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A categorization of equations from expert students

Summary

Secondary school students have difficulties in solving equations, even when they have learned the standard procedures and how to manipulate algebraic expressions. They do not seem to recognize when to use these procedures, because they do not see the structure of equations: they lack symbol sense. Experts categorize problems according to fundamental problem characteristics. Landa suggested analyzing expert knowledge, looking for crucial thinking steps, and teaching these with explicit attention to algorithmization of identification. In our study, four expert students were asked to think aloud when categorizing a set of equations according to the methods for solving equations. Using these results, the author formulated six categories of equations. These categories can be used in teaching the solving of equations. We suggest teachers start early with formulating categories of equations and with recognizing the different categories. Categories formulated in lower secondary school (grades 8 and 9) will form a basis for the categories used in grades 11 and 12.

Introduction

Students often have difficulties with algebra, in particular giving meaning to and grasping the structure of algebraic formulas, and manipulating them (Kieran, 2006; Sfard and Linchevski, 1994).

In lower secondary school (grade 8 and 9) ample attention is paid to different methods of solving linear and quadratic equations. For quadratic equations textbooks formulate typical categories like $ax^2+bx=0$ (e.g. $3x^2-7x=0$), $ax^2+c=0$ (e.g. $3x^2-75=0$), factorization (e.g. $x^2-5x-14=0$), and abc formula (e.g. $3x^2-2x-2=0$). In grades 10 till 12, however, solving of equations is given no structural attention and hardly any categories are formulated. In the current study we looked for categories of equations which can be used throughout secondary school.

Theory

Research has shown that experts categorize problems according to the problem structure (Chi, Feltovich, and Glaser, 1981). In order to be able to categorize equations, students should be able to “see” the structure of the equation. In the literature this is called “symbol sense”. Symbol sense has several aspects, such as the ability to read through algebraic expressions, to see the expression as a whole rather than a concatenation of letters, and to recognize its global characteristics (Arcavi, 1994).

Symbol sense can be seen as complementary to basic algebraic skills. Basic skills are about procedural work (e.g., expanding brackets), local focus, and algebraic calculation; symbol sense is about strategic work, global focus, and algebraic reasoning (Drijvers & Kop, 2008).

Recognizing and using structure may make solving the equation easier and increase success (Hoch & Dreyfus, 2004). For many students the presence of an algebraic fraction in the equation $\frac{1}{4}-\frac{x}{x-1}-x=5+(\frac{1}{4}-\frac{x}{x-1})$ is a signal to multiply by a common denominator leading to a long and error-prone solution (Hoch & Dreyfus, 2004). Hoch and Dreyfus found in their interviews that structure is not something that is in the realm of awareness of high school students.

Wenger (1987) provided a classic example to show that recognizing the structure of an equation is necessary for solving equations: solve $v\cdot\sqrt{u}=1+2v\cdot\sqrt{1+u}$ for v.

Recognizing the structure is not an "all or nothing" process, as Mason (2003) showed in his different states, or structures, of attention: it develops from staring at the whole while hardly knowing how to proceed, to discerning details from which objects and sub-objects can be determined in order to recognize relationships, to perceiving properties, till grasping the essential structure. He suggests that teachers pay attention to classifying and use dimensions-of-possible-variation and range-of-permissible change to capture the variation arising in mathematics. Also Burkhardt and Swan (2013) emphasized the importance of classifying in their formative assessment tasks.

Landa (1983) suggested analyzing expert knowledge, defining experts' crucial thinking steps, and teaching these with explicit attention to algorithmization of identification. According to Landa, students should be confronted with the complexity of the task in a very early stage.

In order to teach the solving of equations effectively and efficiently, it seems reasonable to pay attention to their structure. This requires symbol sense. Expert students are supposed to use this symbol sense. Therefore, we aimed to find out which categories they used when solving equations and to use these categorizations to construct a categorization that could be used in teaching equations.

