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On axiomizing and extending the quasi-arithmetic mean

Maurice Hansen

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Abstract. Quasi-arithmetic means contain many other mean value concepts such as the arithmetic, the geometric or the harmonic mean as special cases. Treating quasi-arithmetic means as sequences of mappings from I^n into I (for some real interval I) this paper shows that under mild additional conditions this mapping is uniquely determined by its values on I^2. This extends a well-known result by Huntington [4] where this claim is proven only for special cases.

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The following symbols are used in this paper:

$\mathbb{N}$	the set of all natural numbers: $\{1, 2, 3, \ldots\}$
$\mathbb{N}_{[m,n]}$	$\{r \in \mathbb{N} \mid m \leq r \leq n\}$
x^T	the transpose of x
$\mathbf{1}_n$	the vector $(1, \ldots, 1)^T \in \mathbb{R}^n$
$e_i^{(n)}$	the unit vector $(a_1, \ldots, a_n)^T \in \mathbb{R}^n$ with $a_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$

If not stated otherwise, the n components of a vector $x \in \mathbb{R}^n$ are denoted by $x_1, \ldots, x_n$, such that $x = (x_1, \ldots, x_n)^T$.

1. Axioms for means

The axiomatization of mean value functions has a long and distinguished history (Cauchy [3], Huntington [4], Kolmogoroff [5], Nagumo [7], Marichal [6], among many others); see also Bullen [2, chapter VI, section 6] for a more recent overview. In compliance with this literature, mean values are defined as a sequence $(M_n)_{n\in\mathbb{N}}$ of mappings $M_n : I^n \to I$ where I is some interval on

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the real line containing more than one element (usually, $I \in \{\mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{R}_{>0}\}$). To qualify as a mean, these functions have to satisfy certain axioms. Foremost among these is the following requirement:

(M1) $\min\{x_1, \ldots, x_n\} \leq M_n(x) \leq \max\{x_1, \ldots, x_n\}$ for all $n \in \mathbb{N}$, $x \in I^n$.

This condition immediately suggests itself; it was already considered by Cauchy [3, p. 14] as fundamental for any mean value concept. Additional conditions, most of them quite obvious, are:

(M2) $M_n(c\mathbf{1}_n) = c$ for all $n \in \mathbb{N}$, $c \in I$ *(idempotency).*

(M3) $M_n(x) \leq M_n(y)$ for all $n \in \mathbb{N}$ and $x, y \in I^n$ with $x_i \leq y_i$ for all $i \in \mathbb{N}_{[1,n]}$ *(isotonicity).*

(M4) $M_n\left((x_1, \ldots, x_n)^T\right) = M_n\left((x_{\pi(1)}, \ldots, x_{\pi(n)})^T\right)$ for all $n \in \mathbb{N}$, $x \in I^n$ and all permutations $\pi : \mathbb{N}_{[1,n]} \to \mathbb{N}_{[1,n]}$ *(symmetry).*

It is easily checked that axioms (M2), (M3) and (M4) are independent, whereas (M1) is implied by (M2) and (M3). Furthermore, the combination of (M1) and (M3) and the combination of (M2) and (M3) are equivalent.

All conventional means obey all these requirements. Additional axioms are:

(M5) $M_n : I^n \to I$ is continuous for all $n \in \mathbb{N}$ *(continuity).*

(M6) $M_n(\lambda x) = \lambda M_n(x)$ for all $n \in \mathbb{N}, x \in I^n, \lambda \in \mathbb{R}$ with $\lambda x \in I^n$ *(homogeneity).*

(M7) $M_n(x + y) = M_n(x) + M_n(y)$ for all $n \in \mathbb{N}$, $x, y \in I^n$, if $a + b \in I$ holds for all $a, b \in I$ *(additivity).*

(M3′) $M_n(x) < M_n(y)$ for all $n \in \mathbb{N}$ and $x, y \in I^n$ with $x_i \leq y_i$ for all $i \in \mathbb{N}_{[1,n]}$ and $x \neq y$ *(strict isotonicity).*

Homogeneity and additivity will play no further role below. Of primary importance, however, will be an axiom which can be viewed as a generalisation of what is usually called *associativity:*

(M8) Let $n \in \mathbb{N}$, $k \in \mathbb{N}_{[1,n]}$, $x \in I^n$ and $M := M_k\left((x_1, \ldots, x_k)^T\right)$. Then

$$M_n\left((x_1, \ldots, x_k, x_{k+1}, \ldots, x_n)^T\right) = M_n\left((M, \ldots, M, x_{k+1}, \ldots, x_n)^T\right).$$

