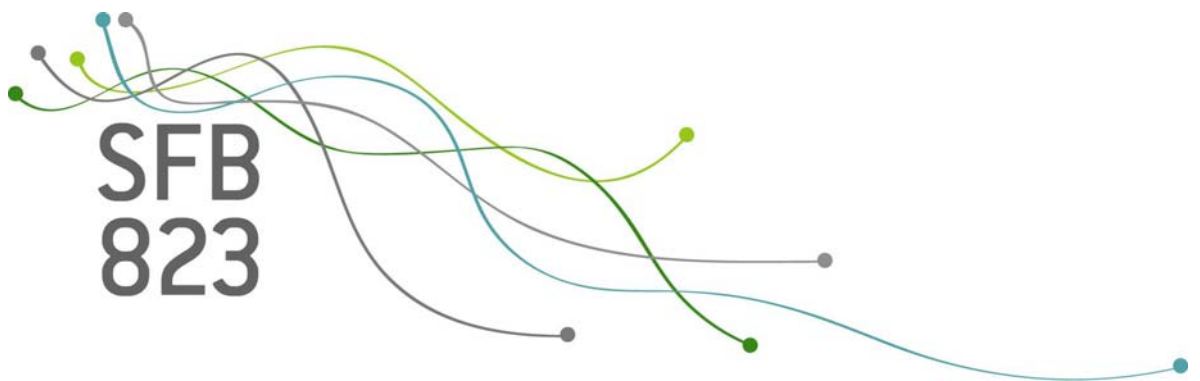# Limit theorems for locally stationary processes

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Limit Theorems for Locally Stationary Processes

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We present limit theorems for locally stationary processes that have a one sided time-varying moving average representation. In particular, we prove a central limit theorem (CLT), a weak and a strong law of large numbers (WLLN, SLLN) and a law of the iterated logarithm (LIL) under mild assumptions that are closely related to those originally imposed by Dahlhaus and Polonik (2006).

1 Introduction

In this paper we consider locally stationary processes, defined via a triangular sequence of stochastic processes $\{\eta_{t,T}\}_{t=1,\dots,T}$ with $T \in \mathbb{N}$, where every $\eta_{t,T}$ has a representation of the form

$$\eta_{t,T} = \sum_{j=0}^{\infty} \psi_{j,t,T}\varepsilon_{t-j}, \quad t = 1, \dots, T. \tag{1}$$

Throughout this paper we impose the following assumption on the error sequence $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ and the moving average coefficients $\psi_{j,t,T}$.

Assumption 1.1. The random variables $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ are independent and identically distributed with $\mathbb{E}\varepsilon_t = 0$, $\mathbb{E}\varepsilon_t^2 = 1$ and $\mathbb{E}|\varepsilon_t|^{2+\kappa} < \infty$ for some $\kappa > 0$. The coefficients $\psi_{j,t,T}$ in the moving average representation (1) fulfill

$$\sup_{t,T} |\psi_{j,t,T}| \leq \frac{K}{l(j)},$$

with constant K independent of T and some positive deterministic sequence $\{l(j)\}_{j\in\mathbb{N}_0}$ satisfying

$$\sum_{j=0}^{\infty} \frac{j}{l(j)} < \infty.$$

Remark 1.2. In contrast to the definition of Dahlhaus and Polonik (2006) we restrict locally stationary processes to have a one-sided moving average representation. Nonetheless, our definition covers most of the important examples of locally stationary processes. For instance, it follows from Dahlhaus and Polonik (2009, Proposition 2.4) that time-varying ARMA processes have a representation of the form (1).

The idea behind locally stationary processes is that, after rescaling the time domain to the unit interval, the process can be approximated by a stationary process. Therefore, one usually assumes that $\psi_{j,t,T} \approx \psi_j(t/T)$ for some well behaving functions ψ_j.

Assumption 1.3. There exist functions $\psi_j : [0,1] \to \mathbb{R}$ with

$$\|\psi_j\|_\infty \leq \frac{K}{l(j)}, \tag{2}$$

$$V(\psi_j) \leq \frac{K}{l(j)} \tag{3}$$

and

$$\sum_{t=1}^{T} \left| \psi_{j,t,T} - \psi_j\left(\frac{t}{T}\right) \right| \leq \frac{K}{l(j)}, \quad \text{for all } T \in \mathbb{N}, \tag{4}$$

where $V(f)$ denotes the total variation of a function f on $[0,1]$.

