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H. Hajduk, D. Kuzmin, V. Aizinger

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Chapter 1
Bathymetry reconstruction using inverse shallow water models: Finite element discretization and regularization

Hennes Hajduk, Dmitri Kuzmin, Vadym Aizinger

Abstract

In the present paper, we use modified shallow water equations (SWE) to reconstruct the bottom topography (also called bathymetry) of a flow domain without resorting to traditional inverse modeling techniques such as adjoint methods. The discretization in space is performed using a piecewise linear discontinuous Galerkin (DG) approximation of the free surface elevation and (linear) continuous finite elements for the bathymetry. Our approach guarantees compatibility of the discrete forward and inverse problems: for a given DG solution of the forward SWE problem, the underlying continuous bathymetry can be recovered exactly. To ensure well-posedness of the modified SWE and reduce sensitivity of the results to noisy data, a regularization term is added to the equation for the water height. A numerical study is performed to demonstrate the ability of the proposed method to recover bathymetry in a robust and accurate manner.

Keywords: *Bathymetry reconstruction, shallow water equations, continuous/discontinuous Galerkin method, inverse problem.*

Hennes Hajduk
TU Dortmund University, Institute of Applied Mathematics (LS III), Vogelpothsweg 87, 44227 Dortmund, Germany, e-mail: `hennes.hajduk@tu-dortmund.de`

Dmitri Kuzmin
TU Dortmund University, Institute of Applied Mathematics (LS III), Vogelpothsweg 87, 44227 Dortmund, Germany, e-mail: `kuzmin@math.uni-dortmund.de`

Vadym Aizinger
Alfred Wegener Institute, Helmholtz Center for Polar and Marine Research, Am Handelshafen 12, 27570 Bremerhaven, Germany, e-mail: `vadym.aizinger@awi.de`
Chair for Applied Mathematics 1, University Erlangen-Nuremberg, Cauerstr. 11, 91058 Erlangen, Germany

1.1 Introduction

The shallow water equations are among the most popular mathematical models for applications in environmental fluid mechanics. The geometry of a computational domain for SWE simulations of coastal, riverine, and estuarine flow problems can be determined using inexpensive and highly accurate measurement techniques for boundaries corresponding to coastlines and the free surface. However, the resolution and accuracy of experimental data for the bottom topography (also called bathymetry) of many regions are very poor. As an alternative to direct measurements, the missing bathymetry data can be reconstructed by solving a (modified) SWE system as originally proposed in [7, 8]. The bathymetry enters the momentum equations as a source term which has a strong influence on the accuracy of simulations. In many applications such as tsunami predictions, numerical results are highly sensitive to errors in bathymetry data. The most common measurement techniques for bathymetry and their respective limitations (see [14, 18]) are as follows:

- Surveys by ships are suitable only for local measurements in small regions;
- LiDAR/LaDAR (**Li**ght/**La**ser **D**etection **A**nd **R**anging) using equipment installed on ships or aircraft is expensive and has limited coverage;
- Multi-spectral satellite imaging is only practical for shallow and clear water.

Discussions of other issues associated with direct bathymetry measurements can be found, e.g., in [14, 16, 20]. In the present paper, we explore the possibility of using SWE-based models for bathymetry reconstruction from the water surface elevation which is much easier to measure remotely (e.g. by satellite altimetry). The proposed approach involves solving a degenerate hyperbolic inverse problem, in which the roles of the free surface elevation and the bottom topography are interchanged [12]. The first proof of concept for bathymetry reconstructions by this technique was proposed by Gessese et al. in [8] using a finite difference discretization of a one-dimensional SWE system for stationary sub- and transcritical configurations. A generalization to the 2D case was presented in [7] and further developed in [9, 10]. The main objective of the present work is the design of a special finite element discretization that ensures compatibility of the forward and the inverse problems. We also address the ill-posedness issue by adding a regularization term which also improves the reconstruction quality in the presence of noise.

