta_1\|^2_{L^2(\Omega)} \tag{3.28}$$

for $\varepsilon > 0$, $C(\Omega) > 0$, see Lemma A.5. We collect expressions in ζ_1 from the right-hand side of (3.23), set $\varepsilon \overset{\text{def}}{=} \frac{D}{2C_q}$ to find the following estimate for the boundary term with (3.28)

$$\begin{aligned}
&D\|\nabla\zeta_1\|^2_{L^2(\Omega)} + \frac{1}{h}\|\zeta_1\|^2_{L^2(\Omega)} - C_q\int_\Gamma \zeta_1^2 \\
&\qquad \geq (D - C_q\varepsilon)\|\nabla\zeta_1\|^2_{L^2(\Omega)} + \left(\frac{1}{h} - C_q C(\Omega)\frac{1}{\varepsilon}\right)\|\zeta_1\|^2_{L^2(\Omega)} \\
&\qquad \geq \frac{D}{2}\|\nabla\zeta_1\|^2_{L^2(\Omega)} + \left(\frac{1}{h} - C_q^2 C(\Omega)\frac{2}{D}\right)\|\zeta_1\|^2_{L^2(\Omega)}.
\end{aligned} \tag{3.29}$$

We choose $h_0 < \frac{D}{2C_q^2 C(\Omega)}$, i.e. let $h < h_0 = h_0(\Omega, C_q, D)$ be sufficiently small to have a positive coefficient for $\|\zeta_1\|^2_{L^2(\Omega)}$. Considering the estimates from (3.24)–(3.29) we derive from (3.23) that

$$\begin{aligned}
\langle\zeta, F(\zeta)\rangle \geq &\int_\Gamma A_u\nabla_\Gamma\zeta_2\cdot\nabla_\Gamma\zeta_2 + \int_\Gamma A_v\nabla_\Gamma\zeta_3\cdot\nabla_\Gamma\zeta_3 \\
&+ \frac{D}{2}\|\nabla\zeta_1\|^2_{L^2(\Omega)} + \left(\frac{1}{h} - C_q^2 C(\Omega)\frac{2}{D}\right)\|\zeta_1\|^2_{L^2(\Omega)} \\
&+ \left(\frac{1}{h} - C_q - C_f\right)\|\zeta_3\|^2_{L^2(\Gamma)} + \left(\frac{1}{h} - C_f\right)\|\zeta_2\|^2_{L^2(\Gamma)} \\
&- \frac{1}{h}\left(\int_\Omega V_{i-1}\zeta_1 + \int_\Gamma u_{i-1}\zeta_2 + v_{i-1}\zeta_3\right)
\end{aligned} \tag{3.30}$$

holds. The boundedness of $\|V_{i-1}\|_{L^2(\Omega)}$, $\|u_{i-1}\|_{L^2(\Gamma)}$ and $\|v_{i-1}\|_{L^2(\Gamma)}$ yields with Cauchy-Schwarz' Inequality, that the last term of (3.30) can be estimated by a constant times the norm of ζ. This will be denoted by $C_{\text{lin}}\|\zeta\|_{M^1}$. Therefore, and with uniform ellipticity for A_u and A_v, see Assumption 1.5, we have that

$$\begin{aligned}\langle \zeta, F(\zeta)\rangle \geq & c_u\|\nabla_\Gamma \zeta_2\|^2_{L^2(\Gamma)} + c_v\|\nabla_\Gamma \zeta_3\|^2_{L^2(\Gamma)} + \frac{D}{2}\|\nabla \zeta_1\|^2_{L^2(\Omega)} \\ & + \left(\frac{1}{h} - C_f\right)\|\zeta_2\|^2_{L^2(\Gamma)} + \left(\frac{1}{h} - C_q^2 C(\Omega)\frac{2}{D}\right)\|\zeta_1\|^2_{L^2(\Omega)} \\ & + \left(\frac{1}{h} - C_q - C_f\right)\|\zeta_3\|^2_{L^2(\Gamma)} - \frac{1}{h}C_{\text{lin}}\|\zeta\|_{M^1} \\ \geq & \frac{D}{2}\|\nabla \zeta_1\|^2_{L^2(\Omega)} + c_u\|\nabla_\Gamma \zeta_2\|^2_{L^2(\Gamma)} + c_v\|\nabla_\Gamma \zeta_3\|^2_{L^2(\Gamma)} \\ & + \frac{1}{2h}\left(\|\zeta_1\|^2_{L^2(\Omega)} + \|\zeta_2\|^2_{L^2(\Gamma)} + \|\zeta_3\|^2_{L^2(\Gamma)}\right) - \frac{1}{h}C_{\text{lin}}\|\zeta\|_{M^1} \\ \geq & c_6\left(\|\nabla \zeta_1\|^2_{L^2(\Omega)} + \|\nabla_\Gamma \zeta_2\|^2_{L^2(\Gamma)} + \|\nabla_\Gamma \zeta_3\|^2_{L^2(\Gamma)}\right) \\ & + \frac{1}{2h}\left(\|\zeta_1\|^2_{L^2(\Omega)} + \|\zeta_2\|^2_{L^2(\Gamma)} + \|\zeta_3\|^2_{L^2(\Gamma)}\right) - \frac{1}{h}C_{\text{lin}}\|\zeta\|_{M^1}\end{aligned}$$

holds. Here, we assumed that $\frac{1}{2h_0} > C_q + C_f$ and $\frac{1}{2h_0} > C_q^2 C(\Omega)\frac{2}{D}$ holds and set $c_6 > 0$ with $c_6 \overset{\text{def}}{=} \min\{c_u, c_v, \frac{D}{2}\}$. Then, for $\frac{1}{2h_0} > c_6$ we find

$$\langle \zeta, F(\zeta)\rangle \geq c_6\,(c_u, c_v, D)\,\|\zeta\|^2_{M^1} - \frac{1}{h}C_{\text{lin}}\|\zeta\|_{M^1}. \tag{3.31}$$

We fix $0 < h < h_0$, then for $\|\zeta\|_{M^1} \to \infty$, the right-hand side of (3.31) tends to infinity. This yields coercivity for F.

□

With the preceding results we state the main proposition of this section.

Proposition 3.5 (Existence of solutions for $(\text{WS})_i^h$)**.** *Choose $h_0 = h_0(\Omega, D, C_q, C_f) > 0$ sufficiently small, namely*

$$h_0 < \min\left\{1, \frac{1}{4(C_f + C_q)}, \frac{D}{2C_q^2 C(\Omega)}, \frac{1}{2c_u}, \frac{1}{2c_v}, \frac{1}{D}\right\}. \tag{3.32}$$

Then, for any $h < h_0$ and any $i \in \{1, \ldots, \lfloor \frac{T}{h} \rfloor + 1\}$ with $V_{i-1} \in L^2(\Omega)$, $u_{i-1}, v_{i-1} \in L^2(\Gamma)$ there exists a triplet $(V_i, u_i, v_i) \in H^1(\Omega) \times H^1(\Gamma) \times H^1(\Gamma)$, such that the weak, implicit Euler scheme $(WS)_i^h$ is solved.

Proof. Lemma 3.1 yields that if one finds a solution $(V_i, u_i, v_i) \in M^1$ to the variational inequality (3.4), then the weak, implicit Euler scheme $(\text{WS})_i^h$ is solved. Proposition 3.2 provides a solution to the variational inequality under appropriate conditions on the structure of the operator F. Let F be defined as in the assumptions of Proposition 3.2 and as in (3.8)–(3.9). Lemma 3.4 (ii) implies that $F = L + K : M^1 \to (M^1)^*$ is bounded, coercivity follows from Lemma 3.4 (vi) We find that F is a compactly perturbed monotone operator according to Lemma 3.3 (v), (ii) and (iii) and Lemma 3.4 (iv) and (v). Proposition 3.2 implies then, that there exists a solution (V_i, u_i, v_i) to the variational inequality (3.4). The conditions on the time-step $0 < h < h_0$ are given in condition (3.32).

□

3.2.4 Nonnegativity of time-discrete solutions

Proposition 3.5 yields the existence of solutions $(V_i, u_i, v_i) \in M^1$ for any $i = 1, \dots, \lfloor \frac{T}{h} \rfloor + 1$ to the time-discrete system $(\mathrm{WS})_i^h$. We want to show that these solutions are nonnegative.

Lemma 3.6 (Nonnegative solutions)**.** *Let solutions in a previous time-step $i-1$ be nonnegative, i.e. $V_{i-1}, u_{i-1}, v_{i-1} \geq 0$ and $0 < h < h_0(\Omega, C_f, C_q)$. Then, the solution (V_i, u_i, v_i) for the time-discrete system $(WS)_i^h$ remains nonnegative.*

Proof. Let $(V_i, u_i, v_i) \in M^1$ be a solution to the weak, time-discrete system $(\mathrm{WS})_i^h$. The triplet $(-V_i^-, -u_i^-, -v_i^-)$ consists of admissible testfunctions for the time-discrete system $(\mathrm{WS})_i^h$ according to Stampacchias Lemma, see [Sch13, Lemma 7.4, p. 146], where the negative parts of the solution (V_i, u_i, v_i) are defined as

$$V_i^- \overset{\text{def}}{=} \max\{-V_i, 0\}, \; u_i^- \overset{\text{def}}{=} \max\{-u_i, 0\} \quad \text{and} \quad v_i^- \overset{\text{def}}{=} \max\{-v_i, 0\}.$$

We test $(\mathrm{WS})_i^h$ with $-V_i^-$, $-u_i^-$ and $-v_i^-$ to find

$$0 = \frac{1}{h} \int_\Omega (V_i - V_{i-1})(-V_i^-) + D \int_\Omega \nabla V_i \cdot \nabla(-V_i^-) + \int_\Gamma q(V_i, u_i, v_i)(-V_i^-), \tag{3.33}$$

$$0 = \frac{1}{h} \int_\Gamma (u_i - u_{i-1})(-u_i^-) + \int_\Gamma A_u \nabla u_i \cdot \nabla(-u_i^-) - \int_\Gamma f(u_i, v_i)(-u_i^-), \tag{3.34}$$

$$\begin{aligned} 0 = \frac{1}{h} \int_\Gamma (v_i - v_{i-1})(-v_i^-) &+ \int_\Gamma A_v \nabla v_i \cdot \nabla(-v_i^-) \\ &- \int_\Gamma (q(V_i, u_i, v_i) - f(u_i, v_i))(-v_i^-). \end{aligned} \tag{3.35}$$

The sum of (3.33)–(3.35) is then given by

$$\begin{aligned} 0 = \frac{1}{h} \bigg(\int_\Omega (V_{i-1} - V_i) V_i^- &+ \int_\Gamma (u_{i-1} - u_i) u_i^- + \int_\Gamma (v_{i-1} - v_i) v_i^- \bigg) \\ &+ \int_\Omega D |\nabla V_i^-|^2 + \int_\Gamma A_u \nabla_\Gamma u_i \cdot \nabla_\Gamma u_i^- + \int_\Gamma A_v \nabla_\Gamma v_i \cdot \nabla_\Gamma v_i^- \\ &+ \int_\Gamma q(V_i, u_i, v_i)(v_i^- - V_i^-) + \int_\Gamma f(u_i, v_i)(u_i^- - v_i^-). \end{aligned} \tag{3.36}$$

Since the previous solution $(V_{i-1}, u_{i-1}, v_{i-1})$ is nonnegative. We estimate (3.36) from above by

$$
\begin{aligned}
&\frac{1}{h}\left(\int_\Omega (V_i^-)^2 + \int_\Gamma (u_i^-)^2 + (v_i^-)^2\right) + \int_\Omega D|\nabla V_i^-|^2 \\
&\quad + \int_\Gamma A_u \nabla_\Gamma u_i \cdot \nabla_\Gamma u_i^- + \int_\Gamma A_v \nabla_\Gamma v_i \cdot \nabla_\Gamma v_i^- \\
&\quad + \int_\Gamma q(V_i, u_i, v_i)(v_i^- - V_i^-) + \int_\Gamma f(u_i, v_i)(u_i^- - v_i^-) \\
&\geq \frac{1}{h}\left(\int_\Omega (V_i^-)^2 + \int_\Gamma (u_i^-)^2 + (v_i^-)^2\right) + \int_\Omega D|\nabla V_i^-|^2 \\
&\quad + \int_\Gamma c_u |\nabla_\Gamma u_i^-|^2 + \int_\Gamma c_v |\nabla_\Gamma v_i^-|^2 \\
&\quad + \int_\Gamma q(V_i, u_i, v_i)(v_i^- - V_i^-) + \int_\Gamma f(u_i, v_i)(u_i^- - v_i^-). \qquad (3.37)
\end{aligned}
$$

In (3.37) we used uniform ellipticity of A_u and A_v, see Assumption 1.5, to have nonnegative gradient terms. We take a closer look at the last two terms in (3.37).

1. For the first term we have with Young's Inequality and $0 \leq q_1(u_i, v_i), q_2(u_i, v_i) \leq C_q$ that

$$
\begin{aligned}
\int_\Gamma q(V_i, u_i, v_i)(v_i^- - V_i^-) &= \int_\Gamma (q_1(u_i, v_i)V_i - q_2(u_i, v_i)v_i)(v_i^- - V_i^-) \\
&= \int_\Gamma q_1(u_i, v_i)V_i v_i^- - \int_\Gamma q_2(u_i, v_i)v_i v_i^- \\
&\quad + \int_\Gamma -q_1(u_i, v_i)V_i V_i^- + \int_\Gamma q_2(u_i, v_i)v_i V_i^- \\
&= \int_\Gamma q_1(u_i, v_i)V_i v_i^- + \int_\Gamma q_2(u_i, v_i)(v_i^-)^2 \\
&\quad + \int_\Gamma q_1(u_i, v_i)(V_i^-)^2 + \int_\Gamma q_2(u_i, v_i)v_i V_i^- \\
&\geq -\int_\Gamma q_1(u_i, v_i)V_i^- v_i^- - \int_\Gamma q_2(u_i, v_i)V_i^- v_i^- \\
&\geq -\int_\Gamma \frac{1}{2}((V_i^-)^2 + (v_i^-)^2)(q_1(u_i, v_i) + q_2(u_i, v_i)) \\
&\geq -\int_\Gamma C_q((V_i^-)^2 + (v_i^-)^2), \qquad (3.38)
\end{aligned}
$$

holds.

2. For the second term we compute

$$\begin{aligned}\int_\Gamma f(u_i,v_i)(u_i^- - v_i^-) &= \int_\Gamma (f_1(u_i,v_i)v_i - f_2(u_i,v_i)u_i)(u_i^- - v_i^-)\\ &\geq -\int_\Gamma f_1(u_i,v_i)u_i^- v_i^- - \int_\Gamma f_2(u_i,v_i)u_i^- v_i^-\\ &\geq -\int_\Gamma \frac{1}{2}((u_i^-)^2 + (v_i^-)^2)(f_1(u_i,v_i) + f_2(u_i,v_i))\\ &\geq -\int_\Gamma C_f((u_i^-)^2 + (v_i^-)^2). \end{aligned} \tag{3.39}$$

The combination of equations (3.38) and (3.39) with (3.37) yield

$$\begin{aligned} 0 &\geq \left(\int_\Gamma (u_i^-)^2 \left(\frac{1}{h} - C_f\right)\right) + \left(\int_\Gamma (v_i^-)^2 \left(\frac{1}{h} - C_q - C_f\right)\right)\\ &\quad + \int_\Omega \frac{1}{h}(V_i^-)^2 + D\int_\Omega |\nabla V_i^-|^2 - \int_\Gamma C_q (V_i^-)^2\\ &\geq \left(\int_\Gamma (u_i^-)^2 \left(\frac{1}{h} - C_f\right)\right) + \left(\int_\Gamma (v_i^-)^2 \left(\frac{1}{h} - C_q - C_f\right)\right)\\ &\quad + \int_\Omega \left(\frac{1}{h} - 2C(\Omega)\frac{C_q^2}{D}\right)(V_i^-)^2 + \frac{D}{2}\int_\Omega |\nabla V_i^-|^2, \end{aligned} \tag{3.40}$$

where we applied Lemma A.5 with $\varepsilon = \frac{D}{2C_q}$ and $C(\Omega) > 0$. We choose h_0 according to (3.32) to find that for any $0 < h < h_0$ all summands on the right-hand side of (3.40) are nonnegative. Thus, we see from (3.37) using (3.38)–(3.40), that all sets where V_i, u_i and v_i are negative have Lebesgue measure, resp. Hausdorff measure zero. This completes the proof. □

3.3 Compactness and limit equations

In this Section we use the time-discrete solutions we have found in Section 3.2 to construct piecewise constant functions and affine linear interpolations on the time-interval $(-\infty, T)$ depending on the time-step h. We introduce energy bounds uniformly in h and appropriate convergence results for a suitable subsequence of $(h_k)_k \to 0$ to find a candidate triplet (V, u, v) to be a solution of the weak system (WS). Then, the main result of this section is that the triplet candidate $(V, u, v) \in L^2(0, T; H^1(\Omega)) \times (L^2(0, T; H^1(\Gamma))^2$ indeed satisfies the weak system (WS).

3.3.1 Formulation as step functions and affine linear interpolations

We consider initial conditions from Assumption 1.3 and a given time-step $0 < h < h_0$ sufficiently small satisfying Condition (3.32). Then, Proposition 3.5 provides that there exists solutions $(V_i, u_i, v_i) \in M$ of $(\mathrm{WS})_i^h$ for all $i \in \{1, \ldots, \lfloor \frac{T}{h} \rfloor + 1\}$.

Piecewise constant approximations We define piecewise constant approximations

$$\overline{V}_h : (-\infty, T) \to H^1(\Omega), \quad \overline{u}_h,\ \overline{v}_h : (-\infty, T) \to H^1(\Gamma)$$

by setting

$$\overline{V}_h(t) \stackrel{\text{def}}{=} \begin{cases} V_0, & \text{for } t \leq 0, \\ V_i, & \text{for } (i-1)h < t \leq ih, \quad i = 1, \ldots, \lfloor \frac{T}{h} \rfloor + 1 \end{cases}$$

$$\overline{u}_h(t) \stackrel{\text{def}}{=} \begin{cases} u_0, & \text{for } t \leq 0, \\ u_i, & \text{for } (i-1)h < t \leq ih, \quad i = 1, \ldots, \lfloor \frac{T}{h} \rfloor + 1 \end{cases}$$

$$\overline{v}_h(t) \stackrel{\text{def}}{=} \begin{cases} v_0, & \text{for } t \leq 0, \\ v_i, & \text{for } (i-1)h < t \leq ih, \quad i = 1, \ldots, \lfloor \frac{T}{h} \rfloor + 1. \end{cases}$$

We set

$$\overline{q}_h \stackrel{\text{def}}{=} q(\overline{V}_h, \overline{u}_h, \overline{v}_h) \quad \text{and} \quad \overline{f}_h \stackrel{\text{def}}{=} f(\overline{u}_h, \overline{v}_h).$$

Claim 3.1. *The triplet* $(\overline{V_h}, \overline{u}_h, \overline{v}_h)$ *satisfies the system*

$$0 = \int_{\Omega_T} \partial_t^{-h} \overline{V}_h \eta_1 + D \int_{\Omega_T} \nabla \overline{V}_h \cdot \nabla \eta_1 + \int_{\Gamma_T} \overline{q}_h \eta_1, \tag{3.41}$$

$$0 = \int_{\Gamma_T} \partial_t^{-h} \overline{u}_h \eta_2 + \int_{\Gamma_T} A_u \nabla_\Gamma \overline{u}_h \cdot \nabla_\Gamma \eta_2 - \int_{\Gamma_T} \overline{f}_h \eta_2, \tag{3.42}$$

$$0 = \int_{\Gamma_T} \partial_t^{-h} \overline{v}_h \eta_3 + \int_{\Gamma_T} A_v \nabla_\Gamma \overline{v}_h \cdot \nabla_\Gamma \eta_3 + \int_{\Gamma_T} (\overline{f}_h - \overline{q}_h) \eta_3 \tag{3.43}$$

for all $\eta_1 \in L^2(0, T; H^1(\Omega)),\ \eta_2, \eta_3 \in L^2(0, T; H^1(\Gamma))$.

