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On the geometrization of problems in electrical measurement

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Abstract. We provide the general solution of problems concerning AC star circuits by turning them into geometric problems. We show that one problem is strongly related to the Fermat-point of a triangle. We present a solution that is well adapted to the practical application the problem is based on. Furthermore, we solve a generalization of the geometric situation and discuss the relation to non-symmetric, unbalanced AC star circuits.

1   The initial problem: the non-symmetric generator

Nowadays the distribution of power in AC systems is not provided by a single power plant anymore. The growth of importance of renewable energies is reflected in an increasing decentralization of energy supply. To guarantee a stable and continuous operation it is important to constantly and precisely measure the involved currents and voltages.

The question that we discuss in our first part of the text came up during the testing of high voltage generators. Its components may independently vary in time, e.g. due to warming, which results in a non-symmetry of the line voltages. In the system at hand these line voltages can not be measured directly for technical reasons, but only the phase-to-phase voltages can be measured. Therefore, the question was how we can get the first ones from the latter ones.

A three-phase AC generator typically is star-shaped. That means that the three coils of the generator are placed around a turning magnet forming a regular three armed star. Of each coil one end is grounded (N) and the free ends are the phases that form the plug socket (A_1, A_2, A_3). This situation yields the star circuit as drawn in Figure 1.

Figure 1: The basic circuit of a star-shaped generator
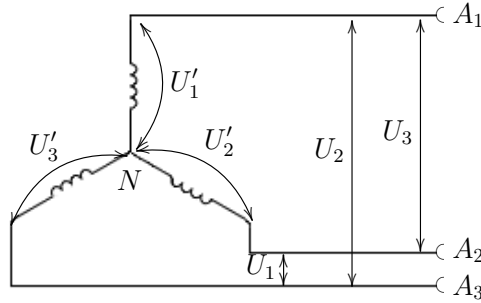
The well known symmetric situation is as follows: Between the points A_i and N we have AC voltages with same amplitudes but phase differences $\psi = 120°$. Then the voltages can be described in terms of harmonic oscillations in the following way:

$$U'_\alpha = \hat{U}'_\alpha \cos(\omega t + \psi_\alpha) = \hat{U}'_\alpha \Re\left(e^{i\omega t + i\psi_\alpha}\right) \quad \alpha = 1, 2, 3\,.$$

with $\hat{U}'_\alpha = \hat{U}$ for all α and $\psi_1 = \psi, \psi_2 = 0, \psi_3 = 2\psi$. The phase-to-phase voltage U_3 between the A_1 and A_2 is given by the difference of the two voltages U'_1 and U'_2, i.e.

$$U_3 = \hat{U}\Re\left(e^{i\omega t + i\psi_2} - e^{i\omega t + i\psi_1}\right)$$

$$= 2\hat{U} \sin \frac{\psi_1 - \psi_2}{2} \cos\left(\omega t + \frac{\psi_1 + \psi_2}{2} - 90°\right).$$

Due to the symmetric situation, $|\psi_i - \psi_j| \sim 120°$, $\hat{U}'_i = \hat{U}'_j$, the amplitudes of U_1, U_2, and U_3 are given by

$$\hat{U}_1 = \hat{U}_2 = \hat{U}_3 = 2\hat{U} \sin 60° = \sqrt{3}\,\hat{U}\,. \tag{1}$$

Using the relation between complex numbers and plane geometry, where addition and multiplication are replaced by vector addition and dilatation rotation, we may translate the above circuit into the plane and get the situation from Figure 2. We emphasize, that in Figure 2 we only draw the amplitudes of the voltages. To see the vector character let U'_1, U'_2, and U'_3 point inwards. Then $U_3 = U'_1 - U'_2$ points south-east with a phase of $\bar{\psi} = 330°$ as angle between the horizontal and $\hat{U}_3$ measured in the upper point.[1] Such diagrams related to AC calculations are called

Figure 2: The phasor diagram of the symmetric star-shaped generator
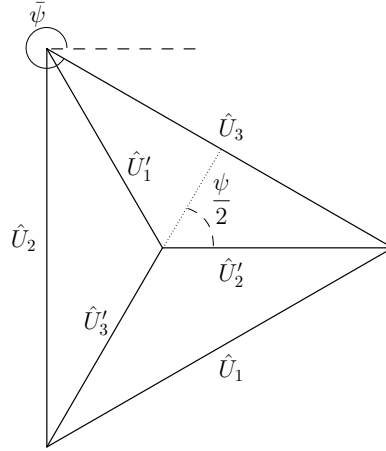
phasor diagrams and a basic introduction can be found in [2, 5, 6] for example. The main tool for the translation from circuit to phasor diagram is *Kirchhoff's mesh rule* or *Kirchhoff's voltage law* that states that the sum of the voltages in a closed loop of a circuit vanishes.

The non-symmetric variant of this situation is as follows: The phase differences of the primed line voltages remain 120° but their amplitudes differ.

Problem 1. We start with the star-shaped generator as given in Figure 1 with non-symmetric line voltages (primed). The configuration of our system only allows to measure the phase-to-phase voltages (non-primed). We need a way to compute the primed quantities from the non-primed ones.

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[1] Whenever we use the term "voltage" from now on, we mean the amplitude of the corresponding physical voltage. Therefore, we will omit the ˆ in the notation.

The geometric reformulation of Problem 1 is as follows:

Problem 2. Given three rays starting from one point M that pairwise form an angle of $120°$. Furthermore, given three points A, B, C each lying on one ray. These points form a triangle $\triangle(ABC)$, see Figure 3. Starting from this configuration and given the lengths of the three edges $a = |\overline{BC}|$, $b = |\overline{AC}|$, $c = |\overline{AB}|$ of the triangle, we like to know the lengths a', b', c' of the segments $\overline{MA}$, $\overline{MB}$, $\overline{MC}$.

