Game Theoretical Analyses of Industrial and Societal Organization - Economic Design with a Focus on Heterogeneity and the Impact of Informational Asymmetries

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Dekan:	Prof. Dr. Andreas Liening
Erstreferent:	Prof. Dr. Lars Metzger
Zweitreferent:	Prof. Wolfgang Leininger, Ph.D.

To my parents, Ursula and Manfred

Publications and Presentations

Publications

An earlier version of Chapter 3 appeared in the Ruhr Economic Papers series as Number 460 under the title "The Recommendation Effect in the Hotelling Game - A New Result for an Old Model". An earlier version of Chapter 4 appeared in the Ruhr Economic Papers series as Number 666 under the title "The Different Effect of Consumer Learning on Incentives to Differentiate in Cournot and Bertrand Competition".

Presentations

This thesis benefitted from many insightful comments and helpful discussions at several conferences. The following is a summary of national and international conference presentations.

Chapter 1 will be presented at

- Econometric Society European Winter Meeting, Naples, Italy (2018).

Chapter 2 was presented at

- Contests: Theory and Evidence, Norwich, United Kingdom (2018),
- International Conference on Game Theory, Stony Brook, USA (2018).

Chapter 3 was presented at

- Spring Meeting of Young Economists, Halle, Germany (2017),
- World Congress of the Game Theory Society, Maastricht, The Netherlands (2016),

- European Economic Association Annual Conference, Mannheim, Germany (2015),
- Augustin Cournot Doctoral Days, Strasbourg, France (2015),
- International Conference on Game Theory, Stony Brook, USA (2014),
- European Meeting on Game Theory [SING 10], Krakow, Poland (2014),
- Royal Economic Society Annual Conference, Manchester, United Kingdom (2014).

Chapter 4 was presented at

- Ruhr Graduate School in Economics Doctoral Conference, Dortmund, Germany (2017).

Chapter 5 was presented at

- Workshop on Cooperative Game Theory in Business Practice, Leipzig, Germany (2016),
- European Meeting on Game Theory [SING 11] and Conference on Game Theory and Management, St. Petersburg, Russia (2015),
- Workshop on Economics and Biology, Toulouse, France (2015),
- Research Seminar at the HHL Graduate School of Management, Leipzig, Germany (2015).

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The shortest poem in the English language is accredited to Muhammad Ali:

Me, We.

Most I want to thank the person who first comes to my mind, when I hear that poem. Teşekkürler, Tuğba, hem iyi hem kötü günlerdeki sevgin, desteğin, sabrın, anlayışın ve yüzümü güldürdüğün sayısız an için. İyi ki varsın.

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Introduction

In many situations, economic agents are heterogeneous. In economic terms, we say that these agents are of different *types*. When consumers go to the movies, they are likely to be *informed* to varying degrees about the quality of the films shown in cinemas. This may, for example, be the case because some types of movie-goers spend more time reading review articles than others. In a firm, the cooperation of different "worker types" may be *productive* to varying degrees. For instance, a firm consisting only of managers or only of workers is usually less productive than a firm which is constituted by some mixture of workers and managers in between those two extremes.

This thesis examines how the heterogeneity of economic agents impacts the shape and design of economic institutions in industrial and societal organization. Part I focusses on the heterogeneity of economic types arising from the fact that *information* is distributed asymmetrically, while Part II focusses on the heterogeneity of economic types arising from different *productive* abilities. Clearly, these distinctions are often intertwined. For instance, buyers of energy-efficient cars may be heterogeneous in how *productive* they are in obtaining extra utility from energy savings. This may be because the annual mileage varies across the population. *Information* about the distance driven per year is known only to the consumers themselves. A government which wants to impose a subsidy scheme for energy-efficient cars needs to take this informational asymmetry regarding the productive heterogeneity among the consumers into account.

This introduction serves to define the main concepts (indicated in **bold letters**) and paradigms used throughout the thesis. It adds a broader perspective to the research presented in the subsequent chapters and highlights its contribution to the literature. An outlook on future research can be found in the conclusions at the end of each chapter.

PART I: OVERVIEW

Informational asymmetries shape a huge variety of economic interactions. Governments have to work out how to obtain information on the above mentioned consumers' productivity (or equivalently, their preferences) when, for example, designing subsidy schemes for energy-efficient cars (see Chapter 1). A hiring committee of a university has to consider which information about its selection criteria for a professorship it should optimally disclose to potential applicants (see Chapter 2). Since many consumers, often termed "laggards", observe the consumption behavior of "early adopters", firms - such as film studios - have to consider the information that is revealed by the early adopters' consumption decisions (see Chapters 3 and 4). All of these situations can be analyzed from an economic design perspective. An economic designer (the government, the hiring committee, the film studio) shapes the economic environment and institutions in which the agents (buyers of energy efficient cars, applicants for a professorship, movie goers) later on interact. The designer does so to achieve a specific goal, such as welfare maximization in a population of car owners, effort maximization by applicants for professorships, or revenue maximization by obtaining a large audience in movie theaters.

All models in Part I of this thesis are characterized by asymmetric information and examine issues of economic design. Such settings can appropriately be analyzed using tools from non-cooperative game theory and from information economics, which we introduce here.

PART I: CONCEPTS

Rational Choice Theory: A Paradigm for Individual Decisions

Standard microeconomic theory analyzes economic and other social interactions under the primacy of methodological individualism.[1,2] Typically, a fundamental building block of microeconomic models is a group of individuals N, consisting either of finitely many individuals (e.g., $N = \{1, 2, ..., n\} \subset \mathbb{N}$), or of infinitely many individuals (e.g., $N = [a, b]$, with $a, b \in \mathbb{R}$, $a < b$, usually $a = 0, b = 1$). For now, we assume that N is finite. Each specific individual $i \in N$ from that group faces a choice set X_i from which he has

[1] Methodological individualism is not undebated in economics, see Gylys (2017) and the sources quoted therein. Chapter 5 discusses eusocial species, which can be considered as an example illustrating that group (and individual) behavior is not always derived from individual rationality. The influential evolutionary theory of the selfish gene presented by Dawkins (1976) is a further alternative to methodological individualism.

[2] Since the spread of the now predominant dynamic stochastic general equilibrium (DSGE) models, standard macroeconomic theory, too, pursues a micro-founded approach based on methodological individualism, see Ljungqvist and Sargent (2000).

to select an action x_i. Rational choice theory developed an approach to systematically tackle such setups within a unified framework.[3] It is based on the axiom of **rationality**, which dictates that the preference relation $\succsim_i$ of an individual i over the elements of the choice set X_i is specifically well-behaved.[4,5] A (Bernoulli) **utility function** $u_i : X_i \to \mathbb{R}$ of individual i can be used to represent the individual's preference relation $\succsim_i$. There are important relations between the rationality of preferences and the existence of utility functions. For instance, utility functions for a preference relation exist (and then are unique up to monotone transformations) only if that relation is rational. Imposing the additional property of continuity onto the preference relation guarantees the existence of a continuous utility function.[6,7] Assuming that individuals seek to maximize their utility implies that the standard individual decision problem of a microeconomic agent can be described by the following problem of constrained mathematical programming:[8]

$$\max_{x_i \in X_i} u_i(x_i). \tag{0.1}$$

These problems can be solved using the Kuhn-Tucker approach of non-linear programming. Usually, an individual faces the choice set $X_i(p, w) \subseteq \mathbb{R}^m$, where $w \in \mathbb{R}$ is the individual's exogenous wealth and $p \in \mathbb{R}^m$ captures the exogenous costs of the choice in each of the m dimensions of the choice set.

Ever since the advancement of rational choice theory, economic scholars have been debating about its empirical validity, as well as the theoretical issues arising from the assumptions of rationality and utility maximization. The literature investigating alternative behavioral paradigms is now subsumed under the term **behavioral economics**. However, due to its advantages concerning tractability compared to many behavioral

[3] Amadae (2003) offers intriguing insights on the intellectual history of rational choice theory and its impact on scientific, political, and social life.

[4] In particular, $\succsim_i$ induces a partial order among the elements of X_i by satisfying the axioms of completeness and transitivity. Preferences $\succsim_i$ are complete iff $x \succsim_i y$, or $x \precsim_i y$, or both for all $x, y \in X_i$ and $\succsim_i$ is transitive iff $x \succsim_i y$ and $y \succsim_i z$ implies $x \succsim_i z$ for all $x, y, z \in X_i$.

[5] One of the most important results of social choice theory proves that the aggregation of individually rational preferences does not necessarily lead to rational preferences on the aggregate level, see Arrow (1950).

[6] Preference relation $\succsim_i$ is continuous at $x \in X$ iff the upper contour set $\{y \in X_i : y \succsim_i x\}$ and the lower contour set $\{y \in X_i : y \precsim_i x\}$ are both closed.

[7] The application of mathematical tools from calculus in economic theory is often based on the tractability of such models, captured usually by the continuity and differentiability of the objective functions, and it reflects the idea that "natura non facit saltus". However, discontinuities are not a marginal phenomenon in economic theory, see Rosser (1991). The models in Chapter 3 and Chapter 5 exhibit discontinuities. In the former, the objective functions of the firms are discontinuous; in the latter, a dynamical system evolves according to differential equation with a discontinuous right-hand side.

[8] We wish to emphasize that this is the *standard* choice problem of a rational microeconomic agent whose utility is independent of the choices of other agents. Nevertheless, rational choice theory also allows us to model interdependent preferences; but this approach to interdependence should be distinguished from the game-theoretic approach to *strategic* interdependence described below.

models - and, more importantly, due to its universal applicability - the rational-choice-theory paradigm of individual behavior remains prevalent in both major approaches to modeling economic and social "interactions", general equilibrium theory, and game theory, which we describe below.

General Equilibrium Theory: A Paradigm for Parametric Interaction

Above, we describe how an individual $i \in N$ autonomously decides under the paradigm of rational choice. We now proceed by introducing general equilibrium theory, which has been the predominant paradigm to model market decisions and market equilibria in economics since it was intiated by Smith (1776) and Walras (1874). A complete formal presentation of this paradigm is outside of the scope of this thesis. Nevertheless, there are two important points to make. Firstly, general equilibrium theory models interaction among individuals in a parametric way, specifically through prices. In a general equilibrium of a market economy, prices for each of the m goods, captured by the price vector p, are such that consumers' demand equals firms' supply. The process of how these equilibrium prices originate, for instance in a bargaining procedure among firms and consumers, is not modeled explicitly. In contrast to this, it is assumed that a Walrasian auctioneer, personalizing Adam Smith's idea of the "invisible hand", sets prices in a way such that an equilibrium emerges. Hence, the decision problem of all individuals $i \in N$ in this environment - taking prices as exogenous - can be described by Equation (0.1).[9]

Secondly, the popularity of the general-equilibrium-theory approach is rooted in the fundamental theorems of welfare economics, which establish that equilibria in general equilibrium models exhibit desirable welfare properties in relation to Pareto efficiency. However, these results crucially rely on three further - often implicitly stated - assumptions: the standard general equilibrium model abstracts from externalities, and there is neither asymmetric information nor market power among the agents.

The thesis at hand elaborates on all of these three aspects explicitly: in all chapters of Part I, agents inflict externalities upon each other and these models additionally contain aspects of asymmetric information. Furthermore, Chapter 3 and Chapter 4 employ benchmark models of market power. Below, we explain why the game-theoretic paradigm of interaction is the adequate framework to examine these settings of interest.

[9] General equilibrium theory distinguishes between two main economic actors: consumers and producers. The producers maximize profit under the constraint of their production technology. However, with some minor exceptions, the mathematical tools of constrained mathematical programming for the consumer side mentioned in this introduction can equivalently be applied to the producer side as well.

Non-Cooperative Game Theory: A Paradigm for Interactive Decisions

A scene in the movie "A Beautiful Mind" iconically pinpoints John Nash's "Copernican moment", which hallmarks the paradigm shift from general equilibrium to game theory, in Nash's, alias Russell Crowe's, exclamation that "Adam Smith was wrong!"[10,11] Game theory supersedes the parametric interaction approach of general equilibrium theory by explicitly modeling the dependence of each agent i's utility on the action of all the other agents in the model, which are denoted by $-i := N\backslash\{i\}$. Therefore, the standard optimization problem of individual i in game theory is depicted by

$$\max_{x_i \in X_i} u_i(x_i, x_{-i}). \tag{0.2}$$

This implies that game theory, in contrast to general equilibrium theory, is decision-theoretically closed and allows for an explicit analysis of social dilemmata which arise due to externalities. It is *the* theory of externalities, since it accounts for the impact of an agent's actions on others.

We proceed with the following instrumental definitions originating from the seminal works of von Neumann and Morgenstern (1944) and Nash (1951). A **normal form game** $G := (N, X, U)$ is given by the set of players N, with elements indexed i, the (pure) strategy space $X := \times_{i \in N} X_i$, and the payoff structure $U := (u_i)_{i \in N}$ with $u_i : X \to \mathbb{R}$ for all $i \in N$. The mixed strategy space of individual i, given by ΔX_i, induces a probability distribution over his pure strategies. Note that all models in this thesis are analyzed in pure strategies.

The **best response** of player $i \in N$ to the other players' actions $x_{-i} \in X_{-i} := \times_{j \in N, j \neq i} X_j$ in pure strategies (and, analogously, for mixed strategies) is given by the correspondence $x_i : X_{-i} \rightrightarrows X_i$ with

$$x_i(x_{-i}) := \arg\max_{x_i \in X_i} u(x_i, x_{-i}). \tag{0.3}$$

Quasi-concavity of the utility function in x_i guarantees the existence and uniqueness of the best response.

A pure-strategy **Nash equilibrium** of game G is given by $x^* := (x_1^*, ..., x_{|N|}^*)$ such that

$$u_i(x_i^*, x_{-i}^*) \geq u_i(x_i', x_{-i}^*) \quad \forall x_i' \in X_i, \ \forall i \in N. \tag{0.4}$$

[10] Ironically, just as Adam Smith was wrong according to Nash, the movie is wrong as well in depicting the appropriate intuition of a Nash equilibrium in exactly this scene.

[11] Amadae (2015) investigates the impact of game theory in social science and political economy and puts its development into a historical context.

A Nash equilibrium in mixed strategies is defined analogously. In a Nash equilibrium, no player has a unilateral incentive to deviate to another strategy, as all players behave according to their best responses. Mathematically, finding a Nash equilibrium boils down to determining the intersection of the best responses. As the seminal work of Nash (1951) points out, existence of (in this case mixed-strategy) Nash equilibria can be proven by applying appropriate fixed-point theorems. These theorems place (usually rather weak) restrictions on the strategy space X_i (or ΔX_i in the case of mixed strategies). For instance, Brouwer's or Kakutani's fixed point theorems demand the strategy space to be a compact and convex set, which is always fulfilled for mixed strategy spaces defined on finite pure strategy spaces.

The taxonomy of Eaton (2004) is a powerful tool to categorize games according to two dimensions, which reflect important economic intuitions. The first dimension is that of externalities. Defining $x := (x_1, ..., x_{|N|}) \in X$, **positive externalities** (of x_j on player i, $i \neq j$) imply $\frac{\partial u_i(x)}{\partial x_j} > 0$ and **negative externalities** imply $\frac{\partial u_i(x)}{\partial x_j} < 0$.[12] This distinction describes how agent i's utility changes in the action of player j. The second dimension is that of strategic incentives, which is a particular focus of the theory of supermodular games and monotone comparative statics. Assume that u_i is increasing and concave in x_i. **Strategic complements** (between x_j and x_i for player i, $i \neq j$) imply $\frac{\partial u_i(x)}{\partial x_i \partial x_j} > 0$ and **strategic substitutes** imply $\frac{\partial u_i(x)}{\partial x_i \partial x_j} < 0$. This distinction depicts how i's utility changes in the action of player j, if i himself varies his choice. The definitions imply that, under strategic complements, best responses increase, i.e., in a two-player game, with $i \neq j$, we have $\frac{\partial x_i(x_j)}{\partial x_j} > 0$. Obviously, under strategic substitutes best responses decrease, i.e., $\frac{\partial x_i(x_j)}{\partial x_j} < 0, i \neq j$.

The following table depicts some canonical games to illustrate these fundamental concepts.[13] All games depict a different social dilemma and in some of them the numerical value of the strategy name, "0" or "1", can be interpreted as the level of aggression of that action against the opponent.

[12] All definitions in this paragraph can be adapted for discrete action spaces using non-marginal changes Δy instead of marginal changes ∂y. Eaton (2004) calls positive (negative) externalities plain complements (substitutes).

[13] In these examples, action spaces are discrete and hence we speak of so-called finite games. The action spaces of such games can be "made continuous" by introducing mixed strategies.

P1 \ P2	0	1
0	$2,2$	$0,\underline{3}$
1	$\underline{3},0$	$\boxed{\underline{1},\underline{1}}$

Prisoners' Dilemma
(Individual vs. Group)

P1 \ P2	0	1
0	$\boxed{\underline{2},\underline{2}}$	$0,1$
1	$1,0$	$\boxed{\underline{1},\underline{1}}$

Stag Hunt
(Safety vs. Coord.)

P1 \ P2	0	1
0	$2,2$	$\boxed{\underline{1},\underline{3}}$
1	$\boxed{\underline{3},\underline{1}}$	$0,0$

Game of Chicken
(Conflict in Anticoord.)

P1 \ P2	0	1
0	$\underline{1},-1$	$-1,\underline{1}$
1	$-1,\underline{1}$	$\underline{1},-1$

Penalty Shootout
(Pure Conflict)

Figure 1: Canonical 2×2 Normal Form Games. Choice sets for the players P1 and P2 are given by $X_{\mathrm{P1}} = X_{\mathrm{P2}} = \{0, 1\}$ and the utility functions are defined by the entries in the matrices (the payoff of P1 is the first entry in a cell, and the payoff of P2 is the second entry in a cell). The social dilemmata are mentioned in brackets. Payoffs of best responses are underlined and payoffs of Nash equilibria (in pure strategies) are boxed.

Coordination games, such as the Stag Hunt Game, are games in which both players' utilities imply strategic complements. Anti-coordination games, such as the Game of Chicken, are games of strategic substitutes for both players. In discoordination games, such as the Penalty Shootout, one player has strategic substitutes, while his opponent has strategic complements. The Prisoners' Dilemma has a dominant strategy, namely "1", for each player.[14] Stag Hunt and the Game of Chicken exhibit negative externalities. Positive externalities can be found in public good provision games, for instance, which, with respect to Table 1, can be deduced from the Game of Chicken by "switching the names of the strategies", "0" and "1" (note that in this context the names of the strategies have a meaning with respect to the difference Δx_i).

An important implicit informational assumption in game theory is that of **common knowledge** of rationality: each player knows that all players in the game are rational, and each player knows that each player knows that each player is rational, and so on. Further assumptions on the information available to the players are discussed below. First, however, we discuss **games of perfect information**. Intuitively, such games depict situations in which no player has any uncertainty about the strategic situation he is in. For instance, he exactly knows which types of players he faces, what the rules of the game are, and what happened previously to his move(s).

All games in Part I are **dynamic games** and thus demand an extension of the framework presented above. Often dynamic, intertemporal decisions involve hyperbolic discounting of future payoffs, as for instance employed in Chapter 4. Dynamic strategic interactions can be modeled by characterizing the appropriate extensive form of such situations. For the sake of brevity, we omit the exact definition of an extensive form

[14] A dominant strategy is defined as a strategy which is a best response to any strategy of the opponent.

game, a subgame, and information sets here. They can be found, for instance, in Maschler et al. (2013, pp. 43). Informally, the **extensive form** Γ of a game specifies the order of moves of players and their information at any given point of the game. A **subgame** roughly comprises a specific decision node in the game such that every player knows that this is the true current decision node, and furthermore a subgame comprises all events or decision nodes which follow in subsequent periods of the game. An **information set** roughly depicts the information available to an agent at a certain decision node. In games of perfect information, all information sets are singletons consisting of a unique decision node.

The most common refinement of the Nash equilibrium suitable for dynamic games with perfect information is the **subgame-perfect (Nash) equilibrium**, which is a set of strategies in game Γ that induces a Nash equilibrium in every subgame of Γ. This concept was developed by Selten (1975) and employs the same insight as the Bellman principle in dynamic optimization: at any decision node of the game, an agent should act optimally (even if the decision node is never reached, i.e., if it is "off-the-equilibrium-path"), taking into account the events which follow on from his decision. In dynamic perfect-information games with a finite time horizon, the subgame-perfect equilibria can be found by the **principle of backward induction**. It suggests solving a game by analyzing the optimal behavior in the subgames starting at the terminal decision nodes and then working through the game tree towards its root, by replacing subgames with the payoffs from optimal play in these subgames.

Product Differentiation

Chapter 3 and Chapter 4 analyze observational social learning in models of product differentiation. However, we examine the issue using two different modeling approaches to product differentiation. To explore this difference, we now provide a broader picture of this topic.

From a theoretical perspective, the idea of product differentiation was introduced mainly in order to cope with the Bertrand paradox, which states that firms with equal marginal costs make zero profit under oligopolistic price competition, even when only two firms compete. By allowing for product differentiation in a Bertrand model, firms may obtain local market power and secure strictly positive profits since they are able to set prices above marginal costs. Additionally, the analysis of product differentiation deals with a general question relevant in any kind of competition: do contestants become more similar (imitation) or more different (differentiation) when they act optimally in a competitive game? Another topic of this literature is examining whether, in market equilibrium, sufficient or excessive (product) diversity is provided.[15]

There are two types of differentiation in economic theory, both of which are present in Chapters 3 and 4. Vertical differentiation is related to product quality: given the same prices, all consumers prefer the same good, namely the higher-quality good.[16] Horizontal differentiation addresses a heterogeneity in tastes among consumers: given the same prices, different consumers prefer different goods, namely each of them prefers the one which best matches his individual preferences.[17]

Offering horizontally differentiated products is profitable if there is demand for different products. The consumer side therefore needs to be modeled accordingly. Chapter 3 pursues the "love-of-proximity approach", also called the address, location, or characteristics approach. There are several heterogeneous consumers, each with an individual ideal consumption point. Each consumer only demands one unit of exactly one of several goods (unit demand), which can be interpreted as a short term perspective (a certain amount of money can only be spent on one of many goods) or the consumption of durable goods, such as cars. The preferences are defined over the characteristics of the goods. In the standard models of this approach, differentiation is usually endogenous and firms interact in oligopolistic competition. Finally, the "love-of-proximity approach" naturally allows modeling of asymmetric differentiation; this makes it easy to distinguish niche and mainstream products.

Chapter 4, in contrast, pursues the "love-of-variety approach",[18] also called representative-consumer approach, in which one representative consumer is equipped with preferences over bundles of goods. In contrast to the "love-of-proximity approach", in this setting, the consumer demands bundles of goods which can be interpreted as a long-term decision perspective. He has to decide on the quantity of each specific good to consume. In the standard version of these models, differentiation is mostly exogenous. Some models characterize the firm side by monopolistic competition and consumers by having CES-utility.[19] These models without strategic interaction mainly analyze whether the market provides the socially beneficial number of products, or equivalently brands, i.e., it examines issues of optimal diversity. Other models characterize the firm side using oligopolistic competition, e.g., a differentiated duopoly, and the consumer

[15] Page (2008) discusses the pitfalls and advantages of a diversity in society in a broader perspective.

[16] See Gabszewicz and Thisse (1986) and Shaked and Sutton (1982).

[17] See Hotelling (1929) and d'Aspremont et al. (1979).

[18] In neoclassical economics, the agents' love of variety is reflected by the convexity of their preferences. Convexity is an important assumption in many fields of economic theory. With respect to mathematical programming, it usually guarantees that the second-order conditions of the maximization problems are satisfied. Probability simplices, which are ubiquitous in information economics, are also convex sets. In these setups, we frequently employ tools of convex analysis and convex optimization.

[19] See Chamberlin (1933), Spence (1976), and Dixit and Stiglitz (1977).

side exhibits linear-quadratic utility.[20] The latter is the love-of-variety approach we apply in Chapter 4. These kinds of models include strategic interaction among firms. They address the Bertrand paradox and compare the social efficiency of the two main models of oligopolistic competition, Bertrand and Cournot competition.[21] In contrast to the love-of-proximity approach, the love-of-variety approach does usually not allow modeling of asymmetric differentiation, i.e., in these models, it is difficult to distinguish between niche and mainstream firms.

The love-of-proximity approach and the differentiated duopoly models of the love-of-variety approach both generate linear demand functions.[22] However, the main technical difference is reflected in the cross elasticities: in love-of-proximity models, consumers would always buy the same product twice if they could (goods are always substitutes; the marginal utility is independent of the amount of the goods consumed), while in the love-of-variety approach, consumers always want to purchase positive quantities of both goods (goods can have different degrees of substitutability or complementarity).

Bayesian Rationality: A Paradigm for the Individual Processing of Information

Random events in economic models usually reflect the fact that the economic agents have **imperfect information** about their environment or the consequences of their actions. Standard theory follows the idea that the agents' beliefs assign subjective probabilities to each possible event or state and that these probabilities satisfy Kolmogorov's axioms.[23]

For instance, let $T : \Omega \to \mathcal{T} \subset \mathbb{R}$ be a discrete, real-valued random variable over the (finite) sample space Ω. The probability $P(T = t)$, i.e., the probability of the realization of T being t, often abbreviated to $P(t)$, is then given using the probability mass function $p_T : \mathcal{T} \to [0, 1]$, with

$$P(T = t) = p_T(t) := P(\{\omega \in \Omega : T(\omega) = t\}). \tag{0.5}$$

Settings with continuous random variables can be analyzed using analogous concepts, such as the density function and the cumulative distribution function.[24]

[20] See Dixit (1979) and Singh and Vives (1984).

[21] Cournot (1838) and Bertrand (1838) can be viewed as the early forefathers of game theory.

[22] Unifying approaches can be found in Perloff and Salop (1985) and Anderson et al. (1992). An approach using networks of substitutability and complementarity between the firms' products can be found in Ushchev and Zenou (2016).

[23] Kolmogorov's three axioms are that 1) the probability of an event is a non-negative real number, 2) unit measure, i.e., the probability that at least one event in the sample space occurs is one, and 3) (sigma) additivity. For details see Gut (2005, pp. 10).

[24] We do not discuss probability spaces and measurability in detail here for the sake of brevity. Details can be found in Gut (2005) and Bertsekas and Tsitsiklis (2008).

Being able to apply standard probability theory is particularly useful when modeling situations in which agents receive new information and adjust their beliefs accordingly. Ross (2007, p. 97) states that, precisely because of its applicability to such settings, the concepts of conditional probability and the related conditional expectation are among the most useful concepts in probability theory. In Part I, we primarily keep to the standard approach and assume that agents are Bayes rational. This implies that they process new information by updating their beliefs, i.e., the subjective probabilities, according to Bayes' rule, which we explain now.

Consider vector (T_i, T), sample space Ω, and $\mathcal{T}_i \times \mathcal{T} \subset \mathbb{R}^2$. Then, let $(T_i, T) : \Omega \to \mathcal{T}_i \times \mathcal{T}$ be a random vector consisting of the two discrete random variables T_i and T. These are completely specified by their joint probability mass function $p_{T_i,T}(t_i, t) = P(T_i = t_i, T = t)$ for all $t_i \in \mathcal{T}_i$, $t \in \mathcal{T}$. By the law of total probability, the marginal probability mass function of T_i (and T analogously) is given by $p_{T_i}(t_i) = \sum_{t \in \mathcal{T}} p_{T_i,T}(t_i, t)$ for $t_i \in \mathcal{T}_i$. As above, we write $P(t_i) = p_{T_i}(t_i)$, and call this the **prior** of the random variable T_i (for T analogously). Note that most economic models assume that agents share a common prior. For $t \in \mathcal{T}$, assuming that $P(t_i) > 0$, we define $P(t|t_i) = p_T(t|T_i = t_i) := \frac{p_{T_i,T}(t_i,t)}{p_{T_i}(t_i)} = \frac{P(t_i,t)}{P(t_i)}$ to be the (conditional) probability of random variable T conditional on the realization $T_i = t_i$. A simple version of **Bayes' rule** is given by[25]

$$P(t|t_i) = P(t) \cdot \frac{P(t_i|t)}{P(t_i)}. \tag{0.6}$$

Assume that an agent $i \in N$ observes a specific signal realization $T_i = t_i$, which is potentially correlated with T. Then, to establish his belief incorporating the new information of the signal, according to Bayes' rule, an agent updates the prior $P(t)$ using the so-called Bayesian multiplier $\frac{P(t_i|t)}{P(t_i)}$.

Bayesian rationality parallels the concept of rationality in decision making and provides a tractable method for modeling information processing. However, the updating process depicted in Equation (0.6) can have counterintuitive implications, as for instance the famous Monty Hall problem illustrates, see Palacios-Huerta (2003). Because of this, similarly to rationality, the plausibility of Bayesian rationality to model human behavior has been a matter of debate, see Binmore (2017), and alternatives have been suggested, see Epstein et al. (2010) and Eyster and Rabin (2005). Nevertheless, similarly to rationality, due to its mathematical tractability and universal applicability, Bayesian rationality remains the workhorse model of information processing in information economics.

[25] Bayes' rule immediately follows from the law of total probability, see Gut (2005, pp. 18). A version of Bayes' rule for multiple events can be derived using the same tools. A version for continuous random variables can be found in Chapter 5.

Social learning deals with the implications of Bayesian rationality on the processing of information in groups. In a setting with an uncertain state, agents observe the behavior of others or communicate with them. Then, they update their belief about the state of nature and act accordingly. Central results in this literature are the formation of information cascades (Bayes rational agents may ignore their own information) and of informational herds (from a certain point onwards, all agents pursue the same action, even though this might be the least beneficial for each of them), see Bikhchandani et al. (1992) and Banerjee (1992).

Information Economics

> *Information is not an ordinary commodity like apples or pears, the use of which alone should be decided upon by market logic. Rather, information creates worldviews and opinions. Google and Facebook have increasingly become the most important windows into the world. They decide how we see things and what things we do not see. (Thomas Beschorner and Martin Kolmar, economists at the University St. Gallen)*[26]

This quote addresses two important issues. First, it emphasizes the prevalent role of information in the economy in times of "big data". There is an ongoing debate on whether "information is the new oil".[27] Undoubtedly, Facebook's most valuable asset is the information it obtains about its customers' preferences, their digital fingerprints, and their social network.

Second, the quote hints at the fact that information is not a typical good. While its producers incur high fixed costs, variable (reproduction) costs are very low. The design of the internet platform Facebook as it is today required many hours of programming, but the data collected on it can be duplicated at almost no cost. The same holds true for many technological inventions and other creative work, such as music. Part I analyzes models in which information is disclosed to the agents without any explicit costs. Instead, a random event, a signal, determines which agent obtains what information, so that players in the game may have different information about their private characteristics, for instance their types, or preferences, or the state of nature.

Since all games in Part I entail asymmetric information, and thus uncertainty, an extension of the game theoretic framework presented above is required at this point.

[26] See http://www.zeit.de/wirtschaft/2018-03/plattformkapitalismus-internetplattformen-regulierung-facebook-cambridge-analytica/komplettansicht (author's translation, last accessed: 28/03/2018)

[27] See for instance https://www.economist.com/news/leaders/21721656-data-economy-demands-new-approach-antitrust-rules-worlds-most-valuable-resource (last accessed: 15/04/2018).

Agents are assumed to be Bayesian rational **expected utility** maximizers,[28] that is an agent $i \in N$ solves

$$\max_{x_i \in X_i} E[u_i(x_i, x_{-i})] := \sum_{t \in \mathcal{T}} \beta_i^t u_i(x_i, x_{-i}(t), t), \tag{0.7}$$

where $t \in \mathcal{T}$ is one of finitely many states of the world and β_i^t is the subjective probability agent i assigns to state t given his belief $\beta_i : \mathcal{T} \to \Delta\mathcal{T}$.[29] Belief β_i may be obtained through an updating process using Bayes' rule as we will see below. The notation indicates that the Bernoulli utility as well as the opponents' actions may vary across states.

The famous Harsanyi transformation, introduced in Harsanyi (1967), allows us to transform **games of incomplete information** into **games of complete, but imperfect information**.[30] In contrast to games of perfect information, information sets in games of imperfect information need not be singletons. Such strategic situations can be analyzed using the tools described in the following.

A **static Bayesian game** $\tilde{G} := (N, X, \mathcal{T}, p, U)$ is given by the set of players N and the (pure) strategy space X defined as before, the type space $\mathcal{T} := \times_i \mathcal{T}_i$ with $i \in N$, a (prior) probability distribution $p_T : \mathcal{T} \to \Delta\mathcal{T}$ over the type space with $t \mapsto p_T(t)$ and $t \in \mathcal{T}$, and the payoff structure $U = (u_i)_{i \in N}$ with $u_i : X \times \mathcal{T} \to \mathbb{R}$. Importantly, a pure strategy of a player now specifies an action for each of his types, $x_i : \mathcal{T}_i \to X_i$. Mixed strategies can be defined, similarly as before, as probability distributions over the pure strategies. The framework captures discrete as well as continuous type spaces.

The idea of a Bayesian game is that nature first draws one realization $T = t \in \mathcal{T}$ from the type space, and then each player in the game updates his belief about the state of nature, captured in the realization of $T = t$, according to the realization of his type

[28] Expected utility, also called von-Neumann-Morgenstern utility, originated in von Neumann and Morgenstern (1944), who establish that the linearity of the expected utility form is fundamentally related to the axiom of independence of irrelevant alternatives. Allais (1953) was the first to demonstrate that, empirically, humans might act contrary to the predictions of expected utility theory.

[29] An analogous version of expected utility for a continuum of (infinitely many) states can be found in Chapter 4.

[30] Generally speaking, games of incomplete information are characterized by the fact that (some) players do not know the rules of the game, i.e., they do not know the extensive form of the game. This may be the case, for example, if the players do not know anything about the payoffs or types of other players in the game. The Harsanyi transformation overcomes implied technical difficulties by introducing an additional player, often called "nature", who draws the players' types (and other relevant random events of nature, which can be interpreted as types of the player "nature") according to specified probability distributions. Since these probability distributions are assumed to be known to all players, each of them knows the extensive form of the game (i.e., the rules of the game) and thus it is a situation of complete information. However, since some players do not observe the realized moves made by nature, the information is imperfect. For details consider Harsanyi (1967).

$T_i = t_i$, using Bayes' rule: $\beta_i : \mathcal{T}_i \to \Delta\mathcal{T}$ with $t_i \mapsto \beta_i(t_i) := p_T(t|T_i = t_i)$. A **Bayesian (Nash) equilibrium** of the game $\tilde{G}$ is a vector of strategies x^* such that for all $i \in N$:

$$E\left[u_i(x^*(T), T)|T_i = t_i\right] \geq E\left[u_i(x_i'(T_i), x_{-i}^*(T), T)|T_i = t_i\right], \quad \forall x_i' \in X_i,\ \forall t_i \in \mathcal{T}_i. \quad (0.8)$$

Note that the expected utilities are defined from an interim-stage perspective using conditional expectations, i.e., each player knows the realization of his type. Additionally, the equilibrium concept specifies type-dependent equilibrium actions.

Since all games in Part I are **dynamic Bayesian games** (where actions from previous periods are observable), the common solution concept suitable to be applied in all models presented here is the **perfect Bayesian (Nash) equilibrium**, which combines the ideas of subgame perfection and Bayesian Nash equilibrium. It specifies the players' strategies *and* beliefs. Strategies need to be sequentially rational, i.e., at each decision node they are optimal in expectation given the players' beliefs. Furthermore, beliefs need to be updated according to Bayes' rule wherever possible, and off-equilibrium-path beliefs may be arbitrary. In dynamic Bayesian games with observable actions (i.e., games in which all players are only uncertain about the other players' types, but not the other players' moves) the perfect Bayesian equilibrium coincides with the widely spread refinement called sequential equilibrium.[31]

Signaling games are an important class of Bayesian games, in which an informed sender can disclose information to a receiver. However, in none of the games presented in Part I does the first-mover in the game have this kind of information advantage at the time of his decision; as such, signaling is not an issue in this thesis.

Economic Design

Economic design analyzes situations in which one agent, the designer (*she*), is able to modify (part of) the economic environment in such a way that other agents, e.g. consumers (all *he*), are more inclined to act in her interest. In most cases, it overlaps with the economics of information because there is an information asymmetry between the designer and the agents.

Market design, which includes **mechanism design**, analyzes situations in which the designer can directly manipulate payoff-relevant aspects of the setup, for instance prices or taxes.[32] This implies changes in the marginal utilities of the affected agents.

[31]For details consider, for instance, Osborne and Rubinstein (1994, pp. 231). The literature has discussed refinements of these concepts which, for instance, specify alternative conditions for off-equilibrium-path beliefs, see also Fudenberg and Tirole (1991, chapter 8).

[32]A distinction in market design is usually made with respect to whether money is explicitly modeled in the agents' utility functions or not. Matching and voting theory analyze settings in which money is not included in the model, while mechanism design explicitly models monetary payoffs.

Information design, in contrast, comprises models in which the designer manipulates the agent's information environment, which is only indirectly relevant to payoffs and instead implies changes in the agents' beliefs.

Typically, information asymmetry in market design consists of the fact that the agents have an informational advantage compared to the designer since they have better knowledge of their own private preferences. The designer has to take this into account. One of the main issues for her is thus to find a design which induces the agents to truthfully reveal their private information. In information design, the information advantage usually is on the designer's side, since she "implicitly" knows the state of nature and has to think about how to optimally disclose or conceal information to the agents.[33]

In market design, the designer typically faces the constraint of having to design an *incentive compatible* - or equivalently strategy-proof - mechanism, while also considering the *participation constraints* of the agents. Incentive compatibility requires that agents truthfully reveal their private information, e.g., their type, (details are presented in Chapter 1). In contrast to this, an information designer faces the constraint that the optimal mechanism should be credible, which translates to the technical conditions of *Bayes plausibility* and *consistency* (details are presented in Chapter 2).

In view of the discussion of economic design from above, mechanism design is also called "reverse game theory": the designer structures the game played among the agents. The methodology of mechanism design can be applied to various different settings, reflected by the fact that economic entities have different objectives. In a typical problem of **public economics**, the objective of a benevolent social planner - for instance when deciding on an optimal tax scheme - is the maximization of social welfare (see Chapter 1). In a typical problem of **industrial organization**, the objective of a firm is the maximization of profit (see Chapters 2 to 4). Nonetheless, both problems can often be solved using the same tools from mechanism design.

PART I: CONTRIBUTIONS

We will now discuss how Part I contributes to the theory of economic design.

[33] Actually, at the time of her decision, the information designer has no information advantage about the state of nature. However, the idea is, that when she communicates information to the other players, she knows the state of nature and discloses information according to a design, which she has credibly committed to before she knows the realization of the state of the world. These issues are discussed in detail in Chapter 2.

Mechanism Design with Non-Bayesian Agents: Correct Me If You Can - Optimal Non-Linear Taxation of Internalities (Chapter 1)

In this chapter, we contribute to the growing research in behavioral public economics by analyzing a mechanism design setting with behavioral consumers. In particular, we examine non-linear commodity taxation in a setting in which consumers systematically misperceive the value of a product characteristic. This is a novel approach to investigating corrective instruments which counteract the negative welfare effects of what are referred to as "internalities", i.e., systematic misperceptions that bias consumers' choices and reduce their welfare

Previous evidence has shown that such behavioral failures affect consumer choices in many settings, such as energy efficiency investments, e.g., concerning vehicles or housing, or the consumption of sugar-sweetened beverages. In this chapter, we characterize the optimal nonlinear tax (or subsidy) for correcting behaviorally biased consumers. Using representative examples we show that it is welfare-enhancing compared to no taxation and linear taxation. We examine how the amount of the welfare improvement delivered depends on informational restrictions affecting the mechanism designer.

Information Design in Multi-Task Contests: Whom to Inform When the Importance of Tasks Is Uncertain (Chapter 2)

Contest theory is an important field of economic research on situations of conflict and competition. It assumes that several contestants compete for one prize (or several possibly differently ranked prizes) by exerting costly efforts, which in turn determine the probabilities of winning. This chapter contributes to the relatively recent research agenda on information design. Its contribution consists in analyzing information disclosure in a multi-task contest.

In many contests, competitors invest effort in different tasks. Ex ante it may not be clear to them how success in the contest depends on the mixture of effort investments in the different tasks. For instance, when applying for a professorship, it may not be clear to applicants how exactly research performances in different fields are weighted against each other by the hiring committee. Nevertheless, the committee usually has the possibility of transmitting information to the contestants before the contest. This chapter addresses the question of how the information structure should be designed in such a setting in order to maximize contestants' joint effort. We show that, in a two-player Tullock contest with an ex-ante uncertain Cobb-Douglas impact function, the designer cannot benefit by transmitting purely public messages to the contestants. However, if the designer asymmetrically discloses information, she can evoke an increase of contestants' efforts. If the designer can send a purely private message to one contestant, depending on the competitiveness of the contest tasks reflected by comparative cost advantages, either no revelation, full revelation, or partial revelation of information may be beneficial for the designer. We show that in some scenarios the principle of "informational favoritism" of an ex-ante disadvantaged player, e.g., disclosing information to the "weak" underdog, increases contestants' efforts. In other scenarios "informational discrimination" of an ex-ante disadvantaged player, e.g., disclosing information only to the "stronger" of two specialists, is better.

Designing Social Learning by Product Differentiation (Chapters 3 and 4)

The research in these chapters examines a novel combination of the mechanism design approach (understood in a broader sense) and the information design approach. Firms design their products via product differentiation and thus determine its characteristics relative to competitors' products in the market. In the terminology of the mechanism design approach, firms *directly* influence the consumers' incentives to buy the product by changing their marginal utility of consumption. However, the consumption behavior of consumers in the first period of the model is observed by consumers who purchase in the second period. This introduces an information-design perspective into the model: when differentiating its product, a firm also has to take into account the *indirect* effect this has on the social learning procedure among consumers. This is because purchase decisions in earlier periods are informative signals about the quality of the firm's product and may thus change beliefs of consumers who purchase in later periods. Chapters 3 and 4 analyze the same general question, namely how product differentiation affects observational social learning among the consumers, and vice versa, how social learning affects the degree of product differentiation.

Differentiate and Conquer - Using Consumer Learning to Grow Out Your Niche (Chapter 3) In Chapter 3, the idea of "Differentiate and conquer" suggests exploiting an a-priori disadvantage, i.e., producing a niche product, to later on gain power over the larger share of the market. The driving mechanism is the recommendation effect, which introduces a new rationale for product differentiation other than the usual motivation to reduce price competition. We incorporate consumer learning in a version of Hotelling's model (1929) of spatial competition with sequential consumer purchases and a second dimension of variation, quality, about which the consumers have differential information. With consumer learning, firms are confronted with two offsetting effects: differentiation decreases the likelihood that a product is bought in earlier periods, but, by making inference more valuable, it also increases the likelihood that later consumers buy the differentiated good. We show that there exists a unique "differentiate-and-conquer equilibrium" in which the second effect dominates, so that the market incumbent locates in the center of the market while the entrant differentiates by producing an ex-ante niche product.[34]

The Different Effect of Consumer Learning on Incentives to Differentiate in Cournot and Bertrand Competition (Chapter 4) As mentioned above, Chapter 4 analyzes the same general question as Chapter 3 about the interplay between consumer learning and product differentiation. However, it applies the love-of-variety approach to product differentiation instead of the love-of-proximity approach. This allows to better analyze price competition and settings with continuous random variables.

We combine two extensions of the differentiated duopoly model of Dixit (1979), namely Caminal and Vives (1996) and Brander and Spencer (2015a,b), to analyze the effect of consumer learning on firms' incentives to differentiate their products in models of Cournot and Bertrand competition.

Products are of different quality, consumers buy sequentially and are imperfectly informed about the quality of the goods. Before simultaneously competing in quantities, firms simultaneously choose their level of investment in differentiation. The more a firm wishes to differentiate its product or, equivalently, the less substitutable it wants the products to be, the higher the investments have to be. Late consumers can observe earlier consumers' decisions and extract information about the quality of the goods. This influences the firms' incentives to differentiate.

If firms compete in quantities, they are more likely to invest in differentiation with consumer learning than without. This is in line with implications of the recommendation effect introduced in Chapter 3. We also examine the case in which firms compete in prices. Here, the effect of consumer learning is reversed, so that differentiation is less likely with consumer learning. Thus, we find an information-based difference between Cournot and Bertrand competition: in the Bertrand setting consumer learning increases competition, i.e., products are more likely to be substitutes; in the Cournot model it weakens it.

PART II: OVERVIEW

Part II abstracts from informational asymmetries and strategic interaction, but maintains the idea that economic agents may be heterogeneous - and that this can have an impact on economic institutions, such as the organizational structure of a firm. On a

[34] Remember that the love-of-proximity approach to product differentiation, as employed, e.g., in Hotelling's model (1929), is the apt model framework to capture the distinction between niche and mainstream product.

methodological basis, this part complements Part I since it introduces tools from the other two main fields of game theory: cooperative and evolutionary game theory.

PART II: CONCEPTS

We proceed by giving an overview over the paradigms of cooperative and evolutionary game theory. All further necessary formal definitions can be found in Chapter 5.

Cooperative Game Theory: An Outcome-Oriented Paradigm

Cooperative game theory shifts the focus from modeling strategic interactions and the agents' strategic reasoning to payoffs. It asks questions about which payoff distribution among the players of a given cooperative game can be considered as fair or stable, for example. Thus, in contrast to non-cooperative game theory, it takes strategic behavior as a black box, looking solely at outcomes. It assumes that there are binding or enforceable agreements among players (cooperation rules). Therefore, individual actions and strategies are usually not explicitly modeled in this framework.

A coalitional game (N, v) with transferable utility[35] is a pair with the (finite) set $N \subset \mathbb{N}$ of all players involved and $v : 2^N \to \mathbb{R}$ being the coalition function that associates for each subset $S \subseteq N$ the worth $v(S)$ of coalition S. The main question of cooperative game theory is as follows: how much of the value generated by all players should be distributed to a specific player in the game under a certain coalition function?

The main concept of fairness is the Shapley value. Its formula is given by Equation (5.5) in Chapter 5. The Shapley value under game (N, v) is a weighted sum of the marginal contributions $MC_i^v(S) := v(S \cup \{i\}) - v(S)$ of a player to each possible coalition $S \subseteq N$, which weighs the marginal contributions of player $i \in N$ to all possible coalitions (in all given rank orders) as if all coalitions were equally likely. In some senses, it gives an ex-ante expected payoff a player may expect to receive in a given cooperative game. In Chapter 5, we apply an extension of the Shapley value to games with infinitely many player types, which better fits the idea of an evolutionary model.

Evolutionary Game Theory: A Paradigm For Boundedly-Rational Interaction

Evolutionary game theory assumes that economic behavior can rather be explained by learning processes among the agents than by rational reasoning. Thus, it postulates evolution as a process which shapes (economic) behavior. For instance, an individual could

[35] Transferable utility implies that players can transfer (part of their) utilities among each other. This is possible, for instance, if players share a currency, with a common value for all.

learn how to play according to a Nash equilibrium in a specific game, but need not know how to do this the first time the game is played. The underlying behavioral assumption is an extreme form of bounded rationality. Thus, it is an alternative approach to behavioral economics in modeling behavioral failures and departures from the rational choice paradigm. The main idea is that agents do not need to have on-equilibrium-path beliefs or any other form of equilibrium knowledge.

Unlike most non-cooperative game theoretic models, which often assume a rather small number of players, evolutionary models entail large populations, whose individuals are assumed to interact repeatedly - something which can be explicitly or implicitly modeled. The main static solution concept is that of an evolutionary stable strategy (ESS), which *implicitly* assumes repeated, yet myopic interaction.[36] Replicator dynamics are the major dynamic modeling framework, which *explicitly* model the repeated interaction process. Replicator dynamics can be considered as "population choice rules", which determine how a population of boundedly rational agents "choosing" among different strategies evolves.

PART II: CONTRIBUTIONS

Stability in Replicator Dynamics Derived from Transferable Utility Games (Chapter 5)

In Chapter 5, we analyze how an evolutionary process, driven by a replicator dynamic derived from a cooperative transferable utility game, results in the creation of specific stable social structures. We propose an approach to derive a population dynamic from an underlying cooperative transferable utility game. To our knowledge, this combination of evolutionary and cooperative game theory is a methodological novelty. Examining the stable points of the dynamical system, we obtain several intuitive results.

Our main result says that a coalition of player types is stable if and only if it implies a higher average productivity than any of its super- or subcoalitions. For instance, in the class of simple monotonic games, only minimal winning coalitions can be stable. Moreover, we can make statements about which player types will vanish and which ones will persist in stable states. Possible applications are the analysis of coalition formation, the population constitution of eusocial species, or the organizational structure in businesses.

[36] For details consider Weibull (1995).

FURTHER LITERATURE

An overview on the mathematical tools employed in economic theory and used throughout this thesis can be found in de la Fuente (2000). Sydsaeter et al. (2008) is especially helpful as an introduction to optimal control and dynamical systems, while Border (1985) is a concise introduction to fixed-point theory for economists. Rockafellar (1970) and Boyd and Vandenberghe (2004) provide an extensive treatment of convex analysis and convex optimization. Useful references for probability theory are Gut (2005) and Bertsekas and Tsitsiklis (2008). In addition, Corfield and Williamson (2001) focus on Bayesianism.

Mas-Colell et al. (1995) is the classical reference for microeconomic theory, especially for rational choice theory and general equilibrium theory. A formal and comprehensive overview on game theory is given in Fudenberg and Tirole (1991) and Osborne and Rubinstein (1994), while Gibbons (1992) complements these books with a less formal perspective. A more recent textbook on game theory is Maschler et al. (2013), which includes an in-depth treatment of modeling information and knowledge in games. Amir (2005) surveys the main results of the theory of supermodular games and monotone comparative statics.

A comprehensive theoretical overview of the economics of uncertainty and information is provided in Hirshleifer and Riley (2013). Stiglitz (2000) surveys the contributions of the economics of information to twentieth century economics, while Brynjolfsson and McAfee (2014) and Shapiro and Varian (1999) provide non-formal, yet very insightful treatments on the role of information in the economy. Chamley (2004) is the primary textbook on social learning. The broadly related book of Surowiecki (2005) discusses the wisdom of crowds and other phenomena of information aggregation in groups in a non-formal way.

Vulkan et al. (2013) is a useful handbook on market design. Börgers (2015) gives a concise and formal, yet illustrative introduction to mechanism design. Roth (2002) and Duflo (2017) discuss the "economic-design perspective". An overview on the insights from the very recent field of information design can be found in Bergemann and Morris (2017).

Konrad (2009) provides an intuitive and very insightful discussion of the central results of contest theory, while Vojnovic (2016) gives an extensive and formal overview on that topic. The textbook of Belleflamme and Peitz (2010) on industrial organization contains many illustrative examples and up-to-date case studies alongside the concise treatment of the theoretical models, while Vives (2000) focusses on the theoretical analysis. Beath and Katsoulacos (2010) and Lancaster (1990) provide an overview of the theory of product differentiation, and, on the same account, Anderson et al. (1992)

specifically treat discrete choice models and the different approaches to modeling the consumer side in models of product differentiation. An introduction to public economics can be found in Atkinson and Stiglitz (2015).

Camerer et al. (2005) provide an overview of models of bounded rationality in game theory. DellaVigna (2009), Kahneman and Tversky (2000), and Rabin (2002) provide general surveys on the different strands of behavioral economics.

Peleg and Sudhölter (2007) provide a formal treatment of cooperative game theory. Weibull (1995) is the classic textbook on evolutionary game theory with a focus on static concepts, while Sandholm (2010) is a formal and comprehensive treatment also including extensive discussions of dynamic and stochastic approaches.

Part I

Economic Design and Information in Non-Cooperative Games

Chapter 1

Correct Me If You Can - Optimal Non-Linear Taxation of Internalities[1]

Consumers' private valuations of a product characteristic can be systematically biased, leading to welfare losses from what are referred to as "internalities". Previous evidence has shown that these kinds of behavioral failures affect consumer choices in many settings such as energy efficiency investments concerning vehicles or housing, for instance, or the consumption of sugar-sweetened beverages. In this chapter, we characterize the optimal non-linear tax (or subsidy) for correcting behaviorally biased consumers. Using representative examples we show that it is welfare-enhancing compared to no taxation and linear taxation. We examine how the amount of welfare improvement delivered depends on informational restrictions affecting the mechanism designer. Furthermore, we briefly contrast our result with subsidy schemes for fostering energy efficiency in the German housing stock and show that the optimal tax is essentially antipodal to current practice.

1.1 Introduction

Consumer sovereignty has long been undebated among economic scholars: the primacy of rationality implies that consumers have well-behaved preferences, and that of Bayes rationality that new information is processed in an optimal way. Over the past decades, researchers discuss the practical and theoretical implications of violations of these fundamental assumptions. Behavioral economics suggests that if agents do not act in their best interest, then there is room for intervention by a benevolent policy maker. The growing literature on behavioral mechanism design and behavioral public economics examines this issue.

[1] This chapter is joint work with Andreas Gerster.

Misoptimizing consumers inflict a so-called "internality" upon themselves, which provides a justification for corrective taxation beyond the classical case of externalities. An important feature of internalities is that they are inherently heterogeneous as they depend on an individual's bias, in contrast to the social cost of global externalities, for example, which do not vary by agent. As a consequence, targeting corrective taxes towards behaviorally biased consumers to improve welfare is particularly important in the case of internalities. Furthermore, in contrast to externalities, individuals' choices reflect their unobserved bias, which can be exploited for targeting.

In this chapter, we explore the potential of non-linear commodity taxation to target behaviorally biased consumers and to increase social welfare. Following Farhi and Gabaix (2017) and Mullainathan et al. (2012), we employ a general model of biases that encompasses a broad class of behavioral failures driving a wedge between "experienced" and "decision utility" (Kahneman et al., 1997), such as limited attention, biased beliefs and present bias. Based on this generic specification of a behavioral bias, we derive the optimal non-linear commodity tax using a mechanism design approach. Our approach allows for commodity taxation in a broad sense that encompasses non-linear taxation of quantities, but also of qualities, which is important as internalities often distort choices between product varieties. For example, consumers fail to take product attributes into account that are less salient than the purchase price, such as co-pays of a life insurance policy and the operating cost of a durable (Abaluck and Gruber, 2011; Allcott and Taubinsky, 2015).

Starting with Pigou (1920), corrective taxation to overcome market failures from externalities has been studied extensively. Recently, a growing literature has studied optimal taxation for behaviorally biased consumers (O'Donoghue and Rabin, 2006; Allcott et al., 2015; Farhi and Gabaix, 2017), yet the potential to target behaviorally biased consumers through non-linear taxation has not been investigated so far. Traditionally, public tax schemes have targeted consumers through "tagging" (Akerlof, 1978), i.e., a conditioning of taxes on observable characteristics. While appealing in theory, it is difficult to avoid strategic behavior and to find immutable characteristics that are socially acceptable. In contrast, a large literature in industrial organization has investigated how firms can use price discrimination to maximize profits (Mussa and Rosen, 1978). We combine these strands of the literature and explore the potential of non-linear tax schemes as a means to correct behaviorally biased consumers and to increase social welfare.

We show that non-linear commodity taxes increase welfare beyond the optimal constant per-unit tax. In particular, when perceived valuations and biases are positively correlated, marginal taxes should be lower for types with a low perceived valuation, and higher for high types. The intuition for this result is straigthforward and parallels the findings in the price discrimination literature: when confronted with a continuous range of quality, consumers partly reveal their type through their position on the demand curve, which can be exploited for targeting. We show how the scope of welfare improvement due to a corrective taxation changes with the information about aggregate consumer misoptimization available to the mechanism designer. Furthermore, we discuss empirical and theoretical findings on the correlation between perceived valuations and biases and contrast our optimal non-linear tax scheme with a behaviorally motivated policy in Germany.

The chapter is structured as follows. In the next section we discuss the related literature, and in Section 1.3 we introduce the model setup. Section 1.4 then contains the analytical characterization of the optimal tax scheme, while Section 1.5 discusses illustrative examples. Section 1.6 discusses our findings and concludes. All proofs and the figures for the numerical simulations discussed in Section 1.5 can be found in the Appendices.

1.2 Related Literature

Our model is related to all three main branches of the research on optimal taxation.[2] First, since we study the taxation of a consumption good such as an electricity-using durable, our analysis examines commodity taxation. This field of research was originated by Ramsey (1927), and its main issue is to study how to distribute linear taxation across different commodities in order to obtain a certain government budget with the least harming consumption distortions.

Second, the study of corrective taxation in order to combat market failures due to externalities was initiated by Pigou (1920). We study corrective taxation with internalities. The role of heterogeneity among consumers in the valuation of a public good and the associated externalities agents might inflict on others by free-riding was recognized for instance in the analysis of optimally individualized Lindahl prices, see Lindahl (1919). This approach did not account for the informational asymmetries, which exist among a social planner designing the tax and the consumers who hold private information on their valuations or abilities.

The third main field of optimal taxation our work relates to - especially on a methodological basis - is that of non-linear income taxation. Mirrlees (1971) was the first to provide a method to cope with the above mentioned informational asymmetries between the designer and consumers. He explicitly accounted for the consumers' strategic responses to taxes and is one of the founding fathers of mechanism design due to examining incentive compatible mechnisms. The literature on optimal income taxation

[2] An excellent overview on the research on optimal taxation is given in Salanié (2011).

focusses on the equity-efficiency trade-off. It analyzes how to redistribute income with imperfect knowledge about the agents' labor skills by taxing income non-linearly.[3] Two major continuations in optimal non-linear taxation are Diamond (1998) and Saez (2001). Non-linear commodity taxation has been studied, for instance, by Mirrlees (1976) and Cremer and Gahvari (1998), and non-linear taxation with externalities has been studied by Cremer et al. (1998), for instance.

The pathbreaking methodological advances of Mirrlees (1971) have also had an immense impact outside the realm of public economics. The theory of non-linear pricing contrasts optimal non-linear taxation in that the designer's objective no longer is the maximization of social welfare, but rather the maximization of profit.[4] However, the fundamental commonality is that the designer has incomplete information about the preferences of the individual consumer and has to take this into account when designing the mechanism.

Behavioral economics and also behavioral game theory have become a growing field of research, see Kőszegi (2014) and Camerer (2003). The same holds true for behavioral public economics, see Oliver (2013), Sunstein (2016), Congdon et al. (2011), Rees-Jones and Taubinsky (2017), and Chetty (2015). Mullainathan et al. (2012) develope a general model for behavioral public policy. Research on behavioral public policy with a focus on environmental economics can be found in Allcott (2016) and Allcott et al. (2017). Behavioral income taxation is studied in Lockwood and Taubinsky (2016) and Gerritsen (2016). A behavioral approach to commodity taxation with internalities and externalities is pursued in Allcott et al. (2014), Allcott et al. (2015), and Lockwood and Taubinsky (2017). The work most closely related to ours is Allcott and Taubinsky (2015), as we discuss below (at the end of Section 1.3 and in Section 1.4 after Proposition 1.1).

1.3 Model Setup

The main goal of this section is to establish a model that allows us to analyze how the social planner (*she*), i.e., the mechanism designer, implements a welfare maximizing tax scheme - or, equivalently, a subsidy scheme - in an economy with a behaviorally biased consumer (*he*).

We model the interaction between the mechanism designer and the consumer as a dynamic Bayesian game with two stages. In period one, the designer commits to

[3]Generally, incentive compatibility and externalities are closely connected. For instance, in the second-price auction, an incentive compatible auction mechanism, the winner has to pay the second-highest bid, which is a measure of how much harm the winner of the auction inflicts upon society, i.e., it takes into account the externalities of the highest bid.

[4]The theory of non-linear pricing is surveyed in Wilson (1997).

a (possibly non-linear) tax regime $t : X \to \mathbb{R}$, where $X \subseteq [0, \infty)$ is the consumer's choice set. In period two, the consumer then chooses his consumption $x \in X$. The game is solved by backward induction. We begin by presenting the characteristics of the consumer side, and then describe the problem of the mechanism designer.

1.3.1 Consumer Side

The representative consumer's choice variable is given by $x \in X \subseteq [0, \infty)$. The consumer's experienced (or "true", or "normative") per-unit valuation of the benefits of consuming good x is captured by the random variable v, which is distributed according to the cumulative distribution function F with support $supp(F) := [\underline{v}, \overline{v}] \subseteq (-\infty, \infty)$, $-\infty < E[v] < \infty$, and density function f. The bias b, reflecting a misperception of the experienced valuation of the consumption choice x, is distributed according to the cumulative distribution function G with support $supp(G) := \left[\underline{b}, \overline{b}\right] \subseteq (-\infty, \infty)$, $-\infty < E[b] < \infty$, and with the density g. Valuation and bias may be correlated.

The consumer's perceived per-unit valuation of the benefits of consuming of good x is given by $\hat{v} : [\underline{v}, \overline{v}] \times \left[\underline{b}, \overline{b}\right] \to \mathbb{R}$, $(v, b) \mapsto \hat{v}(v, b)$, which depends on the experienced valuation v and the bias b. We assume that the perceived valuation increases in the experienced valuation and in the bias. Furthermore, we specify the bias such that $\hat{v}(v, 0) = v$, i.e., a consumer with bias $b = 0$ is unbiased, while $b < 0$ ($b > 0$) imply underestimation (overestimation) of the value of consumption. The perceived valuation $\hat{v}$ is distributed according to the cumulative distribution function P, which is induced by the distributions of v and b. The density is given by p. The support $supp(P) := \left[\underline{\hat{v}}, \overline{\hat{v}}\right]$ is determined by the support of the distributions F and G. We present examples for the dependence of the functional form of P on the functional form of F and G in Section 1.5. The consumer only observes his perceived valuation $\hat{v}(v, b)$ and does not in any form update his belief about his experienced valuation.

Let $z \in \mathbb{R}$ denote the money (numeraire good) the consumer spends for the consumption of other goods. The consumer's objective function is given by his decision utility $u^d : X \times \mathbb{R} \times \left[\underline{\hat{v}}, \overline{\hat{v}}\right] \to \mathbb{R}$ with $(x, z, \hat{v}) \mapsto u^d(x, z, \hat{v})$. The consumer's experienced utility is given by $u^e : X \times \mathbb{R} \times [\underline{v}, \overline{v}] \to \mathbb{R}$ with $(x, z, v) \mapsto u^e(x, z, v)$. Experienced utility increases in the experienced valuation and decision utility increases in the perceived valuation. Experienced and decision utility increase in z and x.

We follow a common routine in the literature and assume quasilinear utility, where the increasing benefits from consuming x are given by $w : X \to \mathbb{R}$. Thus, we can write decision utility as $u^d(x, z, \hat{v}) = z + \hat{v} \cdot w(x)$ and experienced utility as $u^e(x, z, v) = z + v \cdot w(x)$.

The increasing cost function of consuming x is given by $c : X \to \mathbb{R}$. We assume that good x is produced on competitive markets so that the cost function corresponds to the price of consuming x net of taxes. The exogenous, real-valued scalar $m > 0$ denotes the initial endowment with the numeraire. Therefore, the budget constraint is given by $z \leq m - c(x) - t(x)$.

We assume that w is linear and c is convex which implies that $w - c$ is quasiconcave.[5] Overall, this implies that the decision utility can be written as

$$u^d(x, t, \hat{v}) = m + \hat{v}x - t(x) - c(x),$$

and the experienced utility as

$$u^e(x, t, v) = m + vx - t(x) - c(x).^6$$

Furthermore, we assume an additive bias $\hat{v} := v + b$. Note that then $u^e(x, t, v) = u^d(x, t, \hat{v}) - bx$.

The behavior of the biased consumer is captured by

$$x^d(\hat{v}, t) := \arg\max_x u^d(x, t, \hat{v}),$$

and that of the unbiased consumer by

$$x^e(v, t) := \arg\max_x u^e(x, t, v).$$

To simplify notation, we sometimes write x^d instead of $x^d(\hat{v}, t)$, and we proceed in the same manner with x^e. The individual choice of x is observable.

1.3.2 Mechanism Designer

The designer's objective is to elaborate a tax scheme $t : X \to \mathbb{R}$, based on information about the distributions P, F, and G. She observes the consumer's choice x, but not any realization of the random variables v, b, and $\hat{v}$. The designer's objective function consists of the increasing and concave social welfare function $W : \mathbb{R} \to \mathbb{R}$ with $u^e \mapsto W(u^e)$. For simplicity, we assume $u^e \mapsto \alpha(v)u^e$ with $\alpha : [\underline{v}, \overline{v}] \to \mathbb{R}_+$.

[5] As usual, quasiconcavity of $w - c$ can also be guaranteed by assuming that w is concave and c is linear.

[6] With a slight abuse of notation we continue to use these objectives of the unconstrained consumer problem instead of the above introduced objectives of the constrained consumer problem.

Let $\mathbb{T} := \{f|f : X \to \mathbb{R}\}$ denote the function space containing all functions with domain X and codomain $\mathbb{R}$. The designer's objective is then given by

$$\max_{t\in\mathbb{T}} \int_{\hat{v}} E\left[\alpha(v)|\hat{v}\right] \cdot E\left[u^e\left(x^d, t, v\right)|\hat{v}\right] p(\hat{v})d\hat{v} + \int_{\hat{v}} t(x^d)p(\hat{v})d\hat{v}.$$

1.3.3 Discussion of the Model Setup

The perceptional bias manifests itself in the fact that the consumer experiences a heterogeneous internality upon consuming or using the good under consideration, e.g., an energy-using durable or a sugar-sweetened beverage. Although externalities and internalities both imply that a social planner can increase social welfare, for instance, via taxation, targeting through non-linear taxes is more effective for internalities. Welfare losses from externalities arise from the fact that they are not reflected in the individuals' decisions. In contrast, internalities reduce welfare by (wrongly) influencing decision utility. As a consequence, observing consumption choices is more informative about internalities than about externalities.

To fix ideas, assume that a consumer chooses one of several varieties of a horizontally differentiated product, such as an electricity-using durable of varying energy efficiency levels. In this example, x can be interpreted as the energy efficiency of the durable, measured relative to the worst variety on the market, and v corresponds to the actual reduction in operating cost through better energy efficiency, given actual usage and prices. The random variable b can be interpreted as a misperception of the value of energy efficiency at the time of purchase, i.e., a wedge between perceived valuation $\hat{v}$ and the experienced valuation v of energy efficiency. We can thus interpret $\hat{v}x$ as the perceived savings in monetary terms.[7]

We allow the bias to have any (finite) magnitude in expectation, but it is important to note that empirical applications have typically found that consumers are biased downwards, which means that they undervalue certain attributes, for instance due to limited attention (Allcott et al., 2015). An additive specification of the bias is common in the literature, see Farhi and Gabaix (2017) and Mullainathan et al. (2012).[8]

The departure from Bayesian rationality that is inherent in our model, i.e., the fact that the consumer perceives only $\hat{v}$ and does not make inferences on his bias, can be explained by one of the following behavioral failures that found extensive support in the literature. First, present biases can induce consumers to undervalue products that pay out only in the future, such as healthy foods and energy efficient durables (Laibson, 1997; Loewenstein and Prelec, 1992; O'Donoghue and Rabin, 1999, 2006). Second,

[7] Thus, it is also reasonable to assume that w is linear in x.
[8] The main arguments remain the same for a multiplicative specification of the bias.

if consumers pay only limited attention to certain product attributes, decision utility does not match experienced utility. Inattention has been documented in a vast variety of settings including energy efficiency (Allcott and Taubinsky, 2015) and health care choices (Abaluck and Gruber, 2011). Third, biased beliefs can equally drive a wedge between experienced and decision utility. For example, the literature has found evidence for biased beliefs with regard to energy efficiency, the calory content of nutrition, and schooling returns (Attari et al., 2010; Bollinger et al., 2011; Jensen, 2010).