Proof. Choose an arbitrary $\eta_1 \in L^2(0,T;H^1(\Omega))$, then $\eta_1(t) \in H^1(\Omega)$ is valid for almost every $t \in (0,T)$. For every $t \in (0,T)$ there exists a unique $i \in \{1,\dots,\lfloor \frac{T}{h} \rfloor + 1\}$, such that $\overline{q}_h(\overline{V}_h, \overline{u}_h, \overline{v}_h) = q_1(u_i,v_i)V_i - q_2(u_i,v_i)v_i$ ($\overline{f}_h$ analogously). The expression $\partial_t^{-h}\overline{V}_h$ is uniquely given by

$$\partial_t^{-h}\overline{V}_h\Big|_t = \frac{1}{h}(V_i - V_{i-1})$$

almost everywhere in Ω. The time-discretization scheme $(\mathrm{WS})_i^h$ now yields that

$$0 = \int_\Omega \partial_t^{-h}\overline{V}_h\eta_1 + D\int_\Omega \nabla\overline{V}_h \cdot \nabla\eta_1 + \int_\Gamma \overline{q}_h\eta_1$$

pointwise almost everywhere on $(0,T)$. Therefore, equation (3.41) holds. Equations (3.42) and (3.43) follow by the same arguments. ☐

Affine linear interpolations We introduce affine linear interpolations. For given $0 < h < h_0$ we divide the interval $[0,T]$ into subintervals $[t_{i-1}, t_i]$ with $t_i = ih$ for $i = 1,\dots,\lfloor \frac{T}{h} \rfloor + 1$. Let $t_0 = 0$, we define

$$V_h : (-\infty, T] \to H^1(\Omega), \quad u_h,\ v_h : (-\infty, T] \to H^1(\Gamma)$$

by

$$V_h(t) \overset{\text{def}}{=} \begin{cases} V_0, & t < 0, \\ \mu V_{i-1} + (1-\mu)V_i, & t = \mu t_{i-1} + (1-\mu)t_i,\ \mu \in [0,1], \end{cases}$$
$$u_h(t) \overset{\text{def}}{=} \begin{cases} u_0, & t < 0, \\ \mu u_{i-1} + (1-\mu)u_i, & t = \mu t_{i-1} + (1-\mu)t_i,\ \mu \in [0,1], \end{cases}$$
$$v_h(t) \overset{\text{def}}{=} \begin{cases} v_0, & t < 0, \\ \mu v_{i-1} + (1-\mu)v_i, & t = \mu t_{i-1} + (1-\mu)t_i,\ \mu \in [0,1]. \end{cases}$$

Let $t \in (t_{i-1}, t_i)$. Then, the time-derivatives of $V_h,\ u_h$ and v_h satisfy

$$\partial_t V_h(t) = \frac{V_i - V_{i-1}}{t_i - t_{i-1}} = \frac{V_i - V_{i-1}}{h} = \partial_t^{-h}\overline{V}_h,$$
$$\partial_t u_h(t) = \frac{u_i - u_{i-1}}{t_i - t_{i-1}} = \frac{u_i - u_{i-1}}{h} = \partial_t^{-h}\overline{u}_h,$$
$$\partial_t v_h(t) = \frac{v_i - v_{i-1}}{t_i - t_{i-1}} = \frac{v_i - v_{i-1}}{h} = \partial_t^{-h}\overline{v}_h.$$

With this and $(V_i, u_i, v_i) \in M^0$ for all $i \in \{1,\dots,\lfloor \frac{T}{h} \rfloor + 1\}$ we find that

$$V_h \in H^1(0,T;H^1(\Omega)) \quad \text{and} \quad u_h, v_h \in H^1(0,T;H^1(\Gamma))$$

holds. Therefore, in comparison to Claim 3.1 the following equations are satisfied

$$0 = \int_0^T \int_\Omega \partial_t V_h \eta_1 + D \int_0^T \int_\Omega \nabla \overline{V}_h \cdot \nabla \eta_1 + \int_0^T \int_\Gamma \overline{q}_h \eta_1, \tag{3.44}$$

$$0 = \int_0^T \int_\Gamma \partial_t u_h \eta_2 + \int_0^T \int_\Gamma A_u \nabla_\Gamma \overline{u}_h \cdot \nabla_\Gamma \eta_2 + \int_0^T \int_\Gamma (-\overline{f}_h) \eta_2, \tag{3.45}$$

$$0 = \int_0^T \int_\Gamma \partial_t v_h \eta_3 + \int_0^T \int_\Gamma A_v \nabla_\Gamma \overline{v}_h \cdot \nabla_\Gamma \eta_3 + \int_0^T \int_\Gamma (-\overline{q}_h + \overline{f}_h) \eta_3 \tag{3.46}$$

for all $\eta_1 \in L^2(0,T;H^1(\Omega))$, $\eta_2, \eta_3 \in L^2(0,T;H^1(\Gamma))$.

3.3.2 Limit functions and convergence properties

In this Subsection we identify a limit triplet (V,u,v), such that the interpolations from Subsection 3.3.1 converge weakly in suitable spaces of type $L^2(0,T;X)$ for $X = H^1(\Omega)$ and $X = H^1(\Gamma)$, respectively. We apply Lions-Aubin's Lemma to obtain strong convergence in spaces of type $L^2(0,T;Y)$ with $Y = L^2(\Omega)$ and $Y = L^2(\Gamma)$, respectively.

Lions-Aubin's Lemma To pass from time-discrete to time-continuous equations one often uses Lions-Aubin's Lemma which we state next.

Lemma 3.7 (Lions-Aubin). *Let $X \hookrightarrow Y \hookrightarrow X^*$ be Banach spaces with X, X^* reflexive Banach spaces. Let the embedding $X \hookrightarrow Y$ be compact and $Y \hookrightarrow X^*$ be continuous. Then, the embedding*

$$L^2(0,T;X) \cap H^1(0,T;X^*) \hookrightarrow L^2(0,T;Y) \tag{3.47}$$

is compact. In particular, every sequence $(w_k)_k$ being bounded in both spaces on the left-hand side of embedding (3.47) *has a converging subsequence in $L^2(0,T;Y)$.*

Proof. The proof can be found in [Sch13, p. 206 ff.]. □

We first identify suitable triples of type (X,Y,X^*).

Claim 3.2. *The triple $(H^1(\Omega), L^2(\Gamma), (H^1(\Omega))^*)$ satisfies the prerequisites from Lions-Aubin Lemma 3.7.*

Proof. Recall that $H^1(\Omega)$ is compactly embedded into $L^2(\Gamma)$ by Sobolev's Embedding Theorem A.7. For the operator

$$T : L^2(\Gamma) \to (H^1(\Omega))^* \quad \text{with} \quad < Tu, v > \stackrel{\text{def}}{=} \int_\Gamma uv$$

we have for $c_1 > 0$

$$|\langle Tu, v \rangle| \leq \|u\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma)} \leq c_1 \|u\|_{L^2(\Gamma)} \|v\|_{H^1(\Omega)}.$$

Therefore, T is continuous. We have found a suitable triple and the claim is proved. □

We easily find that also

$$(H^1(\Omega), L^2(\Omega), (H^1(\Omega))^*) \text{ and } (H^1(\Gamma), L^2(\Gamma), (H^1(\Gamma))^*)$$

are suitable triples in the sense of Lions-Aubin Lemma 3.7.

Uniform bounds, energy estimates and Convergence results For convenience we introduce the following notation

$$\begin{aligned} L^2M^0 &\stackrel{\text{def}}{=} L^2(0,T;L^2(\Omega)) \times L^2(0,T;L^2(\Gamma)) \times L^2(0,T;L^2(\Gamma)), \\ L^2M^1 &\stackrel{\text{def}}{=} L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Gamma)) \times L^2(0,T;H^1(\Gamma)). \end{aligned}$$

Lemma 3.8 (Uniform bounds, energy estimates)**.** *Let Assumptions 1.3 and 1.4 be statisfied. Moreover, let* (V_h, u_h, v_h) *and* $(\overline{V}_h, \overline{u}_h, \overline{v}_h)$ *be given approximations according to Subsection 3.3.1. Then, the following propositions are satisfied:*

(i) *There exists a constant* $\Lambda_1 = \Lambda_1(T, \Omega, C_q, C_f, D) > 0$ *independent of the time-step* h *and uniformly bounded for* $D \geq 1$*, such that*

$$\begin{aligned} &\operatorname*{ess\,sup}_{t\in[0,T]} \left(\|\overline{V}_h(t)\|^2_{L^2(\Omega)} + \|\overline{u}_h(t)\|^2_{L^2(\Gamma)} + \|\overline{v}_h(t)\|^2_{L^2(\Gamma)} \right) \\ &+ \int_0^T \left(D\|\nabla\overline{V}_h(t)\|^2_{L^2(\Omega)} + 2c_u\|\nabla_\Gamma\overline{u}_h(t)\|^2_{L^2(\Gamma)} + 2c_v\|\nabla_\Gamma\overline{v}_h(t)\|^2_{L^2(\Gamma)} \right) dt \\ &\leq \Lambda_1 \left(\|V_0\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Gamma)} + \|v_0\|^2_{L^2(\Gamma)} \right) \end{aligned}$$

holds. Therefore, $(\overline{V}_h, \overline{u}_h, \overline{v}_h) \in L^2M^1$*.*

(ii) *There exists a constant* $\Lambda_2 = \Lambda_2(T, \Omega, C_q, C_f, D) > 0$ *independent of* h *and uniformly bounded for* $D \geq 1$*, such that the affine linear interpolations* (V_h, u_h, v_h) *are uniformly bounded in* L^2M^1*, i.e.*

$$\|(V_h, u_h, v_h)\|^2_{L^2M^1} \leq \Lambda_2.$$

(iii) *The time-derivative* $\partial_t V_h$ *is uniformly bounded in* $L^2(0,T;(H^1(\Omega))^*)$ *and* $\partial_t u_h$*,* $\partial_t v_h$ *are uniformly bounded in* $L^2(0,T;(H^1(\Gamma))^*)$*, i.e. there exists a constant* $\Lambda_3 = \Lambda_3(T, \Omega, C_q, C_f, D) > 0$*, such that*

$$\|\partial_t V_h\|_{L^2(0,T;(H^1(\Omega))^*)} \leq \Lambda_3,$$

and $\Lambda_3' = \Lambda_3'(T, \Omega, C_q, C_f, C_{A_u}, C_{A_v}, D) > 0$ *uniformly bounded for* $D \geq 1$*, such that*

$$\|\partial_t u_h\|_{L^2(0,T;(H^1(\Gamma))^*)}, \|\partial_t v_h\|_{L^2(0,T;(H^1(\Gamma))^*)} \leq \Lambda_3'.$$

Proof.

(i) We test the sum of $(\mathrm{WS})_i^h$ with $(V_i, u_i, v_i) \in M^1$ and multiply by h. According to the computations about coercivity of the operator F in (3.23)–(3.30), see Lemma 3.4 (vi), we find with $(\zeta_1, \zeta_2, \zeta_3) = (V_i, u_i, v_i)$ that

$$\begin{aligned}
&\|V_N\|^2_{L^2(\Omega)} + \|u_N\|^2_{L^2(\Gamma)} + \|v_N\|^2_{L^2(\Gamma)} \\
&\quad + h\sum_{i=1}^N \Big(D\|\nabla V_i\|^2_{L^2(\Omega)} + 2c_u\|\nabla_\Gamma u_i\|^2_{L^2(\Gamma)} + 2c_v\|\nabla_\Gamma v_i\|^2_{L^2(\Gamma)} \Big) \\
&\le \|V_0\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Gamma)} + \|v_0\|^2_{L^2(\Gamma)} \\
&\quad + h\sum_{i=1}^N \Big(\frac{4C(\Omega)}{D} C_q^2 \|V_i\|^2_{L^2(\Omega)} + 2C_f\|u_i\|^2_{L^2(\Gamma)} + 2(C_q + C_f)\|v_i\|^2_{L^2(\Gamma)} \Big)
\end{aligned} \tag{3.48}$$

holds for $C(\Omega) > 0$. Let $N = \lfloor \frac{T}{h} \rfloor + 1$, then for any $t \in (0,T)$ there exists a uniquely determined $i \in \{1, \dots, N\}$ such that the piecewise constant representation $(\overline{V}_h, \overline{u}_h, \overline{v}_h)$ yields with (3.48) that

$$\begin{aligned}
&\int_\Omega |\overline{V}_h(t)|^2 + \int_\Gamma |\overline{u}_h(t)|^2 + \int_\Gamma |\overline{v}_h(t)|^2 \\
&\quad \le \|V_0\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Gamma)} + \|v_0\|^2_{L^2(\Gamma)} \\
&\qquad + \int_0^t \left(\frac{4C(\Omega)C_q^2}{D} \int_\Omega |\overline{V}_h(s)|^2 + 2C_f \int_\Gamma |\overline{u}_h(s)|^2 \right) \mathrm{d}s \\
&\qquad + \int_0^t \left(2(C_q + C_f) \int_\Gamma |\overline{v}_h(s)|^2 \right) \mathrm{d}s.
\end{aligned}$$

holds. Hence, Gronwall's Lemma (A.9) implies for all $t \in (0,T)$ the uniform bound

$$\begin{aligned}
&\int_\Omega |\overline{V}_h(t)|^2 + \int_\Gamma |\overline{u}_h(t)|^2 + \int_\Gamma |\overline{v}_h(t)|^2 \\
&\qquad \le \Big(\|V_0\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Gamma)} + \|v_0\|^2_{L^2(\Gamma)} \Big) e^{c_1 t},
\end{aligned}$$

where $c_1 = \max\left\{ \frac{4C(\Omega)C_q^2}{D}, 2(C_q + C_f) \right\}$. We notice that c_1 is uniformly bounded for $D \ge 1$. With this and (3.48) we find an energy estimate for piecewise constant approximations, i.e.

$$\begin{aligned}
&\int_\Omega |\overline{V}_h(t)|^2 + \int_\Gamma |\overline{u}_h(t)|^2 + \int_\Gamma |\overline{v}_h(t)|^2 \\
&\qquad + \int_0^t \left(D\int_\Omega |\nabla \overline{V}_h(s)|^2 + 2c_u \int_\Gamma |\nabla_\Gamma \overline{u}_h(s)|^2 + 2c_v \int_\Gamma |\nabla_\Gamma \overline{v}_h(s)|^2 \right) \mathrm{d}s \\
&\quad \le \|V_0\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Gamma)} + \|v_0\|^2_{L^2(\Gamma)} \\
&\qquad + \Big(\|V_0\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Gamma)} + \|v_0\|^2_{L^2(\Gamma)} \Big) e^{c_1 t} \\
&\quad \le \Lambda_1(T, \Omega, C_q, C_f, D) \Big(\|V_0\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Gamma)} + \|v_0\|^2_{L^2(\Gamma)} \Big)
\end{aligned}$$

holds for all $t \in (0,T)$ which implies Lemma 3.8 (i).

(ii) We find that the gradient expressions of affine linear interpolations (V_h, u_h, v_h) are bounded by gradient expressions of piecewise constant approximations. For example we have

$$\frac{1}{6}\int_h^T \|\nabla V_h(t)\|^2_{L^2(\Omega)} \mathrm{d}t \le h \sum_{i=1}^N \|\nabla V_i\|^2_{L^2(\Omega)} = \int_0^T \|\nabla \overline{V}_h(t)\|^2_{L^2(\Omega)} \mathrm{d}t,$$

where the same inequalities holds for $\nabla_\Gamma u_h$ and $\nabla_\Gamma v_h$, see [Sch13, p. 213f]. The same inequality holds when replacing ∇V_h by V_h, u_h and v_h, respectively. Since the initial data is not of class H^1 we shift the time interval $(0,T)$ by h to $(h, T+h)$ and let the shifted space L^2M^1 be denoted by $L^2M^1_h$. Then, we find

$$\begin{aligned}\|(V_h, u_h, v_h)\|^2_{L^2M^1_h} &= \int_h^{T+h} \left(\|V_h(t)\|^2_{H^1(\Omega)} + \|u_h(t)\|^2_{H^1(\Gamma)} + \|v_h(t)\|^2_{H^1(\Gamma)}\right) \mathrm{d}t \\ &\le 6\int_0^{T+h} \|(\overline{V}_h(t), \overline{u}_h(t), \overline{v}_h(t))\|^2_{M^1} \mathrm{d}t \le \Lambda_2\end{aligned}$$

Lemma 3.8 (i) confirms the existence of a constant $\Lambda_2 > 0$ being independent of the time-step h and uniformly bounded for $D \ge 1$, such that the assertion follows.

(iii) For the third part we treat every time-derivative separately, i.e.

$$\begin{aligned}\left|\int_0^T \langle \partial_t V_h, \eta_1\rangle_{L^2(\Omega)}\right| &= \left|\int_0^T \int_\Omega (\partial_t V_h)\eta_1\right| \\ &= \left|\int_0^T \left(-\int_\Omega D\nabla\overline{V}_h \cdot \nabla\eta_1 + \int_\Gamma \left(-q_1(\overline{u}_h, \overline{v}_h)\overline{V}_h + q_2(\overline{u}_h, \overline{v}_h)\overline{v}_h\right)\eta_1\right)\right| \\ &\le D\|\nabla\overline{V}_h\|_{L^2(0,T;L^2(\Omega))}\|\nabla\eta_1\|_{L^2(0,T;L^2(\Omega))} \\ &\quad + C_q\left(\|\overline{V}_h\|_{L^2(0,T;L^2(\Gamma))} + \|\overline{v}_h\|_{L^2(0,T;L^2(\Gamma))}\right)\|\eta_1\|_{L^2(0,T;L^2(\Gamma))} \\ &\le D\|\nabla\overline{V}_h\|_{L^2(0,T;L^2(\Omega))}\|\nabla\eta_1\|_{L^2(0,T;L^2(\Omega))} \\ &\quad + C_qC(\Omega)\left(\|\overline{V}_h\|_{L^2(0,T;L^2(\Omega))} + \|\nabla\overline{V}_h\|_{L^2(0,T;L^2(\Omega))} + \|\overline{v}_h\|_{L^2(0,T;L^2(\Gamma))}\right) \\ &\qquad \cdot C(\Omega)\left(\|\eta_1\|_{L^2(0,T;L^2(\Omega))} + \|\nabla\eta_1\|_{L^2(0,T;L^2(\Omega))}\right) \\ &\le D\|\overline{V}_h\|_{L^2(0,T;H^1(\Omega))}\|\eta_1\|_{L^2(0,T;H^1(\Omega))} \\ &\quad + C_qC(\Omega)\|\overline{V}_h\|_{L^2(0,T;H^1(\Omega))}\|\eta_1\|_{L^2(0,T;H^1(\Omega))} \\ &\quad + C_qC(\Omega)\|\overline{v}_h\|_{L^2(0,T;H^1(\Gamma))}\|\eta_1\|_{L^2(0,T;H^1(\Omega))} \qquad (3.49)\end{aligned}$$

where we used (3.44), Cauchy-Schwarz' Inequqality and the Trace Theorem for $C(\Omega) > 0$ which may change from line to line. We use Lemma 3.8 (i) to obtain from (3.49) that

$$\begin{aligned}\left|\int_0^T \langle \partial_t V_h, \eta_1\rangle_{L^2(\Omega)}\right| &\leq D\|\overline{V}_h\|_{L^2(0,T;H^1(\Omega))}\|\eta_1\|_{L^2(0,T;H^1(\Omega))} \\ &\quad + \left(C_q C(\Omega)\|\overline{V}_h\|_{L^2(0,T;H^1(\Omega))} + C_q C(\Omega)\Lambda_1\right)\|\eta_1\|_{L^2(0,T;H^1(\Omega))} \\ &\leq \Lambda_3\|\eta_1\|_{L^2(0,T;H^1(\Omega))}, \end{aligned} \tag{3.50}$$

where $\Lambda_3 > 0$ depends on T, Ω, C_q, C_f and D, as it was claimed in Lemma 3.8 (iii). With (3.45) we compute

$$\begin{aligned}\left|\int_0^T \langle \partial_t u_h, \eta_2\rangle_{L^2(\Gamma)}\right| &= \left|\int_0^T \int_\Gamma (\partial_t u_h)\eta_2\right| \\ &= \left|\int_0^T \left(-\int_\Gamma A_u \nabla_\Gamma \overline{u}_h \cdot \nabla_\Gamma \eta_2 + \int_\Gamma \left(f_1(\overline{u}_h, \overline{v}_h)\overline{u}_h - f_2(\overline{u}_h, \overline{v}_h)\overline{v}_h\right)\eta_2\right)\right| \\ &\leq C_{A_u}\|\nabla_\Gamma \overline{u}_h\|_{L^2(0,T;L^2(\Gamma))}\|\nabla_\Gamma \eta_2\|_{L^2(0,T;L^2(\Gamma))} \\ &\quad + C_f\left(\|\overline{u}_h\|_{L^2(0,T;L^2(\Gamma))} + \|\overline{v}_h\|_{L^2(0,T;L^2(\Gamma))}\right)\|\eta_2\|_{L^2(0,T;L^2(\Gamma))} \\ &\leq C_{A_u}\|\overline{u}_h\|_{L^2(0,T;H^1(\Gamma))}\|\eta_2\|_{L^2(0,T;H^1(\Gamma))} \\ &\quad + C_f\left(\|\overline{u}_h\|_{L^2(0,T;H^1(\Gamma))} + \|\overline{v}_h\|_{L^2(0,T;H^1(\Gamma))}\right)\|\eta_2\|_{L^2(0,T;H^1(\Gamma))}. \end{aligned} \tag{3.51}$$