I. e. substituting a *leading* subvector of x componentwise by its mean leaves the overall mean unchanged. This condition has first been introduced by Bemporad [1] in a characterization of the arithmetic mean. Marichal [6] points out that (M8) only makes sense in conjunction with (M4) and suggests a generalisation which he calls *strong decomposability:*

(M9) For all $n \in \mathbb{N}$, $k \in \mathbb{N}_{[1,n]}$, $x \in I^n$ and any subset $K = \{i_1, \ldots, i_k\} \subseteq \mathbb{N}_{[1,n]}$ with $i_1 < i_2 < \cdots < i_k$ we have

$$M_n(x) = M_n\left(\sum_{i \in K} M e_i^{(n)} + \sum_{i \in \mathbb{N}_{[1,n]} \setminus K} x_i e_i^{(n)}\right) \quad \text{with} \quad M := M_k\left(\begin{pmatrix} x_{i_1} \\ \vdots \\ x_{i_k} \end{pmatrix}\right).$$

I. e. for *any* subvector y of x (not necessarily leading or connected), M_n remains unchanged if all elements of y are replaced by $M_k(y)$.

2. Characterizing quasi-arithmetic means

Definition 2.1. A *quasi-arithmetic mean* is a sequence $(\mathcal{Q}_n^{[\varphi]})_{n\in\mathbb{N}}$ of mappings $\mathcal{Q}_n^{[\varphi]} : I^n \to I$ with

$$\mathcal{Q}_n^{[\varphi]}(x) = \varphi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\varphi(x_i)\right)$$

for all $n \in \mathbb{N}$, $x \in I^n$, where $\varphi : I \to J$ is a continuous and bijective mapping (see Bullen [2, p. 266]).

There are various characterizations of quasi-arithmetic means. Marichal [6] for instance shows that a sequence $(M_n)_{n\in\mathbb{N}}$ is a quasi-arithmetic mean if and only if it obeys axioms (M2), (M3′), (M5) and (M9).

Below, $\hat{I} \subseteq \mathbb{R}$ is an unbounded interval. The following preliminary result is needed for the central theorem of this paper:

Lemma 2.2. *Let $\varphi : \hat{I} \to J$ be continuous and bijective. Then for all $n \in \mathbb{N}\backslash\{1\}$ and $x \in \hat{I}^n$ there exists an $a \in \hat{I}$ with the following property: For all $k \in \mathbb{N}_{[1,n-1]}$ one has*

$$\left(\underbrace{a,\ldots,a}_{k\text{-times}}, \varphi^{-1}\left(\sum_{i=1}^{k+1}\varphi(x_i) - k\varphi(a)\right), x_{k+2},\ldots,x_n\right)^T \in \hat{I}^n.$$

Proof. We distinguish the following cases:

1. $\hat{I}$ is neither left-bounded nor right-bounded.
2. $\hat{I}$ is left-bounded, but not right-bounded.
3. $\hat{I}$ is right-bounded, but not left-bounded.

In each case, one has to show that all components of the vector belong to $\hat{I}$.

1. If $\hat{I}$ is neither left-bounded nor right-bounded, then $\hat{I} = \mathbb{R}$ and the statement holds for any $a \in \mathbb{R}$.
2. Let $n \in \mathbb{N}$, $x \in \hat{I}^n$, choose $a := \min\{x_1,\ldots,x_n\}$, let $k \in \mathbb{N}_{[1,n-1]}$. By assumption $a, x_{k+2},\ldots,x_n \in \hat{I}$. Show $\varphi^{-1}\left(\sum_{i=1}^{k+1}\varphi(x_i) - k\varphi(a)\right) \in \hat{I}$. If φ is strictly increasing, the following holds:

$$\begin{aligned}
a = \min\{x_1,\ldots,x_n\} \quad &\Rightarrow \quad a \le x_i \quad \text{for all } i \in \mathbb{N}_{[1,k+1]}\\
&\Rightarrow \quad \varphi(a) \le \varphi(x_i) \quad \text{for all } i \in \mathbb{N}_{[1,k+1]}\\
&\Rightarrow \quad (k+1)\varphi(a) \le \sum_{i=1}^{k+1}\varphi(x_i)\\
&\Rightarrow \quad \varphi(a) \le \sum_{i=1}^{k+1}\varphi(x_i) - k\varphi(a)\\
&\Rightarrow \quad a \le \varphi^{-1}\left(\sum_{i=1}^{k+1}\varphi(x_i) - k\varphi(a)\right)
\end{aligned}$$

Hence $\varphi^{-1}\left(\sum_{i=1}^{k+1}\varphi(x_i)-k\varphi(a)\right)\in\hat{I}$, since $a\in\hat{I}$ and $\hat{I}$ is not right-bounded.