Remark 1.4. The coefficient functions are uniquely defined almost everywhere. To see this let $\{\eta_{t,T}\}_{t=1,\dots,T}$ be locally stationary process with moving average coefficients $\psi_{j,t,T}$ and corresponding coefficient functions ψ_j. Let ϕ_j be another set of coefficient functions that fulfills Assumption 1.1. Then it holds that

$$
\begin{aligned}
\|\psi_j - \phi_j\|_{L^1} &= \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \left\| \psi_j\left(\frac{t}{T}\right) - \phi_j\left(\frac{t}{T}\right) \right\| \\
&\leq \lim_{T\to\infty} \frac{1}{T} \left\{ \sum_{t=1}^{T} \left\| \psi_{j,t,T} - \psi_j\left(\frac{t}{T}\right) \right\| + \sum_{t=1}^{T} \left\| \psi_{j,t,T} - \phi_j\left(\frac{t}{T}\right) \right\| \right\} \\
&\leq \lim_{T\to\infty} \frac{2K}{Tl(j)} = 0,
\end{aligned}
$$

implying $\psi_j = \phi_j$ almost everywhere.

For every $u \in [0,1]$ we define the process $\{\eta_t(u)\}_{t\in\mathbb{Z}}$ via

$$\eta_t(u) = \sum_{j=0}^{\infty} \psi_j(u)\varepsilon_{t-j}.$$

By Assumption 1.3 this process is weakly stationary with long-run variance given by $\Psi^2(u)$, where

$$\Psi(u) = \sum_{j=0}^{\infty} \psi_j(u).$$

The main purpose of the process $\{\eta_t(u)\}_{t\in\mathbb{Z}}$ is to approximate $\{\eta_{t,T}\}_{t=1,\dots,T}$. In particular, the process $\{\eta_{t,T}\}_{t=1,\dots,T}$ should approximately behave like $\{\eta_t(u)\}_{t\in\mathbb{Z}}$ in the rescaled time point $u = t/T$. For brevity, we define the auxiliary process $\{\tilde{\eta}_{t,T}\}_{t=1,\dots,T}$ via $\tilde{\eta}_{t,T} = \eta_t(t/T)$, i.e.

$$\tilde{\eta}_{t,T} = \sum_{j=0}^{\infty} \psi_j\left(\frac{t}{T}\right) \varepsilon_{t-j}. \tag{5}$$

Under the stated assumptions it holds that (cf. Lemma A.2 in the Appendix)

$$\frac{1}{T} \sum_{t=1}^{T} (\eta_{t,T} - \tilde{\eta}_{t,T}) \xrightarrow{P} 0.$$

Hence, the process $\{\eta_t(u)\}_{t\in\mathbb{Z}}$ approximates the locally stationary process on average over all rescaled time points $1/T, 2/T \ldots, 1$. Later we will strenghten condition (4) in order to obtain a pointwise approximation.

Remark 1.5. The construction of locally stationary processes with time dependent moving-average coefficients $\psi_{j,t,T}$ on the one hand and approximating functions ψ_j on the other hand looks cumbersome at first glance. It seems more natural to define locally stationary processes directly via (5). However, it was already pointed out by Künsch (1995) and Dahlhaus and Polonik (2009) that this rules out interesting examples such as time-varying autoregressive processes.

2 Main Results

The first limit theorem we present is a CLT for locally stationary processes. To motivate the outcome, we first derive the result for an easy example. Let $\{\eta_{t,T}\}_{t=1,\ldots,T}$ be defined by

$$\eta_{t,T} = \phi\left(\frac{t}{T}\right)\varepsilon_t, \quad t = 1, \ldots, T,$$

for some bounded variation function $\phi : [0,1] \to \mathbb{R}$ and $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ being a sequence of independent and identically $\mathcal{N}(0,1)$ distributed random variables. Then it holds that

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\eta_{t,T} \sim \mathcal{N}\left(0, \frac{1}{T}\sum_{t=1}^{T}\phi^2\left(\frac{t}{T}\right)\right).$$

Since ϕ is of bounded variation it is square-integrable on the unit interval and it holds that

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\phi^2\left(\frac{t}{T}\right) = \int_0^1\phi^2(u)\,du$$

and Lévy's continuity theorem implies that

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\eta_{t,T} \xrightarrow{d} \mathcal{N}\left(0, \int_0^1\phi^2(u)\,du\right). \tag{6}$$

Note that the approximating stationary process $\{\eta_t(u)\}_{t\in\mathbb{Z}}$ is defined by $\eta_t(u) = \phi(u)\varepsilon_t$ with long-run variance given by $\phi^2(u)$. Hence, the variance of the limiting distribution in (6) is equal to the integrated long-run variance of the auxiliary process. This result also holds for arbitrary locally stationary processes.