We find with Lemma 3.8 (i) and (3.51) that

$$\left|\int_0^T \langle \partial_t u_h, \eta_2\rangle_{L^2(\Gamma)}\right| \leq \left(C_{A_u}\Lambda_1 + 2C_f\Lambda_1\right)\|\eta_2\|_{L^2(0,T;H^1(\Gamma))} \tag{3.52}$$

holds, where for $D \geq 1$ the constants appearing on the right-hand side of (3.52) are uniformly bounded. With (3.46) and the Trace Theorem we obtain that

$$\begin{aligned}
\left| \int_0^T \langle \partial_t v_h, \eta_3 \rangle_{L^2(\Gamma)} \right| &= \left| \int_0^T \int_\Gamma (\partial_t v_h) \eta_3 \right| \\
&= \left| \int_0^T \left(- \int_\Gamma A_v \nabla_\Gamma \overline{v}_h \cdot \nabla_\Gamma \eta_3 + \int_\Gamma \left(-f_1(\overline{u}_h, \overline{v}_h)\overline{u}_h + f_2(\overline{u}_h, \overline{v}_h)\overline{v}_h \right) \eta_3 \right.\right. \\
&\quad \left.\left. + \int_\Gamma \left(q_1(\overline{u}_h, \overline{v}_h)\overline{V}_h - q_2(\overline{u}_h, \overline{v}_h)\overline{v}_h \right) \eta_3 \right) \right| \\
&\leq C_{A_v} \|\nabla_\Gamma \overline{v}_h\|_{L^2(0,T;L^2(\Gamma))} \|\nabla_\Gamma \eta_3\|_{L^2(0,T;L^2(\Gamma))} \\
&\quad + C_f \left(\|\overline{u}_h\|_{L^2(0,T;L^2(\Gamma))} + \|\overline{v}_h\|_{L^2(0,T;L^2(\Gamma))} \right) \|\eta_3\|_{L^2(0,T;L^2(\Gamma))} \\
&\quad + C_q \left(\|\overline{V}_h\|_{L^2(0,T;L^2(\Gamma))} + \|\overline{v}_h\|_{L^2(0,T;L^2(\Gamma))} \right) \|\eta_3\|_{L^2(0,T;L^2(\Gamma))} \\
&\leq C_{A_v} \|\overline{v}_h\|_{L^2(0,T;H^1(\Gamma))} \|\eta_3\|_{L^2(0,T;H^1(\Gamma))} \\
&\quad + C_f \left(\|\overline{u}_h\|_{L^2(0,T;H^1(\Gamma))} + \|\overline{v}_h\|_{L^2(0,T;H^1(\Gamma))} \right) \|\eta_3\|_{L^2(0,T;H^1(\Gamma))} \\
&\quad + C_q C(\Omega) \left(\|\overline{V}_h\|_{L^2(0,T;H^1(\Omega))} + \|\overline{v}_h\|_{L^2(0,T;H^1(\Gamma))} \right) \|\eta_3\|_{L^2(0,T;H^1(\Gamma))}
\end{aligned} \tag{3.53}$$

holds. Applying the same argument as in (3.52) to (3.53), we find a constant $\Lambda_3' > 0$ which is uniformly bounded for $D \geq 1$ depending on T, Ω, C_q, C_f, D, C_{A_u} and C_{A_v}, such that

$$\max\{C_{A_v}\Lambda_1 + 2C_f\Lambda_1 + 2C_qC(\Omega)\Lambda_1,\, C_{A_v}\Lambda_1 + 2C_f\Lambda_1\} \leq \Lambda_3'$$

holds. This finishes the proof.

□

With Lemma 3.8 we find the following weak and strong convergence statements.

Lemma 3.9. *There exists a subsequence $(h_k)_k$ with $h_k \to 0$ for $k \to \infty$ and a limit object $(V, u, v) \in L^2M^1$ such that the following assertions hold:*

(i) $(V_{h_k}, u_{h_k}, v_{h_k}) \rightharpoonup (V, u, v)$ *weakly in* L^2M^1,

(ii) $(V_{h_k}, u_{h_k}, v_{h_k}) \to (V, u, v)$ *in* L^2M^0,

(iii) $V_{h_k} \to V$ *in* $L^2(0, T; L^2(\Gamma))$ *and*

(iv) $(\overline{V}_{h_k}, \overline{u}_{h_k}, \overline{v}_{h_k}) \to (V, u, v)$ *in* L^2M^0 *and* $\overline{V}_{h_k} \to V$ *in* $L^2(0, T; L^2(\Gamma))$.

Proof. We choose successively subsequences of $h \to 0$ that, without relabeling, we always denote by $(h_k)_k$, $h_k \to 0$ as $k \to \infty$.

(i) The first assertion is a consequence of the uniform boundedness of $(V_h, u_h, v_h)_h$ in L^2M^1 from Lemma 3.8 (ii) in combination with the observation that the product space L^2M^1 of Bochner-type is reflexive since $H^1(\Omega)$ and $H^1(\Gamma)$ are reflexive Banach spaces. Therefore, bounded sets are weakly precompact in L^2M^1, see [Sch13, Theorem 4.13, p. 79], a limit object $(V, u, v) \in L^2M^1$ and a subsequence $(h_k)_k$ with $h_k \to 0$ for $k \to \infty$ exists such that the assertion follows.

ii) We apply Lions-Aubin's Lemma 3.7 to (V_h, u_h, v_h) to show Lemma 3.9 (ii). With Lemma 3.8 (ii) we find that $(V_h, u_h, v_h) \in L^2M^1$. With Lemma 3.8 (ii) and (iii) we obtain

$$V_h \in H^1(0, T; (H^1(\Omega))^*) \text{ and } u_h, v_h \in H^1(0, T; (H^1(\Gamma))^*).$$

For $(H^1(\Omega), L^2(\Omega), (H^1(\Omega))^*)$ and $(H^1(\Gamma), L^2(\Gamma), (H^1(\Gamma))^*)$ we can apply Lions-Aubin's Lemma 3.7 and find with the compactness of the embedding that there exists a strongly convergent subsequence of $(V_{h_k}, u_{h_k}, v_{h_k})_h$ in L^2M^0, which shows Lemma 3.9 (ii).

(iii) We have that $V_h \in L^2(0, T; H^1(\Omega))$ and $\partial_t V_h \in L^2(0, T; (H^1(\Omega))^*)$. With the triple $(H^1(\Omega), L^2(\Gamma), (H^1(\Omega))^*)$ and Lions-Aubin's Lemma 3.7 we find a subsequence of $(h_k)_k$ such that $V_{h_k} \to V$ in $L^2(0, T; L^2(\Gamma))$. The assertion follows.

(iv) Lemma 3.9 (ii) and (iii) imply the strong convergence for linear interpolations $(V_{h_k}, u_{h_k}, v_{h_k}) \to (V, u, v)$ in L^2M^0 and $V_{h_k} \to V \in L^2(0, T; L^2(\Gamma))$. The comparison of linear affine interpolations and piecewise constant functions will be stated in Lemma 3.10 where either $X = L^2(\Omega)$ or $X = L^2(\Gamma)$. Then, Lemma 3.10 implies the existence of a subsequence $(h_k)_k$ such that piecewise constant step-functions $(\overline{V}_{h_k}, \overline{u}_{h_k}, \overline{v}_{h_k}) \to (V, u, v)$ in L^2M^0 and $\overline{V}_{h_k} \to V \in L^2(L^2(\Gamma))$.

□

For the sake of completeness we state a lemma taken from [Sch13, p. 213ff].

Lemma 3.10. *Let X be a Hilbert space, $T > 0$ and $h \to 0$, such that for every $h > 0$ with $0 = t_0, \dots, t_N = T$ the interval $[0, T]$ is divided into $[t_{i-1}, t_i)$ for $i = 1, \dots, N$. Let $w_i^h \in X$ be the evaluation in t_i. Introduce $\overline{w}_h : [0, T] \to X$ to be the piecewise constant step functions and $w_h : [0, T] \to X$ to be piecewise affine interpolations, such that $w_i = \overline{w}_h(t_i) = w_h(t_i)$ holds. If in this case there is a function $w \in L^2(0, T; X)$ with*

$$w_h \longrightarrow w \text{ in } L^2(0, T; X) \text{ for } h \to 0,$$

then

$$\overline{w}_h \longrightarrow w \text{ in } L^2(0, T; X) \text{ for } h \to 0.$$

Proof. A proof can be found in [Sch13, pp. 213ff]. □

According to Lemma 3.9 we have found a triplet (V, u, v) that is a candidate for a solution of the fully coupled weak system (WS).

3.3.3 Solution of the fully coupled weak system

In this Subsection we find that the triplet (V, u, v) is indeed a solution of the fully coupled weak system given in (WS).

Proposition 3.11 (Limit in the equations)**.** *The triplet* $(V, u, v) \in L^2(0, T; H^1(\Omega)) \times \left(L^2(0, T; H^1(\Gamma))\right)^2$ *is a solution of the fully coupled weak system* (WS)*.*

Proof. The proof is structured as follows: First, we assume to have smooth testfunctions in (3.41)–(3.43). We prove that the limit process $h \to 0$ and integration are commutable by treating every term separately. Afterwards, we relax the smoothness assumption for the testfunctions and show that the fully coupled weak problem (WS) has a solution.

Recall that for $0 < h < h_0$, with h_0 specified in (3.32), the piecewise constant time-discrete approximation satisfies

$$0 = \int_{\Omega_T} \partial_t^{-h} \overline{V}_h \eta_1 + D \int_{\Omega_T} \nabla \overline{V}_h \cdot \nabla \eta_1 + \int_{\Gamma_T} \overline{q}_h \eta_1, \tag{3.41}$$

$$0 = \int_{\Gamma_T} \partial_t^{-h} \overline{u}_h \eta_2 + \int_{\Gamma_T} A_u \nabla_\Gamma \overline{u}_h \cdot \nabla_\Gamma \eta_2 + \int_{\Gamma_T} -\overline{f}_h \eta_2, \tag{3.42}$$

$$0 = \int_{\Gamma_T} \partial_t^{-h} \overline{v}_h \eta_3 + \int_{\Gamma_T} A_v \nabla_\Gamma \overline{v}_h \cdot \nabla_\Gamma \eta_3 + \int_{\Gamma_T} (\overline{f}_h - \overline{q}_h) \eta_3 \tag{3.43}$$

for all $\eta_1 \in L^2(0, T; H^1(\Omega))$, $\eta_2, \eta_3 \in L^2(0, T; H^1(\Gamma))$.

First, we assume that the considered testfunctions are smooth in time with compact support in $[0, T)$ and of class H^1 in space, i.e.

$$\eta_1 \in C_c^\infty([0, T); H^1(\Omega)) \quad \text{and} \quad \eta_2, \eta_3 \in C_c^\infty([0, T); H^1(\Gamma)).$$

We apply a discrete partial integration to find in (3.41)

$$\begin{aligned} \int_{\Omega_T} \partial_t^{-h} \overline{V}_h \eta_1 &= \int_{\Omega_T} \partial_t^{-h} (\overline{V}_h - V_0) \eta_1 \\ &= \int_{\Omega_T} \partial_t^{-h} (\eta_1 (\overline{V}_h - V_0)) - \int_{\Omega_T} (\partial_t^{-h} \eta_1)(\overline{V}_h - V_0)(\cdot - h) \\ &= -\frac{1}{h} \int_{-h}^0 \int_\Omega (\eta_1 (\overline{V}_h - V_0)) + \frac{1}{h} \int_{T-h}^T \int_\Omega (\eta_1 (\overline{V}_h - V_0)) \\ &\quad - \int_{-h}^{T-h} \int_\Omega \partial_t^h \eta_1 (\overline{V}_h - V_0), \end{aligned} \tag{3.54}$$

where we used that the discrete time-derivative of the initial data V_0 is zero. Since $\overline{V}_h = V_0$ for $t \leq 0$ holds, the first term on the right-hand side of (3.54) vanishes. Furthermore, we assume $h > 0$ sufficiently small enough such that $\eta_1 = 0$ on $(T - h, T)$ holds. This implies that the second term of the right-hand side of (3.54) vanishes. With these two arguments we shift the domain of integration from $(-h, T-h)$ to $(0, T)$. This yields the equality

$$\int_{\Omega_T} (\partial_t^{-h} \overline{V}_h) \eta_1 = - \int_{\Omega_T} \partial_t^h \eta_1 (\overline{V}_h - V_0).$$

Lemma 3.9 (iv) yields that $\overline{V}_{h_k} \to V$ in $L^2(\Omega_T)$ holds for a subsequence $h_k \to 0$ for $k \to \infty$. With the convergence $\partial_t^{h_k}\eta_1 \to (\eta_1)_t$ in $C^0([0,T], H^1(\Omega))$ for $h_k \to 0$ with $k \to \infty$ we find

$$-\int_{\Omega_T} (\partial_t^{h_k}\eta_1)(\overline{V}_{h_k} - V_0) \to -\int_{\Omega_T} (\partial_t\eta_1)(V - V_0). \tag{3.55}$$

In the same way we find on Γ_T that

$$-\int_{\Gamma_T} (\partial_t^{h_k}\eta_2)(\overline{u}_{h_k} - u_0)(t) \to -\int_{\Gamma_T} (\partial_t\eta_2)(u - u_0), \tag{3.56}$$

$$-\int_{\Gamma_T} (\partial_t^{h_k}\eta_3)(\overline{v}_{h_k} - v_0)(t) \to -\int_{\Gamma_T} (\partial_t\eta_3)(v - v_0) \tag{3.57}$$

hold for a subsequence $h_k \to 0$ with $k \to \infty$.

The weak convergence statement from Lemma 3.9 (i) implies that for a subsequence $h_k \to 0$ for $k \to \infty$ we find

$$\int_{\Omega_T} D\nabla\overline{V}_{h_k} \cdot \nabla\eta_1 \to \int_{\Omega_T} D\nabla V \cdot \nabla\eta_1, \tag{3.58}$$

$$\int_{\Gamma_T} A_u\nabla\overline{u}_{h_k} \cdot \nabla\eta_2 \to \int_{\Gamma_T} A_u\nabla u \cdot \nabla\eta_2, \tag{3.59}$$

$$\int_{\Gamma_T} A_v\nabla\overline{v}_{h_k} \cdot \nabla\eta_3 \to \int_{\Gamma_T} A_v\nabla v \cdot \nabla\eta_3. \tag{3.60}$$

The nonlinearities are treated in the following way. According to Lemma 3.9 (iv) we find that $(\overline{V}_{h_k}, \overline{u}_{h_k}, \overline{v}_{h_k}) \to (V, u, v)$ is strongly convergent in L^2M^0 and $\overline{V}_{h_k} \to V$ is strongly convergent in $L^2(0,T; L^2(\Gamma))$. After possibly passing to a further subsequence $(h_k)_k$ in addition $\overline{V}_{h_k} \to V$ pointwise a.e. in $\overline{\Omega}_T$ and $\overline{u}_{h_k} \to u$ and $\overline{v}_{h_k} \to v$ pointwise a.e. in Γ_T.

Since the nonlinearities are assumed to be continuous, see Assumption 1.4, we find the following pointwise a.e. convergence statements

$$\begin{aligned}
\overline{q}_{h_k} &= q_1(\overline{u}_{h_k}, \overline{v}_{h_k})\overline{V}_{h_k} - q_2(\overline{u}_{h_k}, \overline{v}_{h_k})\overline{v}_{h_k} \to q_1(u,v)V - q_2(u,v)v,\\
\overline{f}_{h_k} &= f_1(\overline{u}_{h_k}, \overline{v}_{h_k})\overline{u}_{h_k} - f_2(\overline{u}_{h_k}, \overline{v}_{h_k})\overline{v}_{h_k} \to f_1(u,v)u - f_2(u,v)v.
\end{aligned}$$

The nonlinearities $\overline{q}_h$ and $\overline{f}_h$ are bounded which yields

$$\begin{aligned}
|\overline{q}_{h_k}| &= |q_1(\overline{u}_{h_k}, \overline{v}_{h_k})\overline{V}_{h_k} - q_2(\overline{u}_{h_k}, \overline{v}_{h_k})\overline{v}_{h_k}|\\
&\le C_q|\overline{V}_{h_k}| + C_q|\overline{v}_{h_k}| \to C_q(V + v) \text{ in } L^2(\Gamma_T),\\
|\overline{f}_{h_k}| &= |f_1(\overline{u}_{h_k}, \overline{v}_{h_k})\overline{u}_{h_k} - f_2(\overline{u}_{h_k}, \overline{v}_{h_k})\overline{v}_{h_k}|\\
&\le C_f|\overline{v}_{h_k}| + C_f|\overline{u}_{h_k}| \to C_f(u + v) \text{ in } L^2(\Gamma_T),
\end{aligned}$$

where we have used that Lemma 3.9 (iv) holds. Then, with Generalized Lebesgue Theorem A.10 we find strong convergence

$$\begin{aligned}
q_1(\overline{u}_{h_k}, \overline{v}_{h_k})\overline{V}_{h_k} - q_2(\overline{u}_{h_k}, \overline{v}_{h_k})\overline{v}_{h_k} &\to q_1(u,v)V - q_2(u,v)v \text{ in } L^2(\Gamma_T),\\
f_1(\overline{u}_{h_k}, \overline{v}_{h_k})\overline{u}_{h_k} - f_2(\overline{u}_{h_k}, \overline{v}_{h_k})\overline{v}_{h_k} &\to f_1(u,v)u - f_2(u,v)v \text{ in } L^2(\Gamma_T).
\end{aligned}$$

This strong convergence statement implies that for the subsequence chosen above

$$\int_{\Gamma_T} \overline{q}_{h_k}\eta_1 = \int_{\Gamma_T} q(\overline{V}_{h_k}, \overline{u}_{h_k}, \overline{v}_{h_k})\eta_1 \to \int_{\Gamma_T} q(V,u,v)\eta_1, \tag{3.61}$$

$$\int_{\Gamma_T} -\overline{f}_{h_k}\eta_2 = \int_{\Gamma_T} -f(\overline{u}_{h_k}, \overline{v}_{h_k})\eta_2 \to \int_{\Gamma_T} -f(u,v)\eta_2 \tag{3.62}$$

$$\begin{aligned}\int_{\Gamma_T} (\overline{f}_{h_k} - \overline{q}_{h_k})\eta_3 &= \int_{\Gamma_T} (f(\overline{u}_{h_k}, \overline{v}_{h_k}) - q(\overline{V}_{h_k}, \overline{u}_{h_k}, \overline{v}_{h_k}))\eta_3 \\ &\to \int_{\Gamma_T} (f(u,v) - q(V,u,v))\eta_3\end{aligned} \tag{3.63}$$

holds.

With the observations from (3.55)–(3.57), (3.58)–(3.60) and (3.61)–(3.63) we conclude that for the subsequence chosen above equations (3.41)–(3.43) result in the limit equations given by

$$0 = -\int_{\Omega_T} (\partial_t\eta_1)(V - V_0) + D\int_{\Omega_T} \nabla V \cdot \nabla\eta_1 + \int_{\Gamma_T} q(V,u,v)\eta_1, \tag{3.64}$$

$$0 = -\int_{\Gamma_T} (\partial_t\eta_2)(u - u_0) + \int_{\Gamma_T} A_u\nabla_\Gamma u \cdot \nabla_\Gamma\eta_2 + \int_{\Gamma_T} -f(u,v)\eta_2, \tag{3.65}$$

$$0 = -\int_{\Gamma_T} (\partial_t\eta_3)(v - v_0) + \int_{\Gamma_T} A_v\nabla_\Gamma v \cdot \nabla_\Gamma\eta_3 + \int_{\Gamma_T} (f(u,v) - q(V,u,v))\eta_3 \tag{3.66}$$

for all $(\eta_1, \eta_2, \eta_3) \in C_c^\infty([0,T]; H^1(\Omega)) \times \left(C_c^\infty([0,T]; H^1(\Gamma))\right)^2$.