If φ is strictly decreasing, one obtains the conclusion in the same way:

$$\begin{aligned} a=\min\{x_1,\ldots,x_n\} \quad &\Rightarrow \quad a\le x_i \quad \text{for all } i\in\mathbb{N}_{[1,k+1]} \\ &\Rightarrow \quad \varphi(a)\ge\varphi(x_i) \quad \text{for all } i\in\mathbb{N}_{[1,k+1]} \\ &\Rightarrow \quad (k+1)\varphi(a)\ge\sum_{i=1}^{k+1}\varphi(x_i) \\ &\Rightarrow \quad \varphi(a)\ge\sum_{i=1}^{k+1}\varphi(x_i)-k\varphi(a) \\ &\Rightarrow \quad a\le\varphi^{-1}\left(\sum_{i=1}^{k+1}\varphi(x_i)-k\varphi(a)\right) \end{aligned}$$

Again $\varphi^{-1}\left(\sum_{i=1}^{k+1}\varphi(x_i)-k\varphi(a)\right)\in\hat{I}$, since $a\in\hat{I}$ and $\hat{I}$ is not right-bounded.

3. Let $n\in\mathbb{N}$, $x\in\hat{I}^n$, choose $a:=\max\{x_1,\ldots,x_n\}$, let $k\in\mathbb{N}_{[1,n-1]}$. By assumption $a,x_{k+2},\ldots,x_n\in\hat{I}$. Show $\varphi^{-1}\left(\sum_{i=1}^{k+1}\varphi(x_i)-k\varphi(a)\right)\in\hat{I}$. If φ is strictily increasing, the following holds:

$$\begin{aligned} a=\max\{x_1,\ldots,x_n\} \quad &\Rightarrow \quad a\ge x_i \quad \text{for all } i\in\mathbb{N}_{[1,k+1]} \\ &\Rightarrow \quad \varphi(a)\ge\varphi(x_i) \quad \text{for all } i\in\mathbb{N}_{[1,k+1]} \\ &\Rightarrow \quad (k+1)\varphi(a)\ge\sum_{i=1}^{k+1}\varphi(x_i) \\ &\Rightarrow \quad \varphi(a)\ge\sum_{i=1}^{k+1}\varphi(x_i)-k\varphi(a) \\ &\Rightarrow \quad a\ge\varphi^{-1}\left(\sum_{i=1}^{k+1}\varphi(x_i)-k\varphi(a)\right) \end{aligned}$$

Hence $\varphi^{-1}\left(\sum_{i=1}^{k+1}\varphi(x_i)-k\varphi(a)\right)\in\hat{I}$, since $a\in\hat{I}$ and $\hat{I}$ is not left-bounded.

If φ is strictly decreasing, one obtains the conclusion in the same way. □

The following theorem shows that a quasi-arithmetic mean is in a sense uniquely determined by its values on $\hat{I}^2$:

Theorem 2.3. *Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of mappings $M_n:\hat{I}^n\to\hat{I}$. If $(M_n)_{n\in\mathbb{N}}$ satisfies* (M2), (M9) *and $M_2=\mathcal{Q}_2^{[\varphi]}$ for a continuous and bijective mapping $\varphi:\hat{I}\to J$, then $M_n=\mathcal{Q}_n^{[\varphi]}$ for all $n\in\mathbb{N}$.*

Proof. Consider $n = 1$. Because of (M2) the following holds for all $x \in \hat{I}$:

$$M_1(x) = M_1(x\mathbf{1}_1) = x = \mathcal{Q}_1^{[\varphi]}(x)$$

For $n \in \mathbb{N}\backslash\{1\}$, we first show that for all $x \in \hat{I}^n$ one has

$$M_n(x) = M_n\left(\left(\underbrace{a,\dots,a}_{k\text{-times}}, \varphi^{-1}\left(\sum_{i=1}^{k+1}\varphi(x_i) - k\varphi(a)\right), x_{k+2},\dots,x_n\right)^T\right)$$

for all $k \in \mathbb{N}_{[1,n-1]}$, if a is chosen according to lemma 2.2.