Method

We selected four grade 12 students who had done very well in math exams. We asked them to think aloud while solving a variety of equations. The questions were: "Can you sort these equations according to the way you would solve them? You can use as many categories as you wish", and "Can you give a prototypical equation for each category?" Each interview took about one hour and was videotaped.

The equations were chosen from text books and exams. Some of these equations were: $(x^2+2)^{-0,5}=\frac{1}{2}$; $\frac{2x+8}{-3x+10}=3$; $\frac{2}{x}+\frac{3}{x-1}=4$; $\frac{\sin(x)-1}{e^x}=0$; $\sqrt{e^x-2}=0$; $x^3-12x^2+20x=0$; $(\sin(x))^2-\frac{1}{2}\sin(x)=0$; $e^{3x-1}=e^{-2x+10}$; $6=(x^2+1)^3$

Results

The four expert-students all mentioned between 9 and 13 categories. Their categories showed many agreements. For instance, all mentioned "sinus knowledge", "log knowledge", and "multiplying with denominators".

An example of a categorization (in student language), with the prototype examples, is: "expanding brackets (prototype: $2(x^2+3)=6x+2$)", "abc-formula (prototype: $2x^2=6x+7$)", "manipulating, write differently (prototype: : "make denominators equal or multiplying with denominator (prototype: $\frac{2}{x}+\frac{3}{x-1}=4$)", factorizing (prototype: $2x^2-6x=0$)", "working backwards (prototype: $4(e^x+1)^2=40$) ", "sinus knowledge (prototype: $\sin(2x)=\sin(x)$)", "logarithmic knowledge (prototype: ${}^2\log(x)+2={}^2\log(x-1)$", "substituting (prototype: $4(\sin(x))^3+2=12(\sin(x))^3-0{,}5$)", "make zero (prototype: $3(x-1)(x^2+1)=0$"), "thinking away (prototype: $5(x^2+1)=2x(x^2+1)$) ".

Conclusion and implications

From the protocols of the interviews, it was concluded that the expert-students could formulate the categories of the standard equations they used. They worked efficiently, as they used their categorizations and knew algorithms to solve prototype equations. Complex equations were rewritten into more standard equations.

From these categorizations the author formulated six categories.

"thinking away" (e.g. $3^{2x+3}=3^{-2x-6}\rightarrow 2x+3=-2x-6$); "working backwards" (e.g. $e^{2x}=16\rightarrow 2x=\ln(16)$); "substituting"(e.g. $\frac{\sin(x)-(\sin(x))^2}{(\sin(x))^2}=3\rightarrow\frac{p-p^2}{p^2}=3$)

"product equals 0" (e.g. $3(x-4)(x^2-6)=0\rightarrow x=4, x=\sqrt{6},-\sqrt{6}$); "abc formula" (e.g. $2x^2=6x+7$); "eye catchers" (residual category with sinus, logarithm, broken equations) (e.g. $\frac{2x+4}{x^2+1}=2\rightarrow 2x+4=2(x^2+1)$, $\log_2(x)+2=\log_2(x+10)$)

When students in grade 8 or 9 have worked on quadratic equations, this categorization can be used to teach them to recognize the different types of equations. When possible, it is recommended that students make their own categorizations and use Mason's dimensions-of-possible-variation and range-of-permissible change. But it is also possible to follow Landa's idea and to give students a categorization and use an identification algorithm, which will lead to a correct category. Later, standard solving methods for the different categories should be taught. The solving process can be an iterative process, as for instance in $4(\ln^2(x)+\ln(x))^2 = 24$.

Our suggestion is that teachers should pay attention to categories and to recognizing examples of different categories. This would greatly reduce the room for problems in solving equations and give teachers the possibility to explain their own thinking when discussing their solving strategies. A categorization like ours gives teachers and students the possibility to build a repertoire of categories of equations which can be used throughout secondary school.

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