Figure 3: The geometric setup for Problem 2 of the non-symmetric generator according to Problem 1
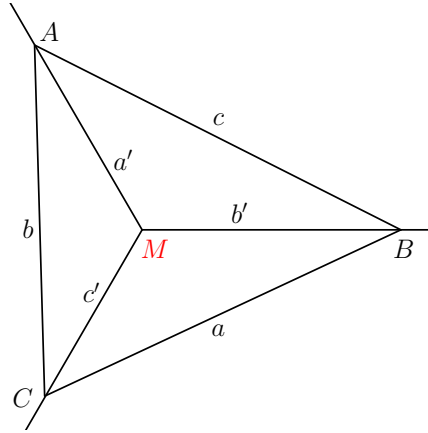
The point M that we introduced above is called Fermat-point and gives the solution of a classical geometric problem. The result can be formulated as follows.

Proposition 3 (The Fermat-point of a triangle). Given a triangle $\triangle(ABC)$ with all angles less than $120°$. Then there is a unique point M in the interior for which the lines from M to the corners form equal angles of $120°$. It can be constructed as drawn in Figure 4 and described as follows:

1. Over each edge of the triangle $\triangle(ABC)$ draw an equilateral triangle: $\triangle(ARB)$, $\triangle(BPC)$ and $\triangle(ACQ)$.

2. Draw straight lines $\overline{BQ}, \overline{AP}$ and $\overline{CR}$.

3. These lines intersect in one point, namely M.

There exists an elementary geometric proof of Proposition 3. The only things that are used are basic geometric ideas such as congruences of triangles and equality of certain angles. This is the proof presented by Evangelista Torricelli, see [15]. It is also published briefly in the English Wikipedia [16] and mentioned in [7].

The Fermat point in addition has the following very interesting minimizing property.

Proposition 4. For the Fermat point M the sum of the distances to the vertices of the triangle $\triangle(ABC)$ attains its minimum.

This property is not so obvious although there is a very short geometric proof, see [16]. For a historical survey of the geometric treatment of this problem see [11] and the wonderful books [3, 4]. In [3, 4] and in [8] the authors also explain the mechanical content of the minimizing property that describes the Fermat point as a point of equilibrium, see also Example 7 for the special situation of an equilateral triangle.

Figure 4: Construction of the Fermat-point M
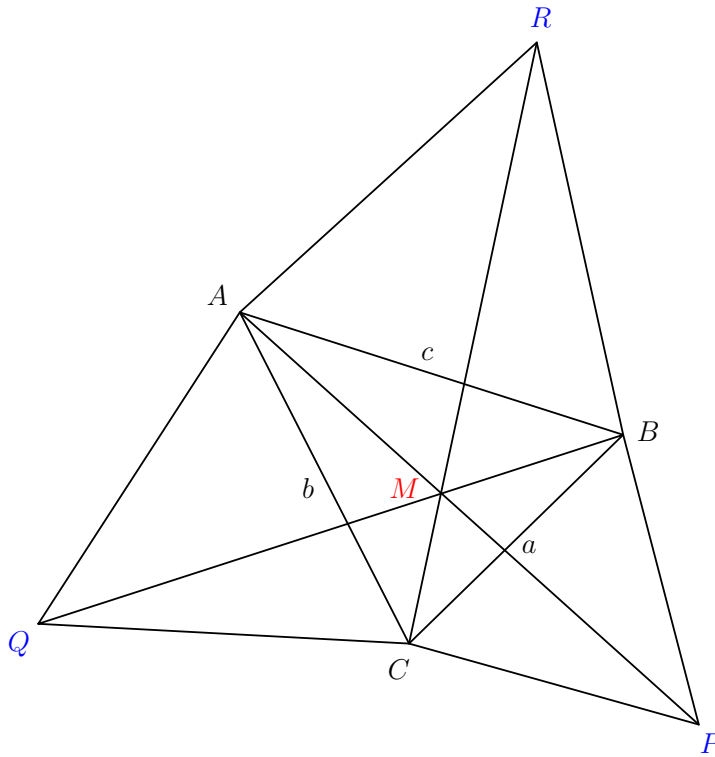
2   The solution of Problem 2: the line voltages of the non-symmetric generator

In this section we give a solution of Problem 1. This is done by presenting formulas for the quantities a', b', c' from Problem 2 that are symmetric as functions of a, b, c.

As a side result we will get a proof of Proposition 3 that explicitly gives the Fermat-point M in terms of the vectors that span the triangle from Figure 3. In Figure 5 we describe this situation by considering C to be the origin of the plane.

Remark 5. When we take a look at the technical literature we see that explicit calculations are usually performed by using complex numbers. Therefore, the question arises if this would be possible and reasonable here, too. Of course, it would be possible. But due to the fact that we look for the intersection of two real lines we would need to consider real and imaginary parts at some point. Geometrically this means that we would consider all quantities with respect to the standard basis of the euclidean plane. In our opinion and concerning to our initial question the use of the vectors that span the triangle is more natural and more reasonable. In fact, from some point the calculations are almost the same but – maybe – a little lengthier when we would use complex numbers.

