SFB 823

Optimal designs for estimating individual coefficients in polynomial regression with no intercept

Holger Dette, Viatcheslav B. Melas, Petr Shpilev

Discussion Paper

Nr. 13/2019

Optimal designs for estimating individual coefficients in polynomial regression with no intercept

Holger Dette
Ruhr-Universität Bochum
Fakultät für Mathematik
44780 Bochum, Germany
e-mail: holger.dette@rub.de

Viatcheslav B. Melas
St. Petersburg State University
Department of Mathematics
St. Petersburg , Russia
email: vbmelas@yandex.ru

Petr Shpilev
St. Petersburg State University
Department of Mathematics
St. Petersburg , Russia
email: pitshp@hotmail.com

June 19, 2019

Abstract

In a seminal paper Studden (1968) characterized c-optimal designs in regression models, where the regression functions form a Chebyshev system. He used these results to determine the optimal design for estimating the individual coefficients in a polynomial regression model on the interval $[-1, 1]$ explicitly. In this note we identify the optimal design for estimating the individual coefficients in a polynomial regression model with no intercept (here the regression functions do not form a Chebyshev system).

AMS subject classification: 62K05
Keywords and phrases: polynomial regression, c-optimal design, Chebyshev system

1 Introduction

Consider the common polynomial regression model of degree n with no intercept

$$Y_i = (x_i, x_i^2, \dots, x_i^n)^\top \theta + \varepsilon_i, \qquad i = 1, \dots, N, \tag{1.1}$$

where $\varepsilon_1, \dots, \varepsilon_N$ denote independent random variables with $\mathbb{E}[\varepsilon_i] = 0$; $\text{Var}(\varepsilon_i) = \sigma^2 > 0$ $(i = 1, \dots, N)$, $\theta = (\theta_1, \dots, \theta_n)^\top \in \mathbb{R}^n$ is a vector of unknown parameters and the explanatory variables $x_1, \dots, x_N$ vary in the interval $[-1, 1]$. An (approximate) optimal design minimizes an appropriate functional of the (asymptotic) covariance matrix of the statistic $\sqrt{N}\hat{\theta}$, where the $\hat{\theta}$ denotes the least squares estimate of the parameter θ in the regression model (1.1) [see Silvey (1980) or Pukelsheim (2006)]. Numerous authors have worked on the problem of determing optimal designs in this model, where the main focus is on the D- and E-optimality criterion corresponding to the minimization of the determinant and maximum eigenvalue of the (asymptotic) covariance matrix of the least squares estimate [see Huang et al. (1995); Chang and Heiligers (1996); Ortiz and Rodríguez (1998); Chang (1999); Fang (2002) or Li et al. (2005)]. While these problems have been nowadays well understood there exist basically no solutions of the optimal design problem for other type of optimality criteria.

In the present note we add to this literature and determine explicitly the approximate (in the sense of Kiefer (1974)) optimal design for estimating the individual coefficients in a polynomial regression model with no intercept on the interval $[-1, 1]$. The corresponding optimality criteria are special cases of the well known c-optimality criterion which seeks for a design minimizing the variance of the best linear unbiased estimate of the linear combination $c^\top \theta$ in model (1.1), where $c \in \mathbb{R}^n$ is a given vector. In a seminal paper Studden (1968) characterizes c-optimal designs in regression models with regression functions forming a Chebyshev system. As an application he found the optimal designs for estimating the individual coefficients in a regression with intercept, that is $Y_i = \sum_{\ell=0}^n \theta_\ell x_i^\ell + \varepsilon_i$. It is also indicated in Studden (1968) that in general the solution of the c-optimal design problem is an extremely difficult one, in particular if the regressions functions do not form a Chebyshev system, such as in model (1.1), if the explanatory variable varies int he interval $[-1, 1]$.

In Section 2 we introduce the basic optimal design problem and review a geometric characterization of c-optimal designs. The main result can be found in Section 3 where the optimal designs for estimating the individual coefficients in polynomial regression model with no intercept are determined explicitly and the theory is illustrated by several examples.

2 c-optimal designs

Following Kiefer (1974) we call a probability measure

$$\xi = \begin{pmatrix} x_1 & x_2 & \cdots & x_m \\ \omega_1 & \omega_2 & \cdots & \omega_m \end{pmatrix} \tag{2.1}$$

with finite support $x_1, \ldots, x_m \in [-1, 1]$ and corresponding weights $\omega_1, \ldots, \omega_m$ an approximate design on the interval $[-1, 1]$. We define

$$f(x) = (x, \ldots, x^n)^\top \tag{2.2}$$

as the vector of regression functions in the polynomial regression model (1.1), and by