The last step in this proof is to change the assumption on the regularity in time from being C^∞ to H^1 for the testfunctions. Consider testfunctions

$$\begin{aligned}\zeta_1 &\in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)), \\ \zeta_2, \zeta_3 &\in H^1(0,T; L^2(\Gamma)) \cap L^2(0,T; H^1(\Gamma))\end{aligned}$$

with vanishing final data, i.e. $\zeta_j(T,\cdot) = 0$ for $j = 1,2,3$. Then, there exist sequences $(\zeta_1^k)_k \in C_c^\infty(0,T; H^1(\Omega))$, $(\zeta_2^l)_l$ and $(\zeta_3^m)_m \in C_c^\infty(0,T; H^1(\Gamma))$ with

$$\zeta_1^k \to \zeta_1 \text{ in } L^2(0,T; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega)), \tag{3.67}$$

$$\zeta_2^l \to \zeta_2,\ \zeta_3^m \to \zeta_3 \text{ in } L^2(0,T; H^1(\Gamma)) \cap H^1(0,T; L^2(\Gamma)) \tag{3.68}$$

for $k, l, m \to \infty$ according to the density property of the given spaces. Since equations (3.64)–(3.66) hold, we find for every ζ_1^k, ζ_2^l and ζ_3^m the following equations

$$0 = -\int_{\Omega_T} (\partial_t\zeta_1^k)(V - V_0) + D\int_{\Omega_T} \nabla V \cdot \nabla\zeta_1^k + \int_{\Gamma_T} q(V,u,v)\zeta_1^k,$$

$$0 = -\int_{\Gamma_T} (\partial_t\zeta_2^l)(u - u_0) + \int_{\Gamma_T} A_u\nabla_\Gamma u \cdot \nabla_\Gamma\zeta_2^l + \int_{\Gamma_T} -f(u,v)\zeta_2^l,$$

$$0 = -\int_{\Gamma_T} (\partial_t\zeta_3^m)(v - v_0) + \int_{\Gamma_T} A_v\nabla_\Gamma v \cdot \nabla_\Gamma\zeta_3^m + \int_{\Gamma_T} (f(u,v) - q(V,u,v))\zeta_3^m.$$

Here, we pass to the limit $k, l, m \to \infty$ using (3.67) and (3.68). We find that

$$\begin{aligned}
\int_{\Omega_T} V(\zeta_1)_t + \int_{\Omega} (\zeta_1)_0 V_0 &= D \int_{\Omega_T} \nabla V \cdot \nabla \zeta_1 + \int_{\Gamma_T} q(V,u,v)\zeta_1, \\
\int_{\Gamma_T} u(\zeta_2)_t + \int_{\Gamma} (\zeta_2)_0 u_0 &= \int_{\Gamma_T} A_u \nabla_\Gamma u \cdot \nabla_\Gamma \zeta_2 + \int_{\Gamma_T} -f(u,v)\zeta_2, \\
\int_{\Gamma_T} v(\zeta_3)_t + \int_{\Gamma} (\zeta_3)_0 v_0 &= \int_{\Gamma_T} A_v \nabla_\Gamma v \cdot \nabla_\Gamma \zeta_3 + \int_{\Gamma_T} (f(u,v) - q(V,u,v))\zeta_3
\end{aligned}$$

holds for all

$$\begin{aligned}
\zeta_1 &\in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)), \\
\zeta_2, \zeta_3 &\in L^2(0,T;H^1(\Gamma)) \cap H^1(0,T;L^2(\Gamma))
\end{aligned}$$

and the assertion follows. □

Remark 3.12. We remark in accordance to [Sch13, Theorem 10.9, p. 201f.] that there exists continuous representants $V, u, v \in C^0([0,T];X)$ for $X = L^2(\Omega)$ or $X = L^2(\Gamma)$, respectively. Due to this fact the initial data V_0, u_0 and v_0 is consistent to the solution (V, u, v) for the limit $t \to 0$, see [Sch13, Bemerkung 11.5, p. 215]. ◁

3.4 Continuous dependence on initial data and uniqueness

In this Section we verify that solutions of (GFCRD) depend continuously on the initial data and satisfy an L^2-continuity property. As a corollary result we find that solutions of (GFCRD) are unique.

Proposition 3.11 yields that there exists a weak solution $(V, u, v) \in L^2M^1$ of (GFCRD), in other words, the fully coupled weak system (WS) given by

$$\text{(WS)}\begin{cases} \int_{\Omega_T} \partial_t \eta_1 (V - V_0) = D \int_{\Omega_T} \nabla V \cdot \nabla \eta_1 + \int_{\Gamma_T} q(V, u, v)\eta_1, \\ \int_{\Gamma_T} \partial_t \eta_2 (u - u_0) = \int_{\Gamma_T} A_u \nabla_\Gamma u \cdot \nabla_\Gamma \eta_2 - \int_{\Gamma_T} f(u, v)\eta_2, \\ \int_{\Gamma_T} \partial_t \eta_3 (v - v_0) = \int_{\Gamma_T} A_v \nabla_\Gamma v \cdot \nabla_\Gamma \eta_3 + \int_{\Gamma_T} f(u, v)\eta_3 - \int_{\Gamma_T} q(V, u, v)\eta_3, \end{cases}$$

is satisfied for all

$$\begin{aligned} \eta_1 &\in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \\ \eta_2, \eta_3 &\in H^1(0, T; L^2(\Gamma)) \cap L^2(0, T; H^1(\Gamma)), \end{aligned}$$

where we assume vanishing final data of the testfunctions. The crucial part concerning the proof of Proposition 3.13 is that we have to test the above system with the solution (V, u, v) while it does not provide a time-derivative at first glance and is a priori not a testfunction since it does not vanish in $t = T$.

Proposition 3.13. *Let $T > 0$, f, q be given as described in the Assumptions in Section 1.1. Let (V, u, v) and $(\tilde{V}, \tilde{u}, \tilde{v})$ be solutions of* (GFCRD) *with initial data (V_0, u_0, v_0) and $(\tilde{V}_0, \tilde{u}_0, \tilde{v}_0)$. Then, there exists a constant $\Lambda = \Lambda(T, D, \Omega, \Gamma) > 0$, such that*

$$\begin{aligned} &\|V - \tilde{V}\|^2_{L^2(0,T;H^1(\Omega))} + \|u - \tilde{u}\|^2_{L^2(0,T;H^1(\Gamma))} + \|v - \tilde{v}\|^2_{L^2(0,T;H^1(\Gamma))} \\ &\qquad \le \Lambda \left(\|V_0 - \tilde{V}_0\|^2_{L^2(\Omega)} + \|u_0 - \tilde{u}_0\|^2_{L^2(\Gamma)} + \|v_0 - \tilde{v}_0\|^2_{L^2(\Gamma)} \right) \end{aligned}$$

holds.

We easily conclude from Proposition 3.13 that solutions are unique.

Corollary 3.14 (Uniqueness of weak solutions)**.** *Let the assumptions of Proposition 3.13 hold. Then, for given initial data (V_0, u_0, v_0) the triplet $(V, u, v) \in L^2M^1$ is a unique solution of* (GFCRD).

Proof of Proposition 3.13. The idea of this proof is the following: for two given solutions (V, u, v) and $(\tilde{V}, \tilde{u}, \tilde{v})$, we build the difference of the given weak systems (WS). We test this system with the difference of the solutions to find that the sum is estimated from above by the initial data. We substract the corresponding equations to obtain

$$
\begin{aligned}
0 = &\int_{\Omega_T} -(V-\tilde V)\partial_t\eta_1 + \int_{\Omega_T} (V_0-\tilde V_0)\partial_t\eta_1 + D\int_{\Omega_T} \nabla(V-\tilde V)\cdot\nabla\eta_1 \\
&+ \int_{\Gamma_T} (q(V,u,v) - q(\tilde V,\tilde u,\tilde v))\eta_1, \qquad (3.69)\\
0 = &\int_{\Gamma_T} -(u-\tilde u)\partial_t\eta_2 + \int_{\Gamma_T} (u_0-\tilde u_0)\partial_t\eta_2 + \int_{\Gamma_T} A_u\nabla_\Gamma(u-\tilde u)\cdot\nabla_\Gamma\eta_2 \\
&+ \int_{\Gamma_T} (-f(u,v) + f(\tilde u,\tilde v))\eta_2, \qquad (3.70)\\
0 = &\int_{\Gamma_T} -(v-\tilde v)\partial_t\eta_3 + \int_{\Gamma_T} (v_0-\tilde v_0)\partial_t\eta_3 + \int_{\Gamma_T} A_v\nabla_\Gamma(v-\tilde v)\cdot\nabla_\Gamma\eta_3 \\
&+ \int_{\Gamma_T} (f(u,v) - f(\tilde u,\tilde v) - q(V,u,v) + q(\tilde V,\tilde u,\tilde v))\eta_3. \qquad (3.71)
\end{aligned}
$$

Following the deductions in [Alt03, p. 286 ff.] we want formally to plug in testfunctions that behave like cut-off functions in $0 < t_0 < T$, i.e.

$$\eta_1 = \chi_{(-\infty,t_0]}(V-\tilde V),\ \eta_2 = \chi_{(-\infty,t_0]}(u-\tilde u) \text{ and } \eta_3 = \chi_{(-\infty,t_0]}(v-\tilde v),$$

where $\chi_{(-\infty,t_0]} : \mathbb{R} \to \{0,1\}$ is a characteristic function in time. As these functions do not have enough regularity in time, in particular are not even continuous, we approximate them with the following setting. The following computations are easily applicable to $(u-\tilde u)$ and $(v-\tilde v)$. We exemplarily perform them for $(V-\tilde V)$ in the latter. Let $0 < t_0 < T$ and $\tau > 0$ sufficiently small. Then, for $\varphi \overset{\text{def}}{=} \chi_{(-\infty,t_0]}$ we approximate in time by

$$\Psi_\tau^V(t) \overset{\text{def}}{=} \frac{1}{\tau}\int_t^{t+\tau} (\varphi(V-\tilde V))(s)\mathrm{d}s.$$

The differentiation of this parameter integral yields that

$$\frac{\mathrm{d}}{\mathrm{dt}}\Psi_\tau^V(t) = \partial_t^\tau(\varphi(V-\tilde V)) \in L^2(0,T;L^2(\Omega))$$

holds. So, $\Psi_\tau^V \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ is an admissible testfunction in equation (3.69). Basically, this is an implication of the fact that this is a convolution of $\varphi(V-\tilde V)$ with a Dirac sequence and has therefore higher regularity. Similarly, we set

$\Psi_\tau^u(t) \stackrel{\text{def}}{=} \frac{1}{\tau}\int_t^{t+\tau}\varphi(u-\tilde{u})$ and $\Psi_\tau^v(t) \stackrel{\text{def}}{=} \frac{1}{\tau}\int_t^{t+\tau}\varphi(v-\tilde{v})$. For the terms in (3.69)–(3.71) including a time-derivative we then compute that

$$\begin{aligned}
-\int_{\Omega_T}\partial_t\Psi_\tau^V\left((V-\tilde{V})-(V_0-\tilde{V}_0)\right) &- \int_{\Gamma_T}\partial_t\Psi_\tau^u\left((u-\tilde{u})-(u_0-\tilde{u}_0)\right)\\
&- \int_{\Gamma_T}\partial_t\Psi_\tau^v\left((v-\tilde{v})-(v_0-\tilde{v}_0)\right)\\
&= -\int_{\mathbb{R}}\left(\int_\Omega \partial_t^\tau(\varphi(V-\tilde{V}))\left((V-\tilde{V})-(V_0-\tilde{V}_0)\right)\right.\\
&\quad + \int_\Gamma \partial_t^\tau(\varphi(u-\tilde{u}))\left((u-\tilde{u})-(u_0-\tilde{u}_0)\right)\\
&\quad \left.+ \int_\Gamma \partial_t^\tau(\varphi(v-\tilde{v}))\left((v-\tilde{v})-(v_0-\tilde{v}_0)\right)\right)\\
&= \int_{\mathbb{R}}\left(\int_\Omega (\varphi(V-\tilde{V}))\partial_t^{-\tau}\left((V-\tilde{V})-(V_0-\tilde{V}_0)\right)\right.\\
&\quad + \int_\Gamma (\varphi(u-\tilde{u}))\partial_t^{-\tau}\left((u-\tilde{u})-(u_0-\tilde{u}_0)\right)\\
&\quad \left.+ \int_\Gamma (\varphi(v-\tilde{v}))\partial_t^{-\tau}\left((v-\tilde{v})-(v_0-\tilde{v}_0)\right)\right)\\
&= \int_0^{t_0}\left(\int_\Omega (V-\tilde{V})\partial_t^{-\tau}(V-\tilde{V}) + \int_\Gamma (u-\tilde{u})\partial_t^{-\tau}(u-\tilde{u})\right.\\
&\quad \left.+ \int_\Gamma (v-\tilde{v})\partial_t^{-\tau}(v-\tilde{v})\right)
\end{aligned} \tag{3.72}$$

holds, where we used discrete partial integration in time, the fact that the characteristic function is defined on $(-\infty, t_0]$ together with $(V-\tilde{V})(t) \stackrel{\text{def}}{=} (V_0-\tilde{V}_0)$ for $t<0$, $u-\tilde{u}$, $v-\tilde{v}$ analogously. We remark that if H is a Hilbert space with an inner product $\langle\cdot,\cdot\rangle_H$, then the following Hilbert space inequality holds:

$$\begin{aligned}
\langle w, w-\tilde{w}\rangle_H &= \frac{1}{2}\langle w+\tilde{w}, w-\tilde{w}\rangle_H + \frac{1}{2}\langle w-\tilde{w}, w-\tilde{w}\rangle_H\\
&\geq \frac{1}{2}\langle w+\tilde{w}, w-\tilde{w}\rangle_H = \frac{1}{2}\left(\|w\|_H^2 - \|\tilde{w}\|_H^2\right),
\end{aligned}$$

where $w, \tilde{w}$ are elements of H, see [Alt03, p. 273]. Then, for any function $t \mapsto w(t) \in H$ we find

$$\langle w(t), \partial_t^{-\tau} w(t)\rangle_H \geq {}^1\!/\!{}_2\,\partial_t^{-\tau}\|w(t)\|_H^2$$

for $\tau > 0$ as introduced above . We notice that $L^2(\Omega)$ is a Hilbert space, respectively $L^2(\Gamma)$. Then, (3.72) is estimated from below by

$$\begin{aligned}
&\frac{1}{2}\int_0^{t_0}\Big(\partial_t^{-\tau}\|V-\tilde{V}\|^2_{L^2(\Omega)}+\partial_t^{-\tau}\|u-\tilde{u}\|^2_{L^2(\Gamma)}+\partial_t^{-\tau}\|v-\tilde{v}\|^2_{L^2(\Gamma)}\Big)\\
&\quad=\frac{1}{2\tau}\int_{t_0-\tau}^{t_0}\Big(\|(V-\tilde{V})(s)\|^2_{L^2(\Omega)}+\|(u-\tilde{u})(s)\|^2_{L^2(\Gamma)}+\|(v-\tilde{v})(s)\|^2_{L^2(\Gamma)}\Big)\,\mathrm{d}s\\
&\quad-\frac{1}{2}\Big(\|V_0-\tilde{V}_0\|^2_{L^2(\Omega)}+\|u_0-\tilde{u}_0\|^2_{L^2(\Gamma)}+\|v_0-\tilde{v}_0\|^2_{L^2(\Gamma)}\Big)
\end{aligned}\tag{3.73}$$

Since $(V-\tilde{V}) \in L^2(0,T;L^2(\Omega))$ and $(u-\tilde{u})$, $(v-\tilde{v}) \in L^2(0,T;L^2(\Gamma))$ we find a sequence $(\tau_k)_k$ with $\tau_k \to 0$ for $k \to \infty$ such that (3.73) converges for almost every $t_0 \in (0,T)$ to

$$\begin{aligned}
&\frac{1}{2}\Big(\|(V-\tilde{V})(t_0)\|^2_{L^2(\Omega)}+\|(u-\tilde{u})(t_0)\|^2_{L^2(\Gamma)}+\|(v-\tilde{v})(t_0)\|^2_{L^2(\Gamma)}\Big)\\
&\qquad-\frac{1}{2}\Big(\|V_0-\tilde{V}_0\|^2_{L^2(\Omega)}+\|u_0-\tilde{u}_0\|^2_{L^2(\Gamma)}+\|v_0-\tilde{v}_0\|^2_{L^2(\Gamma)}\Big).
\end{aligned}\tag{3.74}$$

As we noticed before, the function Ψ_τ^V is defined as a convolution of $\varphi(V-\tilde{V}) \in L^2(0,T;H^1(\Omega))$ and $\delta_\tau \overset{\text{def}}{=} \frac{1}{\tau}\chi_{[0,\tau]}$, a Dirac sequence $(\delta_\tau)_\tau$ for the characteristic function $\chi_{[0,1]}$. This yields $\Psi_\tau^V \to \varphi(V-\tilde{V})$ in $L^2(0,T;H^1(\Omega))$ which causes

$$\begin{aligned}
&D\int_0^T\int_\Omega \nabla(V-\tilde{V})\cdot\nabla\Psi^V_{\tau_k}\\
&\qquad\to D\int_0^T\int_\Omega \varphi|\nabla(V-\tilde{V})|^2 = D\int_0^{t_0}\|\nabla(V-\tilde{V})\|^2_{L^2(\Omega)}(s)\mathrm{d}s
\end{aligned}\tag{3.75}$$

as $\tau_k \to 0$ for $k \to \infty$. Similarly, we find for gradient expressions on Γ that

$$\int_{\Gamma_T} A_u\nabla_\Gamma(u-\tilde{u})\cdot\nabla_\Gamma\Psi^u_{\tau_k} \to \int_0^{t_0}\int_\Gamma A_u\nabla_\Gamma(u-\tilde{u})\cdot\nabla_\Gamma(u-\tilde{u}),\tag{3.76}$$

$$\int_{\Gamma_T} A_v\nabla_\Gamma(v-\tilde{v})\cdot\nabla_\Gamma\Psi^v_{\tau_k} \to \int_0^{t_0}\int_\Gamma A_v\nabla_\Gamma(v-\tilde{v})\cdot\nabla_\Gamma(v-\tilde{v})\tag{3.77}$$

holds as $\tau_k \to 0$ for $k \to \infty$.