The assumption of quasilinear utility implies that the model abstracts from income effects. These are not the focus of this chapter, but are extensively discussed in the literature on optimal income taxation. Note also that the functional form of the consumer's objective with respect to x is endogenous to the model. The related issue of the existence of interior maximizers of consumer utility is discussed in Appendix 1.A.1.

To isolate the corrective nature of taxation in our setting, we assume that the designer can levy lump-sum taxes. As a result, the marginal utility of public funds is unity and the designer has no incentive to distort prices to raise public funds. In addition, owing to our assumption that lump-sum taxation is feasible, we do not need to model a government resource constraint.

As it is common in the mechanism design literature, using the standard solution techniques requires that the hazard rate of the type distribution is increasing. In our setting, the consumer's (perceived) type is given by $\hat{v}$, and its distribution is determined by the joint distribution $h(v, b)$ of v and b according to $p(\hat{v}) = \int_{-\infty}^{\infty} h(v, \hat{v} - v)\mathrm{d}v$. In case v and b are independent, the distribution of $\hat{v}$ is determined by a convolution of the distributions v and b, see Bertsekas and Tsitsiklis (2008, pp. 213). Barlow et al. (1963) show that the set of distributions with an increasing hazard rate is closed under convolution. This implies that if the distributions of the independent random variables v and b have an increasing hazard rate, so does the distribution of $\hat{v}$.[9]

A central distinction to the standard mechanism design problems is the fact that the *experienced utility* is evaluated at x^d, the solution to the maximization problem involving the (possibly biased) *decision utility*. In Section 1.4, we analyze how the designer can possibly correct the internality to increase social welfare using non-linear taxation. Inspired by Allcott and Taubinsky (2015), who examine corrective *linear* commodity taxes in a framework with internalities and a *binary* decision, we investigate the novel question of how *non-linear* tax instruments affect the welfare of heterogeneous consumers facing a *continuous* choice set. In contrast to the existing literature on

[9]In many cases of interest, such as v and b being jointly normal distributed, $p(\hat{v})$ exhibits the increasing hazard rate property. Furthermore, alternative solution techniques (such as the Myerson-ironing approach) are available if the increasing hazard rate property is not satisfied.

behavioral commodity taxation, this automatically introduces the issue of incentive compatibility into our setup, as the consumer has an incentive to misreport his type.

1.4 The Optimal Tax Scheme

In this section, we apply the concept of a perfect Bayesian Nash equilbirium to derive the optimal tax scheme. We proceed in two steps. First, we solve for the optimal behavior of the behaviorally biased consumer that maximizes decision utility (but not necessarily experienced utility). Second, we derive the optimal tax schedule and discuss its properties.

The model will generically imply a loss in consumer surplus compared to a setup with a non-biased consumer, as $u^e(x^d, t, v) \leq u^e(x^e, t, v)$. This implies that due to Bayesian irrationality, even in the case of $E[b] = 0$, the consumer misoptimizes in expectation, and a loss in consumer surplus occurs, as we discuss in Section 1.5.

Börgers (2015) paraphrases the Revelation Principle as: "If an allocation can be implemented through some mechanism, then it can also be implemented through a direct truthful mechanism where the consumer reveals his information about his type". In contrast to a conventional mechanism design approach, the consumer does not know his experienced valuation in the setup at hand, but decides exclusively on the basis of perceived valuation $\hat{v}$. That is, in a straightforward extension of the Revelation Principle to our setup, the type of a consumer is given by his perceived valuation $\hat{v}$, rather than by the tuple $(v,\ b)$. The designer confines herself to designing a direct mechanism $(\xi, \tau) : [\underline{\hat{v}}, \overline{\hat{v}}] \to X \times \mathbb{R}$ under truthtelling to implement the desired outcome. Based on the consumer's strategical report $\tilde{v}$, the direct mechanism then assigns the consumed quantity, $\xi(\tilde{v}) \in X$, and the amount of taxes to be paid, $\tau(\tilde{v}) \in \mathbb{R}$.

Under the direct mechanism, the decision utility for report $\tilde{v}$ given the perceived valuation $\hat{v}$ is

$$u^d(\xi(\tilde{v}), \tau(\tilde{v})|\hat{v}) = m + \hat{v} \cdot \xi(\tilde{v}) - \tau(\tilde{v}) - c(\xi(\tilde{v})).$$

Since the consumer may strategically misreport his perceived valuation, truthtelling can be induced by the designer by implementing an incentive compatible mechanism (in dominant strategies). This implies that the tax scheme must satisfy

$$u^d(\xi(\hat{v}), \tau(\hat{v})|\hat{v}) \geq u^d(\xi(\tilde{v}), \tau(\tilde{v})|\hat{v}) \ \forall \hat{v}, \tilde{v} \in [\underline{\hat{v}}, \overline{\hat{v}}] \,. \tag{IC}$$

As in the standard mechanism design setup, ξ must be increasing in $\hat{v}$ for direct mechanisms to be valid. A standard sufficient condition for ξ to increase is that the hazard rate of P increases.

Optimal strategic reporting of a consumer implies that the solution v^* to the problem $\max_{\tilde{v}} u^d(\xi(\tilde{v}), \tau(\tilde{v})|\hat{v})$ has to satisfy

$$\hat{v}\xi'(v^*) - \tau'(v^*) - \xi'(v^*)c'(\xi(v^*)) \stackrel{!}{=} 0. \tag{1.1}$$

Incentive compatibility implies $v^* = \hat{v}$, and thus equilibrium decision utility in an incentive-compatible direct mechanism is given by $\hat{u}^d(\hat{v}) := u^d(\xi(\hat{v}), \tau(\hat{v})|\hat{v})$, while equilibrium experienced utility is given by $\hat{u}^e(\hat{v}, b) := u^e(\xi(\hat{v}), \tau(\hat{v})|v) = \hat{u}^d(\hat{v}) - b\xi(\hat{v})$. Put differently, incentive compatibility implies that for all $\hat{v} \in \left[\underline{\hat{v}}, \overline{\hat{v}}\right]$ it has to hold that

$$\tfrac{\partial \hat{u}^d(\hat{v})}{\partial \hat{v}} = \xi(\hat{v}) + \hat{v}\xi'(\hat{v}) - \tau'(\hat{v}) - \xi'(\hat{v})c'(\xi(\hat{v})) \stackrel{(1.1)}{=} \xi(\hat{v}). \tag{1.2}$$

We assume that the consumer's outside option is such that he always prefers to participate in the mechanism. Therefore, the participation constraints are fulfilled for any (perceived) consumer type. As usual, when incentive compatibility is satisfied for each perceived type, then it suffices that the participation constraint of the lowest type is satisfied. In particular we can specify the model such that $\hat{u}^d\left(\underline{\hat{v}}\right) = \underline{u} > 0$ and $\hat{u}^d\left(\overline{\hat{v}}\right) \geq \underline{u}$.

The designer solves a dynamic optimization problem which can be analyzed using the optimal control approach.[10] Note that determining the equilibrium values of $\xi(\hat{v})$ and $u^d(\hat{v})$ for all $\hat{v}$ pins down the equilibrium value of $\tau(\hat{v})$ for all $\hat{v}$. Hence, the mechanism design problem of the designer is given by

$$\max_{\xi \in \mathbb{X}} \int_{\hat{v}} \left(E\left[\alpha(v)|\hat{v}\right] \cdot E\left[\hat{u}^e\left(\hat{v}, b\right)|\hat{v}\right]\right) dP(\hat{v}) + \int_{\hat{v}} \tau\left(\hat{v}\right) dP(\hat{v}), \tag{1.3}$$

subject to the condition from Equation (1.2), and where $\mathbb{X} := \{f | f : \left[\underline{\hat{v}}, \overline{\hat{v}}\right] \to X\}$ is the function space containing all functions with domain $\left[\underline{\hat{v}}, \overline{\hat{v}}\right]$ and codomain X. The boundary conditions of the problem are given by $\hat{u}^d\left(\underline{\hat{v}}\right) = \underline{u}$ and $\hat{u}^d\left(\overline{\hat{v}}\right) \geq \underline{u}$. The control variable is ξ and the law of motion of the state variable $\hat{u}^d$ is determined by incentive compatibility and optimal strategic reporting, see Equation (1.2). Using the definition of decision utility to replace the tax and rewriting equilibrium experienced utility in terms of equilibrium decision utility, the Hamiltonian for the above problem for all $\hat{v} \in \left[\underline{\hat{v}}, \overline{\hat{v}}\right]$ is given by

$$H\left(\hat{v}, \xi, \hat{u}^d\right) = \left[E[\alpha(v)|\hat{v}] \cdot \underbrace{\left(\hat{u}^d(\hat{v}) - E[b|\hat{v}]\xi(\hat{v})\right)}_{=E[\hat{u}^e(\hat{v},b)|\hat{v}]} + \underbrace{\left(m + \hat{v}\xi(\hat{v}) - \hat{u}^d(\hat{v}) - c(\xi(\hat{v}))\right)}_{=\tau(\hat{v})}\right] p(\hat{v}) + \mu(\hat{v})\xi(\hat{v}).$$

[10]For details on what follows consider Sydsaeter et al. (2008, chapters 9-10).

Following the standard solution procedure for this kind of mechanism design problem, we employ Pontryagin's Maximum Principle, which yields the following necessary conditions for the optimal tax.[11]

$$\text{FOC on control: } \frac{\partial H}{\partial \xi} = \left[-E[b|\hat{v}] \cdot E[\alpha(v)|\hat{v}] + \hat{v} - c'(\cdot)\right] p(\hat{v}) + \mu\left(\hat{v}\right) \overset{!}{=} 0, \quad (\text{FOC}_\text{x})$$

$$\text{FOC on state: } \frac{\partial H}{\partial \hat{u}^d} = \left[E[\alpha(v)|\hat{v}] - 1\right] p(\hat{v}) \overset{!}{=} -\mu'\left(\hat{v}\right), \quad (\text{FOC}_\text{u})$$

$$\text{transversality cond.: } \mu\left(\underline{\hat{v}}\right) \cdot \hat{u}^d\left(\underline{\hat{v}}\right) = \mu\left(\overline{\hat{v}}\right) \cdot \hat{u}^d\left(\overline{\hat{v}}\right) = 0. \quad (\text{TVC})$$

We obtain the characterization of the optimal non-linear tax scheme in our main result:

Proposition 1.1. *The optimal non-linear commodity tax in the model with internalities and quasilinear utility is implicity given by*

$$t'(x) = \frac{\int_{\hat{v}_x}^{\overline{\hat{v}}} \left(1 - E[\alpha(v)|m]\right) p(m) dm}{p(\hat{v}_x)} + E[b|\hat{v}_x] \cdot E[\alpha(v)|\hat{v}_x] \quad \forall x \in X, \tag{1.4}$$

where $\hat{v}_x$ is the report to be sent by a consumer to obtain the allocation x under the optimal tax scheme.

Proof. See Appendix 1.A.1 □

Remember that our results are independent of whether v and b are correlated or not. The results apply to any distribution of perceived valuations, which satisfies the increasing hazard rate property. In particular, this is the case for the convolution of two uniformly distributed independent random variables v and b, which yields a trapezoid distribution, as we discuss below in Section 1.5. Additionally, this is also the case for the sum of two jointly normal distributed (possibly) correlated random variables v and b, which is itself a normal distribution.

To convey the intuition of the optimal marginal tax we first examine the second summand of Equation (1.4). This term embodies the behavioral aspect of our model and does not appear in the standard literature on non-linear taxation. It corrects for the expected bias of a consumer conditional on the report about his perceived valuation. It may be negative as well as positive depending on the expected conditional bias. The designer uses her potential to help the consumer in correcting his Bayes irrationality with the optimal tax scheme - "she corrects them, if she can".

The optimal tax formula in the binary choice model with linear taxation of Allcott and Taubinsky (2015) does not allow for "price discrimination" among consumers. In

[11] Sufficiency is given if in addition the control region is convex and the Hamiltonian is concave in $(\xi, \hat{u}^d)$ for every $\hat{v}$, see Sydsaeter et al. (2008, page 315). This is satisfied in the setup at hand.

contrast to this, in our model, since there is a correlation between bias and report, the consumer's strategic report about his perceived valuation reveals information about his bias and the designer should exploit this by using Bayesian updating. Details on this updating process can be found in Section 1.5.

If the designer has a utilitarian social welfare function with equal weights for each consumer type independent of his valuation, we can normalize these weights to one without loss of generality, so that $E[\alpha(v)|\hat{v}_x] = 1$ for all $\hat{v}_x \in [\underline{\hat{v}}, \overline{\hat{v}}]$. In this case, the marginal tax rate, given by Equation (1.4), just equals the expected bias *conditional* on the report, $t'(x) = E[b|\hat{v}_x]$. In Appendix 1.A.2, we show that when the designer cannot discriminate among the different consumer types and has to resort to linear taxation, the optimal tax is given by $t^* = E[b]$, that is, the marginal tax rate equals the *unconditional* expected bias. Furthermore, as indicated above, Allcott and Taubinsky (2015) have shown that in a binary investment setting the optimal tax is equal to the average bias of the consumers who are indifferent between both goods at market prices. Again, our result on the non-linear tax differs by its dependence on the consumer's report $\hat{v}$ rather than a fixed market price. In fact, it is exactly the variation in reports that a mechanism designer can exploit with a non-linear tax scheme to improve upon a constant per-unit tax.

Next, we contrast our optimal non-linear tax for behaviorally biased consumers with the famous ABC formula from the theory of optimal non-linear income taxation, derived by Diamond (1998, p. 86, Equation (10)). The ABC formula contains three factors: efficiency considerations (A), redistribution issues (B), and the dependence of the incentive compatibility constraint on the density functions via the hazard rate (C). To start with, part A of the ABC formula is not present in our model, which reflects that our model abstracts from income effects. Yet, redistribution issues (B) and the density of $\hat{v}$ (C) are contained in the first summand of Equation (1.4), although in modified form. The intuition of this summand is as follows. When the designer changes the marginal tax at, say, x, she extracts money from all consumers with $\hat{v} \geq \hat{v}_x$. The change of her objective is captured by the term $\int_{\underline{\hat{v}}}^{\overline{\hat{v}}_x} (1 - E[\alpha(v)|m])\, p(m)dm$, since the marginal value of an additional unit of tax money is one and welfare decreases by $E[\alpha(v)|\hat{v}_x]$ for a consumer with type $\hat{v}_x$. Intuitively, if the average welfare weights for these consumers exceed unity (and thus the marginal value of the tax income to the designer), the designer's objective function decreases and she should reduce the tax. The term is weighted more strongly in the optimal tax formula if the density of the type $\hat{v}_x$ is low, i.e., if it is unlikely that the consumer is marginal to the tax change at x and thus has an incentive to change his behavior. The second summand in our optimal tax formula captures the novel aspect of corrective taxation in our model and does not appear in the standard ABC formula. Another notable difference to the the standard models without behavioral consumers is that our result allows for a non-zero marginal tax rate at the top of the type distribution.

1.5 Illustrative Examples: Updating, Optimal Tax, and Welfare

We now make distributional assumptions and explore the inferences that a designer can make about the consumer type. Furthermore, we derive the optimal non-linear tax schedules for these examples and discuss their properties. For the ease of exposition, we assume that the designer has a utilitarian social welfare function with equal weights normalized to one for each consumer, so that, according to Proposition 1.1, the optimal marginal tax rate just equals the expected bias conditional on the report, i.e., $t'(x) = E[b|\hat{v}_x]$.

1.5.1 Density $p(\hat{v})$ of the Perceived Valuation

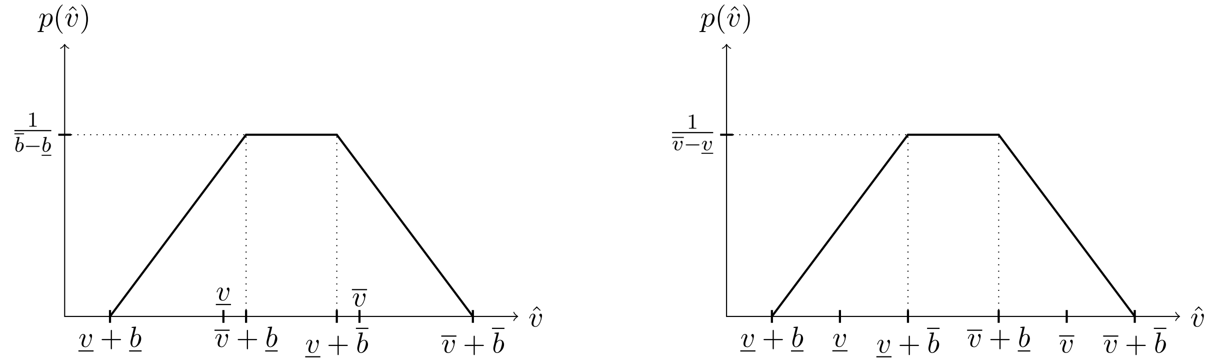
To explore how P is determined by the two random variables v and b, we assume that they are independently and uniformly distributed: $b \sim U\left[\underline{b}, \overline{b}\right]$ and $v \sim U[\underline{v}, \overline{v}]$. Furthermore, we assume that $\overline{b} < \underline{v}$, i.e., the realization of the experienced valuation is always larger than that of the bias. Note that uniform distributions exhibit the increasing-hazard-rate property, and by the above mentioned result of Barlow et al. (1963) this property is inherited by their convolution.

There are two distinct general cases: either the variance, and thus the entropy,[12] of the bias is smaller than that of the experienced valuation, implied by $\overline{v}-\underline{v} \geq \overline{b}-\underline{b}$, or vice versa, $\overline{v}-\underline{v} < \overline{b}-\underline{b}$. Figure 1.1 visualizes that in both cases the induced density p follows a trapezoid distribution (in the case of $\overline{v}-\underline{v} = \overline{b}-\underline{b}$ it follows a triangular distribution, which is a special trapezoid distribution). The trapezoid distribution consists of three ranges: a triangular lower part, a rectangular middle part, and a triangular upper part. If the variance of the bias exceeds the variance of the valuation, the middle range spans over the interval $\left[\overline{v}+\underline{b}, \underline{v}+\overline{b}\right]$, and $\left[\underline{v}+\overline{b}, \overline{v}+\underline{b}\right]$ otherwise. This has important implications for the updating process the designer can exploit, as we discuss below.

1.5.2 Conditional Expectation $E[b|\hat{v}]$ of the Bias

To obtain an estimate for the conditional bias, the designer can calculate the expectation of the bias conditional on the report, $E[b|\hat{v}] = \int_b b \cdot g(b|\hat{v})\ db$, using the conditional

[12] The entropy of a distribution is a measure of its informativeness. In the case of the uniform distribution $b \sim U\left[\underline{b}, \overline{b}\right]$, it is given by $log(\overline{b}-\underline{b})$, and it is positively correlated with the variance $\frac{1}{12}(\overline{b}-\underline{b})^2$.



(a) Variance of valuation smaller than variance of bias: $\overline{v}-\underline{v}<\overline{b}-\underline{b}$:

(b) Variance of valuation greater than or equal to variance of bias: $\overline{v}-\underline{v}\geq\overline{b}-\underline{b}$

Figure 1.1: Density p derived from the convolution of f and g when both, v and b, are i.i.d. and uniformly distributed.

density $g(b|\hat{v})$. For the case of the convolution of two independent uniform random variables, we show in Appendix 1.A.3 that the conditional expectation $E[b|\hat{v}]$ is as follows:

$$E[b|\hat{v}] = \begin{cases} \frac{\hat{v}-\underline{v}+\underline{b}}{2}, & \text{if } \hat{v}<\overline{v}+\underline{b},\ \hat{v}<\underline{v}+\overline{b}, \\ \frac{\hat{v}-\overline{v}+\overline{b}}{2}, & \text{if } \hat{v}\geq\overline{v}+\underline{b},\ \hat{v}\geq\underline{v}+\overline{b}, \\ \frac{2\hat{v}-\underline{v}-\overline{v}}{2}, & \text{if } \hat{v}\geq\overline{v}+\underline{b},\ \hat{v}<\underline{v}+\overline{b}, \\ \frac{\overline{b}+\underline{b}}{2}, & \text{if } \hat{v}<\overline{v}+\underline{b}, \hat{v}\geq\underline{v}+\overline{b}. \end{cases} \tag{1.5}$$

The first row of Equation (1.5) gives the conditional bias on the lower range from both panels of Figure 1.1, while the second row gives the conditional bias for the upper range. Row three gives the conditional bias in the middle range when the variance of the experienced valuation is smaller than the bias ($\overline{v}-\underline{v}<\overline{b}-\underline{b}$), and row four gives the conditional bias in the same range when the opposite holds true ($\overline{v}-\underline{v}\geq\overline{b}-\underline{b}$).

A comparison of the conditional bias with the unconditional bias, $E[b]=(\overline{b}+\underline{b})/2$, reveals that the conditional bias differs from the unconditional bias, which allows the mechanism designer to update her beliefs about the consumer's bias upon receiving a report $\hat{v}$. The only exception is the middle range, when $\overline{v}-\underline{v}\geq\overline{b}-\underline{b}$. For the lower and upper range, the divergence between the conditional and unconditional bias is intuitive, as very low or very high reports imply low and high biases, respectively. Furthermore, as row three in Equation (1.5) shows, a mechanism designer can even extract new information on the bias after a report from the middle range, given that the variance of the experienced valuation is smaller than the variance of the bias ($\overline{v}-\underline{v}<\overline{b}-\underline{b}$). The reason is that these are cases in which $\hat{v}$ can be constituted by any *valuation* $v\in[\underline{v},\overline{v}]$

and a bias from a "restricted" range $b \in [\hat{v} - \overline{v}, \hat{v} - \underline{v}] \subset [\underline{b}, \overline{b}]$. Only when the opposite holds true, the middle range is uninformative for a mechanism designer and does not allow her to update her beliefs beyond the unconditional mean of the bias. The reason is that these are cases in which $\hat{v}$ can be constituted any *bias* $b \in [\underline{b}, \overline{b}]$ and the according valuation v such that $v + b = \hat{v}$.

1.5.3 Specific Numerical Examples

We now make further distributional assumptions, discuss optimal non-linear taxation schedules, and contrast them with both the absence of corrective taxation and the optimal linear tax. Specifically, we consider four scenarios and run a numerical simulation for each of them. Each simulation relies on 100,000 draws.[13] In all scenarios, the experienced valuation is distributed i.i.d. (independent across consumers and with respect to the bias) according to a uniform distribution, $v \sim U[90, 100]$. In the first three scenarios, we furthermore assume that $b \sim U[-10, 10]$, $b \sim U[-5, 5]$, and $b \sim U[-1, 1]$. In all three scenarios, the bias has an expectation of zero, but the variance of the bias is larger than, equal to, and smaller than the variance of the experienced valuation, respectively. In addition, our fourth scenario considers $b \sim U[-8, 4]$, so that the expectation of the bias is negative (-2) and its variance is exactly as in the first scenario. The results are depicted in the figures of Appendix 1.B.

A common observation valid for all figures is that the marginal tax rate (Panel (c) in each figure) is lowest at the lowest consumption level, i.e., the lowest perceived type, and highest at the other end of the consumption (and perceived type) distribution. This seems to be juxtaposed to results in many standard mechanism design problems: in our model the designer should encourage consumption for low types and discourage consumption for high types. The driving force is the functional form of $E[b|\hat{v}]$ (Panel (a) in each figure), which depicts the designer's informational advantage: at the low end of the distribution of $\hat{v}$ she is very certain that consumers are downward biased, and the reverse holds for the upper end of the distribution. This is summarized in the following observation.

Observation 1.1. *In the independent-uniform case*[14] *, the marginal tax rate is lowest at the lowest consumption level, and highest at the highest consumption level.*

We proceed by looking at cases, in which the bias is zero in expectation, i.e., Figures 1.3, 1.4, and 1.5 of Appendix 1.B. Whenever the support of the bias is large compared

[13] Our results hold without loss of generality in a model with a unit mass of consumers who have the same properties as the single representative consumer described in our model setup. True valuation and bias need to be independently and identically distributed across consumers for the results to hold.

[14] By "independent-uniform case" we mean that valuation and bias are independently and uniformly distributed.

to that of the experienced valuation, $\overline{v} - \underline{v} < \overline{b} - \underline{b}$, as for instance in Figure 1.3, then the optimal non-linear tax yields a qualitatively different allocation than a linear tax (see Panel (b)). This effect becomes smaller as the support of the bias becomes smaller (compare Panels (b) of Figures 1.3, 1.4, and 1.5). Note that in many applications it is plausible to assume that (the support of) the bias is large compared to (that of) the experienced valuation, so that optimal non-linear taxation would imply quite different allocations than linear taxation in these cases.

The support of the bias in the scenario with uniform distributions critically determines the shape of the functional form of $E[b|\hat{v}]$, which reflects the designer's information advantage: the smaller the variance of the bias, the more the distribution of $E[b|\hat{v}]$ resembles a uniform distribution itself. Since the uniform distribution has maximum entropy, the information advantage decreases in the relative size of the variance of the bias. Obviously, this has an impact on the possibility to correct the Bayesian irrationality: the welfare improvement induced by non-linear taxation decreases as the support of the bias becomes smaller (compare Panels (e), (f), and (g) of Figures 1.3, 1.4, and 1.5). This is summarized in the following observation.

Observation 1.2. *In the independent-uniform case, the larger the uncertainty about the bias, the larger is the potential for welfare improvement by an optimal non-linear tax scheme compared to no taxation or linear taxation.*

Whenever the variance of the bias is larger than that of the experienced valuation, the optimal mechanism implies bunching contracts for perceived types, which are in the middle range of the distribution. Consider Panel (b) of Figure 1.3, for instance. Remember that whenever the variance of the bias is larger than that of the experienced valuation, $\overline{v} - \underline{v} < \overline{b} - \underline{b}$, the designer obtains new information about the expected bias when she receives a report from the middle range of the domain of P. This is not the case if the variance of the bias is smaller than that of the experienced valuation. Then, reports from the middle range disclose no new information, $E[b|\hat{v}] = E[b]$, see Equation (1.5). However, the bunching in the former case occurs since the designer learns nothing new about v.

Observation 1.3. *In the independent-uniform case, when the uncertainty about the bias is large ($\overline{v} - \underline{v} < \overline{b} - \underline{b}$), the optimal non-linear tax scheme includes bunching contracts for (perceived) types in the middle range.*

In any of the examples depicted in Appendix 1.B, the highest (perceived) type obtains his optimal allocation in the optimal tax scheme. As in most standard mechnism design problems "there is no distortion at the top". However, additionally, in the behavioral mechanism design setup at hand, the designer is certain that the lowest perceived type would be misoptimizing without her intervention. Thus, due to her informational advantage the designer can construct a mechnism such that also the lowest type obtains his optimal allocation, i.e., "there is no distortion at the bottom". These observations concerning the boundary points of the interval of the perceived valuations determine the overall shape of the optimal mechanism.

Observation 1.4. *In the independent-uniform case, the optimal non-linear tax scheme implies "no distortion at the top and at the bottom".*

In all scenarios with $E[b] = 0$, linear taxation does not increase welfare compared to no taxation (Panels (e) - (g) of Figures 1.3, 1.4, and 1.5). This is not the case when $E[b] \neq 0$, as in Figure 1.6: linear taxation increases welfare, since it corrects for the average misoptimization (see Panel (g) of the figure). However, non-linear taxation still improves upon this, since - in addition to correcting for the average misoptimization - it also corrects for the "extreme" misoptimization at the margins of the distribution of the perceived valuation by targeting those consumers with a high degree of Bayesian irrationality. We have already mentioned above that $E[b] < 0$ is a plausible assumption in many real-world applications. Thus, in combination with the observation from above, we see that the designer can notably increase welfare by non-linear taxation in applications with a large bias.

Observation 1.5. *In the independent-uniform case, the optimal non-linear tax scheme induces a higher welfare than no taxation and linear taxation.*

Comparing Figures 1.3 and 1.6, we can also observe that although the marginal tax rates are qualitatively similar (Panel (c)), the divergence of $E[b]$ from zero implies that the tax rate (Panel (d)) becomes less and less symmetric and for a bias with a purely negative support the tax would be monotonically decreasing.

1.6 Discussion and Conclusion

In this chapter, we have derived the optimal non-linear tax to correct misoptimization induced by internalities, i.e., individual welfare losses from behavioral failures. Using a mechanism design approach, we show that consumers' reports contain information that can be employed to improve upon a linear tax. This beneficial effect of corrective taxation increases in the informativeness of the reports available to the designer.

Empirically, there is evidence that the correlation between reports and biases has some regularities that can be exploited by policy. For example, Allcott et al. (2015) show for energy efficiency investments and hybrid car purchases that higher perceived valuations, i.e., higher reports, are positively correlated with the bias. More specifically, for low perceived valuations, the bias is negative and increases as perceived valuations rise. In such a setting, non-linear taxation should give the largest marginal subsidies to participants with a low perceived valuation. This is also what our analysis suggests.

To the extent that the positive correlation between reports and bias holds true, many subsidy schemes employed in practice are effectively antipodal to the optimal non-linear tax derived in this chapter. For example, the German government grants subsidies for energy efficiency in housing only if a newly built (or retrofitted) house meets predefined minimum efficiencies, so-called "KfW-Effizienzhaus" standards. In other words, marginal subsidies are essentially zero if reports are small and increase only as perceived valuations become larger. As a result, the most heavily biased consumers with low perceived valuations receive no subsidies. This observation implies that social welfare can be improved by implementing the optimal non-linear tax, which specifically targets those consumers.

Since internalities and externalities are often intertwined[15] the inclusion of externalities into our model can be part of future research. Modifying the setup to account for income effects would be another sensible extension. Moreover, future research could further investigate empirical applications and tests of the model presented in this chapter.

1.A Appendix: Proofs

1.A.1 Proof of Proposition 1.1: Optimal Non-Linear Tax

The consumer's first-order condition characterizing optimal consumption x^d is given by

$$\left.\frac{\partial u^d(x,t,\hat{v})}{\partial x}\right|_{x=x^d} = \hat{v} - c'(x^d) - t'(x^d) \stackrel{!}{=} 0 \Leftrightarrow c'(x^d) = \hat{v} - t'(x^d). \tag{1.6}$$

The second order condition is satisfied if $-c''(x) - t''(x) \leq 0$ for all $x \in X$. Since the costs are convex in x by assumption, this condition is satisfied, if the optimal tax schedule is convex in x as well.[16] Generally, an interior solution exists, if "c is convex enough compared to t", i.e., $c''(x) \geq -t''(x)$ for all $x \in X$.

[15] For instance the consumption of highly saccherated soft drinks may cause bad health (internality) and higher burdens for health insurances due to diabetes, etc. (externality).

[16] Appendix 1.B illustrates that this is the case in our examples.

As discussed in the text we can always guarantee that $\hat{u}\left(\underline{\hat{v}}\right) = \underline{u} > 0$ and $\hat{u}\left(\overline{\hat{v}}\right) \geq \underline{u}$, so that the transversality condition immediately implies $\mu\left(\underline{\hat{v}}\right) = 0$ and $\mu\left(\overline{\hat{v}}\right) = 0$. We now use Equation ($\text{FOC}_\text{u}$). By integrating and using $\mu\left(\overline{\hat{v}}\right) = 0$ we obtain

$$\int_{\hat{v}}^{\overline{\hat{v}}} -\mu'(n)dn = -\mu\left(\overline{\hat{v}}\right) - \left[-\mu\left(\hat{v}\right)\right] = \mu\left(\hat{v}\right) \stackrel{!}{=} \int_{\hat{v}}^{\overline{\hat{v}}} \left(E[\alpha(v)|m] - 1\right) p(m)dm. \tag{1.7}$$

Using the above equations we rearrange Equation (FOC_x), to obtain the result:

$$\begin{aligned} & \left(\hat{v} - c'(\cdot)\right) \stackrel{!}{=} -\frac{\mu\left(\hat{v}\right)}{p(\hat{v})} + E[b|\hat{v}] \cdot E[\alpha(v)|\hat{v}] \\ \stackrel{(1.6)}{\Leftrightarrow} \quad & t'(x) = -\frac{\mu\left(\hat{v}_x\right)}{p(\hat{v}_x)} + E[b|\hat{v}_x] \cdot E[\alpha(v)|\hat{v}_x] \\ \stackrel{(1.7)}{\Leftrightarrow} \quad & t'(x) = \frac{\int_{\hat{v}_x}^{\overline{\hat{v}}} \left(1 - E[\alpha(v)|m]\right) p(m)dm}{p(\hat{v}_x)} + E[b|\hat{v}_x] \cdot E[\alpha(v)|\hat{v}_x]. \end{aligned}$$

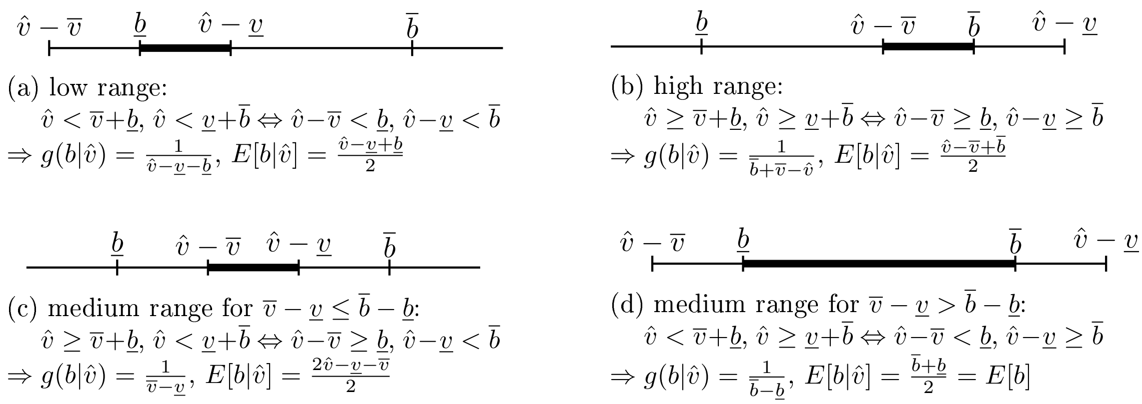
1.A.2 Derivation of the Optimal Linear Tax

In this proof we additionally assume that $c'''(x) = 0$, which simplifies the calculation of the optimal linear tax, but does not change our results on the optimal non-linear tax scheme. Anticipating the behavior on the consumer side, the problem of the designer can be written as $\max_{t \in \mathbb{R}} \int_v \int_b u^e(x^d, t, v) dG(b|v)dF(v) + \int_v \int_b t \cdot x^d dG(b|v)dF(v) =: V(t)$. We evaluate the derivative with respect to the linear tax t:

$$\begin{aligned} \frac{\partial V(t)}{\partial t} &= \int_v \int_b \left[-x^d + \left(v - t - c'(\cdot)\right) \frac{\partial x^d}{\partial t}\right] dG(b|v)dF(v) + \int_v \int_b \left[x^d + t \cdot \frac{\partial x^d}{\partial t}\right] dG(b|v)dF(v) \\ &= \int_v \int_b \left[\left(v - c'(\cdot)\right) \frac{\partial x^d}{\partial t}\right] dG(b|v)dF(v). \end{aligned}$$

The individually optimal consumption is again characterized by Equation (1.6), i.e., $c'(\cdot) = \hat{v} - t'(x) = (v + b) - t$, where the last equality holds since t is linear. Thus, $\frac{\partial V}{\partial t} = \int_v \int_b \left[(t - b)\frac{\partial x^d}{\partial t}\right] dG(b|v)dF(v)$. We can further evaluate $\frac{\partial x^d}{\partial t}$ by differentiating Equation (1.6) with respect to t to obtain $\frac{\partial x^d}{\partial t} = -\frac{1}{c''(x^d)} = a$, for some real-valued constant $a < 0$. The last equality follows from the assumption that $c'''(\cdot) = 0$ and that c is convex. Therefore, the optimal tax t^* is given by $\frac{\partial V}{\partial t}|_{t=t^*} \stackrel{!}{=} 0 \Leftrightarrow t^* = E[b]$.

1.A.3 Indep. Uniform Case: Illustration of the Calculation of the Conditional Density and of the Conditional Expectation of the Bias



(a) low range:
$\hat{v} < \overline{v} + \underline{b}$, $\hat{v} < \underline{v} + \overline{b} \Leftrightarrow \hat{v} - \overline{v} < \underline{b}$, $\hat{v} - \underline{v} < \overline{b}$
$\Rightarrow g(b|\hat{v}) = \frac{1}{\hat{v} - \underline{v} - \underline{b}}$, $E[b|\hat{v}] = \frac{\hat{v} - \underline{v} + \underline{b}}{2}$

(b) high range:
$\hat{v} \geq \overline{v} + \underline{b}$, $\hat{v} \geq \underline{v} + \overline{b} \Leftrightarrow \hat{v} - \overline{v} \geq \underline{b}$, $\hat{v} - \underline{v} \geq \overline{b}$
$\Rightarrow g(b|\hat{v}) = \frac{1}{\overline{b} + \overline{v} - \hat{v}}$, $E[b|\hat{v}] = \frac{\hat{v} - \overline{v} + \overline{b}}{2}$

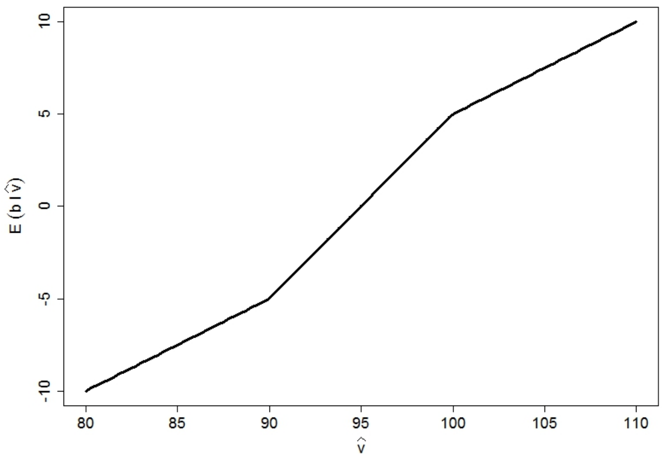
(c) medium range for $\overline{v} - \underline{v} \leq \overline{b} - \underline{b}$:
$\hat{v} \geq \overline{v} + \underline{b}$, $\hat{v} < \underline{v} + \overline{b} \Leftrightarrow \hat{v} - \overline{v} \geq \underline{b}$, $\hat{v} - \underline{v} < \overline{b}$
$\Rightarrow g(b|\hat{v}) = \frac{1}{\overline{v} - \underline{v}}$, $E[b|\hat{v}] = \frac{2\hat{v} - \underline{v} - \overline{v}}{2}$

(d) medium range for $\overline{v} - \underline{v} > \overline{b} - \underline{b}$:
$\hat{v} < \overline{v} + \underline{b}$, $\hat{v} \geq \underline{v} + \overline{b} \Leftrightarrow \hat{v} - \overline{v} < \underline{b}$, $\hat{v} - \underline{v} \geq \overline{b}$
$\Rightarrow g(b|\hat{v}) = \frac{1}{\overline{b} - \underline{b}}$, $E[b|\hat{v}] = \frac{\overline{b} + \underline{b}}{2} = E[b]$

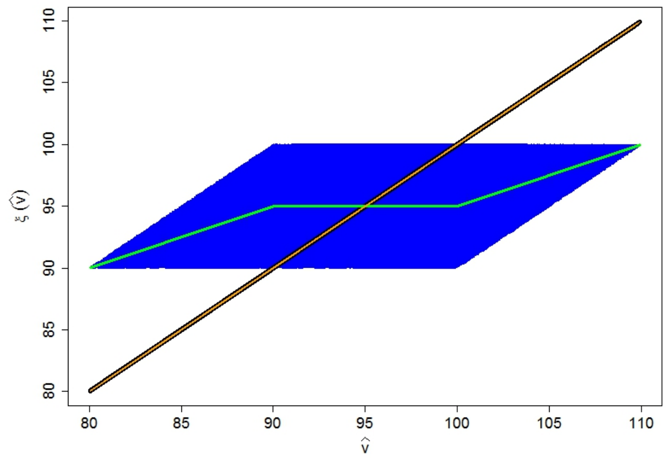
Figure 1.2: Independent Uniform Case: Conditional density $g(b|\hat{v})$ and conditional expectation $E[b|\hat{v}]$.

1.B Appendix: Figures for the Simulations in Section 1.5

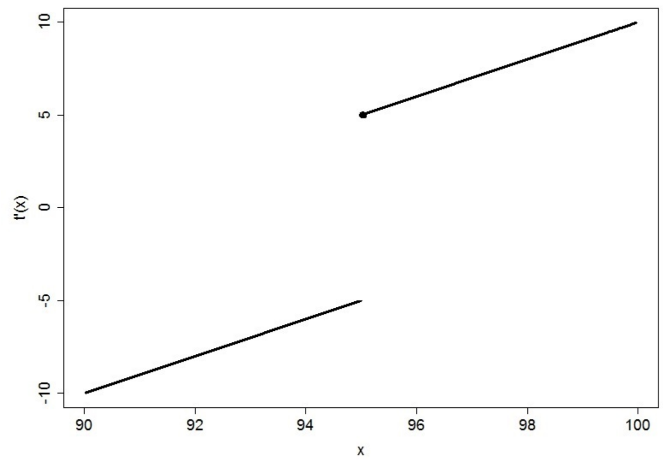
Please turn over! The figures are discussed in Section 1.5.



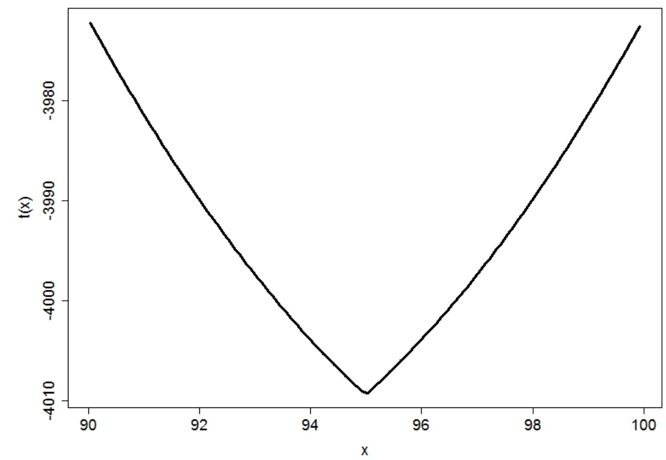
(a) Conditional expectation of the bias $E[b|\hat{v}]$



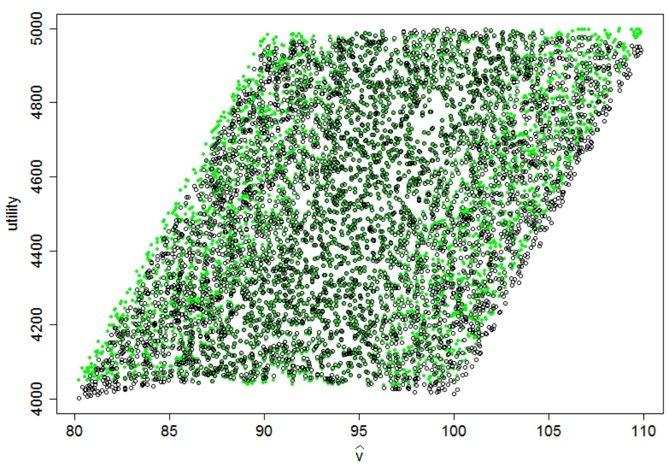
(b) Consumption allocation (black: no tax, blue: optimal [for each possible realization of v, b s.t. $\hat{v} = v + b$], orange: linear tax, green: non-linear tax)



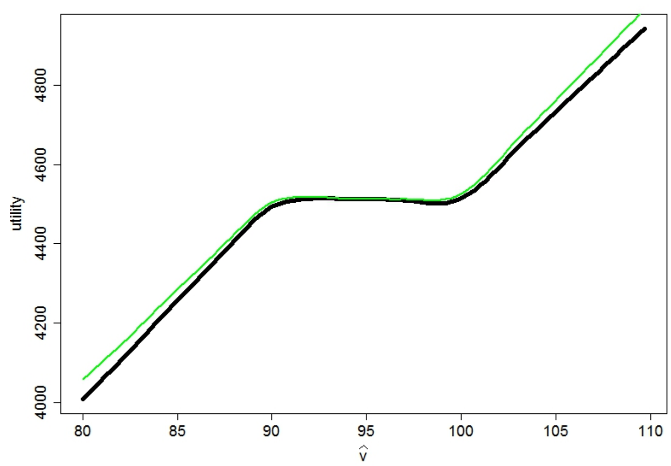
(c) Optimal non-linear marginal tax $t'(x)$



(d) Optimal non-linear tax $t(x)$



(e) Individual experienced utility in equilibrium for each possible realization of v, b s.t. $\hat{v} = v + b$ (black: no tax, green: non-linear tax)

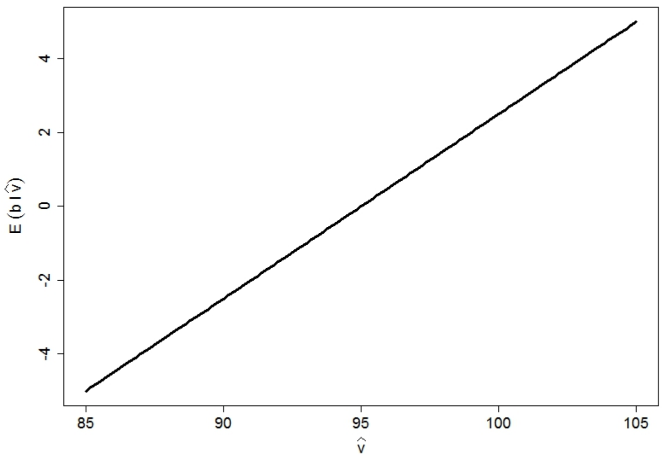


(f) Expected experienced utility in equilibrium (black: no tax, green: non-linear tax)

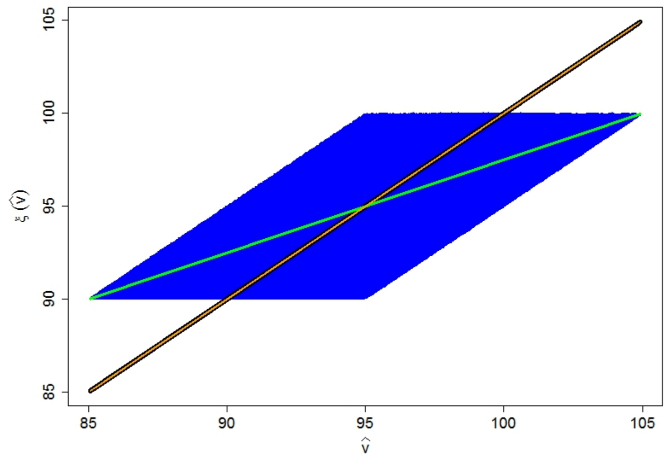
no tax: 4500.528 | linear tax: 4500.528 | non-linear tax: 4514.045

(g) Expected total welfare

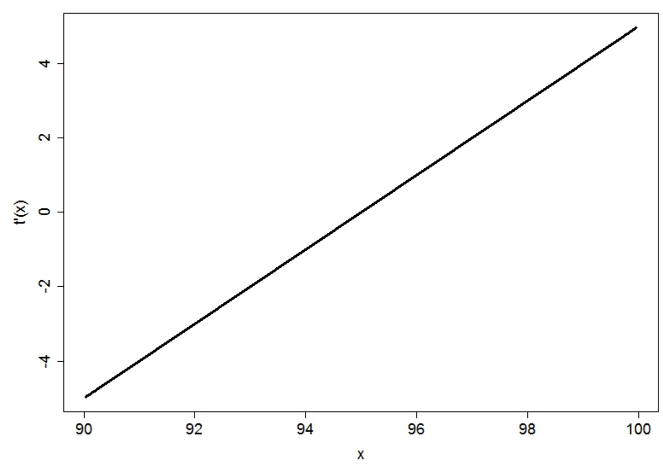
Figure 1.3: Scenario with $v \sim U[90, 100]$ and $b \sim U[-10, 10]$, i.e., $E[b] = 0$, and v, b independent.



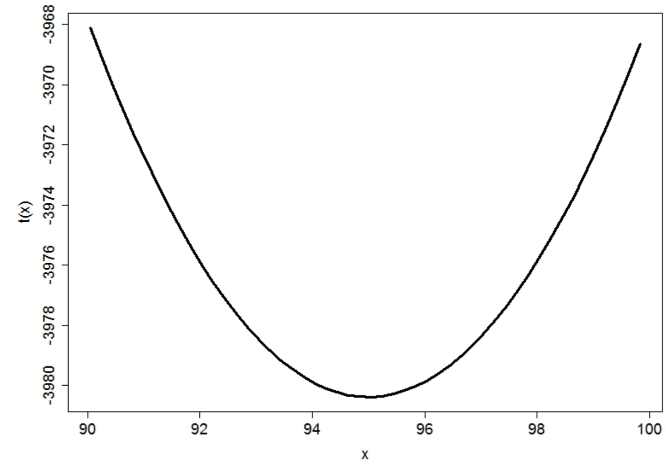
(a) Conditional expectation of the bias $E[b|\hat{v}]$



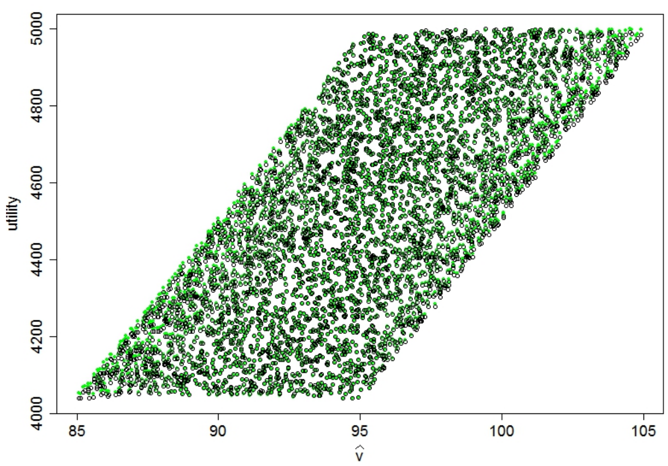
(b) Consumption allocation (black: no tax, blue: optimal [for each possible realization of v, b s.t. $\hat{v} = v + b$], orange: linear tax, green: non-linear tax)



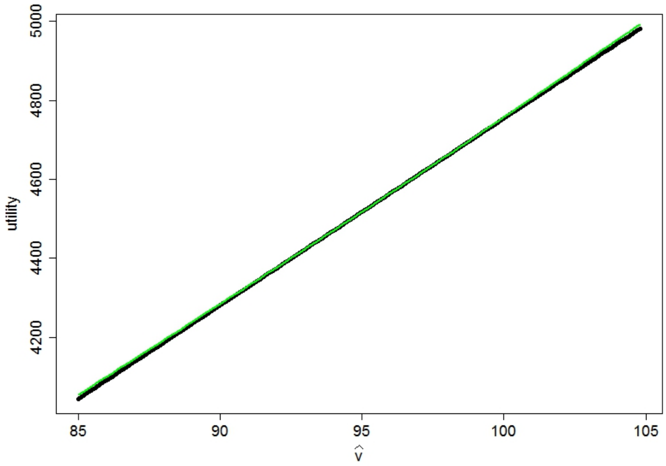
(c) Optimal non-linear marginal tax $t'(x)$



(d) Optimal non-linear tax $t(x)$



(e) Individual experienced utility in equilibrium for each possible realization of v, b s.t. $\hat{v} = v + b$ (black: no tax, green: non-linear tax)

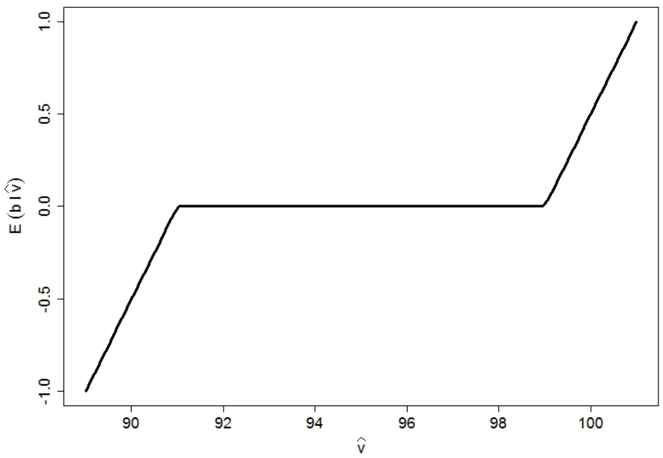


(f) Expected experienced utility in equilibrium (black: no tax, green: non-linear tax)

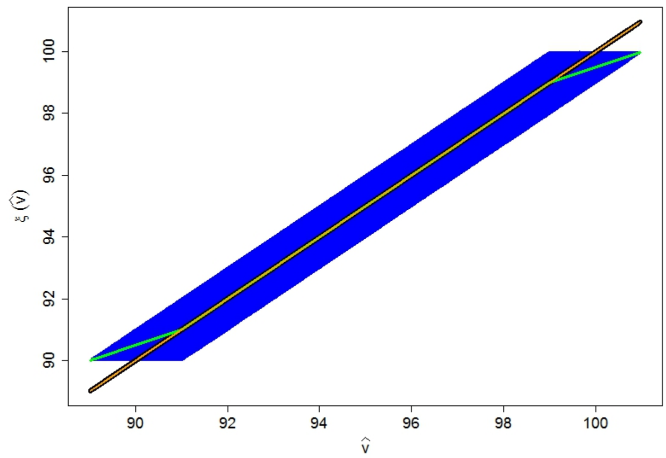
no tax: 4513.158 | linear tax: 4513.158 | non-linear tax: 4515.226

(g) Expected total welfare

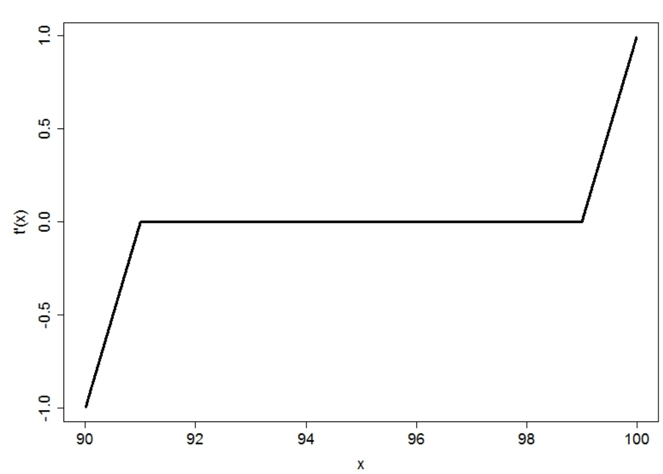
Figure 1.4: Scenario with $v \sim U[90, 100]$ and $b \sim U[-5, 5]$, i.e., $E[b] = 0$, and v, b independent.



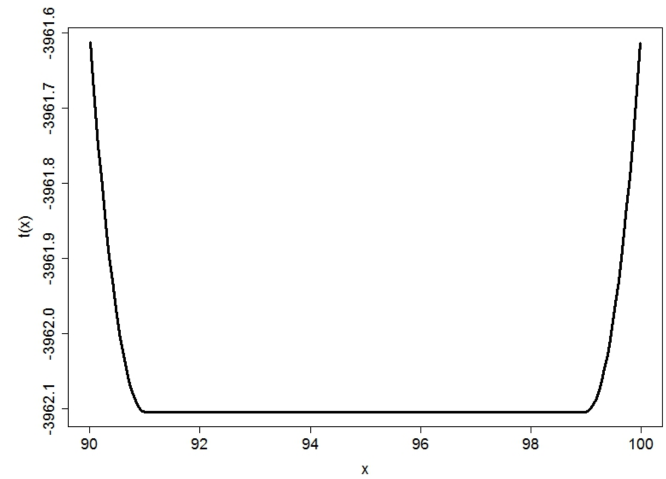
(a) Conditional expectation of the bias $E[b|\hat{v}]$



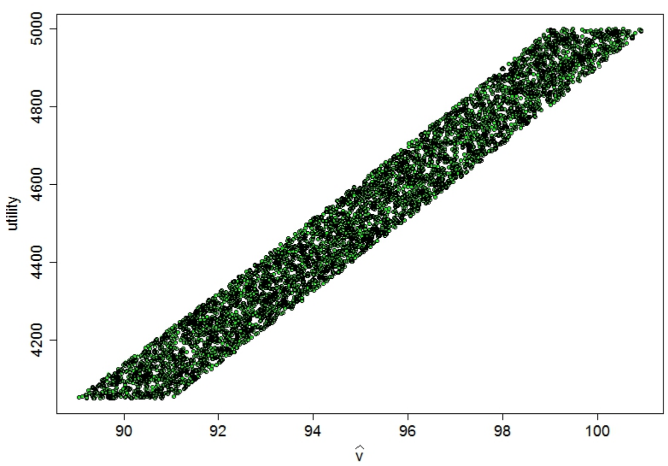
(b) Consumption allocation (black: no tax, blue: optimal [for each possible realization of v, b s.t. $\hat{v} = v + b$], orange: linear tax, green: non-linear tax)



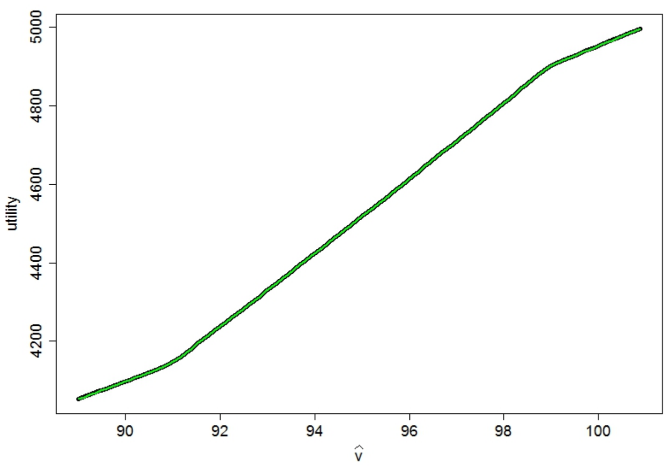
(c) Optimal non-linear marginal tax $t'(x)$



(d) Optimal non-linear tax $t(x)$



(e) Individual experienced utility in equilibrium for each possible realization of v, b s.t. $\hat{v} = v + b$ (black: no tax, green: non-linear tax)

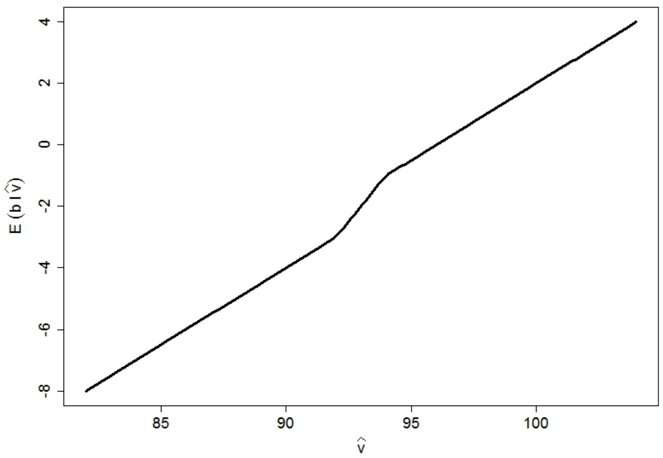


(f) Expected experienced utility in equilibrium (black: no tax, green: non-linear tax)

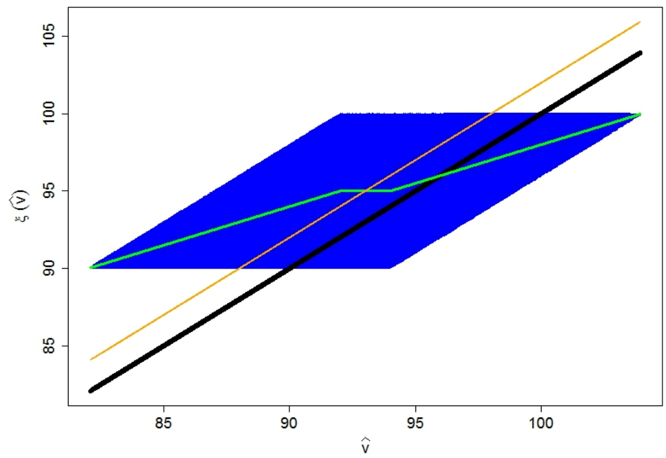
no tax: 4515.744 | linear tax: 4515.744 | non-linear tax: 4515.761

(g) Expected total welfare

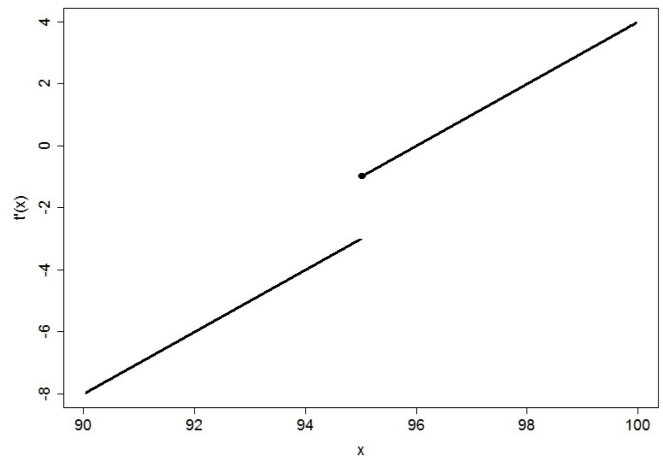
Figure 1.5: Scenario with $v \sim U[90, 100]$ and $b \sim U[-1, 1]$, i.e., $E[b] = 0$, and v, b independent.



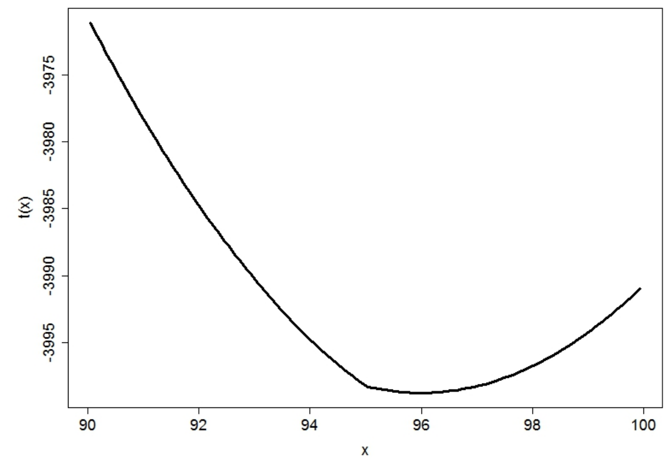
(a) Conditional expectation of the bias $E[b|\hat{v}]$



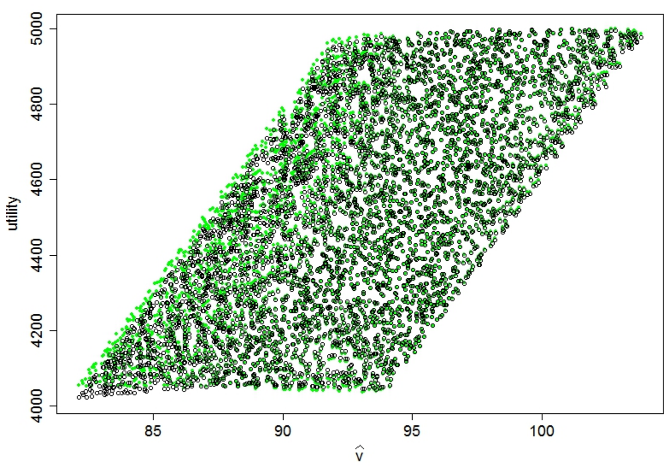
(b) Consumption allocation (black: no tax, blue: optimal [for each possible realization of v, b s.t. $\hat{v} = v + b$], orange: linear tax, green: non-linear tax)



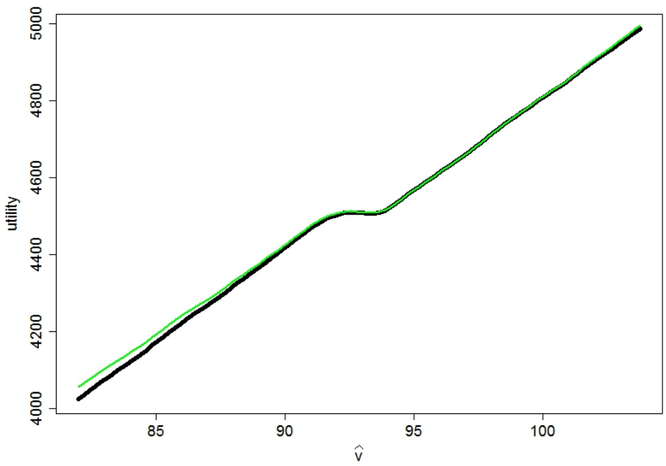
(c) Optimal non-linear marginal tax $t'(x)$



(d) Optimal non-linear tax $t(x)$



(e) Individual experienced utility in equilibrium for each possible realization of v, b s.t. $\hat{v} = v + b$ (black: no tax, green: non-linear tax)



(f) Expected experienced utility in equilibrium (black: no tax, green: non-linear tax)

no tax: 4507.835	linear tax: 4509.821	non-linear tax: 4513.393

(g) Expected total welfare

Figure 1.6: Scenario with $v \sim U[90, 100]$ and $b \sim U[-8, 4]$, i.e., $E[b] = -2$, and v, b independent.