Theorem 2.1 (CLT). *Let $\{\eta_{t,T}\}_{t=1,\ldots,T}$ be a locally stationary process with moving-average representation (1) that satisfies Assumptions 1.1 and 1.3. Then, as $T \to \infty$, it holds that*

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\eta_{t,T} \xrightarrow{d} \mathcal{N}\left(0, \|\Psi\|_{L^2}^2\right),$$

where $\|\Psi\|_{L^2}$ denotes the L^2 norm of Ψ on the unit interval.

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Proof. It suffices to show the claim for the auxiliary process $\{\tilde{\eta}_{t,T}\}_{t=1,\dots,T}$ since

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \eta_{t,T} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \tilde{\eta}_{t,T} + \frac{1}{\sqrt{T}} \sum_{t=1}^{T} (\eta_{t,T} - \tilde{\eta}_{t,T})$$

and the second term goes to zero in probability by Lemma A.2. By Lemma A.1 it holds that

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \tilde{\eta}_{t,T} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \Psi\left(\frac{t}{T}\right) \varepsilon_t - \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \sum_{j=0}^{\infty} \tilde{\psi}_j\left(\frac{t}{T}\right) (\varepsilon_{t-j} - \varepsilon_{t-1-j}). \tag{7}$$

We show that the first term in (7) converges in distribution and the second term vanishes in probability. By the i.i.d. assumption on the innovation terms it holds that

$$\mathrm{Var}\left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \Psi\left(\frac{t}{T}\right) \varepsilon_t\right) = \frac{1}{T} \sum_{t=1}^{T} \Psi^2\left(\frac{t}{T}\right) \to \int_0^1 \Psi^2(u)\, du.$$

Next, we verify the Lyapunov condition. By Assumption 1.1 there exists some $\kappa > 0$ such that $\mathbb{E}|\varepsilon_t|^{2+\kappa}$ is finite. Hence,

$$\lim_{T \to \infty} \frac{\sum_{t=1}^{T} \mathbb{E}\left|\frac{1}{\sqrt{T}} \Psi\left(\frac{t}{T}\right) \varepsilon_t\right|^{2+\kappa}}{\left(\mathrm{Var}\left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \Psi\left(\frac{t}{T}\right) \varepsilon_t\right)\right)} = \lim_{T \to \infty} \frac{\mathbb{E}|\varepsilon_1|^{2+\kappa}}{T^{\kappa/2}} \lim_{T \to \infty} \frac{\frac{1}{T} \sum_{t=1}^{T} \Psi^{2+\kappa}\left(\frac{t}{T}\right)}{\left(\frac{1}{T} \sum_{t=1}^{T} \Psi\left(\frac{t}{T}\right)\right)^{1+\kappa/2}} = 0.$$

From the Lindeberg-CLT for triangular arrays we deduce that

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \Psi\left(\frac{t}{T}\right) \varepsilon_t \xrightarrow{d} \mathcal{N}\left(0, \int_0^1 \Psi^2(u)\, du\right).$$

To finish the proof it remains to show that the second term in (7) goes to zero in probability. It holds that

$$\sum_{j=0}^{\infty} \tilde{\psi}_j\left(\frac{t}{T}\right) (\varepsilon_{t-j} - \varepsilon_{t-1-j}) = \sum_{j=0}^{\infty} \left\{ \tilde{\psi}_j\left(\frac{t}{T}\right) \varepsilon_{t-j} - \tilde{\psi}_j\left(\frac{t}{T}\right) \varepsilon_{t-1-j} \right\}$$

$$= \sum_{j=0}^{\infty} \left\{ \tilde{\psi}_j\left(\frac{t}{T}\right) \varepsilon_{t-j} - \tilde{\psi}_j\left(\frac{t-1}{T}\right) \varepsilon_{t-1-j} + \tilde{\psi}_j\left(\frac{t-1}{T}\right) \varepsilon_{t-1-j} - \tilde{\psi}_j\left(\frac{t}{T}\right) \varepsilon_{t-1-j} \right\}$$

$$= \sum_{j=0}^{\infty} \left\{ \tilde{\psi}_j\left(\frac{t}{T}\right) \varepsilon_{t-j} - \tilde{\psi}_j\left(\frac{t-1}{T}\right) \varepsilon_{t-1-j} \right\} + \sum_{j=0}^{\infty} \left\{ \tilde{\psi}_j\left(\frac{t-1}{T}\right) \varepsilon_{t-1-j} - \tilde{\psi}_j\left(\frac{t}{T}\right) \varepsilon_{t-1-j} \right\}. \tag{8}$$