Let $n \in \mathbb{N}\backslash\{1\}$, $x \in \hat{I}^n$ and choose a according to lemma 2.2. First of all, the term

$$M_n\left(\left(\underbrace{a,\dots,a}_{k\text{-times}}, \varphi^{-1}\left(\sum_{i=1}^{k+1}\varphi(x_i) - k\varphi(a)\right), x_{k+2},\dots,x_n\right)^T\right)$$

is well-defined according to lemma 2.2 for all $k \in \mathbb{N}_{[1,n-1]}$. The claim is shown by induction over $k \in \mathbb{N}_{[1,n-1]}$.

Base case: If $k = 1$, one has

$$\begin{aligned} M_n(x) &= M_n\left((x_1,\dots,x_n)^T\right) \\ &\overset{(M9)}{=} M_n\left(\left(M_2\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}\right), M_2\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}\right), x_3,\dots,x_n\right)^T\right) \\ &= M_n\left(\left(\mathcal{Q}_2^{[\varphi]}\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}\right), \mathcal{Q}_2^{[\varphi]}\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}\right), x_3,\dots,x_n\right)^T\right). \end{aligned} \tag{1}$$

Furthermore

$$\begin{aligned} &M_2\left(\begin{pmatrix}a\\ \varphi^{-1}(\varphi(x_1)+\varphi(x_2)-\varphi(a))\end{pmatrix}\right) \\ &= \mathcal{Q}_2^{[\varphi]}\left(\begin{pmatrix}a\\ \varphi^{-1}(\varphi(x_1)+\varphi(x_2)-\varphi(a))\end{pmatrix}\right) \\ &= \varphi^{-1}\left(\frac{\varphi(a)+\varphi(\varphi^{-1}(\varphi(x_1)+\varphi(x_2)-\varphi(a)))}{2}\right) \\ &= \varphi^{-1}\left(\frac{\varphi(x_1)+\varphi(x_2)}{2}\right) \\ &= \mathcal{Q}_2^{[\varphi]}\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}\right), \end{aligned}$$

hence

$$\begin{aligned} M_n(x) &\overset{(1)}{=} M_n\left(\left(\mathcal{Q}_2^{[\varphi]}\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}\right), \mathcal{Q}_2^{[\varphi]}\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}\right), x_3,\dots,x_n\right)^T\right) \\ &\overset{(M9)}{=} M_n\left(\left(a, \varphi^{-1}(\varphi(x_1)+\varphi(x_2)-\varphi(a)), x_3,\dots,x_n\right)^T\right). \end{aligned}$$

Inductive step: Suppose

$$M_n(x) = M_n\left(\left(\underbrace{a,\dots,a}_{k\text{-times}}, \varphi^{-1}\left(\sum_{i=1}^{k+1}\varphi(x_i) - k\varphi(a)\right), x_{k+2},\dots,x_n\right)^T\right).$$

holds for $k \in \mathbb{N}_{[1,n-2]}$. Since $k \le n-2$, $\varphi^{-1}\left(\sum_{i=1}^{k+1} -k\varphi(a)\right)$ is not the last component of the vector, so x_{k+2} exists. One has