Chapter 2

Information Design in Multi-Task Contests - Whom to Inform When the Importance of Tasks Is Uncertain

In many contests, competitors invest effort in different tasks. Ex ante it may not be clear to them how success in the contest will depend on the mixture of effort investments in the different tasks. For instance, when applying for a professorship, it may not be clear to applicants how exactly research performances in different fields are weighted against each other by the hiring committee. Nevertheless, the committee usually has the possibility of transmitting information to the contestants before the contest. This chapter addresses the question of how the information structure should be designed in this kind of setting in order to maximize contestants' joint effort. We show that, in a two-player Tullock contest with an ex-ante uncertain Cobb-Douglas impact function, the designer cannot benefit by transmitting purely public messages to the contestants. However, if the designer asymmetrically discloses information, she can evoke an increase of contestants' expected efforts. If the designer can send a purely private message to one contestant, depending on the competitiveness of the contest tasks reflected by comparative cost advantages, either no revelation, full revelation, or partial revelation of information may be beneficial for the designer. We show that, in some scenarios the principle of "informational favoritism" of an ex-ante disadvantaged player, e.g., disclosing information to the "weak" underdog, increases contestants' efforts, while in other scenarios "informational discrimination" of an ex-ante disadvantaged player, e.g., disclosing information only to the "stronger" of two specialists, is better.

2.1 Introduction

Information design in games is a recent and very active field of research in information economics, see Bergemann and Morris (2017) and Taneva (2016), for instance. It examines settings in which an information designer ("she" in the following) can manipulate the information structure which generates the messages sent to the receivers (all "he" in the following). These then interact in a so-called base game, which in this chapter is a contest. Models of contest theory, in which contestants compete for prizes by exerting costly efforts, have taken a prominent place in game-theoretic analyses of competition, see Vojnovic (2016), Konrad (2009), and for a non-formal discussion Frank and Cook (1996). A central question in this literature is how the contest designer can structure payoff incentives such that total effort exerted by contestants is maximized. However, how the designer can manipulate contestants' beliefs by information design in contests to achieve the same goal has not yet been studied extensively. This chapter contributes to closing the gap.

Information design in multi-task contests can be applied to many situations. The hiring committee of a university might reveal some information about the demanded expertise in different fields of research when recruiting a new professor. This, of course, generally applies to instances in which enterprises are hiring new employees. During the ice-figure skating competition at the Winter Olympics 2018 in South Korea contestants were unsure about how exactly strength, indicated for example by the hight of jumps, was valued by the judges compared to artistic skills, such as the elegance of the performance.[1] Remarks by members of the judging committee in press conferences might have given some hints on what matters more to convince the judges. Most scientific journals provide rules on what a paper worth to be published generally has to look like. Additionally, the reviewers may have specific selection criteria, and they can reveal some of that information in a review report. Similarly, teachers usually provide information on what is relevant in an exam during a review session at the end of the lecture term.

In all of these situations, the designer of the information structure (hiring committee, judging committee, ...) usually wants to maintain her reputation of being honest, because she represents a company or another kind of institution, which is likely to interact with people observing her current behavior in the future. Therefore, it is plausible to assume that she *credibly commits* to a certain information transmission strategy before even knowing the exact state of the world she can transmit information about. For

[1] See http://www.zeit.de/sport/2018-02/yuzuru-hanyu-eiskunstlaufen-winnie-pu-japan (last accessed 24/02/2018). In ice-figure skating there is no trade-off in judging athletic skill (A score) and artistic skill (B score) for the judges. However, judges might be more inclined to give higher scores in their preferred task (athletic or artistic), or they might be more attentive to the performance in their preferred task compared to the other one during an ice-figure skating competition.

instance, German universities have a guideline for hiring professors, which each hiring committee has to stick to, and which has been written down in a general form before the specific hiring committee meets. Nevertheless, each hiring committee may have specific preferences on how to fill a vacant chair at the faculty and may communicate (part of them) to potential applicants accordingly.

Zhang and Zhou (2016) have analyzed Bayesian persuasion, the single-receiver case of information design, in a Tullock contest. Importantly, this approach abstracts from the distinction between private and public information, which plays a fundamental role in information design in games and which we explicitly elaborate on. The first main contribution of this chapter is to show that the distinction between private and public information is crucial for information design in contests: the designer can never induce higher efforts by disclosing information purely publicly, meaning that both contestants receive the same message and that both of them have common knowledge about this. This result is driven by the fact that no contestant can obtain an informational advantage over his opponent in the purely-public-messages scenario. This, however, is important for information disclosure to be beneficial for the designer.

The next main contribution shows that if, in a two-player contest, the designer sends purely private messages to only one of the contestants, he can benefit from information revelation by adhering to a principle of "informational favoritism": while it is never beneficial to disclose information to a favorite, it is beneficial to disclose information to a ("specifically weak") underdog. Thus, optimal information disclosure in these settings is in favor of players with an ex-ante disadvantage. Furthermore, we illustrate that, in other settings, the designer might optimally reveal information to a contestant who is ex ante more likely to win the contest: if each contestant is a specialist in a different task, then the designer can benefit from disclosing information to the contestant whose preferred task is more likely ex ante. This reflects a principle of "informational discrimination" of ex-ante disadvantaged specialists.

Arbatskaya and Mialon (2010) introduce the multi-task two-player Tullock contest with a Cobb-Douglas impact function, which is the contest framework we employ here. This strand of contest research is related to the famous Colonel Blotto game literature, which is surveyed in Kovenock and Roberson (2017). Contests with multi-tasking have also been analyzed theoretically by Clark and Konrad (2007) and Epstein and Hefeker (2003), and experimentally by Deck et al. (2016).

The literature on optimal favoritism in contests encompasses the manipulation of the rules of the game in favor of ex-ante disadvantaged players. The design of the incentive structure, for instance by using head starts or biases in the contest success function, has been studied extensively, see Franke et al. (2018), for instance.

From the designer's perspective information design in games is a task of belief manipulation. We emphasize the two extreme cases of purely public and purely private messages. In some instances, this implies that we can apply the methodology of the seminal paper in the Bayesian persuasion literature by Kamenica and Gentzkow (2011), who analyze the "decision-theoretic" (i.e. single receiver) information design setup and derive their well-known "concavification results". In the paper "Information Design in Games", Mathevet et al. (2017) analyze "Bayesian persuasion in games" by extending the concavification approach to a framework with multiple strategically interacting receivers. They push forward the composition of the optimal information structure from a public and a private message component.[2] Bergemann and Morris (2016) present an alternative approach to information design using linear programming techniques and Bayes correlated equilibria, where - in contrast to the extension of correlated equilibrium to games of imperfect information introduced by Forges (1993) - one player (the sender of the message) has more information than others (the receivers). A survey on information design can be found in Bergemann and Morris (2017).

Research on issues arising from asymmetric information and information disclosure in contests has been subject of the work by Einy et al. (2017), Epstein and Mealem (2013), Kovenock et al. (2015), Ewerhart and Grünseis (2018), Fu et al. (2013), Denter et al. (2011), Morath and Münster (2013), Münster (2009), Roesler (2015), and Gürtler et al. (2015). Some of these papers examine how a contestant himself should reveal information about his attributes, e.g., effort costs. Others - if they examine how the designer should disclose information - either analyze how the designer should give feedback to contestants during the course of a tournament (by providing preliminary scores, for instance), or they analyze how she should disclose information to an uninformed contestant about his opponent's costs. In contrast, in this chapter both contestants share the same information at the beginning of the game and information is disclosed (potentially) to both of them by the designer, who has an information advantage over the contestants when she sends the messages.

The paper most closely related to the research at hand is Zhang and Zhou (2016).[3] They analyze Bayesian persuasion in a single-activity two-player Tullock contest, where one contestant has imperfect information about the cost function of his opponent and the contest designer decides whether to (partially or fully) disclose information about these costs (which she can observe), or not. In contrast to their analysis, the focus of

[2] Note, that most games discussed in the information design literature so far contain incentives among receivers to "coordinate" in certain situations, see Mathevet et al. (2017), for instance. In contrast to this, the contest model discussed here is a model of pure conflict with a zero-sum character.

[3] Melo Ponce (2017) also discusses information design in contests. The abstract (no complete paper is yet available; 02/06/2018) mainly explains the employed methodological procedure using the Bayes-correlated-equilibrium approach.

this chapter is how a multi-task contest framework can affect information design if *all* contestants have imperfect information. Furthermore, we explicitly account for the fact that when analyzing information design in games the distinction between public and private information is crucial, which has not been accounted for by Zhang and Zhou (2016).

This chapter is structured as follows. In the next section we present an example, which is followed by the model setup and an analysis of the Bayesian updating process. Then, we introduce a setting which we later on explore in more detail, namely the two-extreme-states scenario. In Section 2.6, we analyze the case in which the chosen information design is such that both contestants can perfectly infer the opponent's message. This setting includes non-revealing and purely public information structures. In Section 2.7 the designer can only send purely private messages to one contestant. To isolate the different effects of information design, we analyze this setting for non-Bayesian and for Bayesian agents. Section 2.8 concludes and gives a brief outlook on possible extensions. Most proofs can be found in the Appendix.

2.2 An Example[5]

A contest designer ("she") hosts a contest with two contestants (both "he"), named 1 and 2. Each contestant $i \in \{1,2\} =: N$ can invest effort in two different tasks, A and B. Let x_{it} denote i's effort in task $t \in \{A, B\} =: T$. The effectiveness of the effort in the tasks is ex ante uncertain to the contestants and the designer. There are two states of nature: in state s_A only effort in task A determines success in the contest, while in state s_B only effort in task B matters. The state is drawn randomly according to the commonly known prior and $p_A \in (0,1)$ denotes the ex-ante probability of state s_A. If state $t \in \{A, B\}$ is realized, then contestant i's probability of winning the prize of the contest with homogeneous value $v \equiv 100$ is given by $1/2$ if $x_{it} = x_{jt} = 0$, and by

$$\frac{x_{it}}{x_{it} + x_{jt}}$$

else, for $i, j \in N$, $j \neq i$. Contestants differ in their skills in each of the tasks: i's cost of exerting effort x_{it} is given by the linear cost function $c_{it}x_{it}$ for all $t \in T$, with $c_{it} \in \mathbb{R}_+$, for all $t \in T, i \in N$. The costs are known to each player in the game. Contestants maximize their expected utility (expected gain minus deterministic costs).

The contestants' evaluation of uncertainty in the contest is captured by the belief system $\tilde{\beta} := \left(\tilde{\beta}_1, \tilde{\beta}_2\right)$, where $\tilde{\beta}_i,\ i \in \{1,2\}$ reflects contestant i's belief system. The

[5] The numerical examples in this section are also discussed more formally in Section 2.7, and they are illustrated in the figures of that Section.

designer is able to manipulate the contestants' beliefs about the realization of the state by designing an information structure (M, Q) she commits to. Before she observes the state of nature, she has to decide about the information map Q, which is publicly known to all players. Afterwards, depending on the realization of the state, message $m := (m_1, m_2) \in M$ is sent to the contestants according to Q, where $m_i,\ i \in \{1, 2\}$ is the (component of the) message observed by contestant $i \in \{1, 2\}$, but not by contestant $j \in \{1, 2\}, j \neq i$.[6] For expositional purposes, we focus on two information structures among which the designer may choose in this section: full revelation (Q_i^{FR}), i.e., the designer discloses the true state of the world to contestant i, and non-revelation (Q^{NR}), i.e., no contestant receives any information from the designer.[7]

The "meaning" of a message is determined by Q and upon receiving a message, contestants update their belief system according to the message received. Thus, we denote the contestants' updated belief system after observing m, which is sent according to Q, by $\tilde{\beta}(m, Q)$. The contestants' aggregate equilibrium effort after observing m under Q is denoted by $X^*(\tilde{\beta}(m, Q))$. The contest designer wants to maximize the effort exerted by the contestants via information design.[8] Since the designer does not observe the state according to which messages are sent "via" Q, he solves a problem under uncertainty, where the objective is given by $E\left[X^*(\tilde{\beta}(m, Q))\right](Q)$. The dependence of this expectation on Q reflects the fact, that the choice of Q influences the probability of the realization of message $m \in M$. In short we often write $E[X^*](Q)$.

In any case the recipient of a fully revealing message generically faces two different situations: either he receives an "encouraging" message, i.e., he learns that the relevant task is his "preferred" task, or he receives a "discouraging message", i.e., he learns that the relevant task is the task he "dislikes".[9] Contestants may thus learn the extent to which they have a cost advantage or disadvantage compared to their opponent. Of course, this has an effect on the contestants' effort provisions.

The reaction to information described above takes into account what contestants may learn about the relevant task. In addition to this, the information designer can

[6]Note that in the following we often use the term "message" or "message of contestant i" instead of "component of the message that contestant i can observe". The meaning becomes obvious from the context.

[7]In Section 2.7 we analyze more sophisticated mechanisms, which allow also for partial revelation of information.

[8]Speaking in terms of some of the examples from above, the designer wants to hire the "best behaving" applicant. In this framework, we can think of the cost parameters as representations of the contestants' innate skills. Even if applicant 1 might be more "talented" than applicant 2, it makes sense to hire applicant 2 if he exerts higher efforts. The efforts exerted in the application process (considered also potentially in a broader sense, e.g., as reflected in university grades) are thus a good indicator of performance in the job later on.

[9]The term "preferred task" here means that the contestant has either a higher cost advantage or a lower cost disadvantage compared to his opponent in this task than in the other task. A "disliked task" is defined analogously.

manipulate the contestants' beliefs about the opponent's evaluation of uncertainty in the game, i.e., their higher order beliefs. For instance, suppose that both tasks are ex ante equally likely to matter, $p_A = 0.5$, and Q is chosen such that it is publicly known that the designer fully discloses the true state of the world to contestant 1, while contestant 2 does not receive any relevant information. In this case, upon receiving information, contestant 1 always knows the extent to which contestant 2 underestimates the probability of the relevant task, since the belief of the uninformed contestant is fixed by the prior. Contestants may thus learn that they have an information advantage over their opponent. Furthermore, since Q is public, the uninformed contestant knows about his information disadvantage. These observations obviously have an effect on effort provision as well.

The information designer has to evaluate how to optimally design the contestants' beliefs with respect to the cost advantage and the information advantage described above. In order to "credibly" disclose information to Bayes rational contestants, she faces the following trade-off. Suppose that the designer prefers both contestants to believe that task A is relevant since effort provision is higher in this task. She would want the contestants to always believe that task A is the true state of the world. However, if contestants know that the designer always, that is in states s_A and s_B, informs them that task A is relevant, then they will never believe her and the resulting information structure would be non-revealing. Therefore, to "credibly" disclose information to the contestants about her preferred state s_A being true, the designer sometimes, that is, for instance, whenever s_B is true, has to inform them that her least preferred, since least effort intense, task is relevant.

We now analyze the effects described above in a numerical example. Suppose that the effort costs are given by $c_{1A} = 9$ and $c_{1B} = 4$ for contestant 1, and by $c_{2A} = 1$ and $c_{2B} = 1$ for contestant 2. Due to his cost advantage in both tasks, we call contestant 2 the favorite. The favorite has a cost advantage in any state of the world. Using straightforward calculation of the first-order and equilibrium conditions, we can see that if always task A mattered, equilibrium efforts would be $x^*_{1A} = 1$ and $x^*_{2A} = 9$, so that total efforts in task A would be $X^*_A = 10$. Similarly, we can calculate $x^*_{1B} = 4$, $x^*_{2B} = 16$ and $X^*_B = 20$ for task B. Assuming a prior of $p_A = 0.5$, the expected total effort without any information disclosure is given by $E[X^*](Q^{NR}) = 0.5 \cdot X^*_A + 0.5 \cdot X^*_B = 15$.

Now let us analyze the incentives to transmit information by the designer. First, suppose that the designer is restricted to transmitting a purely public message to both contestants. Then, both Bayes rational contestants share the same belief, and therefore none of them can obtain an information advantage over his opponent. Note also, that the designer obviously would want *both* contestants to believe that the *same* state, s_B, is the real state of the world, as both contestants exert higher effort in task B.

However, due to the above mentioned trade-off inducing a high subjective probability that state B is true among contestants comes at the cost that, in some situations, the designer must reveal that state A is true. In Section 2.6.2 we show that, for the case of purely public information, the benefits of fully revealing a beneficial state for the designer is (in expectation) exactly weighed of by the costs of fully revealing a non-beneficial state. This result holds in a general scenario. Under purely public messages, the designer can never increase contestants' (expected) efforts compared to a situation without information disclosure.

The following discussion illustrates the important insight for information design in contests, which also pertains generally for information design in games. It is crucial to distinguish public and private information: in the multi-activity contest with uncertainty about the tasks, a setting of pure conflict, the designer can only benefit from information transmission if she distributes information asymmetrically, i.e., not purely publicly. Suppose, that the designer can send private messages only to one of the contestants and this is known to all players in the game. The general question then is: "Whom to inform when the importance of tasks is uncertain?", i.e., to which contestant should the designer disclose information to - if she does so at all?

Intuitively, in the numerical example from above it makes no sense for the designer to reveal any information in a private message to the favorite, since - in addition to having a cost advantage - he would also have an informational advantage, which induces him to exert lower effort in expectation. Furthermore, the underdog knows that he has an information disadvantage and thus is discouraged to exert higher effort.

However, if the designer discloses information to the underdog, she basically counterbalances his cost disadvantage and expected equilbrium effort will be higher. We show in Section 2.7 that this "informational favoritism" holds in a more general setting. The increase in effort after the - from the underdog's perspective - encouraging message "Task B is relevant." (the underdog's disadvantage is smaller in task B than in task A) is larger than the decrease in effort after the discouraging message "Task A is relevant.".[10] This is because, in addition to learning that her preferred task is relevant, the information advantage over the favorite, who underestimates the probability of the relevant task, induces the informed underdog to overproportionately increase efforts upon receiving the encouraging message compared to the decrease in effort upon receiving the discouraging message. In addition, the uninformed favorite exerts higher efforts, since he anticipates the underdog's increase in effort. Therefore, overall the designer can increase her expected payoff from $E[X^*](Q^{NR}) = 15$ in a situation without information disclosure to $E[X^*](Q_1^{FR}) = 22.1$ when fully disclosing the true state to the underdog.

[10]This is reflected for instance in the convexity of $\bar{X}^*(\beta_1^A)$ in Figure 2.5.

Now suppose that we change the cost structure in the example by switching the values of c_{1B} and c_{2B}: $c_{1A} = 9$ and $c_{1B} = 1$ are the cost parameters for contestant 1, and $c_{2A} = 1$ and $c_{2B} = 4$ for contestant 2. In aggregate terms nothing has changed: $X_A^* = 10$ and $X_B^* = 20$. However, the underlying strategic situation is fundamentally different: contestant 1 is a specialist in task B (compared to contestant 2), while contestant 2 is a specialist in task A (compared to contestant 1).

Since in this scenario each contestant exerts higher efforts in his preferred task under non-disclosure ($x_{1A}^* = 1$, $x_{1B}^* = 16$, $x_{2A}^* = 9$, $x_{2B}^* = 4$), in contrast to the scenario before, the designer wants *each* contestant to believe that a *different* state is true than his opponent. The issues arising from inducing different beliefs among the contestants, are briefly discussed in Section 2.8. However, the focus of this chapter is the purely-private-message scenario. It serves to analyze which contestant should be adressed with information. It can be shown that, in the scenario with two specialists and a prior $p_A = 0.5$, the designer cannot benefit from information transmission, since in expectation for both contestants the increase in effort after an encouraging message is less than the decrease after a discouraging message. The respective equilibrium efforts for these parameters are $E[X^*](Q^{NR}) = 15$ and $E[X^*](Q_1^{FR}) \approx 14.44$.

However, if $p_A = 0.2$ with the same cost parameters then the designer can benefit from disclosing information to contestant 1. The respective equilibrium efforts are $E[X^*](Q^{NR}) = 18$ and $E[X^*](Q_1^{FR}) \approx 18.99$. Note that, compared to the situation with $p_A = .5$, the designer discloses information to the specialist whose preferred task has become more likely, which is somewhat counterintuitive to the "informational favoritism" discussed above. The detailed intuition for these results will be presented in Section 2.7.

2.3 Model Setup

This section describes the dynamic game of imperfect information involving the designer ("she") and the two contestants (both "he"). We start by presenting the contestants' side and then continue to describe the information designer's side.

2.3.1 A Multi-Task Contest with Uncertainty and the Information Structure

The contestant side is modelled by a base game G, which in this case is a multi-task contest with uncertainty, and an information structure (M, Q).

The contest $G = (N, X, S, p, U)$ with uncertainty about the tasks is played among two contestants, who compete for a prize of homogeneous value v and where $N := \{1, 2\}$

is the set of players. Each contestant $i \in N$ chooses his strategic variables $x_i \in X_i \subseteq \mathbb{R}^k_+$, which specify an investment of effort x_{it} in each of the finitely many tasks $t \in T$, where $|T| = k$, and $X := \times_{i \in N} X_i$ is the action space, with X_i compact for all $i \in N$.

There are $l < \infty$ mutually exclusive σ-states of nature indicated by $s \in S$, and the vector $p := (p_{s_1}, ..., p_{s_l}) \in int\,(\Delta(S))$ depicts the prior probability of each σ-state.[11] The prior is commonly known. The σ-state-dependent weight of task t in σ-state s is captured by the scalar $\alpha_t^s \in \mathbb{R}_+$. The σ-state-dependent Cobb-Douglas impact function transforms effort inputs into output according to

$$f^s(x_i) = \prod_{t=1}^{k} x_{it}^{\alpha_t^s}.$$

The returns to scale parameter $D := \sum_t \alpha_t^s$ is constant across states. As usual, $D = 1$ implies constant returns to scale. Inspired by Tullock (1980), for $i, j \in N, i \neq j$, the σ-state-dependent contest success function, given by

$$CSF^s(x) = \begin{cases} 1/2 & \text{if } x_{it} = x_{jt} = 0, \ \forall t \in T \text{ with } \alpha_t^s > 0, \\ \frac{f^s(x_i)}{\sum_{j \in N} f^s(x_j)} & \text{else,} \end{cases}$$

determines the winning probability of player i in σ-state s given the effort profile $x := (x_1, x_2)$. Costs for effort investments of contestant i are non-stochastic and occur to him according to the linear cost function $\sum_t c_{it} x_{it}$, which might differ across contestants. Contestant i's risk-neutral Bernoulli utility is given by $u_i : X \times S \to \mathbb{R}$, so that the payoff structure of the game is captured by $U := (u_i)_{i \in N}$.

In addition to the base game, the setup involves an information structure (M, Q). It consists of the (finite) message space $M := \times_{i \in N} M_i$ and the information map $Q : S \to \Delta(M)$, which assigns a probability to each message $m \in M$ sent to the contestants in each σ-state.[12] A contestant i only observes the realization of $m_i \in M_i$, but not the (component of the) message sent to his opponent. The messages of the contestants, $m_1 \in M_1$ and $m_2 \in M_2$, may be correlated to a different degree.

Note that the information structure (M, Q) and the base game G constitute a Bayesian game. We call the realization $m^{gh} := (m_1^g, m_2^h) \in M$ the m-state of nature. Then, in the terminology of Harsanyi (1967), we can interpret Q as a probability distribution over the contestants' m-types and the message $m_i \in M_i$ as contestant i's m-type. Note that the m-type is the dimension of his type, that contestant i knows

[11] We can exclude events lying on the boundary of the simplex, since Bayesian updating would assign zero probability to them in any case. We use the terminology σ-state to distinguish it from the other random event in the model, the m-state, which is introduced below.

[12] It is without loss of generality to restrict the message profile to messages which have non-zero marginal probabilities across states under the prior.

when making his decision. He does not know the realization of the σ-state, which reflects the second dimension of his type, capturing the contestants' "competitive abilities". The state of nature in the game is thus captured by the σ-state, which reflects payoff-relevant information via the productivity of a certain effort vector, and the m-state, which shapes the belief hierarchy of the contestants. The latter can be influenced via information design by the contest host, as we discuss below. In Section 2.3.6, we characterize different information maps Q.

It is important to note that the meaning of message m_i is determined by the structure of Q. Contestants use Bayesian updating to process new information that they receive via the messages in order to assign a subjective probability to event e, which in this framework can be a realization of the σ-state, or the m-state, or both:

$$\beta_i^e(g,Q) := P(e|m_i = m_i^g)(Q) = P(e)(Q) \cdot \frac{P(m_i^g|e)(Q)}{P(m_i^g)(Q)}. \tag{2.1}$$

The notation $P(e)(Q)$ indicates that the occurrence of certain events (for instance the realization of specific messages) depends on the information map Q. When contestant i is of m-type m_i, his evaluation of the uncertainty of the σ-state is captured by his first-order belief $\beta_i^\sigma(m_i,Q) = (\beta_i^{\sigma 1}(m_i,Q),...,\beta_i^{\sigma l}(m_i,Q)) \in \Delta(S)$ about the σ-state of nature, so that $\beta_i^{\sigma s}(m_i,Q), s \in S$ is the probability of σ-state s under type m_i's belief. Obviously, $\sum_s \beta_i^{\sigma s}(m_i,Q) = 1$ for each type $m_i \in M_i$ of both contestants $i \in N$.

Additionally, higher-order beliefs in this strategic setting capture i's beliefs about the opponent's beliefs. For instance, the second-order belief of player i about j's first-order belief about the σ-state s is denoted by β_{ij}^σ. The whole system of beliefs of contestant i is captured by his belief hierarchy $\tilde{\beta}_i$, an infinite and recursively defined sequence of higher-order beliefs whose elements are coherent.[13] Let $\tilde{\beta} := (\tilde{\beta}_1, \tilde{\beta}_2)$ denote the belief system of the contest. The message $m_i \in M_i$ shapes contestant i's belief hierarchy. Hence, we write $\tilde{\beta}_i(m_i,Q)$ to denote contestant i's belief hierarchy upon receiving message m_i under Q and $\tilde{\beta}(m,Q)$ to denote the complete belief system in the game upon the realization of message $m \in M$ under Q.

Due to the stochastic impact function, contestant i's ex-ante maximization problem involves an expected utility form and in ex-ante terms is given by

$$\max_{x_i \in X_i} EU_i(x) := \max_{x_i \in X_i} E[u_i(x,s)] = \left[\sum_{s \in S} p_s \sum_{m \in M} P(m|s) \frac{f^s(x_i)}{f^s(x_i) + f^s(x_j(m))}\right] v - \sum_{t \in T} c_{it} x_{it}. \tag{2.2}$$

[13]Intuitively, a belief hierarchy is coherent, if for any level in the hierarchy, the beliefs of that level coincide with all beliefs of lower order on lower order events, see Maschler et al. (2013, pp. 444) for details.

Contestant i's best response as type m_i to his opponent's effort choices $x_{-i} \in \mathbb{R}^k$ given his m_i-type-dependent belief $\beta_i^\sigma(m_i, Q)$ is given by maximizing his interim utility (which we discuss in detail in Section 2.4):

$$x_i(x_{-i}|m_i) := \arg\max_{x_i} EU_i(x|m_i)$$

Contestant i's equilibrium effort given his belief hierarchy is denoted by $x_i^*(\tilde{\beta}_i)$. Finally, $X_t^*(\tilde{\beta}) := \sum_{i \in N} x_{it}^*(\tilde{\beta})$ denotes total equilibrium effort in task t given the belief system $\tilde{\beta}$, and

$$X^*(\tilde{\beta}) := \sum_{t \in T} X_t^*(\tilde{\beta}). \tag{2.3}$$

is the total equilibrium effort exerted by the contestants given their belief systems.

2.3.2 Information Designer

The designer awards the prize v according to the contest success function which determines the contestants' winning probabilities in each σ-state depending on their effort choices. The only strategic task of the contest organizer, however, is to design the information structure (M, Q) according to which information is transmitted to the contestants. The choice set $\mathcal{Q} := \{P | P : S \to \Delta M\}$ is the set of state dependent probability distributions over M. The designer's profit function is given by $\pi : X \times S \to \mathbb{R}$ with $(x, s) \mapsto X^*(\tilde{\beta}(m, Q))$. The objective of the designer in the setup at hand is given by maximizing the contestants' expected aggregate efforts over Q,

$$\max_{Q \in \mathcal{Q}} \sum_{m \in M} P(m)(Q) X^*(\tilde{\beta}(m, Q)), \tag{2.4}$$

where $P(m')(Q) := \sum_{s \in S} p_s Q(m = m')(s)$ for all $m' \in M$ and the notation of the probability of the realization of message $m' \in M$, $P(m')(Q)$, indicates that this probability depends on Q. The tuple (π, G) is also called the information design environment.

2.3.3 Timing of the Game

The contest host designs the information structure (M, Q) and commits to sending the messages to the contestants according to the chosen design.[14] Afterwards, nature randomly determines the production parameters, i.e., the σ-state. Then the contestants each receive a private message leading them to update their beliefs about the state of the

[14] Since Q is a probability distribution over M, in fact nature draws a message according to (M, Q).

world (including beliefs about their oppenent's beliefs).[15] Finally, contestants choose their efforts in each task of the contest and the prize is assigned according to the contest success function.

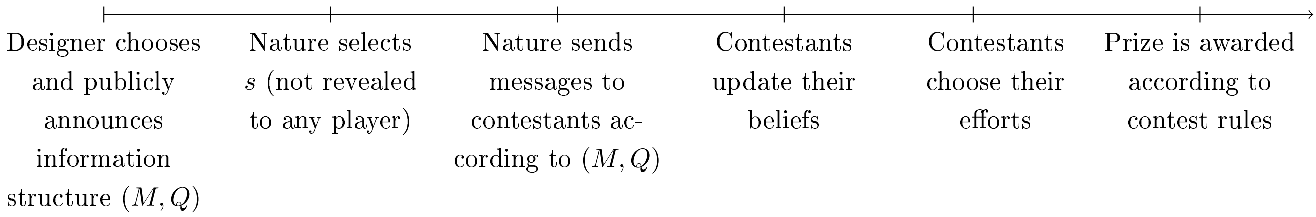


Figure 2.1: Timing of the game with information design in a multi-task constest.

2.3.4 Definitions: Measuring Competitiveness

In the subsequent sections we will see that it makes sense to examine the dependence of the optimal information structure on the contestants' cost structures, which reflect the degree of competitiveness in the different tasks and in the contest overall. Suppose for now that there are only two tasks, i.e., without loss of generality $T = \{A, B\}$. Defining $C_t := \frac{c_{1t}}{c_{2t}}$ with $t \in T$ implies that contestant 1 has a relative (dis)advantage in task t iff $C_t < 1$ ($C_t > 1$). Ceteris paribus, task t is more competitive, the closer C_t is to one. Define $C := (C_A, C_B)$. Additionally, $C_i := \frac{c_{it}}{c_{id}}$ with $i \in N$, $t, d \in T, t \neq d$, similary captures whether contestant i has a relative ("internal") advantage in performing task t or task d.

Contestants are symmetric if $C_A = C_B = 1$. If additionally $C_2 = 1$ then exerting effort in both tasks is equally costly, while if $C_2 > 1$ then task A is more costly. We can distinguish the following cases of interest from the point of view of contestant 1, and which are determined by the comparative cost advantages.

Definition 2.1. *Contestant 1 is a*

1. ***Favorite** iff $C_A, C_B \leq 1$, i.e., $C \in \mathcal{F} := (0, 1]^2$, and an*
2. ***Underdog** iff $C_A, C_B > 1$, i.e., $C \in \mathcal{U} := (1, \infty)^2$, and a*
3. ***Specialist***
   (a) *... **in Task A** iff $C_A \leq 1 < C_B$, i.e., $C \in \mathcal{S}_A := (0, 1] \times (1, \infty)$,*
   (b) *... **in Task B** iff $C_B \leq 1 < C_A$, i.e., $C \in \mathcal{S}_B := (1, \infty) \times (0, 1]$*

The definitions for contestant 2 are analogous.

[15] Maschler et al. (2013) distinguish state of nature (random events in the game excluding beliefs) and state of the world (random events in the game including beliefs). Note that we distinguish σ-states, which determine the effectiveness of efforts, and m-states, which shape the contestants' beliefs.

The according partition of the parameter space is visualized in Figure 2.2. Note that specialists are defined relative to their opponents. Therefore, a specialist may have higher absolute effort costs for what we call his "preferred" task compared to the other task. We call task $t \in T$ contestant 1's "preferred" task if $C_t < C_d$, $t, d \in T, t \neq d$ (and analogously for contestant 2).

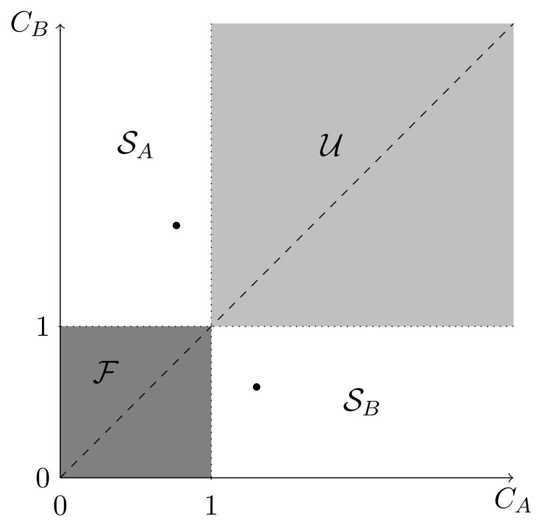


Figure 2.2: Partition of the cost-parameter space with two tasks. The two points indicate identical cost-parameter constellations "from the point of view of the different contestants".

There exists a linear homeomorphism, namely $\xi(C) := (\frac{1}{C_A}, \frac{1}{C_B})$, between $\mathcal{F}$ and $\mathcal{U}$, and similarly between $\mathcal{S}_A$ and $\mathcal{S}_B$, such that a cost-parameter constellation from the point of view of contestant 1 can be identified with a unique point in the topologically isomorphic space (as visualized in Figure 2.2), which "depicts the situation from the perspective of contestant 2". Note that for all $C \in \{C \in \mathbb{R}^2_{++} | C_A \geq C_B\}$, i.e., points below the angle bisector in Figure 2.2, contestant 1 "prefers" task B.

2.3.5 Assumptions and Remarks on the Setup

Note that in a setup with only two tasks, e.g., the two-extreme-states scenario (see Section 2.5), if we normalize the cost structure by, e.g., c_{2A}, the complete (relative) cost structure of the contest is pinned down by C_A, C_B, and C_2, where the last parameter captures the relative costs of task B compared to task A for both contestants.

Assumption 2.1. *We normalize the contestants' homogeneous valuation of the prize, $v \equiv 1$, and assume that it is large enough relative to the cost parameters, so that each contestant will invest a strictly positive effort into any task that is assigned with a positive subjective probability by that contestant.*

This assumption implies that the existence of equilibria described by the analysis below is guaranteed. The following assumption is implicitly made in the discussion above.

Assumption 2.2 (Commitment Assumption)**.** *The designer chooses the information map Q before observing the realization of the σ-state. After observing the realization $s \in S$ she transmits a message according to Q.*

From a model-theoretic perspective, this assumption is restrictive, since an ex-ante optimal information map may not necessarily imply the transmission of a message which is optimal ex post for the designer. For instance, consider a case in which the designer has a preferred σ-state and it is ex-ante optimal to always disclose the true state (full revelation). Due to the trade-off induced by being credible (as discussed in Section 2.2), the designer has to disclose that a less preferred state is realized in some states. If the realization of the σ-state implies that (under full revelation) the designer must reveal that such a realization occurred, she would want to act against her commitment and send a message which induces a belief that she prefers. As discussed in the introduction, it is plausible to assume that social institutions, such as universities or firms, can credibly commit to information maps, since they want to maintain an image of credibility among the economic agents, with whom they interact frequently. A similar argument is made by Zhang and Zhou (2016, p. 2199), who explicitly discuss the plausibility of the commitment assumption in contests. We additionally refer the reader to Kamenica and Gentzkow (2011, Section I.C, pp. 2597), who extensively discuss the commitment assumption and its applicability to real-world scenarios. Their arguments also apply to the setup at hand.

Analogously to the Revelation Principle, which allows the mechanism designer without loss of generality to restrict the choice set of optimal mechanisms to the set of direct mechanisms, Bergemann and Morris (2016) show that the information designer can without loss of generality restrict herself to information structures in which the cardinality of the message space of each individual equals that of the σ-state space when searching for the optimal information structure. Thus, it is without loss of generality to make the following assumption (and to fix M in the optimization problem of the designer described above, as we did).

Assumption 2.3. $|M| = |S| = l$ *and* $M_1 = M_2$.

In mechanism design theory the designer shapes the incentives of the agents. When implementing a truthful (direct) mechanism her freedom in creating incentives is restriced by incentive compatibility and participation constraints. Similarly, Mathevet et al. (2017) show that in information design theory the designer's freedom in shaping contestants' beliefs is restricted to designs which implement consistent belief-hierarchy distributions, i.e., distributions which can be generated from the same prior, and which guarantee Bayes' plausibility for one of the contestant's first-order belief.[16] The latter is satisfied if for some $i \in N$ it holds that for all $s \in S$

$$\sum_{g=1}^{l} P(m_i^g)(Q)\beta_i^{\sigma s}(m_i^g, Q) = p_s. \tag{2.5}$$

This implies that the expected posterior under Q equals the prior. As indicated in the introduction, the consistency criterion is a constraint on the degree to which players may disagree, which is reflected in their belief hierarchies. Both, Bayes plausibility and the consistency criterion, are satisfied in the main analyses of this chapter, Sections 2.6 and 2.7, due to the modeling assumptions.[17]

2.3.6 Definitions: Comparing Information Maps

Define

$$q_{gh}^s := P(m^{gh}|s) = Q(m_1 = m^g, m_2 = m^h)(s)$$

to be the probability of contestant 1 receiving message m^g and contestant 2 receiving m^h in state s under map Q. Additionally, we define

$$\bar{q}_{1g}^s := \sum_{h=1}^{|M_2|} q_{gh}^s \qquad \text{and} \qquad \bar{q}_{2g}^s := \sum_{h=1}^{|M_1|} q_{hg}^s$$

to be the marginal probability for contestant 1, and 2, respectively, to receive message m^g in state s.

Definition 2.2. *For contestant $i \in N$, Q is a*

- *fully-revealing [FR] information map, denoted by Q_i^{FR}, whenever for $s \in S$, $g \in \{1, ..., |M_i|\}$ it holds that $\bar{q}_{ig}^s = 1$ if $s = g$, $\bar{q}_{ig}^s = 0$ else, and a*
- *non-revealing [NR] information map, denoted by Q_i^{NR}, whenever $\bar{q}_{ig}^s = \bar{q}_{ig}^z$ for all $z, s \in S$, $s \neq z$, and for all $g \in \{1, ..., |M_i|\}$, and a*

[16] If there is only one receiver, as in standard Bayesian persuasion, then optimal information design only needs to satisfy Bayes' plausibility, as the consistency criterion is trivially satisfied, see Kamenica and Gentzkow (2011).

[17] Consistency is always satisfied since the designer in the setup either sends the same message to both contestants or she leaves one contestant completely uninformed about the message sent to the other opponent. Thus, contestants cannot agree to disagree. We refer the reader to Mathevet et al. (2017) for a detailed discussion of Bayes plausibility and the consistency criterion. It can easily be verified that Bayes plausibility is always satisfied in the setup at hand. Note that the definition of consistency used in this context is different from that used in the definition of a sequential equilibrium, see Fudenberg and Tirole (1991, pp. 337).

- *partially-revealing [PR] information map, denoted by* Q_i^{PR}*, whenever it is neither NR nor FR for* i*.*

If an information map is FR*,* NR*, or* PR *for both contestants, we write* Q^{FR}*, or* Q^{NR}*, or* Q^{PR}*, respectively. Furthermore, we say that an information map* Q *is*

- *a purely public information map, denoted by* Q^{PUB}*, if* $q_{gh}^s = 0$ *whenever* $h \neq g, g \in \{1, ..., |M_i|\}, h \in \{1, ..., |M_j|\}, i, j \in N, i \neq j$ *for all* $s \in S$*, and*
- *a purely private information map with information (potentially) revealed only to contestant* $i \in N$*, denoted by* Q_i^{PRI}*, if in any state* $s \in S$ *for* $j \in N, j \neq i$*:* $\exists! m_j^g \in M_j$ *s.t.* $\bar{q}_{jg} > 0$*.*[18]

The intuitions are as follows. Under Q_i^{FR}, contestant i is certain about the realization of the σ-state in any state of the world. Under Q_i^{NR}, contestant i does not obtain any new information on the realization of the σ-state in any state of the world. Under Q_i^{PRI}, in any state of the world, contestant $j \in N, j \neq i$ does not obtain any new information on the realization of $m_i \in M_i$, the realization of the opponent's message, or the σ-state. Under Q^{PUB}, both contestants are certain about the realization of the m-state in any state of the world, since they always receive the same message.

2.4 Bayesian Updating and Interim Expected Utility

Upon receiving message $m_i^g \in M_i$ contestant $i \in N$ updates his subjective probabilities for the σ-states according to

$$\beta_i^{\sigma s}(g) = P(s) \cdot \frac{P(m_i^g|s)}{P(m_i^g)} = p_s \cdot \frac{\bar{q}_{ig}^s}{\sum_k p_k \bar{q}_{ig}^k}, \tag{2.6}$$

and for the m-states (with a slight abuse of notation)[19] according to

$$\beta_i^{m_j^h}(g) = P(m_j^h) \cdot \frac{P(m_i^g|m_j^h)}{P(m_i^g)} = \left(\sum_s p_s \bar{q}_{jh}^s\right) \cdot \frac{\sum_w p^w \frac{q_{gh}^w}{\sum_d q_{gd}^w}}{\sum_z p_z \bar{q}_{ig}^z}. \tag{2.7}$$

[18]Note that if no information is revealed in Q_i^{PRI}, then according to the definitions, this information map also is purely public.

[19]For $i = 2$ we need to replace gh with hg and gd with dg in the terms on the RHS of the last equality sign.

From the point of view of contestant i's type $m_i^g \in M_i$ each of the $l \cdot l$ different states of the world is (with a slight abuse of notation)[20] assigned with the probability

$$\beta_i^{m_j^h \wedge s}(g) = P(m_j^h \wedge s) \cdot \frac{P(m_i^g | m_j^h \wedge s)}{P(m_i^g)} = \left(p_s \bar{q}_{jh}^s\right) \cdot \frac{q_{gh}^s}{\sum_k p_k \bar{q}_{ig}^k}. \tag{2.8}$$

Using the updated beliefs the contestant's interim expected utility in information set $m_i^g \in M_i$ can be written as

$$EU_i(x|m_i^g) = \left[\sum_{s\in S} \sum_{h\in\{1,\ldots,l\}} \beta_i^{m_j^h \wedge s}(g) \frac{f^s(x_i)}{f^s(x_i) + f^s(x_j(m_j^h))}\right] - \sum_t c_{it} x_{it}, \tag{2.9}$$

where $x_j(m_j^h)$ denotes contestant i's anticipation of the opponents behavior if he is of type m_j^h. Off-equilibrium beliefs are defined so that they support equilibrium play.[21]

2.5 The Two-Extreme-States Scenario

Since we employ the two-extreme-states scenario in the main part of the analysis, namely in Section 2.7, we will describe it here in more detail. We set $S = \{s_A, s_B\}$ and $M = \{m^A, m^B\} \times \{m^A, m^B\}$. We can without loss of generality assume that message m^g reads "s_g is the true σ-state of the world". The information map is defined by Table 2.1.

$Q(\cdot\|s_A)$	m^A	m^B	
m^A	q_{AA}^A	q_{AB}^A	$\bar{q}_{1A}^A$
m^B	q_{BA}^A	q_{BB}^A	$\bar{q}_{1B}^A$
	$\bar{q}_{2A}^A$	$\bar{q}_{2B}^A$	

$Q(\cdot\|s_B)$	m^A	m^B	
m^A	q_{AA}^B	q_{AB}^B	$\bar{q}_{1A}^B$
m^B	q_{BA}^B	q_{BB}^B	$\bar{q}_{1B}^B$
	$\bar{q}_{2A}^B$	$\bar{q}_{2B}^B$	

Table 2.1: Information map Q for $|S| = 2$. Contestant 1's messages are denoted in the rows, while contestant 2's messages are denoted in the columns. For all $s \in S$ it holds that $\sum_d \sum_t q_{dt}^s = 1$.

With the definitions and conventions from above we can, for instance, see that $\bar{q}_{2A}^A = q_{AA}^A + q_{BA}^A$ is the degree of truthfullness of Q to contestant 2 in state s_A. Thus, if $\bar{q}_{2A}^A = \bar{q}_{2B}^B = 1$ then Q is fully-revealing for contestant 2.[22]

[20] For $i = 2$ we need to replace gh with hg in the terms on the RHS of the last equality sign.

[21] For instance, if $i \in N$ receives a message that is assigned with zero probability under Q, then he interprets this as a random mistake and acts as if he did not receive any new information.

[22] Technically, also $\bar{q}_{2A}^A = \bar{q}_{2B}^B = 0$ yields a fully revealing information map, only that contestant 2 knows that the designer is lying with certainty, if we assume that m^g is to be understood as "s_g is the realized σ-state.". Remember, that generally the "meaning" of the messages is determined endogenously by Q.

Each contestant i can be in one of two distinct information sets, which are uniquely identified by the message he receives: m_i^A or m_i^B (see Figure 2.3). An information set contains four different states of the world, each consisting of different combinations of two different σ- and m-states.

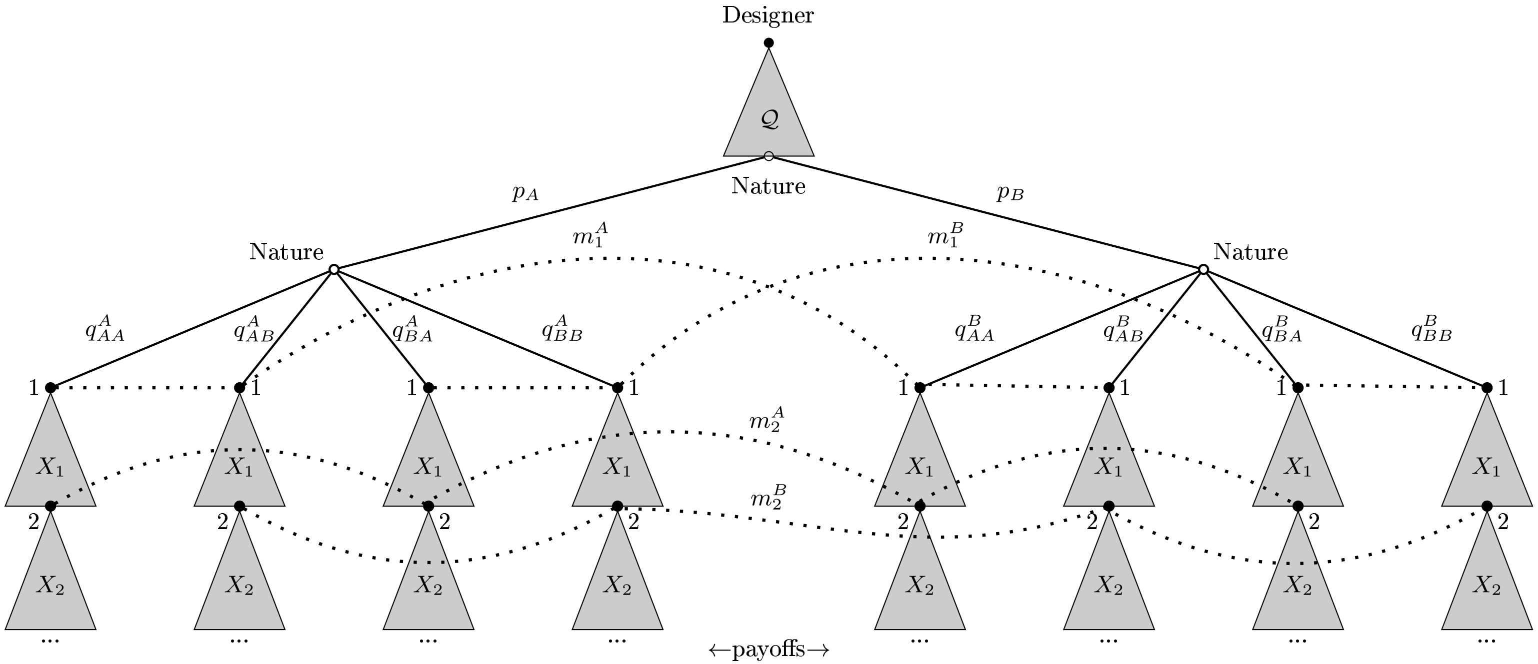


Figure 2.3: Extensive form of the game with two σ-states, i.e. $|S| = 2$.

Suppose now that in addition there are only two tasks, labelled A and B. Furthermore, the impact function exhibits constant returns to scale, $D = 1$.[23] We assume that $\alpha_A^{s_A} = \alpha_B^{s_B} = 1$, and thus $\alpha_B^{s_A} = \alpha_A^{s_B} = 0$. Therefore, it makes sense to label the states, priors, and beliefs according to which task is relevant: $p_A := p_{s_A}$, $p_B := p_{s_B} = 1 - p_A$, and finally $\beta_i^A := \beta_i^{\sigma s_A}$, $\beta_i^B := \beta_i^{\sigma s_B} = 1 - \beta_i^A$.

The interim objective from Equation (2.9) becomes

$$\begin{aligned} EU_i(x|m_i^g) \quad = \quad & \beta_i^{m_j^A \wedge s_A}(g) \cdot \tfrac{x_{iA}}{x_{iA}+x_{jA}(m_j^A)} + \beta_i^{m_j^B \wedge s_A}(g) \cdot \tfrac{x_{iA}}{x_{iA}+x_{jA}(m_j^B)} \\ & + \beta_i^{m_j^A \wedge s_B}(g) \cdot \tfrac{x_{iB}}{x_{iB}+x_{jB}(m_j^A)} + \beta_i^{m_j^B \wedge s_B}(g) \cdot \tfrac{x_{iB}}{x_{iB}+x_{jB}(m_j^B)} \\ & - c_{iA}x_{iA} - c_{iB}x_{iB}. \end{aligned} \tag{2.10}$$

Best responses of contestant i of m-type m_i^g can be calulated by maximizing $EU_i(x|m_i^g)$ with respect to x_i. Contestant i's m_i^g-type equilibrium behavior, denoted by $x_i^*(m_i^g) := (x_{iA}^*(m_i^g), x_{iB}^*(m_i^g))$, is then obtained by replacing the anticipation of the opponents type-dependent behavior, $x_{jt}(m_j^h)$, with their type-dependent best reponses in the resulting system of eight equations (each contestant has two types, and each type has to invest effort in two tasks) and solving. Note that an explicit analytical solution is not always obtainable.

2.6 Analysis: Contestants Know the Opponent's Message

In this section we analyze settings, in which both contestants are certain about m-state in any state of nature. That is, they know the message received by the opponent. We denote information maps which induce such a situation with $\tilde{Q}$. This subsumes cases in which the designer chooses a non-revealing or a purely public information structure. We will discuss these two cases in further detail below.

Without loss of generality we can model $\tilde{Q}$ by assuming that both contestants always receive the same message. Therefore, there is only uncertainty about the σ-state of nature, but not about the m-state of nature:

$$\beta_i^{m_j^h \wedge s}(g) = 0 \quad \forall s \in S, g, h \in \{1, ..., l\},\ h \neq g,\ \forall i \in N.$$

Thus, contestants share the same belief about the σ-state of the world, and they both know this. We can denote the commonly shared belief about the σ-state of nature after receiving message m under $\tilde{Q}$, which pins down the complete system of beliefs $\tilde{\beta}$ in

[23] In contrast to the general setup, in the two-extreme-states scenario the contest can be split into two "parallel" contests, each of which ocurrs with a certain probability. This makes it more tractable for the further analysis.

the game, by $\bar{\beta}^{\sigma}(m,\tilde{Q}) = (\bar{\beta}^{\sigma 1}(m,\tilde{Q}),...,\bar{\beta}^{\sigma l}(m,\tilde{Q}))$, where $\bar{\beta}^{\sigma s}(m,\tilde{Q})$ is defined as the subjective probability assigned to σ-state $s \in S$ given the common belief under Q upon receiving message m.

Proposition 2.1. *The optimal contestant behavior in the equilibrium of the two-player multi-task contest with uncertainty about the tasks and with an information map $\tilde{Q}$, which reveals the m-state to both contestants, is described by*

$$x^*_{it}(\tilde{\beta}(m,\tilde{Q})) = \frac{1}{c_{it}} \sum_{s \in S} \bar{\beta}^{\sigma s}(m,\tilde{Q}) \alpha^s_t \frac{\prod_{z \in T} \left(\frac{c_{iz}}{c_{jz}}\right)^{\alpha^s_z}}{\left[1 + \prod_{z \in T} \left(\frac{c_{iz}}{c_{jz}}\right)^{\alpha^s_z}\right]^2} \quad \forall i,j \in N, j \neq i, \forall t \in T, \forall m \in M. \tag{2.11}$$

Proof. See Appendix 2.A.1. □

2.6.1 Benchmark: Non-Revelation

In this section no contestant obtains new information on the σ-state and both contestants are certain about the m-state. For the two-states scenario a non-revealing information map can, for instance, be characterized by in Table 2.2.

$Q(\cdot\|s_A)$	m^A	m^B
m^A	1	0
m^B	0	0

$Q(\cdot\|s_B)$	m^A	m^B
m^A	1	0
m^B	0	0

Table 2.2: Non-revealing information map Q^{NR} for $|S| = 2$. Contestant 1's messages are denoted in the rows, while contestant 2's messages are denoted in the columns.

For both contestants the posterior on the σ-state equals the prior p. Thus, we obtain the following result.

Corollary 2.1. *Under information map Q^{NR} (non-revealing for both contestants), the contestants' equilibrium efforts are given by replacing $\bar{\beta}^{\sigma s}(m,\tilde{Q})$ with p_s and $\tilde{Q}$ with Q^{NR} in Proposition 2.1. In particular, for the two-extreme-states scenario we obtain*

$$x^*_{it}(\tilde{\beta}(m,Q^{NR})) = \frac{p_t}{c_{jt}\left(1 + \frac{c_{it}}{c_{jt}}\right)^2}, \quad i,j \in \{1,2\}, j \neq i, \forall m \in M, \forall t = \{A,B\}, \tag{2.12}$$

and for all $m \in M$ the aggregate effort in task A and task B is given by,

$$X^*_A(\tilde{\beta}(m,Q^{NR})) = \frac{p_A}{c_{1A} + c_{2A}} \quad \textit{and} \quad X^*_B(\tilde{\beta}(m,Q^{NR})) = \frac{1 - p_A}{c_{1B} + c_{2B}}.$$

Normalizing the cost structure with c_{2A} in the two-extreme-states scenario yields the total equilibrium effort

$$E[X^*](Q^{NR}) = X^*(\tilde{\beta}(m, Q^{NR})) = \frac{p_A(1 + C_B) + C_2(1 - p_A)(1 + C_A)}{(1 + C_A)(1 + C_B)}.$$

Definition 2.3. *Contestant 1 is defined to be the expected winner of the two-extreme-states contest, if under Q^{NR} his winning probability*

$$p_1^* := p_A \cdot \frac{x_{1A}^*(\tilde{\beta}(m, Q^{NR}))}{X_A^*(\tilde{\beta}(m, Q^{NR}))} + (1 - p_A) \cdot \frac{x_{1B}^*(\tilde{\beta}(m, Q^{NR}))}{X_B^*(\tilde{\beta}(m, Q^{NR}))} = \frac{p_A}{1 + C_A} + \frac{1 - p_A}{1 + C_B} \quad (2.13)$$

exceeds one half. The case in which contestant 2 is the expected winner is defined analogously.

Note that the expected winner depends on the parameters p_A, C_A, and C_B only, while C_2 merely influences the costs of winning.

The comparative statics of the equilibrium choices in the two-extreme-states scenario with respect to the cost parameters yield intuitive results considering the terminology introduced in Section 2.3.4: Equation (2.12) implies that ex ante and under any common prior, contestant 1 invests more effort in task $t \in T$ in equilibrium than his opponent, $x_{1t}^*(\tilde{\beta}(m, Q^{NR})) > x_{2t}^*(\tilde{\beta}(m, Q^{NR}))$, iff $C_t < 1$, i.e., iff he is a specialist (or, equivalently, has a relative cost advantage) in that task. Therefore, the favorite has a higher ex-ante winning probability than the underdog in the contest with uncertainty. In contrast to the discussion of whether a contestant invests more or less effort in task $t \in T$ than his opponent, the discussion of whether a contestant invests more in task A than in task B also depends on the relative cost structure between the tasks, captured by C_2, and the relative probability, $\frac{p_A}{1-p_A}$, as, e.g., $x_{1A}^*(\tilde{\beta}(m, Q^{NR})) > x_{1B}^*(\tilde{\beta}(m, Q^{NR}))$ holds, iff $C_1 \frac{(1-p_A)}{p_A} \frac{(1+C_A)^2}{(1+C_B)^2} < 1$. This implies that a specialist does not necessarily need to invest more into his "preferred" task in equilibrium than into the other task. This might for instance be the case when the prior puts more probability on his "non-preferred" task. In a setup with different beliefs among the contestants, the investments in the different tasks obviously also depend on higher order-beliefs about the subjective probabilities the opponent assigns to the different states.

2.6.2 Purely Public Messages: A Non-Improvement Result

In this section we show that if the contest designer always sends identical messages to the contestants, she cannot increase her payoff via information design. For the two-states scenario an information map with purely public messages can be characterized by Table 2.3.

$Q(\cdot\|s_A)$	m^A	m^B
m^A	q_A	0
m^B	0	$1-q_A$

$Q(\cdot\|s_B)$	m^A	m^B
m^A	$1-q_B$	0
m^B	0	q_B

Table 2.3: Purely public information map Q^{PUB} for the case of $|S| = 2$. Contestant 1's messages are denoted in the rows, while contestant 2's messages are denoted in the columns. It holds that $q_A, q_B \in [0, 1]$.

Since both contestants always receive the same message, the next result immediately follows.

Corollary 2.2. *Under information map Q^{PUB}, the contestants' equilibrium efforts are given by replacing $\tilde{Q}$ by Q^{PUB} in Proposition 2.1. Therefore, for all $t \in T, m \in M$, $X_t^*(\tilde{\beta}(m, Q^{PUB}))$ and $X^*(\tilde{\beta}(m, Q^{PUB}))$ are linear functions of the induced subjective probabilities of the states, $\bar{\beta}^{\sigma s}(m, Q^{PUB})$.*

This result has important implications for the optimal information design. Although we analyze a setting with strategically interacting receivers, we can apply the methodology of Bayesian persuasion introduced by Kamenica and Gentzkow (2011) and developed for the non-strategic cases, as the belief system of both contestants is pinned down by the first-order belief $\bar{\beta}^{\sigma}$.

Theorem 2.1. *In the two-player contest with uncertainty about the tasks, the contest designer cannot increase her payoff (in comparison to non-revelation) by sending purely public messages.*

Note that a graphical intuition of the proof is given immediately after the proof.

Proof. The statement follows from two results: Corollary 2.2 from above and Remark 2 in Kamenica and Gentzkow (2011). Corollary 2.2 implies that total equilibrium effort for any message $m \in M$ is a linear function of the induced subjective probabilities, $\bar{\beta}^{\sigma s}$. Note that the complete belief system $\tilde{\beta}$ is pinned down by a first-order belief about the state, $\bar{\beta}^{\sigma}$, which both contestants share. From an information design perspective, since there is no difference in higher-order beliefs among the contestants, this situation is thus equivalent to the situation analyzed in Kamenica and Gentzkow (2011) with a single receiver of the message. This fact is also illustrated by the dependence of $X^*(\tilde{\beta}(m, Q^{PUB}))$ solely on $\bar{\beta}^{\sigma}$, and not on higher order beliefs. Remember that $X^*(\tilde{\beta}(m, Q^{PUB}))$ crucially determines the objective of the designer. The designer "just" has to think about which belief $\bar{\beta}^{\sigma}$ is best to induce. She then knows, according to $X^*(\tilde{\beta}(m, Q^{PUB}))$, how the contestants react in aggregate terms to the induced belief and "does not care about their strategic interaction". Thus, the setting can be analyzed with the tools used in Kamenica and Gentzkow (2011).

The setting in Kamenica and Gentzkow (2011) subsumes the setting at hand, since $u_i : X \times S \to \mathbb{R}$ is continuous for all $i \in N$, $\pi : X \times S \to \mathbb{R}$ is continuous, the designer and the contestants share the same prior, X is compact, S is finite, $Q : S \to \Delta M$ with M finite, the contestants observe Q and the realization of $m \in M$, the contestants are Bayes rational, and the timing of the game is the same in both models. We can thus employ Remark 2 of Kamenica and Gentzkow (2011), which in the terminology of our model setup reads: if $X^*(\tilde{\beta}(m, Q^{PUB}))$ is concave in $\bar{\beta}^\sigma$, the information designer does not benefit from information design for any prior. □

The intuition of the proof of the result can nicely be illustrated and follows the idea of Kamenica and Gentzkow (2011). Suppose that $\bar{\beta}^\sigma$ is uniquelly determined by the parameter $\bar{\beta}$, i.e. there are only two σ-states, which in turn implies that we can without loss of generality assume that $M_1 = M_2 = \{m^A, m^B\}$. We can then write $X^*(\tilde{\beta}(m, Q^{PUB}))$ as a function of $\bar{\beta}$, denoted by $\bar{X}^*(\bar{\beta})$, and visualized in Figure 2.4. Note that in contrast to Corollary 2.2, $\bar{X}^*(\bar{\beta})$ in Figure 2.4 is not a linear function. The function depicts the contestants' aggregate equilibrium effort exerted given they hold belief $\bar{\beta}$. This belief can be manipulated by the designer. Figure 2.4 depicts a situation in which the designer would want to induce the contestants to always believe that $\bar{\beta} = 0$, since then they exert the highest effort, $\bar{X}^*(0) = 18$. However, as mentioned before, the designer faces a trade-off when trying to credibly disclose information: to credibly induce $\bar{\beta} = 0$ in some situations, she has to induce a belief that she "dislikes" in other situations, e.g., here $\bar{\beta} = 1$. The trade-off is formalized by Bayes plausibility, which states that under the optimal information map Q^*, the expected posterior has to equal the prior, in this case $p_A = 0.5$ (see Equation (2.5)). This is indicated in Figure 2.4 by the fact that the dashed horizontal line intersects the gray line exactly in the middle: the optimal information structure Q^* (here full revelation) induces $\bar{\beta}(m^A, Q^*) = 1$ and $\bar{\beta}(m^B, Q^*) = 0$ with equal probability and satisfies Bayes plausibility, since $0.5 \cdot 1 + 0.5 \cdot 0 = 0.5 = p_A$. The gray graph is the concavification of $\bar{X}^*(\bar{\beta})$, which, in cases such as the one depicted in Figure 2.4, is the "upper boundary" of the smallest convex set containing the graph of $\bar{X}^*(\bar{\beta})$. Generally, the intersection of the dashed vertical line with the graph of the concavification determines the weight of the induced subjective probabilities under the optimal information structure and the intersections of the graph of the concavification with the black graph closest to the dashed vertical line determines the induced probabilities under the optimal information structure. This methodology can be used to determine the optimal information map among all possible information maps (FR, NR, and PR). The gray line is also helpful to visualize the designer's linear expected utility (see Equation (2.4)) given the optimal information structure Q^*.

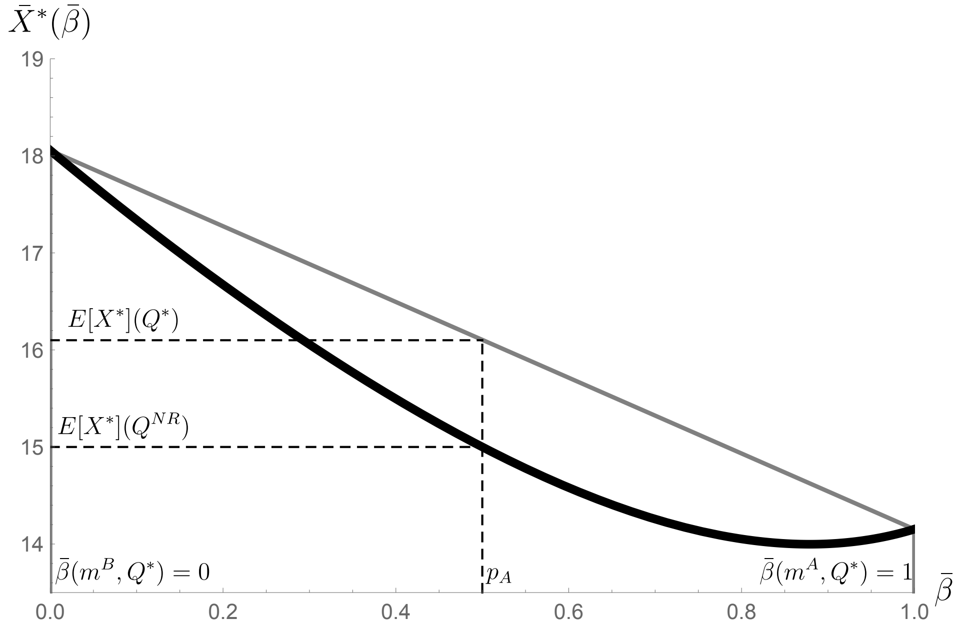


Figure 2.4: Concavification $conc(\bar{X}^*(\bar{\beta}))$ [gray] of $\bar{X}^*(\bar{\beta})$ [black] with the indication of the beliefs and message probabilities induced by the optimal information structure Q^*.

Figure 2.4 hints at the general concavification result of Kamenica and Gentzkow (2011). The information designer can only benefit from information transmission, if evaluated at the prior the concavification $conc(\bar{X}^*(\bar{\beta}))$ of the function $\bar{X}^*(\bar{\beta})$ does not equal $\bar{X}^*(p_A)$. Otherwise, $E[X^*](Q) \leq E[X^*](Q^{NR}) = \bar{X}^*(p_A)$ for all $Q \in \mathcal{Q}$. Since under purely public information total equilibrium effort is a linear function of the induced subjective probabilities, the designer can never increase total expected effort by sending purely public messages.

Note that Theorem 2.1 holds independent of the competitiveness implied by the cost structure, for a setting with more than two states, and also allows for "non-extreme" stochastic impact functions. Remember the intuition of the trade-off the designer has to take into account when credibly disclosing information: inducing a belief, which is beneficial for the designer in one state of nature, comes at the cost of inducing a belief that is not beneficial for the designer in another state. Since aggregate equilibrium efforts are a linear function of the induced subjective probabilities in case of purely public information, the benefits of inducing a profitable belief are exactly weighed off by the costs of it.

2.7 Analysis: Purely Private Messages for One Contestant

The previous section gives rise to the question whether there are any instances in which the designer can benefit from partially or fully revealing information. In this section we show that the answer is affirmative given that we allow the designer to adress different contestants with different information.

An information map with purely private messages to contestant 1 and no information for contestant 2 in the model at hand can be defined as in Table 2.4. A setting in which only contestant 2 receives private information can be analyzed analogously.