For the nonlinear terms we deduce the following: According to Lemma 3.9 (iv), V and $\tilde{V}$ are in $L^2(0,T;L^2(\Gamma))$ and with Proposition 3.11 system (WS) is statisfied. Therefore, for any sequence $(\tau_k)_k$ with $\tau_k \to 0$ for $k \to \infty$ such that $\Psi^V_{\tau_k}(t) \to \varphi(V-\tilde{V})$ in $L^2(0,T;L^2(\Gamma))$, we obtain

$$\int_{\Gamma_T}(q(V,u,v)-q(\tilde{V},\tilde{u},\tilde{v}))\Psi^V_{\tau_k} \to \int_0^{t_0}\int_\Gamma(q(V,u,v)-q(\tilde{V},\tilde{u},\tilde{v}))(V-\tilde{V}).\tag{3.78}$$

In the same way we obtain convergence for the remaining testfunctions $\eta_2 = \Psi_\tau^u$ and $\eta_3 = \Psi_\tau^v$ in the nonlinear equations, i.e.

$$\int_{\Gamma_T} (-f(u,v) - f(\tilde{u},\tilde{v}))\Psi_{\tau_k}^u \to \int_0^{t_0} \int_\Gamma (f(u,v) - f(\tilde{u},\tilde{v}))(u-\tilde{u}), \tag{3.79}$$

$$\begin{aligned} &\int_{\Gamma_T} (-q(V,u,v) + q(\tilde{V},\tilde{u},\tilde{v}) + f(u,v) - f(\tilde{u},\tilde{v}))\Psi_{\tau_k}^v \\ &\qquad \to \int_0^{t_0} \int_\Gamma (-q(V,u,v) + q(\tilde{V},\tilde{u},\tilde{v}) + f(u,v) - f(\tilde{u},\tilde{v}))(v-\tilde{v}). \end{aligned} \tag{3.80}$$

So we have found admissible testfunctions $\eta_1 = \Psi_\tau^V$, $\eta_2 = \Psi_\tau^u$ and $\eta_3 = \Psi_\tau^v$ for (3.69)–(3.71). We use the convergence results (3.74)–(3.80) for $\tau_k \to 0$ as $k \to \infty$ to obtain with Assumption 1.5 on A_u and A_v

$$\begin{aligned} \|V - \tilde{V}\|^2_{L^2(\Omega)}(t_0) + &\|u - \tilde{u}\|^2_{L^2(\Gamma)}(t_0) + \|v - \tilde{v}\|^2_{L^2(\Gamma)}(t_0) + 2D\|\nabla(\tilde{V} - V)\|^2_{L^2(\Omega_{t_0})} \\ &+ 2c_u\|\nabla_\Gamma(\tilde{u} - u)\|^2_{L^2(\Gamma_{t_0})} + 2c_v\|\nabla_\Gamma(\tilde{v} - v)\|^2_{L^2(\Gamma_{t_0})} \\ &\le \left(\|V_0 - \tilde{V}_0\|^2_{L^2(\Omega)} + \|u_0 - \tilde{u}_0\|^2_{L^2(\Gamma)} + \|v_0 - \tilde{v}_0\|^2_{L^2(\Gamma)}\right) \\ &+ 2\int_{\Gamma_{t_0}} (q(\tilde{V},\tilde{u},\tilde{v}) - q(V,u,v))(V - \tilde{V}) \\ &+ 2\int_{\Gamma_{t_0}} (f(u,v) - f(\tilde{u},\tilde{v})(u - \tilde{u}) \\ &+ 2\int_{\Gamma_{t_0}} (f(\tilde{u},\tilde{v}) - f(u,v) + q(V,u,v) - q(\tilde{V},\tilde{u},\tilde{v}))(v - \tilde{v}). \end{aligned} \tag{3.81}$$

The nonlinear parts f_1, f_2, q_1 and q_2 are Lipschitz continuous, see Assumption 1.4. Let $C_L > 0$ denote the maximal Lipschitz-constant for all nonlinearities, then we estimate the nonlinearities of the right-hand side of (3.81) from above by

$$\begin{aligned} &2\int_{\Gamma_{t_0}} \left(C_L|V - \tilde{V}|(|v - \tilde{v}| + |V - \tilde{V}|) + C_L|v - \tilde{v}|(|v - \tilde{v}| + |V - \tilde{V}|)\right) \\ &\quad + 2\int_{\Gamma_{t_0}} (C_L|u - \tilde{u}|(|u - \tilde{u}| + |v - \tilde{v}|) + C_L|v - \tilde{v}|(|u - \tilde{u}| + |v - \tilde{v}|)) \\ &\le 2\int_{\Gamma_{t_0}} \left(\left(C_L + \frac{1}{2}\right)\left(|V - \tilde{V}|^2 + |v - \tilde{v}|^2\right) + \left(C_L + \frac{1}{2}\right)\left(|v - \tilde{v}|^2 + |V - \tilde{V}|^2\right)\right) \\ &\quad + 2\int_{\Gamma_{t_0}} \left(\left(C_L + \frac{1}{2}\right)\left(|u - \tilde{u}|^2 + |v - \tilde{v}|^2\right) + \left(C_L + \frac{1}{2}\right)\left(|u - \tilde{u}|^2 + |v - \tilde{v}|^2\right)\right) \\ &\le (8C_L + 4)\int_{\Gamma_{t_0}} \left(|V - \tilde{V}|^2 + |u - \tilde{u}|^2 + |v - \tilde{v}|^2\right) \end{aligned} \tag{3.82}$$

where we used Young's Inequality. We set $c_1 \overset{\text{def}}{=} 8C_L + 4$ to apply Lemma A.5 on $\int_{\Gamma_{t_0}} |\tilde{V} - V|^2$ with $\varepsilon = \frac{D}{c_1}$ and a constant $C(\Omega) > 0$ to find that (3.81) in combination with (3.82) yields

$$\begin{aligned} &\|V - \tilde{V}\|^2_{L^2(\Omega)}(t_0) + \|u - \tilde{u}\|^2_{L^2(\Gamma)}(t_0) + \|v - \tilde{v}\|^2_{L^2(\Gamma)}(t_0) \\ &\quad + D\|\nabla(\tilde{V} - V)\|^2_{L^2(\Omega_{t_0})} + 2c_u\|\nabla_\Gamma(\tilde{u} - u)\|^2_{L^2(\Gamma_{t_0})} + 2c_v\|\nabla_\Gamma(\tilde{v} - v)\|^2_{L^2(\Gamma_{t_0})} \\ &\leq \Big(\|V_0 - \tilde{V}_0\|^2_{L^2(\Omega)} + \|u_0 - \tilde{u}_0\|^2_{L^2(\Gamma)} + \|v_0 - \tilde{v}_0\|^2_{L^2(\Gamma)}\Big) \\ &\quad c_1 \int_0^{t_0} \Big(\|u - \tilde{u}\|^2_{L^2(\Gamma)} + \|v - \tilde{v}\|^2_{L^2(\Gamma)}\Big) + \frac{c_1^2 C(\Omega)}{D} \int_0^{t_0} \|V - \tilde{V}\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.83}$$

Since the norm is a nonnegative function we find for almost every $t_0 \in (0, T)$ that the pointwise evaluation in t_0 is less or equal the integral expression. This is related to the fact that almost every $t_0 \in (0, T)$ is a Lebesgue point for L^1-summable functions. Therefore, we write the left-hand side of (3.83) in terms of corresponding H^1-norms to obtain

$$\begin{aligned} &\|(V - \tilde{V})(t_0)\|^2_{H^1(\Omega)} + \|(u - \tilde{u})(t_0)\|^2_{H^1(\Gamma)} + \|(v - \tilde{v})(t_0)\|^2_{H^1(\Gamma)} \\ &\leq \Big(\|V_0 - \tilde{V}_0\|^2_{L^2(\Omega)} + \|u_0 - \tilde{u}_0\|^2_{L^2(\Gamma)} + \|v_0 - \tilde{v}_0\|^2_{L^2(\Gamma)}\Big) \\ &\quad c_1 \int_0^{t_0} \Big(\|u - \tilde{u}\|^2_{H^1(\Gamma)} + \|v - \tilde{v}\|^2_{H^1(\Gamma)}\Big) + \frac{c_1^2 C(\Omega)}{D} \int_0^{t_0} \|V - \tilde{V}\|^2_{H^1(\Omega)}. \end{aligned}$$

With Gronwall's Lemma A.9 we find for almost every $t_0 \in (0, T)$ and a constant $c_2 > 0$ depending on D, C_f, C_q and Ω, such that

$$\begin{aligned} &\|(V - \tilde{V})(t_0)\|^2_{H^1(\Omega)} + \|(u - \tilde{u})(t_0)\|^2_{H^1(\Gamma)} + \|(v - \tilde{v})(t_0)\|^2_{H^1(\Gamma)} \\ &\qquad \leq \Big(\|V_0 - \tilde{V}_0\|^2_{L^2(\Omega)} + \|u_0 - \tilde{u}_0\|^2_{L^2(\Gamma)} + \|v_0 - \tilde{v}_0\|^2_{L^2(\Gamma)}\Big) \exp(c_2 t_0) \\ &\qquad \leq \Big(\|V_0 - \tilde{V}_0\|^2_{L^2(\Omega)} + \|u_0 - \tilde{u}_0\|^2_{L^2(\Gamma)} + \|v_0 - \tilde{v}_0\|^2_{L^2(\Gamma)}\Big) \exp(c_2 T) \end{aligned}$$

holds. With $\Lambda = \exp(c_2 T)$ the continuous dependence on initial data and an L^2-continuity property follows. □

3.5 Uniform boundedness

In this Section we introduce a maximum principle for (GFCRD) to afterwards apply it in Chapter 4. We work in a rescaled framework: We multiply (V, u, v) with an exponential factor in time $e^{-\lambda t}$ for $\lambda > 0$ to be specified later, apply the technique and auxiliary problems we already used in Section 2.1.2 and deduce maximum bounds. For convenience, we state the rescaling procedure once again. Moreover, in the weak regime it is not obvious that solutions are admissible testfunctions, see Section 3.4. Therefore, we recall the correct testfunctions and results in the proof of Proposition 3.16 again.

We multiply (GFCRD) with $e^{-\lambda t}$, $\lambda > 0$ and find with partial integration and (2.1)–(2.3) that

$$
\begin{cases}
\int_{\Omega_T} \partial_t \eta_1 (\tilde{V} - V_0) = D \int_{\Omega_T} \nabla \tilde{V} \cdot \nabla \eta_1 + \lambda \int_{\Omega_T} \tilde{V} \eta_1 \\
\qquad\qquad + \int_{\Gamma_T} \left(\tilde{q}_1(t, \tilde{u}, \tilde{v}) \tilde{V} - \tilde{q}_2(t, \tilde{u}, \tilde{v}) \tilde{v} \right) \eta_1, & (3.84) \\
\int_{\Gamma_T} \partial_t \eta_2 (\tilde{u} - u_0) = \int_{\Gamma_T} A_u \nabla_\Gamma \tilde{u} \cdot \nabla_\Gamma \eta_2 \\
\qquad\qquad - \int_{\Gamma_T} \left(\tilde{f}_1(t, \tilde{u}, \tilde{v}) \tilde{v} - \tilde{f}_2(t, \tilde{u}, \tilde{v}) \tilde{u} \right) \eta_2 + \lambda \int_{\Gamma_T} \tilde{u} \eta_2, & (3.85) \\
\int_{\Gamma_T} \partial_t \eta_3 (\tilde{v} - v_0) = \int_{\Gamma_T} A_v \nabla_\Gamma \tilde{v} \cdot \nabla_\Gamma \eta_3 + \int_{\Gamma_T} \left(\tilde{f}_1(t, \tilde{u}, \tilde{v}) \tilde{v} - \tilde{f}_2(t, \tilde{u}, \tilde{v}) \tilde{u} \right) \eta_3 \\
\qquad\qquad - \int_{\Gamma_T} \left(\tilde{q}_1(t, \tilde{u}, \tilde{v}) \tilde{V} - \tilde{q}_2(t, \tilde{u}, \tilde{v}) \tilde{v} \right) \eta_3 + \lambda \int_{\Gamma_T} \tilde{v} \eta_3, & (3.86)
\end{cases}
$$

holds for all

$$
\eta_1 \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)),
$$
$$
\eta_2, \eta_3 \in H^1(0, T; L^2(\Gamma)) \cap L^2(0, T; H^1(\Gamma)).
$$

The functions $\tilde{f}_i$ and $\tilde{q}_i$ for $i = 1, 2$ are defined as in Section 2.1.2. In $t = 0$ the initial data for the rescaled functions and (V, u, v) coincide.

Remark 3.15 (Invariant Region Principle)**.** Another canonical approach in Reaction-Diffusion Systems is to apply an invariant region approach, see for example [Smo83, p. 192ff.]. An invariant region is a bounded, in our case rectangular region in the phase space such that the boundary has a repelling property. Whenever initial data lies in that region, one can show that the evolution in time does not leave this region. In general, for our system, this cannot be easily deduced from an invariant region principle. Instead we use a different technique on finding L^∞-a priori estimates as in Chapter 2, namely, based on appropriate testfunctions and comparison functions. ◁

Proposition 3.16. *Let Assumptions 1.3–1.4 hold and $\tilde{\Lambda}_2$, C_{fq} be given as in* (2.8)*. Let*

$$
\lambda \geq \max \left\{ \frac{C_{fq}^2}{D} C(\Omega), 4C_{fq}, C_{fq} \left(1 + C_1 C_{fq} + \frac{2\|V_0\|_{L^\infty(\Omega)} + 1}{\tilde{\Lambda}_2} \right) \right\}, \tag{3.87}
$$

be a fixed constant, where $C(\Omega) > 0$ and $C_1(T,\Omega,D) > 0$ is uniformly bounded for $D \geq 1$.

Then, any solution of (GFCRD) *is nonnegative and essentially bounded, i.e.*

$$\|u\|_{L^\infty(\Gamma_T)},\ \|v\|_{L^\infty(\Gamma_T)} \leq e^{\lambda T}\tilde{\Lambda}_2,$$
$$\|V\|_{L^\infty(\overline{\Omega}_T)} \leq e^{\lambda T}\left(2\|V_0\|_{L^\infty(\Omega)} + 1 + C_1 C_{fq}\tilde{\Lambda}_2\right).$$

Proof. For both, the nonnegativity and uniform boundedness, we have to test the given weak regime (3.84)–(3.86) with testfunctions that consist basically of a solution triplet $(\tilde{V},\tilde{u},\tilde{v})$ or variations of it. According to Section 3.4 we set $\varphi \overset{\text{def}}{=} \chi_{(-\infty,t]}$, where $\chi_{(-\infty,t]} : \mathbb{R} \to \{0,1\}$ is a characteristic function in time for $t \in (0,T)$. Then, we have justified that the formal choice $(\varphi\tilde{V},\varphi\tilde{u},\varphi\tilde{v})$ as testfunction is justified with an approximation of admissible testfunctions. At this point we avoid to repeat the proof in detail. As a result we find for example for $\tilde{V}$ that the term including the time-derivative can be estimated in the following way:

$$\int_{\Omega_T} \partial_t \eta_1 (\tilde{V} - V_0) \leq \frac{1}{2}\left(\|V_0\|^2_{L^2(\Omega)} - \|\tilde{V}(t)\|^2_{L(\Omega)}\right)$$

for $\eta_1 = \varphi\tilde{V}$. For the remaining terms there are no changes except that the cut-off functions determines the time-interval. With this observation we begin the proof and show that weak solutions (V,u,v) of (GFCRD) are nonnegative.

We consider $(\tilde{V},\tilde{u},\tilde{v})$ and test the corresponding rescaled equations (3.84)–(3.86) formally with $(-\varphi\tilde{V}^-,-\varphi\tilde{u}^-,-\varphi\tilde{v}^-)$. We find for any $t \in (0,T)$ with integration over Ω_T and Γ_T and the fact that $\tilde{v} = \tilde{v}^+ - \tilde{v}^- \geq -\tilde{v}^-$ holds, $\tilde{u} \geq -\tilde{u}^-$ respectively, that

$$\begin{aligned}
&\frac{1}{2}\left(\|\tilde{V}_0^-\|^2_{L^2(\Omega)} + \|\tilde{u}_0^-\|^2_{L^2(\Gamma)} + \|\tilde{v}_0^-\|^2_{L^2(\Gamma)}\right) \\
&\qquad - \frac{1}{2}\left(\|\tilde{V}^-(t)\|^2_{L^2(\Omega)} + \|\tilde{u}^-(t)\|^2_{L^2(\Gamma)} + \|\tilde{v}^-(t)\|^2_{L^2(\Gamma)}\right) \\
&\geq \int_{\Omega_t}\left(\lambda(\tilde{V}^-)^2 + D|\nabla\tilde{V}^-|^2\right) + \int_{\Gamma_t}\left(\lambda((\tilde{u}^-)^2 + (\tilde{v}^-)^2) + c_u|\nabla_\Gamma\tilde{u}^-|^2 + c_v|\nabla_\Gamma\tilde{v}^-|^2\right) \\
&\quad + \int_{\Gamma_t}\left(-\tilde{q}_1(\cdot,\tilde{u},\tilde{v})(\tilde{V}^-)^2 + \tilde{q}_2(\cdot,\tilde{u},\tilde{v})\tilde{v}\tilde{V}^-\right) \\
&\quad + \int_{\Gamma_t}\left(\tilde{q}_1(\cdot,\tilde{u},\tilde{v})\tilde{V}\tilde{v}^- - \tilde{q}_2(\cdot,\tilde{u},\tilde{v})(\tilde{v}^-)^2\right) \\
&\quad + \int_{\Gamma_t}\left(-\tilde{f}_1(\cdot,\tilde{u},\tilde{v})(\tilde{v}^-)^2 + \tilde{f}_2(\cdot,\tilde{u},\tilde{v})\tilde{u}\tilde{v}^-\right) \\
&\quad + \int_{\Gamma_t}\left(\tilde{f}_1(\cdot,\tilde{u},\tilde{v})\tilde{v}\tilde{u}^- - \tilde{f}_2(\cdot,\tilde{u},\tilde{v})(\tilde{u}^-)^2\right) \\
&\geq \int_{\Omega_t}\left(\lambda(\tilde{V}^-)^2 + D|\nabla\tilde{V}^-|^2\right) + \int_{\Gamma_t}\left(\lambda((\tilde{u}^-)^2 + (\tilde{v}^-)^2) + |\nabla_\Gamma\tilde{u}^-|^2 + d|\nabla_\Gamma\tilde{v}^-|^2\right) \\
&\quad + \int_{\Gamma_t}\left(-2C_q(\tilde{V}^-)^2 - 2C_q(\tilde{v}^-)^2 - 2C_f(\tilde{v}^-)^2 - 2C_f(\tilde{u}^-)^2\right), \qquad (3.88)
\end{aligned}$$

where we applied Young's Inequality, estimates for the negative parts and Assumption 1.4 which remains valid for $\tilde{f}_i$ and $\tilde{q}_i$ for $i = 1, 2$. The Trace Theorem implies that the right-hand side of (3.88) is estimated from below by

$$\int_{\Omega_t} \left(\lambda - \frac{C(\Omega)C_q^2}{D}\right)(\tilde{V}^-)^2 + \frac{D}{2}\int_{\Omega_t} |\nabla \tilde{V}^-|^2 + \int_{\Gamma_t} (\lambda - 2C_f)(\tilde{u}^-)^2 + \int_{\Gamma} |\nabla_\Gamma \tilde{u}^-|^2$$
$$+ \int_{\Gamma_t} (\lambda - 2\,(C_q + C_f))\,(\tilde{v}^-)^2 + \int_{\Gamma_t} d|\nabla_\Gamma \tilde{v}^-|^2 \tag{3.89}$$

with a constant $C(\Omega) > 0$. We obtain from (3.88) and (3.89) for any $\lambda > 0$ satisfying

$$\lambda \geq \max\left\{\frac{C_{fq}^2}{D}C(\Omega), 4C_{fq}\right\},$$

that

$$\frac{1}{2}\left(\|\tilde{V}^-(t)\|_{L^2(\Omega)}^2 + \|\tilde{u}^-(t)\|_{L^2(\Gamma)}^2 + \|\tilde{v}^-(t)\|_{L^2(\Gamma)}^2\right)$$
$$\leq \frac{1}{2}\left(\|\tilde{V}_0^-\|_{L^2(\Omega)}^2 + \|\tilde{u}_0^-\|_{L^2(\Gamma)}^2 + \|\tilde{v}_0^-\|_{L^2(\Gamma)}^2\right)$$

holds. The initial data was assumed to be nonnegative, therefore the Lebesgue measure and respectively Hausdorff measure of the negative parts $(\tilde{V}, \tilde{u}, \tilde{v})$ are zero for every $t \in (0, T)$. By rescaling we obtain that also (V, u, v) is nonnegative for all times.

To derive a maximum principle we follow the proof of Lemma 2.3. We restate an auxiliary problem given by a heat equation with constant boundary flux to control the Robin-boundary condition of the bulk equation we introduced in Subsection 2.1.2. Let $\tilde{\Psi} : \overline{\Omega} \times (0, T) \to \mathbb{R}$ be the classical solution of

$$\begin{cases} \partial_t \tilde{\Psi} = D\Delta\tilde{\Psi} - \lambda\tilde{\Psi} & \text{in } \Omega \times (0, T), \quad (2.11a) \\ -D\nabla\tilde{\Psi}\cdot\nu = -\tilde{\mu} & \text{on } \Gamma \times (0, T), \quad (2.11b) \\ \tilde{\Psi}(\cdot, 0) = \tilde{\Psi}_0 & \text{on } \overline{\Omega}, \quad (2.11c) \end{cases}$$

where $\tilde{\mu} > 0$ is a constant to be specified later and $\tilde{\Psi}_0$ will be chosen below satisfying the compatibility condition

$$-D\nabla\tilde{\Psi}_0 \cdot \nu = -\tilde{\mu}. \tag{3.91}$$

Lemma 2.1 implies that

$$\tilde{\Psi}(x, t) \leq \sup_{\overline{\Omega}} \tilde{\Psi}_0 + \tilde{\mu}e^{c_0 T} \tag{3.92}$$

where $c_0 = c_0(\Omega, D) > 0$ is uniformly bounded for $D \geq 1$.