1.2 Formulation of the forward and inverse problems

The shallow water equations are derived from the incompressible Navier-Stokes equations using the hydrostatic pressure assumption and averaging in the vertical direction [5, 19]. The result is the system of conservation laws

$$\frac{\partial H}{\partial t} + \nabla\cdot(H\boldsymbol{u}) \;=\; 0\,, \tag{1.1}$$

$$\frac{\partial(Hu)}{\partial t} + \nabla\cdot(Hu\boldsymbol{u}) + \frac{g}{2}\frac{\partial H^2}{\partial x} + g\,H\,\frac{\partial b}{\partial x} + \tau_{bf}\,Hu - f_c\,Hv \;=\; 0\,, \tag{1.2}$$

$$\frac{\partial(Hv)}{\partial t} + \nabla\cdot(Hv\boldsymbol{u}) + \frac{g}{2}\frac{\partial H^2}{\partial y} + g\,H\,\frac{\partial b}{\partial y} + \tau_{bf}\,Hv + f_c\,Hu \;=\; 0\,, \tag{1.3}$$

where $\boldsymbol{u} = [u,v]^T$ is the depth-averaged velocity, and $H = \xi - b$ is the total water height, that is, the difference between the free surface elevation ξ and the bathymetry b both measured with respect to the same level. The terms depending on τ_{bf} and f_c are due to the bottom friction and the Coriolis force, respectively. In a compact form, the SWE system can be written as

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla\cdot\mathbf{F}(\mathbf{U}) \;=\; \mathbf{S}(\mathbf{U},\nabla b), \tag{1.4}$$

where

$$\mathbf{U} = \begin{bmatrix} H \\ H\boldsymbol{u} \end{bmatrix},\; \mathbf{F}(\mathbf{U}) = \begin{bmatrix} H\boldsymbol{u} \\ H\boldsymbol{u}\otimes\boldsymbol{u} + \frac{gH^2}{2}\mathbb{1} \end{bmatrix},\; \mathbf{S}(\mathbf{U},\nabla b) = \begin{bmatrix} 0 \\ f_c\,Hv - \tau_{bf}\,Hu - gH\frac{\partial b}{\partial x} \\ -f_c\,Hu - \tau_{bf}\,Hv - gH\frac{\partial b}{\partial y} \end{bmatrix}.$$

In the context of bathymetry reconstruction, the forward and inverse problems require numerical solution of system (1.1)–(1.3). In the forward problem, the free surface elevation is given by $\xi = H + b$, where b is a known bathymetry. The bathymetry gradient which appears in (1.2), (1.3) is known as well, so the source term $g\,H\,\nabla b$ depends linearly on H. In the inverse problem, the bathymetry is unknown and defined by $b = \xi - H$, where ξ is given. Since the gravitational force term of the inverse momentum equation contains the unknown bathymetry gradient, the inverse problem exhibits entirely different mathematical behavior. As a consequence of swapping the roles of ξ and b, system (1.1)–(1.3) becomes degenerate hyperbolic and more difficult to solve (see [12] for an in-depth analysis of the potentially ill-posed inverse problem). The possible lack of uniqueness can be cured by adding a regularization term of the form $\epsilon\Delta b$ to (1.1). This regularization resembles the Brezzi-Pitkäranta stabilization method [3] for equal-order numerical approximations to the velocity and pressure in the incompressible Navier-Stokes equations; however, in the setting of the shallow water equations, such a term would induce artificial currents in the presence of bathymetry gradients. The Laplacian can be replaced by a total variation regularization term [4] or another anisotropic diffusion operator. In this work, we define the regularization term at the discrete level in terms of finite element matrices (see below).

1.3 Discretization of the SWE system

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain with a polygonal boundary. Given a conforming triangulation $\mathcal{T}_h \coloneqq \{T_1, \dots, T_K\}$ of Ω, we define the DG space $\mathbb{V}_h \coloneqq \{v_h \in L^2(\Omega) : v_h|_T \in \mathbb{P}_1(T)\ \forall T \in \mathcal{T}_h\}$ and the corresponding continuous Galerkin (CG) space $\mathbb{W}_h \coloneqq \mathbb{V}_h \cap C^0(\overline{\Omega})$. The space $\mathbb{V}_h$ is spanned by $3K$ piecewise-linear basis functions $\psi_{kj},\ k = 1,\dots,K,\ j = 1,2,3$. The dimension of $\mathbb{W}_h$ equals the number of vertices $\mathbf{x}_1,\dots,\mathbf{x}_L$ of $\mathcal{T}_h$. The Lagrange basis functions $\varphi_1,\dots,\varphi_L$ have the property that $\varphi_i(\boldsymbol{x}_j) = \delta_{ij},\ i,j = 1,\dots,L$.
System (1.1)–(1.3) is discretized using the space $\mathbb{V}_h$ for $H, H\boldsymbol{u}$, ξ and the space $\mathbb{W}_h$ for b. Since the number of equations must be equal to the number of unknowns, the variational forms of the discrete forward and inverse problems differ in the choice of the test function space for the continuity equation. Time integration is performed using an explicit second-order SSP Runge-Kutta scheme [11], i.e., Heun's method.