$Q(\cdot\|s_A)$	m^A	m^B
m^A	q_A	0
m^B	$1-q_A$	0

$Q(\cdot\|s_B)$	m^A	m^B
m^A	$1-q_B$	0
m^B	q_B	0

Table 2.4: Purely private information map Q_1^{PRI} for the case of $|S| = 2$, when only contestant 1 receives (potentially) informative messages. Contestant 1's messages are denoted in the rows, while contestant 2's messages are denoted in the columns. It holds that $q_A, q_B \in [0, 1]$.

In contrast to the purely public information setting, contestants do not necessarily share the same belief. With respect to Equations (2.6) to (2.8) for the informed contestant 1 it holds for all $g \in \{A, B\}$, and q_g^s as defined by Table 2.4, that

$$\beta_1^{\sigma s}(g) = p_s \cdot \frac{\bar{q}_{1g}^s}{\sum_k p_k \bar{q}_{1g}^k} = \frac{p_s q_g^s}{\sum_k p_k q_g^k} \ \forall s \in S, \quad \text{and} \quad \beta_1^{m_2^h}(g) = \begin{cases} 1 & \text{if } h = A, \\ 0 & \text{if } h = B, \end{cases} \quad \text{and}$$

$$\beta_1^{m_2^h \wedge s}(g) = \begin{cases} \beta_1^{\sigma s}(g) & \text{if } h = A, \\ 0 & \text{if } h = B. \end{cases}$$

Although contestant 2 does not obtain any information on the σ-state, he observes the information map Q. He should use this information and adapt his behavior accordingly. In the following we analyze the setup for two different assumptions on how the uninformed contestant processes the information on Q. In the first case, he is non-Bayesian and his complete system of beliefs is pinned down by the prior.[24] In the second case, he is a Bayesian updater and uses all information available to him. Obviously, these are two extreme cases demanding either very sophisticated Bayesian reasoning or implying that the contestant does not use part of the information available to him at all. Nevertheless, this distinction allows us to examine the effects of belief manipulation separately and many of the observations that follow are qualitatively the same in both

[24]Literature on non-Bayesian behavior is cited on page 10.

settings. It is important to note that we find striking similarities in the results derived for the different setups and many of the intuitions in the non-Bayesian setup also apply to the Bayesian setup.

In order to obtain explicit solutions for the equilibrium effort we restrict the analysis in this chapter to the two-extreme-states scenario introduced in Section 2.5, so that each state can uniquely be associated with one "relevant" task, i.e. we write β^t to indicate the subjective probability that the belief assigns to the realization of the state in which only task $t \in T$ matters. Note that the parameter β^t fully pins down the first-order belief of the contestant, since there are only two states.

2.7.1 Case 1: Non-Bayesian Uninformed Contestant

Here we assume that the uninformed contestant's belief system is determined by the prior, i.e. $\beta_2^t = p_t$ for all $t \in T$. The informed Bayesian contestant knows this, so that for his second-order belief we have $\beta_{12}^t = p_t$ for all $t \in T$. Furthermore, the second-order belief of contestant 2 about contestant 1's belief, denoted by β_{21}^t, also equals the prior, and thus also $\beta_{212}^t = \beta_{121}^t = p_t$. Therefore, higher-order beliefs in this scenario are pinned down in a convenient way for the analysis. Furthermore, contestant 2 ignores the fact that he faces two different types of contestant 1, i.e., $\beta_2^{m_1 \wedge s} = \beta_2^s$ for all $s \in S, m_1 \in M_1$. Contestant 1 knows this. Instead of the superscript "$*$", we use "$\star$" in this section to indicate equilibrium behavior under the above behavioral assumptions. The respective definitions are analogous.

Contestant Behavior

First, note that, when replacing "$*$" with "$\star$" in Corollary 2.1, it is straightforward to show, that the result then also describes the contestants' equilibrium behavior under non-revelation in the setup with a non-Bayesian uninformed contestant. Furthermore, we can derive the following result.

Proposition 2.2. *The contestants' equilibrium behavior in the two-extreme-states scenario of the multi-task contest with uncertainty, in which a purely private message is sent only to contestant* 1*, and contestant* 2 *is non-Bayesian, is described by*

$$x_{it}^\star(\tilde{\beta}(m, Q_1^{PRI})) = \frac{\beta_{ij}^t(m, Q_1^{PRI})}{c_{jt}\left(1 + \frac{c_{it}\beta_{ij}^t(m,Q_1^{PRI})}{c_{jt}\beta_i^t(m,Q_1^{PRI})}\right)^2}, \quad i, j \in \{1, 2\}, j \neq i, t = \{A, B\}, m \in M_i, \tag{2.14}$$

$$\bar{X}_t^\star(\beta_1^A) := X_t^\star(\tilde{\beta}(m, Q_1^{PRI})) = \frac{p_t}{c_{2t}\left(1 + \frac{c_{1t}p_t}{c_{2t}\beta_1^t(m,Q_1^{PRI})}\right)^2} + \frac{p_t}{c_{1t}\left(1 + \frac{c_{2t}}{c_{1t}}\right)^2}, \quad t = \{A, B\}, m \in M_1, \tag{2.15}$$

$$\begin{aligned}
\bar{X}^\star(\beta_1^A) &:= X^\star(\tilde{\beta}(m, Q_1^{PRI})) = \\
&\frac{p_A}{c_{2A}\left(1+\frac{c_{1A}p_A}{c_{2A}\beta_1^A(m,Q_1^{PRI})}\right)^2} + \frac{p_A}{c_{1A}\left(1+\frac{c_{2A}}{c_{1A}}\right)^2} \\
&+ \frac{(1-p_A)}{c_{2B}\left(1+\frac{c_{1B}(1-p_A)}{c_{2B}(1-\beta_1^A(m,Q_1^{PRI}))}\right)^2} + \frac{(1-p_A)}{c_{1B}\left(1+\frac{c_{2B}}{c_{1B}}\right)^2}, \quad m \in M_1.
\end{aligned} \tag{2.16}$$

Proof. See Appendix 2.A.2. □

Note that we write the aggregate terms $\bar{X}_t^\star(\beta_1^A), t \in \{A, B\}$ and $\bar{X}^\star(\beta_1^A)$ as functions of β_1^A, which indicates that the complete belief system $\tilde{\beta}$ is pinned down by the (manipulable) belief of contestant 1, captured by β_1^A, and the (exogenous) prior p_A.

As mentioned above, we employ the fact that with respect to the second-order beliefs $\beta_{12}^t(m) = \beta_{21}^t(m) = p_t$ holds for any message $m \in M_1$ and for all $t \in T$. As a result, and in contrast to the analogous function in Corollary 2.2, $\bar{X}^\star(\beta_i^A)$ now is a non-linear function of the manipulable subjective probability β_i^A. This can be seen in Figures 2.5, 2.6, and 2.7, which depict the situations from the examples in Section 2.2. These figures allow important insights for the optimal information design and the intuitions explained in Section 2.2 by analyzing the functional form of $\bar{X}^\star(\beta_i^A)$ (remember also the discussion below the proof of Theorem 2.1).

The left panel in Figure 2.5 shows that in the depicted scenario with a (weak) underdog the optimal design is given by $q_A^\star = q_B^\star = 1$, i.e. full revelation. The right panel visualizes Bayesian plausibility (see Equation (2.5)) and the designer's linear expected utility (see Equation (2.4)). Furthermore, the equilibrium effort as a function of the induced belief, $\bar{X}^\star(\beta_i^A)$, is convex: the increase in effort after the - from the underdog's perspective - encouraging message m^B (his disadvantage is smaller in task B than in task A) is larger than the decrease in effort after the discouraging message m^A. In the situation depicted in Figure 2.5, the designer can increase her expected payoff from $E[X^\star](Q^{NR}) = \bar{X}^\star(p_A) = 15$ in a situation without information disclosure to $E[X^\star](Q^\star) = 16.1$.

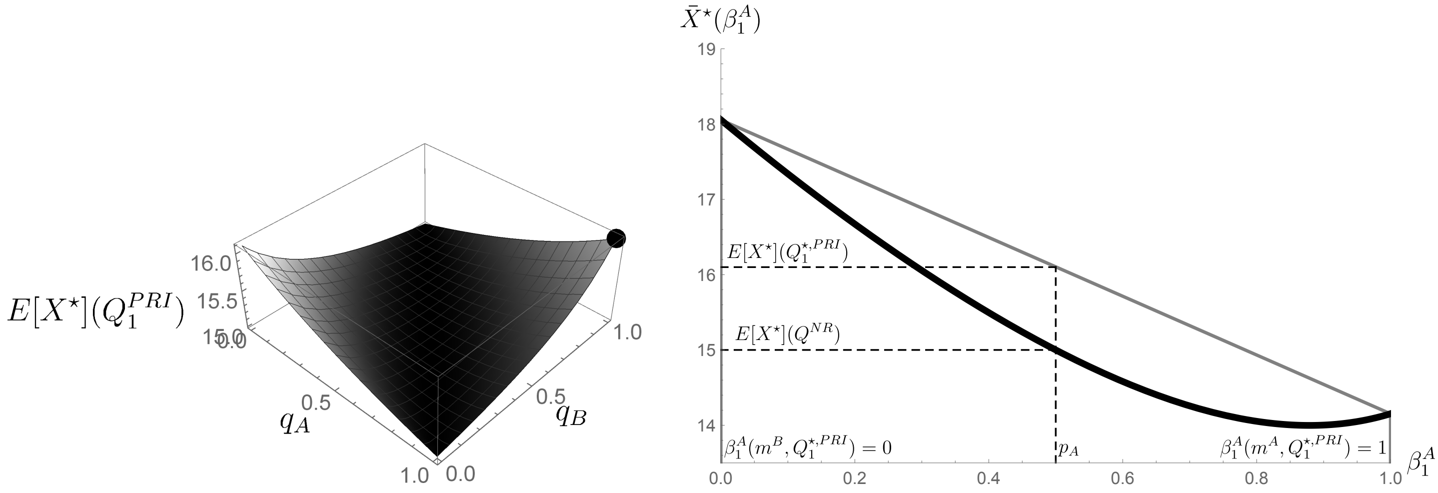


Figure 2.5: The scenario $c_{1A} = 9$, $c_{1B} = 4$, $c_{2A} = 1$, $c_{2B} = 1$ with prior $p_A = 0.5$ and purely private messages to the underdog (contestant 1), when the favorite (contestant 2) is non-Bayesian. Left Panel: $E[X^\star](Q_1^{PRI})$ as a function of the information map Q_1^{PRI}. The optimal information structure is indicated by the dot, i.e. $q_A^\star = q_B^\star = 1$ (full revelation). Right Panel: Concavification [gray] of $\bar{X}^\star(\beta_1^A)$ [black] with the indication of the beliefs and message probabilities induced by the optimal information structure $Q^\star$.

Figure 2.6 depicts a scenario with two specialists. The designer cannot increase her expected payoff by information revelation. This is reflected in the concavity of $\bar{X}^\star(\beta_i^A)$.

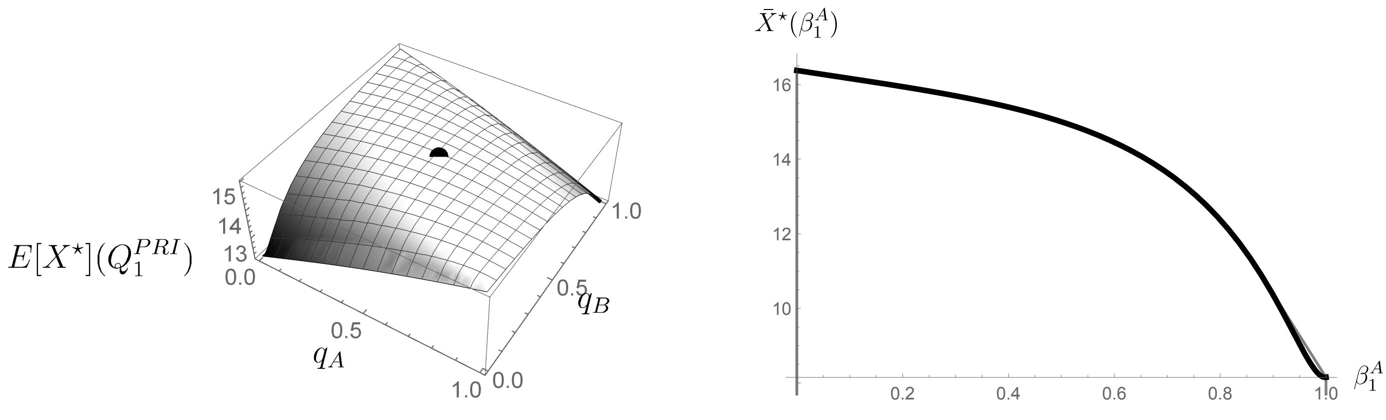


Figure 2.6: The scenario $c_{1A} = 9$, $c_{1B} = 1$, $c_{2A} = 1$, $c_{2B} = 4$ with prior $p_A = 0.5$ and purely private messages to the specialist in task B (contestant 1), when the specialist in task A (contestant 2) is non-Bayesian. Left Panel: $E[X^\star](Q_1^{PRI})$ as a function of the information map Q_1^{PRI}. The optimal information structure is indicated by the dot, i.e. $q_A^\star = q_B^\star = .5$ (non-revelation). Right Panel: Concavification [gray] of $\bar{X}^\star(\beta_1^A)$ [black].

The right panel of Figure 2.7 shows that, if - compared to the cost structure from Figure 2.6 - the relative costs of task B compared to task A increase for both contestants, then partial revelation is optimal, since the function $\bar{X}^\star(\beta_i^A)$ neither is "globally convex enough" (full revelation) nor "globally concave enough" (non-revelation).

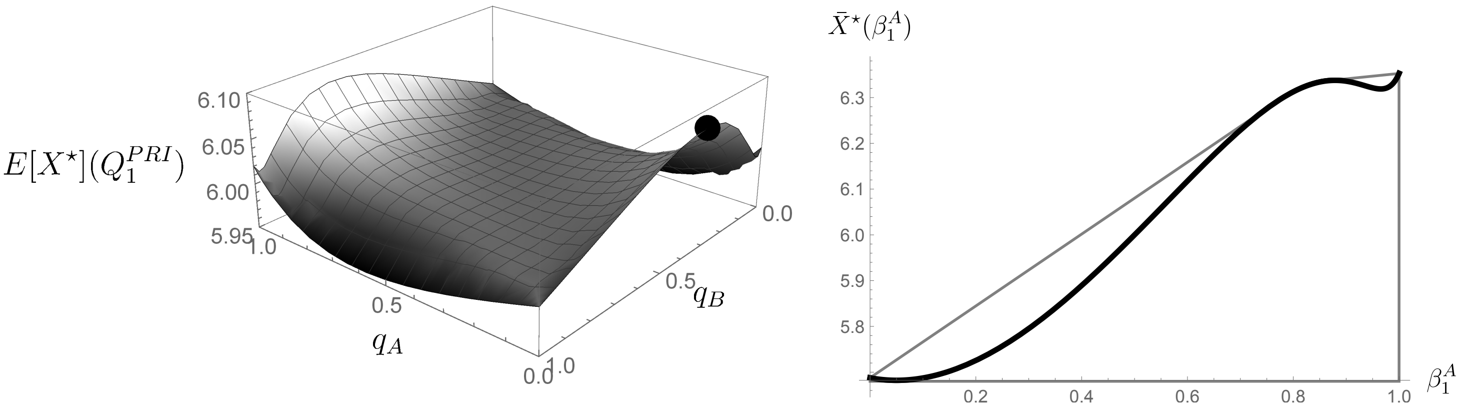


Figure 2.7: The scenario $c_{1A} = 9$, $c_{1B} = 10$, $c_{2A} = 1$, $c_{2B} = 40$ with prior $p_A = 0.5$ and purely private messages to the specialist in task B (contestant 1), when the specialist in task A (contestant 2) is non-Bayesian. Left Panel: $E[X^\star](Q_1^{PRI})$ as a function of the information map Q_1^{PRI}. The optimal information structure is indicated by the dot, i.e. $q_A^\star = 0$ and $q_B^\star = .47$ (partial revelation). Right Panel: Concavification [gray] of $\bar{X}^\star(\beta_1^A)$ [black].

In the next section we generalize the above observations concerning the optimal information structure.

Optimal Information Design

Suppose for a moment that the designer is completely free to shape beliefs. The designer would want each contestant to always believe that the state in which he exerts more effort is more likely. In the scenario with purely private messages for contestant 1 only and a non-Bayesian contestant 2, contestant 2's effort is constant in the information map Q. Thus, the designer can only shape contestant 1's first-order belief about the σ-state and she wants him to believe that the task is more likely to matter in which he exerts more effort. Additionally, the designer does not have to consider whether the induced beliefs are consistent, since contestant 2's beliefs (also at higher orders) always equal the prior, and contestant 1 knows this. In other words, when wanting to induce a specific belief for contestant 1, this does not come at the potential costs of being restricted when wanting to shape contestant 2's belief in an opposite manner. The objective function of the designer therefore becomes

$$E[X^\star](Q) = \sum_{m_1 \in M_1} P(m_1)(Q) \cdot \bar{X}^\star(\beta_1^A(m_1, Q)),$$

which is maximized over $Q \in \mathcal{Q}_1^{PRI}$, where $\mathcal{Q}_1^{PRI}$ denotes the set of all information maps, which imply purely private messages to contestant 1.

The following results illustrate the dependence of the optimal information design on the cost structure. Remember that in the two-extreme-states scenario with higher-order beliefs being structured as described above, contestant 1's belief (and thus in this case all *manipulable* beliefs) is fully pinned down by β_1^A. We say that an action Q is compatible with the designer's equilibrium play, if - anticipating the contestants' optimal expected efforts under Q - it yields a payoff which is at least as high as that of all the other actions availble to the designer.

Lemma 2.1. *In the two-extreme-states scenario of the two-player multi-activity contest, when the designer is restricted to send purely private messages to contestant* 1*, and contestant* 2 *is non-Bayesian,* ***non-revelation [NR]*** *is compatible with the designer's equilibrium play, iff* $\forall \beta_1^A \in [0,1]$*:*

$$\begin{aligned}(p_A - \beta_1^A)^2 \Bigg[& \frac{C_A\left(C_A(1-C_A)p_A + 2\beta_1^A\right)}{(1+C_A)^3\left(\beta_1^A + C_A p_A\right)^2} \\ & + \frac{C_2 C_B\left(C_B(1-C_B)(1-p_A) + 2(1-\beta_1^A)\right)}{(1+C_B)^3\left((\beta_1^A - 1) + C_B(p_A - 1)\right)^2}\Bigg] \geq 0,\end{aligned} \tag{2.17}$$

and ***full revelation [FR]*** *is compatible with the designer's equilibrium play, iff* $\forall \beta_1^A \in [0,1]$*:*

$$\begin{aligned}& p_A \beta_1^A \left[\frac{(1-\beta_1^A)\left(p_A^2 C_A^2 - \beta_1^A\right)}{\left(\beta_1^A + p_A C_A\right)^2 (1 + p_A C_A)^2}\right] \\ + C_2(1-p_A)(1-\beta_1^A) & \left[\frac{\beta_1^A\left[(1-p_A)^2 C_B^2 - (1-\beta_1^A)\right]}{\left((1-\beta_1^A) + (1-p_A)C_B\right)^2 (1 + (1-p_A)C_B)^2}\right] \geq 0.\end{aligned} \tag{2.18}$$

Partial revelation [PR] *is compatible with the designer's equilibrium play, iff neither Condition (2.17), nor Condition (2.18) is satisfied.*

Proof. See Appendix 2.A.3. □

Generically, the Conditions (2.17) and (2.18) are mutually exclusive by the definition of the auxiliary functions in Appendix 2.A.3, i.e., they only hold simultaneously if

$\bar{X}_1^\star(\beta_1^A)$ is a linear function, that is if the designer can not benefit from information disclosure. It is helpful to note that the Conditions (2.17) and (2.18) exhibit a similar structure: they consist of two summands, each describing the competitive structure in either task A or task B, weighted with the prior probability, p_A, the comparative cost structure across tasks, C_2, and the belief of the receiver, β_1^A. Lemma 2.1 is helpful to derive the following proposition.

Proposition 2.3. *In the two-extreme-states scenario of the two-player multi-activity contest, when the designer is restricted to send purely private messages to contestant* 1*, and contestant* 2 *is non-Bayesian, a sufficient condition is satisfied for*

1. ***NR** to be compatible with the designer's equilibrium play, if $C_A \leq 1$ and $C_B \leq 1$, and for*

2. ***NR** not to be compatible with the designer's equilibrium play, if $C_A \geq 2$ and $C_B \geq 2$, and for*

3. ***FR** to be compatible with the designer's equilibrium play, if $C_A \geq \frac{1}{p_A} > 1$ and $C_B \geq \frac{1}{1-p_A} > 1$.*

Proof. See Appendix 2.A.4. □

The sufficient conditions involving non-revelation only depend on the parameters reflecting the relative cost structure among the contestants, C_A and C_B, and hold for any prior, p_A, and relative costs across tasks, C_2. Considering the sufficient conditions for FR, note that if p_A increases the condition on C_A becomes less restrictive, while the condition on C_B becomes more restrictive. The Figures 2.8 and 2.9 depict the necessary and sufficient conditions from Lemma 2.1 and Proposition 2.3.

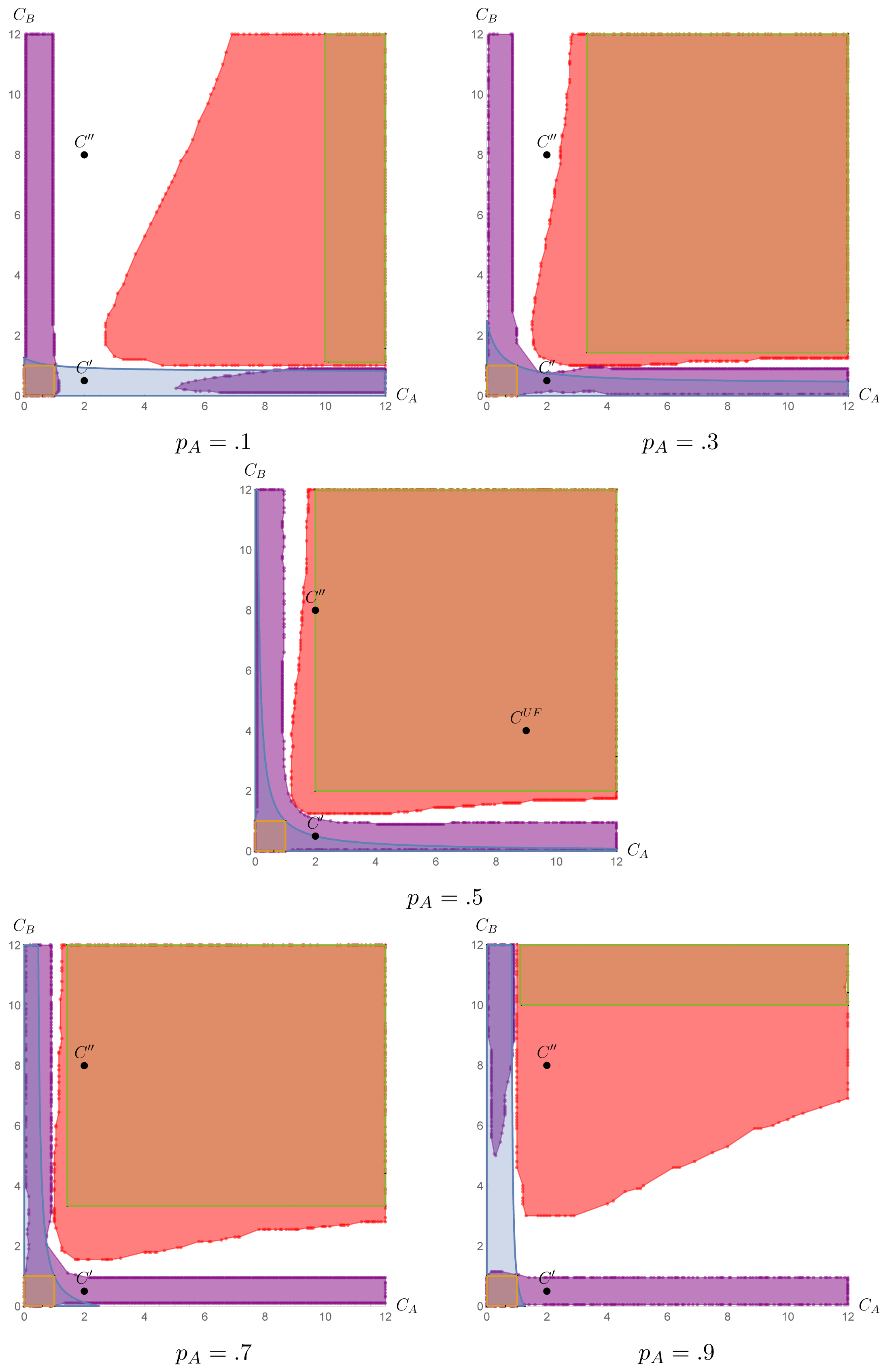


Figure 2.8: Comparative statics of the optimal information structure with respect to p_A for $C_2 = 1$. Orange rectangle = sufficient condition NR, green rectangle = sufficient condition FR, purple area = numerical test of necessary condition NR positive, red area = numerical test of necessary condition FR positive, white area = PR, blue area = cost structures under which contestant 1 is the expected winner (see Equation (2.13)). The point C^{UF} depicts the situation with contestant 1 being the underdog as discussed in Figure 2.5 and the introduction. The points C' and C'' are discussed in the current section.

The next theorem immediately follows from the above results:

Theorem 2.2 (Informational Favoritism of the Ex-Ante Disadvantaged Player (Weak Underdog))**.** *In the two-extreme-states scenario of the two-player multi-activity contest, if the designer is restricted to send purely private messages to one player, and the other player is non-Bayesian,*

1. *it is never optimal to reveal information to a favorite, and*
2. *it is always optimal to reveal information (partially or fully) to a weak underdog, i.e., a contestant whose effort costs in each task are at least twice as high as those of his competitor.*

The theorem gives an easy-to-follow rule when thinking about who to disclose information to in this setup. It is effort-maximizing to even out the cost disadvantage of the weak underdog by disclosing information to him. This "informational favoritism" induced by designing the information structure parallels favoritism induced by designing incentives via the introduction of head starts or biases, which is discussed in the literature, see Franke et al. (2018) and Fu (2006).

The above results also imply that any information regime (NR, PR, or FR) might be optimal when thinking about transmitting information to a "strong" underdog or to specialists. However, non-revelation is always optimal when facing a favorite. The Figures 2.8 and 2.9 allow more detailed insights into further dependencies of the optimal information structure on the cost structure, captured by C_A, C_B, and C_2, and the uncertainty about the σ-state, captured by p_A.

It is useful to remember the mechanism at work when disclosing information: in any (generic) setting each contestant has a preferred task; a task in which he has a higher cost advantage (or a lower cost disadvantage) compared to his opponent. This implies that when receiving an informative message, one of these reveals encouraging information to the contestant, and the other one discouraging information. The contestant's sensitivity to these messages determines whether it is beneficial for the designer to disclose information to the contestant. The linear homeomorphism ξ between the different elements of the partitions discussed in Section 2.3.4, reflects the fact that the designer can face two different situations depending on the cost parameters: either he is in a setting with an underdog and a favorite, or he faces two specialists - each preferring a different σ-state to be relevant. When facing an underdog and a favorite the decision of whom to adress with new information (if she does so at all) is straight forward according to Theorem 2.2, since the favorite, a contestant who already has a cost advantage, never increases his effort intensely enough after messages encouraging him to do so to outweigh the decrease in effort after different messages. When facing two specialists, the optimal information design in the current setup can be characterized by the following results. We can observe the mechanism of the results in Figures 2.8 and 2.9.

Observation 2.1 (Informational Discrimination or Favoritism of the Ex-Ante Disadvantaged Specialist)**.** *In the two-extreme-states scenario of the two-player multi-activity contest, when the designer is restricted to send purely private messages to one player, and the other player is non-Bayesian, depending on the cost-parameter C_2, the designer may either prefer to disclose information to the ex-ante disadvantaged specialist, or to the ex-ante advantaged specialist.*

These observations are also reflected in the following result.

Proposition 2.4. *If contestant 1 is a specialist in task B, i.e. $C_B < 1 < C_A$, and when contestant 2 is non-Bayesian, the propensity of the designer to reveal information to contestant 1*

- *increases in the probability of task B, measured by $1 - p_A$, and*
- *decreases in the comparative costs of task B relative to task A, measured by C_2.*

In view of the informational favoritism mentioned above, the result might seem surprising, since it implies that the designer should in some cases reveal information to the ex-ante stronger of the two specialists, i.e., the one who has a higher ex-ante winning probability (see Equation (2.13)). Thus, we observe a form of "informational discrimination" against the ex-ante weaker contestant. This is illustrated in Figure 2.8, which also depicts the partition of the cost parameter space according to the winning probability: in the left panel of the first row the cost structure associated with the point $C' = (2, \frac{1}{2})$ (specialist in task A: contestant 2, specialist in task B: contestant 1) implies partial revelation to contestant 1 under prior $p_A = 0.1$, while the other panels imply no revelation of information at the respective priors p_A. Thus "the informativeness at C' decreases in p_A". Note that for the same panels in Figure 2.8 for the cost structure $C'' = (2, 8)$ (weak underdog and favorite) we observe "an increase in informativeness in p_A". For that weak underdog, who prefers task A over task B, receiving a positive message becomes more likely when moving from $p_A = .1$ to $p_A = .5$, so this encourages him to exert more effort and the designer is more inclined to reveal information. For the same reason, as Proposition 2.4 states, the designer should disclose information to the "stronger" among the two specialists, i.e. the one who is less likely to receive a discouraging message. For contestant 1, being the specialist in task B under the cost structure $C' = (2, \frac{1}{2})$, a positive message becomes less likely when moving from $p_A = .1$ to $p_A = .5$. Thus, less information is disclosed to him. The driving force behind this is illustrated in the third condition of Proposition 2.3: ceteris paribus, $C_A \geq 1/p_A$ is "easier to be satisfied", if $C_A > C_B$, i.e., if contestant 1 has a larger relative cost disadvantage in task A than in task B. Furthermore, as discussed above, if p_A increases, the condition on C_A stated in Proposition 2.3 becomes less restrictive, while the condition on C_B becomes more restrictive.

Whether a specialist is favored depends on his ex-ante winning probability (as discussed above), and - as stated in Observation 2.1 - on the cost parameter C_2. Remember that if, ceteris paribus, C_2 increases task B becomes relatively more costly compared to task A for both contestants. Comparing Figures 2.8 and 2.9, we observe that (as Proposition 2.4 states), the designer is more inclined to disclose information to the specialist whose preferred task becomes relatively more costly. Remember that this does not affect the winning probabilities (see Equation (2.13) and the discussion below it). However, from the designer's point of view this is desirable, since the trade-off of an effort increase after receiving a positive message and an effort decrease after receiving a negative message becomes smaller for that specialist. This is also reflected by the change of the slope of the concavification when comparing the Figures 2.6 and 2.7. Thus, the designer is, cetris paribus, inclined to disclose more information to the specialists whose preferred task is relatively more costly.

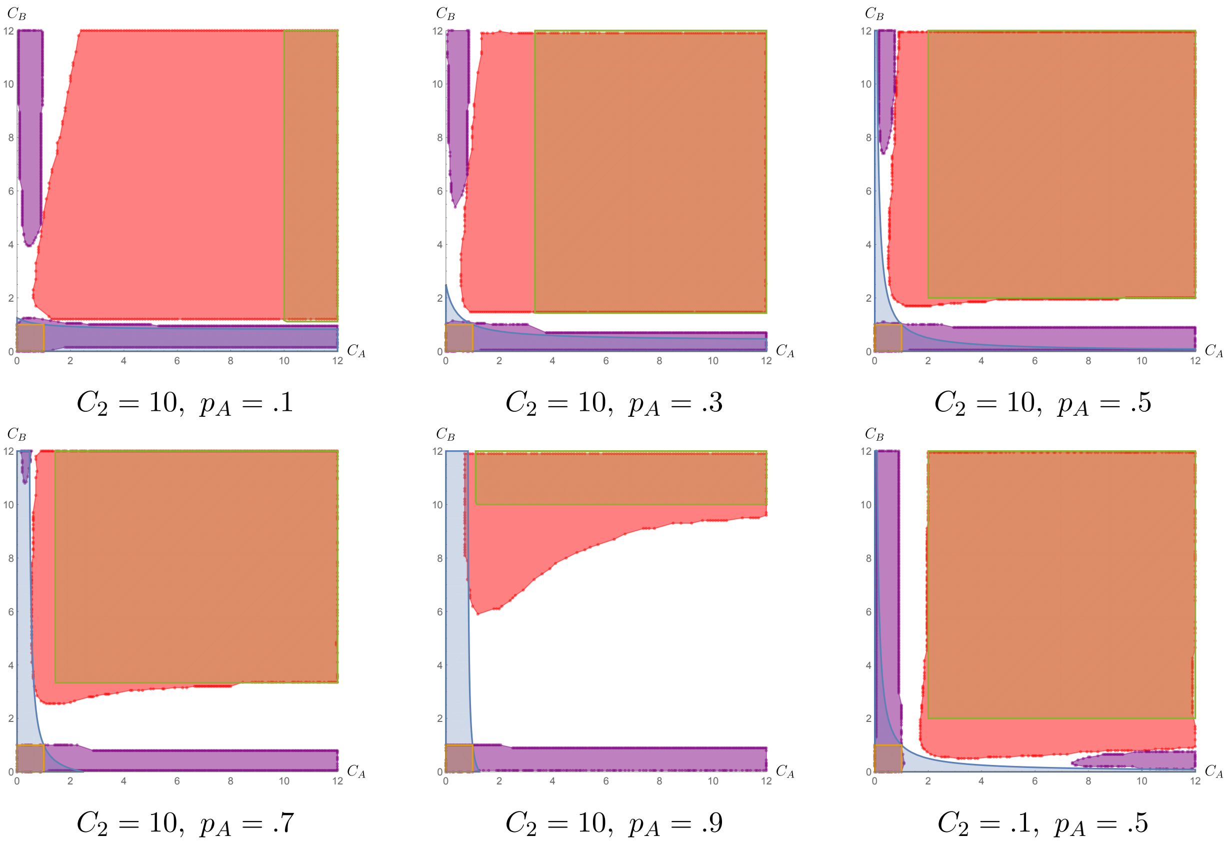


Figure 2.9: Comparative statics of the optimal information structure with respect to p_A for $C_2 = .1$ and $C_2 = 10$. Orange rectangle = sufficient condition NR, green rectangle = sufficient condition FR, purple area = numerical test of necessary condition NR positive, red area = numerical test of necessary condition FR positive, white area = PR, blue area = cost structures, under which contestant 1 is the expected winner (see Equation (2.13)). Comparing the right panels in both rows indicates that the situations with $C_2 = 1/10$ and $C_2 = 10$ are symmetric.

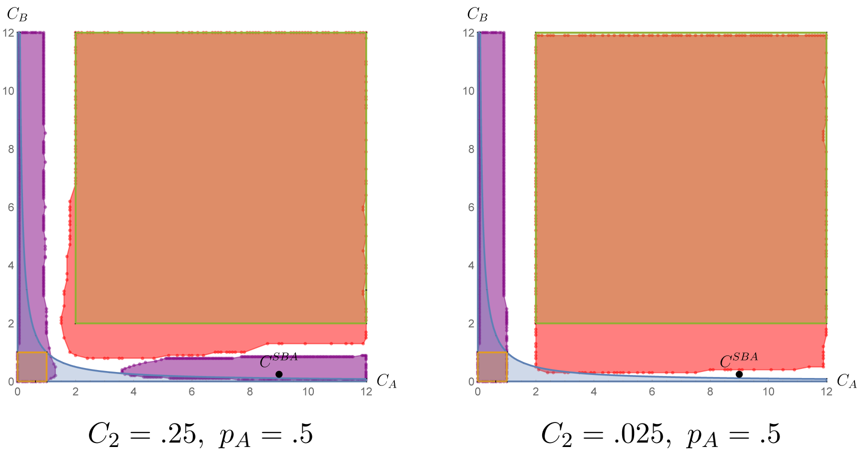


Figure 2.10: Comparative statics of the optimal information structure for the examples from Figure 2.6 and 2.7, indicated by C^{SBA}. Orange rectangle = sufficient condition NR, green rectangle = sufficient condition FR, purple area = numerical test of necessary condition NR positive, red area = numerical test of necessary condition FR positive, white area = PR, blue area = cost structures, under which contestant 1 is the expected winner (see Equation (2.13)).

2.7.2 Case 2: Bayesian Uninformed Contestant

We now assume that contestant 2 is Bayes rational as well. Thus, in the setup with purely private messages only for contestant 1 (as defined in Table 2.4), the uninformed contestant 2 holds the updated belief, that

$$\beta_2^{gs}(A) = p_s \ \forall s \in S, \quad \text{and} \quad \beta_2^{m_1^h}(A) = \begin{cases} p_A q_A + (1-p_A)(1-q_B) & \text{if } h = A \\ p_A(1-q_A) + (1-p_A)q_B & \text{if } h = B \end{cases}, \quad \text{and}$$

$$\beta_2^{m_1^h \wedge s}(A) = \frac{p_s q_{hA}^s}{\sum_s p_s q_{hA}^s} \ \forall s \in S, g \in \{1, ..., l\}.$$

In contrast to a non-Bayesian player, contestant 2 now interprets Q and anticipates his opponent's (potential) information advantage. An analytical solution of a general setup placing no constraints on Q is not obtainable in a manner "digestable" for the reader, since it involves solving quartic equations. Thus, we focus on comparing non-revelation, Q^{NR}, to full revelation to contestant 1, Q_1^{FR}.

Optimal Contestant Behavior

Note that Corollary 2.1 also describes optimal contestant behavior under Q^{NR}, given by $E[X^*](Q^{NR})$ in this setup. The next proposition describes contestant behavior under Q_1^{FR}, which is given by $q_A = q_B = 1$ as defined in Table 2.4. Note that in this scenario, contestant 1 can be of two m-types, each of which is fully informed about the true state of the world. Contestant 2 is always of m-type m^A and remains uninformed. Nevertheless, he knowns that contestant 1 has an information advantage.

Proposition 2.5. *The contestants' aggregate equilibrium effort in the two-extreme-states scenario of the multi-task contest with uncertainty, in which a fully revealing purely private message is sent only to contestant* 1*, is described by*

$$\begin{aligned} E[X^*](Q_1^{FR}) &= E[X_A^*](Q_1^{FR}) + E[X_B^*](Q_1^{FR}) \\ &= \frac{(1+C_A)p_A^2}{(1+p_A C_A)^2} + \frac{C_2(1+C_B)(1-p_A)^2}{((1+(1-p_A)C_B)^2}. \end{aligned} \tag{2.19}$$

Proof. See Appendix 2.A.6. □

Getting back to our introductory example of an underdog (contestant 1) and a favorite (contestant 2) with the cost structure given by $c_{1A} = 9, c_{1B} = 4, c_{2A} = 1, c_{2B} = 1$, we can now isolate two effects of disclosing information to a contestant: the direct effect of information disclosure describes that due to better knowing which task is relevant in the contest, an informed contestant will put more effort into this task. Depending on the cost structure, the increase may be overproportionally large (e.g., if the underdog learns that his "preferred" task is relevant) compared to the decrease in effort upon receiving a "discouraging" message. This effect is described in Section 2.7.1, where we analyze the setting with a non-Bayesian uninformed contestant, who does not use the information provided by Q. In the example we observed an overall increase from $E[X^*](Q^{NR}) = 15$ to $E[X^\star](Q_1^{FR}) \approx 16.1$.

In addition to this direct effect there is also an indirect effect, which becomes obvious when comparing the situation with a non-Bayesian uninformed contestant to that with a Bayesian uninformed contestant: a Bayesian uninformed contestant uses the information provided by Q and adapts his effort choices vis-a-vis the information advantage of the opponent. In the example with an underdog and a favorite, the uninformed favorite thus increases his effort choices. Obviously, knowing that the favorite increases his effort is disadvantageous for the informed underdog (compared to a situation with a non-Bayesian favorite), but in the example the overall effect is positive: the expected effort increases to $E[X^*](Q_1^{FR}) \approx 22.1$.

Information Design: Full Revelation vs. Non-Revelation

We now use the results from Proposition 2.5 and Corollary 2.1 to compare full revelation and non-revelation.

Proposition 2.6. *In the two-extreme-states scenario of the two-player multi-activity contest, when the designer is restricted to send purely private messages to contestant* 1*, a sufficient condition is satisfied for*

1. ***NR** to be (weakly) preferred to FR by the designer, if* $C_A \leq 1$ *and* $C_B \leq 1$*, and for*
2. ***FR** to be (weakly) preferred to NR by the designer, if* $C_A \geq \sqrt{\frac{1}{p_A}} > 1$ *and* $C_B \geq \sqrt{\frac{1}{1-p_A}} > 1$.

Proof. See Appendix 2.A.7. □

Note the similarity of the conditions in Proposition 2.6 to those stated in Proposition 2.3. Thus, the main mechanisms are similar in the model with Bayesian rational agents (Section 2.7.2) and in the model with a non-Bayesian agent (Section 2.7.1). Therefore, many of the intuitions we discuss in Section 2.7.1 can be applied also to the current setup.

Definition 2.4. *Contestant* 1 *is a "probablistically-weak" underdog, iff* $C_A \geq \sqrt{\frac{1}{p_A}} > 1$ *and* $C_B \geq \sqrt{\frac{1}{1-p_A}} > 1$*. The definition for contestant* 2 *is analogous.*

The next theorem immediately follows from the above results:

Theorem 2.3 (Informational Favoritism of the Ex-Ante Disadvantaged Player (Non-Bayesian Case))**.** *In the two-extreme-states scenario of the two-player multi-activity contest, the designer can never increase her payoff by disclosing information only to the favorite. However, she can (weakly) increase her payoff by disclosing the true state of the world fully only to a probablistically-weak underdog (see Definition 2.4).*

These results are in line with the idea of informational favoritism we discuss above: it can never be (strictly) better to reveal information only to a favorite, but the designer can increase her payoff by disclosing the σ-state of nature to an underdog, if the cost structure satisfies the condition stated in Proposition 2.6.

Analyzing the parameter constellations from the introductory example, we observe that the proposition predicts what we have already calculated. With $c_{1A} = 9, c_{1B} = 4, c_{2A} = 1, c_{2B} = 1$, which implies $C_A = 9$ and $C_B = 4$, and the prior $p_A = 0.5$, the sufficient condition for full revelation to be beneficial for the designer is satisfied ($C_A = 9 > \sqrt{1/0.5} \approx 1.414$, and $C_B = 4 > \sqrt{1/0.5} \approx 1.414$). Thus, she can benefit from informational favoritism for the underdog.

With $C_A = 2$ and $C_B = 5$, and the prior $p_A = 0.1$, the sufficient condition for full revelation to be beneficial for the designer is not satisfied ($C_A = 2 < \sqrt{1/0.1} \approx 3.162$, and $C_B = 5 > \sqrt{1/0.9} \approx 1.054$), and indeed non-revelation is more profitable than revelation, since $E[X^*](Q^{NR}) = 18.333$ and $E[X^*](Q_1^{FR}) \approx 18.149$. However, when the prior changes to $p'_A = 0.9$, then the sufficient condition for full revelation to be beneficial for the designer is satisfied ($C_A = 2 > \sqrt{1/0.9} \approx 1.054$, and $C_B = 5 > \sqrt{1/0.1} \approx 3.162$) and thus revelation is more profitable than non-revelation, with $E[X^*](Q^{NR}) = 31.667$ and $E[X^*](Q_1^{FR}) \approx 33.661$. This is an instance of the mechanism driving informational discrimination against the ex-ante weaker contestant, which we already discussed in Section 2.7.1. The probability of the underdogs "preferred", less cost-disadvantageous task A increases from $p_A = 0.1$ to $p'_A = 0.9$. The designer is more inclined to disclose information to the (ex-ante) more advantaged underdog, since this contestant is more likely to receive an encouraging message, which increases expected efforts.

2.8 Conclusion and Outlook

This chapter analyzes how the contest host and designer of an information structure can manipulate the contestants' beliefs about the relevance of the different tasks in such a way that it increases their (expected) exerted efforts. The economics of information makes a central distinction between public and private information. In the model at hand this distinction plays a crucial role: if information is disclosed purely publicly, which implies that all contestants share the same belief, then the designer cannot benefit from information design. The result is different when information is distributed asymmetrically, as the designer can benefit from information design by disclosing information to the ("probablistically-weak") underdog, i.e., the contestant who most benefits from an information advantage vis-a-vis his opponent.

We have found that in some cases the designer's self interest to maximize profit induces her to level the playing field among contestants (informational favoritism for the underdog), and sometimes it induces her to further enhance existing asymmetries (informational discrimination of the ex-ante weaker contestant).

In non-strategic settings the value of information to Bayes rational agents is always positive, see Blackwell (1951). This need not be the case in strategic settings, see Morris and Shin (2002) and Einy et al. (2017). Specifically, in the setup at hand the designer can "increase the (expected) perceived competitiveness" of the contest. Thus, it would be interesting to examine to which extent the disclosure of information is beneficial to the contestants.

An important assumption in our model is the commitment assumption. This assumption is not made in the cheap talk models of Crawford and Sobel (1982) for the single receiver case, and Farrell and Gibbons (1989) and Goltsman and Pavlov (2011) for the multiple receiver case. Extending the model to settings in which the designer cannot commit could be a fruitful path of future research. Furthermore, it is not clear why contestants should not share their private information with their opponent, especially in view of the discussion of the value of information from above. Relaxing this assumption could also yield new insights.

An obvious extension of the model would be to discuss a setup with more than two contestants and to allow the asymmetric distribution of information in settings with more than two states (compare Section 2.7). The assumption of "extreme states" in the two-extreme states scenario is made mainly for expositional purposes, i.e., to obtain explicit solutions. We conjecture that the results qualitatively also hold in more general settings without constant returns to scale and in non-extreme scenarios, i.e., both task are relevant with a positive probability in both states. We also conjecture that the results qualitatively hold for a multi-task version of the other workhorse model in contest theory, the first-price all-pay auction.

The effect of a "problematic" change in higher-order beliefs induced by belief manipulation and the related issue of consistency does not play a role in the models discussed in Section 2.6 and 2.7. When the designer is restricted to induce consistent beliefs, she has to consider the second-order effects of manipulating beliefs, i.e., how a change in the information design affects beliefs about the opponent's belief, and thus she has to consider the interplay of the different order-effects of belief manipulation. Consider the introductory example of Section 2.2. We discuss that in the scenario with one underdog and one favorite ($c_{1A} = 9$, $c_{1B} = 4$, $c_{2A} = 1$, $c_{2B} = 1$) the designer wants to induce both contestants to believe that task B is more likely, since they both exert higher effort in this task. In contrast to this, in the scenario with one specialist in task A and one specialist in task B ($c_{1A} = 9$, $c_{1B} = 1$, $c_{2A} = 1$, $c_{2B} = 4$), which yields the same aggregate equilibrium effort if contestants knew the relevant task ($X_A^* = 10$, $X_B^* = 20$), the designer wants to induce different beliefs among the contestants. Each contestant should believe that his preferred task is more likely, since he exerts higher effort in that task. However, the designer cannot manage to induce completely opposing beliefs even if she sends private messages to the contestants. Since Q is public, this would imply that contestants would agree to disagree about the state of nature, which cannot be the case if they share the same prior and are Bayesian updaters, as Aumann (1976) shows. Their beliefs induced by Q have to be consistent. The competitiveness implied by the underlying cost structure in the multi-task contest with uncertainty about the tasks determines the degree to which the designer wants to induce different beliefs among the contestants. In consideration of the results from Section 2.7, if for the designer it is optimal to induce different beliefs among the contestants, then she must improve upon her profit obtained when only purely private messages to one contestant are allowed. When generalizing our results, it is thus open to further research to examine whether the additional freedom of shaping beliefs of *both* contestants is outweighed by the restriction imposed by consistency or not.

2.A Appendix: Proofs

2.A.1 Proof of Proposition 2.1: Both Contestants Know the Opponent's Message

Proof. Note that all that follows is true for any message that a contestant receives and any $\tilde{Q}$, so that we drop the dependence of β^σ on m and $\tilde{Q}$ for the ease of notation. Contestant i's first order condition with respect to task t of can be written as

$$\sum_{s \in S} \bar{\beta}^{\sigma s} \frac{\alpha_t^s}{x_{it}} \cdot \frac{f^s(x_i) \sum_{j \neq i} f^s(x_j)}{\left[\sum_j f^s(x_j)\right]^2} \stackrel{!}{=} c_{it}.$$

Since there are only two contestants, we can write

$$x_{it} c_{it} \stackrel{!}{=} \sum_s \bar{\beta}^{\sigma s} \alpha_t^s \cdot \frac{f^s(x_i) f^s(x_j)}{\left[\sum_j f^s(x_j)\right]^2}.$$

In equilibrium the right-hand side of this equation has the same value for both contestants and thus it must hold in equilibrium that $x_{jt}^* = \frac{c_{it}}{c_{jt}} \cdot x_{it}^*$. Plugging this into the first-order condition and noting that

$$f^s(x_j^*) = \prod_{t \in T} \left(\frac{c_{it}}{c_{jt}} \cdot x_{it}^* \right)^{\alpha_t^s} = \prod_{t \in T} \left(\frac{c_{it}}{c_{jt}} \right)^{\alpha_t^s} f^s(x_i^*),$$

gives the resulting equilibrium effort in task t:

$$x_{it}^* = \frac{1}{c_{it}} \sum_{s \in S} \bar{\beta}^{\sigma s} \alpha_t^s \cdot \frac{\prod_k \left(\frac{c_{ik}}{c_{jk}}\right)^{\alpha_k^s} [f^s(x_i^*)]^2}{\left[f^s(x_i^*) \left(1 + \prod_k \left(\frac{c_{ik}}{c_{jk}} \right)^{\alpha_k^s} \right) \right]^2}.$$

Equilibrium effort in each task and total effort can easily be calculated by summing up the individual efforts. □

2.A.2 Proof of Proposition 2.2: Private Messages (Non-Bayesian Case) - Contestants

Proof. In the proof we drop the dependence of β_1 on the message received, m_1. Each message induces a posterior and the calculations below are valid for both realizations of m_1. Using standard calculus and the corresponding beliefs we obtain the first-order conditions:

$$\frac{\partial EU_i(x)}{\partial x_{it}} \overset{!}{=} 0 \Leftrightarrow \frac{\beta_i^t \cdot \sum_{j\neq i} x_{jt}}{\left(\sum_j x_{jt}\right)^2} = c_{it} \Leftrightarrow x_{it}(x_{jt}) = \sqrt{\frac{\beta_i^t}{c_{it}} x_{jt}} - x_{jt} \text{ for } i,j \in \{1,2\}, j \neq i, t = \{A,B\}.$$

Best responses intersect in the Nash equilibrium and the higher-order beliefs are pinned down by $\beta_{12}^t = \beta_{21}^t = \beta_2^t = p_t$ for all $t \in T$, so we can write

$$\begin{aligned} x_{it}^\star(\tilde{\beta}) = \sqrt{\frac{\beta_i^t}{c_{it}}\left[\sqrt{\frac{\beta_{ij}^t}{c_{jt}} x_{it}^\star} - x_{it}^\star\right]} - \left[\sqrt{\frac{\beta_{ij}^t}{c_{jt}} x_{it}^\star} - x_{it}^\star\right] &\Leftrightarrow 0 = \sqrt{\frac{\beta_i^t}{c_{it}}\left[\sqrt{\frac{\beta_{ij}^t}{c_{jt}} x_{it}^\star} - x_{it}^\star\right]} - \sqrt{\frac{\beta_{ij}^t}{c_{jt}} x_{it}^\star} \\ &\Leftrightarrow \frac{\beta_{ij}^t}{c_{jt}} x_{it}^\star = \frac{\beta_i^t}{c_{it}}\left[\sqrt{\frac{\beta_{ij}^t}{c_{jt}} x_{it}^\star} - x_{it}^\star\right] \Leftrightarrow \left(1 + \frac{c_{it}\beta_{ij}^t}{c_{jt}\beta_i^t}\right)^2 (x_{it}^\star)^2 = \frac{\beta_{ij}^t}{c_{jt}} x_{it}^\star \\ &\Leftrightarrow (x_{it}^\star)^2 - \frac{\beta_{ij}^t}{c_{jt}\left(1 + \frac{c_{it}\beta_{ij}^t}{c_{jt}\beta_i^t}\right)^2} \cdot x_{it}^\star = 0 \Leftrightarrow x_{it}^\star(\tilde{\beta}) = \frac{\beta_{ij}^t}{c_{jt}\left(1 + \frac{c_{it}\beta_{ij}^t}{c_{jt}\beta_i^t}\right)^2}. \end{aligned}$$

In the last line we use, that by our assumptions on the parameters, $X_t^\star = (0,0)$ cannot be an equilibrium. Equilibrium effort in task t and total equilibrium effort are given by

$$\begin{aligned} X_t^\star(\tilde{\beta}) = \frac{\beta_{12}^t}{c_{2t}\left(1 + \frac{c_{1t}\beta_{12}^t}{c_{2t}\beta_1^t}\right)^2} + \frac{\beta_{21}^t}{c_{1t}\left(1 + \frac{c_{2t}\beta_{21}^t}{c_{1t}\beta_2^t}\right)^2} &= \frac{p_t}{c_{2t}\left(1 + \frac{c_{1t}p_t}{c_{2t}\beta_1^t}\right)^2} + \frac{p_t}{c_{1t}\left(1 + \frac{c_{2t}}{c_{1t}}\right)^2} =: \bar{X}_t^\star(\beta_1^A), \\ \bar{X}^\star(\beta_1^A) = \bar{X}_A^\star(\beta_1^A) + X_B^\star(\beta_1^A) &= \frac{p_A}{c_{2A}\left(1 + \frac{c_{1A}p_A}{c_{2A}\beta_1^A}\right)^2} + \frac{p_A}{c_{1A}\left(1 + \frac{c_{2A}}{c_{1A}}\right)^2} \\ &+ \frac{(1-p_A)}{c_{2B}\left(1 + \frac{c_{1B}(1-p_A)}{c_{2B}(1-\beta_1^A)}\right)^2} + \frac{(1-p_A)}{c_{1B}\left(1 + \frac{c_{2B}}{c_{1B}}\right)^2}. \end{aligned}$$

□

2.A.3 Proof of Lemma 2.1: Private Messages (Non-Bayesian Case) - Characterization

Proof. As we mention above, only contestant 1's effort can be influenced by belief manipulation. Rewriting the equilibrium effort of contestant 1 given belief β_1^A from Proposition 2.2 in terms of the cost-structure parameters introduced in Section 2.3.4 yields

$$\bar{X}_1^\star(\beta_1^A) = \frac{p_A}{c_{2A}\left(1 + C_A \frac{p_A}{\beta_1^A}\right)^2} + \frac{(1-p_A)}{\frac{c_{2A}}{C_2}\left(1 + C_B \frac{(1-p_A)}{(1-\beta_1^A)}\right)^2}.$$

We can without loss of generality normalize with c_{2A}. The first derivative can be evaluated as

$$\frac{\partial \bar{X}_1^\star(\beta_1^A)}{\partial \beta_1^A} = \frac{2p_A^2 C_A}{(\beta_1^A)^2\left(1+C_A\frac{p_A}{\beta_1^A}\right)^3} - \frac{2(1-p_A)^2 C_2 C_B}{(1-\beta_1^A)^2\left(1+C_B\frac{(1-p_A)}{(1-\beta_1^A)}\right)^3},$$

and the second derivative as

$$\begin{aligned}\frac{\partial^2 \bar{X}_1^\star(\beta_1^A)}{\partial\left(\beta_1^A\right)^2} =& \frac{6p_A^3 C_A^2}{(\beta_1^A)^4\left(1+C_A\frac{p_A}{\beta_1^A}\right)^4} - \frac{4p_A^2 C_A}{(\beta_1^A)^3\left(1+C_A\frac{p_A}{\beta_1^A}\right)^3} \\ &+ \frac{6(1-p_A)^3 C_2 C_B^2}{(1-\beta_1^A)^4\left(1+C_B\frac{(1-p_A)}{(1-\beta_1^A)}\right)^4} - \frac{4(1-p_A)^2 C_2 C_B}{(1-\beta_1^A)^3\left(1+C_B\frac{(1-p_A)}{(1-\beta_1^A)}\right)^3}.\end{aligned}$$

Evaluating both derivatives at the prior yields

$$\left.\frac{\partial \bar{X}_1^\star(\beta_1^A)}{\partial \beta_1^A}\right|_{\beta_1^A=p_A} = \frac{2C_A}{(1+C_A)^3} - \frac{2C_2 C_B}{(1+C_B)^3},$$

and

$$\left.\frac{\partial^2 \bar{X}_1^\star(\beta_1^A)}{\partial\left(\beta_1^A\right)^2}\right|_{\beta_1^A=p_A} = \frac{6C_A^2}{p_A(1+C_A)^4} - \frac{4C_A}{p_A(1+C_A)^3} + \frac{6C_2 C_B^2}{(1-p_A)(1+C_B)^4} - \frac{4C_2 C_B}{(1-p_A)(1+C_B)^3}.$$

We need two linear auxiliary functions to find necessary and sufficient conditions that characterize NR, FR, and PR equilibria. They are depicted in Figure 2.11.

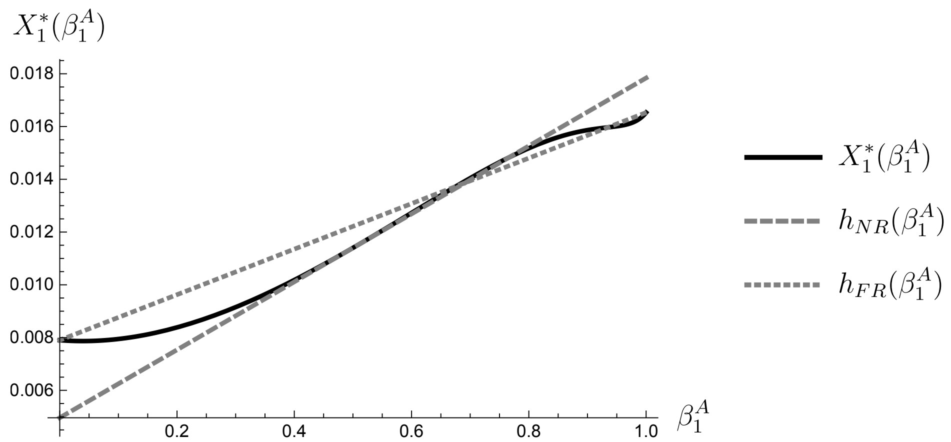


Figure 2.11: Auxiliary Functions for $p_A = .5,\ C_A = 9,\ C_B = .25,\ C_2 = .02.$

The first function is a tangent to the graph of $\bar{X}_1^\star(\beta_1^A)$ at the point $\big(p_A, \bar{X}_1^\star(p_A)\big)$. Using the results from above, we know it is given by

$$h_{NR}(\beta_1^A) := \left[\frac{p_A}{(1+C_A)^2} + \frac{(1-p_A)C_2}{(1+C_B)^2}\right] + \left[\frac{2C_A}{(1+C_A)^3} - \frac{2C_2C_B}{(1+C_B)^3}\right](\beta_1^A - p_A).$$

A necessary and sufficient condition for NR to be compatible with equilibrium is given if $h_{NR}(\beta_1^A)$ is a supporting hyperplane of the graph of $\bar{X}_1^\star(\beta_1^A)$, i.e., if

$$h_{NR}(\beta_1^A) - \bar{X}_1^\star(\beta_1^A) \geq 0 \quad \forall \beta_1^A \in [0,1]. \tag{2.20}$$

Reformulating this inequality yields that for all $\beta_1^A \in [0,1]$ it must be that

$$(p_A - \beta_1^A)^2 \left[\frac{C_A\left(C_A(1-C_A)p_A + 2\beta_1^A\right)}{(1+C_A)^3\left(\beta_1^A + C_Ap_A\right)^2} + \frac{C_2C_B\left(C_B(1-C_B)(1-p_A) + 2(1-\beta_1^A)\right)}{(1+C_B)^3\left((\beta_1^A - 1) + C_B(p_A - 1)\right)^2}\right] \geq 0.$$

Note that, by the definition of $h_{NR}(\beta_1^A)$, this is trivially satisfied if $\beta_1^A = p_A$.

The second auxiliary function is a secant to the graph of $\bar{X}_1^\star(\beta_1^A)$ through the points $\big(0, \bar{X}_1^\star(0)\big)$ and $\big(1, \bar{X}_1^\star(1)\big)$. Using limits in the calculation, we can see that it is given by

$$h_{FR}(\beta_1^A) := \left[\frac{(1-p_A)C_2}{(1+C_B(1-p_A))^2}\right] + \left[\frac{p_A}{(1+C_Ap_A)^2} - \frac{(1-p_A)C_2}{(1+C_B(1-p_A))^2}\right]\beta_1^A.$$

A necessary and sufficient condition for FR to be compatible with equilibrium is given by

$$h_{FR}(\beta_1^A) - \bar{X}_1^\star(\beta_1^A) \geq 0 \quad \forall \beta_1^A \in [0,1]. \tag{2.21}$$

Reformulating this inequality yields that for all $\beta_1^A \in [0,1]$ it must be that

$$\begin{aligned} &p_A\beta_1^A\left[\frac{\left(\beta_1^A + p_AC_A\right)^2 - \beta_1^A\left(1+p_AC_A\right)^2}{\left(\beta_1^A + p_AC_A\right)^2\left(1+p_AC_A\right)^2}\right] \\ +C_2(1-p_A)(1-\beta_1^A)&\left[\frac{\left((1-\beta_1^A) + (1-p_A)C_B\right)^2 - (1-\beta_1^A)\left(1+(1-p_A)C_B\right)^2}{\left((1-\beta_1^A) + (1-p_A)C_B\right)^2\left(1+(1-p_A)C_B\right)^2}\right] \geq 0. \end{aligned}$$

Note that by the definition of $h_{FR}(\beta_1^A)$ this is trivially satisfied if $\beta_1^A = 0$ or $\beta_1^A = 1$. $\square$

2.A.4 Proof of Proposition 2.3: Private Messages (Non-Bayesian Case) - Sufficiency

Proof. It is easy to see that for Condition (2.17) to hold it suffices that $C_A, C_B \leq 1$. Furthermore, it is easy to see that for Condition (2.18) to hold it suffices that $C_A \geq \frac{1}{p_A} > 1$ and $C_B \geq \frac{1}{(1-p_A)} > 1$.