$$\begin{aligned}
&M_n(x)\\
&= M_n\left(\left(\underbrace{a,\dots,a}_{k\text{-times}}, \varphi^{-1}\left(\sum_{i=1}^{k+1}\varphi(x_i) - k\varphi(a)\right), x_{k+2},\dots,x_n\right)^T\right)\\
&\overset{(\mathrm{M9})}{=} M_n\left(\left(\underbrace{a,\dots,a}_{k\text{-times}}, M_2\left(\begin{pmatrix}\varphi^{-1}\left(\sum_{i=1}^{k+1}\varphi(x_i) - k\varphi(a)\right)\\ x_{k+2}\end{pmatrix}\right),\right.\right.\\
&\qquad\left.\left. M_2\left(\begin{pmatrix}\varphi^{-1}\left(\sum_{i=1}^{k+1}\varphi(x_i) - k\varphi(a)\right)\\ x_{k+2}\end{pmatrix}\right), x_{k+3},\dots,x_n\right)^T\right)\\
&= M_n\left(\left(\underbrace{a,\dots,a}_{k\text{-times}}, \mathcal{Q}_2^{[\varphi]}\left(\begin{pmatrix}\varphi^{-1}\left(\sum_{i=1}^{k+1}\varphi(x_i) - k\varphi(a)\right)\\ x_{k+2}\end{pmatrix}\right),\right.\right.\\
&\qquad\left.\left. \mathcal{Q}_2^{[\varphi]}\left(\begin{pmatrix}\varphi^{-1}\left(\sum_{i=1}^{k+1}\varphi(x_i) - k\varphi(a)\right)\\ x_{k+2}\end{pmatrix}\right), x_{k+3},\dots,x_n\right)^T\right)\\
&= M_n\left(\left(\underbrace{a,\dots,a}_{k\text{-times}}, \varphi^{-1}\left(\frac{\varphi\left(\varphi^{-1}\left(\sum_{i=1}^{k+1}\varphi(x_i) - k\varphi(a)\right)\right) + \varphi(x_{k+2})}{2}\right),\right.\right.\\
&\qquad \varphi^{-1}\left(\frac{\varphi\left(\varphi^{-1}\left(\sum_{i=1}^{k+1}\varphi(x_i) - k\varphi(a)\right)\right) + \varphi(x_{k+2})}{2}\right),\\
&\qquad\left.\left. x_{k+3},\dots,x_n\right)^T\right)\\
&= M_n\left(\left(\underbrace{a,\dots,a}_{k\text{-times}}, \varphi^{-1}\left(\frac{\sum_{i=1}^{k+2}\varphi(x_i) - k\varphi(a)}{2}\right),\right.\right.\\
&\qquad\left.\left. \varphi^{-1}\left(\frac{\sum_{i=1}^{k+2}\varphi(x_i) - k\varphi(a)}{2}\right), x_{k+3},\dots,x_n\right)^T\right). \qquad (2)
\end{aligned}$$

(Clearly, the components $x_{k+3},\dots,x_n$ at the end of the vectors are omitted if $k = n-2$.)

Furthermore

$$
\begin{aligned}
&M_2\left(\begin{pmatrix} a \\ \varphi^{-1}\left(\sum_{i=1}^{k+2}\varphi(x_i)-(k+1)\varphi(a)\right)\end{pmatrix}\right)\\
&=\mathcal{Q}_2^{[\varphi]}\left(\begin{pmatrix} a \\ \varphi^{-1}\left(\sum_{i=1}^{k+2}\varphi(x_i)-(k+1)\varphi(a)\right)\end{pmatrix}\right)\\
&=\varphi^{-1}\left(\frac{\varphi(a)+\varphi\left(\varphi^{-1}\left(\sum_{i=1}^{k+2}\varphi(x_i)-(k+1)\varphi(a)\right)\right)}{2}\right)\\
&=\varphi^{-1}\left(\frac{\sum_{i=1}^{k+2}\varphi(x_i)-k\varphi(a)}{2}\right),
\end{aligned}
$$

hence

$$
\begin{aligned}
&M_n(x)\\
&\overset{(2)}{=} M_n\Bigg(\Bigg(\underbrace{a,\dots,a}_{k\text{-times}},\varphi^{-1}\left(\frac{\sum_{i=1}^{k+2}\varphi(x_i)-k\varphi(a)}{2}\right),\\
&\qquad\qquad \varphi^{-1}\left(\frac{\sum_{i=1}^{k+2}\varphi(x_i)-k\varphi(a)}{2}\right),x_{k+3},\dots,x_n\Bigg)^T\Bigg)\\
&\overset{(M9)}{=} M_n\left(\left(\underbrace{a,\dots,a}_{(k+1)\text{-times}},\varphi^{-1}\left(\sum_{i=1}^{k+2}\varphi(x_i)-(k+1)\varphi(a)\right),x_{k+3},\dots,x_n\right)^T\right).
\end{aligned}
$$

This shows that

$$
M_n(x)=M_n\left(\left(\underbrace{a,\dots,a}_{k\text{-times}},\varphi^{-1}\left(\sum_{i=1}^{k+1}\varphi(x_i)-k\varphi(a)\right),x_{k+2},\dots,x_n\right)^T\right)
$$

holds for all $k\in\mathbb{N}_{[1,n-1]}$ and $n\in\mathbb{N}\backslash\{1\}$, $x\in\hat{I}^n$.