Figure 5: The construction of the Fermat-point: the vector formulation
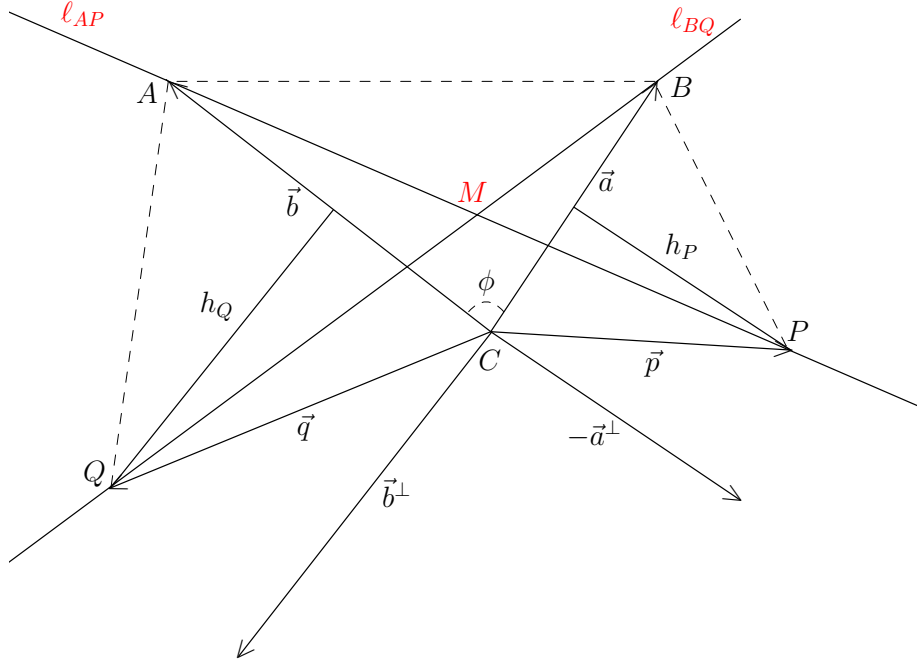
Before starting the calculations we recall some useful facts about vectors of the euclidean plane.[1] We consider two non vanishing plane vectors $\vec{a}$ and $\vec{b}$ drawing an angle $\phi = \angle(\vec{a}, \vec{b})$.[2] It is given by $\langle \vec{a}, \vec{b} \rangle = ab \cos \phi$ with $\langle \cdot, \cdot \rangle$ being the euclidean product and $a := \|\vec{a}\| = \sqrt{\langle \vec{a}, \vec{a} \rangle}$.

For any vector $\vec{a}$ there exists an unique perpendicular vector $\vec{a}^\perp$ that has length as $\|\vec{a}^\perp\| = a$ and both vectors form a positive basis of the plane w.r.t. the order

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[1]For more details on basics in linear algebra see [12, 13], for example.

[2]By $\angle(\vec{a}, \vec{b})$ we will always mean the oriented angle $0° \leq \phi < 360°$ that goes from $\vec{a}$ in counterclockwise direction to $\vec{b}$. The angle between $\vec{b}$ and $\vec{a}$ is then $\angle(\vec{b}, \vec{a}) = 360° - \phi$ if $\phi \neq 0°$ and $\angle(\vec{b}, \vec{a}) = 0°$ if $\phi = 0°$. If we do not care about the orientation we write e.g. $\angle(\vec{a}, \vec{b}) \sim 45°$, i.e. $\angle(\vec{a}, \vec{b}) \sim 45° \sim 315°$.

$\{\vec{a}, \vec{a}^{\perp}\}$. We have $\langle \vec{a}^{\perp}, \vec{b} \rangle = -\langle \vec{a}, \vec{b}^{\perp} \rangle = ab \sin \phi$.

In our situation $\vec{a}$ and $\vec{b}$ are linearly independent such that we can expand $\vec{a}^{\perp}$ and $\vec{b}^{\perp}$ as linear combinations. With $\vec{a}^{\perp} = \alpha \vec{a} + \beta \vec{b}$ we get $\langle \vec{a}^{\perp}, \vec{b}^{\perp} \rangle = \alpha \langle \vec{a}, \vec{b}^{\perp} \rangle$ and $\langle \vec{a}^{\perp}, \vec{a}^{\perp} \rangle = \beta \langle \vec{b}, \vec{a}^{\perp} \rangle$. This yields $\alpha = -\cot \phi$ and $\beta = \frac{a}{b \sin \phi}$. Doing the same for $\vec{b}^{\perp}$ we get

$$\vec{a}^{\perp} = \frac{a}{\sin \phi}\left(-\cos \phi \frac{\vec{a}}{a} + \frac{\vec{b}}{b}\right), \quad \vec{b}^{\perp} = \frac{b}{\sin \phi}\left(\cos \phi \frac{\vec{b}}{b} - \frac{\vec{a}}{a}\right). \tag{2}$$

Let us turn to our situation from Figure 5 and include the additional perpendicular vectors $\vec{a}^{\perp}$ and $\vec{b}^{\perp}$ into our discussion. We note that the lengths h_P and h_Q of the two heights of the equilateral triangles are given by $h_P = \frac{\sqrt{3}}{2} a$ and $h_Q = \frac{\sqrt{3}}{2} b$. With $\sin 60° = \frac{\sqrt{3}}{2}$, $\cos 60° = \frac{1}{2}$ and $\sqrt{3} \cos \phi + \sin \phi = 2 \sin(\phi + 60°)$ the position vectors $\vec{p}$ and $\vec{q}$ of P and Q are given by

$$\begin{aligned}\vec{p} &= \frac{1}{2}\vec{a} - \frac{\sqrt{3}}{2}\vec{a}^{\perp} = \frac{\sin(\phi + 60°)}{\sin \phi}\vec{a} - \frac{\sqrt{3}a}{2b \sin \phi}\vec{b}, \\ \vec{q} &= \frac{1}{2}\vec{b} + \frac{\sqrt{3}}{2}\vec{b}^{\perp} = \frac{\sin(\phi + 60°)}{\sin \phi}\vec{b} - \frac{\sqrt{3}b}{2a \sin \phi}\vec{a}.\end{aligned} \tag{3}$$