We test (2.11a)–(2.11c) with a testfunction $\eta_1 \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ that will be specified later to find

$$0 = -\int_{\Omega_T} \partial_t \eta_1 (\tilde{\Psi} - \tilde{\Psi}_0) + D \int_{\Omega_T} \nabla \eta_1 \cdot \nabla \tilde{\Psi} + \lambda \int_{\Omega_T} \tilde{\Psi} \eta_1 - \int_{\Gamma_T} \tilde{\mu} \eta_1. \tag{3.93}$$

We find that the difference of (3.93) and (3.84) is given by

$$\begin{aligned} 0 = &-\int_{\Omega_T} \partial_t \eta_1 (\tilde{V} - \tilde{\Psi}) + \int_{\Omega_T} \partial_t \eta_1 (\tilde{V}_0 - \tilde{\Psi}_0) + D \int_{\Omega_T} \nabla \eta_1 \cdot \nabla (\tilde{V} - \tilde{\Psi}) \\ &+ \int_{\Gamma_T} (\tilde{q}_1(\cdot,\tilde{u},\tilde{v})\tilde{V} - \tilde{q}_2(\cdot,\tilde{u},\tilde{v})\tilde{v})\eta_1 + \lambda \int_{\Omega_T} \eta_1 (\tilde{V} - \tilde{\Psi}) + \tilde{\mu} \int_{\Gamma_T} \eta_1. \end{aligned} \tag{3.94}$$

We claim that $V_0 \leq \tilde{\Psi}_0$ holds. Then, with the deductions of Section 3.4 we find for $\eta_1 = \varphi(\tilde{V} - \tilde{\Psi})_+$ from (3.94) that

$$\begin{aligned} 0 \geq &\frac{1}{2} \|(\tilde{V} - \tilde{\Psi})_+(t)\|^2_{L^2(\Omega)} + D \|\nabla (\tilde{V} - \tilde{\Psi})_+\|^2_{L^2(\Omega_t)} + \tilde{\mu} \int_{\Gamma_t} (\tilde{V} - \tilde{\Psi})_+ \\ &+ \int_{\Gamma_t} (\tilde{q}_1(\cdot,\tilde{u},\tilde{v})\tilde{V} - \tilde{q}_2(\cdot,\tilde{u},\tilde{v})\tilde{v})(\tilde{V} - \tilde{\Psi})_+ + \lambda \int_{\Omega_t} (\tilde{V} - \tilde{\Psi})^2_+, \end{aligned} \tag{3.95}$$

since the initial values are equal. Analogously, we find for (3.85) and (3.86) tested with $(\tilde{u} - \tilde{\Lambda}_2)_+$ and $(\tilde{v} - \tilde{\Lambda}_2)_+$ that

$$\begin{aligned} 0 \geq &\frac{1}{2} \|(\tilde{u} - \tilde{\Lambda}_2)_+(t)\|^2_{L^2(\Gamma)} + \int_{\Gamma_t} A_u \nabla_\Gamma \tilde{u} \cdot \nabla_\Gamma (\tilde{u} - \tilde{\Lambda}_2)_+ + \lambda \int_{\Gamma_t} \tilde{u} (\tilde{u} - \tilde{\Lambda}_2)_+ \\ &+ \int_{\Gamma_t} (-\tilde{f}_1(\cdot,\tilde{u},\tilde{v})\tilde{v} + \tilde{f}_2(\cdot,\tilde{u},\tilde{v})\tilde{u})(\tilde{u} - \tilde{\Lambda}_2)_+, \end{aligned} \tag{3.96}$$

$$\begin{aligned} 0 \geq &\frac{1}{2} \|(\tilde{v} - \tilde{\Lambda}_2)_+(t)\|^2_{L^2(\Gamma)} + \int_{\Gamma_t} A_v \nabla_\Gamma \tilde{v} \cdot \nabla_\Gamma (\tilde{v} - \tilde{\Lambda}_2)_+ + \lambda \int_{\Gamma_t} \tilde{v} (\tilde{v} - \tilde{\Lambda}_2)_+ \\ &+ \int_{\Gamma_t} \left(\tilde{f}_1(\cdot,\tilde{u},\tilde{v})\tilde{v} - \tilde{f}_2(\cdot,\tilde{u},\tilde{v})\tilde{u} - \tilde{q}_1(\cdot,\tilde{u},\tilde{v})\tilde{V} + \tilde{q}_2(\cdot,\tilde{u},\tilde{v})\tilde{v} \right) (\tilde{v} - \tilde{\Lambda}_2)_+. \end{aligned} \tag{3.97}$$

We drop the arguments of $\tilde{f}_i$ and $\tilde{q}_i$ for $i = 1, 2$ and add (3.95)–(3.97) to find

$$\begin{aligned} &\frac{1}{2} \left(\|(\tilde{V} - \tilde{\Psi})_+(t)\|^2_{L^2(\Omega)} + \|(\tilde{u} - \tilde{\Lambda}_2)_+(t)\|^2_{L^2(\Gamma)} + \|(\tilde{v} - \tilde{\Lambda}_2)_+(t)\|^2_{L^2(\Gamma)} \right) \\ &\leq -D \int_{\Omega_t} |\nabla (\tilde{V} - \tilde{\Psi})_+|^2 - c_u \int_{\Gamma_t} |\nabla_\Gamma (\tilde{u} - \tilde{\Lambda}_2)_+|^2 - c_v \int_{\Gamma_t} |\nabla_\Gamma (\tilde{v}_k - \tilde{\Lambda}_2)_+|^2 \\ &\quad - \lambda \int_{\Omega_t} (\tilde{V} - \tilde{\Psi})^2_+ - \lambda \int_{\Gamma_t} \tilde{u} (\tilde{u} - \tilde{\Lambda}_2)_+ - \lambda \int_{\Gamma_t} \tilde{v} (\tilde{v} - \tilde{\Lambda}_2)_+ \\ &\quad + \int_{\Gamma_t} \left(\tilde{f}_1 \tilde{v} - \tilde{f}_2 \tilde{u} \right) (\tilde{u} - \tilde{\Lambda}_2)_+ + \int_{\Gamma_t} \left(-\tilde{f}_1 \tilde{v} + \tilde{f}_2 \tilde{u} + \tilde{q}_1 \tilde{V} - \tilde{q}_2 \tilde{v} \right) (\tilde{v} - \tilde{\Lambda}_2)_+ \\ &\quad + \int_{\Gamma_t} \left(-\tilde{q}_1 \tilde{V} + \tilde{q}_2 \tilde{v} \right) (\tilde{V} - \tilde{\Psi})_+ - \tilde{\mu} \int_{\Gamma_t} (\tilde{V} - \tilde{\Psi})_+. \end{aligned} \tag{3.98}$$

We have to control the right-hand side of (3.98) appropriately. To obtain a higher clarity we demonstrate the arising computations separately. With a constant $C_{fq} > 0$ defined above, we basically copy the estimates from the proof of Lemma 2.3 with one exception, i.e.

$$\begin{aligned}
&\int_{\Gamma_t} \left(-\tilde{f}_1\tilde{v} + \tilde{f}_2\tilde{u} + \tilde{q}_1\tilde{V} - \tilde{q}_2\tilde{v}\right)(\tilde{v} - \tilde{\Lambda}_2)_+ \\
&\quad \le \int_{\Gamma_t} \left(\frac{C_{fq}}{2}(\tilde{u} - \tilde{\Lambda}_2)_+^2 + C_{fq}(\tilde{v} - \tilde{\Lambda}_2)_+^2 + \frac{C_{fq}}{2}(\tilde{V} - \tilde{\Psi})_+\right) \\
&\qquad + \int_{\Gamma_t} C_{fq}\left(\tilde{\Lambda}_2 + \tilde{\Psi}\right)(\tilde{v} - \tilde{\Lambda}_2)_+,
\end{aligned} \tag{3.99}$$

where we used Young's Inequality. We modify (2.27), (2.29)–(2.31) by additionally integrating over $(0, t)$. Then, (3.98) is estimated from above by using the aforementioned inequalities and (3.99) by

$$\begin{aligned}
-\frac{D}{2}&\int_{\Omega_t} |\nabla(\tilde{V} - \tilde{\Psi})_+|^2 + \left(\frac{C_{fq}^2}{2D}C(\Omega) - \lambda\right)\int_{\Omega_t}(\tilde{V} - \tilde{\Psi})_+^2 \\
&+ (C_{fq} - \lambda)\int_{\Gamma_t}(\tilde{u} - \tilde{\Lambda}_2)_+^2 + (2C_{fq} - \lambda)\int_{\Gamma_t}(\tilde{v} - \tilde{\Lambda}_2)_+^2 \\
&+ (C_{fq} - \lambda)\tilde{\Lambda}_2\int_{\Gamma_t}(\tilde{u} - \tilde{\Lambda}_2)_+ + \int_{\Gamma_t}(C_{fq}(\tilde{\Psi} + \tilde{\Lambda}_2) - \lambda\tilde{\Lambda}_2)(\tilde{v} - \tilde{\Lambda}_2)_+ \\
&+ (C_{fq}\tilde{\Lambda}_2 - \tilde{\mu})\int_{\Gamma_t}(\tilde{V} - \tilde{\Psi})_+,
\end{aligned} \tag{3.100}$$

where we dropped gradient expressions on Γ and $C(\Omega) > 0$ is a given constant. We set $\tilde{\mu} = C_{fq}\tilde{\Lambda}_2$ and choose λ according to (3.87). We choose $\tilde{\Psi}_0$, such that $V_0 \le \tilde{\Psi}_0$ holds, the compatibility condition stated in (3.91) is satisfied and

$$\|\tilde{\Psi}_0\|_{L^\infty(\Omega)} \le 2\|V_0\|_{L^\infty(\Omega)} + 1$$

holds. Then, $\tilde{\Psi}$ is determined and we set $C_1(T, \Omega, D) \overset{\text{def}}{=} e^{c_0 T}$ being uniformly bounded for $D \ge 1$. Let

$$\tilde{\Lambda}_1 \overset{\text{def}}{=} 2\|V_0\|_{L^\infty(\Omega)} + 1 + C_1 C_{fq}\tilde{\Lambda}_2.$$

These choices yield that all coefficients of (3.100) are less or equal zero. The settings above and the estimates from (3.98)–(3.100) imply that

$$\frac{1}{2}\left(\|(\tilde{V} - \tilde{\Psi})_+(t)\|_{L^2(\Omega)}^2 + \|(\tilde{u} - \tilde{\Lambda}_2)_+(t)\|_{L^2(\Gamma)}^2 + \|(\tilde{v} - \tilde{\Lambda}_2)_+(t)\|_{L^2(\Gamma)}^2\right) \le 0$$

for $t \in (0, T)$. For the rescaled functions we conclude that there exists a finite $\lambda > 0$, such that

$$\begin{aligned}
\|\tilde{u}\|_{L^\infty(\Gamma_T)},\ \|\tilde{v}\|_{L^\infty(\Gamma_T)} &\le \tilde{\Lambda}_2, \\
\|\tilde{V}\|_{L^\infty(\Omega_T)} \le \|\tilde{\Psi}\|_{L^\infty(\Omega_T)} &\le 2\|V_0\|_{L^\infty(\Omega)} + 1 + C_1 C_{fq}\tilde{\Lambda}_2,
\end{aligned}$$

where we used (3.92). In turn, we conclude for the rescaled variables that

$$\begin{aligned} \|u\|_{L^\infty(\Gamma_T)},\ \|v\|_{L^\infty(\Gamma_T)} &\leq e^{\lambda T}\tilde{\Lambda}_2, \\ \|V\|_{L^\infty(\Omega_T)} &\leq e^{\lambda T}\left(2\|V_0\|_{L^\infty(\Omega)} + 1 + C_1 C_{fq}\tilde{\Lambda}_2\right). \end{aligned}$$

holds, which finishes the proof. □

Consequences for generalized nonlinearities The nonlinearities f_1, f_2, q_1 and q_2 throughout this Chapter are assumed to be Lipschitz continuous on $\mathbb{R}^2$ and are therefore bounded on compact subsets of $\mathbb{R}^2$, see Assumption 1.4. An example for $f_1(u,v)$ is given in [RR12] by $f_1 : \mathbb{R}^2 \to \mathbb{R}$ with

$$f_1(u,v) = \gamma a_1 + \gamma(a_3 - a_1)\frac{u}{a_2 + u},$$

being independent of v with constants $a_1, a_2, a_3, \gamma > 0$ satisfying $a_3 > a_1$. Here, as in many other cases, the choice of constitutive functions is only derived for the biological relevant domain of nonnegative values. In this particular case f_1 becomes unbounded as $u \to -a_2$ and is therefore not covered by Assumption 1.4. We modify f_1 to find that systems of type (GFCRD) incorporating such kind of nonlinearity are also covered by Theorem 1.2.

Let $\Pi : \mathbb{R}^2 \to \mathbb{R}^2_{\geq 0}$ be the closest-point projection. We set

$$\tilde{f}(x,y) \stackrel{\text{def}}{=} f(\Pi(x,y)) \tag{3.101}$$

to find that $\tilde{f}$ leaves f untouched on $\mathbb{R}^2_{\geq 0}$, while it is Lipschitz continuous on $\mathbb{R}^2$ satisfying Assumption 1.4. Theorem 1.2 yields that this modified system of type (GFCRD) has a unique, nonnegative solution (V,u,v). The nonnegativity of u guarantees that f does not blow up in the biological nonrelevant sector. This observation implies that Theorem 1.2 can be extended to nonlinearities f_1, f_2, q_1 and q_2 only being Lipschitz-continuous on $\mathbb{R}^2_{\geq 0}$. This corollary result weakens Assumption 1.4.

4 Application: A Shadow System reduction

In this Chapter we discuss an asymptotic model reduction of our fully coupled model (GFCRD) that in particular covers the model in [RR12]. According to the observation that the cytosolic diffusion is typically much larger than on the membrane, the authors assume an infinite cytoplasmic diffusion and spatially constant concentrations in Ω. This leads to a reduced system where only partial differential equations on the membrane Γ remain and the time-dependent but spatially constant concentration in Ω is determined just by a mass conservation condition. On the other hand the evolution on the membrane now includes a nonlocal contribution. The resulting system can be compared to so-called *Shadow Systems* in the analysis of two variable Reaction-Diffusion equations in flat space, see [Ni11; Kee78]. Here, we present a rigorous proof of a convergence to such a nonlocal reduction of the system (GFCRD).

Rigouros proof of the existence of a nonlocal functional

We consider system (GFCRD), where D is replaced by D_k, $k \in \mathbb{N}$, where $(D_k)_k$ is any sequence with $D_k \to \infty$. This results in a family of systems of type (GFCRD) for every $k \in \mathbb{N}$. The corresponding weak solutions (V_k, u_k, v_k), $k \in \mathbb{N}$, are characterized by

$$\int_{\Omega_T} \partial_t \eta_1 \, (V_k - V_0) = D_k \int_\Omega \nabla V_k \cdot \nabla \eta_1 + \int_\Gamma \left(q_1(u_k, v_k) V_k - q_2(u_k, v_k) v_k\right) \eta_1, \tag{4.1}$$

$$\int_{\Gamma_T} \partial_t \eta_2 \, (u_k - u_0) = \int_{\Gamma_T} A_u \nabla_\Gamma u_k \cdot \nabla_\Gamma \eta_2 + \int_{\Gamma_T} \left(-f_1(u_k, v_k) v_k + f_2(u_k, v_k) u_k\right) \eta_2, \tag{4.2}$$

$$\begin{aligned} \int_{\Gamma_T} \partial_t \eta_3 \, (v_k - v_0) = \int_{\Gamma_T} A_v \nabla_\Gamma v_k \cdot \nabla_\Gamma \eta_3 + \int_{\Gamma_T} \big(&f_1(u_k, v_k) v_k - f_2(u_k, v_k) u_k \\ &- q_1(u_k, v_k) V_k + q_2(u_k, v_k) v_k\big) \eta_3 \end{aligned} \tag{4.3}$$

for all

$$\eta_1 \in L^2(0,T; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega)) \text{ and}$$
$$\eta_2, \eta_3 \in L^2(0,T; H^1(\Gamma)) \cap H^1(0,T; L^2(\Gamma))$$

with $\eta_i(T) \equiv 0$ for $i = 1, 2, 3$, compare (WS) in Section 3.1.

Weak solutions and uniform boundedness Theorem 1.2 guarantees the existence and uniqueness of a family of nonnegative weak solutions (V_k, u_k, v_k) in L^2M^1 for every $k \in \mathbb{N}$. We have to ensure that the respective bounds are uniform in k.

In Proposition 3.16 have we found that the weak solutions are bounded uniformly. According to Proposition 3.16 the maximum bounds remain uniform in k. Hence, there exist $\Lambda_1, \Lambda_2 > 0$ depending on T and system constants but independent of k, such that

$$\|u_k\|_{L^\infty(\Gamma_T)}, \|v_k\|_{L^\infty(\Gamma_T)} \leq \Lambda_1 \text{ and } \|V_k\|_{L^\infty(\Omega_T)} \leq \Lambda_2 \quad \text{for all } k \in \mathbb{N}. \tag{4.4}$$

Moreover, by Lemma 3.8 we find uniform energy estimates for u_k and v_k, i.e. u_k, v_k are of class $L^2(0,T;H^1(\Gamma))$. By Lemma 3.8 (iii) we find that there exists a constant $\Lambda_3' = \Lambda_3'(\Omega, C_q, C_f, C_{A_u}, C_{A_v}, T) > 0$, such that

$$\begin{aligned} &u_k, v_k \in H^1(0,T;(H^1(\Gamma))^*), \\ &\|\partial_t u_k\|_{L^2(0,T;(H^1(\Gamma))^*)}, \|\partial_t v_k\|_{L^2(0,T;(H^1(\Gamma))^*)} \leq \Lambda_3' \end{aligned} \tag{4.5}$$

holds for all $k \in \mathbb{N}$. Since $(L^2(0,T;H^1(\Gamma)))^2$ is a reflexive Banach space we find that bounded sets are weakly precompact in $(L^2(0,T;H^1(\Gamma)))^2$, see [Sch13, Theorem 4.13, p. 79]. This holds also true for $L^2(0,T;L^2(\Omega))$. Hence, there exists a limit object we denote with $(V_\infty, u_\infty, v_\infty)$ in $L^2(0,T;H^1(\Omega)) \times (L^2(0,T;H^1(\Gamma)))^2$ and a subsequence $k \to \infty$ (not relabled), such that

$$(V_k, u_k, v_k) \rightharpoonup (V_\infty, u_\infty, v_\infty) \text{ for } k \to \infty \text{ in } L^2(0,T;H^1(\Omega)) \times (L^2(0,T;H^1(\Gamma)))^2. \tag{4.6}$$

Spatial homogeneity With the deductions in Section 3.4 we obtain the following inequality that can be formally obtained by the choice $\eta_1 = V_k$ in (4.1), i.e.

$$\begin{aligned} D_k \int_{\Omega_T} |\nabla V_k|^2 &\leq \frac{1}{2}\left(\|V_0\|^2_{L^2(\Omega)} - \|V_k(T)\|^2_{L^2(\Omega)}\right) \\ &\quad + \int_{\Gamma_T} \left(-q_1(u_k, v_k)V_k + q_2(u_k, v_k)v_k\right) V_k \\ &\leq \frac{1}{2}\left(\|V_0\|^2_{L^2(\Omega)} - \|V_k(T)\|^2_{L^2(\Omega)}\right) + \frac{1}{2}\int_{\Gamma_T} C_q \left((v_k)^2 + (V_k)^2\right) \end{aligned} \tag{4.7}$$

holds, where we used Young's Inequality and the nonnegativity of the first nonlinear expression, see Assumption 1.4. With $\varepsilon = \frac{D_k}{C_q}$ in Lemma A.5 we estimate (4.7) from above and absorb the respective gradient into the right-hand side. Therefore

$$\begin{aligned} D_k \|\nabla V_k\|^2_{L^2(\Omega_T)} &\leq \|V_0\|^2_{L^2(\Omega)} - \|V_k(T)\|^2_{L^2(\Omega)} + C_q \|v_k\|^2_{L^2(\Gamma_T)} + \frac{C(\Omega)C_q}{D_k}\|V_k\|^2_{L^2(\Omega_T)} \\ &\leq \|V_0\|^2_{L^2(\Omega)} + C_q|\Gamma|\,|T|\,\Lambda_1^2 + \frac{C(\Omega)C_q}{D_k}|\Omega|\,|T|\,\Lambda_2^2 \stackrel{\text{def}}{=} \hat{\Lambda} \end{aligned} \tag{4.8}$$

holds with a constant $C(\Omega) > 0$, according to (4.4). Using weak lower semi-continuity of the norm and (4.6), we find

$$\int_0^T \int_\Omega |\nabla V_\infty|^2 \leq \liminf_{k\to\infty} \int_0^T \int_\Omega |\nabla V_k|^2 \leq \liminf_{k\to\infty} \frac{\hat{\Lambda}}{D_k} = 0,$$

i.e. $\nabla V_\infty = 0$ almost everywhere in $\Omega \times (0,T)$. This implies that the cytosolic concentration is spatially constant.