Space discretization of the forward problem
In the semi-discrete forward problem, we seek the coefficients of surface elevation $\xi_h \in \mathbb{V}_h$ and momentum $(H\boldsymbol{u})_h \in (\mathbb{V}_h)^2$. In practice, it is more convenient to formulate the semi-discrete forward problem in terms of $\mathbf{U}_h = [H_h, (H\boldsymbol{u})_h]^T \in (\mathbb{V}_h)^3$ and calculate the surface elevation $\xi_h = H_h + b_h \in \mathbb{V}_h$ by adding the known continuous bathymetry $b_h \in \mathbb{W}_h$ to the discontinuous water height $H_h \in \mathbb{V}_h$. For any element $T^- \in \mathcal{T}_h$ and any test function $\mathbf{v}_h \in (\mathbb{V}_h)^3$, the (element-local) DG form of system (1.4) is given by [1, 12]

$$\int_{T^-} \mathbf{v}_h \cdot \partial_t \mathbf{U}_h \,\mathrm{d}\boldsymbol{x} \;-\; \int_{T^-} \boldsymbol{\nabla}\mathbf{v}_h : \mathbf{F}(\mathbf{U}_h)\,\mathrm{d}\boldsymbol{x} \\ + \int_{\partial T^-} \mathbf{v}_h \cdot \widehat{\mathbf{F}}(\mathbf{U}_h^-, \mathbf{U}_h^+; \boldsymbol{\nu}_{T^-})\,\mathrm{d}s = \int_{T^-} \mathbf{v}_h \cdot \mathbf{S}(\mathbf{U}_h, \nabla b_h)\,\mathrm{d}\boldsymbol{x}\,, \tag{1.5}$$

where $\boldsymbol{\nu}_{T^-}$ is the unit outward normal and $\widehat{\mathbf{F}}(\mathbf{U}_h^-, \mathbf{U}_h^+; \boldsymbol{\nu}_{T^-})$ is a numerical flux defined in terms of the one-sided limits $\mathbf{U}_h^\pm$ (see [12] for details). In the numerical study below, we use the Roe flux or the Lax-Friedrichs flux.
Summing over all elements, we obtain a semi-discrete problem of the form

$$(v_h, \partial_t H_h) + a_H(v_h, \mathbf{U}_h) = f_H(v_h) \quad \forall v_h \in \mathbb{V}_h, \tag{1.6}$$

$$(\mathbf{v}_h, \partial_t (H\boldsymbol{u})_h) + a_{\boldsymbol{u}}(\mathbf{v}_h, \mathbf{U}_h) = (\mathbf{v}_h, \mathbf{S}_h(\mathbf{U}_h, \nabla b_h)) + \mathbf{f}_{\boldsymbol{u}}(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in (\mathbb{V}_h)^2, \tag{1.7}$$

where $(\cdot,\cdot)$ is the L^2 scalar product on Ω. The forms $a_H(\cdot,\cdot)$ and $a_{\boldsymbol{u}}(\cdot,\cdot)$ consist of volume integrals depending on $\boldsymbol{\nabla}\mathbf{v}_h : \mathbf{F}(\mathbf{U}_h)$ and jump terms depending on $(\mathbf{v}_h^+ - \mathbf{v}_h^-) \cdot \widehat{\mathbf{F}}(\mathbf{U}_h^-, \mathbf{U}_h^+; \boldsymbol{\nu}_{T^-})$. The linear forms $f_H(\cdot)$ and $\mathbf{f}_{\boldsymbol{u}}(\cdot)$ contain the contribution of weakly imposed boundary conditions.

Space discretization of the inverse problem
In the inverse problem, the water height $H_h \in \mathbb{V}_h$ is uniquely determined by the $3K$ known coefficients of the surface elevation $\xi_h \in \mathbb{V}_h$ and L unknown co-

efficients of the bathymetry $b_h \in \mathbb{W}_h$. Hence, the dimension of the test function space for system (1.6),(1.7) exceeds the number of unknowns. Substituting $\xi_h - b_h$ for H_h, we replace the continuity equation (1.6) by

$$(w_h, \partial_t b_h) - a_H(w_h, \mathbf{U}_h) = (w_h, \partial_t \xi_h) - f_H(w_h) \qquad \forall w_h \in \mathbb{W}_h, \tag{1.8}$$

while keeping the momentum equation (1.7) unchanged. This yields a system of $L + 6K$ equations for $L + 6K$ unknowns. Since $\mathbb{W}_h$ is a subspace of $\mathbb{V}_h$, a DG approximation $\mathbf{U}_h = [H_h, (H\boldsymbol{u})_h]^T$ satisfying (1.6),(1.7) will satisfy (1.7),(1.8) as well. If the given surface elevation ξ_h corresponds to a solution of the discrete forward problem, the underlying bathymetry b_h must be a (possibly non-unique) solution of the discrete inverse problem.