Therefore, the lines ℓ_{AP} and ℓ_{BQ} that contain the segments $\overline{AP}$ and $\overline{BQ}$ are parametrized by

$$\begin{aligned}\ell_{AP}(\tau) &= \vec{b} + \tau(\vec{p} - \vec{b}) = \left(1 - \tau - \frac{\sqrt{3}a}{2b \sin \phi}\tau\right)\vec{b} + \frac{\sin(\phi + 60°)}{\sin \phi}\tau\,\vec{a}, \\ \ell_{BQ}(\sigma) &= \vec{a} + \sigma(\vec{q} - \vec{a}) = \left(1 - \sigma - \frac{\sqrt{3}b}{2a \sin \phi}\sigma\right)\vec{a} + \frac{\sin(\phi + 60°)}{\sin \phi}\sigma\,\vec{b}.\end{aligned} \tag{4}$$

The intersection point of ℓ_{AP} and ℓ_{BQ} is determined by the solution (τ_0, σ_0) of the equation $\ell_{AP}(\tau) = \ell_{BQ}(\sigma)$ that is equivalent to

$$\left(\frac{\sin(\phi + 60°)}{\sin \phi}\tau - \left(1 - \sigma - \frac{\sqrt{3}b}{2a \sin \phi}\sigma\right)\right)\vec{a}$$

$$= \left(\frac{\sin(\phi + 60°)}{\sin \phi}\sigma - \left(1 - \tau - \frac{\sqrt{3}a}{2b \sin \phi}\tau\right)\right)\vec{b}$$

$$\Longleftrightarrow \underbrace{\begin{pmatrix} \sin(\phi + 60°) & \sin \phi + \frac{\sqrt{3}b}{2a} \\ \sin \phi + \frac{\sqrt{3}a}{2b} & \sin(\phi + 60°) \end{pmatrix}}_{= \Omega} \begin{pmatrix} \tau \\ \sigma \end{pmatrix} = \sin \phi \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The determinant of the coefficient matrix is

$$\det \Omega = \; = -\frac{\sqrt{3} \sin \phi}{2ab}\left(a^2 + b^2 - 2ab \cos(\phi + 60°)\right)$$

such that the solution (τ_0, σ_0) is given by

$$
\begin{pmatrix} \tau_0 \\ \sigma_0 \end{pmatrix} = \frac{-\sin\phi}{\det\Omega} \begin{pmatrix} -\sin(\phi+60°) & \sin\phi + \frac{\sqrt{3}b}{2a} \\ \sin\phi + \frac{\sqrt{3}a}{2b} & -\sin(\phi+60°) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
$$
$$
= \begin{pmatrix} \dfrac{\sqrt{3}\,b^2 + 2ab\sin(\phi-60°)}{\sqrt{3}(a^2+b^2) - 2\sqrt{3}\,ab\cos(\phi+60°)} \\[2ex] \dfrac{\sqrt{3}\,a^2 + 2ab\sin(\phi-60°)}{\sqrt{3}(a^2+b^2) - 2\sqrt{3}\,ab\cos(\phi+60°)} \end{pmatrix} \tag{5}
$$

We use this result to calculate the position vector $\vec{m}$ of the Fermat-point M in the situation of Figure 5.

$$
\vec{m} = \ell_{AP}(\tau_0) = \ell_{BQ}(\sigma_0) = (1-\tau_0)\vec{b} + \tau_0\vec{p} = (1-\sigma_0)\vec{a} + \sigma_0\vec{q}. \tag{6}
$$

We use $\langle \vec{b}, \vec{p} \rangle = \frac{1}{2}\langle \vec{b}, \vec{a} \rangle - \frac{\sqrt{3}}{2}\langle \vec{b}, \vec{a}^\perp \rangle = ab\cos(\phi+60°)$ such that in terms of τ_0

$$
\begin{aligned}
(c')^2 &= \|\vec{m}\|^2 = (1-\tau_0)^2 b^2 + \tau_0^2 \|\vec{p}\|^2 + 2\tau_0(1-\tau_0)\langle \vec{b}, \vec{p} \rangle \\
&= (1-\tau_0)^2 b^2 + \tau_0^2 a^2 + 2\tau_0(1-\tau_0)ab\cos(\phi+60°).
\end{aligned} \tag{7}
$$

Similarly we get

$$
(a')^2 = |AM|^2 = \tau_0^2\big(a^2 + b^2 - 2ab\cos(\phi+60°)\big), \tag{8}
$$
$$
(b')^2 = |BM|^2 = \sigma_0^2\big(a^2 + b^2 - 2ab\cos(\phi+60°)\big). \tag{9}
$$

To get formulas for $a', b',$ and c' that are symmetric with respect to $a, b,$ and c we recall the cosine-theorem that says

$$
2ab\cos\phi = a^2 + b^2 - c^2. \tag{10}
$$

In the same way we express $2ab\sin\phi$ as

$$
4a^2b^2\sin^2\phi = 4a^2b^2(1 - \cos^2\phi) = 4a^2b^2 - (a^2 + b^2 - c^2)^2 = \Theta^4 \tag{11}
$$

where we use the abbreviation

$$
\Theta^2 = \sqrt{(c+a+b)(a+c-b)(b+c-a)(a+b-c)} \tag{12}
$$

that is invariant under relabeling the three edges.[1]

The denominator of $a', b',$ and c' agrees – up to a factor of 3 – with the denominator of τ_0 and σ_0. It can be written as

$$
a^2 + b^2 - 2ab\cos(\phi+60°) = \frac{1}{2}(a^2+b^2+c^2) + \frac{\sqrt{3}}{2}\Theta^2.
$$

---

[1]Formulas (11) with (12) recall the famous Heron formula that gives the area of an triangle in terms of its three edges. This area is given by $4 \cdot \mathrm{area}\big(\triangle(ABC)\big) = \Theta^2$.