$$M(\xi) = \int_{-1}^{1} f(x) f^\top(x) \xi(dx)$$

the information matrix of the design ξ. The interpretation of ξ and $M(\xi)$ is as follows. If an experimenter takes $n_1, \ldots, n_m$ observations at the experimental conditions $x_1, \ldots, x_m$, respectively, $N = \sum_{i=1}^m n_i$ denotes the total sample size and n_i/N converge to ω_i ($i = 1, \ldots, m$), then the asymptotic covariance matrix of the scaled least squares estimate $\sqrt{N}\hat{\theta}$ in the regression model (1.1) is given by $\sigma^2 M^{-1}(\xi)$, where σ^2 is the variance of the errors. An approximate optimal design minimizes a functional of the matrix $M^{-1}(\xi)$ (or more generally of a generalized inverse $M^-(\xi)$), which is called optimality criterion in the literature [see Silvey (1980) or Pukelsheim (2006)].

In this paper we investigate a special case of the c-optimality criterion, which is defined by

$$\Phi_c(\xi) = \begin{cases} c^\top M^-(\xi) c & \text{if there exists a vector} v \in \mathbb{R}^n \text{ such that } c = M(\xi)v; , \\ \infty, & \text{otherwise} \end{cases} \tag{2.3}$$

for a given vector $c \in \mathbb{R}^n$. In the first case the design ξ is called *admissible for estimating the linear combination* $c^\top\theta$ in the regression model (1.1) and the value of the quadratic form does not depend on the choice of the generalized inverse [see Pukelsheim (2006)]. The criterion (2.3) corresponds to the minimization of the asymptotic variance of the best linear unbiased estimate for the linear combination $c^\top\theta$. In particular for the pth unit vector $e_p = (0, \ldots, 0, 1, 0, \ldots, 0)^\top \in \mathbb{R}^n$ we obtain $e_p^\top\theta = \theta_p$ and the e_p-optimal design minimizes the asymptotic variance of the best linear unbiased estimate for the coefficient θ_p corresponding to the monomial x^p in the polynomial regression model with no intercept ($p = 1, \ldots, n$). Throughout this paper we denote the optimal design with respect to the criterion Φ_{e_p}, which is obtained from (2.3) for $c = e_p$ as e_p-optimal design or optimal design for estimating the coefficient θ_p in the polynomial regression model with no intercept.

We conclude this section with a geometric characterization of c-optimal designs called Elfving's theorem [see Elfving (1952)], which will be used in Section 3. A proof can be found in Dette et al. (2004).

Theorem 2.1 *An admissible design ξ^* for estimating the linear combination $c^\top\theta$ with support points $x_1, x_2, \ldots, x_m$ and weights $\omega_1, \omega_2, \ldots, \omega_m$ is c-optimal if and only if there exists a vector $u \in \mathbb{R}^d$ and a constant h such that the following conditions are satisfied:*

(1) $|u^\top f(x)| \le 1$ for all $x \in \mathcal{X}$;

(2) $|u^\top f(x_i)| = 1$ for all $i = 1, 2, \ldots, m$;

(3) $c = h \sum_{i=1}^m f(x_i)\omega_i u^\top f(x_i)$.

Moreover, in this case we have $c^\top M^-(\xi^*)c = h^2$.

3 Optimal designs for estimating individual coefficients in models with no intercept

For the polynomial regression model with no intercept the function $u^\top f$ in Theorem 2.1 is of the form $u^\top f(x) = \sum_{\ell=1}^n b_\ell x^\ell$. This function will be called extremal polynomial throughout this paper. From Theorem 2.1 it follows that the support points of the e_p-optimal design are the extremal points of a - in some sense - optimal polynomial. In fact it is possible to identify these optimal polynomials explicitly. For this purpose let

$$T_s(x) = \cos(s \arccos(x))$$

denote the sth Chebyshev polynomial of the first kind [see Szegö (1975)] and consider the polynomials

$$T_{2k-1}(x)\ , \quad T_{2k+1}(x) \tag{3.1}$$

and the polynomial

$$E_{2k}(x) = T_k\Big((x^2(1+\cos\frac{\pi}{2k}) - \cos\frac{\pi}{2k})\Big). \tag{3.2}$$

It is easy to see that T_{2k-1} and T_{2k+1} have exactly $2k$ and $2k+2$ extremal points, which are denoted by $s_1 < s_2 < \ldots < s_{2k}$ and $x_1 < x_2 < \ldots < x_{2k+2}$, respectively. Note that these points are given explicitly by

$$s_i = \cos\left(\tfrac{(2k-i)\pi}{n}\right) \ \ (i = 1, 2, \ldots, 2k), \qquad x_i = \cos\left(\tfrac{(2k+2-i)\pi}{2k+1}\right) \ \ (i = 1, 2, \ldots, 2k+2). \tag{3.3}$$