Pseudo-time stepping and regularization
Any explicit SSP Runge-Kutta time discretization of system (1.8),(1.7) can be expressed as a convex combination of forward Euler updates. Let $\mathsf{M} = (m_{ij})$ denote the consistent mass matrix with entries $m_{ij} = (\varphi_i, \varphi_j),\ i, j = 1, \ldots, L$, where φ_i are the continuous Lagrange basis functions spanning the space $\mathbb{W}_h$. Row-sum mass lumping yields the diagonal approximation $\mathsf{M}_L = (m_i \delta_{ij})$, where $m_i = (\varphi_i, 1)$. To deal with the issue of ill-posedness, we march the bathymetry $b_h \in \mathbb{W}_h$ to a steady state using the regularized matrix form

$$\mathsf{M}_L \frac{b^{n+1} - b^n}{\Delta t} = R(\mathbf{U}_h^n) + \epsilon(\mathsf{M} - \mathsf{M}_L) b^n \tag{1.9}$$

of a generic forward Euler step for pseudo-time integration of (1.8). The first term on the right-hand side of the above linear system is defined by

$$R_i(\mathbf{U}_h^n) = \left(\varphi_i, \frac{\xi_h^{n+1} - \xi_h^n}{\Delta t}\right) + a_H(\varphi_i, \mathbf{U}_h^n) - f_H(\varphi_i), \tag{1.10}$$

where Δt is the pseudo-time step. The regularization term $\epsilon(\mathsf{M} - \mathsf{M}_L) b^n$ has the same form as the pressure stabilization term proposed by Becker and Hansbo [2] for a finite element discretization of the Stokes system. In our experience, the use of a discrete Laplace operator in place of $\mathsf{M} - \mathsf{M}_L$ produces similar results. We envisage that the use of anisotropic diffusion operators such as the one employed in [4] for total variation-based image denoising purposes can lead to more accurate reconstructions of small-scale features.

1.4 Numerical results for the inverse problem

Our numerical discretization utilizes the FESTUNG toolbox [6, 17, 15] and is described in detail in [13]. We consider the domain $\Omega = (0, 1\,\text{km}) \times (0, 1\,\text{km})$ and utilize a triangular unstructured grid with $\Delta x = 40\,m$ to solve the forward and inverse problems on a time interval of three hours with $\Delta t = 0.1\,s$. The employed parameter settings are $g = 9.81\,m/s^2, f_c = 3\cdot 10^{-5}s^{-1}, c_f = 10^{-3}s^{-1}$. Bathymetry for solving the forward problem is specified as a rather complex yet smooth function (see [12] and Fig. 1.1). The boundary conditions are as

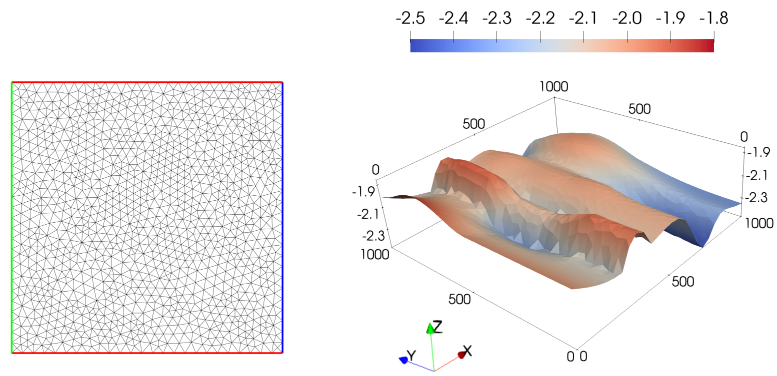

Fig. 1.1: Computational domain and mesh (left), exact bathymetry (right).

follows: in both problems, the normal fluxes are set to zero on the upper and lower boundary, and the flux $H\boldsymbol{u} = [4\ 0]^T$ is prescribed on the left (or inflow) boundary. In the forward problem, $\xi \equiv 0$ is used at the outlet, whereas in the inverse problem, the bathymetry is prescribed at the inlet.