The numerator of a' is given by the numerator of τ_0^2. It differs from that of b' or σ_0^2 only by interchanging a and b and is given by

$$\left(\sqrt{3}b^2 + 2ab\sin(\phi - 60°)\right)^2 = \frac{1}{4}\left(\sqrt{3}(b^2 + c^2 - a^2) + \Theta^2\right)^2 .$$

We insert these expressions into (8) and (9) and get expressions for a' and b'. A similar calculation yields the remaining length c' from (7).

Proposition 6. Given a triangle $\triangle(ABC)$ and the Fermat-point M as given in Figure 3. Then a', b', and c' are given in terms of a, b, and c by

$$(a')^2 = \frac{1}{6} \cdot \frac{\left(\sqrt{3}(b^2 + c^2 - a^2) + \Theta^2\right)^2}{a^2 + b^2 + c^2 + \sqrt{3}\,\Theta^2} , \tag{13}$$

$$(b')^2 = \frac{1}{6} \cdot \frac{\left(\sqrt{3}(a^2 + c^2 - b^2) + \Theta^2\right)^2}{a^2 + b^2 + c^2 + \sqrt{3}\,\Theta^2} , \tag{14}$$

$$(c')^2 = \frac{1}{6} \cdot \frac{\left(\sqrt{3}(a^2 + b^2 - c^2) + \Theta^2\right)^2}{a^2 + b^2 + c^2 + \sqrt{3}\,\Theta^2} . \tag{15}$$

This also yields the solution of initial Problem 1 but we postpone the formulation to the summary, see Section 4.

Example 7. As a first example and also a first check of our result we consider $a = b = c$, i.e. an equilateral triangle. In this case we have $\Theta^2 = \sqrt{3}a^2$ and $a' = b' = c' = \frac{1}{\sqrt{3}}a$ which is exactly the result from (1). In particular $\vec{m} = \frac{1}{3}(\vec{a} + \vec{b})$ is the position vector of the geometric center of the triangle.

Remark 8. To finish a proof of Proposition 3 we have to do some more calculations:

Firstly, we have to show that M lies on the line that connects the origin and R, i.e. there exists λ_0 such that $\vec{m} = \ell_{CR}(\lambda_0) = \lambda_0\vec{r} = \frac{\lambda_0}{2}\left(\vec{a} + \vec{b} + \sqrt{3}(\vec{a}^\perp - \vec{b}^\perp)\right)$.

Secondly, we have to show that the angles $\angle(\vec{p} - \vec{a}, \vec{q} - \vec{b})$, $\angle(\vec{m}, \vec{p} - \vec{a})$, and $\angle(\vec{m}, \vec{q} - \vec{b})$ coincide and, therefore, are given by $120°$.

Remark 9. Starting from our results (13)-(15) we can prove the minimizing property from Proposition 4. For this we look for critical points of the function $f(a', b', c', \psi, \phi) = a' + b' + c'$ w.r.t. the three constraints $g_x(x, y, z, \phi, \psi) = y^2 + z^2 - 2yz\cos\phi - a^2 = 0$, $g_y(x, y, z, \phi, \psi) = x^2 + z^2 - 2xz\cos\psi - b^2 = 0$, and $g_z(x, y, z, \phi, \psi) = x^2 + y^2 - 2xy\cos(\phi + \psi) - c^2 = 0$ by using the Lagrange method.

We have to show that our solution yields a critical point of the Lagrange function $L = f + \sum_{\omega \in \{x,y,z\}} \lambda_\omega g_\omega$ for Lagrange parameters $2(xy + xz + yz)\lambda_\omega = \omega$.

Moreover, we have to show that this critical point is indeed a minimum, for example by using the rendered Hessian, see [9].

3   A generalization: the unbalanced star circuit

We consider a 3-phase AC star circuit with unbalanced star point, i.e. the line between the star point N of the generator and the star point M of the circuit is missing, see Figure 6.

Figure 6: The unbalanced star circuit
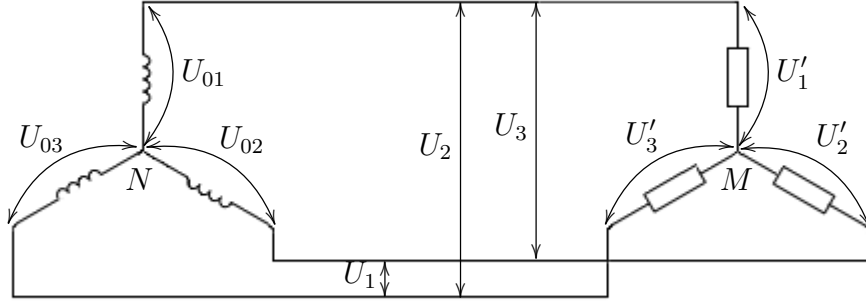
Let us first assume that the perfect generator provides three equal line-voltages $U_{01} = U_{02} = U_{03} = \frac{1}{\sqrt{3}}U$ with a phase difference of 120°. Then the phase-to-phase voltages fulfill $U_1 = U_2 = U_3 = U$ and they have a phase difference of 120°, too, resulting in an equilateral phasor diagram.

The unbalanced configuration typically yields $U'_1 \neq U'_2 \neq U'_3$ for the primed line voltages of the load. This is reflected in the phasor diagram in such a way that the star point is displaced in the equilateral triangle defined by U_1, U_2, U_3, see Figure 7. We emphasize the fact that the phase-to-phase voltages of the generator and the load are the same due to the mesh rule. Such unbalanced star circuits have been considered in [14] for special almost symmetric configurations, e.g. $U'_1 = U'_2$.