In particular, setting $k=n-1$, one has

$$
M_n(x)=M_n\left(\left(\underbrace{a,\dots,a}_{(n-1)\text{-times}},\varphi^{-1}\left(\sum_{i=1}^{n}\varphi(x_i)-(n-1)\varphi(a)\right)\right)^T\right). \tag{3}
$$

This implies that

$$\begin{aligned} M_n(x) &\overset{(3)}{=} M_n\left(\left(\underbrace{a,\dots,a}_{(n-1)\text{-times}}, \varphi^{-1}\left(\sum_{i=1}^{n}\varphi(x_i)-(n-1)\varphi(a)\right)\right)^T\right) \\ &= M_n\left(\left(\underbrace{a,\dots,a}_{(n-1)\text{-times}}, \varphi^{-1}\left(n\varphi\left(\mathcal{Q}_n^{[\varphi]}(x)\right)-(n-1)\varphi(a)\right)\right)^T\right) \\ &= M_n\left(\left(\underbrace{a,\dots,a}_{(n-1)\text{-times}}, \varphi^{-1}\left(\sum_{i=1}^{n}\varphi\left(\mathcal{Q}_n^{[\varphi]}(x)\right)-(n-1)\varphi(a)\right)\right)^T\right). \end{aligned}$$

Moreover, one has

$$\begin{aligned} &M_n(\mathcal{Q}_n^{[\varphi]}(x)\mathbf{1}_n) \\ &\overset{(3)}{=} M_n\left(\left(\underbrace{a,\dots,a}_{(n-1)\text{-times}}, \varphi^{-1}\left(\sum_{i=1}^{n}\varphi\left(\mathcal{Q}_n^{[\varphi]}(x)\right)-(n-1)\varphi(a)\right)\right)^T\right), \end{aligned}$$

so

$$\begin{aligned} M_n(x) &= M_n(\mathcal{Q}_n^{[\varphi]}(x)\mathbf{1}_n) \\ &\overset{(\mathrm{M2})}{=} \mathcal{Q}_n^{[\varphi]}(x). \end{aligned}$$

Hence, $M_n = \mathcal{Q}_n^{[\varphi]}$ also holds for all $n \in \mathbb{N}\backslash\{1\}$. □

Theorem 2.3 easily allows for a characterization of specific quasi-arithmetic means by independent axioms, as shown by Huntington [4] for the arithmetic, geometric and harmonic mean and the root mean square. As Huntington operated without the strong decomposability axiom, he had to rely on a weaker symmetry axiom instead. With strong decomposability, the symmetry requirement is redundant. The following theorem can therefore be viewed as a generalisation of Huntington's result:

Theorem 2.4.

1. *Let* $(M_n)_{n\in\mathbb{N}}$ *be a sequence of mappings* $M_n : \hat{I}^n \to \hat{I}$, *let* $\varphi : \hat{I} \to J$ *be continuous and bijective. Then* $(M_n)_{n\in\mathbb{N}}$ *satisfies* (M2), (M9) *and* $M_2 = \mathcal{Q}_2^{[\varphi]}$ *if and only if* $M_n = \mathcal{Q}_n^{[\varphi]}$ *for all* $n \in \mathbb{N}$.
2. *Let* $\varphi : I \to J$ *be continuous and bijective. Then the three properties* (M2), (M9) *and* $M_2 = \mathcal{Q}_2^{[\varphi]}$ *are independent (i. e none of the axioms is implied by the others).*

Proof.

1. Necessity: See Theorem 2.3.
   Sufficiency: Trivial, since $(\mathcal{Q}_n^{[\varphi]})_{n\in\mathbb{N}}$ satisfies (M2) and (M9).

2. Choose any fixed $c \in I$, then consider the sequences $(M_n)_{n\in\mathbb{N}}$, $(N_n)_{n\in\mathbb{N}}$ and $(O_n)_{n\in\mathbb{N}}$ of mappings $M_n, N_n, O_n : I^n \to I$ with

$$M_n(x) = \begin{cases} \mathcal{Q}_n^{[\varphi]}(x) & , n \in \{1,2\} \\ c & , n \geq 3 \end{cases}$$

$$N_n(x) = \begin{cases} \mathcal{Q}_n^{[\varphi]}(x) & , n \in \{1,2\} \\ \mathcal{M}_n(x) & , n \geq 3 \end{cases}$$

$$O_n(x) = x_1$$

for all $n \in \mathbb{N}$, $x \in I^n$. Here, $\mathcal{M}_n(x)$ denotes the median of x.