A sufficient condition for NR not to be an equilibrium is given if $\exists \beta_1^A \in [0,1]$ s.t. in Condition (2.17) both nominators in the brackets are negative, i.e. if $C_A > \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\beta_1^A}{p_A}} := \tilde{C}(\beta_1^A)$ and $C_B > \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(1-\beta_1^A)}{1-p_A}} := \hat{C}(\beta_1^A)$. For $\beta_1^A \in [0,1]$, $\tilde{C}(\beta_1^A)$ and $\hat{C}(\beta_1^A)$ are positive-valued functions. The former increases, while the latter decreases in β_1^A. Their unique intersection is given when the argument satisfies $\beta_1^A = p_A$ and $\tilde{C}(p_A) = \hat{C}(p_A) = 2$. □

2.A.5 Proof of Proposition 2.4: Private Messages (Non-Bayesian Case) - Specialists

Proof. The first part can be shown by proving that we can find a $p_A \in (0,1)$ such that there exists a $\beta_1^A \in [0,1]$ for which Condition (2.17) is violated. Let us fix $\beta_1^A = 0$, $C_A > 1$ and $C_B < 1$, i.e., contestant 1 is the specialist in task B, and thus at $\beta_1^A = 0$ the first summand in the second factor of Condition (2.17) is negative, while the second summand is positive. We now show that by decreasing p_A arbitrarily close to zero, we can always guarantee that the sum of these two terms is negative. Evaluate the relevant terms by applying l'Hôpital's Rule (its premises are satisfied):

$$\begin{aligned} \lim_{p_A \to 0} \frac{C_A\left(C_A(1-C_A)p_A\right)}{\left(1+C_A\right)^3\left(C_A p_A\right)^2} &= \lim_{p_A \to 0} \frac{\partial\left[C_A\left(C_A(1-C_A)p_A\right)\right]/\partial p_A}{\partial\left[\left(1+C_A\right)^3\left(C_A p_A\right)^2\right]/\partial p_A} \\ &= \lim_{p_A \to 0} \frac{C_A^2(1-C_A)}{2\left(1+C_A\right)^3 C_A^2 p_A} = -\infty \; \forall C_A > 1, \end{aligned}$$

and

$$\begin{aligned} \lim_{p_A \to 0} \frac{C_2 C_B\left(C_B(1-C_B)(1-p_A)+2\right)}{\left(1+C_B\right)^3\left(C_B(p_A-1)-1\right)^2} &= \lim_{p_A \to 0} \frac{\partial\left[C_2 C_B\left(C_B(1-C_B)(1-p_A)+2\right)\right]/\partial p_A}{\partial\left[\left(1+C_B\right)^3\left(C_B(p_A-1)-1\right)^2\right]/\partial p_A} \\ &= \frac{C_B^2(1-C_B)}{2\left(1+C_B\right)^4} \in (0,\infty) \; \forall C_B < 1. \end{aligned}$$

The second part of the proposition can be verified by analyzing Condition (2.18). Suppose that $C_A \geq \frac{1}{p_A}$ and $C_B < 1 < \frac{1}{1-p_A}$, i.e., contestant 1 is a specialist in task B. Additionally, the first summand in Condition (2.18) is positive, while the second is negative. Now we can arbitrarily decrease C_2 until the inequality is satisfied for all

$\beta_1^A \in [0,1]$. A similar procedure can be applied to construct a violation of Condition (2.17) by increasing / decreasing C_2. □

2.A.6 Proof of Proposition 2.5: Private Messages (Bayesian Case) - Contestants

Proof. Contestant 2 is always of m-type m^A. Thus, he has to solve the following problem:

$$\max_{x_{2A}, x_{2B}} \left[p_A \cdot \frac{x_{2A}}{x_{2A} + x_{1A}(m^A)} + (1 - p_A) \cdot \frac{x_{2B}}{x_{2B} + x_{1B}(m^B)} \right] - c_{2A}x_{2A} - c_{2B}x_{2B}, \quad (2.22)$$

where $x_{1t}(m)$ denotes the choice of contestant 1's type $m \in M_1$ in task $t \in T$. The first order conditions yield the best responses

$$x_{2t}(x_{1t}(m^t)) = \sqrt{\frac{p_t}{c_{2t}} \cdot x_{1t}(m^t)} - x_{1t}(m^t), \quad \forall t \in \{A, B\}. \quad (2.23)$$

Contestant 2 can be of two types, m^A and m^B. As type m^t he knows that he is in a state, in which only task $t \in T$ matters. Thus, he has to solve the following problem:

$$\max_{x_{1A}(m^t), x_{1B}(m^t)} \left[\frac{x_{1t}(m^t)}{x_{1t}(m^t) + x_{2t}} \right] - c_{1A}x_{1A}(m^t) - c_{1B}x_{1B}(m^t). \quad (2.24)$$

Applying the Kuhn-Tucker approach with the constraint that $x_{1t}(m^z) \geq 0 \ \forall t, z \in T$ yields the best responses for all $t \in T$:

$$x_{1t}(x_{2t}|m^z) = \begin{cases} \sqrt{\frac{1}{c_{1t}} \cdot x_{2t}} - x_{2t} & \text{if } z = t, \\ 0 & \text{if } z \neq t. \end{cases} \quad (2.25)$$

This system of six best response equations with six unknowns can be solved and we obtain the equilibrium behavior given by

$$x^*_{1A}(\tilde{\beta}_1(m^A, Q_1^{FR})) = \frac{p_A}{c_{2A}\left(1 + \frac{p_A c_{1A}}{c_{2A}}\right)^2}, \qquad x^*_{1B}(\tilde{\beta}_1(m^A, Q_1^{FR})) = 0,$$

$$x^*_{1A}(\tilde{\beta}_1(m^B, Q_1^{FR})) = 0, \qquad x^*_{1B}(\tilde{\beta}_1(m^B, Q_1^{FR})) = \frac{(1 - p_A)}{c_{2B}\left(1 + \frac{(1-p_A)c_{1B}}{c_{2B}}\right)^2},$$

$$x^*_{2A}(\tilde{\beta}_2(m^A, Q_1^{FR})) = \frac{p_A\left(\frac{p_A c_{1A}}{c_{2A}}\right)}{c_{2A}\left(1 + \frac{p_A c_{1A}}{c_{2A}}\right)^2}, \qquad x^*_{2B}(\tilde{\beta}_2(m^A, Q_1^{FR})) = \frac{(1 - p_A)\left(\frac{(1-p_A)c_{1B}}{c_{2B}}\right)}{c_{2B}\left(1 + \frac{(1-p_A)c_{1B}}{c_{2B}}\right)^2}.$$

Normalizing with c_{2A}, summing up, and reformulating, we obtain the expected equilibrium efforts in each task, given by

$$\begin{aligned} E[X_A^*](Q_1^{FR}) &= P(m^A)(Q_1^{FR}) \cdot x_{1A}^*(\tilde{\beta}_1(m^A, Q_1^{FR})) + x_{2A}^*(\tilde{\beta}_2(m^A, Q_1^{FR})) \\ &= \frac{(1+C_A)p_A^2}{(1+p_A C_A)^2}, \end{aligned} \tag{2.26}$$

$$\begin{aligned} E[X_B^*](Q_1^{FR}) &= P(m^B)(Q_1^{FR}) \cdot x_{1B}^*(\tilde{\beta}_1(m^B, Q_1^{FR})) + x_{2B}^*(\tilde{\beta}_2(m^A, Q_1^{FR})) \\ &= \frac{C_2(1+C_B)(1-p_A)^2}{[(1+(1-p_A)C_B]^2}. \end{aligned} \tag{2.27}$$

□

2.A.7 Proof of Proposition 2.6: Private Messages (Bayesian Case) - Full Vs. Non-Revelation

Proof. From Proposition 2.5 and Corollary 2.1 we obtain the difference we have to evaluate:

$$\begin{aligned} E[X^*](Q_1^{FR}) - E[X^*](Q^{NR}) &= \frac{(1+C_A)p_A^2}{(1+p_A C_A)^2} + \frac{C_2(1+C_B)(1-p_A)^2}{((1+(1-p_A)C_A)^2} \\ &\quad - \frac{p_A(1+C_B) + C_2(1-p_A)(1+C_A)}{(1+C_A)(1+C_B)} \\ &= (p_A - 1)p_A \left[\frac{C_2(1-C_B^2(1-p_A))}{(1+C_B)(C_B(p_A-1)-1)^2\}} \right. \\ &\quad \left. + \frac{1-C_A^2 p_A}{(1+C_A)(1+C_A p_A)^2} \right] \end{aligned}$$

Evaluating whether the nominators in the huge bracket are positive or negative at the same time gives the conditions stated in the proposition. □

Chapter 3

Differentiate and Conquer - Using Consumer Learning to Grow Out Your Niche[1]

"Differentiate and conquer" suggests exploiting an a-priori disadvantage, i.e., producing a niche product, to later on gain power over the larger share of the market. The driving mechanism is the recommendation effect, which introduces a new rationale for product differentiation other than the usual motivation to reduce price competition. We incorporate consumer learning in a model of spatial competition with sequential consumer purchases and a second dimension of variation, quality, about which the consumers have differential information. With consumer learning, firms are confronted with two mutually offsetting effects: differentiation decreases the likelihood that a product will be bought in earlier periods, but, by making inference more valuable, it also increases the likelihood that later consumers may buy the differentiated good. We show that there exists a unique "differentiate-and-conquer equilibrium" in which the second effect dominates, so that the market incumbent locates in the center of the market, while the entrant differentiates by producing an ex-ante niche product.

3.1 Introduction

10th rule of building a successful business: Swim upstream. Go the other way. Ignore the conventional wisdom. If everybody is doing it one way,

[1]This chapter is joint work with Maximilian Conze.

> *there's a good chance you can find your niche by going exactly in the opposite direction. (Sam Walton, founder of Walmart)*[2]

When introducing a new product, one of the most important questions a firm faces is how to design its product with respect to the products already offered by its competitors and with respect to consumer taste: shall it produce more of a mainstream product or shall it occupy a niche in the market, i.e., offer a product differentiated from its competitors and preferred ex-ante only by a minority of the consumers?

This question seems to gain even more importance as early adopters (agents consuming in earlier periods) find a growing number of opportunities to publicly announce their choice behavior using Internet platforms such as Yelp or the recommendation opportunities on the online market place Amazon, for instance. Platforms like foursquare, Google and Facebook explicitly keep track of "check-ins" in restaurants, bars and many other venues to provide this data online for undecided consumers. These sources of information influence the choice behavior of laggards (agents consuming in later periods).

Such effects are not considered in earlier research dealing with the incentives to offer differentiated products. It has instead focused on the fact that price competition may yield incentives to differentiate one's product: by offering a differentiated product, firms are able to set a price above marginal costs and thus obtain a positive profit, although they serve a smaller market share. In those situations, the competition effect, i.e., the ability to raise prices because of the "local monopoly power" obtained by offering a differentiated product, dominates the market size effect, i.e., the possibility to serve a larger market share when offering a product similar to that of the competitor (see, e.g., d'Aspremont et al., 1979).

We establish a fundamentally different rationale for offering differentiated products. The effect driving our result is one of informational nature and arises due to the possibility of consumer learning. In our model, laggards can observe the behavior of the early adopters. When adding vertical differentiation, e.g., quality, about which the consumers have different knowledge, into a model of (spatial) product differentiation, the choice behavior of early adopters contains information influencing the choice behavior of laggards. A firm can influence and exploit consumer learning using its location choice. This may yield incentives to offer a differentiated (niche) product, as from a laggard's perspective, a purchase of a niche product by an early adopter is more likely based on its high quality than on a good match of consumer taste and product characteristic. We call this the "recommendation effect".

A recent event in the movie industry fits our model very well. The movie "The Artist", which aired in cinemas in 2011, was a major success of that year and, in addition

[2]See `http://corporate.walmart.com/our-story/history/10-rules-for-building-a-business` (last accessed: 02/11/2017).

to receiving mainly positive critique, it won numerous prizes, including five Oscars.[3] It brought in almost \$133.5M worldwide, while being produced with a \$15M budget.[4] So on both counts - artistically and economically - it was a major success. What makes this movie especially interesting for our case, is that, compared to the advanced techniques commonly used in the movie industry nowadays with its 3D-effects and Dolby Surround, the means used for the shooting of "The Artist" were rather unconventional: it was entirely shot in black-and-white and mainly abstracted from dialogues, almost making it a silent movie. Compared to the other blockbusters at that time, "The Artist" definitely was a "niche product". Yet, it may well be that the high popularity of this unconventional movie among the early adopters in the first weeks of broadcasting induced the laggards to attribute the reason for that choice behavior to the high cinematographic quality of "The Artist". It is likely that the producers anticipated just this reasoning and therefore decided to dive into this unorthodox project. Indeed, the director of "The Artist", Michel Hazanavicius, said that when he presented his idea "[he'd] only get an amused reaction - no one took this seriously".[5]

We use the term "niche product" in the sense that such a product is of relatively low appeal to uninformed consumers ex ante. As our model shows, and the example of the movie "The Artist" illustrates, a niche product according to this definition can still generate a larger demand than a mainstream product ex-post. Thus, niche firms applying the "differentiate and conquer" strategy may in the end dominate the market.

Our contribution to the literature and the main goal of this chapter is to show in a theoretical model how the firms' incentives to differentiate are affected by social learning among heterogeneous consumers. The chapter is structured as follows. In Section 3.2 we review the related literature. Section 3.3 presents a short and simplified example, while the full model is introduced in Section 3.4. In Section 3.5 we analyze the optimal consumer behavior. A benchmark model and the main model are solved in Sections 3.6 and 3.7, respectively. Welfare comparisons are made in Section 3.7.5. Finally, Section 3.8 concludes. All proofs are relegated to the Appendix.

[3] See `http://articles.economictimes.indiatimes.com/2012-02-27/news/31104573_1_oscars-foreign-language-category-actor-race` and `http://www.theguardian.com/film/2011/dec/08/artist-silent-film-michel-hazanavicius` (last accessed: 02/11/2017).

[4] See `http://www.boxofficemojo.com/movies/?id=artist.htm` and `http://www.imdb.com/title/tt1655442/business?ref_=ttrel_ql_4` (last accessed: 02/11/2017).

[5] See page 5 of the official press kit at: `http://www.festival-cannes.com/en/films/the-artist` (last accessed: 02/11/2017). Additionally, the success of the movie was called "surprising" by the media, see, e.g., `http://www.theguardian.com/film/2012/feb/04/hollywood-nostalgia-chaplin-valentino` (last accessed: 02/11/2017).

3.2 Literature

In his seminal paper on *spatial competition* and product differentiation, Hotelling (1929) proposed that, when choosing locations (which can be interpreted as representing the product's characteristics) on a linear bounded market - where consumers are uniformly distributed - before setting prices, firms choose the same location, namely the center, and set the same price in equilibrium. It has later been shown by D'Aspremont et al. (1979), that the celebrated Principle of Minimum Differentiation is only valid in this framework if prices are exogenously fixed and equal; we will refer to this setup as Hotelling's "pure spatial competition model".[6] As in any model of spatial competition with endogenous prices, there exist two offsetting effects in Hotelling's setup. On the one hand, firms have an incentive to increase the distance, thereby relaxing competition ("competition effect"). On the other hand, decreasing the distance allows to serve a larger share of the market ("market size effect").

In contrast to most models of spatial competition, we find asymmetric pure strategy equilibria, for both cases - where firms differentiate and where they do not. Tabuchi and Thisse (1995) assume a non-uniform distribution of consumers, sequential location choice and simultaneous price setting and also find asymmetric pure strategy equilibria. However, differing from their results, serving a smaller ex-ante market share - that is being a niche producer - need not be disadvantageous in our model, i.e., there is no second-mover disadvantage as it happens to be the case in Tabuchi and Thisse (1995).

Among others,[7] Economides (1989) combines *price setting, horizontal and vertical differentiation*. In both versions of his model - price competition followed by quality choice, or both choices of these strategic variables happening simultaneously (location choice occurs at the first stage in both versions) - maximum horizontal differentiation and minimal differentiation in quality and prices is obtained in equilibrium. Bester (1998) differs from Economides (1989) in assuming quadratic instead of linear transport costs (which strengthens incentives to differentiate) and, more importantly, in the consumer's imperfect knowledge about qualities. He shows that this imperfect knowledge mitigates product differentiation: as consumers associate low prices with a low quality, there is an endogenous lower bound to prices. Thus, price competition is already relaxed, making it less necessary to horizontally differentiate in order to decrease price competition.

In the literature on *social learning*, Bikhchandani et al. (1992) and Banerjee (1992) are the first to examine the phenomena of *information cascades* and *herding*. They

[6] With fixed prices the firms' goals narrow down to serving the largest possible market share. Then, the only situation without an incentive to relocate is the one where both firms are located at the center.

[7] See, e.g., Gabszewicz and Thisse (1986), Dos Santos Ferreira and Thisse (1996), and Gabszewicz and Wauthy (2011).

show that with sequential consumer choice, Bayes rational inference from the previous behavior of others may guide consumers to ignore their own (imperfect) private signal on the quality of a firm; a behavior which in the end may result in herding, driving all subsequent consumers to buy only from one firm. Smith and Sørensen (2000) deliver the most complete analysis of this setup of social learning.

Ridley (2008) combines the ideas of *Hotelling and herding.* Nevertheless, his research question is fundamentally different to ours: he models two firms with different information levels about market demand and - as they sequentially decide about entering the market - the second mover can possibly deduce information from the other firm's decision.

Our model is also related to the recent literature on *Bayesian persuasion* as introduced by Kamenica and Gentzkow (2011): they analyze in which way the sender (in our case the firms) can influence the updating of the receiver (in our case the laggards) in their favor by choosing the "sender-optimal" information structure of the signal. As in our model, the sender does not know the realization of the state (in our case the product's quality), when taking his decision. In the analysis at hand the early adopters are so to speak "used" by the firm to signal the quality of its product and differentiation in some cases is advantageous for the market entering firm, as via the Bayesian updating mechanism producing a niche product distorts the signal (i.e., the purchase decision of the early adopter) in its favor.

The strand of literature that is closest to our approach has taken a look at the *impact of consumers' social learning on competition among firms* producing horizontally and vertically differentiated products. In Caminal and Vives (1996) two firms compete for homogeneous consumers by setting prices. The authors formulate two models, and, in one of these, firms do not know the quality of their product, just as it is the case in this chapter. Consumers have different information about the products' qualities and observe the history only partially. Given the incomplete observation of the history, consumers are led to believe that a good is of higher quality whenever its market share is high. The authors show that this leads to a strategic incentive for the firms to generate a higher demand in early periods by setting a low price. However, Caminal and Vives do not analyze incentives to differentiate.

Miklos-Thal and Zhang (2013) model a *monopolistic market with consumer learning* showing that "demarketing [i.e. visibly toning down the marketing efforts] lowers expected sales ex ante but improves the product quality image ex post, as consumers attribute good sales to superior quality" (Miklos-Thal and Zhang, 2013, p. 55). In the same vein, Parakhonyak and Vikander (2016) show that a monopolist may have an incentive to reduce its capacity in order to make laggards infer a high product quality from sold out capacities in earlier periods and thus induce herding in its favor.

In the model at hand we assume that prices are exogenous.[8] The assumption is appropriate for situations in which prices are actually fixed, or where price differences among products are perceived as too small to influence the consumption decision (see, for example, Courty (2000)), or markets where prices are chosen in a long term perspective. Additionally, there are many markets where consumers do not pay any price at all, but firms nevertheless compete in order to maximize their demand. This is true for many media markets like TV or radio stations, as discussed in Gabszewicz et al. (2001) and Loertscher and Muehlheusser (2011). The exogeneity of prices can be seen as a reduced form model for markets which are better characterized by competition in market shares than by competition in prices. Theoretically, we are interested in identifying the effect consumer learning has on the differentiation incentives of firms and by abstracting from price competition isolating this effect becomes easier.[9]

Tucker and Zhang (2011) show in an *empirical* paper[10] that - in line with the intuition of our theoretical results - popularity information (indicated by the choice of previous consumers) is especially beneficial for niche products, because for the same popularity, niche products are more likely to be of superior quality than mainstream products.

Another empirical paper is even more suitable for our analysis - and especially for our example above on the movie industry: Moretti (2011) is among the first researchers to empirically analyze real world data on social learning. He investigates in how far it influences movie sales. The results show that social learning indeed matters and that "surprise" in the early demand increases later demand for a movie.[11] That is, if a movie was seen by surprisingly many consumers (compared to the prior) in the first weeks

[8] In Chapter 4, we show that similar effects also arise in a setup of non-spatial differentiation with endogenous prices. Nevertheless, in some applications the assumption of fixed prices may be plausible. Consider the movie industry, where the entrance fees for blockbusters of the same length at cinemas are usually the same. See Orbach and Einav (2007), for instance. Many people arguably decide on which movie to watch before seeing the prices. Furthermore, they most probably do not revise their decision when finding out that prices are slightly different than expected. De Vany (2006) discusses the three different pricing levels of the movie industry (producers, distributors, box offices) extensively and shows that empirically box office prices are fixed - which indeed is an economic puzzle. Additionally, it is shown that the producers obtain a contractually regulated share of the revenues generated by the box offices. This implies that the only way producers can influence their revenue is by generating a larger audience.

[9] A research field in which exogenous prices are frequently assumed is health economics, as medical treatment is reimbursed to consumers by their health insurance. Several research projects in this area use models of spatial competition with fixed prices and discuss differences in quality among hospitals, see, e.g., Brekke et al. (2006), Brekke et al. (2011), and Gravelle and Sivey (2010).

[10] In a working paper version, they also include a theoretical model in which location, however, is given exogenously.

[11] While Sorensen (2007) and Chen (2008) support the social learning argumentation in another setting, Gilchrist and Sands (2016) attribute the fact, that a positive "shock" to early demand for cinema movies (e.g. by bad weather) increases demand in later periods, to network externalities, that is, "people have something in common to talk about".

of airing, this will have the additional (indirect) effect of a social multiplier: while it immediately increases profits to the cinemas, it also generates a higher demand in the following periods. We can infer that this yields an incentive for movie producers to create "surprising" movies in the sense that they are very successful in the first weeks compared to the expectations. This may just be the reason to produce a black-and-white silent movie nowadays.

3.3 Illustrative Example With Discrete Strategy Spaces

Before presenting the full model, we demonstrate the effects at force in a small illustrative example with discrete action spaces.

There are two firms producing an ex-ante homogeneous good: firm A produces a good with a deterministic value of $v = 20€$, and firm B's good B is either of value $v_B = 30€$ or $v_B = 10€$ with probability 0.5 each. Neither firm knows the realized value of firm B's product. The price of all goods is 5€. Firms A and B sequentially choose their locations a and b along a road at one of three locations: kilometer 0, 0.5 or 1. Firm A is the first mover, and as $a = 0$ is equivalent $a = 1$, we restrict firm A to the right part of the interval. There are two consumers (each independently located at 0, or 0.5, or 1 with probability 1/3): an early adopter who with probability 0.5 is either completely informed or uninformed, and a laggard, who is completely uninformed about product B's value and the early adopter's location, but observes the choice behavior of the early adopter. A consumer has to pay 4.5€ for traveling the 0.5 km to the next location, and costs are linear. Consumers maximize their expected utility and the firms compete over the two consumers by their location choice.

Let β denote the consumer's belief that firm B's product is superior. The expected difference in value between B's and A's product is then given by

$$E[v_B - v_A] = \beta \cdot 30€ + (1 - \beta) \cdot 10€ - 20€ = (2\beta - 1) \cdot 10€. \quad (3.1)$$

Consumers compare this difference to the different transport costs between the firms. An uninformed early adopter perceives the goods' values to be the same ($\beta = 0.5$), and so chooses the closer firm. We assume that if firms locate at the same position, an uninformed early adopter at the same position chooses each firm with probability 0.5 and chooses B (A) if he is located left (right) of the firms. In this example the tie-breaking rules out equilibria arising only due to the discrete action space.[12]

[12]Obviously, there are several other tie-breaking rules that are also compatible with rationality of consumers, but any other tie-breaking rule would still lead to the result of one firm not positioning at the center of the market, which is a result driven solely by the recommendation effect described later. The tie-breaking rule at hand is the discrete analogue of the one we employ in our model with

For an informed consumer, i.e., $\beta \in \{0,1\}$, the sure gain of buying the superior product (10€) is always higher than possible transport costs (9€ at the maximum), so she always buys at the better firm, no matter where she is located.

The belief of the laggard is of more interest, since she uses Bayes' rule to calculate the probability of each firm offering the superior product as follows. If firm B was chosen in the first period the updated belief is given by

$$\beta_B^u := Pr(v_B = 30 \mid C_1 = B) = \frac{Pr(C_1 = B \mid v_B = 30)}{Pr(C_1 = B)} \cdot Pr(v_B = 30).$$

The probability that B is superior given A was bought in the first period, β_A^u, is derived analogously as

$$\beta_A^u := Pr(v_B = 30 \mid C_1 = A) = \frac{Pr(C_1 = A \mid v_B = 30)}{Pr(C_1 = A)} \cdot Pr(v_B = 30).$$

In general, these probabilities depend on the firms' locations, and because of the possibility that the first period consumer was informed, the probability that a product is bought is always higher if it is superior, so that updating is informative and $\beta_B^u > 0.5$ (and $\beta_A^u < 0.5$). When firms are located at the same spot, every consumer has to incur the same transport costs for both firms, and so the laggard always follows the decision of the early adopter. If firms are not located at the same spot, consumers compare the expected additional value of the goods, as stated in Equation (3.1) to the additional transport costs. For all symmetric positions of the two firms, the choice probabilities and thus the beliefs are the same and are calculated as $\beta_B^u = 0.75 = 1 - \beta_A^u$.

If, however, $b = 0$ and $a = 0.5$, i.e., the firms locations are asymmetric, $\beta_B^u = \frac{0.5 \cdot 1 + 0.5 \cdot 1/3}{0.5 \cdot 0.5 + 0.5 \cdot 1/3} \cdot 0.5 = 4/5$ and $\beta_A^u = \frac{0.5 \cdot 0 + 0.5 \cdot 2/3}{0.5 \cdot 0.5 + 0.5 \cdot 2/3} \cdot 0.5 = 2/7$, so that in this case $\beta_B^u = 4/5 > 1 - \beta_A^u = 5/7$. A choice of B in period 1 increases the laggards confidence in this product more than a choice of product A, in particular a laggard is willing to travel an additional distance of 0.5 to obtain product B instead of A if B was chosen in period 1, but she is unwilling to travel the same distance to buy A instead of B if A was chosen.

a continuous action space (see Assumption 3.2 in Section 3.5). Basically, the tie-breaking rule is a selection mechanism which rules out other possible equilibria. In particular, in the discrete model there are equilibria in which both firms locate at the same end of the market (locations 0 and 0 as choices of firm A and B, respectively, or 1 and 1) or at different ends of the market (0 and 1 or 1 and 0 as choices of firms A and B, respectively). This is the case for instance under the "equal-split" tie-breaking rule, in which each consumer (independent of his location) randomizes with equal probabilities whenever the firms are located at the same position. In addition to its plausibility in view of the model setup, the tie-breaking rule at hand also supports the most plausible of the possible equilibria: the market incumbent chooses the central position and the entrant chooses the niche position. A tie-breaking rule opposed to the one used in the example at hand might favor the niche firm in an analogous way and would result in a second mover advantage, which also seems implausible considering the setup at hand.

This is easily seen by using Equation (3.1) to compare the expected additional valuation to the additional transport costs:

$$(2\beta_B^u - 1) \cdot 10€ = 7€ > 4.5€ > (2 \cdot (1 - \beta_A^u) - 1) \cdot 10€ \approx 4.29€$$

All other beliefs can directly be obtained using these calculations because of the symmetry of the model. The induced asymmetry in the beliefs and the consequences for the consumers' behavior are the driving effects for the results to follow.

The beliefs and the resulting behavior of the laggard partition the action space of the firms as visualized in Figure 3.1. In situations labeled $\mathfrak{D}^2$ firms locate at the maximal distance from each other, and it will never be the case that all types of laggards follow the behavior of the early adopter, as $(2\beta_B^u - 1) \cdot 10€ = 7€ < 9€$. In situations labeled $\mathfrak{D}^4$ firms locate at the same position and a laggard always follows the behavior of the early adopter.

In situation $\mathfrak{D}^{3B}$ ($\mathfrak{D}^{3A}$) the firms positions are different and asymmetric, furthermore B (A) is the niche firm here, thus the notation. As shown above, the laggards behavior now depends on the history: If, for instance, B is the niche firm, i.e., $(a, b) \in \mathfrak{D}_L^{3B}$, all consumers located at $x = 0.5$ will consume at firm B after observing it was chosen in the first period.[13] The same holds true for consumers located at $x = 1$, as they have the same (additional) travel costs as the consumers located at $x = 0.5$, once they traveled to location 0.5. Consumers located at $x = 0$ will still consume at firm B, even if they observed $C_1 = A$. Consumer updating is beneficial for firm B, as it is the niche firm. The reverse is true in situations labeled with $\mathfrak{D}^{3A}$.

[13]Note, that we use subscripts L and R to indicate whether firm B is positioned left or right of firm A.

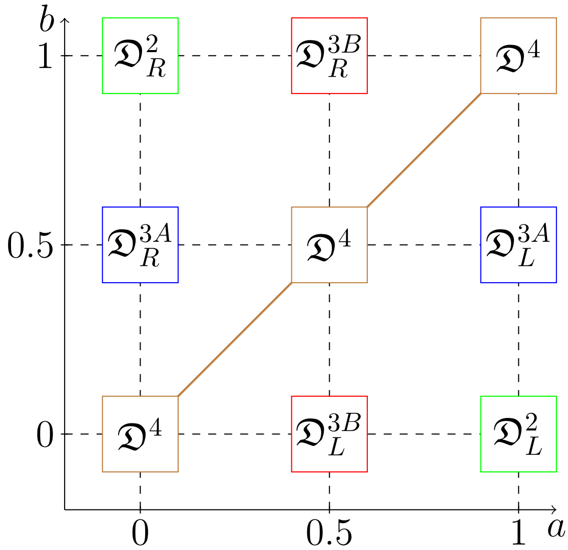


Figure 3.1: Partition of the action space for the discrete example.

Let $D(a,b)$ denote B's (expected) demand if locations are (a,b). Firm A serves the remainder of the market, so its demand is $\tilde{D}(a,b) = 2 - D(a,b)$. To obtain the equilibrium we need to calculate $D(0.5,0)\left[= \tilde{D}(1,0.5)\right]$, $D(1,1)\,[= D(0,0)]$ and $D(0.5,0.5)\,[= D(1,0)]$. It is easy to see that both firms split the market equally in the latter cases and the resulting demand equals $D(0.5,0.5) = 1$. Additionally, we have

$$D(0.5,0) = \underbrace{Pr(C_1 = B)}_{\text{1st period}} + \underbrace{P(C_1 = B)\cdot 1 + P(C_1 = A)\cdot 1/3}_{\text{2nd period}} = 5/12 + 5/12 + 7/36 = 37/36,$$

$$D(1,1) = 1/2 \cdot 1/2 + 1/2 \cdot 5/6 + P(C_1 = B)\cdot 1 + P(C_1 = A)\cdot 0 = 2/3 + 2/3 \cdot 1 = 1.33.$$

This implies that firm B's best response to a, $b^*(a)$, is given by $b^*(1) = 1$ and $b^*(0.5) = 0$. Using backward induction firm A then chooses its best point of B's best response function, and will locate at $a^* = 0.5$, so that resulting equilibrium locations are $a^* = 0.5, b^* = 0$, i.e. an equilibrium in which "differentiate and conquer" prevails.[14] Without consumer learning, both firms would locate at the market center, i.e., in an equilibrium with symmetric minimum differentiation.

[14] Obviously, the same reasoning applies for situations with $b \geq a$, i.e., those indexed with R in the partition of the action space, and thus another equilibrium with differentiation is given by $(a^*, b^*) = (0.5, 1)$.

3.4 Model Setup

To generalize the results obtained in the example with a discrete action space, we extend the pure spatial competition model by introducing vertical differentiation, letting firms and consumers decide sequentially and letting consumers observe previous purchase decisions. The full model is detailed below.

Since we are interested in the effect of the possibility to learn from other consumers' purchases, we also present a benchmark model where consumer learning is not possible and we demonstrate that without consumer learning there would be no product differentiation in our model.

Firms Two firms A and B (both: "it") produce (potentially) differentiated goods at zero costs. The retail price is regulated and set to $p > 0$. The firms' locations describing their products' characteristics are confined to the unit interval and are denoted by a and b for firm A and B respectively, so that $a \in \mathcal{A} := [0, 1]$, and $b \in \mathcal{B} := [0, 1]$. The location choice of the firms occurs sequentially, with firm A choosing its location first, and firm B following. The situation if A chooses $a \leq 0.5$ is equivalent to a situation where A chooses $a' = 1 - a$ instead. Thus, it is without loss of generality to restrict $a \in [0.5, 1]$, and $b \in \mathcal{B}$. Note that a situation with $b \geq a$ can be treated as the situation where $a' = 1 - a$ and $b' = 1 - b$. The presentation will therefore be based on the case $b \leq a$ unless stated otherwise. If firm B chooses a position, such that $0.5 < a < b$, we say that it positions on the short side of the market. The firm that is closer to any of boundaries of the interval $[0, 1]$ will be called the niche firm. Thus, B is the niche firm whenever $b < a$ and $b < 1 - a$, or $b > a$ and $b > 1 - a$.

A firm's profit simply is the number of consumers served, multiplied with the exogenous price p. Firms are risk neutral and since discounting future profits does not alter the results qualitatively, it is left out for simplicity.

Besides the horizontal differentiation as measured by the firms' locations, the goods are also of different "value" to consumers. This value can be thought of as representing a good's quality. There is uncertainty about the quality differential between the firms' products, which is randomly determined after the firms have chosen their locations: the value of firm A's product is common knowledge and given by $v_A = v > 0$, while the second firm's quality v_B is either $v_B = v + \delta$ or $v_B = v - \delta$ with $\delta > 0$, both of which occur with probability 0.5.[15] The realized value of v_B is unknown to both firms. Thus, producers possess the following information when choosing their location: firm A has no information, so its information set is given by $\mathcal{I}_A = \{\emptyset\}$, and firm B knows $\mathcal{I}_B = \{a\}$.

[15]This modeling of the supply side can be considered as the first mover A being the market incumbent with a known quality, while firm B with an unknown quality is a new entrant to the market.

Firms play pure strategies. The strategy of firm A is the choice of its location a, while the strategy of the second mover B maps the location a of its competitor into its own location, i.e., $b : \mathcal{A} \to \mathcal{B}$ with $a \mapsto b(a)$.

All cases with $b > a$ can be described with the formulas derived for $b \leq a$, so that in the following we focus on this case and specifically analyze situations with $b > a$ only when necessary.

Consumers On the other side of the market, there are two consumers (both: "she") with heterogeneous preferences, who sequentially make their purchase decisions in periods $t = 1$ and $t = 2$. Consumers are exogenously sorted into being a laggard or an early adopter. Each consumer buys at most one good and will be referred to by the period she has the opportunity to make a purchase (an early adopter in period $t = 1$ and a laggard in period $t = 2$).

Heterogeneity is modeled by assuming that each consumer t is described by a location on the unit interval. In every period, the location of consumer purchasing in period t, denoted by x_t, is independently drawn from a uniform distribution on $\mathcal{X} := [0, 1]$. It measures consumer t's preference towards a good of a firm $F \in \{A, B\}$ located at $f \in [0, 1]$. The closer the location of the consumer to the firm where she buys (holding everything else constant), the higher is the resulting utility. The ex-post utility of a consumer located at x when buying the product from firm $F \in \{A, B\}$ located at $f \in [0, 1]$ is given by

$$u(x, F) = v_F - p - \tau|x - f|,$$

where the last term with the real-valued scalar $\tau > 0$ captures the transport costs. We normalize utility to zero for the case in which a consumer does not buy any of the two goods. Note that with v_B being stochastic this Bernoulli utility function implies risk-neutrality in money. As long as preferences are quasilinear and the ex-ante expected utility of both products (gross of transportation costs) is the same, the results would not change if consumers were risk-averse.

While it is generally possible that a consumer abstains from buying, we will make the following assumption for convenience:

Assumption 3.1. *Every consumer prefers to buy one good to not buying any good, i.e.,* $v - \delta > p + \tau$.

In addition to the heterogeneous preferences, consumers differ in their expertise $\phi \in \{u, i\}$ about firm B's product. Informed consumers ($\phi = i$) observe the realization of v_B, whereas uninformed consumers ($\phi = u$) only have the prior information that $v_B = v + \delta$ or $v_B = v - \delta$, each with probability 0.5. In each period, the consumer is informed with probability $q \in (0,1)$ and uninformed with probability $(1-q)$.[16] A consumer's expertise is independent of her location x and of the expertise of the other consumer. A consumer's type in period t is thus given by (x_t, ϕ_t).

In the second period, the laggard observes the action taken by the early adopter, but neither the early adopter's location nor whether she was informed. Formally, let $C_0 = \emptyset$ and let $C_1 \in \{A, B\}$ be the choice of an early adopter, then the information set of an uninformed consumer is given by $\mathcal{I}_t^u = \{a, b, x_t, v_A, C_{t-1}\}$ and that of an informed consumer by $\mathcal{I}_t^i = \mathcal{I}_t^u \cup \{v_B\}$, for $t = 1, 2$.

Consumers form beliefs β about the probability of firm B offering the product of higher quality by mapping the available information $\mathcal{I}$ into the probability space, $\beta := Pr(v_B > v|\mathcal{I}) \in [0,1]$. The belief of an uninformed consumer is the function $\beta^u : \mathcal{A} \times \mathcal{B} \times \{\emptyset, A, B\} \to [0,1]$, and that of an informed consumer is the function $\beta^i : \mathcal{A} \times \mathcal{B} \times \{\emptyset, A, B\} \times \{v-\delta, v+\delta\} \to [0,1]$. Note that we assume that a consumer's location does not influence her belief.

The strategy of a consumer is a mapping $C_t : \mathcal{A} \times \mathcal{B} \times \mathcal{X} \times [0,1] \to \{A, B\}$ from public and her private information into a purchase decision, where $C(a, b, x, \beta)$ is the choice of a consumer with location $x \in \mathcal{X}$, belief $\beta \in [0,1]$, and firms locations $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Solution Concept and Timing Because of the uniform distribution of consumers, the situation in which first-mover A chooses $a \geq 0.5$ is equivalent to a situation where A chooses $a' = 1 - a$ instead. Therefore, we focus on identifying equilibria with $a \geq 0.5$ in the following and keep in mind that for each of these equilibria an analogous equilibrium exists for the case that $a \leq 0.5$.[17]

We employ the concept of a perfect Bayesian Nash equilibrium in pure strategies to solve the game. We assume that there are only "second order effects" of the firms' locations on the consumers' belief β via the interpretation of the early adopter's choice C_1. This assumption fixes off-equilibrium beliefs and is plausible, as firms have no information about the quality differential.

The timing of the game is depicted in the figure below:

[16] In contrast to the above example with a discrete action space, we allow for *informed* laggards in the second period in the model with continuous action spaces. The minor modeling difference in the example was merely introduced to simplify calculations.

[17] We do not impose any further restrictions, such as the usual assumption $b \leq a$. The coordination issue of this assumption discussed in Bester et al. (1996) does not arise in our setup due to the sequential location choice.

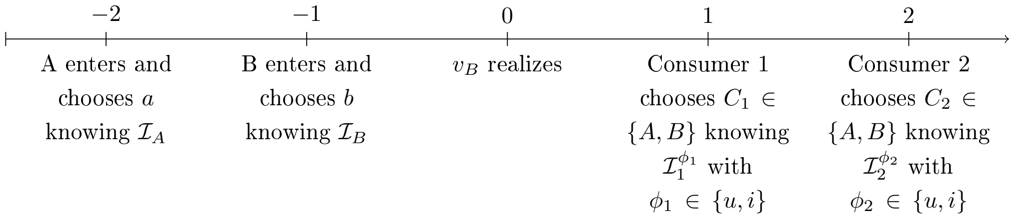


Figure 3.2: Timing of the differentiate-and-conquer game

Discussing the Assumptions

Before solving the model, some of the assumptions deserve additional thought. Similar effects as in this model are also be obtained in a model with endogenous prices, simultaneous choice of product differentiation, a continuously distributed quality differential, and a continuous information structure (see Chapter 4).

Many other unit demand models, where the consumer's type is drawn from some distribution $G(\cdot)$, are equivalent to models with a continuum of consumers of mass one distributed according to $G(\cdot)$, because both formulations yield the same probabilistic demand and thus lead to the same behavior of the firms. However, since observing choices of the whole population is completely informative in our model, this equivalence does not hold. Nevertheless, introducing additional uncertainty to the model, for instance, by assuming that the distribution of the first period consumers and the value of q is uncertain reestablishes this connection even if the laggard observes choices of the whole population. Alternatively, we could reinterpret our modeling assumption as each laggard being drawn from a unit mass of consumers observing only one particular early adopter also drawn from a (different) unit mass of consumers.

In order to simplify the updating of an uninformed laggard, we restrict ourselves to binary signals. One could well assume more than two possible levels of expertise so that consumers would not either be completely informed or completely uninformed. We leave out such specifications as this complicates the Bayesian updating and distracts from the main issue under consideration.

Since the updated probabilities are different for each history of the game, the (sets of) indifferent consumer types (see Assumption 3.2 in Section 3.5 and the discussion below it) are also potentially different for each history, meaning that in each period $t > 1$, 2^{t-1} indifferent consumers have to be determined, quickly making the model intractable. The effects we wish to characterize are already apparent with one period of updating, i.e., with two consumers, which is why we concentrate on this case.

Different cost functions than the linear one applied here do not eliminate the underlying effects of our model, as long as costs are increasing in distance.[18] In the common Hotelling model, quadratic costs enhance the incentive to differentiate and would thus probably make the detection of the driving forces yielding the differentiation result in our model more complicated.

As mentioned above, the assumption of exogenous prices is applicable for markets in which the main endogenous determinant of the firms' profit is the market share. We show that, while without consumer learning the Principle of Minimum Differentiation prevails, consumer learning is sufficient for the existence of equilibria with differentiated products - even when prices are fixed. From a modeling perspective the assumption of exogenous prices has two advantages: on the one hand it assures the existence of a pure-strategy equilibrium (non-existence is a common problem in the original Hotelling model with linear transport costs), and on the other hand it eliminates the competition effect, which allows to more clearly identify the source of differentiation. Thus, the assumption of exogenous prices is in favor of our focus on the effect of consumer learning.

The exogeneity of the quality differential seems plausible in many cases. For instance, concerning the example of the movie "The Artist" and the movie industry in general, it may well be that producers can not completely influence the (perceived) quality of a movie (see De Vany (2006) on this aspect of the movie industry). Also, note that movies - and many other goods - are experience goods (see, e.g., De Vany, 2006), whose value is revealed to consumers only after their consumption. Thus, in many cases even firms arguably do not know their product's relative quality (or the consumers' perceived quality) ex ante. This directly implies that the firms can not signal information about the realized quality to consumers. If their location choice would signal information to consumers, i.e., in a separating equilibrium, the information would already be revealed before the first consumer's choice and social learning would not occur. Since the effects of social learning by the consumers on the firms' location choices are exactly what we are interested in, situations with separating equilibria are not of our primary concern. We thus impose the assumption that firms are unaware of their quality in order to make sure that social learning is possible.

Overall, while the assumptions may seem restrictive first, they allow us to fully characterize the equilibria of this game. We conjecture that the underlying effects identified in this chapter emerge in many different settings.

[18] We show later that Bayesian updating is unaffected by the cost function. Obviously, the demand functions - and thus in our model the best response functions of the firms - are not discontinuous for quadratic costs. Nevertheless, they entail regions, in which the recommendation effect makes it profitable to differentiate.

In the following sections it will become clear that the assumed setup is the most conservative one leading to product differentiation: abstracting from consumer learning leads to the usual result of (symmetric) minimal differentiation.

3.5 General Analysis of Optimal Consumer Behavior

The expected utility of a consumer with location type x and belief β is given by $u(x, \beta, a, b, A) = v - p - \tau|a - x|$, if she buys from firm A, and by $E[u(x, \beta, a, b, B)] = v + (2\beta - 1)\delta - p - \tau|x - b|$, if she buys from B. Clearly, the expected utilities depend on a consumer's belief β and location x. For any consumer type $x \in (b, a)$, the expected utility from B's (A's) product decreases (increases) in x. For all types x that are not located between the firms, changing x affects both expected utilities in exactly the same way, so that the difference of expected utilities is constant. The reason for this is that for all these consumers the difference in distances to the two firms is the same and so is the difference in transportation costs between the firms. This means that all consumers with the same belief β located left of b or right of a must prefer the same firm or are indifferent.

These observations imply that whenever there exists an unique indifferent consumer type, it must be located in the interval (b, a). A consumer located at x holding belief β is indifferent between the products of A and B if

$$E[u(x, \beta, a, b, B)] - u(x, \beta, a, b, A) = (2\beta - 1)\delta - \tau(|x - b| - |a - x|) = 0. \tag{3.2}$$

We define

$$\bar{x}(\beta) := \frac{a + b}{2} + \frac{\delta}{\tau}\Big(\beta - \frac{1}{2}\Big), \tag{3.3}$$

which, for a given belief β, coincides with the consumer type x solving Equation (3.2) whenever Equation (3.2) has a solution $x \in (b, a)$. In this case, there exists an unique indifferent consumer type $x \in (b, a)$ and it is given by $\bar{x}(\beta)$. Then, all consumers with the same belief β and a location left (right) of $\bar{x}(\beta)$ must prefer B (A).

If $\bar{x}(\beta) = b$ ($\bar{x}(\beta) = a$) the consumer with belief β and type $x = b$ ($x = a$) is indifferent between both products and so are all types with the same belief located left of b (right of a).

Inequality $\bar{x}(\beta) < b$ means that the consumer with belief β and located at $x = b$ prefers good A over B and thus this must be true for all consumers located left of $x = b$ and in fact for any consumer type with belief β. Similarly, if $\bar{x}(\beta) > a$, all consumer types x with belief β prefer good A.

We impose the following assumption on the behavior of indifferent consumers:

Assumption 3.2. *If $a \neq b$, then all indifferent consumer types buy from the firm that is located closer to them. If $a = b \geq 1/2$, then types $x \leq b$ of indifferent consumers buy from firm B and the remaining indifferent consumers buy from firm A.*

In the standard model of spatial competition à la Hotelling and in our model, situations where sets of consumer types are indifferent may emerge. Assumption 3.2 deals with those cases. In the standard Hotelling model such a situation can only arise if firms locate at the same place. In contrast, in our model multiple consumer types may be indifferent if the distance between firms is sufficiently small. The behavior described in Assumption 3.2 is obtained as the limiting case of situations in which the distance between the firms' locations is marginally greater. The second part of Assumption 3.2 seems plausible in that light: as firm B is the second mover, it can always choose to locate infinitesimally close to firm A on each side, so that B intuitively chooses which side to position itself on, even if both firms are located at the same spot. If indifferent consumers behave otherwise than assumed, best responses may not be defined and a pure-strategy equilibrium may cease to exist, so that only this behavior is compatible with equilibrium. This argument is also put forward by Simon and Zame (1990) for general discontinuous games involving "sharing rules" such as Assumption 3.2. Thus, we only postulate it here to avoid complications due to off-equilibrium-path behavior.[19]

With Assumption 3.2 and with $b \leq a$, the above observations imply that there must be one highest consumer type for a given belief that purchases from firm B. We denote by $\tilde{x}(\beta)$ the threshold, such that all consumers with location $x \leq \tilde{x}(\beta)$ and belief β choose the product of firm B. Threshold $\tilde{x}(\beta)$ equals the indifferent consumer type in (b, a) with belief β whenever it exists. Otherwise, $\tilde{x}(\beta) = 0$, if all consumers with belief β prefer A, and in the analogous case, when all consumers with β prefer to buy from B, $\tilde{x}(\beta) = 1$. Thus, the threshold type can be calculated as

$$\tilde{x}(\beta) = \begin{cases} 0 & \text{if} \quad \beta < \frac{1}{2} - \frac{\tau}{\delta} \cdot \frac{a-b}{2} & \Leftrightarrow \bar{x}(\beta) < b, \\ \bar{x}(\beta) & \text{if} \quad \beta \in \left[\frac{1}{2} - \frac{\tau}{\delta} \cdot \frac{a-b}{2}, \frac{1}{2} + \frac{\tau}{\delta} \cdot \frac{a-b}{2}\right] & \Leftrightarrow \bar{x}(\beta) \in [b, a], \\ 1 & \text{if} \quad \beta > \frac{1}{2} + \frac{\tau}{\delta} \cdot \frac{a-b}{2} & \Leftrightarrow \bar{x}(\beta) > a. \end{cases} \tag{3.4}$$

As consumers are uniformly distributed over $[0, 1]$, $\tilde{x}(\beta)$ is constructed such that it equals the probability of a consumer with belief β buying product B .

[19] In the light of our results, the recommendation effect, i.e., firm B's incentive to differentiate, is independent of Assumption 3.2, which merely determines the equilibrium behavior of firm A.

The consumer's optimal strategy with location x and belief β is always characterized by

$$C(x,\beta) = \begin{cases} B & \text{if} \quad x \leq \tilde{x}(\beta), \\ A & \text{if} \quad x > \tilde{x}(\beta). \end{cases}$$

Thus, to obtain the optimal consumer behavior we just need to find the relevant threshold types $\tilde{x}(\beta)$.

3.6 Equilibrium Analysis Without Consumer Learning (Benchmark)

In our benchmark model, consumers are unable to infer information from the other consumer's action. This is essentially the same as having two independent consumers purchasing in period one. In all other aspects, the model remains unchanged. Our benchmark model differs from Hotelling's original model in that prices are fixed, firms move sequentially, products are of different quality, and some consumers possess information about the quality differential. Proposition 3.1 shows that this leads to the "symmetric minimum differentiation" result also obtained by Hotelling. The result is easily obtained by the following steps.

3.6.1 Consumer Behavior

Let us start with assuming $b \leq a$. With the assumptions from above, an uninformed consumer's belief in the first period must equal the prior, and we denote it by $\beta^u_\emptyset := Pr(v_B = v + \delta) = 1/2$. As there is no updating involved in the benchmark model, according to Equation (3.4) the uninformed indifferent consumer type

$$\tilde{x}(\beta^u_\emptyset) = \frac{a+b}{2},$$

i.e., the midpoint between the firms, characterizes the behavior of uninformed consumers. Note that, given $b \leq a$, whenever $\tilde{x}(\beta^u_\emptyset) > 0.5$ ($\tilde{x}(\beta^u_\emptyset) < 0.5$), firm B (A) is the niche firm, i.e., it serves the smaller ex-ante market share.

Informed consumers possess all relevant information and hence their beliefs are the same in all periods, depending only on which firm's product is of higher quality. Thus, from now on we write $\beta^i_A := Pr(v_B = v + \delta | v_B < v_A) = 0$ to denote the belief of an informed consumer when A is the better firm, and $\beta^i_B := Pr(v_B = v + \delta | v_B > v_A) = 1$

analogously for the case where B sells the superior product. Using Equation (3.4) and defining

$$b_1(a) := a - \frac{\delta}{\tau},$$

which solves the equations

$$b \quad s.t. \quad \bar{x}(\beta^i_A) = b \qquad \text{and} \qquad b \quad s.t. \quad \bar{x}(\beta^i_B) = a, \tag{3.5}$$

threshold types for informed consumers can easily be calculated as

$$\tilde{x}(\beta^i_A) = \begin{cases} \bar{x}(\beta^i_A) := \frac{a+b}{2} - \frac{\delta}{2\tau} & \text{if} \quad b \leq b_1(a) \Leftrightarrow \bar{x}(\beta^i_A) \in [b,a], \\ 0 & \text{if} \quad b > b_1(a) \Leftrightarrow \bar{x}(\beta^i_A) < b, \end{cases} \tag{3.6}$$

if A is the superior product, and

$$\tilde{x}(\beta^i_B) = \begin{cases} \bar{x}(\beta^i_B) := \frac{a+b}{2} + \frac{\delta}{2\tau} & \text{if} \quad b \leq b_1(a) \Leftrightarrow \bar{x}(\beta^i_B) \in [b,a], \\ 1 & \text{if} \quad b > b_1(a) \Leftrightarrow \bar{x}(\beta^i_A) > a, \end{cases} \tag{3.7}$$

in the case that B's product is of higher quality. Intuitively, if $b > b_1(a)$, i.e., firm B locates relatively close to firm A, the additional transport costs when traveling to the better firm matter less than the additional value to informed consumers, and thus all of them buy according to their signal. It can directly be seen that $\bar{x}(\beta^i_A) \in [b,a] \Leftrightarrow \bar{x}(\beta^i_B) \in [b,a]$, and furthermore that $\bar{x}(\beta^i_A) < b \Leftrightarrow \bar{x}(\beta^i_B) > a$. These two cases distinguish whether firms are sufficiently close to each other or not, such that all informed consumers follow their signal.

3.6.2 Firm Behavior and Equilibrium

Combining the thresholds from above, firm B's demand, given $b \leq a$, calculates as

$$\begin{aligned} D_L(a,b) &= 2 \cdot \left[\frac{q}{2}(\tilde{x}(\beta^i_A) + \tilde{x}(\beta^i_B)) + (1-q)\tilde{x}(\beta^u_\emptyset)\right] \\ &= \begin{cases} a+b & \text{if} \quad b \leq b_1(a), \\ q + (1-q)(a+b) & \text{if} \quad b > b_1(a), \end{cases} \end{aligned}$$

where the subscript L denotes that firm B is located left of A. As stated in the model setup, it is without loss of generality to concentrate on situations where $a \geq 0.5$. Clearly, B's demand is increasing in b in both cases, so that with $a \geq 0.5$, B can never profit from choosing $b > a$, implying that B's best response $b^*(a)$ is given by one of the two maximal points of each segment, i.e., $b^*(a) \in \{b_1(a), a\}$, where $b_1(a)$ is a feasible choice whenever $b_1(a) \geq 0$.

Firm A's goal is to minimize B's demand by choosing its optimal point of B's best response function. In order for B to prefer $b = b_1(a)$ to $b = a$, a must be sufficiently high, in particular, as shown in Appendix 3.A.1 it must exceed 0.5. For any $a > 0.5$, B's demand is higher than 1. By choosing $a = 0.5$, A induces B to choose $b = a$, which leads to a demand of 1 for each firm, the highest demand A can generate in this model. The result in the benchmark model is thus as follows:

Proposition 3.1 (Symmetric Minimum Differentiation)**.** *In the unique equilibrium of the model without consumer learning, firms do not differentiate their products and equilibrium locations are* $a = b = 0.5$*.*

Proof. See Appendix 3.A.1. □

3.7 Equilibrium Analysis With Consumer Learning

This section shows that the possibility to learn from previous consumers' actions can drastically change the outcome of the game compared to the benchmark model. With the consumer's ability to observe her predecessor's action, new effects arise in the model and Proposition 3.1 will not generally hold. Instead, our main result, Proposition 3.3, shows that for particular values of the parameters at least one firm moves away from the center, and differentiation can arise in equilibrium. We postpone this result to the end of the section in order to now guide the reader through its construction. We again let the subscript L indicate that firm B positions left of firm A, i.e., $b \leq a$, whereas the subscript R represents the opposite case. We then make use of the fact that all situations $b > a$ can be described using the formulas obtained for $b \leq a$.

3.7.1 Informed Consumers and Uninformed Early Adopters

The decision of the consumer in the first period does not differ from the benchmark model, so that their behavior is fully characterized by $\tilde{x}(\beta^u_\emptyset)$ if uninformed and - depending on which firm is superior - by $\tilde{x}(\beta^i_A)$ or $\tilde{x}(\beta^i_B)$ if informed. As mentioned above, because informed consumers already have perfect information about both goods, an informed consumer in $t = 2$ behaves as one in period $t = 1$. In what follows, we discuss peculiarities only occurring in the model with consumer learning.

3.7.2 Uninformed Laggards: Updating and the Recommendation Effect

An uninformed laggard uses her information to update her belief $\beta^u(a, b, C_1) : \mathcal{A} \times \mathcal{B} \times \{A, B\} \to [0, 1]$. Let $\beta^u_{C_1} := \beta^u(a, b, C_1)$ denote the belief of an uninformed laggard given that $C_1 \in \{A, B\}$ was chosen in the first period. Although we assume that the beliefs do not depend on the firms' locations directly, a and b have an indirect effect via the interpretation of the predecessor's action, C_1. Observing C_1 becomes useful for uninformed consumers, because of the possibility that the previous consumer was informed. Hence, history C_1 can now contain information that allows an uninformed laggard to update her estimate of which firm produces the good of higher value. Using Bayes' rule she will calculate her belief $\beta^u_{C_1}$ of the probability that firm B is the higher quality firm as

$$\beta^u_{C_1} = Pr(v_B > v|C_1) = \frac{Pr(C_1|v_B > v)}{Pr(C_1)} \cdot Pr(v_B > v).$$

In the first period, the products of both firms have the same expected utility (gross transportation costs) for uninformed consumers. This however is not the case in the second period, as the updated probability $\beta^u_{C_1}$ must be used to calculate expected utilities when comparing the utility of buying good A to the expected utility from purchasing firm B's product. The updating introduces an asymmetry in the expected valuations of the products, implying that in contrast to period $t = 1$, it is possible that no type of uninformed consumer is indifferent between the products.

It is thus necessary to distinguish three cases for any given belief β^u_{C1}. Either there is an unique indifferent consumer type, meaning it is in the interval $[b, a]$,[20] or the consumer located at a prefers B or the consumer type at b prefers A. In the latter two cases the same holds for all types right of a, respectively left of b; the intuition behind this was described in Section 3.5. Using Equation (3.4) we have

$$\tilde{x}(\beta^u_{C_1}) = \begin{cases} \bar{x}(\beta^u_{C_1}) := \tilde{x}(\beta^u_{\emptyset}) + \frac{\delta}{\tau}\left(\beta^u_{C_1} - \beta^u_{\emptyset}\right) & \text{if} \quad \bar{x}(\beta^u_{C_1}) \in [a, b], \\ 0 & \text{if} \quad \bar{x}(\beta^u_{C_1}) < b, \\ 1 & \text{if} \quad \bar{x}(\beta^u_{C_1}) > a. \end{cases} \tag{3.8}$$

The uninformed indifferent consumer for the case that it is in $[b, a]$, i.e., $\bar{x}(\beta^u_{C_1}) \in [a, b]$, can nicely be interpreted, in that it is the first period's uninformed indifferent type, shifted to the left (right) by a term that weighs the product of the additional likeliness

[20] For the case of sets of indifferent consumers their behavior is determined by Assumption 3.2.

that B is the superior firm, if the choice in the first period was firm B (firm A), and the excess utility from choosing the better product against the additional transport costs.

In the literature on social learning (see, e.g., Smith and Sørensen (2000)), "herding" is defined as a behavior, where an agent's action is independent of her private signal: all information she uses comes from the (possibly updated) public belief derived from the behavior of others. The situation where an uninformed laggard always follows the early adopter can be viewed from a similar perspective: an agent chooses to buy from one firm using information which only comes from the observed behavior of other consumers. Thus, in our model herding does not mean that imitation dominates private information, but rather that imitation dominates own (ex-ante) tastes. We could extend our model to the case where signals are not completely informative and "herding" consumers additionally ignore the information revealed by their own private signal.

The results that follow crucially depend on the behavior of an uninformed laggard, which in turn is dictated by her belief. Bayes' rule is used to calculate the updated probability that B is the superior firm given $C_1 = B$ as

$$\begin{aligned}\beta^u_B &= Pr(v_B > v|C_1 = B) = \frac{Pr(C_1 = B|v_B > v)}{Pr(C_1 = B)} \cdot Pr(v_B > v) \\ &= \frac{q\tilde{x}(\beta^i_B) + (1-q)\tilde{x}(\beta^u_\emptyset)}{q(\tilde{x}(\beta^i_A) + \tilde{x}(\beta^i_B)) + (1-q)2\tilde{x}(\beta^u_\emptyset)}.\end{aligned} \tag{3.9}$$

The updated probability that $v_B > v_A$ after observing $C_1 = A$ is calculated similarly, and given by

$$\begin{aligned}\beta^u_A &= Pr(v_B > v|C_1 = A) = \frac{Pr(C_1 = A|v_B > v)}{Pr(C_1 = A)} \cdot Pr(v_B > v) \\ &= \frac{q[1-\tilde{x}(\beta^i_B)] + (1-q)[1-\tilde{x}(\beta^u_\emptyset)]}{q\{[1-\tilde{x}(\beta^i_A)] + [1-\tilde{x}(\beta^i_B)]\} + (1-q)\cdot 2\cdot [1-\tilde{x}(\beta^u_\emptyset)]}.\end{aligned} \tag{3.10}$$

It was shown before, that $\tilde{x}(\beta^i_A) < \tilde{x}(\beta^i_B)$, so that we can see from Equations (3.9) and (3.10) that $\beta^u_B > 0.5 > \beta^u_A$, meaning that observing $C_1 = B$ ($C_1 = A$) increases (decreases) the probability that B sells the good of higher value, just as one would expect.

Letting the fraction of informed consumers approach zero, that is $q \to 0$, the "updated" probabilities approach the prior: $\beta^u_A, \beta^u_B \to \frac{1}{2}$. Overall, we can order all relevant beliefs according to $0 = \beta^i_A < \beta^u_A < \beta^u_\emptyset = 0.5 < \beta^u_B < \beta^i_B = 1$.

An interesting observation that can be made with regard to the updated probabilities, is that product differentiation has two effects for a firm. Suppose that both firms are symmetrically positioned around 0.5, i.e. $a + b = 1$. In this case a purchase of each good is equally informative to uninformed laggards as $\beta^u_B = 1 - \beta^u_A$. Now consider firm

Bs' incentives to increase the differentiation to A's product. If $b \leq a$, this means that B considers decreasing b. Increasing the product differentiation, which means that B is now producing a "niche product", makes it less likely that product B is chosen by uninformed consumers in the first period, thus $Pr(C_1 = B)$ and $Pr(C_1 = B|v_B > v)$ and therefore the nominator and the denominator of Equation (3.9) get smaller. Since b affects all threshold types $\tilde{x}(\cdot)$ in Equation (3.9) in the same way, the effect on the denominator is twice as large as the one on the nominator and the updated probability that B's product is superior given it was chosen in the first period, β_B^u, increases, i.e., $\partial\beta_B^u/\partial b < 0$. This mechanism lays the foundation for the "recommendation effect". Intuitively, since a niche product is a good match to relatively few consumer types (compared to a mainstream product), if it was chosen in $t = 1$, it is more likely that this was due to superior information about the quality than due to a better match of the product's characteristic and the consumer's taste.

An opposing effect is created regarding the updated probability, β_A^u. With A being the mainstream product (compared to B), an observed choice of it in the first period was more likely induced by a good match (a consumer located close to a) than by an informed consumer.

Put differently, the increased confidence that a product is of superior quality after having observed that it was bought in the first period, is higher for a niche product than for a mainstream product. Those are precisely the effects for which Tucker and Zhang (2011) find empirical evidence by examining the usefulness of popularity information for what they call products of "narrow" and "broad appeal".

The next subsections will show that, because of the recommendation effect, the (expected) demand of firm B will not be monotonically increasing the smaller the distance to firm A gets. Clearly, in the benchmark model it is monotonically increasing in that distance. It will be shown that the introduction of the recommendation effect can dramatically change the equilibria of the game, not only for the above model, but also for a hypothetical model where the two firms choose their locations simultaneously (but the rest of the model is unchanged).

3.7.3 Firms' Expected Demand

Having calculated the threshold types of the consumers, B's demand for $b \leq a$ is given by

$$\begin{aligned} D_L(a,b) = q\Big[\tilde{x}(\beta_A^i) + \tilde{x}(\beta_B^i)\Big] \\ + (1-q)\Big[\tilde{x}(\beta_\emptyset^u) + Pr(C_1 = A)\tilde{x}(\beta_A^u) + Pr(C_1 = B)\tilde{x}(\beta_B^u)\Big]. \end{aligned} \tag{3.11}$$

It was shown in the previous sections that the threshold for a given belief β, $\tilde{x}(\beta)$, can either be at an interior value, meaning in the interval $[b, a]$, or it equals 0 or 1. As $\tilde{x}(\beta)$ is shifted away from $\tilde{x}(\beta^u_\emptyset) = (a+b)/2$, whether the threshold for some belief β is at an interior level, or not, depends on the distance between the two firms. Only for a sufficiently large distance, $\tilde{x}(\beta)$ can be at an interior level. Equations (3.4) and (3.8) show that the necessary distance between firms' locations implying $\tilde{x}(\beta)$ to be interior is increasing with the belief, and, by Equation (3.3), also $\partial\bar{x}(\beta)/\partial\beta > 0$. Hence, the threshold type for informed consumers is always shifted further away from $\tilde{x}(\beta^u_\emptyset)$ than the one of the uninformed laggards, meaning that $\bar{x}(\beta^u_A) \geq \bar{x}(\beta^i_A)$ and $\bar{x}(\beta^u_B) \leq \bar{x}(\beta^i_B)$. Thus, $\bar{x}(\beta^u_B) < a$ or $\bar{x}(\beta^u_A) < b$ directly imply $\bar{x}(\beta^i_A) < b, \bar{x}(\beta^i_B) > a$. It was already argued that $\bar{x}(\beta^i_A) \notin [b, a] \Leftrightarrow \bar{x}(\beta^i_B) \notin [b, a]$ and because a first period purchase from one firm increases the belief that it offers the superior product, a situation with $\bar{x}(\beta^u_B) = b$ or $\bar{x}(\beta^u_A) = a$ can not occur.

This leaves us with five qualitatively different combinations of threshold types induced by different tuples (a, b). More precisely, in Proposition 3.4 in Appendix 3.A.2 we show that for each configuration of the values of the parameters δ, τ and q there is a unique partition $\mathfrak{D}_L$ of $\{(a, b) \in \mathcal{A} \times \mathcal{B} | b \leq a\}$ given by

1) $\bar{x}(\beta^i_A), \bar{x}(\beta^i_B) \in [b, a], \quad \bar{x}(\beta^u_B),\ \bar{x}(\beta^u_A) \in [b, a],$

2) $\bar{x}(\beta^i_A), \bar{x}(\beta^i_B) \notin [b, a], \quad \bar{x}(\beta^u_B),\ \bar{x}(\beta^u_A) \in [b, a],$

3A) $\bar{x}(\beta^i_A), \bar{x}(\beta^i_B) \notin [b, a], \quad \bar{x}(\beta^u_B) \in [b, a],\ \bar{x}(\beta^u_A) \notin [b, a],$

3B) $\bar{x}(\beta^i_A), \bar{x}(\beta^i_B) \notin [b, a], \quad \bar{x}(\beta^u_B) \notin [b, a],\ \bar{x}(\beta^u_A) \in [b, a],$

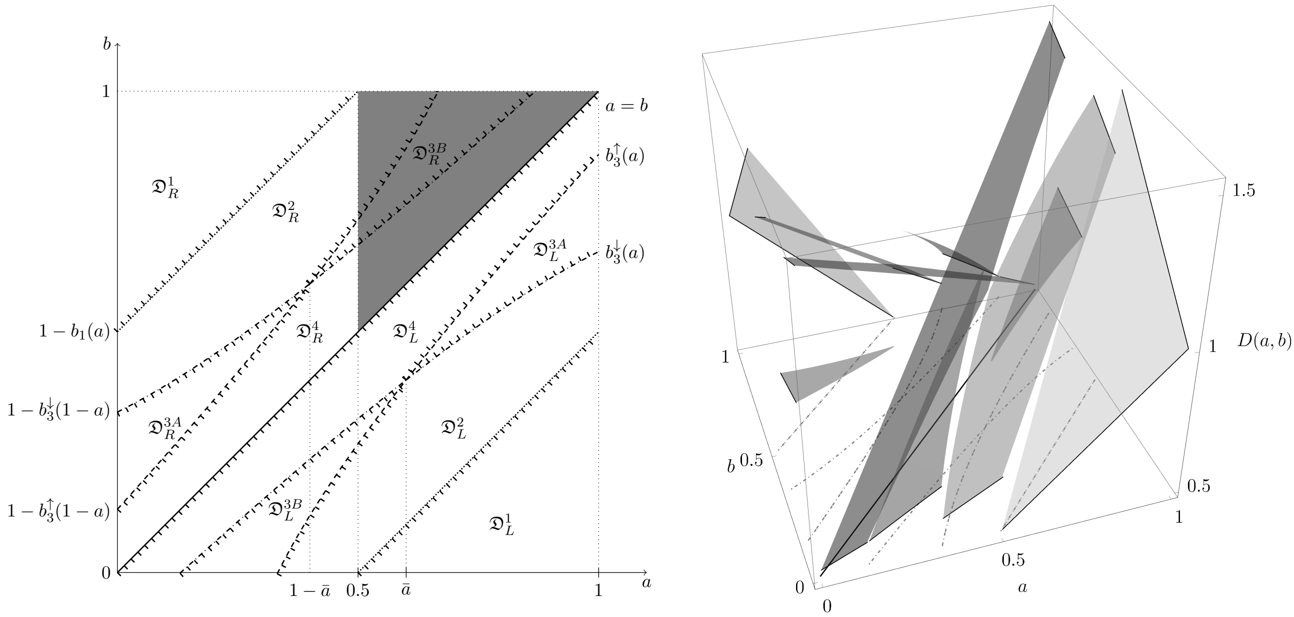
4) $\bar{x}(\beta^i_A), \bar{x}(\beta^i_B) \notin [b, a], \quad \bar{x}(\beta^u_B),\ \bar{x}(\beta^u_A) \notin [b, a].$

We use the notation $\mathfrak{D}_L$ to indicate that B is located left of A and $\mathfrak{D}^{3F}$ to imply that firm $F \in \{A, B\}$ is the niche firm in this region.

For $b > a$ there also exist five qualitatively different and mutually exclusive situations described by the threshold types $\bar{x}(\cdot)' := 1 - \bar{x}(\cdot)$, which induce a partition $\mathfrak{D}_R$ of $\{(a, b) \in \mathcal{A} \times \mathcal{B} | b > a\}$, where the R symbolizes B being located right of A. The partition is given by $\mathfrak{D}^j_R$ obtained from the definition of $\mathfrak{D}_L$, by replacing all a, b and $\bar{x}(\cdot)$ by $(a', b') := (1-a, 1-b)$ and $\bar{x}(\cdot)'$. Thus, $\mathfrak{D} := \mathfrak{D}_L \cup \mathfrak{D}_R$ describes a partition of the whole action space $\mathcal{A} \times \mathcal{B}$. Further below we will describe the boundaries of the respective elements of the partition in more detail.