We will now consider non-equal phase-to-phase-voltages U_α that are provided by a non-symmetric generator according to Section 1. We now ask the following question: Knowing the phase-to-phase voltages of the load/generator we want to recover the primed line voltages of the load.

Of course, the phase-to-phase voltages alone do not contain enough information to obtain a solution. Typically, you know about the technical configuration of the load, for example about resistors, capacities or inductances, see [14]. To create a purely geometric problem, we assume the load to be a black box of which we do not know the exact components but we know about the phase differences of the primed voltages.

Problem 10. Given the phase-to-phase-voltages U_1, U_2, U_3 of the load provided by a non-symmetric generator as well as the phase-differences ψ_1 and ψ_2 of the primed line voltages of the load: What are the values of the line voltages U'_1, U'_2, and U'_3?

Figure 7: The phasor diagram of the load of an unbalanced star circuit for a symmetric generator
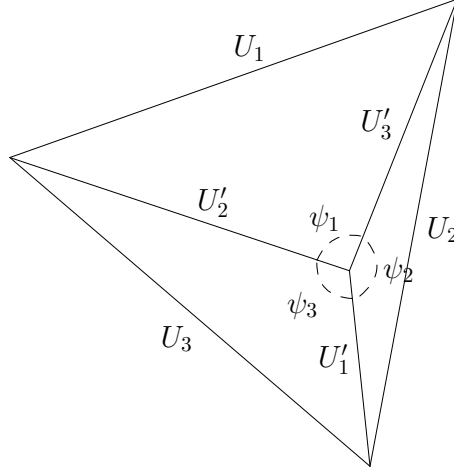
The geometric reformulation is as follows.

Problem 11. Given a plane triangle $\triangle(ABC)$ with lengths a, b, c of its edges. Furthermore, given an unknown point X in the interior of $\triangle(ABC)$ of which we know the angles $\psi_a = \angle(BXC)$ and $\psi_b = \angle(CXA)$: What are the lengths of the connecting edges $a' = |AX|, b' = |BX|$, and $c' = |CX|$?

In fact, for phase differences $\psi_a = \psi_b = \psi_c = 120°$ this yields another formulation of Problem 1 in terms of load instead of generator.

For the discussion of Problem 11 we consider Figure 7 with non equal phase-to-phase voltages and translate it to the vector picture from Figure 8. We will make use of the preliminaries and the notation from Section 2 and add a few more quantities that we describe next.

Due to the inscribed angle theorem, all points X that draw an angle ψ_b with the endpoints of the segment $\overline{AC}$ lie on a circle with center S, the circumcircle of $\triangle(XAC)$, see [1] for example. Suppose $\psi_b \geq 90°$ then S and X lie on different sides of $\overline{AC}$ and the central angle is given by $2(180° - \psi_b) = 360° - 2\psi_b$. If the angle ψ_a obeys the restriction $\psi_a \geq 90°$, too, the point X is the intersection of the two circles with centers R and S that contain the two chords $\overline{AC}$ and $\overline{BC}$, respectively.[1]

The radii of the circumcircles of $\triangle(XCB)$ and $\triangle(XAC)$ are given by $\rho_a = \frac{a}{2\cos(\psi_a - 90°)} = \frac{a}{2}\sqrt{1 + \cot^2 \psi_a}$ and $\rho_b = \frac{b}{2}\sqrt{1 + \cot^2 \psi_b}$, respectively. The heights

---

[1]The restriction on the two angles ψ_a, ψ_b before is actually no restriction, because due to $\psi_a, \psi_b, \psi_c < 180°$ at least two of the three angles ψ_a, ψ_b, and $\psi_c = 360° - \psi_a - \psi_b$ are of this form. Therefore, Figure 8 describes the general situation, at least after renaming the points and edges of the triangle.

Figure 8: The geometric description of the load of an unbalanced star circuit with non-symmetric generator

of the corresponding triangles $\triangle(BCR)$ and $\triangle(ASC)$ are $h_R = \frac{a}{2}\tan(\psi_a - 90°) = -\frac{a}{2}\cot\psi_a$ and $h_S = -\frac{b}{2}\cot\psi_b$, respectively. Therefore, the position vectors of the centers of the circumcircles are $\vec{r} = \frac{1}{2}\vec{a} - \frac{h_R}{a}\vec{a}^\perp = \frac{1}{2}(\vec{a} + \cot\psi_a\,\vec{a}^\perp)$ and $\vec{s} = \frac{1}{2}(\vec{b} - \cot\psi_b\,\vec{b}^\perp)$.

We will calculate the position vector $\vec{x}$ of X whose length is given by c'. For this, we write $\vec{x}$ as a linear combination of the two vectors that span the triangle:

$$\vec{x} = \frac{\alpha}{2}\,\vec{a} + \frac{\beta}{2}\,\vec{b}.$$

As said before, X is given as an intersection point of the two circles

$$\left\{\vec{y}\ \middle|\ \|\vec{y} - \vec{r}\|^2 = \rho_a^2\right\} \quad \text{and} \quad \left\{\vec{y}\ \middle|\ \|\vec{y} - \vec{s}\|^2 = \rho_b^2\right\}$$

such that the coefficients of $\vec{x}$ obey

$$\begin{cases} \left\|(\alpha - 1)\vec{a} - \cot\psi_a\,\vec{a}^\perp + \beta\,\vec{b}\right\|^2 = a^2(1 + \cot^2\psi_a) \\ \left\|(\beta - 1)\vec{b} + \cot\psi_b\,\vec{b}^\perp + \alpha\,\vec{a}\right\|^2 = b^2(1 + \cot^2\psi_b) \end{cases}$$