Taking partial sum of the first term and dividing by $T^{1/2}$ leads to:

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \sum_{j=0}^{\infty} \left\{ \tilde{\psi}_j\left(\frac{t}{T}\right) \varepsilon_{t-j} - \tilde{\psi}_j\left(\frac{t-1}{T}\right) \varepsilon_{t-1-j} \right\} = \frac{1}{\sqrt{T}} \sum_{j=0}^{\infty} \left\{ \tilde{\psi}_j(1)\, \varepsilon_{T-j} - \tilde{\psi}_j(0)\varepsilon_{-j} \right\}, \tag{9}$$

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as the sum over t is telescopic. Since

$$\mathbb{E}\left|\sum_{j=0}^{\infty}\sup_{u\in(0,1)}|\tilde{\psi}_j(u)|\varepsilon_{t-j}\right| \leq \sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty}\sup_{u\in(0,1)}|\psi_k(u)|\mathbb{E}|\varepsilon_1| \leq \sum_{j=0}^{\infty}\frac{jK\mathbb{E}|\varepsilon_1|}{l(j)} < \infty$$

for an arbitrary $t \in \{1, \ldots, T\}$ it follows that the term on the right hand side of (9) converges to zero in probability.

It remains to prove that the scaled partial sum of the second term in (8) also vanishes asymptotically. It holds that

$$\mathbb{E}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\sum_{j=0}^{\infty}\left\{\tilde{\psi}_j\left(\frac{t}{T}\right)-\tilde{\psi}_j\left(\frac{t-1}{T}\right)\right\}\varepsilon_{t-1-j}\right|$$

$$\leq \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\sum_{j=0}^{\infty}\left|\tilde{\psi}_j\left(\frac{t}{T}\right)-\tilde{\psi}_j\left(\frac{t-1}{T}\right)\right|\mathbb{E}|\varepsilon_1|$$

$$\leq \frac{1}{\sqrt{T}}\sum_{j=0}^{\infty}V(\tilde{\psi}_j)\mathbb{E}|\varepsilon_1|,$$

which converges to zero if the $V(\tilde{\psi}_j)$ are summable. Using the definition of the total variation we obtain

$$\sum_{j=0}^{\infty}V(\tilde{\psi}_j) = \sum_{j=0}^{\infty}\sup_{\substack{0\leq x_1<\ldots<x_M\leq 1\\M\in\mathbb{N}}}\sum_{i=1}^{M}|\tilde{\psi}_j(x_{i+1})-\tilde{\psi}_j(x_i)|$$

$$\leq \sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty}\sup_{\substack{0\leq x_1<\ldots<x_M\leq 1\\M\in\mathbb{N}}}\sum_{i=1}^{M}|\psi_j(x_{i+1})-\psi_j(x_i)|$$

$$= \sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty}V(\psi_j)$$

$$\leq \sum_{j=0}^{\infty}\frac{jK}{l(j)},$$

which is finite by Assumption 1.3, condition (3). $\qquad\square$

From Theorem 2.1 we immediately obtain a WLLN.

Corollary 2.2 (WLLN). *Let $\{\eta_{t,T}\}_{t=1,\ldots,T}$ be a locally stationary process defined via its moving-average representation (1) with Assumptions 1.1 and 1.3 in place. Then, as $T \to \infty$, it holds that*

$$\frac{1}{T}\sum_{t=1}^{T}\eta_{t,T} \overset{P}{\to} 0.$$

In order to prove a SLLN and a LIL we require a stronger assumption that connects the coefficient functions $\psi_{j,t,T}$ and the approximating functions ψ_j. The following assumption, that immediately implies condition (4), corresponds to assumption (69) in Dahlhaus (2012).