Each element of the partition leads to a specific form of B's demand, and we will denote B's demand in any given part by $D(a, b) = D^j_S(a, b)$ iff $(a, b) \in \mathfrak{D}^j_S \in \mathfrak{D}$ for $j \in \{1, 2, 3A, 3B, 4\}$ and $S \in \{L, R\}$.

If firm B chooses a position such that $b > a$ and additionally $a \geq 0.5$, we say that it positions on the short side of the market. Figure 3.3 depicts the partition of the action space and visualizes many results we obtain in the following.



(a) Generic partition of the action space. Note that the graphs for $a < 0.5$ are point reflection at $(0.5, 0.5)$ of the graphs for $a \geq 0.5$. The gray area depicts the short side. The little ticks indicate which region the respective boundaries belong to.

(b) Firm B's demand as an area over the action space.

Figure 3.3: Visualization of the partition of the action space and the implied form of firm B's expected demand. (Parameters: $q = 0.4, \tau = 2, \delta = 1$)

For a fixed a, we can calculate the boundaries between the different regions by choosing b such that the threshold consumer types of the respective parts of the partition are at the location of one of the firms. In addition to $b_1(a)$ as defined in the benchmark model, there are two further points of discontinuity of B's demand, namely the two points where, for each history C_1 and for a given a, the unique indifferent uninformed laggard ceases to exist. Whereas $\tilde{x}(\beta_A^i) = 0$ implies $\tilde{x}(\beta_B^i) = 1$ and vice versa, this generally is not the case for $\tilde{x}(\beta_A^u)$ and $\tilde{x}(\beta_B^u)$. Here, one threshold type may still be interior while the other already is at a corner value. Whenever firms are not symmetrically positioned around 0.5, the updating is asymmetric, so that the thresholds' distances from $(a+b)/2$ are not the same. Those two discontinuity points are implicitly characterized by the following equations stating that the indifferent type in period 2 after A (B) was chosen in the first period is located at b (a):

$$\begin{aligned} b \quad \text{s.t.} \quad \bar{x}(\beta_A^u) = b \quad &\Leftrightarrow \quad b = \frac{a+b}{2} + \frac{\delta}{2\tau}(2\beta_A^u - 1) \\ &\Leftrightarrow \quad \frac{\delta q}{\tau} = (2-q)(a-b) - (1-q)\left(a^2 - b^2\right), \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} b \quad \text{s.t.} \quad \bar{x}(\beta_B^u) = a \quad &\Leftrightarrow \quad a = \frac{a+b}{2} + \frac{\delta}{2\tau}(2\beta_B^u - 1) \\ &\Leftrightarrow \quad \frac{\delta q}{\tau} = q(a-b) + (1-q)\left(a^2 - b^2\right). \end{aligned} \tag{3.13}$$

Since both equations are quadratic in b, they both have two solutions. As shown in Lemma 3.2 in Appendix 3.A.2, at most one of those solutions, for each equation, lies in the permissible range of $[0, 1]$. Those permissible solutions will in the following be referred to as $b_3^{\downarrow}(a)$ for Equation (3.12) and $b_3^{\uparrow}(a)$ for Equation (3.13). They are the discontinuity points in B's demand and mark the boundaries between demand parts D_L^2, D_L^{3B}, and D_L^4, or between D_L^2, D_L^{3A}, and D_L^4.

To illustrate the underlying mechanism of the discontinuity in the demand of firm B, suppose that b is chosen such that $\bar{x}(\beta_B^u) = a$, meaning that firm B positions itself at the location $b_3^{\uparrow}(a)$, and all uninformed laggards to the right of a are indifferent between both firms. If B instead chose a location slightly smaller than $b_3^{\uparrow}(a)$, this would increase the transport costs of all consumers located at or to the right of a, meaning that those uninformed laggards would then prefer A's product. On the other hand, a location b slightly larger than $b_3^{\uparrow}(a)$ would induce all those uninformed laggards to buy the product from firm B. Taken together this implies that at $b_3^{\uparrow}(a)$, B's demand has an upward jump. Thus, we use the notation "↑".

Similar reasoning leads to the observation that B's demand jumps downward at $b_3^{\downarrow}(a)$, as at this point the whole mass of consumers to the left of b switches from preferring B's good to preferring the one of firm A if $C_1 = A$. Thus, overall we have

$$\tilde{x}(\beta_A^u) = \begin{cases} \bar{x}(\beta_A^u) := \frac{a+b}{2} + \frac{\delta}{2\tau}(2\beta_A^u - 1) & \text{if} \quad b \leq b_3^{\downarrow}(a) \Leftrightarrow \bar{x}(\beta_A^u) \in [a, b], \\ 0 & \text{if} \quad b > b_3^{\downarrow}(a) \Leftrightarrow \bar{x}(\beta_A^u) < b. \end{cases} \tag{3.14}$$

if A is the superior product, and

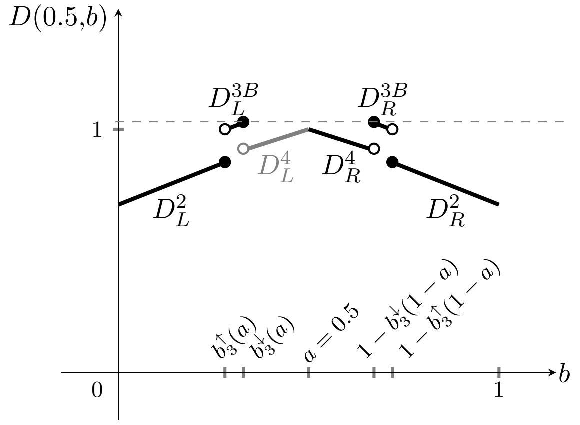
$$\tilde{x}(\beta_B^u) = \begin{cases} \bar{x}(\beta_B^u) := \frac{a+b}{2} + \frac{\delta}{2\tau}(2\beta_B^u - 1) & \text{if} \quad b \leq b_3^{\uparrow}(a) \Leftrightarrow \bar{x}(\beta_B^u) \in [a, b], \\ 1 & \text{if} \quad b > b_3^{\uparrow}(a) \Leftrightarrow \bar{x}(\beta_B^u) > a. \end{cases} \tag{3.15}$$

The calculations in Appendix 3.A.2 show that for small a we have $b_3^{\uparrow}(a) < b_3^{\downarrow}(a)$ and $b_3^{\uparrow}(a) > b_3^{\downarrow}(a)$ for larger a. By $a = \bar{a}$ we denote the location of firm A, so that $b_3^{\downarrow}(a) = b_3^{\uparrow}(a)$. In the case of $b_3^{\uparrow}(a) < b_3^{\downarrow}(a)$, B's "middle" demand is characterized by D_L^{3B} and "starts" with an "upward jump". In the case of $b_3^{\uparrow}(a) > b_3^{\downarrow}(a)$, B's "middle" demand is characterized by D_L^{3A} and "starts" with a "downward jump". To understand these points intuitively, recall, that in order for $\tilde{x}(\beta_B^u)$ to be at an interior value, the distance between a and b must be large enough and that updating is more favorable for a niche firm, meaning that $\tilde{x}(\beta_B^u)$ is shifted further from $(a+b)/2$ than $\tilde{x}(\beta_A^u)$ whenever firm B is the niche firm. Depending on a, it can be the case that the distance between a and b at which $\tilde{x}(\beta_B^u)$ stops being at an interior level is obtained with B being the niche firm, i.e., $b_3^{\uparrow}(a) < 1 - a$, or not. Whenever $\tilde{x}(\beta_B^u) = 0$ and B is the niche firm, we are in region $\mathfrak{D}_L^{3B}$ with resulting demand D_L^{3B}. This will happen if a is sufficiently small. If a is large enough so that $\tilde{x}(\beta_B^u)$ stays at interior levels as long as A is the niche firm, $\tilde{x}(\beta_A^u)$ changes to its corner value "before" $\tilde{x}(\beta_B^u)$ does, and the applicable region is $\mathfrak{D}_L^{3A}$ with the corresponding demand. The transition between those two situations is obtained for $a = \bar{a}$, which plays a crucial role in the following equilibrium characterization. At this point, $b_3^{\downarrow}(a) = b_3^{\uparrow}(a)$, meaning that (only with this location a) $\tilde{x}(\beta_A^u) = 0$ implies $\tilde{x}(\beta_B^u) = 1$ and vice versa, hence, no firm is a niche firm when either $\tilde{x}(\beta_F^u), F \in \{A, B\}$ stops being at an interior level. If $a = \bar{a}$, then $(a, b) \notin \mathfrak{D}_L^{3B}$ and $(a, b) \notin \mathfrak{D}_L^{3A}$ for all $b \leq a$, and so B's demand does not contain demand part 3, for smaller (larger) a, $(a, b) \in \mathfrak{D}_L^{3B}$ ($(a, b) \in \mathfrak{D}_L^{3A}$) for any $b \leq a$.

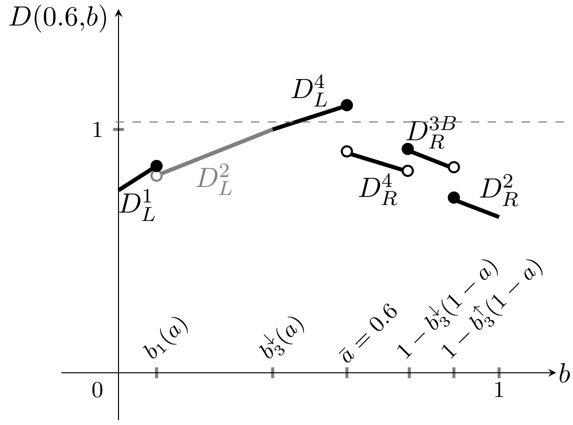
The behavior of consumer types $\tilde{x}(\cdot)$ determines which part of the demand functions the discontinuity points belong to (see Lemma 3.6 in Appendix 3.A.2 for a complete description of the boundary points in the unique partition of the action space). By construction these types are indifferent and their behavior is pinned down in Assumption 3.2 to guarantee the existence of equilibrium.

From Equation (3.11) it is easy to see that the demand for $b \leq a$ is increasing in a and b as long as no threshold type changes from being interior to 0 or 1, which implies that the demand in each part $D_L^j(a, b)$ is increasing in a and b (see also Lemma 3.5 in Appendix 3.A.2). By symmetry, demand in each part $D_R^j(a, b)$ is then decreasing in a and b.

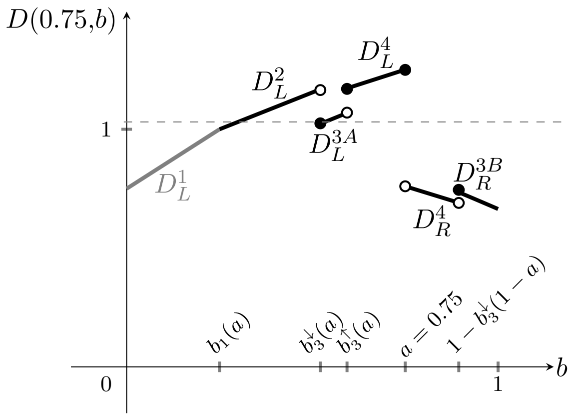
With these observations and the above description of the demand parts, we can depict B's demand for a fixed a and varying choices of b, as is done in Figure 3.4, which shows the three generic situations that may occur in our model. Each of the different panels in this figure can be viewed as "slice" cut out of the demand depicted in the right panel of Figure 3.3 for a fixed a.



(a) B's expected demand given $a = 0.5$ (exploiting the recommendation effect is advantageous for firm B). Note: $b_1(0.5) = 0$, i.e., D^1 is not available for B.



(b) B's expected demand given $a = \bar{a} = 0.6$.



(c) B's expected demand given $a = 0.75$ (exploiting the recommendation effect is not advantageous for firm B).

Figure 3.4: B's expected demand as a function of the chosen location b for three different locations of A. In Panel (a), $b_3^{\downarrow} > b_3^{\uparrow}$, so that D^3 of the demand is an upward step. Reversed situation in Panel (c) and no jump in Panel (b). The parameters yield an equilibrium with asymmetric central differentiation (see Proposition 3.3), which is visualized also by the dashed line. (Parameters: $q = 0.4, \tau = 2, \delta = 1$)

3.7.4 Firms' Best Responses and Equilibrium

In the description of the different parts of the demand it was argued that B's demand is increasing in b in each part. Thus, for a given $a \geq 0.5$, the demand of firm B is maximized by setting b equal to one of the points $b = b_1(a)$, $b = b_3^{\downarrow}(a)$, $b = a$, or by $b = 1 - b_3^{\downarrow}(1-a) > a$. Point $b = 1 - b_3^{\downarrow}(1-a) > a$ is the only point right of a that can ever be optimal, since for the highest points in the other demand parts with $b > a$, there is a corresponding point left of a yielding a higher demand given that $a \geq 0.5$. For ease of notation we define the value functions $V_S^j(a)$ with $j \in \{1,2,3,4\}, S \in \{L,R\}$ to equal B's demand if it locates optimally in part j of its demand left ($S = L$) or right ($S = R$) of a. Let the function $V_S(a)$ denote the maximum of all value functions $V_S^j(a)$ and let $V(a)$ be the overall maximum, that is the maximum of $V_R^3(a)$ and $V_L(a)$.[21] B's best response to any a can then be written as

$$b^*(a) \in \underset{b \in \{b_1(a), b_3^{\downarrow}(a), a, 1-b_3^{\downarrow}(1-a)\}}{\arg\max} V(a)$$

We are interested in equilibria where the outcome is not symmetric minimal differentiation, in particular when firm B prefers to differentiate from the center, which follows whenever

$$V_L^3(0.5) > V_L^4(0.5). \tag{BDC}$$

Given that demand is shared equally when both firms locate at the center (i.e. $V_L^4(0.5) = 1$), Condition (BDC) implies that, whenever A locates at the center, B's demand is higher if $b = b_3^{\downarrow}(0.5)$ than if $b = a = 0.5$. This directly implies, that whenever Condition (BDC) holds, both firms locating at the center can not be the outcome of any equilibrium in this game.

In a simultaneous model, both firms' reaction function would be given by the one of firm B in our model. For any division of market shares when both locate at the center, at least one firm has a total demand that is not greater than one, so that at least this firm has an incentive to deviate to $b_3^{\downarrow}(0.5)$. Thus, the result of "symmetric minimum differentiation" would also not be obtained in a model where the two firms choose their locations simultaneously. In such a model no equilibrium in pure strategies exists, which is why we concentrate on the model with sequential location choice of the firms.[22]

The following proposition summarizes those first findings, the full equilibrium characterization for the sequential location choice model follows in the next proposition.

Proposition 3.2. *For sufficiently small values of $\frac{\delta q}{\tau}$, Condition* (BDC) *holds. In this case, the strategies from the benchmark model do not constitute an equilibrium, irrespectively of whether the firms choose their locations simultaneously or sequentially.*

[21] As indicated above, we show in Lemma 3.7 in Appendix 3.A.3, that B's demand on the short side obtains its maximal value via $V_R^3(a)$. The formal definition of $V(\cdot)$ can be found in Corollary 3.3 in Appendix 3.A.3.

[22] As symmetric minimum differentiation is no equilibrium in a simultaneous model, in a hypothetical pure strategy equilibrium at least one firm must be located at a location different from 0.5. Firms would either share the market equally or one firm would serve more than half of the demand. In any case, at least one firm would have an incentive to relocate.

With simultaneous location choice, no equilibrium in pure strategies exists.

Firm B's demand for all $b \leq a$ consists of the same parts for any $a < \bar{a}$ or for any $a > \bar{a}$, and since every single $V_L^j(a)$ is increasing in a, the same must be true for $V_L(a)$ for any point but $\bar{a}$. Clearly, $V_R^3(a)$ is decreasing in a. Firm A's problem is given by $\min_a V(a)$, which is either obtained by equalizing $V_L(a)$ and $V_R^3(a)$ or by setting $a = \bar{a}$, where $V_L(a)$ potentially has a downward jump. Setting $a = 0.5$ equalizes $V_L(a)$ and $V_R^3(a)$, but there might also be some $a > \bar{a}$ equalizing the two value functions, so that

$$a^* \in \underset{a \in \{\bar{a}, a'\}}{\arg\min}\, V(a),$$

where $a' = \max_a \text{s.t. } V_L(a) = V_R^3(a)$.

Firm A thus decides between inducing two qualitatively different situations. By setting $a = a'$, it induces B to differentiate and exploit the recommendation effect, either left of a, then $a = a' = 0.5$, or right of a, then $a = a' > \bar{a}$. For some particular values of the parameters, A can keep B from differentiating by granting a relatively high (ex-ante) market share to firm B, which might be obtained by $a = \bar{a}$.

Which of the situations is preferred by firm A depends on whether

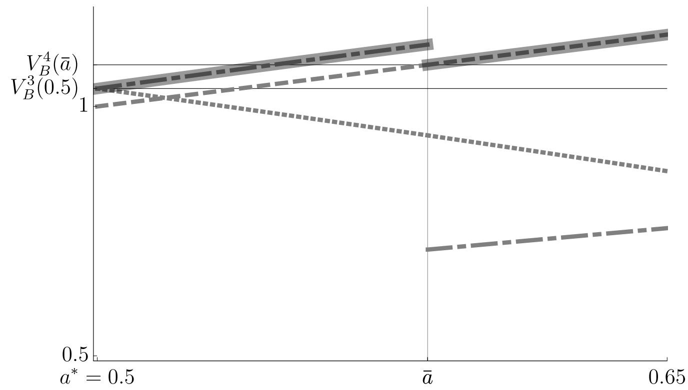
$$V_L^3(0.5) \leq V_L^4(\bar{a}), \tag{ADC}$$

which states that firm <u>A</u> prefers to induce <u>d</u>ifferentiation from the <u>c</u>enter. If firm A does not prefer to induce differentiation from the center, i.e., for its optimal decision a^* it holds that $a^* > 0.5$, firm <u>B</u> might still want to <u>d</u>ifferentiate on the <u>s</u>hort side:

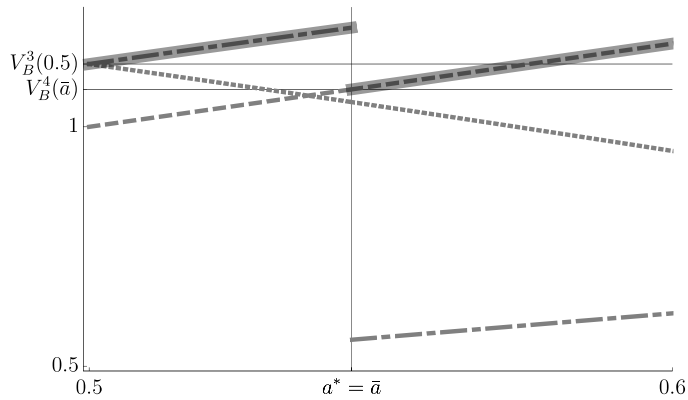
$$V_L^4(\bar{a}) \leq V_R^3(\bar{a}), \tag{BDS}$$

states that firm B prefers to locate at $1 - b_3^{\downarrow}(\bar{a})$ instead of $b = a = \bar{a}$.

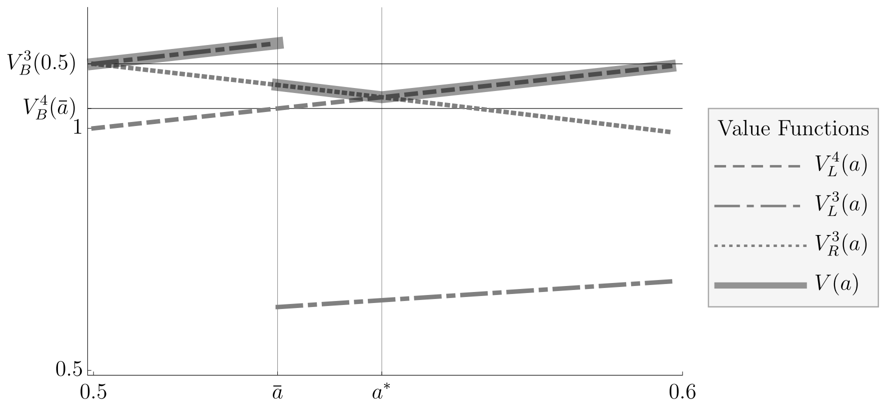
The three different situations are depicted in Figure 3.5. The complete parameter space is characterized in Figure 3.6 further below.



(a) $\delta/\tau = 0.5, q = 0.4$



(b) $\delta/\tau = 1, q = 0.1$



(c) $\delta/\tau = q = 0.25$

Figure 3.5: Combinations of the value functions induced by particular values of the parameters. In Panels (a) and (b), B never profits from differentiating on the short side of the market. In Panel (a), $V_L^4(\bar{a}) > V_L^3(0.5)$, so that $a^* = 0.5$. Reversed situation in Panel (b), and $a^* = \bar{a}$. In Panel (c), $V_R^3(\bar{a}) > V_L^4(\bar{a})$, and $a^* = \max a$ s.t $V_R^3(a) = V_L^4(a)$.

Conditions (BDC), (ADC) and (BDS) distinguish the equilibria and are used to derive the respective parameter restrictions stated in Proposition 3.3. Note that (BDS) implies that (ADC) is violated, which in turn implies (BDC).

Proposition 3.3. *In the model with consumer learning, we obtain the following results.*

1. *Conditions* (BDC) *and* (ADC) *are necessary and sufficient conditions so that the locations are $a^* = 0.5$ and $b^* < 0.5$ in the unique equilibrium (Central Differentiation Equilibrium).*

2. *If Conditions* (ADC) *and* (BDS) *do not hold the locations are $a^* = b^* > 0.5$ in the unique equilibrium (Asymmetric Minimum Differentiation Equilibrium).*

3. *Condition* (BDS) *is a necessary and sufficient condition so that there exist an equilibrium (not necessarily unique) with locations* $b^* > a^* > \bar{a} > 0.5$ *(Short Side Differentiation Equilibrium).*

Uniqueness is up to symmetry, as to any equilibrium with (a^*, b^*) *there exists an analogous equilibrium with* $(1 - a^*, 1 - b^*)$.

Proof. See Appendix 3.A.3. □

The proposition shows that in two of the described equilibria the strategy "differentiate-and-conquer" prevails. Sufficient conditions for the equilibria can be calculated as follows (see Corollary 3.4 of Appendix 3.A.3 for details):

$$\frac{\delta q}{\tau} < 0.192 \Rightarrow V_L^3(0.5) > V_L^4(0.5) \tag{BDC}$$

$$\frac{\delta q}{\tau} > 0.166, q < 0.4 \Rightarrow V_L^3(0.5) \leq V_L^4(\bar{a}) \tag{ADC}$$

If both those sufficient conditions are fulfilled, equilibrium locations are $b^* = b_3^{\downarrow}(0.5) < a^* = 0.5$. If

$$\frac{\delta q}{t} < 0.079 \Rightarrow V_R^3(\bar{a}) > V_L^4(\bar{a}) \tag{BDS}$$

holds, there is an equilibrium with $b^* = 1 - b_3^{\downarrow}(1 - a) > a > \bar{a}$. We only state the sufficient conditions here, as there exist no closed form solutions to the necessary and sufficient conditions, which are implied by the inequalities.

The following figure depicts the necessary and sufficient conditions on the values of the parameters inducing the equilibria, as well as the (weaker) sufficient conditions for Conditions (BDC) and (ADC), and additionally (BDS).

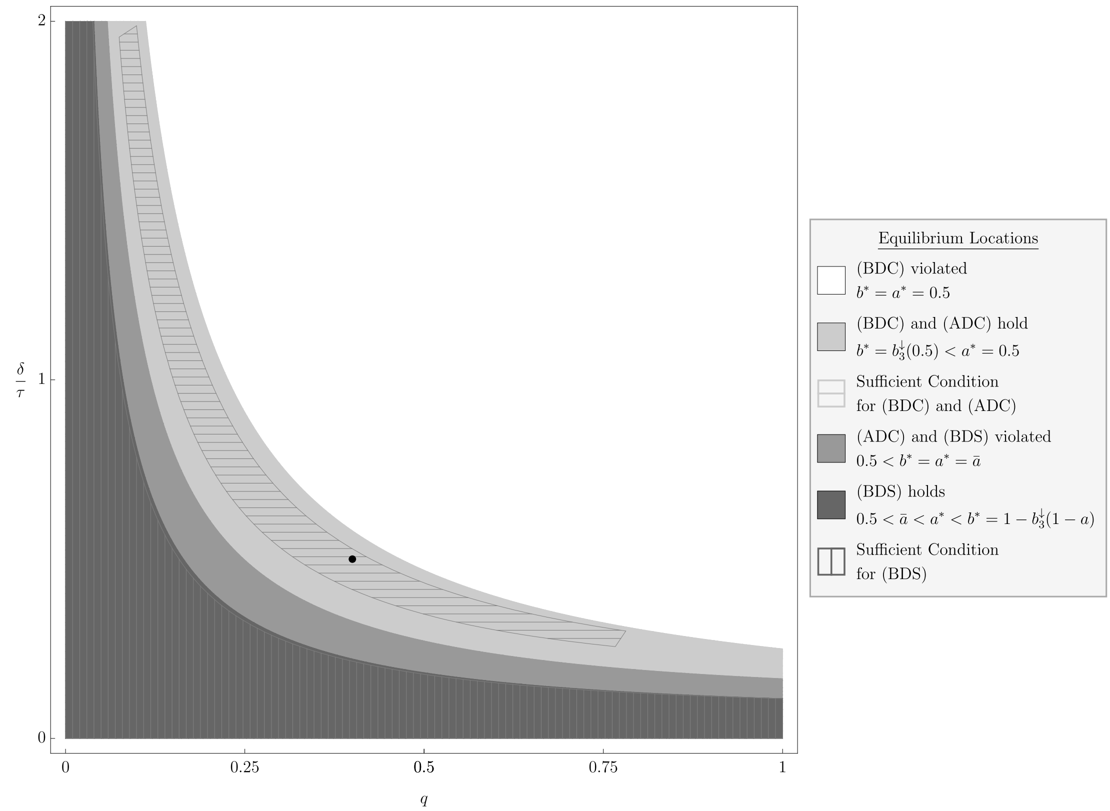


Figure 3.6: The figure depicts the necessary and sufficient parameter restrictions of the equilibria stated in Proposition 3.3. The black dot shows that the parameter combinations used in Figure 3.4 fulfills the conditions stated in Conditions (BDC) and (ADC).

Figure 3.6 shows that $\frac{\delta q}{\tau}$ has to be sufficiently small for either of the "new" equilibria (with differentiation) to exist. The reason is that for all equilibria, firm B must want to deviate from a situation with symmetric minimum differentiation, which is formalized by condition (BDC). For differentiation to matter, $\frac{\delta q}{\tau}$ must not be too large. A large fraction implies that either the relative gain from choosing the higher quality product, $\frac{\delta}{\tau}$, or the likelihood that the first consumer is informed, q, is high. This makes it especially promising for uninformed consumers to follow the previous consumers' behavior even for large distances between the two firms' locations, in turn making differentiation unattractive for the firms.

As $\frac{\delta q}{\tau}$ decreases, $\bar{a}$ approaches 0.5, which has two effects. Firstly, this makes it less costly for firm A to locate at $\bar{a}$, thereby granting a relatively high ex-ante market share to firm B. Secondly, as $\bar{a} \to 0.5$, demand at the sides left and right of $\bar{a}$ are getting more and more alike, implying that it is "more likely" that B also wants to differentiate on the short side of the market, given that it prefers to do so if $a = 0.5$.

In contrast to Hotelling's result of "symmetric minimum differentiation" where both firms choose to locate at the center, the firms' positions are not symmetric in any of the "new" equilibria from above. The doubly sequential nature of the game clearly makes firm A worse off compared to the situation where consumers decided simultaneously (or were unable to observe the others' decisions). This also contrasts Tabuchi and Thisse (1995), who find asymmetric pure strategy equilibria with a first-mover advantage.[23]

3.7.5 Welfare

We use an utilitarian approach to compare the welfare induced by the three new equilibria (central differentiation [$b^* < a^* = 0.5$], short side differentiation [$0.5 < a^* < b^*$] and asymmetric minimum differentiation [$0.5 < a^* = b^*$]) in the model with consumer learning to that of the equilibrium with symmetric minimum differentiation [$a^* = b^* = 0.5$] in the benchmark model.

We start by comparing the equilibrium with central differentiation under consumer learning with the symmetric minimum differentiation result in the benchmark model. First, note that producer surplus is the same in both equilibria. When analyzing consumer surplus it is convenient to distinguish agents according to their different information. An informed consumer will have the same expected gain in both equilibria, which is given by $g^i = v + \delta$. However, in the differentiation equilibrium she has additional expected transport costs due to the fact that she might need to travel to niche firm B instead of the market center with probability 0.5.[24] These additional costs are given by $\Delta c^i = M/2$ with M as visualized in Figure 3.7 below.

[23] Bester et al. (1996) find asymmetric equilibria in mixed strategies in a game with simultaneous location choice (followed by simultaneous price setting).

[24] Note that when comparing expected transport, any location different from the center of the market, i.e., $f \neq 0.5, f \in \{a, b\}$, implies additional expected transport costs compared to those of a firm located at the center, i.e. $f = 0.5, f \in \{a, b\}$: the decrease in expected costs for the consumers located closer to the firm with $f \neq 0.5, f \in \{a, b\}$ does not outweigh the increase in expected costs of the consumers located further away from that firm.

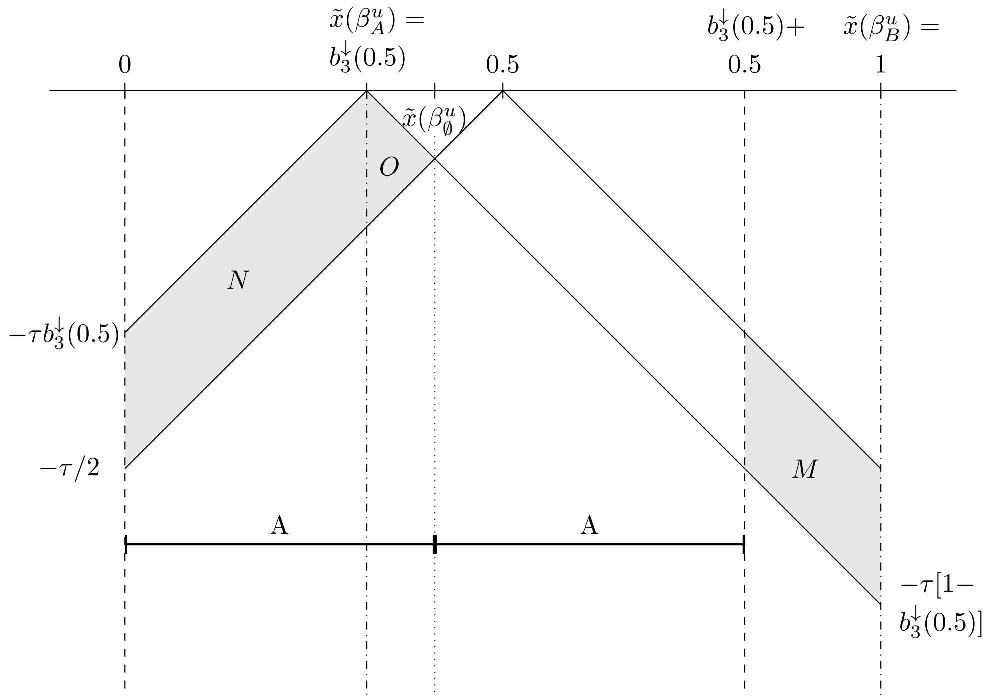


Figure 3.7: The figure depicts transport costs in the equilibrium of the benchmark model ($b^* = a^* = 0.5$) and the central differentiation result of the model with consumer learning ($b^* = b_3^\downarrow(0.5) < a^* = 0.5$). Indifferent types are depicted for the model with learning.

In the equilibrium with central differentiation, firm A is at the same location as in the benchmark model, namely at $a = 0.5$. For (completely) uninformed consumers, both firms have the same expected quality and firm A's quality is deterministic (and thus the same in both models regardless of the available information). This implies that <u>uninformed consumers</u> can obtain the same expected utility in the model with learning as in the model without learning, whenever they choose to purchase good A in the model with learning. A revealed preference argument then implies that all uninformed consumers must be (weakly) better off in the model with learning.

Although the simple argument is enough to show that uninformed consumers (weakly) benefit from the outcome of the model with learning, we can go into further detail, by examining the uninformed consumers in the two periods and the different possible histories separately.

The uninformed early adopter ($t = 1$) unambiguously benefits in the central differentiation equilibrium: while her expected gain remains the same ($g_1^u = v$), the expected transport costs decrease by $\Delta c_1^u = \tilde{x}(\beta_\emptyset^u) \cdot [N + O]$, with N and O as visualized in Figure 3.7, due to the fact that differentiation allows some consumer types to travel to the niche firm which is located closer to them and has the same expected quality as the main stream firm.

Additionally, for an uninformed laggard ($t = 2$), we can distinguish three cases.[25] Either she buys good A (then A must have been bought in period 1), or she buys B which can occur in both occasions, when A or B was bought in the first period.

Whenever the uninformed laggard buys A her utility is exactly equal to the one in the benchmark model. When she buys B after observing a purchase of A she also obtains exactly the same utility as in the benchmark model. This is because of the way the equilibrium is constructed: firm B chose its location precisely to make the uninformed laggard after history $C_1 = A$ indifferent. For the last possible situation, that is, a purchase of B in both periods, the revealed preference argument again implies that all consumers must be weakly better off compared to the benchmark model. But now some consumer types $0 < x < \frac{a+b}{2}$ are strictly better off in terms of expected utility. Take the consumer located at the same spot as firm B, for example. She has less distance to travel and since B was bought in the first period, she expects the product B to be of better quality. Thus, she has less transport costs and a higher expected valuation when compared to her situation in the benchmark model.

Hence, uninformed consumers are unambiguously better off in the equilibrium with central differentiation and benefit from the fact that observing informed consumers provides additional information. Informed consumers, on the other hand, prefer the result of the benchmark model without consumer learning. Which of these opposing effects on the consumers dominates, depends on the share of informed consumers, q, and the excess utility when consuming the superior product, δ/τ. Since the explicit formulation of the answer to the question under which parameter restrictions the welfare is enhanced is too complex, the numerical condition on the parameters is depicted in Figure 3.8 below.

[25] Overall, there are two effects on the expected transport costs of uninformed laggards, which stems from the fact that, given the history of the game, either the threshold type satisfies $\tilde{x}(\beta_B^u) = 1$ (just as the threshold type of informed consumers given $v_B > v$) or the threshold type $\tilde{x}(\beta_A^u)$ is in the interval (b, a) (just as the threshold type of uninformed early adopters). In the first case, expected costs increase, as she might have to travel further to the firm, which she perceives to be superior. In the second case, expected costs decrease. Overall, we have $\Delta c_2^u = P(C_1 = A) \cdot \tilde{x}(\beta_A^u) \cdot N + P(C_1 = B) \cdot M$.

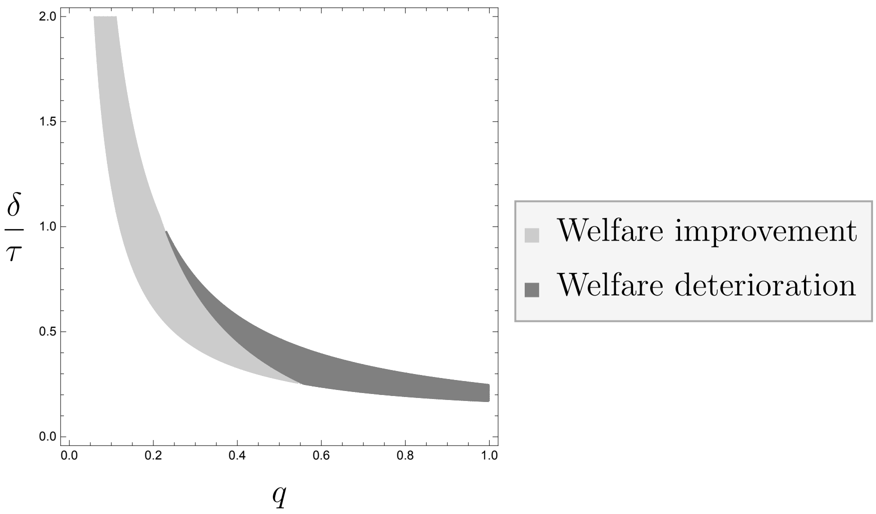


Figure 3.8: The shaded areas depict values of the parameters implying the existence of the central differentiation equilibrium ($b^* < a^* = 0.5$) in the model with consumer learning. The differently shaded areas show whether the central differentiation equilibrium yields a welfare improvement / deterioration compared to the symmetric minimum differentiation result ($b^* = a^* = 0.5$) in the model without consumer learning.

A social planner might consider two different transparency enhancing policies: on the one hand, she could force the firms to provide more information on the product, so that q increases.[26] Figure 3.8 then shows that welfare is decreasing, simply because the share of informed consumers (which bear higher expected transportation costs in the model with learning than uninformed consumers) increases. On the other hand, the social planner could increase market transparency by making the firms provide information about previous sales, i.e., it could induce a transition from a world without consumer learning to a world with consumer learning. In this case, the result is ambiguous, i.e., welfare might increase or decrease depending on the particular values of the parameters $\frac{\delta}{\tau}$ and q, as Figure 3.8 shows.

The welfare analysis of the equilibrium with differentiation on the short side is similar to the equilibrium with central differentiation. However, expected transport costs (for informed and uninformed consumers) now increase even more due to the fact that $a \neq 0.5$.

In the equilibrium with asymmetric minimum differentiation of the model with consumer learning all consumers incur higher expected transportation costs and uninformed early adopters and informed consumers obtain the same expected gain as in the equilibrium of the benchmark model. Because of the updating the expected gain of uninformed

[26] Even if firms do not know the quality (differential), the information provided by them might be helpful to evaluate the products' quality, as it might be the case for experience goods.

laggards is higher. Overall, the comparison of consumer surplus depends on which effect dominates.

3.8 Conclusion

This chapter gives an information-related explanation for why a firm may want to produce a product which appeals to relatively few consumers ex-ante. In our variant of the classical model of spatial competition due to Hotelling (1929), the effect emerges because consumers, who are heterogeneous with respect to their preferred good and with respect to the level of information they possess, make their purchase decisions sequentially and are able to observe which good previous consumers bought. Uninformed consumers in later periods rationalize the choice of other consumers by considering that earlier consumers possibly made their decision because they were better informed about the quality of the different goods. An uninformed consumer thus updates her estimate about the difference in the good's quality after observing previous consumer choices using Bayes' rule.

This updating is especially favorable for niche products. Niche products are not as appealing as mainstream products to a broad range of consumers. Therefore, later consumer's reasoning after having observed an earlier consumer's purchase of a niche product puts more weight on the possibility that this purchase was due to the earlier consumer being informed instead of being due to a good match of the earlier consumer's preference and the good's characteristic.

When deciding about the good's characteristics, i.e., how much to differentiate from the opponent's product, a firm has to take into account two offsetting effects. On the one hand, producing a niche product decreases the product's overall appeal to consumers, hence the expected demand in early periods is decreased. But on the other hand, exactly because the overall appeal is decreased, an early purchase of the niche product leads to a larger boost of later uninformed consumers' confidence in the niche products superior quality. As this chapter shows, the second effect can dominate, making the "differentiate-and-conquer strategy" profitable for a market entrant, and leading to an equilibrium with differentiated goods. The underlying effect, the "recommendation effect", is different from what is generally called the "competition effect", which goes into a similar direction as it makes differentiation profitable for firms, but in the latter the driving force is that it relaxes price competition, thus increasing possible markups.

In biology there is an effect similar to the one described in this chapter: the Handicap Principle (see e.g. Zahavi, 1975) explains why some animals have certain features which at a first glance seem to be an evolutionary disadvantage. A popular example is the tail of the peacock. This tail is a huge obstacle when being hunted by predators. But if one such peacock survives and is chosen by a mate to pass on his genes, then the (probably even bigger) tail of the offspring can work as a strong signal for its high (evolutionary) quality.[27]

On a broader view, our research is related to the literature which emphasizes the (evolutionary) value of diversity in the need for innovation (see, e.g., Page, 2008 or Surowiecki, 2005): when "solutions" (e.g. products) to "problems" are more diverse, then the process of finding out which is the "better" one is more effective.[28] Our model shows that this "social rationale" for diversity can be compatible with the decisions of "individually rational" firms. Future research could investigate this relationship more explicitly.

3.A Appendix: Proofs

3.A.1 Proof of Proposition 3.1 for the Benchmark Model

Proof. In the following, we start with assuming $b \leq a$, but similar arguments apply to $a' := 1 - a$ and $b' := 1 - b$ with $\tilde{x}' := 1 - \tilde{x}$ if $b > a$.

In the benchmark model there are two qualitatively different positions firm B can choose for a given a: either it chooses a small b which implies a high degree of differentiation and also that both, $\bar{x}(\beta^i_A)$ and $\bar{x}(\beta^i_B)$ are in the interval $[b, a]$ and $\bar{x}(\beta^i_A) + \bar{x}(\beta^i_B) = a + b$, or it chooses a location relatively close to a so that $\bar{x}(\beta^i_A) > b$ and $\bar{x}(\beta^i_B) < a$. We define $b_1(a)$ to be the largest b where the first case occurs. It is calculated as $b_1(a) = a - \frac{\delta}{\tau}$ from b s.t. $\bar{x}(\beta^i_B) = a$. Note that if $b_1(a) < 0$, firm B cannot induce a case in which all informed consumers follow their signal and the result follows immediately.

Let $D_L(a, b)$ denote B's expected demand depending on firm A's location a and firm B's location b whenever $b \leq a$. In the benchmark model it is given by

$$D_L(a, b) = 2 \cdot \left[\frac{q}{2}(\tilde{x}(\beta^i_A) + \tilde{x}(\beta^i_B)) + (1 - q)\tilde{x}(\beta^u_\emptyset)\right]$$
$$= \begin{cases} a + b & \text{if} \quad b \leq b_1(a), \\ q + (1 - q)(a + b) & \text{if} \quad b > b_1(a). \end{cases}$$

[27] In contrast to our model, this theory however attributes the effect to the underlying mechanism of costly signaling.

[28] The related issue (and the importance) of the speed of social learning and its convergence "to the truth" is discussed, for instance, in Gale (1996).

As B's demand is increasing in b (there are no "information effects" in the model without consumer learning), a profit-maximizing firm B chooses between the locations $b = b_1(a)$ and $b = a$. B will prefer $b = b_1(a)$ to $b = a$ if

$$\begin{aligned} D_L(a, b_1(a)) &\geq D_L(a, a) \\ \Leftrightarrow 2 \cdot \frac{a + b_1(a)}{2} &= 2a - \frac{\delta}{\tau} \geq 2[(1-q)a + \frac{q}{2}] \\ \Leftrightarrow a &\geq \frac{1}{2} + \frac{\delta}{2\tau q} =: \breve{a}. \end{aligned}$$

Thus, firm B's best response is given by

$$b^*(a) = \begin{cases} b_1(a) & \text{if} \quad a \geq \breve{a}, \\ a & \text{if} \quad a < \breve{a}. \end{cases}$$

Note that firm A could always guarantee a demand of $2 - D_L(0.5, 0.5) = 1$ to itself. We have established that A would have to choose a location further to the right ($a = \breve{a}$ instead of $a = 0.5$), if it would want to induce $b = b_1(a)$. The smallest a that would induce $b_1(a)$ is given by $\breve{a}$. Firm A's expected demand $\tilde{D}_L(a, b) := 2 - D_L(a, b)$ anticipating the behavior of firm B is

$$\tilde{D}_L(a, b^*(a)) = \begin{cases} 2(1-a) + \frac{\delta}{\tau} & \text{if} \quad a \geq \breve{a}, \\ 2 - q - 2a(1-q) & \text{if} \quad a < \breve{a}. \end{cases}$$

For firm A to prefer inducing $b = b_1(a)$, we would need:

$$\begin{aligned} \tilde{D}_L(\breve{a}, b_1(\breve{a})) &\geq \tilde{D}_L(0.5, 0.5) \\ \Leftrightarrow 2\left[1 - \frac{\breve{a} + b_1(\breve{a})}{2}\right] &\geq 1 \\ \Leftrightarrow 1 &\leq 1 + \frac{\delta}{\tau} - \frac{\delta}{\tau q}, \end{aligned}$$

which never holds. Thus, it will always be the case that $\bar{x}(\beta_A^i) < b$ and $\bar{x}(\beta_B^i) > a$ in equilibrium. It also follows immediately, that in equilibrium we have $a = b = 0.5$. □

3.A.2 Generic Properties of Firm B's Expected Demand

Proposition 3.4. *For each particular configuration of the values of the parameters δ, τ and q, the demand of both firms is characterized by the constellation of the different threshold types $\bar{x}(\cdot)$. For $b \leq a$, five qualitatively different and mutually exclusive cases can occur:*

1) $\bar{x}(\beta_A^i), \bar{x}(\beta_B^i) \in [b,a], \quad \bar{x}(\beta_B^u),\ \bar{x}(\beta_A^u) \in [b,a],$

2) $\bar{x}(\beta_A^i), \bar{x}(\beta_B^i) \notin [b,a], \quad \bar{x}(\beta_B^u),\ \bar{x}(\beta_A^u) \in [b,a],$

3A) $\bar{x}(\beta_A^i), \bar{x}(\beta_B^i) \notin [b,a], \quad \bar{x}(\beta_B^u) \in [b,a],\ \bar{x}(\beta_A^u) \notin [b,a],$

3B) $\bar{x}(\beta_A^i), \bar{x}(\beta_B^i) \notin [b,a], \quad \bar{x}(\beta_B^u) \notin [b,a],\ \bar{x}(\beta_A^u) \in [b,a],$

4) $\bar{x}(\beta_A^i), \bar{x}(\beta_B^i) \notin [b,a], \quad \bar{x}(\beta_B^u),\ \bar{x}(\beta_A^u) \notin [b,a].$

For $b > a$, there also exist five qualitatively different and mutually exclusive cases described by the threshold types $\bar{x}(\cdot)' := 1 - \bar{x}(\cdot)$, and replacing all a, b and $\bar{x}(\cdot)$ by $(a', b') := (1-a, 1-b)$ and $\bar{x}(\cdot)'$ in the above cases.

These cases translate to the following unique partition $\mathfrak{D} := \mathfrak{D}_L \cup \mathfrak{D}_R$ of the action space $\mathcal{A} \times \mathcal{B}$ with (possibly empty) elements $\mathfrak{D}_L^j$ and $\mathfrak{D}_R^j$:

$$\begin{aligned}
\mathfrak{D}_L^1 &= \{(a,b) \in \mathcal{A} \times \mathcal{B} \mid b \le b_1(a)\},\\
\mathfrak{D}_L^2 &= \{(a,b) \in \mathcal{A} \times \mathcal{B} \mid b_1(a) < b \le b_3^{\uparrow}(a)\} \cap \{(a,b) \in \mathcal{A} \times \mathcal{B} \mid b_1(a) < b < b_3^{\downarrow}(a)\},\\
\mathfrak{D}_L^{3A} &= \{(a,b) \in \mathcal{A} \times \mathcal{B} \mid b_3^{\downarrow}(a) \le b < b_3^{\uparrow}(a)\},^{29}\\
\mathfrak{D}_L^{3B} &= \{(a,b) \in \mathcal{A} \times \mathcal{B} \mid b_3^{\uparrow}(a) < b \le b_3^{\downarrow}(a)\},\\
\mathfrak{D}_L^4 &= \{(a,b) \in \mathcal{A} \times \mathcal{B} \mid b_3^{\downarrow}(a) < b_3^{\uparrow}(a) \le b \le a\} \cup \{(a,b) \in \mathcal{A} \times \mathcal{B} \mid b_3^{\uparrow}(a) \le b_3^{\downarrow}(a) < b \le a\} \cup \{b_3^{\downarrow}(\bar{a}), \bar{a}\},\\
\mathfrak{D}_R^4 &= \{(a,b) \in \mathcal{A} \times \mathcal{B} \mid a < b \le 1 - b_3^{\uparrow}(a) < 1 - b_3^{\downarrow}(a)\}\\
&\quad \cup \{(a,b) \in \mathcal{A} \times \mathcal{B} \mid a < b < 1 - b_3^{\downarrow}(a) \le 1 - b_3^{\uparrow}(a)\},\\
\mathfrak{D}_R^{3B} &= \{(a,b) \in \mathcal{A} \times \mathcal{B} \mid 1 - b_3^{\downarrow}(a) \le b < 1 - b_3^{\uparrow}(a)\},\\
\mathfrak{D}_R^{3A} &= \{(a,b) \in \mathcal{A} \times \mathcal{B} \mid 1 - b_3^{\uparrow}(a) < b \le 1 - b_3^{\downarrow}(a)\},\\
\mathfrak{D}_R^2 &= \{(a,b) \in \mathcal{A} \times \mathcal{B} \mid 1 - b_3^{\uparrow}(a) \le b < 1 - b_1(a)\} \cap \{(a,b) \in \mathcal{A} \times \mathcal{B} \mid 1 - b_3^{\downarrow}(a) < b < 1 - b_1(a)\},\\
\mathfrak{D}_R^1 &= \{(a,b) \in \mathcal{A} \times \mathcal{B} \mid 1 - b_1(a) \le b\}.
\end{aligned}$$

The generic partition is depicted in Figure 3.3. The functional form of demand is different in each element of the partition, but in each element, firm B's demand is increasing in a and b if $b \le a$ and decreasing in a and b otherwise.[30]

Proof. The proof is constructed using a succession of lemmata.

Lemma 3.1. *For $b \le a$, firm B's demand, and thus also firm A's demand, consists of five qualitatively different and mutually exclusive cases described by the threshold types $\bar{x}(\cdot)$.*

[29] As $b_3^{\uparrow}(a)$ might be complex valued for values of a where $b_3^{\downarrow}(a) > 0$, the precise formulation of this conditions reads $\mathfrak{D}_L^{3A} = \{(a,b) \in \mathcal{A} \times \mathcal{B} \mid 0 > b > b_3^{\downarrow}(a)\} \cap \{(a,b) \in \mathcal{A} \times \mathcal{B} \mid \neg[b_3^{\downarrow}(a) < b_3^{\uparrow}(a)]\}$

[30] We deal with the fact that $\mathfrak{D}^{3A}$ and $\mathfrak{D}^2$ have no maximizers by defining the value function in these parts in terms of limits.

Proof. For $b \leq a$, firm B's demand is given by

$$\begin{aligned}D_L(a,b) = & q\Big[\tilde{x}(\beta_A^i) + \tilde{x}(\beta_B^i)\Big] \\ & + (1-q)\Big[\tilde{x}(\beta_\emptyset^u) + Pr(C_1 = A)\tilde{x}(\beta_A^u) + Pr(C_1 = B)\tilde{x}(\beta_B^u)\Big].\end{aligned}$$

Case distinctions depending on whether the thresholds $\bar{x}(\cdot)$, determining the value of $\tilde{x}(\cdot)$, are at interior or corner values, which in turn depends on the locations a and b, have to be made. As a first period purchase from one firm increases the belief that it offers the superior product, a case with $\bar{x}(\beta_B^u) < b$ or $\bar{x}(\beta_A^u) > a$ can not occur. It can directly be seen that $\bar{x}(\beta_A^i) \in [b,a] \Leftrightarrow \bar{x}(\beta_B^i) \in [b,a]$, and furthermore that $\bar{x}(\beta_A^i) \notin [b,a] \Leftrightarrow \bar{x}(\beta_B^i) \notin [b,a]$. Another crucial insight is that the threshold type for informed consumers is always shifted further apart from $\bar{x}(\beta_\emptyset^u)$ than the one of the uninformed laggards, meaning that $\bar{x}(\beta_A^u) \geq \bar{x}(\beta_A^i)$ and $\bar{x}(\beta_B^u) \leq \bar{x}(\beta_B^i)$. This can easily be seen in Equation (3.8) with 0 or 1 put in place of the belief $\beta_{C_1}^u$. Thus, $\bar{x}(\beta_B^u) \notin [b,a]$ or $\bar{x}(\beta_A^u) \notin [b,a]$ directly imply $\bar{x}(\beta_A^i) \notin [b,a]$ and $\bar{x}(\beta_B^i) \notin [b,a]$. These five cases describe all cases. Note that $\bar{x}(\beta_\emptyset^u) = (a+b)/2$ in any case. □

Lemma 3.2. *Equations* (3.5)*, implicitly defining the boundaries between regions* $\mathfrak{D}_L^1$ *and* $\mathfrak{D}_L^2$ *have the solution* $b_1(a) = a - \frac{\delta}{\tau}$*. Equations* (3.12) *and* (3.13)*, implicitly defining the boundaries between regions* $\mathfrak{D}_L^2, \mathfrak{D}_L^3$ *and* $\mathfrak{D}_L^4$ *have at most one solution in* $[0,1]$*. Call these solutions* $b_3^{\downarrow}(a)$ *and* $b_3^{\uparrow}(a)$*, respectively. If* a *is s.t.* $b_3^{\uparrow}(a), b_3^{\downarrow}(a) \in [0,1]$*, then the respective function is continuous in* a*. Function* $b_1 : \mathcal{A} \to [0,1]$ *is continuous in* a*. For any given* a*, the function values* $b_3^{\uparrow}(a)$*,* $b_3^{\downarrow}(a)$ *and* $b_1(a)$ *correspond to the discontinuity point of the demand.*

Proof. The first statement can be proven by straightforward calculations. The possibly valid solutions to Equations (3.12) and (3.13) are given by

$$b_3^{\downarrow}(a) = \frac{(2-q) - \sqrt{(2-q)^2 - 4(1-q)\Big[(2-q)a - (1-q)a^2 - \frac{\delta q}{\tau}\Big]}}{2(1-q)}$$

for (3.12), and

$$b_3^{\uparrow}(a) = \frac{q - \sqrt{q^2 + 4(1-q)\Big[qa + (1-q)a^2 - \frac{\delta q}{\tau}\Big]}}{-2(1-q)}$$

for (3.13). A bit of calculation shows that both discontinuity points exist for a such that

$$b_3^{\downarrow}(a) \in [0,1] \Leftrightarrow a \in \left[\frac{(2-q) - \sqrt{(2-q)^2 - 4(1-q)\frac{\delta q}{\tau}}}{2(1-q)}, \frac{(2-q) - \sqrt{(2-q)^2 - 4(1-q)\frac{\delta q + \tau}{\tau}}}{2(1-q)} \right],$$

and

$$b_3^{\uparrow}(a) \in [0,1] \Leftrightarrow a \in \left[\frac{-q + \sqrt{q^2 + 4(1-q)\frac{\delta q}{\tau}}}{2(1-q)}, \frac{-q + \sqrt{q^2 + 4(1-q)\frac{\delta q + \tau}{\tau}}}{2(1-q)} \right].$$

Note that these two restrictions on a imply that the radicands in the definition of $b_3^{\uparrow}(a)$ and $b_3^{\downarrow}(a)$ are in the interval $[q^2, (2-q)^2] \subset \mathbb{R}_{++}$, which in turn implies that $b_3^{\uparrow}(a)$ and $b_3^{\downarrow}(a)$ are real valued if they are in $[0,1]$.[31] Continuity can easily be seen and the last statement in the lemma follows by construction of $b_1(a)$, $b_3^{\downarrow}(a)$ and $b_3^{\uparrow}(a)$. □

Lemma 3.3. $\frac{\partial b_3^{\uparrow}(a)}{\partial a} > 1 > \frac{\partial b_3^{\downarrow}(a)}{\partial a} > 0$ *for* $b \leq a$ *and* $a, b \in [0,1]$. *Furthermore,* $b_3^{\downarrow}(a)$ *and* $b_3^{\uparrow}(a)$ *cross only once at* $\bar{a} = \frac{1}{2} + \frac{\delta q}{2\tau}$, *and* $b_3^{\downarrow}(a)$ *is concave. Hence* $a < \bar{a} \Leftrightarrow b_3^{\uparrow}(a) < b_3^{\downarrow}(a)$ *and* $a > \bar{a} \Leftrightarrow b_3^{\uparrow}(a) > b_3^{\downarrow}(a)$. *In addition* $b_3^{\downarrow}(\bar{a}) = b_3^{\uparrow}(\bar{a}) = 1 - \bar{a}$.

Proof. Derivatives can be calculated directly from Equations (3.12) and (3.13) via implicit differentiation as:

$$\frac{\partial b_3^{\downarrow}(a)}{\partial a} = \frac{(2-q) - 2(1-q)a}{(2-q) - 2(1-q)b_3^{\downarrow}(a)},$$

and

$$\frac{\partial b_3^{\uparrow}(a)}{\partial a} = \frac{q + 2(1-q)a}{q + 2(1-q)b_3^{\uparrow}(a)}.$$

Simple calculations show that the stated inequalities hold, given that $b_3^{\downarrow}(a), b_3^{\uparrow}(a) < a$. Continuity of $b_3^{\uparrow}(a)$ and $b_3^{\downarrow}(a)$ allows the application of the intermediate value theo-

[31] To handle situations in which they are smaller than zero and complex-valued, which is a relevant situation for the definition of the partition in the proposition, we could define $b_3^{\uparrow}(a)$ and $b_3^{\downarrow}(a)$ as functions with the same positive slope intersecting the horizontal axis at $b_3^{\uparrow}(a) = 0$, and $b_3^{\downarrow}(a) = 0$ (as defined above), respectively.

rem, and monotonicity implies the uniqueness of the intersection point $\bar{a}$. The second derivative of $b_3^{\downarrow}(a)$ is negative whenever

$$0 > -2(1-q)[2-q-2(1-q)b_3^{\downarrow}(a)] + 2(1-q)\frac{\partial b_3^{\downarrow}(a)}{\partial a}[2-q-2(1-q)a].$$

The right hand side of this expression is smaller than

$$-2(1-q)[2-q-2(1-q)b_3^{\downarrow}(a)] + 2(1-q)[2-q-2(1-q)a] = 2(1-q)2(1-q)(b_3^{\downarrow}(a)-a) < 0,$$

so that $b_3^{\downarrow}(a)$ is indeed concave.

The rest of the lemma is obtained by simply equalizing Equations (3.12) and (3.13), which gives the condition $b_3^{\downarrow}(\bar{a}) = b_3^{\uparrow}(\bar{a}) = 1-\bar{a}$. Plugging in one of the values of $b_3^{\downarrow}(a)$ or $b_3^{\uparrow}(a)$ for b allows to calculate $\bar{a}$ as stated in the lemma. ☐

Lemma 3.4. *Let* $\underline{b_3^{\downarrow}}(a) := a - \frac{\delta q}{\tau}$ *and* $\overline{b_3^{\downarrow}} := b_3^{\downarrow}(\bar{a}) = \frac{1}{2} - \frac{\delta q}{2\tau}$. *For any* $a \leq \bar{a}$, $b_3^{\downarrow}(a)$ *lies in the interval* $\left[\underline{b_3^{\downarrow}}(a), \overline{b_3^{\downarrow}}\right]$. *If* $a > \bar{a}$, *then* $b_3^{\downarrow}(a) \in \left[\overline{b_3^{\downarrow}}, \underline{b_3^{\downarrow}}(a)\right]$. *Furthermore, it holds that* $b_3^{\uparrow}(a) > b_1(a)$ *and* $b_3^{\uparrow}(a) < a$, *if* $b \leq a$.

Proof. Lemma 3.3 implies that the distance $a - b_3^{\downarrow}(a)$ is increasing in a for all $a \leq \bar{a}$, hence

$$a - b_3^{\downarrow}(a) \leq \bar{a} - b_3^{\downarrow}(\bar{a}) = 2\bar{a} - 1,$$

which, for any $a \leq \bar{a}$ gives a lower bound on $b_3^{\downarrow}(a)$ as $\underline{b_3^{\downarrow}}(a) := a - \frac{\delta q}{\tau}$. Since $b_3^{\downarrow}(a)$ is increasing in a, $\overline{b_3^{\downarrow}} := b_3^{\downarrow}(\bar{a})$ is an upper bound on $b_3^{\downarrow}(a)$ for all $a \leq \bar{a}$.

Define a'' as a s.t. $b_3^{\uparrow}(a) = 0$ and a''' as a s.t. $b_1(a) = 0$. As $\frac{\partial b_3^{\uparrow}(a)}{\partial a} > 1 = \frac{\partial b_1(a)}{\partial a}$, we need to show that $a'' < a'''$, in order to show $b_3^{\uparrow}(a) > b_1(a)$.

$$b_3^{\uparrow}(a) = 0 \Leftrightarrow a = \sqrt{\frac{\delta q}{\tau(1-q)} + \frac{q^2}{4(1-q)^2}} - \frac{q}{2(1-q)} =: a'',$$

and

$$a'' < a''' := \frac{\delta}{\tau} \Leftrightarrow 0 < \frac{\delta}{\tau},$$

which always holds under the assumptions.

As $\frac{\partial b_3^{\uparrow}(a)}{\partial a} > 1$, and as the slope of the diagonal through the action space is 1, we need to show that $b_3^{\uparrow}(1) < 1$ in order to show $b_3^{\uparrow}(a) < a$.

$$b_3^{\uparrow}(1) = \frac{q - \sqrt{q^2 + 4(1-q)(1-\frac{\delta q}{\tau})}}{2q-2} < 1 \Leftrightarrow \frac{4\delta q}{\tau}(q-1) < 0,$$

which always holds under the assumptions. □

Corollary 3.1. *$b_1(a) < b_3^{\uparrow}(a) < a$ and $b_1(a) < b_3^{\downarrow}(a) < a$, $\forall (a,b) \in \mathcal{A} \times \mathcal{B}$ with $b \leq a$.*

Proof. This follows from Lemma 3.3 and 3.4 . □

Corollary 3.2. *There is an equivalence between the five cases of Proposition 3.4 and the partition described therein.*

Proof. The result follows immediately from Equations (3.6), (3.7), (3.14) and (3.15) and the previous lemmata. □

Lemma 3.5. *If $b \leq a$, firm B's expected demand is strictly increasing in a and b in each element of partition $\mathfrak{D}_L$. The opposite is true for $\mathfrak{D}_R$, i.e., when $b > a$.*

Proof. The following reformulation of the second period parts of demand of an uninformed consumer will prove to be useful:

$$\begin{aligned} Pr(C_1)\tilde{x}(\beta^u_{C_1}) &= Pr(C_1)\left(\frac{a+b}{2} + \frac{\delta}{2\tau}\left(2\frac{Pr(C_1|v_L > v) \cdot Pr(v_L > v)}{Pr(C_1)} - 1\right)\right) \\ &= Pr(C_1)\tilde{x}(\beta^u_{\emptyset}) + \frac{\delta}{2\tau}\Big(Pr(C_1|v_L > v) - Pr(C_1)\Big). \end{aligned}$$

Note that a enters each of the demand parts in the same way as b (this will become even more obvious below), and thus whenever demand is increasing in b it also increases in a.

- Part 1: $\mathfrak{D}^1_L(a,b) := \{(a,b) \in \mathcal{A}\times\mathcal{B} \mid \tilde{x}(\beta^u_B) \in (b,a), \tilde{x}(\beta^u_A) \in (b,a), \tilde{x}(\beta^i_A) + \tilde{x}(\beta^i_B) = a+b\}$ With all indifferent types being at interior levels, they are symmetrically spread around $\tilde{x}(\beta^u_{\emptyset})$ and $D^1_L(a,b)$ simplifies to:

  $$D^1_L(a,b) = q[2\tilde{x}(\beta^u_{\emptyset})] + (1-q)[2\tilde{x}(\beta^u_{\emptyset})] = 2\tilde{x}(\beta^u_{\emptyset}) = a+b.$$

  Obviously, $D^1_L(a,b)$ is increasing in b.

- Part 2: $\mathfrak{D}^2_L(a,b) := \{(a,b) \in \mathcal{A}{\times}\mathcal{B} \mid \tilde{x}(\beta^u_B) \in (b,a), \tilde{x}(\beta^u_A) \in (b,a), \tilde{x}(\beta^i_A){+}\tilde{x}(\beta^i_B) = 1\}$ If the uninformed indifferent laggard, and thus $\tilde{x}(\beta^u_{C1})$, is always in between the location of both firms, demand can be written as follows:

$$D^2_L(a,b) = q + (1-q)\left\{\tilde{x}(\beta^u_\emptyset) + Pr(C_1 = B)\tilde{x}(\beta^u_\emptyset) + \frac{\delta}{2\tau}\Big(Pr(C_1 = B|v_L > v) - Pr(C_1 = B)\Big)\right.$$
$$\left. + Pr(C_1 = A)\tilde{x}(\beta^u_\emptyset) + \frac{\delta}{2\tau}\Big(Pr(C_1 = A|v_L > v) - Pr(C_1 = A)\Big)\right\}$$

With $Pr(C_1 = A) = 1 - Pr(C_1 = B)$ and $Pr(C_1 = A|v_L > v) = 1 - Pr(C_1 = B|v_L > v)$, this simplifies to

$$D^2_L(a,b) = q + (1-q)(2 \cdot \tilde{x}(\beta^u_\emptyset)) = q + (1-q)(a+b).$$

Hence, for the case where $\tilde{x}(\beta^u_{C1})$ is between both firms' locations, B's demand increases linearly in b.

- Part 3A: $\mathfrak{D}^{3A}_L(a,b) := \{(a,b) \in \mathcal{A}{\times}\mathcal{B} \mid \tilde{x}(\beta^u_B) \in (b,a), \tilde{x}(\beta^u_A) = 0, \tilde{x}(\beta^i_A){+}\tilde{x}(\beta^i_B) = 1\}$ In this part, uninformed laggards follow the choice if the early adopter chose A. If $C_1 = B$, the indifferent consumer in period 2 lies between the two firm's locations. Hence, B's demand calculates as

$$\begin{aligned} D^{3A}_L(a,b) &= q + (1-q)\Big[\tilde{x}(\beta^u_\emptyset) + Pr(C_1 = B)\tilde{x}_2(B)\Big] \\ &= q + (1-q)\left[\tilde{x}(\beta^u_\emptyset)\left(1 + \frac{q + (1-q)(a+b)}{2}\right) + \frac{q\delta}{4\tau}\right], \end{aligned}$$

and the following derivative shows that $D^{3A}_L(a,b)$ is strictly increasing in b:

$$\frac{\partial D^{3A}_L}{\partial \tilde{x}(\beta^u_\emptyset)} = (1-q)\Big[\tfrac{1}{2}q + 1 + 2(1-q)\tfrac{a+b}{2}\Big].$$

- Part 3B: $\mathfrak{D}_L^{3B}(a,b) := \{(a,b) \in \mathcal{A} \times \mathcal{B} \mid \tilde{x}(\beta_B^u) = 1, \tilde{x}(\beta_A^u) \in (b,a), \tilde{x}(\beta_A^i) + \tilde{x}(\beta_B^i) = 1\}$ If a purchase of B in the first period is always followed by an uninformed laggard, but not a purchase of A, demand of B is given by

$$\begin{aligned} D_L^{3B}(a,b) &= q + (1-q)\Big[\tilde{x}(\beta_\emptyset^u) + Pr(C_1 = B) + Pr(C_1 = A)\tilde{x}(\beta_A^u)\Big] \\ &= q + (1-q)\Big\{\tilde{x}(\beta_\emptyset^u) + \frac{q}{2} + (1-q)\tilde{x}(\beta_\emptyset^u) + Pr(C_1 = A)\tilde{x}(\beta_\emptyset^u) \\ &+ \frac{\delta}{2\tau}\Big(Pr(C_1 = A|v_L > v) - Pr(C_1 = A)\Big)\Big\} \\ &= q + (1-q)\left\{\tilde{x}(\beta_\emptyset^u)\Big[2 - \tfrac{q}{2} + (1-q)(1-\tilde{x}(\beta_\emptyset^u))\Big] - \frac{q\delta}{4\tau} + \frac{q}{2}\right\}, \end{aligned}$$

which is quadratic in $\tilde{x}(\beta_\emptyset^u)$ and thus in b. Nevertheless, the derivative:

$$\frac{\partial D_L^{3B}}{\partial \tilde{x}(\beta_\emptyset^u)} = \Big[3 - \tfrac{3}{2}q - 2(1-q)\tfrac{a+b}{2}\Big](1-q)$$

shows that it is strictly increasing in b for the relevant values of a and b.