Strong convergence and a limit of V_k With (4.5) we have found that u_k and v_k are in $H^1(0,T;(H^1(\Gamma))^*)$. According to Lions-Aubin's Lemma 3.7 applied to the triple $(H^1(\Gamma), L^2(\Gamma), (H^1(\Gamma))^*)$ we find a subsequence $k \to \infty$ (not relabled), such that u_k and v_k converge strongly in $L^2(0,T;L^2(\Gamma))$, i.e.

$$u_k \to u_\infty \text{ and } v_k \to v_\infty \text{ in } L^2(0,T;L^2(\Gamma)). \tag{4.9}$$

The strong convergence for V_k in $L^2(0,T;L^2(\Omega))$ cannot be shown with equation (4.1), since the expression $D_k \int_{\Omega_T} \nabla V_k \cdot \nabla \eta_1$ is apparently not under control for $k \to \infty$. We follow a different strategy to obtain a convergence result and claim the following.

Lemma 4.1. *For the subsequence $k \to \infty$, such that (4.6) holds and the weak L^2-limit V_∞ we have*

$$V_\infty \in W^{1,\infty}(0,T) \quad \text{and} \quad \int_\Omega V_k(t)\mathrm{d}x \to V_\infty(t)\,|\Omega| \quad \text{for any} \quad t \in (0,T),$$

where we have chosen a continuous representative of V_∞.

Proof. We use spatially constant testfunctions $\eta_1 = \eta_1(t)$ in equation (4.1) to find that $t \mapsto \int_\Omega V_k(x,t)\mathrm{d}x$ is weakly differentiable with

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega V_k(x,t)\mathrm{d}x = \int_\Gamma q_1(u_k,v_k)V_k + q_2(u_k,v_k)v_k.$$

In addition, by Assumption 1.4 and (4.4)

$$\left|\int_\Gamma q_1(u_k(x,\cdot),v_k(x,\cdot))V_k(x,\cdot) - q_2(u_k(x,\cdot),v_k(x,\cdot))v_k(x,\cdot))\mathrm{d}\sigma(x)\right| \le c_1(C_q,\Lambda_1,\Lambda_2)T.$$

is uniformly bounded in $L^\infty(0,T)$ and $c_1 > 0$. This implies that $\int_\Omega V_k(x,\cdot)\mathrm{d}x$ is in $W^{1,\infty}(0,T)$ and is uniformly bounded. With Sobolev's Embedding Theorem A.7, we find for every $\alpha \in (0,1)$ that there is a compact embedding from $W^{1,\infty}(0,T)$ into $C^{0,\alpha}([0,T])$. The compactness implies that there exists a subsequence $k \to \infty$ and

$$w \in C^{0,\alpha}([0,T]) \cap W^{1,\infty}(0,T),$$

such that

$$\int_\Omega V_k(x,t)\mathrm{d}x \to w(t) \quad \text{holds for any} \quad t \in [0,T].$$

Here, we have to check if the limit objects $w(t)$ and $\int_\Omega V_\infty(t)\mathrm{d}x$ are the same. We use the weak convergence $V_k \rightharpoonup V_\infty$ in $L^2(0,T;L^2(\Omega))$ and the spatial homogeneity of V_∞ to compute for every $\eta \in L^2(0,T)$

$$\begin{aligned}\int_0^T \eta(t)\left(w(t) - V_\infty|\Omega|\right)\mathrm{d}t \\ &= \lim_{k\to\infty}\int_0^T \eta(t)\int_\Omega V_k(x,t)\mathrm{d}x\mathrm{d}t - \int_0^T\int_\Omega \eta(t)V_\infty(x,t)\mathrm{d}x\mathrm{d}t \\ &= \lim_{k\to\infty}\left(\int_0^T \eta(t)\int_\Omega V_k(x,t)\mathrm{d}x\mathrm{d}t - \int_0^T \eta(t)\int_\Omega V_k(x,t)\mathrm{d}x\mathrm{d}t\right) = 0.\end{aligned}$$

All together, this shows that the limits are the same. □

By similar arguments we also observe that

$$t \mapsto \int_\Gamma u_\infty(x,t)\mathrm{d}x, \qquad t \mapsto \int_\Gamma v_\infty(x,t)\mathrm{d}x$$

have continuous representatives.

Mass conservation We set $\eta_i \equiv 1$ and find by adding equations (4.1)–(4.3) that for every k the mass is being conserved, i.e.

$$\int_\Omega V_k(x,t)\mathrm{d}x + \int_\Gamma (u_k + v_k)(x,t)\mathrm{d}\sigma(x) = \int_\Omega V_0\mathrm{d}x + \int_\Gamma (u_0+v_0)\mathrm{d}\sigma(x) \tag{4.10}$$

for almost every $t \in (0,T)$. Passing to the limit $k \to \infty$ in equation (4.10), we find with the deductions from above that

$$V_\infty(t)|\Omega| + \int_\Gamma (u_\infty + v_\infty)(x,t)\mathrm{d}\sigma(x) = \int_\Omega V_0\mathrm{d}x + \int_\Gamma (u_0+v_0)\mathrm{d}\sigma(x). \tag{4.11}$$

holds.

The nonlocal functional We conclude from (4.11) that

$$V_\infty(t) = \frac{1}{|\Omega|}m_0 - \frac{1}{|\Omega|}\int_\Gamma (u_\infty + v_\infty)(t)\mathrm{d}\sigma(x)$$

for every $t \in (0,T)$ for m_0 being the mass stored in the system, see (1.1). We remark that if the initial spatial concentration V_0 is constant as in [RR12], then

$$V_\infty(t) = V_0 + \frac{1}{|\Omega|}\int_\Gamma (u_0+v_0)\mathrm{d}\sigma(x) - \frac{1}{|\Omega|}\int_\Gamma (u_\infty + v_\infty)(t)\mathrm{d}\sigma(x)$$

holds.

Limit in the equations We follow the deductions in Subsection 3.3.3. The convergence results for u_k and v_k according to (4.5) and (4.6) imply that we are allowed to pass to the limit in (4.2) to find that (1.2a) holds.

For a convergence result for (4.3) we have to control V_k on Γ and have to ensure that V_k converges to V_∞. We consider the following estimate for $w \in H^1(\Omega)$ with mean value zero. A boundary type inequality holds,

$$\frac{1}{|\Gamma|^{1/2}}\|w\|_{L^2(\Gamma)} \leq \|w\|_{L^4(\Gamma)} \leq c_2\|\nabla w\|_{L^2(\Omega)}, \tag{4.12}$$

where $c_2 > 0$ depends on Ω, see Lemma A.4 for $\gamma = 1$. We apply (4.12) to V_k decreased by its mean value to obtain

$$\begin{aligned}\int_0^T \|V_k(\cdot,t) - \frac{1}{|\Omega|}\int_\Omega V_k(x,t)\mathrm{d}x\|^2_{L^2(\Gamma)}\mathrm{d}t &\leq c_3\int_0^T \|\nabla V_k(\cdot,t)\|^2_{L^2(\Omega)}\mathrm{d}t \\ &\leq c_3\frac{\hat{\Lambda}}{D_k} \to 0 \text{ as } k\to\infty,\end{aligned}$$

for a constant $c_3(\Omega) > 0$, where we used (4.8). Then, for V_k on Γ we obtain

$$\lim_{k\to\infty} V_k\Big|_\Gamma = \lim_{k\to\infty}\frac{1}{|\Omega|}\int_\Omega V_k = V_\infty$$

in $L^2(\Gamma_T)$. Then, the convergence results for u_k, v_k and V_k imply that we are allowed to pass to the limit in (4.3), see the arguments in the proof of Proposition 3.11. We obtain that (1.2b) holds and the limit function of V_k on Γ is the same as in the bulk Ω. This completes the proof of Theorem 1.3.

5 Summary

The main purpose of this thesis was the mathematical analysis of certain spatially coupled Reaction-Diffusion Systems arising in the description of Signaling Networks in biological cells. The spatial coupling between a diffusion e.g. in the cytosolic bulk and reaction and diffusion processes on the boundary surface is given by a Robin-type boundary condition that introduces a source term in one of the membrane GTPase equations given by the outflux from the bulk of cytosolic GTPase. There is a great interest in understanding the implications of such type of spatially coupled systems. Our main goal here was to provide existence and well-posedness results for prototypes of such models, like (FCRD) and (GFCRD), as they are not covered by standard theory. These main results can now be used as a starting point to examine more complex spatially coupled systems and to rigorously investigate qualitative properties. The applied methods to prove the main results were the following:

In the second chapter we have considered regular data and classical diffusion operators (in case of the surface equations expressed by the Laplace-Beltrami operator). The result on existence of classical solutions for (FCRD) in Theorem 1.1 was based on an operator splitting approach that decouples bulk and surface equations. With the help of L^∞-a priori estimates, short-time existence for Reaction-Diffusion Systems on manifolds, regularity results for scalar parabolic equations on manifolds and nonlinear theory of parabolic initial-boundary problems we were able to obtain Schauder estimates in Hölder spaces of different orders. It turned out that the combination of such estimates was sufficient to apply Schauder's Fixed-Point Theorem to an updating procedure. The uniqueness of the resulting fixed-point was independently achieved with weak theory from Chapter 3; bootstrapping arguments led to a classical solution triplet (V, u, v) of the fully coupled Reaction-Diffusion System (FCRD) of parabolic Hölder order $(2 + \alpha)$ for given $\alpha \in (0, 1)$ on a given time-interval $[0, T]$, for $T > 0$. Moreover, we here obtained that solutions depend continuously on the initial data and are essentially bounded and nonnegative. The system (FCRD) is well-posed.

Complementary to the classical setting we have also considered possibly nonsmooth diffusion operators that for example can model heterogeneous domains with specific properties on the membrane. The existence and well-posedness result from Theorem 1.2 for (GFCRD) was based on an implicit discretization in time which reduced the given nonlinear parabolic system to a sequence of nonlinear recursive elliptic problems. This was then solved by an application of the theory of monotone perturbed operators. We furthermore here proved continuous dependence on the initial data, an L^2-continuity property and that solutions are essentially bounded on bounded time intervals. In general, for our system, this cannot be deduced from an invariant region principle. Instead we use a different technique on finding L^∞-a priori estimates as in Chapter 2, namely, based on appropriate testfunctions and comparison functions.

Furthermore, in Chapter 4 we applied the results from the weak theory to find that an asymptotic model reduction for (GFCRD) can be deduced rigorously, cf. Theorem 1.3. In fact, when the cytosolic diffusion constant formally tends to infinity, then the concentration in the bulk is spatially constant. A reduced system of partial differential equations remains on the boundary incorporating a nonlocal functional; a so-called Shadow System appears.

Let us finally point out two directions of more complex models motivated by Signaling Networks in biological cells. Firstly, one could integrate directed signaling pathways on the inside of a cell, i.e. information is being distributed along microtubulis or the cell skeleton. In this case, drift terms occur in the bulk equations, which need further investigation, see [Cal+12] and the recent contribution [AR16] for some simplified models in that direction. Secondly, the assumption that the cell has a fixed geometry is very restrictive and in many cases not realistic. Blood cells, human skin cells and pancreas cells bend and stretch forced by external and internal influences. Therefore, it becomes necessary to consider a moving membrane surface coupled to the bulk-surface Reaction-Diffusion System. Under proper assumptions on the geometric evolution, like a preserved regularity of the manifold, and reasonable governing equations for bending and stretching operations, our results and methods of proof may be transferred to such extended systems.

Appendix: background material and auxiliary results

In this Chapter we state and prove auxiliary results which we have referred to in the other Chapters of this thesis.

A.1 Inequalities, notation and function spaces

A.1.1 Inequalities

For convenience, we state the following inequalities.

- For $\varepsilon > 0$, each of the inequalities

  $$2ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon} \quad \text{and} \quad ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}$$

  is called *Young's Inequality*, see for example [Eva10, B.2., p. 706ff.].

- Let $m \in \mathbb{N}$ and $w_i \in L^{p_i}(\mu)$ for $i = 1, \dots, m$ with $p_i,\ q \in [1, \infty]$ satisfying $\sum_{i=1}^m \frac{1}{p_i} = \frac{1}{q}$, then the product $w_1 \cdots w_m$ is in $L^q(\mu)$ and *Hölder's Inequality*

  $$\left\| \prod_{i=1}^m w_i \right\|_{L^q(\mu)} \leq \prod_{i=1}^m \|w_i\|_{L^{p_i}(\mu)}$$

  holds. Here, μ is an appropriate measure, for example the Lebesgue measure on Ω or the surface area measure on Γ, see [Alt12, p. 54ff.].

A.1.2 Function spaces on flat domains

In this subsection we introduce Hölder spaces, parabolic Hölder spaces, Sobolev and Bochner spaces for a domain $\Omega \subset \mathbb{R}^n$ or $\Omega_T \subset \mathbb{R}^{n+1}$, respectively.

Hölder spaces and parabolic Hölder spaces The set of continuously differentiable functions $w : \overline{\Omega} \to \mathbb{R}$ of order k shall be denoted by $C^k(\overline{\Omega})$, such that there exists a continuation on the boundary of Ω for derivatives up to order k. We define a corresponding norm by

$$\|w\|_{C^k(\overline{\Omega})} \stackrel{\text{def}}{=} \sum_{j=0}^{k} \|D^j w\|_{C^0(\overline{\Omega})}$$

with supremum norm $\|w\|_{C^0(\overline{\Omega})} = \sup\{|w(x)| : x \in \overline{\Omega}\}$. For $0 < \beta < 1$ and $w : \overline{\Omega} \to \mathbb{R}$ let the quantity

$$\text{höl}_\beta(w, \Omega) \overset{\text{def}}{=} \sup\left\{ \frac{|w(x) - w(y)|}{|x - y|^\beta} \ , \ x, y \in \Omega, \ x \neq y \right\} \in [0, \infty]$$

be the *Hölder constant* of w. The Banach space given by

$$C^{k+\beta}(\overline{\Omega}) \overset{\text{def}}{=} \{w \in C^k(\overline{\Omega}) \ , \ \text{höl}_\beta(\partial^s w, \overline{\Omega}) < \infty, |s| = k\},$$
$$\|w\|_{C^{k+\beta}(\overline{\Omega})} \overset{\text{def}}{=} \sum_{|s| \leq k} \|\partial^s w\|_{C^0(\overline{\Omega})} + \sum_{|s|=m} \text{höl}_\beta(\partial^s w, \overline{\Omega})$$

is called *Hölder space* of order $(k + \beta)$ for $\beta \in (0, 1)$, for further details and properties see [Alt12, p. 46]. We denote the space of *Lipschitz continuous functions* by $C^{0,1}(\overline{\Omega})$.

A function $w : \overline{\Omega}_T \to \mathbb{R}$ lies in the space $H^{k+\beta,(k+\beta)/2}(\overline{\Omega_T})$ if the corresponding norms we denote with $|w|^{(k+\beta)}_{\Omega_T}$ are finite. The cases where $k = 0, 1, 2$ are of interest in this thesis. We write the norms out in full for $k = 1, 2$, i.e.

$$\begin{aligned}
|w|^{(1+\beta)}_{\Omega_T} &= \|w\|_{C^0(\overline{\Omega_T})} + \|D_x w\|_{C^0(\overline{\Omega_T})} + \langle D_x w\rangle^{(\beta)}_{x,\overline{\Omega_T}} \\
&\quad + \langle w\rangle^{((1+\beta)/2)}_{t,\overline{\Omega_T}} + \langle D_x w\rangle^{(\beta/2)}_{t,\overline{\Omega_T}}, \qquad \text{(A.1)}\\
|w|^{(2+\beta)}_{\Omega_T} &= \|w\|_{C^0(\overline{\Omega_T})} + \|D_x w\|_{C^0(\overline{\Omega_T})} + \|D^2_x w\|_{C^0(\overline{\Omega_T})} + \|D_t w\|_{C^0(\overline{\Omega_T})} \\
&\quad + \langle D^2_x w\rangle^{(\beta)}_{x,\overline{\Omega_T}} + \langle D_t w\rangle^{(\beta)}_{x,\overline{\Omega_T}} + \langle D_t w\rangle^{(\beta/2)}_{t,\overline{\Omega_T}} \\
&\quad + \langle D_x w\rangle^{((1+\beta)/2)}_{t,\overline{\Omega_T}} + \langle D^2_x w\rangle^{(\beta/2)}_{t,\overline{\Omega_T}}, \qquad \text{(A.2)}\\
\langle w\rangle^{(\beta)}_{x,\overline{\Omega_T}} &= \sup_{\substack{(x,t),(x',t)\in\overline{\Omega_T} \\ |x-x'|\leq\varrho_0}} \frac{|w(x,t) - w(x',t)|}{|x - x'|^\beta}, \\
\langle w\rangle^{(\beta)}_{t,\overline{\Omega_T}} &= \sup_{\substack{(x,t),(x,t')\in\overline{\Omega_T} \\ |t-t'|\leq\varrho_0}} \frac{|w(x,t) - w(x,t')|}{|t - t'|^\beta},
\end{aligned}$$

where we adopted the notation from [LSU68, p. 7f.]. Here, $\varrho_0 > 0$ is a fixed constant. Remark that

$$|w|^{(1+\beta)}_{\Omega_T} \leq C(\varrho_0, \beta)|w|^{(2+\beta)}_{\Omega_T}$$

holds.

Sobolev and Bochner spaces Consider the measure space $(\Omega, \mathcal{B}, \mathcal{L}^3)$ for a σ-algebra $\mathcal{B}$ on Ω.

$$L^p(\Omega) = \{w : \Omega \to \mathbb{R} \ , \ w \text{ is } \mathcal{L}^3 \text{ measurable and } \int_\Omega |w|^p \mathrm{d}\mathcal{L}^3 < \infty\},$$

where $p \in [1, \infty)$. For $p = \infty$, let $L^\infty(\Omega)$ be the set of essentially bounded and $\mathcal{L}^3$-measurable functions. For $p \in [1, \infty)$ let $L^p(\Omega)$ be the Banach space consisting of all equivalence classes of functions $w : \Omega \to \mathbb{R}$, such that $\int_\Omega |w|^p \mathrm{d}\mathcal{L}^3 < \infty$ holds . The corresponding norm is given by

$$\|w\|_{L^p(\Omega)} = \left(\int_\Omega |w|^p \mathrm{d}\mathcal{L}^3 \right)^{1/p}.$$

Let $L^\infty(\Omega)$ with the norm $\|w\|_{L^\infty(\Omega)} \overset{\text{def}}{=} \sup_{x \in \Omega} |w(x)|$ denote the space of essentially bounded functions on Ω. With these definitions we are able to define *Sobolev spaces* on Ω. For $k \in \mathbb{N}$ and $p \in [1, \infty]$ we set

$$W^{k,p}(\Omega) = \{w \in L^p(\Omega)\ ,\ D^\beta w \in L^p(\Omega) \text{ for } \beta \in \mathbb{N}^3,\ |\beta| \le k\},$$

$$\|w\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\beta| \le k} \|D^\beta w\|^p_{L^p(\Omega)} \right)^{1/p} & \text{for } 1 \le p < \infty \\ \sum_{|\beta| \le k} \|D^\beta w\|_{L^\infty(\Omega)} & \text{for } p = \infty. \end{cases}$$

For $p = 2$, we write $H^k(\Omega) = W^{k,2}(\Omega)$ for convenience. The dual space of $H^k(\Omega)$ shall be denoted by $(H^k(\Omega))^*$. For further results on Sobolev spaces we refer to [Bre11] or [MS11].

For a proper formulation of time and space dependent functions it is convenient to define *Bochner spaces*. Let X be a Banach space, $T > 0$ and $p \in [1, \infty)$. For strongly measurable functions $w : [0, T] \to X$ we set the space $L^p(0, T; X)$ equipped with the norm

$$\|w\|_{L^p(0,T;X)} \overset{\text{def}}{=} \left(\int_0^T \|w(t)\|^p_X \mathrm{d}t \right)^{1/p} < \infty$$

to be a Banach space of *Bochner*-type. Here, w is a representant of an equivalence class such that all functions in this class coincide for almost every $t \in [0, T]$. This definition was taken from [Sch13, p. 191ff.], where further properties of Bochner spaces can be found.