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Assumption 2.3. The functions ψ_j and the moving average coefficients $\psi_{j,t,T}$ satisfy

$$\sup_{1 \leq t \leq T} \left| \psi_{j,t,T} - \psi_j \left(\frac{t}{T} \right) \right| \leq \frac{K}{Tl(j)}, \quad \text{for all } T \in \mathbb{N}.$$

Previously, we observed that the stationary process $\{\eta_t(u)\}_{t\in\mathbb{Z}}$ approximates the locally stationary process on average over the series. Under Assumption 2.3 we have a better approximation as it now holds that $\eta_{t,T} = \eta_t(t/T) + \mathcal{O}_P(T^{-1})$. This follows from the fact that

$$\lim_{T \to \infty} \mathbb{E} \left| \eta_{t,T} - \eta_t \left(\frac{t}{T} \right) \right| \leq \lim_{T \to \infty} \sum_{j=0}^{\infty} \left| \psi_{j,t,T} - \psi_j \left(\frac{t}{T} \right) \right| \mathbb{E}|\varepsilon_{t-j}| \leq \lim_{T \to \infty} \sum_{j=0}^{\infty} \frac{K\mathbb{E}|\varepsilon_0|}{Tl(j)} = 0.$$

Consequently, the stationary process $\{\eta_t(u)\}_{t\in\mathbb{Z}}$ approximates the locally stationary process $\{\eta_{t,T}\}_{t=1,\ldots,T}$ in every rescaled time point $u = t/T$. In fact, we even have a strong approximation, i.e. $\eta_{t,T} = \eta_t(t/T) + o_{a.s}(1)$ (cf. Lemma A.3 in the appendix).

Theorem 2.4 (SLLN). *Let $\{\eta_{t,T}\}_{t=1,\ldots,T}$ be a locally stationary process defined via its moving-average representation* (1) *with Assumptions 1.1, 1.3 and 2.3 in place. Then, as $T \to \infty$, it holds that*

$$\frac{1}{T} \sum_{t=1}^{T} \eta_{t,T} \overset{a.s.}{\to} 0.$$

Proof. It suffices to show the claim for the auxiliary process $\{\tilde{\eta}_{t,T}\}_{t=1,\ldots,T}$ since by Lemma A.4,

$$\frac{1}{T} \sum_{t=1}^{T} \eta_{t,T} = \frac{1}{T} \sum_{t=1}^{T} \tilde{\eta}_{t,T} + o_{a.s.}(1).$$

Using Lemma A.1 we first need to verify that

$$\frac{1}{T} \sum_{t=1}^{T} \Psi \left(\frac{t}{T} \right) \varepsilon_t \overset{a.s.}{\to} 0. \tag{10}$$

It holds that

$$\sup_{u \in (0,1)} |\Psi(u)| \leq \sup_{u \in (0,1)} \sum_{j=0}^{\infty} |\psi_j(u)| \leq \sum_{j=0}^{\infty} \sup_{u \in (0,1)} |\psi_j(u)| \leq \sum_{j=0}^{\infty} \frac{K}{l(j)} < \infty.$$

Since the ε_t's are independent and identically distributed with $\mathbb{E}(\varepsilon_1) = 0$ and $\mathbb{E}\varepsilon_1^2 < \infty$ almost sure convergence of (10) follows from Cuzick (1995, Theorem 1.1) or Choi and Sung (1987, Theorem 5).

It remains to show that

$$\frac{1}{T} \sum_{j=0}^{\infty} \tilde{\psi}_j(1)\varepsilon_{T-j} \overset{a.s.}{\to} 0, \tag{11}$$

$$\frac{1}{T} \sum_{j=0}^{\infty} \tilde{\psi}_j(0)\varepsilon_{-j} \overset{a.s.}{\to} 0 \tag{12}$$

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and

$$\frac{1}{T} \sum_{t=1}^{T} \sum_{j=0}^{\infty} \left\{ \tilde{\psi}_j \left(\frac{t-1}{T} \right) - \tilde{\psi}_j \left(\frac{t}{T} \right) \right\} \varepsilon_{t-1-j} \overset{a.s.}{\to} 0. \tag{13}$$

It holds that

$$\mathbb{E} \left(\frac{1}{T} \sum_{j=0}^{\infty} \tilde{\psi}_j(1) \varepsilon_{T-j} \right)^2 = \frac{1}{T^2} \sum_{j=0}^{\infty} \tilde{\psi}_j(1)^2 = \frac{1}{T^2} \sum_{j=0}^{\infty} \left(\sum_{i=j+1}^{\infty} \psi_i(1) \right)^2 = \mathcal{O} \left(\frac{1}{T^2} \right),$$

since

$$\sum_{j=0}^{\infty} \sum_{i=j+1}^{\infty} \psi_i(1) \leq \sum_{j=0}^{\infty} \sum_{i=j+1}^{\infty} \|\psi_i\|_\infty = \sum_{j=0}^{\infty} j \|\psi_j\|_\infty < \infty.$$