- Part 4: $\mathfrak{D}_L^4(a,b) := \{(a,b) \in \mathcal{A} \times \mathcal{B} \mid \tilde{x}(\beta_B^u) = 1,\ \tilde{x}(\beta_A^u) = 0,\ \tilde{x}(\beta_A^i) + \tilde{x}(\beta_B^i) = 1\}$ $\tilde{x}(\beta_B^u) = 1$ and $\tilde{x}(\beta_A^u) = 0$ means that an uninformed laggard always follows the lead of the early adopter. The demand in such a case is described by

$$\begin{aligned} D_L^4(a,b) &= q + (1-q)\Big[\tilde{x}(\beta_\emptyset^u) + Pr(C_1 = B)\Big] \\ &= q + (1-q)\Big[(2-q)\Big(\frac{a+b}{2}\Big) + \frac{q}{2}\Big]. \end{aligned}$$

Demand in this case is linear, and increasing in b.

Inspection of the different demand parts shows that updating of the uninformed laggards, and thus the shifting of the indifferent consumer types, is symmetric in parts D^2 and D^4 and asymmetric in parts D^{3B} and D^{3A}. Only in the latter cases does the demand depend on the parameters δ and τ. Furthermore, $\partial D^j/\partial b \downarrow$ as "$j \uparrow$" with $j \in \{1, 2, 3B, 3A, 4\}$.

□

Combining the lemmata yields the result.

□

Lemma 3.6. *$b_3^{\downarrow}(a)$ and $b_3^{\uparrow}(a)$ converge to $b_1(a)$, as $q \to 1$. Also, $b_3^{\downarrow}(a)$ and $b_3^{\uparrow}(a)$ converge to a, as $q \to 0$. Thus, increasing (decreasing) $\frac{\delta}{\tau}$ "stretches" ("compresses") the graphs of $b_3^{\uparrow}(a)$ and $b_3^{\downarrow}(a)$ (compare Figure 3.3).*

Proof. Straightforward calculations show that the derivative of the numerator of $b_3^{\downarrow}(a)$ w.r.t. q is given by

$$-1-\frac{(2a-1)[2a(q-1)-q+2]\tau+\delta(2-4q)}{\tau\sqrt{\frac{\tau[q-2-2a(q-1)]^2-4\delta(q-1)q}{\tau}}}.$$

Similarly, the derivative of the numerator of $b_3^{\uparrow}(a)$ w.r.t. q can be calculated as

$$1-\frac{(2a-1)[2a(q-1)-q]\tau+\delta(4q-2)}{\tau\sqrt{\frac{4(q-1)\{a\tau[a(q-1)-q]+\delta q\}}{\tau}+q^2}}.$$

Applying l'Hopital's rule then yields the results for $q \to 1$. The limits for $q \to 0$ can be obtained directly.

Using these results, and as $b_1(a) = a - \frac{\delta}{\tau}$, $b_1(a) < b_3^{\uparrow} < a$, and $b_1(a) < b_3^{\downarrow} < a$, one can easily observe that increasing (decreasing) $\frac{\delta}{\tau}$ "stretches" ("compresses") the graphs of $b_3^{\uparrow}(a)$ and $b_3^{\downarrow}(a)$. □

3.A.3 Proof of Proposition 3.3 (Main Result)

Remember that for any equilibrium with $b^* < a^*$ there exists an analogous equilibrium with $1 - b^* > 1 - a^*$. Thus, if we assume $a \geq 0.5$ in the following, this is without loss of generality.

Lemma 3.7. *B's best response $b^*(a)$ to any $a \in [0.5, 1]$ is a subset of $\{b_1(a), b_3^{\downarrow}(a), a, 1 - b_3^{\downarrow}(1-a)\}$.*

Proof. By Lemma 3.5 all parts of B's demand are increasing in b. The highest demand in each part is thus obtained at the highest possible value, belonging to this part. Clearly, it can never be optimal to choose $b = b_3^{\uparrow}(a)$, since B's demand has an upward jump at this point, so any slightly larger b will increase the demand. For $0.5 \leq a \leq \bar{a}$, the demands $D_L(a,b)$ and $D_R(a,b)$ consist of the same parts, so the optimum of B's demand must be obtained for some $b \in \mathfrak{D}_L$, i.e. $b^*(a) \leq a \leq \bar{a}$, as B's demand increases in a. If, however, $0.5 \leq \bar{a} \leq a$, then $D_L(a,b)$ and $D_R(a,b)$ do not consist of the same parts anymore. On the left side, part 3 of B's demand is given by $D_L^{3A}(a,b)$, but on the right, the demand part 3 consists of $D_R^{3B}(a,b)$. Hence, the demand maximizing location for firm B might be at the optimum of $D_R^{3B}(a,b)$, which is calculated as $1 - b_3^{\downarrow}(1-a)$. □

Corollary 3.3. *The value function of firm B in the respective regions of the action space is given by*

$$V_L^1(a) = \begin{cases} 0 & \text{if} \quad b_1(a) < 0, \\ D_L^1(a, b_1(a)) & \text{else.} \end{cases}$$

$$V_L^2(a) = \begin{cases} D_L^2(a, b_3^{\uparrow}(a)) & \text{if} \quad a \le \bar{a}, \\ \lim\limits_{b \to b_3^{\downarrow}(a)} D_L^2(a, b) & \text{if} \quad a > \bar{a}, \\ 0 & \text{if} \quad \min\{b_3^{\uparrow}(a),\ b_3^{\downarrow}(a)\} < 0. \end{cases}$$

$$V_L^3(a) = \begin{cases} D_L^{3B}(a, b_3^{\downarrow}(a)) & \text{if} \quad a \le \bar{a}, \\ \lim\limits_{b \to b_3^{\uparrow}(a)} D_L^{3A}(a, b) & \text{if} \quad a > \bar{a}, \\ 0 & \text{if} \quad \max\{b_3^{\uparrow}(a),\ b_3^{\downarrow}(a)\} < 0. \end{cases}$$

$$V_L^4(a) = D_L^4(a, a)$$

$$V_R^3(a) = \begin{cases} 0 & \text{if} \quad 1 - b_3^{\downarrow}(a) > 1, \\ D_R^{3B}(a, 1 - b_3^{\downarrow}(1-a)) & \text{else.} \end{cases}$$

Lemma 3.8. *For $a \le \bar{a}$, $V_L^4(a) > V_L^2(a) > V_L^1(a)$. This implies that for $a \le \bar{a}$, firm B will choose a location such that $(a, b) \notin \mathfrak{D}_L^1$ and $(a, b) \notin \mathfrak{D}_L^2$.*

Proof. Remember that all demand parts are increasing in a and b. With $b_3^{\downarrow}(\bar{a}) = 1 - \bar{a}$, we can calculate $V_L^1(\bar{a}) = D_L^1(\bar{a}, b_1(\bar{a})) = 2\bar{a} - \delta/\tau = 1 + \delta q/\tau - \delta/\tau < 1$ and $V_L^2(\bar{a}) = D_L^2(\bar{a}, 1-\bar{a}) = q + (1-q)(\bar{a} + 1 - \bar{a}) = 1$. For V_L^4 we know $D_L^4(\bar{a}, 1-\bar{a}) = q + (1-q)(\bar{a} + 1 - \bar{a}) = 1 < D_L^4(\bar{a}, \bar{a}) = V_L^4(\bar{a})$. As $\partial D_4(a,b)/\partial a = (1-q)(2-q)/2 < \partial D_1(a,b)/\partial a = 1$, and $V_L^2(0.5) = D_L^2(0.5, b_3^{\uparrow}(0.5)) < D_L^2(0.5, 0.5) = D_L^4(0.5, 0.5) = V_L^4(0.5)$, the result follows. □

Lemma 3.9. *B's demand at the optimum of any part of D_L is increasing in a, i.e., $\partial V_L^j(a)/\partial a > 0\ \forall j \in \{1, 2, 3, 4\}$. Furthermore, $D_L^{3B}(a, b_3^{\downarrow}(a))$ is concave, while $V_L^j(a)$ is linear in a for $j \in \{1, 4\}$.*

Proof. By Lemma 3.5 all parts of $D_L(a, b)$ are increasing in a and b. Since all potentially optimal locations of B, i.e., $b_1(a), b_3^{\downarrow}(a), b_3^{\uparrow}(a)$ and a, are increasing in a, the first part of the lemma follows. For B's demand $D_L^{3B}(a, b_3^{\downarrow}(a))$, we can calculate the derivative w.r.t. a as follows

$$\frac{\partial D_L^{3B}(a, b_3^{\downarrow}(a))}{\partial a} = (1-q)\left(1 + \frac{\partial b_3^{\downarrow}(a)}{\partial a}\right)\left[\frac{4-q}{4} + (1-q)\frac{1-a-b_3^{\downarrow}(a)}{2}\right].$$

This derivative is positive if

$$b_3^{\downarrow}(a) \leq \frac{6-3q}{2-2q} - a,$$

which always holds for $a \in [0, 1]$.

The second derivative of $D_L^{3B}(a, b_3^{\downarrow}(a))$ w.r.t a is negative if

$$\frac{\partial^2 b_3^{\downarrow}(a)}{\partial a^2}\left(\frac{4-q}{4} + (1-q)\frac{1-a-b_3^{\downarrow}(a)}{2}\right) + \left(1 + \frac{\partial b_3^{\downarrow}(a)}{\partial a}\right)\left(-\frac{1-q}{2}\left(1 + \frac{\partial b_3^{\downarrow}(a)}{\partial a}\right)\right) < 0,$$

which can be verified using the results of Lemma 3.3. Linearity can directly be seen from the linearity of $b_1(a), b = a$ and from the demand D_L^j, $j \in \{1, 4\}$ calculated in Lemma 3.5. □

Lemma 3.10. *For $a \geq 0.5$, $V_L(a) := \max V_L^j(a), j \in \{1, 2, 3, 4\}$ is increasing in a for all $a \in \mathcal{A} \setminus \bar{a}$ and the function $V_R^{3B}(a)$ is decreasing in a for all $a \geq 0.5$.*

Proof. By Lemma 3.9, given $b \leq a$, B's demand is increasing in a at the maximizing b in any single part. Below and above $a = \bar{a}$, the maximum taken over increasing functions must thus be increasing in a. Note that, whenever $V_L^4(\bar{a}) \geq V_L^{3B}(\bar{a})$, $V_L(a)$ is increasing in a for all $a \in \{\mathcal{A} \cup \bar{a} \mid a \geq b\}$. Since $\bar{a} > 0.5$, symmetry implies that $V_R^{3B}(a)$ is decreasing for all $b > a$. □

Lemma 3.11.

$$V_L^4(0.5) < V_L^{3B}(0.5) \tag{BDC}$$

implies that $\bar{a} < 1$ and $b_3^{\downarrow}(0.5) > 0$.

Proof. An upper bound of $V_L^{3B}(0.5) = D_L^{3B}(0.5, b_3^{\downarrow}(0.5))$ is given by

$$\begin{aligned} D_L^{3B}(0.5, 0.5(1-\delta q/\tau)) &= q + (1-q)\left\{\left(\frac{1}{2} - \frac{\delta q}{4\tau}\right)\left(2 - \frac{q}{2} + (1-q)\left(\frac{1}{2} + \frac{\delta q}{4\tau}\right)\right) - \frac{\delta q}{4\tau} + \frac{q}{2}\right\} \\ &= q + (1-q)\left\{\frac{5}{4} - \frac{\delta q}{4\tau}\left(3 - \frac{q}{2} + (1-q)\frac{\delta q}{4\tau}\right)\right\}. \end{aligned}$$

In order for this upper bound to be larger than 1 for some $q \in (0,1)$, $\delta q/\tau$ must be smaller than 2/5. By Lemma 3.4, $b_3^{\downarrow}(a) \in \left[a - \frac{\delta q}{\tau}, \frac{1}{2} - \frac{\delta q}{2\tau}\right]$, meaning that, given $\delta q/\tau < 2/5$, $b_3^{\downarrow}(a) > 0$ if $a > 2/5$. Additionally, $\bar{a} = 0.5 + \delta q/(2\tau) < 7/10$. □

Lemma 3.12. *If* (ADC) *does not hold, that is,*

$$V_L^{3B}(0.5) > V_L^4(\bar{a}),$$

$\bar{a} < 2/3 < 1$ *and* $1 - b_3^{\downarrow}(1-\bar{a}) < 1$*. Furthermore,* $\exists a' < 2/3$ *s.t.* $V_L^4(a') = V_R^{3B}(a')$*.*

Proof. If (ADC) is violated, it must be that

$$\begin{aligned} V_L^4(\bar{a}) = D_L^4(\bar{a},\bar{a}) = q + (1-q)\left(1 + (1-0.5q)\frac{\delta q}{\tau}\right) &< V_L^3(0.5) = D_L^{3B}(0.5, b_3^{\downarrow}(0.5)) \\ &< D_L^{3B}(0.5,0.5) = q + (1-q)\left\{\frac{1}{2}\left(2 - \frac{q}{2} + (1-q)\frac{1}{2}\right) - \frac{\delta q}{4\tau} + \frac{q}{2}\right\} \\ &= q + (1-q)\left\{\frac{5}{4} - \frac{\delta q}{4\tau}\right\}, \end{aligned}$$

which implies

$$1 + \frac{\delta q}{\tau}(1 - 0.5q) < \frac{5}{4} - \frac{\delta q}{4\tau} \Leftrightarrow \frac{\delta q}{\tau}\left(\frac{5}{4} - \frac{q}{2}\right) < \frac{1}{4}.$$

For this equation to be fulfilled for some $q \in (0,1)$, it must be that $\frac{\delta q}{\tau} < \frac{1}{3}$. In this case, $\bar{a} = 0.5 + \delta q/2t < 2/3$ and $b_3^{\downarrow}(a) > 0$ if $a > 1/3$, so that $1 - b_3^{\downarrow}(1-\bar{a}) < 1$.
The demand $D_L^4(2/3, 2/3)$ calculates as

$$\begin{aligned} D_L^4(2/3,2/3) &= q + (1-q)\left((2-q)\frac{2}{3} + \frac{q}{2}\right) = q + (1-q)\left(\frac{4}{3} - \frac{q}{6}\right) \\ &> q + (1-q)\left(\frac{5}{4} - \frac{\delta q}{4\tau}\right) = D_L^{3B}(0.5,0.5) > D_L^{3B}(1-2/3, b_3^{\downarrow}(1-2/3)). \end{aligned}$$

As $V_L^4(a) = D_L^4(a,a)$ is increasing and $V_R^{3B}(a) = D_L^{3B}(1-a, b_3^{\downarrow}(1-a))$ is decreasing in a, the remainder of the lemma follows. □

Proposition 3.3. *In the model with consumer learning, we obtain the following results.*

1. *Conditions* (BDC) *and* (ADC) *are necessary and sufficient conditions so that the locations are* $a^* = 0.5$ *and* $b^* < 0.5$ *in the unique equilibrium (Central Differentiation Equilibrium).*
2. *If Conditions* (ADC) *and* (BDS) *do not hold the locations are* $a^* = b^* > 0.5$ *in the unique equilibrium (Asymmetric Minimum Differentiation Equilibrium).*

3. *Condition* (BDS) *is a necessary and sufficient condition so that there exist an equilibrium (not necessarily unique) with locations* $b^* > a^* > \bar{a} > 0.5$ *(Short Side Differentiation Equilibrium).*

Uniqueness is up to symmetry, as to any equilibrium with (a^*, b^*) *there exists an analogous equilibrium with* $(1 - a^*, 1 - b^*)$.

Proof. A's objective is given by $\min_a V(a) := \max\{V_L(a), V_R(a)\}$. It is useful to notice, that whenever $V_R^{3B}(\bar{a}) \leq V_L^4(\bar{a})$, as the former is decreasing in a and the latter is increasing in a, it must be that $V(a) = V_L(a)$.

- Central Differentiation Equilibrium: $b^* = b_3^{\downarrow}(0.5) < a^* = 0.5$

  $V_L^{3B}(0.5) \leq V_L^4(\bar{a})$ implies $D_R^{3B}(\bar{a}) < V_L^{3B}(0.5) \leq V_L^4(\bar{a})$, and thus $V(a) = V_L(a)$. By Lemma 3.10, $V_L(a)$ is increasing in a if $V_L^{3B}(0.5) \leq V_L^4(\bar{a})$, so A's optimal choice is the smallest possible a, given by $a^* = 0.5$. As $V_L^4(0.5) < V_L^{3B}(0.5)$, $b^*(0.5) = b_3^{\downarrow}(0.5)$.

- Asymmetric Minimum Differentiation Equilibrium: $b^* = a^* = \bar{a}$

  $V_L^{3B}(0.5) > V_L^4(\bar{a}) > D_R^{3B}(\bar{a})$ again implies $V(a) = V_L(a)$. In contrast to the previous case, $V_L(a)$ has a downward jump at $a = \bar{a}$. By Lemma 3.10, it is increasing in a at all other points. As $V_L^{3B}(0.5) > V_L^4(\bar{a})$, $a = \bar{a}$ is the unique minimizer of $V_L(a)$ in this case. Since demand part $D^{3B}(a, b)$ exists only for $a < \bar{a}$, B's best response is given by $b^*(\bar{a}) = \bar{a}$.

- Short Side Differentiation Equilibrium: $b^* = 1 - b_3^{\downarrow}(1 - a) > a^* > \bar{a}$

  If $V_R^{3B}(\bar{a}) > V_L^4(\bar{a})$, there is some $a' > \bar{a}$ is such that $V(a') = V_R^{3B}(a') = V_L^4(a')$, as V_R^{3B} decreases in a and $V_L^4(a)$ increases in a. Clearly, a' minimizes $V(a)$. B's best response to this a' is not unique, since, by construction, B is indifferent between choosing either the $b \leq a$ maximizing demand, or the $b > a$ maximizing demand. Furthermore, the optimal $b \leq a$ can be any of the set $\{b_1(a'), b_3^{\downarrow}(a'), a'\}$. The optimal $b > a$, however, is given by $1 - b_3^{\downarrow}(1 - a')$, which by Lemma 3.12 exists in the action space.

Note that what distinguishes the last two equilibria, is whether $a' \leq \bar{a}$ or $a' > \bar{a}$. □

Corollary 3.4. *Sufficient conditions for the conditions of Proposition 3.3 can be calculated as follows:*

$$\frac{\delta q}{\tau} < 0.192 \Rightarrow D_L^{3B}(0.5, b_3^{\downarrow}(0.5)) > D_L^4(0.5, 0.5), \tag{BDC}$$

$$\frac{\delta q}{\tau} > 0.166, q < 0.4 \Rightarrow D_L^{3B}(0.5, b_3^{\downarrow}(0.5)) \leq D_L^4(\bar{a}, \bar{a}), \tag{ADC}$$

$$\frac{\delta q}{t} < 0.079 \Rightarrow D_L^{3B}(1-\bar{a}, b_3^{\downarrow}(1-\bar{a})) > D_L^4(\bar{a}, \bar{a}). \tag{BDS}$$

Proof. Using the upper and lower bound of $b_3^{\downarrow}(a)$ as calculated in Lemma 3.4, sufficient conditions for the conditions of Proposition 3.3 can be calculated as follows:

$$\frac{\delta q}{\tau} < 0.192 \Rightarrow \frac{\delta q}{\tau}\left((1-q)\frac{\delta q}{\tau} + 5 - q\right) < 1 \Leftrightarrow D_L^{3B}(0.5, \underline{b_3^{\downarrow}}(a)) > 1$$
$$\Rightarrow D_L^{3B}(0.5, b_3^{\downarrow}(0.5)) > D_L^4(0.5, 0.5), \tag{BDC}$$

$$\frac{\delta q}{\tau} > 0.166, q < 0.4 \Rightarrow 1 \leq \frac{\delta q}{2\tau}\left[\frac{\delta q}{2\tau}(1-q) + 14 - 5q\right] \Leftrightarrow D_L^{3B}(0.5, \overline{b_3^{\downarrow}}(a)) \leq D_L^4(\bar{a}, \bar{a})$$
$$\Rightarrow D_L^{3B}(0.5, b_3^{\downarrow}(0.5)) \leq D_L^4(\bar{a}, \bar{a}). \tag{ADC}$$

If both those sufficient conditions are fulfilled, equilibrium locations are $b = b_3^{\downarrow}(0.5) < a = 0.5$.

$$\frac{\delta q}{t} < 0.079 \Rightarrow 1 > \frac{\delta q}{\tau}\left(13 - 4q + 4(1-q)\frac{\delta q}{\tau}\right) \Leftrightarrow D_L^{3B}(1-\bar{a}, 1-\bar{a}) > D_L^4(\bar{a}, \bar{a})$$
$$\Rightarrow D_L^{3B}(1-\bar{a}, b_3^{\downarrow}(1-\bar{a})) > D_L^4(\bar{a}, \bar{a}) \tag{BDS}$$

If this holds, there is an equilibrium with $b = 1 - b_3^{\downarrow}(1-a) > a > \bar{a}$. Note that we do not derive sufficient conditions for the equilibrium with asymmetric minimum differentiation, i.e., $b = a = \bar{a}$, as these would be "too small" in the parameter space. □

Chapter 4

The Different Effect of Consumer Learning on Incentives to Differentiate in Cournot and Bertrand Competition[1]

We combine two extensions of the differentiated duopoly model of Dixit (1979), namely Caminal and Vives (1996) and Brander and Spencer (2015a,b), to analyze the effect of consumer learning on firms' incentives to differentiate their products in models of Cournot and Bertrand competition. Products are of different quality, consumers buy sequentially and are imperfectly informed about the quality of the goods. Before simultaneously competing in quantities, firms simultaneously select their level of investment in differentiation. The more a firm wishes to differentiate its product or, equivalently, the less substitutable it wants the products to be, the higher the investments have to be. Late consumers can observe earlier consumers' decisions and extract information about the quality of the goods from it. This influences the firms' incentives to differentiate. If firms compete in quantities, they are more likely to invest in differentiation with consumer learning than without. This is in line with implications of the recommendation effect introduced in Chapter 3 in a model of spatial differentiation. We also examine cases in which firms compete in prices. Here, the effect of consumer learning is reversed, so that differentiation is less likely with consumer learning. Thus, we find an information-based difference between Cournot and Bertrand competition: in the Bertrand setting consumer learning increases competition, i.e., products are more likely to be substitutes; in the Cournot model, it weakens it.

[1] This chapter is joint work with Maximilian Conze.

4.1 Introduction

Most of the literature dealing with firms' incentives to differentiate characterizes two different and opposing effects. The competition effect induces firms to differentiate their products from each other, since they then obtain local monopoly power and are able to charge higher prices. On the other hand, differentiating decreases the market share and the amount of goods that the firm in question is able to sell. This is the so-called market-size effect (see e.g. Belleflamme and Peitz, 2010, Chapter 5.2).

In the related Chapter 3 we use a spatial model of product differentiation à la Hotelling (1929) to establish a new effect that may incentivize firms to differentiate. The effect arises because of the possibility of consumer learning, and is called the recommendation effect. Let us recall the intuition of that effect. In the model, two firms A and B compete by choosing their locations on the unit interval, representing the choice of the goods' characteristics. Two consumers sequentially choose between the two goods that are of different quality. Consumers are heterogeneous with respect to their preference towards the goods and to their information about the quality differential. The late consumer (laggard) observes the purchase decision of her predecessor (early adopter), which may contain valuable information. Neither the information, nor the preference of the early adopter is observed by the laggard. The laggard then uses Bayes' rule to update her belief on the good's qualities. In this setup, it is the case that a purchase of a niche (i.e. a differentiated) product in the first period is more likely based on its high quality than on a good match of consumer taste and product characteristic. A firm can influence and exploit consumer learning using its location choice (mainstream vs. niche), which yields incentives to offer a differentiated (niche) product.

In order to make the above mentioned model tractable, it abstracts from endogenous prices by assuming that they are regulated to the same value for both firms. Additionally, there are only two consumers and two situations concerning the stochastic quality (either good A's quality exceeds the one of good B by a fixed amount, or vice versa). Although there are situations plausibly described by these assumptions, the goal of the research at hand is to show that similar effects as described above also arise in a model where these assumptions are relaxed. More explicitly, the model in this chapter differs from Chapter 3 in that it entails endogenous prices, a simultaneous choice of product differentiation, a continuously distributed quality differential, a continuous information structure and the fact that firms may reset one of their choice variables (quantities in Cournot, prices in Bertrand competition), which would be equivalent to allowing firms to relocate in Chapter 3.

The underlying model in this chapter is the standard model of differentiated duopoly introduced by Dixit (1979). Dixit's model was extended in various ways. In particular, Caminal and Vives (1996) formulate a model of Bertrand competition with two firms who compete for a continuum of consumers in each of two periods by setting prices. The goods of the two firms are horizontally and vertically differentiated, but firms can not control either dimension. Firms also do not know the quality of their good when setting their prices. Consumers receive signals about the goods' qualities, and the consumers in the second period observe past market shares but not past prices. The authors show that such a situation leads to lower prices in the first period - compared to a situation without consumer learning - as each firm has an incentive to decrease its price in order to obtain a higher market share. This is because a high market shares serves as a signal of high quality to consumers in the second period. The authors state that "[...] an increase in the degree of product substitutability [...] increases the effectiveness of the manipulation by firms" (Caminal and Vives, 1996, p. 228). As the model does not allow for endogenous levels of differentiation or substitutability, Caminal and Vives do not elaborate on this insight any further.

This is where the model of Brander and Spencer (2015a,b), also based on Dixit's model, comes in. In their papers, the authors analyze the competition between two firms in the common differentiated duopoly setup without vertical differentiation. The goal of their article is to compare the firms' incentives to differentiate in Cournot and Bertrand competition setups. To endogenize the levels of differentiation, the authors assume that firms can make costly investments in order to increase the differentiation between the products. It is shown that differentiation is more likely to occur in Bertrand than in Cournot setups.

We combine the approaches of Caminal and Vives (1996) and Brander and Spencer (2015a) to explore how the possibility of consumer learning influences firms incentives to differentiate their products. Our model thus makes the following changes to the differentiated duopoly setup of Dixit (1979). First, we allow the firms' products to be of different, random, and a priori unknown quality, which introduces information asymmetries. The model consists of three stages. Before firms compete in quantities, they decide on their investment into differentiation. This stage is followed by two stages in which firms set quantities and consumers buy the goods, based on imperfect signals about the goods qualities. In the last stage, consumers additionally observe past prices (but not market shares).

Additionally, we analyze an analogous Bertrand model with consumer learning and compare the implications of Bayesian updating among consumers between the two models of price and quantity competition. We start with analyzing Cournot competition as in this model the endogenous variable of interest, namely the product differentiation, appears more intuitively in the consumers' utility function.

4.2 Model Setup: Quantity Competition

Consumers In two periods, $t = 1, 2$, consumption decisions are made for two (potentially) differentiated goods, A and B, with the respective quantities denoted by $x^j, j \in \{A, B\}$. In each period there is a (different) continuum of consumers with mass one uniformly distributed on $[0, 1]$ and indexed by i. Note that each consumer only purchases goods once. The utility of a consumer when consuming any real-valued amount $x = (x^A, x^B, x^0) \in \mathbb{R}^3$ is given by

$$U(x^A, x^B, x^0) = (\alpha + q)x^A + (\alpha - q)x^B - 0.5\left[(x^A)^2 + 2\gamma x^A x^B + (x^B)^2\right] + x^0, \quad (4.1)$$

with $\alpha > 0$. All consumers have the same utility function, which can be derived from quasilinear preferences. Quantity x^0 captures the consumption of a composite good containing all other goods different from x^A and x^B, and its price is normalized to $p^0 = 1$.

The feasible range of the measure of product differentiation or substitutability is $\gamma \in (0, 1]$. The higher γ, the higher is the substitutability between the products. Goods are perfect substitutes if $\gamma = 1$, and they would be independent if $\gamma = 0$.

Consumers have access to distinct information. The (relative) quality of the goods is measured by the random variable q, which is normally distributed with mean zero and variance $1/\tau_q$.[2] The realization of variable q is unknown to firms and consumers, but each consumer receives a signal $s_t^i = q + \epsilon_t^i$ about it, where ϵ also is an independent and normally distributed random variable with zero mean and variance $1/\tau_\epsilon$. Both variances, $1/\tau_q$ and $1/\tau_\epsilon$, are known to the players in the model. We assume $\int_0^1 s_t^i di \xrightarrow{a.s.} q,\ t \in \{1, 2\}$, analogously to a version of the Strong Law of Large Numbers.[3] The consumers in period two also observe past prices, but not sold quantities or previous signals. Let I_t^i denote all information available to consumer i in period t, and let

$$\eta_t := \int_0^1 E(q|I_t^i)di$$

denote the aggregate belief on quality of all rational consumers in period $t \in \{1, 2\}$. Rational consumers maximize their expected utility subject to the budget constraint $m = x^0 + p^A \cdot x^A + p^B \cdot x^B$.

In addition to the rational consumers, in each period, there are also consumers who ignore prices and whose utility for both products is the same irrespectively of the realized quality. Those consumers purchase from both firms randomly (see also Caminal

[2] As will become clear later on, when dealing with the Bayesian updating of the consumers, it is easier to work with precisions τ than with variances.

[3] This assumption needs to be made due to a related issue pointed out by Judd (1985).

and Vives, 1996). Their impact on the inverse demand at firm j in period t is the given by the random variable u_t^j. These random variables are i.i.d. draws from $\mathcal{N}(0, 1/\tau_u)$.

Firms In two periods, $t = 1, 2$, two firms, $j = A, B$, compete by producing quantities x_t^j. For both firms marginal production costs are zero and no fixed costs occur.

The product substitutability γ is endogenous and chosen by the firms via their investment decision. At the beginning of the game, in period $t = 0$, firms can make monetary investments $k^A \in \mathbb{R}_+$ and $k^B \in \mathbb{R}_+$ in order to increase the differentiation between their products, according to the following functional form

$$\gamma = e^{-\lambda(k^A + k^B)}.$$

If both firms make zero investments, then $\gamma = 1$ and the goods become perfect substitutes. Larger investments of any firm decrease γ, which approaches zero if the investments approach infinity. As $\gamma \in (0, 1]$, the goods are between the extremes of independence or perfect substitutability. The parameter $\lambda \in \mathbb{R}_+$ measures the technology (or, equivalently, effectiveness) of how the firms' investments translate into increased differentiation. The higher λ, the smaller are the necessary investments to increase the differentiation by a certain amount.

Firms maximize their expected total profits and second period profits are discounted with the factor $\delta \in (0, 1)$. Firms do not know the quality (differential) when making their differentiation investments and when choosing their quantities. Thus, their information set in the first two periods is given by $I_0 = I_1 = \{\emptyset\}$, and by $I_2 = \{x_1, p_1\}$ in the last period, where $x_t := (x_t^A, x_t^B)$ and $p_t := (p_t^A, p_t^B)$ denote the quantities and prices of the respective period $t \in \{1, 2\}$.

Auctioneer, Timing, and Solution Concept As it is common for Cournot models, the process of price formation can be modeled via an auctioneer. She knows quality q, which allows her to calculate the consumers' aggregate belief on quality η_t, and she also knows the realization of u_t^A and u_t^B for $t = 1, 2$. In each period, the firms inform the auctioneer about the quantity produced and the auctioneer calculates the prices of the two goods, p_t^A and p_t^B, with $t \in \{1, 2\}$. These prices are announced to consumers and each of them purchases his optimal quantity of each of the two goods. As announced prices contain information on the quality, we assume that consumers' beliefs are not affected by announced prices.

For convenience, we let firm specific variables without superscript denote the vector of the two variables of both firms and by Δy we describe the difference of variable y^A

and y^B, so for example $x_1 = (x_1^A, x_1^B)$ and $\Delta u_1 = u_1^A - u_1^B$. The following graphic depicts the timing of the game:

$t = 0$
- Quality q realizes.
- Firms make their differentiation investments k^A and k^B simultaneously knowing $I_0 = \emptyset$.

$t = 1$
- Firms set quantities x_1^A, x_1^B knowing $I_1 = \emptyset$.
- Consumer i receives signal s_1^i
- (Auctioneer announces prices p_1^A and p_1^B knowing η_1 and u_1)
- Consumer i purchases optimal quantity of goods knowing $I_1^i = \{s_1^i, x_1\}$.

$t = 2$
- Firms set quantities x_2^A, x_2^B knowing $I_2 = \{p_1, x_1\}$.
- Consumer j receives signal s_2^j
- (Auctioneer announces prices p_2^A and p_2^B knowing η_2 and u_2)
- Consumer j purchases optimal quantity of goods knowing $I_2^j = \{s_2^j, \Delta p_1, x_2\}$.

Figure 4.1: Timing of the Differentiated Duopoly Game with Consumer Learning

We employ the solution concept of perfect Bayesian Nash equilibrium. To avoid complications "off the equilibrium path", it is assumed that consumers' beliefs are constant with respect to observed current-period quantities, i.e. $\partial\eta_t^i/\partial x_t = 0$.

Discussion of the Model Setup

In addition to heterogeneous information levels among the consumers, we could introduce heterogeneity in the utility by choosing an individual parameter α_i with an appropriate distribution such that the results continue to hold. One could interpret each representative agent with a certain information level as representing a group of consumers with that same information level.

The fact that consumers in period two observe past prices, but not sold quantities or previous signals, is plausible, for instance, at online platforms on which consumers can observe the history of past prices and current quantities, but not quantities sold in previous periods.

As in many economic models, the application of normally distributed random variables has economically implausible consequences: depending on the received signal, individual demand might become negative or tend to infinity for a fixed set of prices.

Analogous consequences can be found for the individual inverse demand for a fixed set of quantities in Bertrand competition. To rule out such cases we could alternatively assume that q is distributed according to a truncated normal distribution on an interval $[-Y, Y]$ with $0 < Y < \alpha/(1-\gamma^2)$. One would then only need to incorporate the changed variance of q, the rest of the Bayesian updating described below remains unchanged. Another possibility to decrease the probability of the above mentioned "unwanted" events would be to increase the precision of the random variables. Our results hold if we approach the model without asymmetric information and consumer learning, i.e., when $\tau_\epsilon \to \infty$. If $\gamma = 1$, the optimal consumer behavior implies that individual demands (see Equation 4.2 in Section 4.3 below) become infinitely large or small. Furthermore, in the analogous model of Bertrand competition, the consumers' first order conditions are not invertible, such that there are problems with the microfoundation of the aggregate model described in Section 4.4 below. However, when starting directly with the standard aggregated form of the oligopoly models as represented in Equations (4.3) and (4.9) stated in Section 4.4, such issues are avoided.

The fact that irrational consumers may have a negative impact on (inverse) demand might be interpreted as some part of the rational consumers refraining from a purchase of the good, although rational utility optimization implies the opposite. Random shoppers are included in the model to prevent that Bayes rational consumers purchasing in the second period can perfectly infer the product qualities by observing the decisions of consumers who bought in the first period. The assumption that consumers' beliefs are not affected by announced prices can be justified by the fact that consumers do not understand the (informationally complex) process of price formation implemented by the auctioneer.

In the model at hand, the substitutability parameter γ can be manipulated by firms via their investment decision into differentiation. An example is the investment of Coca Cola and Pepsi into advertisement in order to emphasize the differences between the two products, although their taste is indistinguishable for consumers (see Brander and Spencer, 2015a,b). Of course there are cases in which firms invest into making their products complementary to each other, which however is not modeled in our setup. Our model is a special case of a specification using the utility function $U(x^A, x^B, x^0) = (\alpha+q)x^A + (\alpha-q)x^B - 0.5\left[\beta(x^A)^2 + 2\gamma x^A x^B + \beta(x^B)^2\right] + x^0$ with $\beta = 1$. In the more general model, goods may also be perfect complements, i.e. $\gamma = -1$. Cases where the goods are (or can become) complements are left out here, as our focus are situations like the Coke-Vs-Pepsi example mentioned above.

Note that firms do not know the quality (differential) when making their differentiation investments and choosing their quantities, which is plausible in the case of experience goods, for instance. A strand of recent literature deals with the implications of unknown quality to both, buyers and sellers, see Szentes and Roesler (2017).

Kreps and Scheinkman (1983) employ a model with a stage where both firms choose capacities before competing in prices, and which under certain assumptions leads to "standard" Cournot outcomes This framework of the "justification" of Cournot models can not be applied here, because introducing demand uncertainty in a capacity-then-price-competition model leads to non-existence of a pure strategy equilibrium, in particular also to the absence of the equilibrium with Cournot quantities (see for example Hviid, 1991 and Behrens and Lijesen, 2012). As the case of homogeneous goods is nested in our model, the same applies here.

4.3 Solving the Model with Quantity Competition

We first replicate and adapt the results for Bertrand competition of Caminal and Vives (1996) for our Cournot framework. Then we analyze the effect of consumer learning on product differentiation in this framework.

4.3.1 Consumers

Optimal behavior of the rational consumer $i \in [0,1]$ implies that his demand in period $t \in \{1,2\}$ is given by

$$\begin{aligned} x_t^{A,i} &= \frac{\alpha}{1+\gamma} + \frac{s_t^i}{1-\gamma} - \frac{p^A}{1-\gamma^2} + \frac{\gamma p^B}{1-\gamma^2}, \\ x_t^{B,i} &= \frac{\alpha}{1+\gamma} - \frac{s_t^i}{1-\gamma} - \frac{p^B}{1-\gamma^2} + \frac{\gamma p^A}{1-\gamma^2}, \\ x_t^{0,i} &= m - p_A \cdot x^{A,i} - p_A \cdot x^{B,i}. \end{aligned} \tag{4.2}$$

Aggregating the rational consumers' demands via $x_t^j := \int_0^1 x_t^{j,i} di,\ j \in \{A,B\}$, and inverting yields the total inverse demand for the product of firm $j \in \{A,B\}$ generated by rational consumers. Combining the demand of rational consumers and random shoppers leads to the following aggregated inverse demand functions

$$\begin{aligned} p_t^A &= \alpha + \eta_t - x_t^A - \gamma x_t^B + u_t^A, \\ p_t^B &= \alpha - \eta_t - x_t^B - \gamma x_t^A + u_t^B. \end{aligned} \tag{4.3}$$

In order to fully characterize the optimal behavior of consumers in each period, we now need to calculate how they use the available information to update their beliefs about the goods' qualities, captured by η_t in aggregate terms, and can then use the aggregated inverse demand calculated in Equation (4.3). We exploit the properties of the normal distribution, in particular the fact that the updating rules for both, mean and variance, are linear (see e.g. Section 2.2.2 of Chamley, 2004). Details on the calculations can be found in Appendix 4.A.1.

First Period Early adopter i's belief about the quality is given by $\eta_1^i := E[q|I_1^i] = m_1 s_1^i$ where $m_1 := \frac{\tau_\epsilon}{\tau_\epsilon+\tau_q}$ weighs the precisions of the distributions of the signal the consumers receive against the precision of the quality. The aggregate expectation in the first period can be calculated as

$$\eta_1 = \int_0^1 \eta_1^i di = m_1 q.$$

As $\text{Var}(\eta_1) < \text{Var}(q)$, some uncertainty is resolved in the aggregate and in aggregate the consumers' belief is closer to the true value of q than the unconditional expectation $E[q]$. In equilibrium, beliefs are correct, such that the consumers' first period equilibrium belief is $\eta_1^* = m_1 q$.

The utility maximization of the rational consumers and the behavior of the random shoppers results in the following inverse demand

$$\begin{aligned} p_1^A &= \alpha + m_1 q - x_1^A - \gamma x_1^B + u_1^A, \\ p_1^B &= \alpha - m_1 q - x_1^B - \gamma x_1^A + u_1^B. \end{aligned}$$

Second Period In addition to the signal and the quantities x_2, the laggard i's information set now also contains the observed price difference in period 1, i.e., her information set is $I_2^i = \{s_2^i, \Delta p_1, x_2\}$. The price differential Δp_1 contains information about q:

$$\begin{aligned} &\Delta p_1 := p_1^A - p_1^B = 2m_1 q - (1-\gamma)\Delta x_1 + \Delta u \\ &\Leftrightarrow q = [\Delta p_1 + (1-\gamma)\Delta x_1 - \Delta u_1]/2m_1. \end{aligned}$$

As Δx_1 is not observed by consumers in the second period and $E[\Delta u_1] = 0$, a laggard's best estimate of the quality is $q^e = [\Delta p_1 + (1-\gamma)\Delta x_1^e]/2m_1$, where Δx_1^e is the conjectured difference in quantities. Because actual first period quantities are not observed by consumers in the second period, they have to make conjectures about them, that is, they need to interpret past prices as signals of the chosen quantities, which is formalized by Δx_1^e. Thus, q^e is obtained by solving the observed price difference Δp_1 for q and replacing the unknown variables from the perspective of the consumer by their expected value (Δu_1) and the conjecture about the played strategy (Δx_1). Inserting the realized price difference Δp_1, it equals

$$q^e = q + [\Delta u_1 - (1-\gamma)(\Delta x_1 - \Delta x_1^e)]/2m_1. \tag{4.4}$$

This expression contains the two random variables q and Δu_1, and the second summand captures the error the consumers make in conjecturing past market shares.

The laggard now combines his observation extracted from the price difference with his signal using Bayesian updating, so that her belief is given by $\eta_2^i := E[q|I_2^i] = m_2 s_2^i + n_2 q^e$, with $m_2 = \tau_\epsilon/\tau_2^i$, $n_2 = 2\tau_u m_1^2/\tau_2^i$, and $\tau_2^i = \tau_\epsilon + \tau_q + 2\tau_u m_1^2$. The aggregate belief then is given by

$$\eta_2 = \int_0^1 \eta_2^i di = m_2 q + n_2 q^e.$$

As in equilibrium beliefs are correct we obtain the equilibrium belief $\eta_2^* = m_2 q + n_2 \tilde{q}$ with $\tilde{q} = q + \Delta u_1/2m_1$. Note that $\tilde{q}$ equals q^e with correctly conjectured first period quantities, i.e. with $\Delta x_1^e = \Delta x_1$.

Clearly,

$$\frac{\partial \eta_2}{\partial x_1^A} = -\frac{\partial \eta_2}{\partial x_1^B} = n_2 \cdot \frac{\partial q^e}{\partial x_1^A} = -\frac{(1-\gamma)n_2}{2m_1}. \tag{4.5}$$

The derivative shows that the effect of a change in the first period quantity on the consumer belief in the second period is higher, the smaller the substitutability between products, γ. Phrased differently, the more differentiated the goods are, the higher is the impact of a firm's change of its choice variable in the first period on the laggards' belief. The decreased price p_1^A induced by a higher quantity x_1^A decreases the belief that product A is of superior quality because first period quantities are not observed by laggards, so that consumers can not be certain whether the price decrease was due to a low quality product, or due to a high volume of sales. This reasoning is analogous to the recommendation effect introduced in Chapter 3.

Similarly as in period one, the utility maximization of the rational consumers and the behavior of the random shoppers results in the following inverse demand in period two:

$$p_2^A = \alpha + m_2 q + n_2 q^e - x_2^A - \gamma x_2^B + u_2^A,$$
$$p_2^B = \alpha - m_2 q + n_2 q^e - x_2^B - \gamma x_2^A + u_2^B.$$

4.3.2 Firms

Firm behavior is analyzed via backward induction, but we start with analyzing the firms' information processing.

Bayesian Updating

In order to optimally set quantities, firms need to forecast the consumers' beliefs on quality, so they form a belief about the consumers' (aggregate) belief on quality denoted by $\theta_t := E[\eta_t|I_t]$. Both firms have identical information, so that they cannot manipulate each other.

In period 1, the firms do not have any information about the consumers' belief, and thus $\theta_1 = E[m_1 q|I_1] = 0$.

In period 2, in contrast to the consumers, the firms can extract $\tilde{q}$ (the consumers' second period estimate of the quality extracted from the price difference in the previous period with correctly conjectured quantities) from past prices and quantities, so that $\theta_2 = E[m_2 q + n_2 q^e|I_2] = m_2 E[q|\tilde{q}] + n_2 E[q^e|\tilde{q}]$. Equilibrium beliefs are satisfy $\theta_t^* = E[\eta_t^*|I_t]$.

Optimal Quantities

Given the beliefs about consumers' beliefs, firms choose optimal quantities. Details on the calculations can be found in Appendix 4.A.2. In the second period, firms take the differentiation parameter and first period quantities and prices as given so that their optimization problem boils down to maximizing the profit $\pi_2^j = x_2^j \cdot p_2^j(x_2)$ by the choice of x_2^j. Best responses are given by $x_2^A(x_2^B) = \frac{\alpha+\theta_2-\gamma x_2^B}{2}$ and $x_2^B(x_2^A) = \frac{\alpha-\theta_2-\gamma x_2^A}{2}$. Equilibrium quantities are

$$x_2^{A*} = \frac{\alpha}{2+\gamma} + \frac{\theta_2^*}{2-\gamma},$$
$$x_2^{B*} = \frac{\alpha}{2+\gamma} - \frac{\theta_2^*}{2-\gamma}.$$

In the first period, firms take into account the indirect effect of their quantity choice via Bayesian updating among the consumers on the profit in period 2. Thus, the objective function of firm A is given by

$$\pi_1^A(x_1) = x_1^A(\alpha + \theta_1 - x_1^A - \gamma x_1^B) + \delta E[\pi_2^A|I_1], \tag{4.6}$$

where π_2^A is firm A's second period profit and π_1^A is the total revenue of firm A, that is the profit from periods one and two, ignoring potential investments in differentiation.

Remember that $\theta_1 = 0$. Furthermore, note that $E[p_2^A|I_1] = \frac{\alpha}{2+\gamma} + \frac{\theta_2}{2-\gamma} = x_2^{A*}$, which implies that $E[\pi_2^A|I_1] = E[(x_2^{A*})^2|I_1]$. Firm A's best response is then $x_1^A(x_1^B) = \alpha/2 - x_1^B\gamma/2 + \frac{\delta x_2^{A*}}{2-\gamma} \cdot \frac{\partial\theta_2}{\partial x_1^A}$. Firm B's best response can be calculated analogously, and using Equation (4.5) shows that equilibrium quantities are

$$x_1^{A*} = x_1^{B*} = \frac{\alpha}{2+\gamma} \cdot \left(1 + \frac{2\delta}{4-\gamma^2}\frac{\partial\theta_2}{\partial x_1^A}\right).$$

Overall, we obtain the following result, which is analogous to the proposition for Bertrand competition in Caminal and Vives (1996).

Lemma 4.1. *In the equilibrium of our model, optimal quantities in period* 2 *are given by*

$$x_2^{A*} = \frac{\alpha}{2+\gamma} + \frac{\theta_2^*}{2-\gamma} \quad \text{and} \quad x_2^{B*} = \frac{\alpha}{2+\gamma} - \frac{\theta_2^*}{2-\gamma}. \tag{4.7}$$

Optimal quantities in period 1 are given by

$$x_1^{A*} = x_1^{B*} = \frac{\alpha}{2+\gamma} \cdot \left(1 - \frac{\delta(1-\gamma)}{4-\gamma^2} \cdot \frac{n_2}{m_1}\right). \tag{4.8}$$

The optimal second-period quantity of firm j is higher (lower) than in a standard differentiated Cournot model ($\alpha/(2+\gamma)$), if the expectation of the consumer belief is (not) in favor of firm j.[4] That is, the firm which is expected to be preferred by consumers sells a higher quantity.

As $\gamma \leq 1$, first period quantities are (weakly) lower than those without consumer learning, meaning that first period prices exceed those of the standard differentiated Cournot model. This is due to the fact that consumers in period 2 only observe past prices but not quantities. A higher price thus leads them to expect the good to be of higher quality.

Optimal Differentiation Investments

Forecasting the resulting optimal quantities, firms choose the investment into differentiation in period zero. There exist no closed-form solutions to derive the optimal investment in differentiation, k^{j*}, and furthermore, conventional comparative static tools such as the implicit function theorem or approaches via lattice theory involve calculations, which are too computationally complex. Thus, to compare the differentiation incentives without relying on the full solution, we use the technique of Brander and Spencer (2015a), who compare the minimal effectiveness of investments in differentia-

[4]Equilibrium quantities are positive whenever $n_2/m_1 < 4$, which is always fulfilled.

tion needed to induce firms to invest, that is, we derive and compare the thresholds λ so that firms investments become positive.

Without Consumer Learning (Benchmark) If second period consumers were to have a belief of $\eta_2 = 0$, the model in the second period is the same as the standard model of Dixit (1979), and the resulting optimal quantities would be $x_{NL}^{A*} = x_{NL}^{B*} = \alpha/(2+\gamma)$. The profit of a benchmark model with two periods without consumer learning is thus given by

$$\pi_{NL}^{j} = (1+\delta) \cdot E[\pi_2^j | I_1] - k^j = (1+\delta) \cdot (x_{NL}^{j*})^2 - k^j.$$

The derivative of the objective function is then

$$\begin{aligned} \partial \pi_{NL}^{j} / \partial k^j &= (1+\delta) \cdot \frac{d(x_{NL}^{j*})^2}{d\gamma} \cdot \frac{\partial \gamma}{\partial k^j} - 1 = (1+\delta) \cdot \frac{-2\alpha^2}{(2+\gamma)^3} \cdot (-\lambda\gamma) - 1 \\ &= (1+\delta) \cdot \frac{2\lambda\gamma\alpha^2}{(2+\gamma)^3} - 1. \end{aligned}$$

Firm j will invest in differentiation in equilibrium if

$$\partial \pi_{NL}^{j} / \partial k^j \big|_{\gamma=1} > 0 \Leftrightarrow \lambda > \frac{27}{2(1+\delta)\alpha^2} := \bar{\lambda}_{NL}^{C}.$$

The threshold without learning can also be obtained as a corollary of Proposition 4 from Brander and Spencer (2015a) by extending their model to two periods. The threshold decreases in α, as the positive effect of increased differentiation on profit is higher the higher α, so that the necessary technology ($\bar{\lambda}$) is decreasing in α. Additionally, differentiation incentives are stronger, as δ increases. This is because the gain from differentiation is higher than the costs compared to a situation with a lower δ.

With Consumer Learning With $\pi_L^j(\cdot) := \pi_1^j(\cdot) - k_j$, $j \in \{A, B\}$ using Equation (4.6) and the results mentioned thereafter, implies that the derivative of the objective function is given by

$$\partial \pi_L^j(x^*) / \partial k^j = \frac{\partial \left[x_1^{j*} \{ \alpha - (1+\gamma) x_1^{j*} \} \right]}{\partial k^j} + \delta \cdot \frac{d(x_2^{j*})^2}{d\gamma} \cdot \frac{\partial \gamma}{\partial k^j} - 1.$$

Firm j will invest into differentiation in equilibrium if

$$\begin{aligned} \partial\pi_L^j/\partial k^j\big|_{\gamma=1} &= \frac{\alpha^2(n_2\delta+2m_1(1+\delta))\lambda}{27m_1} - 1 > 0 \\ \Leftrightarrow \qquad \lambda &> \frac{27m_1}{\alpha^2(2m_1+2\delta m_1+\delta n_2)} := \bar{\lambda}_L^C. \end{aligned}$$

We can easily see that $\bar{\lambda}_L^C < \bar{\lambda}_{NL}^C$, as

$$\bar{\lambda}_L^C = \bar{\lambda}_{NL}^C \frac{2\alpha^2(1+\delta)}{2\alpha^2(1+\delta)+\alpha^2\delta n_2/m_1} < \bar{\lambda}_{NL}^C,$$

which leads to our first main result:

Proposition 4.1. *In the Cournot model, in equilibrium firms offer perfect substitutes for a smaller range of parameters λ with consumer learning than without. That is, the threshold λ above which firms invest in differentiation is lower with consumer learning than without, $\bar{\lambda}_L^C < \bar{\lambda}_{NL}^C$.*

The comparative statics of threshold $\bar{\lambda}_L$ with respect to α and δ are the same as those of threshold $\bar{\lambda}_{NL}^C$, but the extent of the changes on the threshold induced by changes in α and δ now depends on the parameters introduced by consumer learning, m_1 and n_2. Furthermore, as $\frac{\partial n_2}{\partial \tau_u} > 0$ and $\frac{\partial \bar{\lambda}_L}{\partial n_2} < 0$, the critical value $\bar{\lambda}_L$ decreases in τ_u, $\frac{\partial \bar{\lambda}_L}{\partial \tau_u} < 0$. Intuitively, the more noise caused by the random shoppers is contained in the observed statistic about the quality, the smaller are the incentives to differentiate.

4.4 Solving the Model with Price Competition

As shown by Singh and Vives (1984), with a linear quadratic utility function as given in Equation (4.1), there is a close relationship between Cournot and Bertrand competition. In their words:

> *Cournot (Bertrand) competition with substitutes is the dual of Bertrand (Cournot) competition with complements. This means that they share similar strategic properties. For example, with linear demand, reaction functions slope downwards (upwards) in both cases. It is a matter of interchanging prices and quantities. (Singh and Vives, 1984, p. 547)*

Indeed, using the following utility function from Caminal and Vives (1996) which slightly differs from the one of the previous sections,

$$U^B(x^A,x^B,x^0) = (\alpha+(1-\gamma)q)x^A + (\alpha-(1-\gamma)q)x^B - 0.5\left[(x^A)^2+2\gamma x^A x^B+(x^B)^2\right] + x^0,$$

and including random shoppers similarly to the model before, we obtain the following set of demand functions

$$\begin{aligned} x_t^A &= a + \eta_t - bp_t^A + cp_t^B + u_t^A, \\ x_t^B &= a - \eta_t - bp_t^B + cp_t^A + u_t^B, \end{aligned} \tag{4.9}$$

where $a = \alpha/(1+\gamma)$, $b = 1/(1-\gamma^2)$ and $c = \gamma/(1-\gamma^2)$. Variable η_t again is the aggregate consumers' belief about the quality in period t. Besides the slightly different utility function and induced demands, all variables remain as in the previous section.

Comparing the above system of direct demands in Equation (4.9) to the inverse demand system from Equation (4.3), we can obtain one from the other by simply exchanging quantities and prices and replacing a by α, b by $\beta = 1$, and c by $-\gamma$.

Using Equation (4.5), this implies that the recommendation effect in the Bertrand model is formalized by

$$\frac{\partial \eta_2}{\partial p_1^A} = -\frac{\partial \eta_2}{\partial p_1^B} = n_2 \cdot \frac{\partial q^e}{\partial p_1^A} = -\frac{n_2}{2(1-\gamma)m_1}. \tag{4.10}$$

This shows that, in Bertrand competition, the parameter of substitution (γ), has the inverse impact on the magnitude of the recommendation effect compared to Cournot competition.

Additionally, using the above shortcut, we know from Lemma 4.1 that the equilibrium prices in this setting are given by the following lemma.

Lemma 4.2 (Caminal and Vives, 1996)**.** *In the equilibrium of the model with price setting, optimal prices in period* 2 *are given by*

$$p_2^{A*} = \frac{a}{2b-c} + \frac{\theta_2^*}{2b+c} \quad \textit{and} \quad p_2^{B*} = \frac{a}{2b-c} - \frac{\theta_2^*}{2b+c}. \tag{4.11}$$

Optimal prices in period 1 are given by

$$p_1^{A*} = p_1^{B*} = \frac{a}{2b-c} \cdot \left(1 - \frac{(b+c)b\delta}{4b^2-c^2} \cdot \frac{n_2}{m_1}\right). \tag{4.12}$$

Without learning, optimal prices of both firms are calculated as $p_{NL}^{A*} = p_{NL}^{B*} = \frac{a}{2b-c}$. We see that the firm with the higher perceived quality charges a higher price, and in the first period both firms charge a lower price than in a model without learning.

As in the previous section, we can use the equilibrium prices to calculate equilibrium profits for a fixed γ and solve the derivative of the profit with respect to the investment k_j evaluated at $\gamma = 1$ for the threshold λ above which firms make their investments in differentiation. Details on the calculations can be found in Appendix 4.A.3. We obtain the following result:

Proposition 4.2. *In the Bertrand model, in equilibrium firms offer perfect substitutes for a smaller range of parameters λ without consumer learning than with consumer learning. That is, the threshold λ above which firms invest in differentiation with consumer learning ($\bar{\lambda}_L^B$) is higher than the threshold without learning ($\bar{\lambda}_{NL}^B$):*

$$\bar{\lambda}_L^B = \frac{2}{\alpha^2\big[(1+\delta) - 2\delta n_2/(3m_1)\big]} > \frac{2}{\alpha^2(1+\delta)} = \bar{\lambda}_{NL}^B.$$

4.5 Informational Incentives to Differentiate: Bertrand Vs. Cournot

While we should keep in mind, that the parameter of substitution (γ) is incorporated in a different manner in the microfoundation of the Bertrand and the Cournot model,[5] it is nevertheless worthwhile to compare the influence of consumer learning on the incentives to differentiate of the two models. Comparing our findings in the different models yields our final main result:

Proposition 4.3. *The effect of consumer learning on the firms' incentives to differentiate their products is different in the Cournot model and in the Bertrand model. In contrast to quantity competition, consumer learning in a model with price setting decreases the firms' incentives to differentiate:*

$$\bar{\lambda}_L^B - \bar{\lambda}_{NL}^B > 0 > \bar{\lambda}_L^C - \bar{\lambda}_{NL}^C.$$

Consumer learning thus tends to increase the competition in the Bertrand setting, and it weakens it in the Cournot model.

In order to understand this result in more detail, it is useful to compare the equilibrium choices from the models with learning to those without. From the perspective of period zero, where firms choose their differentiation investments, and given the equilibrium strategies for periods one and two, the expected optimal quantities in period 2 are the same in the models with and without learning, as $E(\theta_2) = 0$. Thus, the second period affects the differentiation incentives only through its influence on the optimal

[5] The different utility functions in the two models are employed, as they allow to compare the impact of γ on the aggregated (inverse) demand systems in Equations (4.3) and (4.9) more easily.

first period choices of the firms. In the Cournot game, equilibrium quantities in the first period are given by

$$x_1^{j*} = x_{NL}^{j*} \cdot \left(1 - \frac{\delta(1-\gamma)}{4-\gamma^2} \cdot \frac{n_2}{m_1}\right) = \frac{\alpha}{2+\gamma} \cdot \left(1 - \frac{\delta(1-\gamma)}{4-\gamma^2} \cdot \frac{n_2}{m_1}\right),$$

and equilibrium prices in the Bertrand model are

$$\begin{aligned} p_1^{j*} = p_{NL}^{j*} \cdot \left(1 - \frac{(b+c)b\delta}{4b^2-c^2} \cdot \frac{n_2}{m_1}\right) \qquad &= \frac{a}{2b-c} \cdot \left(1 - \frac{(b+c)b\delta}{4b^2-c^2} \cdot \frac{n_2}{m_1}\right) \\ = \frac{\alpha(1-\gamma)}{2-\gamma} \cdot \left(1 - \frac{(1+\gamma)\delta}{4-\gamma^2} \cdot \frac{n_2}{m_1}\right) \end{aligned}$$

for j $\in \{A, B\}$. In both models, the first factor, gives the optimal choices in a model without consumer learning. The second factor in both cases is (weakly) smaller than one, so that quantities in a Cournot style competition and prices in the Bertrand variant of our model are (weakly) decreased by the introduction of consumer learning. Only if $\gamma = 1$, the optimal choices in the models with learning and those without learning coincide.

Starting with the Cournot model and comparing the marginal profit of investing in differentiation (increasing k^A or k^B) in a situation where $\gamma = 1$ ($k^A = k^B = 0$), the previous calculations showed that the marginal profit is higher with learning than without, leading to the lower threshold value in the model with learning compared to the model without.

The mechanisms behind this difference are as follows. In the Cournot model without learning, and in any similar model, the two firms could increase their first period profit by reducing their quantities. In the model without learning, decreasing one's quantity below $x_{NL}^{A*} = x_{NL}^{B*}$ is not individually rational. If $\gamma < 1$ consumer learning however introduces an incentive to decrease first period quantities below the level of a model without learning due to the recommendation effect, meaning that at $\gamma = 1$, consumer learning generates an additional incentive to invest in differentiation, as this enhances the impact of the recommendation effect.

The situation in the Bertrand setup is different in that prices are already too low in the model without learning if the goal is to maximize the joint first period profit of the firms. Firms could therefore increase their profits if they were to jointly raise their prices. With $\gamma = 1$, prices in the model with and without learning coincide and equal zero. Decreasing γ, that is increasing the differentiation, increases the optimal first period price, but the increase is smaller with consumer learning. The marginal profit of increasing $k^A = k^B (= 0)$, is thus higher in the model without learning than it is in the model with learning, explaining the ordering of the thresholds in this setup.

Finally, we can elaborate on the result of Brander and Spencer (2015a,b), who showed that firms are more likely to invest in differentiation in Bertrand than in Cournot competition. As our benchmark models without learning are two-period extensions of their models, we obtain the same result if we compare the models without learning, that is $\bar{\lambda}^C_{NL} > \bar{\lambda}^B_{NL}$. Consumer learning has been shown to decrease the threshold in the Cournot model and to increase it in the Bertrand setting, but even then, the ranking of the two models is maintained, i.e., we also have $\bar{\lambda}^C_L > \bar{\lambda}^B_L$.

4.6 Conclusion

Differentiating one's product from those of a competitor results in a weaker competition and thus allows for higher prices and profits. By introducing consumer learning in a duopoly model with vertically differentiated goods and by endogenizing the horizontal differentiation between the products, we have shown that the incentives to differentiate are changed when consumer learning about the size of the vertical differentiation, i.e., the difference in quality, is introduced. Furthermore, the effects created by consumer learning differ between a model of quantity and a model of price competition.

In each of the two models, consumers learn from observed previous purchasing decisions. As their observations are not fully revealing all information, firms can manipulate the inference of late consumers by influencing the purchase decisions of early consumers. When setting their prices or quantities in early periods, firms take this effect of their choices on the inference of later consumers into account. Only when the two products are perfect substitutes ($\gamma = 1$), the presence of consumer learning does not change the firms optimal behavior compared to a model without consumer learning.

For quantity competition with differentiated products, firms optimally choose lower quantities in a model with consumer learning than in a model without. Low quantities lead to higher prices which tend to signal higher quality to later consumers. If firms compete in prices, optimal first period prices are below those of a model without consumer learning, as here higher sold quantities signal high quality to later consumers.

These "distortions" of the optimal choices in early periods lead to different effects on the differentiation incentives of the firms induced by consumer learning. Because profits in a Cournot model can typically be increased by reducing the produced quantities, which is precisely the effect consumer learning has in our model of quantity competition, consumer learning increases the incentives to differentiate above the incentives generated by the desire to relax competition. The reverse is true in a model of Bertrand style price competition: here increasing prices would increase the profit of the firms, but consumer learning reduces the prices even further than the already strong competition in a Bertrand setup. The introduction of consumer learning thus decreases the incentives to invest in differentiation if firms compete in prices.

The presented results seem to support the notion that price competition, i.e., a game with strategic complements, leads to a stronger competition than quantity competition, that is competition with strategic substitutes, as, in the setting discussed here, products are more likely to be substitutes (in the equilibrium) in the former oligopoly model.

4.A Appendix: Proofs

4.A.1 Bayesian Updating Among Consumers

Bayes' rule in the context at hand can be formulated as

$$f(q|o) = \frac{\phi(o|q) \cdot f(q)}{\int \phi(o|q) \cdot f(q)dq},$$

where $f(\cdot)$ is the density of q and $\phi(\cdot)$ is the density of some observation o containing information on quality q, i.e., in our case signal s_i^t or the estimate of q extracted from the price difference in the first period, q^e. Gaussian models as the one at hand (i.e., Bayesian updating over normally distributed random variables and observations), are particularly tractable, as the posterior distribution is also normal and the updating rules for mean and variance are linear: the posterior mean is the weighted average of the prior mean and that of the observation weighted with the respective precisions, while the posterior variance is that of the prior increased by that of the observation.

In our model, consumers want to best estimate q from their observations. Consumers have the prior knowledge that $q \sim N\left(\mu_q, \frac{1}{\tau_q}\right)$ and they make one or two additional observations o_r, $r \in \{1, 2\}$, with information about q. All consumers receive a signal about q, and consumers in period two additionally observe past prices. Both, the signal and the information extracted from past prices can be reformulated to observation $o_{r,t}^i$ of consumer i in period t in the following form:

$$o_{r,t}^i = q + v_{r,t}^i \quad \text{where} \quad v_{r,t}^i \sim N\left(0, \frac{1}{\tau_{v_{r,t}^i}}\right)$$

Using Bayesian updating as described above, this leads to the following distribution of q conditional on the available observations, for $t \in \{1, 2\}$

$$q|I_t^i \sim N\left(\frac{\tau_q \mu_q + \sum_{r=1}^t \tau_{v_{r,t}^i} o_{r,t}^i}{\tau_q + \sum_{r=1}^t \tau_{v_{r,t}^i}}, \frac{1}{\tau_q + \sum_{r=1}^t \tau_{v_{r,t}^i}}\right)$$

Let $\eta_1^i := E[q|I_1^i]$ be the updated belief of consumer i about q in period 1 after receiving signal s_1^i, then

$$\eta_1^i \sim N\Big((1-m_1)\cdot 0 + m_1 \cdot s_1^i, \tfrac{1}{\tau_q+\tau_\varepsilon}\Big).$$

with $m_1 := \frac{\tau_\varepsilon}{\tau_q+\tau_\varepsilon}$. Using the assumption on the average signal, the aggregate belief is given by

$$\eta_1 := \int_0^1 \eta_1^i di = \int_0^1 m_1 s_1^i di = m_1 \int_0^1 s_1^i di \to m_1 q.$$

The information a consumer i in period 2 can extract about q from the observed price difference only is given by

$$\begin{aligned} q^e &= [\Delta p_1 + (1-\gamma)\Delta x_1^e]/2m_1 \\ &= q + [\Delta u_1 - (1-\gamma)(\Delta x_1 - \Delta x_1^e)]/2m_1. \end{aligned} \tag{4.13}$$

This expression contains the two random variables $q \sim N(0, \frac{1}{\tau_q})$ and $\frac{\Delta u_1}{2m_1} \sim N(0, \frac{2}{4m_1^2\tau_u})$. When combining this with the signal, $\eta_2^i := E[q|I_2^i]$, the updated belief of consumer i about q in period 2, is normally distributed with

$$\eta_2^i \sim N\Big((1-m_2-n_2)\cdot 0 + m_2 s_2^i + n_2 q^e, \tfrac{1}{\tau_2^i}\Big),$$

with $\tau_2^i = \tau_\epsilon + \tau_q + 2\tau_u m_1^2$, $m_2 = \tau_\epsilon/\tau_2^i$, and $n_2 = 2\tau_u m_1^2/\tau_2^i$. Thus, the aggregate belief is given by

$$\eta_2 := \int_0^1 \eta_2^i di = \int_0^1 (m_2 s_2^i + n_2 q^e) di = m_2 \int_0^1 s_2^i di + n_2 q^e \to m_2 q + n_2 q^e,$$

again making use of the assumption on the average signal.