- $(M_n)_{n\in\mathbb{N}}$ satisfies (M9): Let $n \in \mathbb{N}$, $k \in \mathbb{N}_{[1,n]}$, $x \in I^n$ and $K = \{i_1, \dots, i_k\} \subseteq \mathbb{N}_{[1,n]}$ with $i_1 < i_2 < \dots < i_k$. If $n \in \{1,2\}$, it is clear that

$$M_n(x) = M_n\left(\sum_{i\in K} M_k\left(\begin{pmatrix} x_{i_1} \\ \vdots \\ x_{i_k}\end{pmatrix}\right) e_i^{(n)} + \sum_{i\in\mathbb{N}_{[1,n]}\setminus K} x_i e_i^{(n)}\right)$$

holds, since $M_n(x) = \mathcal{Q}_n^{[\varphi]}(x)$ for all $x \in I^n$ in this case and the quasi-arithmetic mean is strictly decomposable. If $n \geq 3$, then $M_n(x) = c$ and

$$M_n\left(\sum_{i\in K} M_k\left(\begin{pmatrix} x_{i_1} \\ \vdots \\ x_{i_k}\end{pmatrix}\right) e_i^{(n)} + \sum_{i\in\mathbb{N}_{[1,n]}\setminus K} x_i e_i^{(n)}\right) = c$$

by definition, thus $(M_n)_{n\in\mathbb{N}}$ satisfies (M9). Furthermore $M_2 = \mathcal{Q}_2^{[\varphi]}$. However, property (M2) is not satisfied: Choose $d \in I$ with $d \neq c$, then

$$M_3(d\mathbf{1}_3) = c \neq d.$$

- $(N_n)_{n\in\mathbb{N}}$ statisfies (M2) and $N_2 = \mathcal{Q}_2^{[\varphi]}$. However, property (M9) is not satisfied: Choose $a, b, c, d, e \in I$ with $a < b < c < d < e$, then

$$N_5\left((a,b,c,d,e)^T\right) = c \neq N_5\left((b,b,b,d,e)^T\right) = b.$$

- $(O_n)_{n\in\mathbb{N}}$ satisfies (M2), which is clear, and also (M9): Let $n \in \mathbb{N}$, $k \in \mathbb{N}_{[1,n]}$, $x \in I^n$ and $K = \{i_1, \dots, i_k\} \subseteq \mathbb{N}_{[1,n]}$ with $i_1 < i_2 <$

$\cdots < i_k$. Then

$$
\begin{aligned}
&O_n\left(\sum_{i\in K} O_k\left(\begin{pmatrix} x_{i_1}\\ \vdots \\ x_{i_k}\end{pmatrix}\right)e_i^{(n)} + \sum_{i\in\mathbb{N}_{[1,n]}\setminus K} x_i e_i^{(n)}\right)\\
&= \begin{cases} O_k\left(\begin{pmatrix} x_{i_1}\\ \vdots \\ x_{i_k}\end{pmatrix}\right) & , i_1 = 1\\ x_1 & , i_1 \neq 1\end{cases}\\
&= \begin{cases} x_{i_1} & , i_1 = 1\\ x_1 & , i_1 \neq 1\end{cases}\\
&= x_1\\
&= O_n(x).
\end{aligned}
$$

However, there is no continuous and bijective mapping $\varphi : I \to J$ such that

$$\varphi^{-1}\left(\frac{\varphi(x_1)+\varphi(x_2)}{2}\right) = x_1$$

holds for all $x_1, x_2 \in I$. Since φ is a bijection, the following holds for all $x_1, x_2 \in I$ with $x_1 \neq x_2$:

$$
\begin{aligned}
\varphi(x_1) \neq \varphi(x_2) \quad &\Rightarrow \quad \frac{\varphi(x_1)+\varphi(x_2)}{2} \neq \varphi(x_1)\\
&\Rightarrow \quad \varphi^{-1}\left(\frac{\varphi(x_1)+\varphi(x_2)}{2}\right) \neq x_1. \qquad \square
\end{aligned}
$$

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Maurice Hansen
Technische Universität Dortmund
Fakultät Statistik
CDI-Gebäude
44221 Dortmund
Germany
e-mail: maurice.hansen@tu-dortmund.de