Hence, the term in (11) converges sufficiently fast to zero in probability and almost sure convergence follows from the Borel-Cantelli lemma. The proof of (12) is identical. At last we have to show (13). It holds that

$$\mathbb{E} \left(\frac{1}{T} \sum_{t=1}^{T} \sum_{j=0}^{\infty} \left\{ \tilde{\psi}_j \left(\frac{t-1}{T} \right) - \tilde{\psi}_j \left(\frac{t}{T} \right) \right\} \varepsilon_{t-1-j} \right)^2$$

$$\leq \frac{1}{T^2} \sum_{j_1,j_2=0}^{\infty} \sum_{t_1,t_2=1}^{T} \left| \tilde{\psi}_{j_1} \left(\frac{t_1-1}{T} \right) - \tilde{\psi}_{j_1} \left(\frac{t_1}{T} \right) \right| \left| \tilde{\psi}_{j_2} \left(\frac{t_2-1}{T} \right) - \tilde{\psi}_{j_2} \left(\frac{t_2}{T} \right) \right|$$

$$\leq \frac{1}{T^2} \sum_{j=0}^{\infty} V(\tilde{\psi}_j) \sum_{k=0}^{\infty} V(\tilde{\psi}_k) = \mathcal{O} \left(\frac{1}{T^2} \right).$$

Hence, the second moment of the term in (13) converges sufficiently fast to zero implying almost sure convergence. $\square$

Our last result is a LIL. In order to prove the theorem we impose some additional moment condition on the sequence $\{\varepsilon_t\}_{t \in \mathbb{Z}}$. In particular, we assume that at least the fourth moment of ε_t is finite.

Theorem 2.5 (LIL). *Let $\{\eta_{t,T}\}_{t=1,\dots,T}$ be a locally stationary process with Assumptions 1.1, 1.3 and 2.3 in place and let $d_T = T \log \log T$. Assume further that the innovation sequence $\{\varepsilon_t\}_{t \in \mathbb{Z}}$ satisfies $\mathbb{E}\varepsilon_t^4 = \mu_4 < \infty$. Then, as $T \to \infty$, it holds that*

$$\limsup_{T \to \infty} \frac{1}{\sqrt{d_T}} \sum_{t=1}^{T} \eta_{t,T} \overset{a.s.}{=} \sqrt{2} \|\Psi\|_{L^2}.$$

Proof. Again, by Lemma A.4 it suffices to show the claim for the auxiliary process $\{\tilde{\eta}_{t,T}\}_{t=1,\dots,T}$. Following the lines of the proof of Theorem 2.4 we first prove

$$\limsup_{T \to \infty} \frac{1}{\sqrt{d_T}} \sum_{t=1}^{T} \Psi \left(\frac{t}{T} \right) \varepsilon_t \overset{a.s.}{=} \sqrt{2} \|\Psi\|_{L^2}.$$

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Since $\{\varepsilon_t\}_{t \in \mathbb{Z}}$ is a sequence of independent random variables with finite variance the claim follows immediately from Tomkins (1975, Theorem 1) and Wichura (1973, page 279). Therefore, it remains to prove

$$\frac{1}{\sqrt{d_T}} \sum_{j=0}^{\infty} \tilde{\psi}_j(1) \varepsilon_{T-j} \overset{a.s.}{\to} 0, \tag{14}$$

$$\frac{1}{\sqrt{d_T}} \sum_{j=0}^{\infty} \tilde{\psi}_j(0) \varepsilon_{-j} \overset{a.s.}{\to} 0 \tag{15}$$

and

$$\frac{1}{\sqrt{d_T}} \sum_{t=1}^{T} \sum_{j=0}^{\infty} \left(\tilde{\psi}_j\left(\frac{t-1}{T}\right) - \tilde{\psi}_j\left(\frac{t}{T}\right) \right) \varepsilon_{t-1-j} \overset{a.s.}{\to} 0. \tag{16}$$