First, we solve the forward problem with the initial condition $\xi \equiv 0$ and $H\boldsymbol{u} \equiv [4\ 0]^T$. The steady-state result (as presented in [12]) is subsequently used as input for the inverse problem, where the initial bathymetry is set to $b \equiv -2$, and initial momentum is as in the forward problem. Running the code until the change in bathymetry between pseudo-time steps becomes sufficiently small, we obtain a very accurate reconstruction (the L^∞ error is $7.85\cdot 10^{-6}\,m$). Excellent results are also obtained if a non-stationary free surface elevation is used as input for the inverse problem, similarly to the example considered in [12].

To study the effect of noisy input data on the free surface elevation, we add random perturbations ranging in $(-10^{-4}m, 10^{-4}m)$ to the free surface values in each grid vertex. Fig. 1.2 (left) shows a typical reconstruction result for such a case. The amplification of data errors in the reconstruction indicates the ill-posedness of the inverse problem, and further study shows that the reconstruction error is even worse on refined grids. However, interesting effects can be observed if one substitutes the 'noisy' result shown in Fig. 1.2 (left) as the bathymetry for the forward problem: a surface elevation field differing from the original perturbed steady state is produced. Remarkably, using this new steady state as input for the inverse problem results in the exact same oscillatory steady-state bathymetry as in Fig. 1.2 (left). We attribute this phenomenon to the space relation $\mathbb{W}_h \subset \mathbb{V}_h$: proper reconstruction of free surface elevation $\xi_h \in \mathbb{V}_h$ from a solution $b_h \in \mathbb{W}_h$ of the inverse problem may be impossible due to the larger DG space $\mathbb{V}_h$. On the other hand, the continuous bathymetry seems to be uniquely determined by the free surface and the inflow boundary condition as long as the velocities are non-zero – which is an encouraging result.

Finally, we demonstrate how to improve the oscillatory reconstruction via inclusion of diffusive terms: so far the parameter ε was set to zero. From heuristic testing we found the best possible reconstruction is possible with $\varepsilon = 0.08\,m^2/s$. The result can be seen in Fig. 1.2 (right); it indicates that the influence of flawed data can be filtered out by our regularization approach. Furthermore, we are able to reduce the required number of pseudo-time-stepping iterations by a factor of around 15 due to the smoothing properties of the artificial diffusive term. Corresponding results for reconstruction from noisy

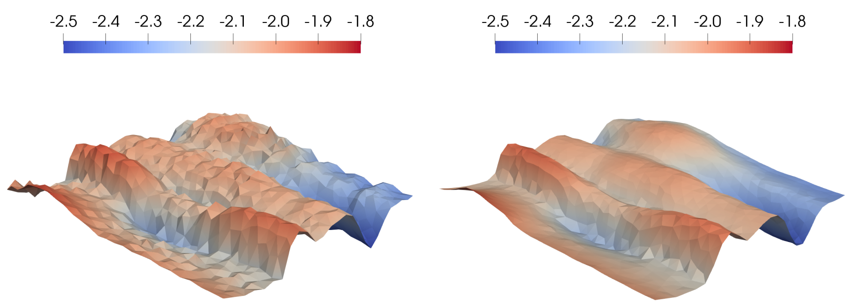

Fig. 1.2: Bathymetry reconstruction from noisy data without (left) and with (right) artificial diffusion.

surface elevation were obtained on refined grids. However, the regularization parameter ε has to be chosen much larger. In some cases, the steady-state convergence behavior can also be improved by decreasing the pseudo-time step. Promising results can be obtained even for $\varepsilon \to 0$ by gradually decreasing the value of ϵ.

1.5 Conclusion and outlook

The main highlight of this work is a combined CG-DG finite element method for SWE-based reconstruction of bottom topography from surface elevation data. The use of a continuous finite element space $\mathbb{W}_h \subset \mathbb{V}_h$ for the bathymetry produces a realistic number of constraints and ensures compatibility to the DG scheme for the hyperbolic forward problem. A regularization term is added to the discretized continuity equation of the inverse problem to to obtain stable steady state solutions. The presented numerical examples demonstrate the potential of the proposed methodology. Further work is required to study the sensitivity of results to the choice of the regularization parameter and definition of the artificial diffusion operator. These studies may involve theoretical investigations, as well as applications to rivers with well-explored bathymetry and/or comparison to laboratory experiments.

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