Similarly, the polynomial E_{2k} in (3.2) has $2k$ extremal points $t_1, \ldots, t_{2k}$, which are given by

$$t_i = -\sqrt{\frac{\cos\frac{(i-1)\pi}{k} + \cos\frac{\pi}{2k}}{1 + \cos\frac{\pi}{2k}}}\ ,\ t_{2k+1-i} = \sqrt{\frac{\cos\frac{(i-1)\pi}{k} + \cos\frac{\pi}{2k}}{1 + \cos\frac{\pi}{2k}}}\ ,\ i = 1, \ldots, k \tag{3.4}$$

Finally for a given set of support points of a design, say $t_1^*, \ldots, t_m^*$, we define for $i = 1, \ldots, m$

$$\bar{L}_i(x) = \frac{x \prod_{j \neq i}(x - t_j^*)}{t_i^* \prod_{j \neq i}(t_i^* - t_j^*)} \tag{3.5}$$

as the ith Lagrange basis interpolation polynomial without intercept corresponding to the nodes $t_1^*, \ldots, t_m^*$ (note that the degree of $\bar{L}_i(x)$ is m). The main result of this paper is the following.

Theorem 3.1 *Consider the polynomial regression model of degree $n \geq 1$ with no intercept.*

(a) If $n = 2k+1$ or $n = 2k$ for some $k \geq 1$ and p is even, then there exists an e_p-optimal design supported at the extremal points $t_1, \ldots, t_{2k}$ of the polynomial $E_{2k}(x)$ defined in (3.4).

(b) If $n = 2k$ and p is odd, then there exists an e_p-optimal design supported at the extremal points $s_1, \ldots, s_{2k}$ of the polynomial $T_{2k-1}(x)$ defined in (3.3).

(c) If $n = 2k+1$ and $p = 1$ then there exist exactly two e_p-optimal designs with $2k+1$ support points: one design with support $x_2, \ldots, x_{2k+2}$ and the other design with support points $x_1, \ldots, x_{2k+1}$.
If $n = 2k+1$ and p is odd, $p > 1$ then there exist exactly two e_p-optimal designs with $2k+1$ support points. One design with support points $x_1, \ldots, x_k, x_{k+2} \ldots, x_{2k+2}$ and the other design with support points $x_1, \ldots, x_{k+1}, x_{k+3} \ldots, x_{2k+2}$.

The weights $\omega_1, \ldots, \omega_m$ at the support points $t_1^*, \ldots, t_m^*$ of the e_p-optimal design are given by the formula

$$\omega_i = \frac{|a_{i,p}|}{\sum_{j=1}^m |a_{j,p}|}\ ,\ i = 1, \ldots, m, \tag{3.6}$$

where $m = 2k$ in cases (a) and (b), $m = 2k+1$ in case (c) and $a_{p,i}$ is the coefficient of the monomial x^p in the polynomial $\bar{L}_i$ defined in (3.5) $(i = 1, \ldots, m)$.

Proof. We first consider assertion (a) and use Theorem 2.1 with the polynomial $u^\top f(x) = E_{2k}(x)$ defined in (3.2). The properties (1) and (2) are obviously fulfilled and it remains to show that condition (3) holds for some nonnegative weights ω_i, $i = 1, 2, \ldots, 2k$. This condition reads as follows

$$\delta_{qp} = h \sum_{i=1}^{2k} t_i^q \omega_i E_{2k}(t_i) \ , \quad q = 1, \ldots, 2k+1, \tag{3.7}$$

where δ_{qp} denotes Kronecker's symbol. We show that a solution is in fact possible under the symmetry assumption $\omega_{2k-i+1} = \omega_i$, $i = 1, 2, \ldots, k$. Observing that

$$E_{2k}(t_i) = E_{2k}(t_{2k-i+1}), \tag{3.8}$$

$$t_i^{2q+1} = -\left(t_{2k-i+1}\right)^{2q+1}, \ q = 0, 1, \ldots, k \tag{3.9}$$

we see that the condition (3.7) is obviously satisfied for odd exponents (note that p is even) Consequently, it remains to show that there exist nonnegative weights $\omega_1, \ldots, w_{2k}$ such that

$$h \sum_{i=1}^{2k} t_i^{2q} \omega_i E_{2k}(t_i) = \delta_{2q,p},$$

which reduces using the symmetries in (3.8) and (3.9) to

$$h \sum_{i=1}^{k} t_i^{2q} \omega_i E_{2k}(t_i) = \frac{1}{2} \delta_{2q,2p}, \quad q = 1, \ldots, k \tag{3.10}$$