In contrast to the proof of Theorem 2.4 it is not sufficient to investigate the second moments of these terms, as d_T^{-1} decays too slowly. However, we adapt the proof using fourth moments. For the term in (14) it holds that

$$\mathbb{E}\left(\sum_{j=0}^{\infty} \tilde{\psi}_j(1) \varepsilon_{T-j} \right)^4 = \sum_{j_1,\dots,j_4=0}^{\infty} \left\{ \prod_{m=1}^{4} \tilde{\psi}_{j_m}(1) \right\} \mathbb{E}\left(\prod_{m=1}^{4} \varepsilon_{T-j_m} \right) \leq \mu_4 \left(\sum_{j=0}^{\infty} \tilde{\psi}_j(1) \right)^4,$$

implying

$$\sum_{T=1}^{\infty} \mathbb{E}\left(\frac{1}{\sqrt{d_T}} \sum_{j=0}^{\infty} \tilde{\psi}_j(1) \varepsilon_{T-j} \right)^4 \leq \sum_{T=1}^{\infty} \frac{C}{T^2 (\log \log T)^2} < \infty$$

and, by the Borel-Cantelli Lemma, almost sure convergence. The claim in (15) is proven in exactly the same way. To show (16) consider

$$\mathbb{E}\left(\frac{1}{\sqrt{d_T}} \sum_{t=1}^{T} \sum_{j=0}^{\infty} \left(\tilde{\psi}_j\left(\frac{t-1}{T}\right) - \tilde{\psi}_j\left(\frac{t}{T}\right) \right) \varepsilon_{t-1-j} \right)^4$$

$$= \frac{1}{d_T^2} \sum_{t_1,\dots,t_4=1}^{T} \sum_{j_1,\dots,j_4=0}^{\infty} \left\{ \prod_{m=1}^{4} \left(\tilde{\psi}_j\left(\frac{t-1}{T}\right) - \tilde{\psi}_j\left(\frac{t}{T}\right) \right) \right\} \mathbb{E}\left(\prod_{m=1}^{4} \varepsilon_{t_m-1-j_m} \right)$$

$$\leq \frac{1}{d_T^2} \sum_{j_1,\dots,j_4=0}^{\infty} \left\{ \prod_{m=1}^{4} \sum_{t_m=1}^{T} \left| \tilde{\psi}_j\left(\frac{t_m-1}{T}\right) - \tilde{\psi}_j\left(\frac{t_m}{T}\right) \right| \right\} \mu_4$$

$$\leq \frac{1}{d_T^2} \sum_{j_1,\dots,j_4=0}^{\infty} \left\{ \prod_{m=1}^{4} V(\tilde{\psi}_{j_m}) \right\} = \frac{1}{d_T^2} \left(\sum_{j=0}^{\infty} V(\tilde{\psi}_j) \right)^4 \mu_4.$$

The claim follows by the same arguments as above. $\qquad\square$

Obviously, if the coefficients $\psi_{j,t,T}$ are not time-dependent, the statement of Theorem 2.5 coincides with the LIL for linear processes which was proven by Phillips and Solo (1992).

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Acknowledgements

Appendix: Auxiliary Lemmata

Consider the stationary approximating process $\{\eta_t(u)\}_{t \in \mathbb{Z}}$ for some fixed $u \in [0, 1]$. By the Beveridge-Nelson decomposition (cf. Phillips and Solo, 1992) it holds that

$$\eta_t(u) = \sum_{j=0}^{\infty} \psi_j(u)\varepsilon_{t-j} = \sum_{j=0}^{\infty} \psi_j(u)\varepsilon_t - \sum_{j=0}^{\infty} \left(\sum_{k=j+1}^{\infty} \psi_k(u) \right) (\varepsilon_{t-j} - \varepsilon_{t-1-j}),$$

which is well defined due to Assumption 1.3, condition (2). Setting $u = t/T$ we obtain a time-varying Beveridge-Nelson decomposition for the auxiliary process $\{\tilde{\eta}_{t,T}\}_{t=1,\dots,T}$.

Lemma A.1 (Time-varying Beveridge-Nelson decomposition)**.** *The auxiliary process $\{\tilde{\eta}_{t,T}\}_{t=1,\dots,T}$ exhibits a representation of the form*

$$\tilde{\eta}_{t,T} = \Psi\left(\frac{t}{T}\right)\varepsilon_t - \sum_{j=0}^{\infty} \tilde{\psi}_j\left(\frac{t}{T}\right)(\varepsilon_{t-j} - \varepsilon_{t-1-j})$$

with

$$\tilde{\psi}_j(u) = \sum_{k=j+1}^{\infty} \psi_j(u).$$

Lemma A.2. *Let $\{\eta_{t,T}\}_{t=1,\dots,T}$ be a locally stationary process defined via its moving average representation (1) with Assumptions 1.1 and 1.3 in place and let $\{\tilde{\eta}_{t,T}\}_{t=1,\dots,T}$ be the corresponding auxiliary process, defined by (5). Then it holds that*

$$\sum_{t=1}^{T} |\eta_{t,T} - \tilde{\eta}_{t,T}| = \mathcal{O}_P(1).$$

Proof. It holds that

$$\sum_{t=1}^{T} |\eta_{t,T} - \tilde{\eta}_{t,T}| \leq \sum_{j=0}^{\infty}\sum_{t=1}^{T} \left| \psi_{j,t,T} - \psi_j\left(\frac{t}{T}\right) \right| |\varepsilon_{t-j}|.$$