A.1.3 Definitions and properties from differential geometry

The following general definitions and properties can be found in [Aub98].

Smooth manifolds We assume that $\Gamma \subset \mathbb{R}^3$ is a Riemannian manifold of dimension 2 or, in a more abstract sense, is a topological Hausdorff space, such that for every point $p \in \Gamma$, there exists a neighbourhood homeomorphic to $\mathbb{R}^2$. We assume that Γ is of class C^∞, i.e. local charts parametrize smoothly, coordinate changes are smooth. We remark that the C^∞-assumption is not necessary and one could assume a less regular Riemannian manifold. Moreover, we assume that Γ is *orientable*, *compact* and *closed* that is represented as boundary $\Gamma = \partial\Omega$ of an open domain $\Omega \subset \mathbb{R}^3$. In particular, there exist exactly two different continuous normal fields, there exists a finite atlas and Γ has no boundary.

Tangential space, Riemannian metric Let $p \in \Gamma$, then $\tilde{v} \in \mathbb{R}^3$ is called *tangential vector* at p, if there exists a C^1-curve $\gamma : (-\varepsilon, \varepsilon) \to \Gamma$ for $\varepsilon > 0$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = \tilde{v}$. The space of all tangential vectors in p is denoted by $\mathrm{T}_p\Gamma$ and called *tangential space in p*. Let $\varphi : U \subset \mathbb{R}^2 \to W \subset \Gamma$ be a local chart and $x_0 \in U$, then the matrix

$$g(x_0) \stackrel{\text{def}}{=} D\varphi^T(x_0)D\varphi(x_0) \in \mathbb{R}^{2\times 2},$$

which we identify with its induced bilinear form $g_{x_0}(v_1, v_2) \stackrel{\text{def}}{=} v_1 \cdot g(x_0)v_2$ for $v_1, v_2 \in \mathbb{R}^2$, is the *pullback metric* in x_0. The matrix $g_{ij}(x_0)$ is symmetric and positive definite, the inverse shall be denoted by $(g^{ij})_{1\leq i,j\leq 2} = g^{ij}$. This gives rise to the definition of the *first fundamental form* on Γ as a scalar product of two vectors of the tangential plane, i.e. $I_p(\tilde{v}_1, \tilde{v}_2) = \tilde{v}_1 \cdot \tilde{v}_2 = g_{x_0}(v_1, v_2)$ where $D\varphi_{x_0}v_1 = \tilde{v}_1$ and $D\varphi_{x_0}v_2 = \tilde{v}_2$. In our case $g(x_0)$ exists for every $x_0 \in \Gamma$. Therefore, we say that (Γ, g) is a *smooth differentiable Riemannian manifold*.

Covariant derivative and surface gradient The motivation to introduce covariant derivatives is that one wants to measure the derivative on non-flat spaces, lateral derivatives, which is different to the usual derivative in space. There are several definitions to determine covariant derivatives, namely for functions, for vector fields, covector fields and tensor fields on Γ. We focus on the first case and begin with the following: for $w \in C^1(\Gamma)$, we define the map $D_\Gamma w_p : \mathrm{T}_p\Gamma \to \mathbb{R}$ via $\mathrm{d}(w \circ \varphi)_x(v) = D_\Gamma w_{\varphi(x)} D\varphi(x)v$ for all $v \in T_p\Gamma$ and all local charts φ with $\varphi(x) = p$. Here, we demand that the usual chain rule holds. The operator D_Γ on the right-hand side is called the *covariant derivative* of w in p. For $w \in C^1(\Gamma)$ the *surface gradient* $\nabla_\Gamma w(p)$ for $p \in \Gamma$ is given as the unique vector $\nabla_\Gamma w(p) \in \mathrm{T}_p\Gamma$, such that

$$\nabla_\Gamma w(p) \cdot \tilde{v} = D_\Gamma w_p \tilde{v}$$

holds for all $\tilde{v} \in \mathrm{T}_p\Gamma$.

Divergence and the Laplace-Beltrami operator Since we are interested in second order partial differential equations, we introduce higher order surface derivatives by inductively applying the surface gradient ∇_Γ to functions w. Analogously, higher order covariant derivatives D_Γ^j are defined. If these derivatives of order k exist, then $w \in C^k(\Gamma)$ and the corresponding norm is given by

$$\|w\|_{C^k(\Gamma)} = \sum_{j=0}^{k} \sup_{x\in\Gamma} |\nabla_\Gamma^j w(x)|.$$

For a vector field $X \in C^1(\Gamma; \mathbb{R}^3)$ the *tangential divergence* is defined by $\mathrm{div}_\Gamma X \stackrel{\text{def}}{=} \mathrm{tr}\nabla_\Gamma X$. For $w \in C^2(\Gamma)$ the *Laplace-Beltrami operator* $\Delta_\Gamma w \in C^0(\Gamma)$ is defined by

$$\Delta_\Gamma w \stackrel{\text{def}}{=} \mathrm{div}_\Gamma \nabla_\Gamma w.$$

Integration on manifolds For the oriented manifold Γ we define

$$\mathrm{d}v_g \overset{\text{def}}{=} \sqrt{\det g}\ \mathrm{d}x^1 \mathrm{d}x^2,$$

where $\{x^1, x^2\}$ is the local coordinate system corresponding to a local chart φ. This *oriented Riemannian volume element* is compatible to the surface area measure $\mathrm{d}\sigma$, the Hausdorff measure on Γ. For a local chart $\varphi : U \to W$ and a continuous function $w : W \to \mathbb{R}$ with compact support on W, we set

$$\int_{\Gamma \cap W} w \mathrm{d}\sigma(x) = \int_{\Gamma \cap W} w \mathrm{d}\sigma = \int_{\Gamma \cap W} w \mathrm{d}v_g = \int_U (w \circ \varphi)\sqrt{\det g}\ \mathrm{d}x^1 \mathrm{d}x^2.$$

The map $w \longmapsto \int_\Gamma w \mathrm{d}v_g$ defines a positive Radon measure, see [Aub98, p. 29ff.]. We remark that Gauss' Divergence Theorem holds.

Theorem A.1 (Gauss' Divergence Theorem)**.** *For Γ compact, a tangent vector field $X \in C^1(\Gamma, \mathbb{R}^3)$ and for all $\eta \in C^1(\Gamma)$ we find*

$$\int_\Gamma \eta \,\mathrm{div}_\Gamma X \mathrm{d}\sigma = -\int_\Gamma \nabla_\Gamma \eta \cdot X \mathrm{d}\sigma.$$

Function spaces on manifolds In the last paragraph we already defined spaces of continuous differentiable functions up to order k on Γ. Here, we are introducing Hölder spaces, parabolic Hölder spaces and Sobolev spaces on Γ. We begin with the definition of a distance function. We define

$$d_g(p,q) \overset{\text{def}}{=} \inf\{L(\gamma) \mid \gamma : [0,1] \to \Gamma,\ \gamma(0) = p,\ \gamma(1) = q\},$$

with $L(\gamma) = \int_0^1 \sqrt{g(\gamma'(t), \gamma'(t))}\mathrm{d}t$ for $p, q \in \Gamma$. For $\beta \in (0,1)$ the function w is in $C^\beta(\Gamma)$ is the corresponding norm

$$\|w\|_{C^\beta(\Gamma)} = \sup |w| + \sup_{p \neq q} \frac{|w(p) - w(q)|}{(d_g(p,q))^\beta}$$

is bounded. Then, $C^{k+\beta}(\Gamma)$ for $k \in \mathbb{N}_0$ consists of functions, such that the k-th derivative is β-Hölder continuous. Vice versa, parabolic Hölder spaces take the corresponding time-derivatives and some mixed derivatives into account. In the manifold case, we denote these spaces by $H^{k+\beta,(k+\beta)/2}(\overline{\Gamma_T})$. The first number $k + \beta$ denotes the order of the Hölder norm in space, $(k+\beta)/2$ denotes the order of time-derivatives. We denote the supremum norm with $\|\cdot\|_{C^0(\Gamma)}$. A function $w : \Gamma_T \to \mathbb{R}$ is in $H^{k+\beta,(k+\beta)/2}(\overline{\Gamma_T})$, if the quantity we denote with

$$\begin{aligned} |w|^{(k+\beta)}_{\Gamma_T} = \|w\|_{C^0(\overline{\Gamma_T})} &+ \sum_{j=1}^{k} \sum_{2r+s=j} \|D_t^r D_{\Gamma,x}^s w\|_{C^0(\overline{\Gamma_T})} \\ &+ \sum_{2r+s=k} \langle D_t^r D_{\Gamma,x}^s w \rangle^{(\beta)}_{x,\overline{\Gamma_T}} + \sum_{0<k+\beta-2r-s<2} \langle D_t^r D_{\Gamma,x}^s w \rangle^{\left(\frac{k+\beta-2r-s}{2}\right)}_{t,\overline{\Gamma_T}} \end{aligned} \tag{A.3}$$

is finite. Here, for some function $\tilde{w} : \Gamma \times (0,T) \to \mathbb{R}$ we have

$$\langle \tilde{w} \rangle^{(\beta)}_{x,\overline{\Gamma_T}} = \sup_{\substack{(x,t),(x',t)\in\overline{\Gamma_T} \\ d_g(x,x')<\varrho_0}} \frac{|\tilde{w}(x,t)-\tilde{w}(x',t)|}{(d_g(x,x'))^{\beta}},$$

$$\langle \tilde{w} \rangle^{(\beta/2)}_{t,\overline{\Gamma_T}} = \sup_{\substack{(x,t),(x,t')\in\overline{\Gamma_T} \\ |t-t'|<\varrho_0}} \frac{|\tilde{w}(x,t)-\tilde{w}(x,t')|}{(|t-t'|)^{(\beta/2)}},$$

for $\beta \in (0,1)$, where we used the notation in [LSU68, p. 7f]. The parameter $\varrho_0 > 0$ is an injectivity radius which is strictly greater than zero since we are on a compact manifold and geodesics are uniquely defined. For convenience, we used the notation $D_{\Gamma,x}$ and D_t standing for the covariant derivative D_Γ for spatial dependency on Γ and the time-derivative. These norms are quite similar to the flat case of an open domain $\Omega \in \mathbb{R}^n$.

We denote by $\mathcal{L}^p(\Gamma)$ for $p \in [1,\infty)$ the class of all measurable functions w on Γ, such that the quantity $(\int_\Gamma |w|^p \mathrm{d}\sigma)^{1/p}$ is finite. Then, $L^p(\Gamma)$ is defined in the common Lebesgue sense endowed with norm

$$\|w\|_{L^p(\Gamma)} = \left(\int_\Gamma |w|^p \mathrm{d}v_g\right)^{\frac{1}{p}}.$$

Moreover, the space $L^\infty(\Gamma)$ shall consists of all measurable functions that are essentially bounded with respect to the Hausdorff measure.

Let ϕ be of class $C^\infty(\Gamma)$. Let $\mathcal{C}^p_k(\Gamma)$ be the space of functions, such that $\nabla^j_\Gamma \phi \in L^p$ for $0 \le j \le k$. The completion of $\mathcal{C}^p_k(\Gamma)$ with respect to the norm

$$\|\phi\|_{W^{k,p}(\Gamma)} = \sum_{j=0}^{k} \|\nabla^j_\Gamma \phi\|_{L^p(\Gamma)}$$

is called *Sobolev space* $W^{k,p}(\Gamma)$ on Γ. For $p = 2$, we write $H^k(\Gamma)$.

A.2 Gagliardo-Nirenberg and boundary estimates

We aim on finding estimates for functions $w : \overline{\Omega} \to \mathbb{R}$, such that expressions $w \in L^p(\Gamma)$ are estimated by terms defined in Ω. We begin with a variation of Gagliardo-Nirenberg estimates which is taken from [LU68, p. 45].

Lemma A.2 (Gagliardo-Nirenberg-type inequality)**.** *For a bounded domain $\Omega \subset \mathbb{R}^n$, let $w \in W^{1,m}(\Omega)$, $m \in \mathbb{N}$, have a mean-value of zero, i.e.*

$$\int_\Omega w(x)\,dx = 0,$$

then the inequality

$$\|w\|_{L^p(\Omega)} \leq c\|\nabla w\|^{\gamma}_{L^m(\Omega)}\|w\|^{1-\gamma}_{L^r(\Omega)}$$

holds, whenever $\frac{n}{p} = \gamma\frac{n-m}{m} + (1-\gamma)\frac{n}{r}$, $r \geq 1$ *and* $\gamma \in [0,1]$ *for* $c = c(m,p,r,n,\alpha) > 0$.

Proof. The proof of this classical interpolation estimate can be found in several books, for example in DiBenedetto [DiB02, p. 423ff.]. The ideas stem from the papers of Gagliardo [Gag58] and Nirenberg [Nir59]. □

As a corollary we formulate this statement for a compact, smooth 2-manifold Γ.

Corollary A.3. *Let* $\Gamma \subset \mathbb{R}^3$ *be a smooth, compact 2-manifold. Let* $w \in H^1(\Gamma)$ *have a mean value of zero. Then there exists a constant* $c > 0$*, such that*

$$\|w\|_{L^p(\Gamma)} \leq c\|\nabla_\Gamma w\|^{\gamma}_{L^2(\Gamma)}\|w\|^{1-\gamma}_{L^r(\Gamma)}, \tag{A.4}$$

where $\frac{1}{p} = \frac{1-\gamma}{r}$ *for* $\gamma \in (0,1)$.

Proof. See [DHV99]. □

The connection of boundary values and values inside a domain is described by the following lemma.

Lemma A.4 (Multiplicative Gagliardo-Nirenberg-type inequality)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded Lipschitz domain and* Γ *be its boundary. Then for* $w \in H^1(\Omega)$ *with mean value zero we have*

$$\|w\|_{L^p(\Gamma)} \leq c\|\nabla w\|^{\gamma}_{L^2(\Omega)}\|w\|^{1-\gamma}_{L^2(\Omega)} \tag{A.5}$$

with $\gamma \in [0,1]$, $p = \frac{2(n-1)}{n-2\gamma}$ *and* $c = c((\Omega), p) > 0$.

Proof. This assertion follows for example with [MS11, Corollary 2, p. 82]. In the corresponding notation we set $\mu = \mathcal{H}^2$, the Hausdorff measure on Γ, $k = 0$, $l = 1$ and $s = 2$. After rearranging the notation we find

$$\|w\|_{L^p(\Gamma)} \leq c\|w\|^{\gamma}_{H^1(\Omega)}\|w\|^{1-\gamma}_{L^2(\Omega)}, \tag{A.6}$$

where $c = c(\Omega) > 0$. Since the mean value of w was claimed to be zero, Poincaré's Inequality implicates the desired inequality. □

The statements above lead to an L^p-boundary estimation by arbitrarily small amounts of the L^2-gradient norm and the L^2-norm.

Lemma A.5 (ε-estimation on the boundary)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain and* Γ *be its Lipschitz boundary. For an arbitrary function* $w \in H^1(\Omega)$ *and* $0 < \varepsilon < 1$ *we have*

$$\|w\|^2_{L^p(\Gamma)} \leq \varepsilon\|\nabla w\|^2_{L^2(\Omega)} + \varepsilon^{\frac{-\beta}{1-\beta}}C(\Omega)\|w\|^2_{L^2(\Omega)} \tag{A.7}$$

with $p = \frac{2(n-1)}{n-2\beta}$ *and* $\beta \in (0,1)$.

Proof. A proof can be found in [Gri85, Theorem 1.5.1.10, p. 41]. □

Corollary A.6. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and Γ be its boundary being a compact, smooth manifold. For $w \in H^1(\Omega)$ with $0 < \varepsilon < 1$ we have*

$$\|w\|_{L^2(\Gamma)} \leq C(\Omega) \left(\sqrt{\varepsilon} \|\nabla w\|_{L^2(\Omega)} + \frac{1}{\sqrt{\varepsilon}} \|w\|_{L^2(\Omega)} \right)$$

with $C(\Omega) > 0$.

A.3 Sobolev Embeddings

In this subsection we are restating Sobolev's Embedding Theorem and in particular results about compact embeddings. For the case of a domain with Lipschitz boundary we have the following result.

Theorem A.7 (Sobolev's Embedding Theorem, flat case)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain and Γ be its Lipschitz boundary.*

1. *To embed Sobolev spaces into Hölder spaces, let $k \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $1 \leq p < \infty$.*
   a) *If $k - \frac{n}{p} \geq m + \gamma$ holds for $0 < \gamma < 1$, then $\mathrm{Id} : W^{k,p}(\Omega) \hookrightarrow C^{m+\gamma}(\Omega)$ is a continuous embedding, i.e. there exists $c_1 \overset{def}{=} c_1(\Omega, n, m, p, k, \gamma)$, such that for $w \in W^{k,p}(\Omega)$ the inequality*
   $$\|w\|_{C^{m+\gamma}(\Omega)} \leq c_1 \|w\|_{W^{k,p}(\Omega)}$$
   holds.
   b) *If $k - \frac{n}{p} > m + \gamma$, then $\mathrm{Id} : W^{k,p}(\Omega) \hookrightarrow C^{m+\gamma}(\Omega)$ is a continuous and compact embedding.*
2. *To embed Sobolev spaces into Sobolev spaces, let $k_1, k_2 \in \mathbb{N}_0$ and $1 \leq p_1, p_2 < \infty$.*
   a) *If $k_1 - \frac{n}{p_1} \geq k_2 - \frac{n}{p_2}$ with $k_1 \geq k_2$, then $\mathrm{Id} : W^{k_1,p_1}(\Omega) \hookrightarrow W^{k_2,p_2}(\Omega)$ is a continuous embedding and there exists a constant $c_2 \overset{def}{=} c_2(n, \Omega, k_1, p_1, k_2, p_2)$, such that for $w \in W^{k_1,p_1}(\Omega)$ the inequality*
   $$\|w\|_{W^{k_2,p_2}(\Omega)} \leq c_2 \|w\|_{W^{k_1,p_1}(\Omega)}$$
   holds.
   b) *If $k_1 - \frac{n}{p_1} > k_2 - \frac{n}{p_2}$ with $k_1 > k_2$, then $\mathrm{Id} : W^{k_1,p_1}(\Omega) \hookrightarrow W^{k_2,p_2}(\Omega)$ is a continuous and compact embedding.*

Proof. The proof is given, for example, in [Alt12, p. 345ff., 350ff.]. □

For functions defined on a compact 2-manifold Γ we cite the following embedding theorem taken from [Aub82, 2.20 Theorem, p. 44].

Theorem A.8 (Sobolev's Embedding Theorem on compact manifolds)**.** *For compact manifolds Sobolev's Embeddings Theorem A.7 holds.*

A.4 Gronwall's Lemma

Here we restate a differential and integral formulation of Gronwall's Lemma.

Lemma A.9 (Gronwall's Lemma). *Let w, β and γ be nonnegative continuous functions defined on $[0,T]$.*

(i) *If w satisfies the integral inequality*

$$w(t) \leq \beta(t) + \int_0^t \gamma(s) w(s)\, ds \quad \text{for all} \quad t \in [0,T],$$

then

$$w(t) \leq \beta(t) + \int_0^t \beta(r)\gamma(r) \exp\left(\int_r^t \gamma(s)\, ds\right) dr \quad \text{for} \quad t \in [0,T].$$

(ii) *If w is differentiable in $(0,T)$ satisfying*

$$w'(t) \leq \beta(t) + \gamma(t) w(t) \quad \text{for all} \quad t \in (0,T),$$

then

$$w(t) \leq w(0) \exp\left(\int_0^t \gamma(s)\, ds\right) + \int_0^t \beta(r) \exp\left(\int_r^t \gamma(s)\, ds\right) dr.$$

Proof. The proof for the integral form can be found for example in [Pac98, Theorem 1.3.2, p. 13] and the second assertion can be found for example in [Eva10, p. 708]. □

A.5 Generalized Lebesgue Convergence Theorem

Here we restate a generalized Lebesgue Convergence Theorem. This standard result can for example be found in [Alt12, 1.25, p. 62].

Theorem A.10 (Generalized Lebesgue Convergence Theorem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $p \in [1,\infty)$ and $g_k, h_k : \Omega \to \mathbb{R}$, $k \in \mathbb{N}$, and $g, h : \Omega \to \mathbb{R}$ for $k \to \infty$. If $g_k \to g$ pointwise a.e. and $|g_k| \leq h_k$ with $h_k \to h$ in $L^p(\Omega)$, then*

$$g \in L^p(\Omega) \quad \text{and} \quad g_k \to g \text{ in } L^p(\Omega)$$

holds.

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