$$\Leftrightarrow \begin{cases} 0 = \alpha^2 a^2 + \beta^2 b^2 + 2\alpha\beta\langle\vec{a},\vec{b}\rangle - 2\alpha a^2 - 2\beta\langle\vec{a},\vec{b}\rangle - 2\beta\cot\psi_a\langle\vec{a}^\perp,\vec{b}\rangle \\ 0 = \alpha^2 a^2 + \beta^2 b^2 + 2\alpha\beta\langle\vec{a},\vec{b}\rangle - 2\beta b^2 - 2\alpha\langle\vec{a},\vec{b}\rangle - 2\alpha\cot\psi_b\langle\vec{a}^\perp,\vec{b}\rangle \end{cases}$$

We subtract the two equations and get

$$\alpha\left(a^2 - \langle\vec{a},\vec{b}\rangle - \cot\psi_b\langle\vec{a}^\perp,\vec{b}\rangle\right) - \beta\left(b^2 - \langle\vec{a},\vec{b}\rangle - \cot\psi_a\langle\vec{a}^\perp,\vec{b}\rangle\right) = 0\,.$$

We write this as $\beta = t\alpha$ with $t = \frac{a^2 - \langle\vec{a},\vec{b}\rangle - \cot\psi_b\langle\vec{a}^\perp,\vec{b}\rangle}{b^2 - \langle\vec{a},\vec{b}\rangle - \cot\psi_a\langle\vec{a}^\perp,\vec{b}\rangle}$. We introduce the length of the third edge of the triangle, $c = \|\vec{a} - \vec{b}\|$, and use $2\langle\vec{a},\vec{b}\rangle = a^2 + b^2 - c^2$ as well as $2\langle\vec{a}^\perp,\vec{b}\rangle = \Theta^2$, see (10)-(12), and write

$$t = \frac{c^2 + a^2 - b^2 - \cot\psi_b\Theta^2}{c^2 + b^2 - a^2 - \cot\psi_a\Theta^2}\,, \quad t_* = \frac{c^2 + b^2 - a^2 - \cot\psi_a\Theta^2}{c^2 + a^2 - b^2 - \cot\psi_b\Theta^2}\,. \tag{16}$$

We insert this into the quadratic equations and get for $\alpha, \beta \neq 0$

$$\begin{aligned}
\alpha &= \frac{2a^2 + t(a^2 + b^2 - c^2) + t\cot(\psi_a)\Theta^2}{a^2 + t^2 b^2 + t(a^2 + b^2 - c^2)}\,, \\
\beta &= \frac{2b^2 + t_*(a^2 + b^2 - c^2) + t_*\cot(\psi_b)\Theta^2}{b^2 + t_*^2 a^2 + t_*(a^2 + b^2 - c^2)}\,.
\end{aligned} \tag{17}$$

The length of $\vec{x}$,

$$(c')^2 = \frac{1}{4}\left(\alpha^2 a^2 + \beta^2 b^2 + 2\alpha\beta\langle\vec{a},\vec{b}\rangle\right) = \frac{1}{4}\left(\alpha^2 a^2 + \beta^2 b^2 + \alpha\beta(a^2 + b^2 - c^2)\right)\,,$$

is now obtained by a lengthy calculation. In particular, we use

$$1 - \cot\psi_a\cot\psi_b = -\cot(\psi_a + \psi_b)(\cot\psi_a + \cot\psi_b) = \cot\psi_c(\cot\psi_a + \cot\psi_b)\,.$$

The result is formulated in the next Proposition.

Proposition 12. We consider the situation from Problem 11. Then the length c' of the connecting edge is given by

$$\begin{aligned}
(c')^2 &= \frac{\frac{1}{4}\left((1 - \cot\psi_a\cot\psi_b)\Theta^2 - (\cot\psi_a + \cot\psi_b)(a^2 + b^2 - c^2)\right)^2}{\left(c^2 + a^2\cot^2\psi_a + b^2\cot^2\psi_b\right) + \cot\psi_a\cot\psi_b(a^2 + b^2 - c^2) - (\cot\psi_a + \cot\psi_b)\Theta^2} \\
&= \frac{\frac{1}{4}\left(\cot\psi_a + \cot\psi_b\right)^2\left((a^2 + b^2 - c^2) - \Theta^2\cot\psi_c\right)^2}{a^2(1 + \cot^2\psi_a) + b^2(1 + \cot^2\psi_b) - (\cot\psi_a + \cot\psi_b)\left((a^2 + b^2 - c^2)\cot\psi_c + \Theta^2\right)}\,.
\end{aligned} \tag{18}$$

By interchanging the roles of $a, b,$ and c we get the results for a' and b'. To end up this section we will check our result by discussing some special examples:

- The equilateral triangle with $a = b = c$ yields $\Theta^2 = \sqrt{3}a^2$ and

$$\begin{aligned}
c' &= \frac{a}{2}\cdot\frac{(1 - \cot\psi_a\cot\psi_b)\sqrt{3} - (\cot\psi_a + \cot\psi_b)}{\sqrt{1 + \cot^2\psi_a + \cot^2\psi_b + \cot\psi_a\cot\psi_b - \sqrt{3}(\cot\psi_a + \cot\psi_b)}} \\
&= \frac{a}{2}\cdot\frac{(\cot\psi_a + \cot\psi_b)(\sqrt{3}\cot\psi_c - 1)}{\sqrt{2 + \cot^2\psi_a + \cot^2\psi_b - (\cot\psi_a + \cot\psi_b)\left(\cot\psi_c + \sqrt{3}\right)}}\,.
\end{aligned}$$

- The situation of equal angles, $\psi_a = \psi_b = \psi_c = 120°$, with $\cot\psi_a = -\frac{\sqrt{3}}{3}$ yields

$$(c')^2 = \frac{\frac{1}{6}\left(\Theta^2 + \sqrt{3}(a^2 + b^2 - c^2)\right)^2}{c^2 + a^2 + b^2 + \sqrt{3}\Theta^2}$$

which is exactly the result we obtained in (15).