Hence,

$$\sum_{t=1}^{T} \mathbb{E}\, |\eta_{t,T} - \tilde{\eta}_{t,T}| \leq \sum_{j=0}^{\infty}\sum_{t=1}^{T} \left| \psi_{j,t,T} - \psi_j\left(\frac{t}{T}\right) \right| \mathbb{E}|\varepsilon_1| \leq \sum_{j=0}^{\infty} \frac{K\mathbb{E}|\varepsilon_1|}{l(j)} < \infty,$$

implying the claim. $\qquad\square$

Lemma A.3. *Let $\{\eta_{t,T}\}_{t=1,\ldots,T}$ be a locally stationary process with Assumptions 1.1, 1.3 and 2.3 in place. Further, let $\{\tilde{\eta}_{t,T}\}_{t=1,\ldots,T}$ be the corresponding auxiliary process, defined by (5). Then, it holds that*

$$\eta_{t,T} = \tilde{\eta}_{t,T} + o_{a.s.}(1), \quad t = 1, \ldots, T.$$

Proof. It holds that

$$(\eta_{t,T} - \tilde{\eta}_{t,T})^2 = \sum_{j_1,j_2=0}^{\infty} \prod_{l=1}^{2} \left\{ \psi_{j_l,t,T} - \psi_{j_l}\left(\frac{t}{T}\right) \right\} \varepsilon_{t-j_1} \varepsilon_{t-j_2}$$

Since $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a sequence of independent random variables we obtain

$$\mathbb{E}\,(\eta_{t,T} - \tilde{\eta}_{t,T})^2 \leq \sum_{j=0}^{\infty} \left\{ \sup_{1\leq t\leq T} \left| \psi_{j,t,T} - \psi_j\left(\frac{t}{T}\right) \right| \right\}^2 \leq \sum_{j=0}^{\infty} \frac{K^2}{T^2 l^2(j)}.$$

Hence, for all $\alpha > 0$,

$$\sum_{T=1}^{\infty} P\left(|\eta_{t,T} - \tilde{\eta}_{t,T}| > \alpha\right) \leq \sum_{T=1}^{\infty} \mathbb{E}\,(\eta_{t,T} - \tilde{\eta}_{t,T})^2 \leq \sum_{T=1}^{\infty} \frac{\tilde{K}}{T^2} < \infty$$

and the claim follows from the Borel-Cantelli Lemma. $\qquad\square$

Lemma A.4. *Under the Assumptions of Lemma A.3 it holds that*

$$\frac{1}{\sqrt{d_T}} \sum_{t=1}^{T} (\eta_{t,T} - \tilde{\eta}_{t,T}) \stackrel{a.s.}{\to} 0,$$

where $d_T = T \log \log T$.

Proof. It holds that

$$\mathbb{E}\left(\sum_{t=1}^{T} (\eta_{t,T} - \tilde{\eta}_{t,T})\right)^2 \leq \sum_{t_1,t_2=1}^{T} \sum_{j_1,j_2=0}^{\infty} \prod_{l=1}^{2} \left\{ \psi_{j_l,t_l,T} - \psi_{j_l}\left(\frac{t_l}{T}\right) \right\} \mathbb{E}[\varepsilon_{t_1-j_1} \varepsilon_{t_2-j_2}]$$

$$\leq \sum_{t=1}^{T} \left(\sum_{j=0}^{\infty} \sup_{t,T} \left| \psi_{j,t,T} - \psi_j\left(\frac{t}{T}\right) \right| \right)^2$$

$$\leq \sum_{t=1}^{T} \frac{1}{T^2} \left(\sum_{j=0}^{\infty} \frac{K}{l(j)} \right)^2 = \frac{\tilde{K}}{T}.$$

Hence, for some arbitrary $\alpha > 0$,

$$\sum_{T=1}^{\infty} P\left(\left| \frac{1}{\sqrt{d_T}} \sum_{t=1}^{T} (\eta_{t,T} - \tilde{\eta}_{t,T}) \right| > \alpha \right) \leq \sum_{T=1}^{\infty} \frac{\tilde{K}}{\alpha^2 d_T T} < \infty,$$

implying the claim. $\qquad\square$

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