Ana DONEVSKA-TODOROVA & Maryna VIAZOVSKA, Berlin

Higher Dimensional Geometry from a Didactical Perspective

This paper considers questions related to geometry in higher dimensions, as: (1) whether and why it could be important for students' engagements in prosperous mathematical activities and (2) how could geometrical objects in higher dimensions be explored by different groups of students. We consider that regular polytopes and the sphere packing problem are two examples that could make the learning of multi-dimensional geometry attractive and accessible even at school level mathematics. From a didactical point of view, they both offer mathematical content for developing geometric, algebraic and formal-axiomatic way of thinking. Our main focus is on the development of spatial abilities and spatial thinking supported by a usage of hand-held or digitally-based manipulatives. Our initial empirical trials took place in a special summer school program.

1. Introduction and Rationale

In the spirit of the joint meeting GDMV 2018, this paper tries to bridge the latest scientific findings in mathematics and mathematics education. The study is relevant for all specified sections on the interface between science and education which were offered at the meeting in the following sense:

- *Transition from school to university*, because it considers the development of analogies and generalizations of mathematical terms from lower to higher dimensions and shows a geometric and algebraic concretization of abstract mathematical concepts e.g. through a nested model of three modes of thinking (Donevska-Todorova, 2017) and allows a development of different facets of mathematical thinking, e.g. reasoning by analogy or visual reasoning (Dreyfus & Eisenberg, 1996).
- *Teaching and learning mathematics at university level*, because it connects the contents of geometry with concepts in measure theory such as distance and volume; abstract algebra, group theory and linear algebra, such as vector spaces, Euclidean spaces, linear dependence and independence of vectors and the dot product.
- *Mathematics for future teachers*, because it opens questions if the content could be relevant for pre-service teachers' education and in-service teachers' sustainable continuous professional development.
- *History and philosophy of mathematics and of mathematics education*, because it considers mathematical knowledge which is

out of the current scope of school curricula, but is related to their dynamic evolution.

Interesting research questions concerning the third section could be, for example, how teachers approach to solve a particular task, geometrically, algebraically or apply formal-axiomatic way of thinking. Should research efforts be invested in developing an intuitive visualization of 4-dimensional geometry by teachers in a similar way as the representations of geometrical objects (e.g. a point, a line) in lower dimensions, i.e. zero and one? It is not rare the case that pre-service teachers perceive points as components of the drawing plane without dimension. Need teachers understanding of the abstract existence of geometrical objects in dimensions different than two and three, even in case they are not an explicit part of school curricula? For which exact group of students could the teaching and learning geometry in higher dimensions be meaningful? Answering these questions certainly requires serious research. Notwithstanding, in the frame of this paper, we have decided to concentrate on the first two of the four above aspects, i.e. teaching and learning mathematics at university and transitional issues.

2. Theoretical Considerations

Geometric objects in two or three dimensions are relatively easy to visualize and understand because of their frequent everyday usage, aside from those in higher dimensions that exist exclusively in the mathematical abstractness. Yet, the mathematical knowledge of higher dimensions has a significant role in studying digital signal processing, cryptography and physics. In fact, school mathematics education considers only two- and three-dimensional geometry. Regular polygons and platonic solids have historically been, and still are part of the contemporary school curricula even in the early grades. The question is whether *4D* geometry for example could be a considerable content for upper high school mathematics to a certain extent, (1) why, in which form and for which particular focus group of learners. We do not claim that *4D*, i.e. n-dimensional ($n=0,1$ or $n>3$) geometry has to become an obligatory part of the general curricula for school mathematics. Our aim is rather to offer some insights into several important mathematical topics, which challenge math researchers and bring them closer to the younger population.

Though the research in pure mathematics has exploited the topic to a high extent, little has been done in the research from an educational perspective.

Certain argumentations referring the question (1) why could the study of geometry in higher dimensions be beneficial in mathematics instruction are the following. There is no doubt that the goals of the geometry education are to support a development of spatial abilities. Authors, e.g. Maier (1994), have perceived them as having five constituents: spatial perception, spatial visualization, mental rotations, spatial relations and spatial orientation. We consider that studying the content of higher dimensional geometry is relevant for the improvement of the first three components and the balance between visual-spatial and verbal-logical thought. The development of spatial thinking described as "a form of mental activity which makes it possible to create spatial images and manipulate them in the course of solving various practical and theoretical problems" (Yakimanskaya, 1991, p. 21) can also take advantage through this theme.

The question that naturally follows is (2) how to offer such highly abstract mathematical knowledge to students in an appropriate way to keep them motivated, actively engaged and finally gain mathematical knowledge. The efectivness of the usage of manipulatives in comparison to material-free instruction on the development of spatial skills has been confirmed in literature (Bishop, 1980). Involvement in inquiry-based activities seems to be a nice possibility which could include usage of hand-held materials, e.g. the Zometool or technology-enhanced materials. Paper-pencil i.e. chalk-board learning environments may not be sufficient for exploiting the most of the potentials that the content has. Which are those didactical materials that offer the most advantages for achieving learning goals? Dynamic visualizations, tangible models, audio-visuals and haptic technologies may support the learning and understanding of higher dimensional solids and the exploration of their properties. Multi-modal environments e.g. dynamic haptic geometry, appear to have high potential in supporting a variety of modalities and communication channels (such as vision, gesture and speech) for interaction with mathematical objects, through different interfaces (Güçler et al., 2013). *3D* pens may in this sense be used as tools for meaningful hands-on/minds-on acivities and learning through the sences for touch and vision.

Related to these considerations, we have designed a teaching unit: Higher dimensional geometry constructing a hypercube by drawing with *3D* pens which involves: classroom activities (descriptions and instructions for teachers), a worksheet (activities and tasks for students) and a template/ paper-based model for drawing a hypercube with *3D* pens. The design challenges the didactical issue of visualization of *nD* objects in *(n-1)D* interface notified in literature (Gutiérrez, 1996).

3. Practical Implementation

The empirical part of the study took place during a summer school 2016 for upper high school students from five schools with a focus on natural sciences and mathematics in the Berlin network of schools with the Institute of Mathematics at the Humboldt-University of Berlin. A specifically designed program included six themes and one of them was “How can we 'feel' higher dimension?” regarding the Euclidian space, groups and subgroups, polytopes and Platonic solids, and Schlegel diagrams.

4. Conclusions and Remaining Issues

The theoretical discussion above shows didactical potentials of the theme about higher dimensional geometry in school mathematics. Our initial empirical findings have shown that during participation in a special summer school program, upper high school students not only have a particular interest in the topic but are also successful in spatial perception and visualization when solving the tasks. It is our ambition to implement an adapted design with students in the begging classes of the lower secondary education. This may lead us to new inputs for further research about development of spatial abilities and thinking during students’ engagement with different manipulatives which are appropriate for the their age.

Literature

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