- The isosceles triangle with $a = b$, $\psi_a = \psi_b$ yields $\Theta^2 = c\sqrt{4a^2 - c^2}$. This is the case mainly discussed in [14]. In this case the formulas for a' and b' analogue to (18) coincide such that $a' = b'$. Moreover, we have

$$c' = \frac{c}{2} \cdot \frac{(1 - \cot^2\psi_a)\Theta^2 - 2\cot\psi_a(2a^2 - c^2)}{c^2 - \Theta^2\cot\psi_a}\ .$$

The limiting situation $c' = 0$ is obtained if and only if $\cot^2\psi_a + \frac{2(2a^2-c^2)}{\Theta^2}\cot\psi_a - 1 = 0$. Because in this situation ψ_a obeys $180° > \psi_a \geq 90°$ we have $\cot\psi_a \leq 0$. Therefore, the remaining negative solution the quadratic equation is

$$\cot\psi_a = -\frac{\Theta^2}{c^2} = -\frac{\sqrt{4a^2 - c}}{c} = -\frac{h_c}{c/2}$$

where h_c is length of $\triangle(ABC)$ over its edge c. If we denote half the angle of $\triangle(ABC)$ at C by ϕ, then $\cot\phi = \frac{h_c}{c/2}$ such that $\psi_a = 180° - \phi$. As expected, we see that in the limiting case ψ_a coincides with the angle between the lines extending a and h_c. Moreover, again as expected, $\psi_c = 2\phi$ and $a' = b' = a$.

For the explicit translation of the result to a solution of Problem 10, see again the summarizing Section 4.

4   Summary: The solutions of Problems 1 and 10

The solution of the initial Problem 1.

We consider a generator with non-symmetric phase-to-phase voltages U_1, U_2, U_3 as described in Figure 1. Then the line voltages U_1', U_2', U_3' are given by

$$U_1' = \frac{1}{\sqrt{6}} \cdot \frac{\sqrt{3}\left(U_2^2 + U_3^2 - U_1^2\right) + \Theta^2}{\sqrt{U_1^2 + U_2^2 + U_3^2 + \sqrt{3}\,\Theta^2}}$$

$$U_2' = \frac{1}{\sqrt{6}} \cdot \frac{\sqrt{3}\left(U_3^2 + U_1^2 - U_2^2\right) + \Theta^2}{\sqrt{U_1^2 + U_2^2 + U_3^2 + \sqrt{3}\,\Theta^2}}$$

$$U_3' = \frac{1}{\sqrt{6}} \cdot \frac{\sqrt{3}\left(U_1^2 + U_2^2 - U_3^2\right) + \Theta^2}{\sqrt{U_1^2 + U_2^2 + U_3^2 + \sqrt{3}\,\Theta^2}}$$

The solution of the general Problem 10

Given an unbalanced star circuit according to Figure 6. We know the non-symmetric phase-to-phase voltages U_1, U_2, U_3 of the load – or the generator. Furthermore, we know the phase differences ψ_1, ψ_2, and $\psi_3 = 360° - \psi_1 - \psi_2$ of the load. Then the line voltages U_1', U_2', U_3' of the load are given by

$$U_1' = \frac{1}{2} \cdot \frac{\left|\left(\cot\psi_2 + \cot\psi_3\right)\left(U_2^2 + U_3^2 - U_1^2 - \Theta^2 \cot\psi_1\right)\right|}{\sqrt{U_2^2(1 + \cot^2\psi_2) + U_3^2(1 + \cot^2\psi_3) - (\cot\psi_2 + \cot\psi_3)\left((U_2^2 + U_3^2 - U_1^2)\cot\psi_1 + \Theta^2\right)}}$$

$$U_2' = \frac{1}{2} \cdot \frac{\left|\left(\cot\psi_3 + \cot\psi_1\right)\left(U_3^2 + U_1^2 - U_2^2 - \Theta^2 \cot\psi_2\right)\right|}{\sqrt{U_3^2(1 + \cot^2\psi_3) + U_1^2(1 + \cot^2\psi_1) - (\cot\psi_3 + \cot\psi_1)\left((U_3^2 + U_1^2 - U_2^2)\cot\psi_2 + \Theta^2\right)}}$$

$$U_3' = \frac{1}{2} \cdot \frac{\left|\left(\cot\psi_1 + \cot\psi_2\right)\left(U_1^2 + U_2^2 - U_3^2 - \Theta^2 \cot\psi_3\right)\right|}{\sqrt{U_1^2(1 + \cot^2\psi_1) + U_2^2(1 + \cot^2\psi_2) - (\cot\psi_1 + \cot\psi_2)\left((U_1^2 + U_2^2 - U_3^2)\cot\psi_3 + \Theta^2\right)}}$$

The special case $\psi_1 = \psi_2 = \psi_3 = 120°$ coincides with the solution of our initial Problem 1. This situation in particular occurs when we consider the star circuit from Figure 6 to be balanced.

In both sets of formulas we use the abbreviation

$$\Theta^2 = \sqrt{\left(U_1 + U_2 + U_3\right)\left(U_2 + U_3 - U_1\right)\left(U_3 + U_1 - U_2\right)\left(U_1 + U_2 - U_3\right)}$$

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