4.A.2 Firm Behavior in the Cournot Model

Firm behavior is analyzed via backward induction.

Quantity Setting in Stage $t = 2$

Firm A's profit in stage $t = 2$ is given by

$$\pi_2^A = x_2^A \cdot p_2^A = x_2^A \cdot (\alpha + \theta_2 - x_2^A - \gamma x_2^B).$$

Best responses are obtained by the FOCs $\partial\pi_2^j/\partial x_2^j = 0$ with $j \in \{A, B\}$, which gives

$$x_2^A(x_2^B) = \frac{\alpha + \theta_2 - \gamma x_2^B}{2}, \qquad \text{and similarly} \qquad x_2^B(x_2^A) = \frac{\alpha - \theta_2 - \gamma x_2^B}{2}.$$

In equilibrium best responses intersect, so we obtain the equilibrium quantities

$$x_2^{A*} = \frac{\alpha}{2+\gamma} + \frac{\theta_2}{2-\gamma} \qquad \text{and} \qquad x_2^{B*} = \frac{\alpha}{2+\gamma} - \frac{\theta_2}{2-\gamma}.$$

Quantity Setting in Stage $t = 1$

Firm A's expected profit considered in stage $t = 1$ is given by

$$\pi_1^A = x_1^A p_1^A + \delta E[\pi_2^A | I_1] = x_1^A \cdot (\alpha + \theta_1 - x_1^A - \gamma x_1^B) + \delta E[p_2^A x_2^A | I_1].$$

In period 1, firms anticipate the equilibrium quantities from period 2, so that $E[p_2^A(x_2^*)|I_1] = \frac{\alpha}{2+\gamma} + \frac{\theta_2}{2-\gamma} = x_2^{A*}$, and thus $E[\pi_2^A|I_1] = (x_2^{A*})^2$. We can additionally use the observations that $\theta_1 = E[\theta_1^*|I_1] = 0$ and

$$\partial\theta_2/\partial x_1^A = -\partial\theta_2/\partial x_1^B = \partial\eta_2/\partial x_1^A = (\gamma - 1)\tfrac{n_2}{2m_1}.$$

Using $\partial E[\pi_2^A|I_1]/\partial x_1^A = 2x_2^A \cdot \frac{1}{2-\gamma} \cdot \frac{\partial\theta_2}{\partial x_1^A}$, we obtain the FOC of firm A, given by

$$\frac{\partial\pi_1^A}{\partial x_1^A} = \alpha - 2x_1^A - \gamma x_1^B + \delta\left(\frac{2x_2^A}{2-\gamma} \cdot \frac{\partial\theta_2}{\partial x_1^A}\right) = 0.$$

This yields the best responses

$$x_1^A(x_1^B) = \frac{\alpha}{2} - \frac{\gamma x_1^B}{2} + \delta\left(\frac{2x_2^A}{2-\gamma} \cdot \frac{\partial\theta_2}{\partial x_1^A}\right),$$

and similarly

$$x_1^B(x_1^A) = \frac{\alpha}{2} - \frac{\gamma x_1^A}{2} + \delta\left(\frac{2x_2^B}{2-\gamma} \cdot \frac{\partial\theta_2}{\partial x_1^A}\right).$$

In equilibrium best responses intersect, and using $E[\theta_2^*|I_1] = m_1 E[q] + n_2 E[\tilde{q}] = 0$ the equilibrium quantities are given by

$$x_1^{A*} = x_1^{B*} = \frac{\alpha}{2+\gamma}\left(1 + \frac{\delta(\gamma-1)}{4-\gamma^2} \cdot \frac{n_2}{m_1}\right).$$

Differentiation Investment in Stage $t = 0$

It holds that for $j \in \{A, B\}$

$$\frac{d(x_{NL}^{j*})^2}{d\gamma} \cdot \frac{\partial \gamma}{\partial k^j}\bigg|_{\gamma=1} = \frac{d(x_2^{j*})^2}{d\gamma} \cdot \frac{\partial \gamma}{\partial k^j}\bigg|_{\gamma=1} = \frac{2\delta\lambda\alpha^2}{27}.$$

Further helpful results for the calculation of the model with consumer learning are

$$\begin{aligned}\frac{dx_1^{j*}}{d\gamma} \cdot \frac{\partial \gamma}{\partial k^j} = \Bigg[&\frac{-\alpha}{(2+\gamma)^2} \\ &+ \frac{\alpha\delta n_2}{m_1}\left\{\frac{(2+\gamma)^2(2-\gamma) - (\gamma-1)[2(2+\gamma)(2-\gamma) - (2+\gamma)^2]}{(2+\gamma)^4(2-\gamma)^2}\right\}\Bigg] \cdot (-\lambda\gamma)\end{aligned}$$

and

$$\frac{\partial[\alpha - (1+\gamma)x_1^{j*}]}{\partial k^j} = \lambda\gamma x_1^{j*} - (1+\gamma)\left[\frac{dx_1^{j*}}{d\gamma} \cdot \frac{\partial \gamma}{\partial k^j}\right].$$

Evaluating at $\gamma = 1$, using the sum rule in differentiation and the above results, we obtain

$$\begin{aligned}\frac{\partial\{x_1^{j*} \cdot [\alpha - (1+\gamma)x_1^{j*}]\}}{\partial k^j}\bigg|_{\gamma=1} &= \frac{\lambda\alpha^2}{27}\left(1 - \frac{\delta n_2}{m_1}\right) + \frac{\lambda\alpha^2}{9} \cdot \left(1 - \frac{2}{3}\left\{1 - \frac{\delta n_2}{m_1}\right\}\right) \\ &= \frac{\lambda\alpha^2}{9} \cdot \left(\frac{2}{3} + \frac{\delta n_2}{3m_1}\right).\end{aligned}$$

Overall, the derivative of profit w.r.t. investment in differentiation, evaluated at $\gamma = 1$, is

$$\begin{aligned}\partial\pi_L^j/\partial k^j\big|_{\gamma=1} &= \left(\frac{\partial\left[x_1^{j*}\{\alpha - (1+\gamma)x_1^{j*}\}\right]}{\partial k^j} + \delta \cdot \frac{d(x_2^{j*})^2}{d\gamma} \cdot \frac{\partial \gamma}{\partial k^j} - 1\right)\bigg|_{\gamma=1} \\ &= \frac{\lambda\alpha^2}{9} \cdot \left(\frac{2}{3} + \frac{\delta n_2}{3m_1} + \frac{2\delta}{3}\right) - 1.\end{aligned}$$

4.A.3 Firm Behavior in the Bertrand Model

The calculations on the price setting in stages one and two of the Bertrand model can be done analogously to the quantity setting in the Cournot model (see Appendix 4.A.2), and thus we will only calculate the optimizing behavior for differentiation investment in stage $t = 0$. In the following, the profit functions Π_k^j represent the same profits as in the Cournot model, only adapted to the Bertrand setting.

Differentiation Investment Without Consumer Learning (Benchmark)

No consumer learning implies $\theta_2 = 0$, such that the resulting optimal quantities and prices for firm A are

$$E[x_2^A(p_2^{A*})|I_1] = \frac{\alpha}{(2-\gamma)(1+\gamma)},$$
$$E[p_2^{A*}|I_1] = \frac{\alpha(1-\gamma)}{(2-\gamma)}.$$

The profit of firm A in a benchmark model with two periods without consumer learning is thus given by

$$\Pi_{NL}^j = (1+\delta)\cdot E[\Pi_2^A|I_1] - k^A = (1+\delta)E[x_2^A(p_2^{A*})|I_1]\cdot E[p_2^{A*}|I_1] - k^A.$$

The derivative of the objective function is then

$$\begin{aligned}\partial\Pi_{NL}^j/\partial k^j &= (1+\delta)\cdot\frac{d(E[x_2^A(p_2^{A*})|I_1]\cdot E[p_2^{A*}|I_1])}{d\gamma}\cdot\frac{\partial\gamma}{\partial k^j} - 1\\ &= (1+\delta)\cdot\frac{-\alpha^2(\gamma^3-3\gamma^2+4)+(3\gamma^2-6\gamma)(\alpha^2(1-\gamma))}{(\gamma^3-3\gamma^2+4)^2}\cdot(-\lambda\gamma) - 1.\end{aligned}$$

Firm A will invest in differentiation in equilibrium if

$$\partial\Pi_{NL}^j/\partial k^j\big|_{\gamma=1} > 0 \Leftrightarrow \lambda > \frac{2}{\alpha^2(1+\delta)} = \bar{\lambda}_{NL}^B.$$

Differentiation Investment With Consumer Learning

We can write

$$\begin{aligned}p_1^{A*} &= \frac{a}{2b-c}\cdot\left(1-\frac{(b+c)b\delta}{4b^2-c^2}\cdot\frac{n_2}{m_1}\right)\\ &= \frac{\alpha(1-\gamma)}{2-\gamma}\cdot\left(1-\frac{(1+\gamma)\delta}{4-\gamma^2}\cdot\frac{n_2}{m_1}\right)\\ &= \frac{\alpha(1-\gamma)}{2-\gamma}-\frac{(1-\gamma^2)\delta\alpha}{(4-\gamma^2)(2-\gamma)}\cdot\frac{n_2}{m_1}.\end{aligned}$$

The profit of firm A in the model with consumer learning is given by

$$\begin{aligned}\Pi_L^j &= p_1^{A*}\cdot\left[a+(c-b)p_1^{A*}\right] + \delta\cdot E[x_2^A(p_2^{A*})|I_1]\cdot E[p_2^{A*}|I_1] - k^A\\ &= p_1^{A*}\cdot\left[\frac{\alpha}{1+\gamma}+\frac{\gamma-1}{1-\gamma^2}\cdot p_1^{A*}\right] + \delta\cdot E[x_2^A(p_2^{A*})|I_1]\cdot E[p_2^{A*}|I_1] - k^A.\end{aligned}$$

Further helpful results are

$$\frac{\partial p_1^{A*}}{\partial \gamma} = \frac{-\alpha(2-\gamma)+\alpha(1-\gamma)}{(2-\gamma)^2} - \frac{(-2\gamma\delta\alpha)(2-\gamma)(4-\gamma^2)-(3\gamma^2-4\gamma-4)\alpha\delta(1-\gamma^2)}{(2-\gamma)^2(4-\gamma^2)^2},$$

$$\begin{aligned}
\frac{\partial\left[\frac{\alpha}{1+\gamma}+\frac{\gamma-1}{1-\gamma^2}\cdot p_1^{A*}\right]}{\partial k^A} &= -\alpha + \frac{(1-\gamma^2)+2\gamma(\gamma-1)}{(1-\gamma^2)^2}\cdot p_1^{A*} + \frac{\gamma-1}{1-\gamma^2}\cdot\frac{\partial p_1^{A*}}{\partial\gamma},\\
\left[\frac{\alpha}{1+\gamma}+\frac{\gamma-1}{1-\gamma^2}\cdot p_1^{A*}\right]\Bigg|_{\gamma=1} &= \alpha/2,\\
p_1^{A*}\Big|_{\gamma=1} &= 0,\\
\frac{\partial p_1^{A*}}{\partial\gamma}\Bigg|_{\gamma=1} &= -\alpha + \frac{2\alpha\delta n_2}{3m_1}.
\end{aligned}$$

Overall, the derivative of profit w.r.t. investment in differentiation, evaluated at $\gamma = 1$, is

$$\begin{aligned}
\partial\Pi_L^j/\partial k^j\big|_{\gamma=1} &= \alpha\lambda\left(1-\frac{2\delta n_2}{3m_1}\right)\cdot\frac{\alpha}{2} + 0\cdot\frac{\partial\left[\frac{\alpha}{1+\gamma}+\frac{\gamma-1}{1-\gamma^2}\cdot p_1^{A*}\right]}{\partial k^A}(-\lambda) + \delta\lambda\frac{\alpha^2}{2} - 1\\
&= \left[\frac{\alpha^2}{2}(1+\delta) - \frac{\alpha^2\delta n_2}{3m_1}\right]\lambda - 1.
\end{aligned}$$

Firm j will invest in differentiation in equilibrium if

$$\begin{aligned}
& \partial\Pi_L^j/\partial k^j\big|_{\gamma=1} > 0\\
\Leftrightarrow\quad & \lambda > \frac{2}{\alpha^2\left[(1+\delta)-2\delta n_2/(3m_1)\right]} := \bar{\lambda}_L^B.
\end{aligned}$$

Part II

Evolution and Cooperative Games

Chapter 5

Stability in Replicator Dynamics Derived from Transferable Utility Games[1]

We propose an approach to derive a population dynamic from an underlying cooperative transferable utility game. To our knowledge, this combination of evolutionary and cooperative game theory is a methodological novelty. Examining the stable points of the dynamical system, we obtain several intuitive results. Our main result says that a coalition of player types is stable if and only if it implies a higher average productivity than any of its super- or subcoalitions. For instance, in the class of simple monotonic games, only minimal winning coalitions can be stable. Moreover, we can make statements about which player types will vanish and which ones will persist in stable states. Possible applications are the analysis of coalition formation, the population constitution of eusocial species, or the organizational structure in businesses.

5.1 Introduction

In non-cooperative game theory, the evolutionary approach has long been used to model the behavior of boundedly rational agents and to analyze the relation between the stable outcomes of the dynamical process generated by their repeated interaction and the common static solution concept of Nash equilibrium, see Weibull (1995) and Sandholm (2010), for instance. In contrast to non-cooperative game theory, cooperative game theory shifts the focus from modeling strategic interactions and the agents' strategic reasoning to payoffs: it asks questions about which payoff distribution among the players of a given cooperative game can, for instance, be considered as fair or stable, see Peleg and Sudhölter (2007), for instance.

[1] This chapter is joint work with André Casajus and Harald Wiese.

Our approach is to introduce the evolutionary methodology to cooperative game theory in order to answer questions on the relation between the properties of the underlying cooperative game, which also determines the attributes of (coalitions of) players, and the stable outcomes of the dynamical evolutionary process. In non-cooperative game theory evolutionary pressures work against strategies, for instance, dominated strategies may die out (see, e.g., Hofbauer and Weibull, 1996), while in our approach these pressures work against certain types of coalitions, like, for instance, non-minimal winning coalitions, or against certain player types, like null players or dominated players (see Section 5.5). That is, in our setup evolution takes place over player types and not over strategies. We can thus answer questions on the connection between asymptotically stable coalitions and the underlying transferable utility (TU) game, as well as what characterizes aymptotically stable coalitions or which player types survive the evolutionary process.

Aside the new methodological approach, a contribution of our research is to model the dynamic process of coalition formation. The common interpretation of the Shapley value is that it distributes the worth of the grand coalition among all players in the cooperative game according to their individual marginal contributions to each possible coalition. This ex-ante perspective is implausible in many settings: it often seems more suitable to assume that - once a coalition is formed - its structure plays a crucial role for how wealth is allocated among members of society. Thus, it is important to examine the process of coalition formation and the stability of its outcomes, as it has been done, for instance, by Hart and Kurz (1983) or Ray and Vohra (1999).[2] Moreover, in view of the fact that coalitions often are an outcome of an "evolutionary" process, the dynamic nature of the model is a desirable feature.

The work at hand is related to the concept of dynamic cooperative games introduced by Filar and Petrosjan (2000). They define a sequence of games so that one TU game is determined by the previous one and by the payoffs achieved under some solution concept. The players obtain the sum of payoffs for this sequence of coalition functions and the problem of whether the payoffs satisfy a consistency criterion is discussed.

Nash (2008) also adresses the issue of a "cooperative evolutionary game theory", but takes a completely different approach, as he models the formation of cooperative coalitions by allowing non-cooperative players to decide repeatedly whether or not to completely follow another players agency, i.e., to be cooperative.

In contrast to the finite population models of Nax (2015) and Nax and Pradelski (2015, 2016) we assume an infinite population of agents of different player types of the coalitional game. To our knowledge we are the first to derive an evolutionary dynamic from a cooperative game in a similar fashion as it is done in non-cooperative evolutionary

[2]Ray and Vohra (2015) survey game theoretic approaches to coalition formation.

game theory. By this we mean that there is a fraction of the population (consisting of infinitely many agents) that belong to a certain player type of the underlying TU game. This is similar to a fraction of the population in non-cooperative evolutionary game theory playing one of the strategies of the underlying non-cooperative game.

The rest of this chapter is structured as follows. In Section 5.2 we use a simple example to illustrate our results. In Section 5.3 we build up the framework in which coalitions of players generate worth, and in Section 5.4 we introduce the evolutionary model. Section 5.5 contains results on the asymptotic stability. Section 5.6 finally concludes.

5.2 Illustrative Example

Triadic coalitions - i.e., situations in which two players can unite against one other player, all three players can unite, or each of them can stay autonomous - have long been at the center of attention in the study of coalition formation, see Mesterton-Gibbons et al. (2011). Consider, for instance, the cooperative game defined by the coalition function $v(\cdot)$ for the player set $N = \{1, 2, 3\}$ as defined by Table 5.1.

$v(1) = 3$	$v(1,2) = 4$	$v(N) = 7$
$v(2) = 2$	$v(1,3) = 5$	$v(\emptyset) = 0$
$v(3) = 1$	$v(2,3) = 7$	

Table 5.1: A three-player cooperative game with transferable utility

The core of this game is empty, so that an important concept of stability does not yield any predictions on the realization of the final allocation. The Shapley value, the main fairness-related concept in cooperative game theory, yields the payoffs $Sh(v, N) = \left(\frac{12}{6}, \frac{15}{6}, \frac{15}{6}\right)$ for players one, two, and three, respectively.[3] A frequently discussed assumption of the Shapley value is the symmetry of players, i.e., equally productive players obtain the same payoff. This implies that, in the underlying calculations, symmetric players are not distinguished when considering coalition formation. However, certain political parties - although symmetric on a payoff basis - might be inclined to a different degree to form a coalition with a third party, for instance, due to historical reasons. Additionally, the Shapley value satisfies the axiom of efficiency, that is, the worth to be distributed is that generated by the grand coalition. This implies that the grand coalition is actually formed. These assumptions seem to be restrictive and unrealistic in many settings.

[3] We maintain this order of players when using vector notation in what follows.

We propose a framework in which only the worth generated by actually formed coalitions is redistributed. We can interpret the players of the above example as investors, each having a certain endowment of time (e.g., working hours per day) at their disposal, depicted by the vector $s = \left(\frac{2}{6}, \frac{1}{6}, \frac{3}{6}\right)$, where the first entry denotes the (absolute) amount of time of the first investor, and so on. This implies each investor can invest one sixth of a working hour per day into the project "grand coalition" to generate worth. Investors 1 and 3 have enough time left to form a coalition, i.e., to cooperate in a project consisting only of them both, and finally one sixth of a working hour is left for investor 3 to produce worth alone. Thus, for the initial investment state s and the coalition function v defined by Table 5.1 the overall worth generated by the society is $\bar{v}(s) = \frac{1}{6} \cdot 7 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 1$.[4]

The generated worth is distributed according to the Lovász-Shapley value $LS(s, v)$, which has plausible axiomatic properties in the setup at hand, see Casajus and Wiese (2016). The value pays each investor according to his marginal contribution to the coalition in which players are ordered with respect to their agent population sizes, starting with the largest population, i.e., here they are ordered according to $\rho(s) = (3, 1, 2)$.[5] The calculated marginal productivity is then multiplied with the initial endowment of the investor. Thus, we obtain $LS(s, v) = \left(\frac{8}{6}, \frac{2}{6}, \frac{3}{6}\right)$.

If we assume that the Lovász-Shapley value equals the growth of the investors' disposable working hours in each state (e.g., they might hire new employees with the income generated by the executed projects), then we obtain the following dynamic depicting the evolution of the investors' relative shares in the working time available for society:

[4] The term $\bar{v}$ is the Lovász-extension of the game (v, N). Details on this follow below. The described way of forming coalitions to create worth is not efficient for society and may also not be individually rational. It can however be justified by the fact that investors are boundedly rational, which is the common intuition for evolutionary models. We further discuss this issue in Section 5.6.

[5] Intuitively, investor 2's time is most scarce in the initial state s and thus his investment is in some way "most crucial" to form the grand coalition, so that he should obtain a payoff proportional to his marginal contribution to the grand coalition, $MC_2^v(3, 1, 2) = 2$. Next, investor 1's time is most crucial to form the coalition of investors 1 and 3, so he should obtain a payoff proportional to $MC_1^v(3, 1, 2) = 4$.

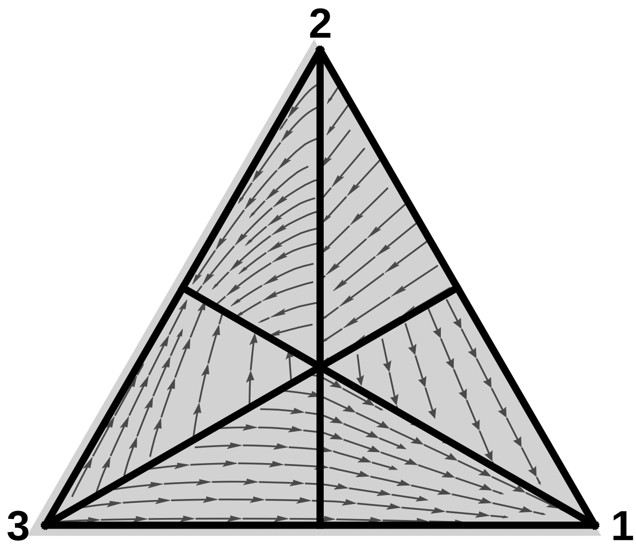


Figure 5.1: Vector field for the TU game described in Table 5.1 assuming that the growth of disposable working time (population sizes) equals the Lovász-Shapley value. The figure depicts the dynamics “in relative terms”, i.e., each point in the simplex determines the (relative) population shares (or working-hour shares relative to the time worked in total) of the three different investors. For instance, the vertex labeled “1” depicts a state in which only investor 1 has a positive population share.

There exist two asymptotically stable distributions of wealth: one characterized by the coalition of investors 2 and 3, and one is the singleton containing only investor 1. These coalitions have in common that they yield a higher average payoff for the investors than each of its sub- or supercoalitions, which is one main result of this chapter.

An alternative interpretation of our framework suits the usual evolutionary interpretation very well: society consists of infinitely many agents (vector s) of the distinct player types or casts (the three different investors). Our model then determines which are the evolutionary stable population profiles of the player types for a given initial population state.

5.3 Populations of Players Generating Worth

In this section, we first provide the foundation of cooperative game theory needed for what follows. Then, we present the Lovász-Shapley value, which will be used to derive the fitness of type populations from cooperative games.

5.3.1 Basic Definitions and Notation

A (finite) cooperative game with transferable utility (TU game) for a non-empty and finite set of players N is given by a coalition function $v \in \mathbb{V} := \left\{ f \mid f : 2^N \to \mathbb{R},\ f(\emptyset) = 0 \right\}$. The latter describes the worths $v(S)$ that can be generated by individuals who cooperate within coalitions $S \subseteq N$. A solution is a mapping $\varphi : \mathbb{V} \to \mathbb{R}^N$, which assigns a payoff $\varphi_i(v)$ to any player $i \in N$ for any game $v \in \mathbb{V}$.

For $v, w \in \mathbb{V}$, $\alpha \in \mathbb{R}$, the coalition functions $v + w \in \mathbb{V}$ and $\alpha \cdot v \in \mathbb{V}$ are given by $(v + w)(S) = v(S) + w(S)$ and $(\alpha \cdot v)(S) = \alpha \cdot v(S)$ for all $S \subseteq N$. The game $\mathbf{0} \in \mathbb{V}$ given by $\mathbf{0}(S) = 0$ for all $S \subseteq N$ is called the **null game**. For $T \subseteq N$, $T \neq \emptyset$, the game $u_T \in \mathbb{V}$, $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise, is called an **unanimity game**. Any $v \in \mathbb{V}$ can uniquely be represented by unanimity games, i.e.,

$$v = \sum_{T \subseteq N : T \neq \emptyset} \lambda_T(v) \cdot u_T, \tag{5.1}$$

where the coefficients $\lambda_T(v)$, the so-called Harsanyi dividends (Harsanyi, 1959), can be determined recursively via

$$v(S) = \sum_{T \subseteq S : T \neq \emptyset} \lambda_T(v), \qquad S \subseteq N. \tag{5.2}$$

Player $i \in N$ is called a **null player** in v iff $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$. Players $i, j \in N$ are called **symmetric** in v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.

A **rank order** of a set N is a bijection $\rho : N \to \{1, \ldots, |N|\}$ with the interpretation that i is the $\rho(i)$th player in ρ. The set of rank orders of N is denoted by $R(N)$. The **marginal contribution** of i to $S \subseteq N \setminus \{i\}$ is denoted by

$$MC_i^v(S) := v(S \cup \{i\}) - v(S) \overset{(5.2)}{=} \sum_{T \subseteq S \setminus \{i\}} \lambda_{T \cup \{i\}}(v), \tag{5.3}$$

and the marginal contribution of i under ρ is denoted by

$$MC_i^v(\rho) := v(\{j : \rho(j) \leq \rho(i)\}) - v(\{j : \rho(j) < \rho(i)\}). \tag{5.4}$$

The **Shapley value** (Shapley, 1953), Sh, is given by

$$\mathrm{Sh}_i(v) := |R(N)|^{-1} \cdot \sum_{\rho \in R(N)} MC_i^v(\rho) = \sum_{T \subseteq N : i \in T} |T|^{-1} \cdot \lambda_T(v), \qquad v \in \mathbb{V},\ i \in N. \tag{5.5}$$

We add some further defintions needed for our analysis in Section 5.5. Let $S, T \subseteq N$. Game (v, N) is a **simple game** iff for all S we have $v(S) \in \{0,1\}$ and $v(N) = 1$, and it is a **monotonic game** iff we have $v(S) \leq v(T)$ for all $S \subseteq T$. Coalition T is a **minimal winning coalition** in a simple game iff we have $v(T) = 1$ and $v(S) = 0$ for all $S \subsetneq T$. For $i \in N$ with $|N| \geq 2$, the **apex game** (v, N) is defined by $v(S) = 1$ if $i \in S \wedge S \backslash \{i\} \neq \emptyset$, $v(S) = 1$ if $S = N \backslash \{i\}$, $v(S) = 0$ otherwise. Player i is called the apex player. Player i **strictly dominates** player j in v iff $v(K \cup \{i\}) > v(K \cup \{j\})$ holds for all $K \subseteq N \backslash \{i, j\}$. Player i **weakly dominates** player j in v iff $v(K \cup \{i\}) \geq v(K \cup \{j\})$ holds for all $K \subseteq N \backslash \{i, j\}$ and there is a coalition $\hat{K} \subseteq N \backslash \{i, j\}$ such that $v(\hat{K} \cup \{i\}) > v(\hat{K} \cup \{j\})$.

5.3.2 The Lovász-Shapley Value and the Fitness of Populations

Casajus and Wiese (2016) suggest to interpret the players in a TU game as **types of agents**. The **population sizes** of the types are given by a vector $s \in \mathbb{R}_+^N$ of non-negative weights, where s_i denotes the size of the population (of agents) of type $i \in N$. We address pairs $(v, s) \in \mathbb{V} \times \mathbb{R}_+^N$ as **population games**. A **population solution** is a mapping $\varphi : \mathbb{V} \times \mathbb{R}_+^{\mathrm{N}} \to \mathbb{R}^N$, which assigns a payoff $\varphi_i(v, s)$ to any population of type $i \in N$ for any population game $(v, s) \in \mathbb{V}$.

Specifically, Casajus and Wiese (2016, Theorem 1) advocate a particular population solution called the **Lovász-Shapley value**, LS. It is given by

$$\mathrm{LS}_i(v, s) := |R(s)|^{-1} \cdot \sum_{\rho \in R(s)} s_i \cdot MC_i^v(\rho), \qquad \text{for all } v \in \mathbb{V},\ s \in \mathbb{R}_+^N, \text{ and } i \in N, \tag{5.6}$$

where

$$R(s) := \{\rho \in R(N) \mid \rho(i) < \rho(j) \text{ for all } i, j \in N \text{ with } s_i > s_j\} \text{ for all } s \in \mathbb{R}_+^N. \tag{5.7}$$

Note that $R(s)$ contains those rank orders for which types with a greater population size preceed types with a smaller population size. This definition allows for states with equal population sizes, and, as a consequence, for such population states $R(s)$ is not a singleton, but rather a set consisting of all rank orders that order the types with *different* population sizes accordingly.

Alternatively, the Lovász-Shapley value can be expressed in terms of the Harsanyi dividends (Casajus and Wiese, 2016, Equation 13). In particular, we have

$$\mathrm{LS}_i(v, s) = s_i \cdot \sum_{T \subseteq N : i \in \operatorname{argmin}_T(s)} \frac{\lambda_T(v)}{|\operatorname{argmin}_T(s)|}, \qquad \text{for all } v \in \mathbb{V},\ s \in \mathbb{R}_+^N, \text{ and } i \in N, \tag{5.8}$$

where

$$\min_T(s) := \min_{i \in T} s_i \quad \text{and} \qquad \text{argmin}_T(s) := \{i \in T \mid s_i = \min_T(s)\} \qquad \text{for all } s \in \mathbb{R}_+^N.$$

Equations (5.6) and (5.8) indicate a number of crucial properties of the Lovász-Shapley value. For a more detailed account and motivation of these, we refer to Casajus and Wiese (2016, Section 3).

(i) From Equation (5.8), one can infer that the fitness generated with respect to the dividend $\lambda_T(v)$ is restricted by the smallest population from T, i.e., the smallest population involved in the creation of this dividend. This indicates that the types generate fitness via a Leontief-type technology. In particular, the total fitness in the population game (v, s) amounts to the worth generated by the population state represented by s under the **Lovász extension**[6]

$$\bar{v}(v,s) := \sum_{l=1}^{q(s)} (\bar{s}_l - \bar{s}_{l-1}) \cdot v(\{i \in N | s_i \geq \bar{s}_l,\ l \in \{1, ..., q(s)\}\}) \qquad \forall s \in \mathbb{R}_+^N \quad (5.9)$$

of the game v (see Casajus and Wiese (2016), Lovász (1983) and Algaba et al. (2004) for details). This Leontief-type process of generating worth is described in the introductory example of Section 5.2.

(ii) Further, Equations (5.6) and (5.8) imply that type populations of the same size that are equally productive with respect to the underlying game generate the same amount of fitness. The types i and j are equally productive in the game v whenever they are symmetric in v, that is, whenever their marginal contributions to coalitions not containing them coincide, i.e., we have $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ for all $S \subseteq N \setminus \{i, j\}$.

(iii) Equation (5.6) entails that, for given population sizes as in s, a type i's fitness does not decrease whenever this type's productivity in the underlying game does not decrease. A type i's productivity does not decrease from game v to game w whenever this type's marginal contributions to coalitions not containing it does not decrease, i.e., we have $v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \setminus \{i\}$.[7]

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[6]Let $q(s) := |\{s_i | i \in N\}|$ and mapping $\bar{s} : \{1, ..., q(s)\} \to \{s_i | i \in N\}$ s.t. $\forall k, l \in \{1, ..., q(s)\}$ with $k > l$ this implies $\bar{s}(k) > \bar{s}(l)$, i.e., the mapping orders types in the reverse order of $\rho \in R(s)$. Set $\bar{s}_l := \bar{s}(l)$ and $\bar{s}_0 := 0$.

[7]Note, that in their Remark 5, Casajus and Wiese (2016) mention that in the axiomatization of the Lovász-Shapley value, one can replace strong monotonicity by additivity in the game and the null player property for population games.

(iv) In view of Equation (5.8), the fitness generated with respect to the dividend $\lambda_T(v)$ is only attributed to the scarce types from T. This can be interpreted as a kind of competitive remuneration of types, i.e., remuneration according to their marginal contribution to fitness driven by the pressure to increase the total fitness of the population. This fits, for example, the interpretation that types represent the casts of a eusocial species[8] as the European hornet (Vespa crabro) or the naked mole rat (Heterocephalus glaber).

For generic[9] weight vectors, that is $s_i \neq s_j$ for all $i, j \in N, i \neq j$, the Lovász-Shapley value can be expressed in a particularly simple way (Casajus and Wiese, 2016, Remark 5). From Equation (5.6) it immediately follows that for all $\rho \in R$, $v \in \mathbb{V}$, $i \in N$, we have

$$\mathrm{LS}_i(v, s) = s_i \cdot MC_i^v(\rho) \qquad \text{for all } s \in \mathbb{R}_+^N(\rho), \tag{5.10}$$

where for each $\rho \in R(N)$ we define the set of population states

$$\mathbb{R}_+^N(\rho) := \left\{ s \in \mathbb{R}_+^N \mid \text{for all } i, j \in N,\ \rho(i) < \rho(j) \text{ implies } s_i > s_j \right\}. \tag{5.11}$$

Note that $\mathbb{R}_+^N(\rho)$ does not contain population states in which any two player types $i, j \in N, i \neq j$ have an equal population size. This implies, that $R(s)$ is a singleton if $s \in \mathbb{R}_+^N(\rho)$. Equation (5.10) has important implications. We interpret the term $s_i \cdot MC_i^v(\rho)$ as the fitness of the population of type i. It is obtained by multiplying the fitness of a single agent, $MC_i^v(\rho)$, by the population size s_i. Now observe that the fitness of a single agent does depend on the population size only with respect to the (strict) order of the population sizes. This fact will prove to be very useful later on, when we study the stability of population states.

5.4 A Framework for Evolutionary Cooperative Game Theory

The general idea is to establish an evolutionary setup in which the population dynamics are determined by an underlying TU game (N, v), where $i \in N$ is one of the player-types from the finite and non-empty type set N, and the coalition function $v \in \mathbb{V}$ determines the worth generated by any set of players $S \subseteq N$. As the population dynamics are

[8]A common "criterion for eusociality is the presence of castes, which are groups of individuals that become irreversibly behaviorally distinct at some point prior to reproductive maturity. Eusocial societies are characterized by two traits: (1) helping by individuals of the less-reproductive caste, and (2) either behavioral totipotency of only the more reproductive caste (facultative eusociality) or totipotency of neither caste (obligate eusociality)", see Crespi and Yanega (1995).

[9]Throughout the chapter we use a measure-theoretic definition for a generic property, i.e., a generic property holds almost everywhere except on a set of measure zero.

evolving in a continuous space of the types' population sizes, we use the Lovász-Shapley value as a plausible concept to determine the fitness of a type in each population state $s \in \mathbb{R}_+^N$ from the underlying TU game (N, v).

5.4.1 Replicator Dynamics Derived From a TU Game

If in total a society in population state s generates worth according to the Lovász extension $\bar{v}$ of TU game (v, N), one may view $\bar{v}(s)$ as the total additional fitness of the society to be distributed among its members. We can thus interpret $LS_i(s)$ as the growth of type i's (absolute) population size in state s, that is, the population dynamic is generally described by

$$\dot{s}_i = LS_i(v, s), \qquad \forall s \in \mathbb{R}_+^N, \tag{5.12}$$

where throughout we use the notation $\dot{f} := \partial f / \partial t$ for the first derivative of function f with respect to (continuous) time t. We can reformulate these dynamics in relative terms, as the vector field of the population size dynamics of s is positive homogeneous in s:[10]

$$\dot{x}_i = g_i(x), \qquad \forall x \in \Delta_+^N, \tag{DYN}$$

where $x_i := s_i / \sum_{j \in N} s_j$ is the population share of type i, that is $x \in \Delta_+^N := \{x \in \mathbb{R}_+^N | \sum_{i \in N} x_i = 1\}$. The real valued, measurable and bounded function $g_i(x)$ in Equation (DYN) is implicitly defined by Equation (5.12) and will be discussed explicitly further below. Note that the right-hand side of Equation (5.12) is discontinuous (indicated by the minimum operator in Equation (5.8)), which obviously implies that also $g_i(x)$ is discontinuous.

Equation (5.12) implies that for generic population states, i.e., $s \in \mathbb{R}_+^N(\rho)$, by Equation (5.10) we obtain the population dynamics in absolute terms for some given $\rho \in R(N)$ by

$$\dot{s}_i = s_i \cdot MC_i^v(\rho), \qquad \forall s \in \mathbb{R}_+^N(\rho), i \in N. \tag{5.13}$$

Therefore, the growth rate of types i's population size is $\dot{s}_i / s_i = MC_i^v(\rho)$ for $\rho \in R(s)$. Equation (5.13) defines an autonomous, linear and homogeneous dynamical system and the local solution $\chi : I \to \mathbb{R}_+^N$ to the initial value problem $\dot{s} = s \cdot MC^v(\rho)$ with the initial value condition $s(0) = s_0 \in \mathbb{R}_+^N(\rho)$ is given by $\chi(t) = exp[MC^v(\rho) \cdot t] \cdot s_0$, where $exp[\cdot]$ denotes the matrix exponential, $MC^v(\rho)$ denotes the vector whose entries are the

[10]See Remark 3 in Casajus and Wiese (2016) on the positive homogeneity of the Lovász-Shapley value in the population states.

marginal contributions of all types under ρ in the game $v(\cdot)$, and $I := [0,T]$ denotes the (time) interval of definition.

Looking at this setup in relative terms yields the common replicator dynamics (see e.g. Taylor and Jonker, 1978), which we focus on in the following. For $\rho \in R(N)$ and $x \in \Delta_+^N(\rho)$ it follows from Equation (5.13) and (DYN) that

$$\dot{x}_i = g_i(x) := x_i \cdot \left(MC_i^v(\rho) - \overline{MC}^v(\rho, x) \right), \quad \forall i \in N, \tag{RD}$$

where the weighted average of the population fitness is given by $\overline{MC}^v(\rho, x) := \sum_{j \in N} x_j \cdot MC_j^v(\rho)$, and $\Delta_+^N(\rho) := \Delta_+^N \cap \mathbb{R}_+^N(\rho)$. Hence, the growth rate of type i's population share, $\dot{x}_i / x_i$, is given by the difference of its marginal contribution and the weighted average marginal contribution in population state x. Note that $\Delta_+^N(\rho)$ is a region in Δ_+^N in which population shares are *strictly* ordered according to $\rho \in R(N)$.

5.4.2 The Dynamics as a Differential Inclusion and Its Filippov Solution

As mentioned above, Equation (DYN) generally describes a system with a discontinuous right-hand side, so that we use the specific techniques for the analysis of such setups. Fixing $\rho \in R(N)$, from the definition of $MC_i^v(\rho)$ in Equation (5.4) we can see that this fitness indicator is constant on $\Delta_+^N(\rho)$ for each type $i \in N$. This obviously simplifies the behavior of the solution to Equation (DYN) for $x \in \Delta_+^N(\rho), \rho \in R(N)$, which is described by Equation (RD): the function $g_i(x)$ is Lipschitz continuous in each region $\Delta_+^N(\rho)$, which guarantees the existence and uniqueness of the solution according to the Picard-Lindelöf theorem (see de la Fuente (2000, pp. 433)), when "looking at each region $\Delta_+^N(\rho), \rho \in R(N)$ separately".

For $J \subseteq N$ with $|J| \geq 2$, let

$$\Sigma_J := \{x \in \Delta_+^N | \; x_i = x_j, \; \forall i, j \in J \subseteq N\} \subset \Delta_+^N \tag{5.14}$$

be the set of population shares, where all players $j \in J$ have the same population share. For $|J| > 1$, the set

$$D := \cup_{J \subseteq N} \Sigma_J, \tag{5.15}$$

which is closed relative to Δ_+^N, then is a union of a finite number (of subsets) of hyperplanes between the (open) regions $\Delta_+^N(\rho), \rho \in R(N)$, and it is of measure zero relative to Δ_+^N.[11] While the replicator dynamic is Lipschitz continuous on

$$C := \Delta_+^N \backslash D = \cup_{\rho \in R(N)} \Delta_+^N(\rho), \tag{5.16}$$

the Lovázs-Shapley value, which determines the behavior of Equation (DYN), has discontinuity points on D, so that below we describe the dynamics on the complete simplex $\Delta_+^N = C \cup D$ as a differential inclusion.

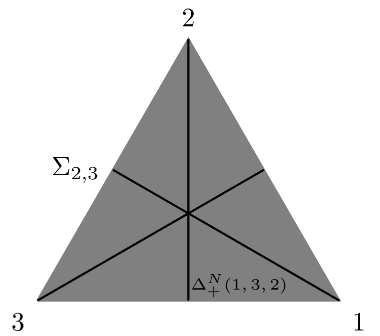


Figure 5.2: Areas of continuity, C (gray), and of discontinuity points, D (black), in the simplex Δ_+^3.

A system of differential equations with a discontinuous right-hand side as in Equation (DYN) can be adequately dealt with using a differential inclusion, which in our case yields the set valued vector field $G : \Delta_+^N \rightrightarrows \mathbb{R}^N$, so that for all $i \in N, \rho \in R(x)$

$$\dot{x}_i \in G_i(x) := con \left\{ \underset{\substack{(x_k) \in \Delta_+^N(\rho): \\ (x_k) \to x}}{lim} g_i(x_k) \right\} = \begin{cases} \{g_i(x)\} & \text{if } x \in C, \\ con \left\{ \underset{\substack{(x_k) \in \Delta_+^N(\rho): \\ (x_k) \to x}}{lim} g_i(x_k) \right\} & \text{if } x \in D, \end{cases} \tag{DI}$$

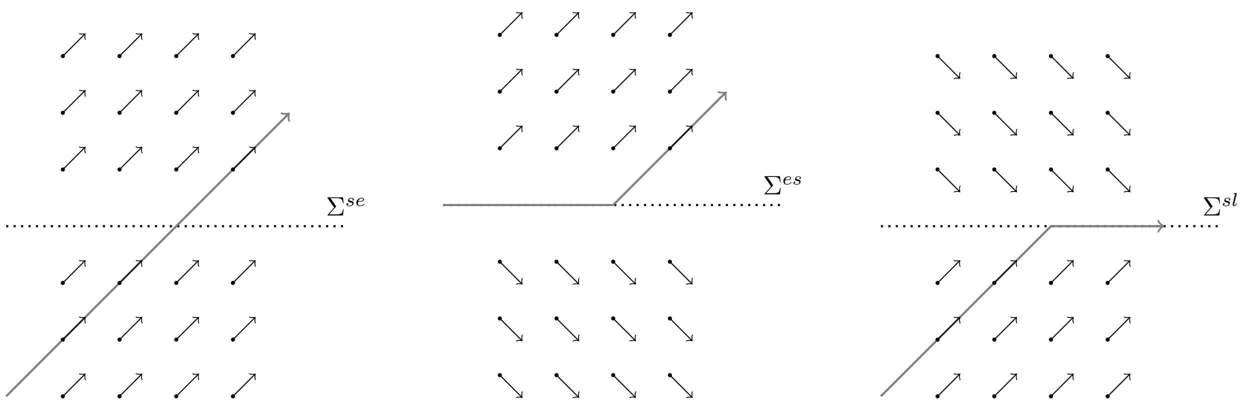
where $con\{p\}$ is defined as the convex hull of vectors in the set p and $(x_k) \to x$ means that the sequence (x_k) converges to x.[12] The above differential inclusion depicts a crucial insight, on which we further elaborate below: the collection of hyperplanes, D, being of measure zero can be interpreted as being "too small to matter for the overall vector field". This is reflected by the fact that in Equation (DI) the vector field on D is

[11] For the case where $|N| = 2$, we define $D := (\cup_{J \subseteq N} \Sigma_J) \cup \{x \in \Delta_+^N | x_i = 1 \text{ for some } i \in N\}$.
[12] Note that all points $x \in \Delta_+^N$ are regular, i.e., $\sum_i \dot{x}_i = 0$, even if $x \in D$, as the vector field $G_i(x)$ is determined by a convex combination of regular points.

determined by the points lying close to it in the adjacent regions. The Filippov solution (Filippov, 1988) uses this insight:[13]

Definition 5.1 (Filippov Solution). *A Filippov solution at x^* of the differential equation* (DYN) *with a discontinuous right-hand side is an absolutely continuous function $\chi : I \to \Delta_+^N$ for which $\chi(0) = x^*$ and with respect to the differential inclusion* (DI) *it satisfies $\chi(t) \in G(x)$ almost everywhere on some set $I := [0, T] \subseteq \mathbb{R}$.*

The trajectory of the Filippov solution to (DYN) is equal to that of the "usual" solution in the region C, where the vector field is Lipschitz continuous. The flow on the measure-zero region D can intuitively be distinguished according to three generic cases, which can be visualized for $|N| \in \{2, 3\}$: in the sewing mode, the vector field points towards the hyperplane $\Sigma^{se} \subseteq \Sigma_J$ on one side and it points away from $\Sigma^{se} \subseteq \Sigma_J$ on the other side, while, in the escaping mode, the vector field points away from $\Sigma^{es} \subseteq \Sigma_J$ on both sides. In the sliding mode, the vector field points towards $\Sigma^{sl} \subseteq \Sigma_J$ on both sides.



(a) Sewing mode (b) Escaping mode (c) Sliding mode

Figure 5.3: Different modes of the vector field with respect to the hyperplane (dotted) and Filippov solution (gray arrow).

As depicted in Figure 5.3, in the sewing mode, we can just appose the trajectories of the solutions of the adjacent Lipschitz continuous regions to obtain the trajectory of the Filippov solution. In the escaping mode, the solution to (DYN) may be non-unique, as we can not always determine at which point it will leave D. At a point in a sliding mode region, $x \in \Sigma^{sl}$, the vector field points into a direction which is lying in the convex hull of all flow vectors which are generated by approaching x in the limit and whose vector is tangent to Σ^{sl}. Thus, the Filippov solution will move along Σ^{sl}.

[13] These techniques have been applied to economic theory for instance in Honkapohja and Ito (1983), Ito (1979) and Mohlin (2012), while other solution concepts such as the Caratheodory solution to differential inclusions have been applied to economics for instance in Lahkar and Sandholm (2008).

5.5 Stability Analysis

First, we analyze where stable profiles lie in the simplex, and then we show how these profiles are related to the underlying TU game.

5.5.1 General Stability Results

Equation (RD) describes a system of non-linear, coupled first order ordinary differential equations in region C. Therefore, an exact solution is difficult to determine for the general case and we use the insights from Lyapunov theory to analyze the stability of the system.[14]

Definition 5.2 (Lyapunov function). *The C^1-function $L : \mathbb{R}^N \to \mathbb{R}$ is an (increasing) strict Lyapunov function for the differential equation $\dot{x} = g(x)$ if $\dot{L}(x) \geq 0$ for all $x \in \Delta_+^N$, with equality only at stationary points of $g(x)$.*

As it is commonly known, this implies that $L(\cdot)$ increases along all solutions and is positive semi-definite, see Sydsaeter et al. (2008, pp. 273). Remember that D is a set of hyperplanes of measure zero and thus the dynamics in this region are determined by the dynamics in the adjacent regions of continuity. The case of $MC_1^v(\rho) = MC_2^v(\rho) = ... = MC_N^v(\rho)$ for $\rho \in R(N)$ is non-generic and trivial, as each point $x \in \Delta_+^N(\rho)$ then is stationary. Thus, we leave it out in the discussions below.

Lemma 5.1. *A Lyapunov function of the dynamical system (DI) is given by*

$$L(x) := \sum_{i \in N} MC_i^v(\rho) x_i \quad for\ \rho \in R(x).$$

Proof. Using vector notation, due to the chain rule we have $\dot{L}(x) = \nabla L(x) \cdot \dot{x} = \sum_i MC_i^v(\rho)\dot{x}_i$. Now, take some state x, where $\dot{x} \neq 0$. Such a state generically exists. We can always divide the types into the two sets $N_+(x) := \{i \in N | \dot{x}_i > 0\}$ and

[14] For the case where $|N| = 2$ the dynamics in a region $\Delta_+^N(\rho)$ can be broken down to a Bernoulli differential equation and an exact solution can be found according to the common procedure used for such equations. For $|N| = 3$ the system resembles a competitive Lotka-Volterra equation for two species (due to $\dot{x}_1 + \dot{x}_2 + \dot{x}_3 = 0$ we have one degree of freedom). Abdelkader (1974) shows that an exact solution can be determined for a model equivalent to the three-type-case at hand, that is his "case 3".

$N_-(x) := \{i \in N | \dot{x}_i < 0\}$ according to their growth rate in state x. Rewriting the above condition yields

$$\begin{aligned}\sum_{i\in N} MC_i^v(\rho)\dot{x}_i &= \sum_{i\in N_+(x)} MC_i^v(\rho)\dot{x}_i + \sum_{i\in N_-(x)} MC_i^v(\rho)\dot{x}_i \\ &\geq \left[\min_{i\in N_+(x)} MC_i^v(\rho)\right] \sum_{j\in N_+(x)} \dot{x}_j + \left[\max_{i\in N_-(x)} MC_i^v(\rho)\right] \sum_{j\in N_-(x)} \dot{x}_j \\ &= \underbrace{\left(\min_{i\in N_+(x)} MC_i^v(\rho) - \max_{i\in N_-(x)} MC_i^v(\rho)\right)}_{>0} \underbrace{\sum_{j\in N_+(x)} \dot{x}_j}_{>0} > 0,\end{aligned}$$

using $-\sum_{i\in N_+(x)} \dot{x}_i = \sum_{i\in N_-(x)} \dot{x}_i$ in the last step, which holds, as the population states have to lie in the simplex. This implies that $L(\cdot)$ increases along all Filippov solutions to $\dot{x} = g(x)$ with the starting condition $x^0 \in \Delta_+^N(\rho)$. Define $\underline{0} \in \mathbb{R}^{|N|}$ to be a vector with all entries being zeros. Obviously, $\dot{L}(x) = 0$ if $\dot{x} = \underline{0}$.

Note, that $L(x)$ is single valued, even if $R(x)$ is not a singleton. To see this, fix $y \in \Delta_+^N$ to be such a population state where $R(y)$ is not a singleton for the rest of the proof. In this case, there exist $i, j \in N,\ i \neq j$, such that $y_i = y_j$. Assume for now that there exists exactly one group of players $Q \subseteq N$, such that for all $i, j \in Q,\ i \neq j$ it holds that $y_i = y_j =: z(y)$. Therefore, $z(y)$ is uniquely defined. If there would be several types with the same population share differing from $z(y)$, the same reasoning as below applies by adding further summands of types with the same population share in the summation below. Define sets of player types according to

$$\begin{aligned}Z(y) &= \{i \in N : y_i = z(y)\}, \\ \underline{Z}(y) &= \{i \in N : y_i < z(y)\}, \\ \overline{Z}(y) &= \{i \in N : y_i > z(y)\}.\end{aligned}$$

For all $\rho \in R(y)$ we can write

$$\sum_{i\in N} MC_i(\rho)y_i = \sum_{i\in \underline{Z}(y)} MC_i(\rho)y_i + \sum_{i\in Z(y)} MC_i(\rho)y_i + \sum_{i\in \overline{Z}(y)} MC_i(\rho)y_i.$$

Let the set of players weakly coming before i in ρ be denoted by

$$B_i(\rho) = \{j \in N : \rho(j) \leq \rho(i)\}.$$

In the expression above, the first and the third summation term on the right-hand side of the equality sign are constant for all $\rho, \rho' \in R(y)$. This is, because elements of $\underline{Z}(y)$

have the same position in ρ and ρ', and, in addition, the elements of $B_i(\rho)$ and $B_i(\rho')$ are the same for al $i \in \underline{Z}(y)$. The same reasoning applies for $i \in \overline{Z}(y)$. Note that $v(S)$ is independent of the rank order of the elements of S for all $S \subseteq N$. Define $\mathfrak{z}(y) := |Z(y)|$. For all $\rho \in R(y)$ we have

$$\sum_{i \in Z(y)} MC_i(\rho) y_i = z(y) \sum_{i \in Z(y)} MC_i(\rho) = z(y)\mathfrak{z}(y) \left[v\left(Z(y) \cup \underline{Z}(y)\right) - v\left(\underline{Z}(y)\right)\right],$$

where the last factor is equal for all $\rho \in R(y)$, as $v(S)$ is independent of the rank order. If $\underline{Z}(y)$ or $\overline{Z}(y)$ have elements with an equal population share, the same reasoning applies. □

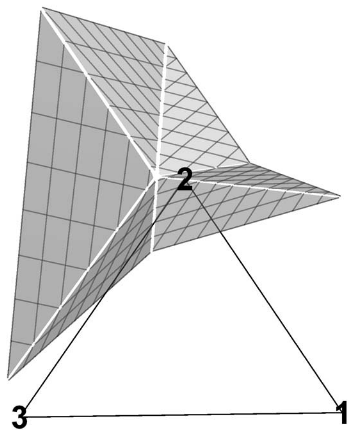


Figure 5.4: Region-wise Lyapunov functions for the illustrative example in Section 5.2. The white lines are not to be understood as areas, in which the function is not defined, but rather indicate the points in which the graph of the *continuous* function has kinks.

The Lyapunov function is a piecewise linear function on Δ_+^N, it is non-differentiable (but continuous) at $x \in D$, and it is an indicator of the fitness of the whole population, as it coincides with the Lovász extension. Intuitively, the evolutionary process will drive the population to states with larger attainable fitness for a given starting point.

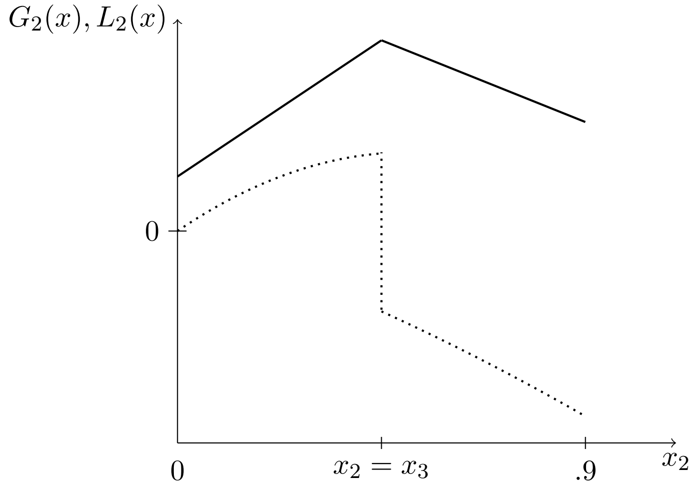


Figure 5.5: The graph of the growth of the population share, $G_2(x)$ [dotted], and $L_2(x)$ [straight] for the example in Section 5.2 (The population share of type 1 is fixed to $\bar{x}_1 = 0.1$).

Due to the possibility of the non-uniqueness of the Filippov solution in the escaping mode we need to use the adapted definition of asymptotic stability accordingly.

Definition 5.3 (Asymptotic Stability). *A population profile $x^* \in \Delta_+^N$ is asymptotically stable in (DI), if both of the following properties hold:*

1. *For any $\epsilon > 0$ there is some $\delta > 0$ such that for any solution χ to (DI) and any $x_0 \in \Delta_+^N$ such that $||x^* - x_0|| < \delta$ and $\chi(x_0, 0) = x_0$, we have $||\chi(x_0, t) - x^*|| < \epsilon$ for all $t \in [0, \infty)$.*
2. *$\lim_{t\to\infty} ||\chi(x_0, t) - x^*|| = 0$ for any solution χ to (DI) and any $x_0 \in \Delta_+^N$.*[15]

Note that $x \in \Delta_+^N$ is a strict local maximizer of the Lyapunov function iff it is an asymptotically stable point.

Theorem 5.1. *An asymptotically stable population profile generically exists.*

Proof. As Δ_+^N is a compact set in a metric space (here, equipped with the Euclidean metric) and $L(\cdot)$ is a continuous, real-valued function defined on it, the existence of a maximum follows from Weierstrass' extreme-value theorem. Additionally, as for all $x \in \Delta_+^N$ it holds that $L(x) \neq L(y)$ for all $y \in B_\epsilon(x)$, each extremum is strict.[16] □

[15] In difference to the usual definition of stability, the properties have to hold for *any* solution to (DI).

[16] $B_\epsilon(x) := \{y \in \Delta_+^N | d(x, y) < \epsilon\}$ is the epsilon ball around x with radius $\epsilon > 0$, and $d(x, y)$ denotes the Euclidean distance between the points x and y.

The linear functional form of L does not allow for strict local maxima lying in the region C, so that the next lemma immediately follows.

Lemma 5.2. *No population profile $x \in C$ is asymptotically stable.*

We now analyze the stability of the system including the area of discontinuity D. For $S \subseteq N$, let

$$\xi^S := \left\{ x \in \Delta_+^N \middle| \xi_i^S = 0 \ \forall i \notin S \text{ and } \xi_i^S = \frac{1}{|S|} \ \forall i \in S \right\}, \tag{5.17}$$

that is, in population state ξ^S only types $i \in S$ have a positive (and equally split) population share.

Theorem 5.2. *If x is asymptotically stable, then there exists some $S \subseteq N$ such that $x = \xi^S \in D$.*

Proof. Let x be stable. From Lemma 5.2 we know that asymptotically stable profiles can only lie in areas of discontinuity, i.e., $x \in D$. We know $x \in D$ is an asymptotically stable point, iff the different parts of the piecewise defined Lyapunov function in each adjacent region attain a strict local maximum in x. Remember that we have shown that $L(x)$ is single-valued even if $x \in D$. The set

$$\Xi(y) := \cup_{\rho \in R(y)} \Delta_+^N(\rho) \tag{5.18}$$

is the set of all regions $\Delta_+^N(\rho)$ which are adjacent to some point $y \in \Delta_+^N$. Obviously, $\Xi(y)$ is equivalent to $\Delta_+^N(\rho)$ if $y \in \Delta_+^N(\rho)$, i.e., if $R(y)$ is a singleton. Thus, by stability,

$$L(x) > L(y), \quad \forall y \in \Xi(x).$$

As the Lyapunov function is piecewise linear and non-constant in the generic case, this implies that $x \notin D \setminus \left(\bigcup_{S \subseteq N} \xi^S \right)$, and hence there exists some $S \subseteq N$ such that $x = \xi^S$. □

It is easy to find examples of underlying TU games where point $x = \xi^S$ for some $S \subseteq N$ is asymptotically stable, as can be seen in the illustrative example of Section 5.2. We can interpret this result in the following way: either player types survive the evolutionary process and are part of one single coalition (possibly consisting of only one single member), or they die out. While the above findings suggest that states with different positive population shares are not asymptotically stable, such states may nevertheless be a stationary point of the evolutionary process.[17] This is the case for

[17] In the replicator dynamic, stationary points trivially exist; these are the corners of the simplex, i.e., states with $x_i = 1$ for some $i \in N$.

instance if $L(x)$ is constant on some hyperplane Σ_J with $|J| \geq 2$. Stable states however consist of only one coalition, meaning that in it all player types have the same population share. It is a natural question to ask which properties such a surviving coalition and its members must have considering the underlying TU game. We will adress these questions in the following.

5.5.2 Relating Asymptotically Stable Profiles to the Underlying TU Game

The following result shows that "inefficient" coalitions will not persist in the replicator dynamics derived from TU games.

Theorem 5.3. *For $S \subseteq N, S \neq \emptyset$, ξ^S is asymptotically stable iff $\frac{v(S)}{|S|} > \frac{v(T)}{|T|}$ for all $T \subseteq N$ such that $T \subsetneq S$ or $T \supsetneq S$.*

Proof. By Theorem 5.2 we know that we only have to consider points $x = \xi^S$ where $\xi_i^S = 0$ for all $i \notin S$ and $\xi_i^S = \frac{1}{|S|}$ for all $i \in S$. Iff ξ^S is a strict local maximizer of $L(\cdot)$, it must in our case hold for all $T \subseteq N$ with $T \subsetneq S$ or $T \supsetneq S$ and $\rho \in R(N)$ such that $\Delta_+^N(\rho) \subseteq \Xi(x)$, that

$$L(\xi^S) > L(\xi^T) \Leftrightarrow \sum_{i \in S} MC_i(\rho)\xi_i^S > \sum_{i \in T} MC_i(\rho)\xi_i^T$$

$$\Leftrightarrow \sum_{i \in S} \frac{MC_i(\rho)}{|S|} > \sum_{i \in T} \frac{MC_i(\rho)}{|T|} \Leftrightarrow \frac{v(S)}{|S|} > \frac{v(T)}{|T|}.$$

□

The result says that a coalition of player types is stable if and only if it implies a higher average productivity than any of its super- or subcoalitions. This reflects the idea, that starting from a population profile, those types with the highest marginal productivity in that profile proliferate with the largest growth rate, and as the population as a whole strives for the maximum fitness, the dynamics end up in asymptotically stable population profiles as characterized in the theorem.

We say that a state $x \in \Delta_+^N$ is characterized by a coalition S iff $x_i = 0$ for all $i \notin S$ and $x_i \neq 0$ for all $i \in S$, that is if $S = \{i \in N : x_i > 0\}$. The next statements on the general properties of the stable population profiles follow immediately from Theorem 5.3.

Corollary 5.1 (General Properties of Asymptotically Stable Population Profiles)**.**

- *Asymptotically stable population profiles are robust to small perturbations to the coalition function of the underlying TU game.*

- *Asymptotically stable population profiles which are characterized by the grand coalition imply that no other population state (characterized by another coalition) is asymptotically stable in that population game.*

The interpretation of the first statement is that small "mutations" do not have an impact on the constellation of aymptotically stable profiles. Next, we state results for important classes of TU games, which also immediately follow from Theorem 5.3.

Corollary 5.2 (Asymptotically Stable Population Profiles in Specific Games)**.**

- *If x is an asymptotically stable point in a simple monotonic game, then it is characterized by a minimal winning coalition.*
- *If x is an asymptotically stable point in an apex game, then it is characterized by a coalition of the apex player with one small player or by the coalition of all small players.*

The first statement reflects the message of the previous theorem: only "efficient" coalitions are stable. An apex game is a special form of a simple monotonic game in which only a coalition of the apex player and one "small" player or the coalition of all small players are the minimal coalitions that can generate positive value. Thus, the second statement follows from the first.

Below, we present some insights on which kind of player types can be members of surviving coalitions. Persistence and extinction are to be understood in reference to stable states: if a player is extinct in a certain stable state, this means that he is not a member of that stable coalition, while if he persists in a certain stable state, he is the member of that stable coalition.

Corollary 5.3 (Persistence and Extinction of Player Types in Stable States)**.**

1. *A sufficient condition for player i to be extinct in any stable state is that for all $\rho \in R(N)$: $MC_i(\rho) < MC_j(\rho)$ for all $j \in N$.*
2. *A necessary condition for player i to persist in some stable state is that for some $\rho \in R(N)$: $MC_i(\rho) > MC_j(\rho)$ for some $j \in N$.*
3. *A weakly dominating player can be extinct in a stable state, while the players dominated by him need not.*
4. *Strictly dominated players may persist in some stable state.*
5. *A necessary condition for a null player $i \in N$ to persist in some stable state which is characterized by coalition S is that $v(S \backslash \{i\}) < 0$. Therefore, strictly dominated null players will be extinct in any stable state.*

Proof.

1. $MC_i(\rho) < MC_j(\rho)$ for all $j \in N$ and all $\rho \in R(N)$ implies $\frac{v(S)}{|S|} < \frac{v(S \setminus \{i\})}{|S|-1}$ for all S with $i \in S$ and $\frac{v(\{i\})}{1} < \frac{v(T)}{|T|}$ for all $T \supset \{i\}$.

2. Follows from the converse of *1.*

3. Follows from Corollary 5.2, as the apex player weakly dominates the "small" players.

4. Consider the game $N = \{1, 2\},\ v(1) = 1,\ v(2) = 0,\ v(1, 2) = 3$, where player 2 is strictly dominated, and apply the previous results.

5. For a null player, Condition *2.* of the corollary can only be satisfied if some other player type generates "negative worth".

□

5.6 Conclusion

We have introduced an approach to derive an evolutionary dynamic from an underlying cooperative transferable utility game. This allows us to relate the survival of certain types of coalitions to the payoff structure of the cooperative game. For instance, we obtain the plausible result that in simple monotonic games only minimal winning coalitions, that is, so to say, "efficient" coalitions, can be asymptotically stable under the replicator dynamic. Besides these results, our model also adds to the literature on dynamic models of coalition formation. Furthermore, we derive a straightforward method to determine stability of coalitions even in rather complicated games as for instance the one defined by Table 5.1 in Section 5.2.

Future research could exploit the fact that the model at hand allows to evaluate the "evolutionary plausibility" of cooperative solution concepts like the Shapley value or the Aumann-Dreze value (Aumann and Dreze, 1974) in the same way as non-cooperative evolutionary game theory allows to examine the "evolutionary plausibility" of Nash equilibria. Thus, it could also be viewed as a complementary approach to the Nash program, which tries to establish non-cooperative foundations of cooperative solution concepts.

An extension of the model at hand would be to add to the selection process implied by the replicator dynamics some form of mutation - either via small changes in the coalition function or by introducing a small fraction of a new player type.

Obviously, the dynamics of the model we discuss are driven by how worth is generated. This is captured here by the Lovász extension, telling the story of a society in which scarcity and competition are driving forces for the production of worth. A straightforward extension of the model would be to incorporate different and more general production functions such as, for instance, the common constant-elasticity-of-substitution function.

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Declaration

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Arbeit selbständig verfasst habe und alle in Anspruch genommenen Quellen und Hilfen in der Dissertation vermerkt wurden. Diese Dissertation ist weder in der gegenwärtigen noch in einer anderen Fassung oder in Teilen an der Technischen Universität Dortmund oder an einer anderen Hochschule im Zusammenhang mit einer staatlichen oder akademischen Prüfung vorgelegt worden.

Dortmund, Juli 2018 Michael Kramm