for some constant h .
For this purpose we introduce the notation $\tilde{\beta} = (\beta_1, \ldots, \beta_k)^\top$, where $\beta_i = h\omega_i E_{2k}(t_i)$, and $\tilde{e}_{p/2} = (0, \ldots, 0, 1/2, 0, \ldots, 0)^\top \in \mathbb{R}^k$, where $1/2$ is in the $p/2$ position (recall that p is even) and rewrite the equations in (3.10) as follows

$$F\tilde{\beta} = \tilde{e}_{p/2},$$

where the matrix F is defined by $F = \left(t_i^{2q}\right)_{q,i=1}^k$. Because the functions $t^2, t^4, \ldots, t^{2k}$ generate a Chebyshev system on the interval $(-1, 0)$, the matrix F is non-singular and the elements of F^{-1} are alternating in sign. Consequently, the components of the vector

$$\tilde{\beta} = F^{-1} \widetilde{e}_{p/2}$$

are also alternating in sign and the corresponding weights $\omega_i = \beta_i / (hE_{2k}(t_i))$ are positive, which completes the proof of assertion (a).

Next we consider assertion (b) , where $n = 2k$ and p is odd. A direct calculation shows that properties (1) and (2) are fulfilled for the polynomial $u^\top f(x) = T_{2k-1}(x)$. Again we have to prove the existence of nonnegative weights ω_i, $i = 1, \ldots, 2k$ satisfying part (3) of

Theorem 2.1. We consider first the equations corresponding to even exponents and note that for arbitrary ω_j, $i = 1, \dots, 2k$, satisfying $\omega_{2k-i+1} = \omega_i$, $i = 1, \dots, k$ we have

$$\sum_{i=1}^{2k} s_i^{2q} \omega_i T_{2k-1}(s_i) = 0, \quad q = 1, \dots, k,$$

where we used the symmetry properties

$$T_{2k}(s_i) = -T_{2k-1}(s_{2k-i+1}), \;\; s_i^{2q} = (s_{2k-i+1})^{2q} \;, \;\; q = 0, \dots, k.$$

Therefore it remains to consider the equations corresponding to odd exponents, i.e. there exist nonnegative weights $\omega_i, \dots, \omega_{2k}$ such that $\omega_i = \omega_{2k-i+1}$, $i = 1, \dots, k$ and

$$h \sum_{i=1}^{2k} s_i^{2q-1} \omega_i T_{2k-1}(s_i) = \delta_{2q-1,p}, \quad q = 1, \dots, k,$$

which reduce (observing the symmetry properties) to

$$h \sum_{i=1}^{k} s_i^{2q-1} \omega_i T_{2k-1}(s_i) = \frac{1}{2} \delta_{2q-1,p}$$

for some nonnegative ω_i, $i = 1, \dots, k,$. With the notation $\tilde{\beta} = (\tilde{\beta}_1, \dots, \tilde{\beta}_k)$, where $h\tilde{\beta}_i = \omega_i T_{2k-1}(s_i)$, and $\tilde{e}_{(p-1)/2} = (0, \dots, 0, 1/2, 0, \dots, 0)^\top \in \mathbb{R}^k$, where the non-vanishing entry $1/2$ is in the $(p-1)/2$ position, we rewrite these equations in matrix form

$$F\tilde{\beta} = \tilde{e}_{(p-1)/2},$$

where $F = \left(s_i^{2q-1}\right)_{q,i=1}^{k}$. Note that the functions $t, t^3, \dots, t^{2k-1}$ generate a Chebyshev system on the interval $(-1, 0)$. Consequently, the matrix F is non-singular and the elements of F^{-1} are alternating in sign. This implies that the components of the vector

$$\tilde{\beta} = F^{-1} \tilde{e}_{(p-1)/2}$$

are also alternating in sign and the corresponding weights $\omega_i = \beta_i/(hT_{2k-1}(s_{2i-1}))$ are positive.

In order to prove part (c) we use the polynomial $u^\top f(x) = T_{2k+1}(x)$ as an extremal polynomial in Theorem 2.1 as it satisfies conditions (1) and (2) of this theorem. Consequently, the points $x_1, \dots, x_{2k+2}$ in (3.3) are potential support points of the e_p-optimal design. We now choose $2k+1$ points $t_1^*, t_2^*, \dots, t_{2k+1}^*$ from the extremal points as described in part (c) of Theorem 3.1.

By Theorem 2.1 a design with weights $\omega_1, \omega_2, \ldots, \omega_{2k+1}$ at the points $t_1^*, t_2^*, \ldots, t_{2k+1}^*$ is e_p-optimal if

$$e_p = hF\beta, \tag{3.11}$$

for some constant h, where β is a $(2k+1)$-dimensional vector with components $\beta_i = u^\top f(t_i^*)\omega_i = T_{2k+1}(t_i^*)\omega_i$ $(i = 1, \ldots, 2k+1)$ and $F = (f(t_1^*), \ldots, f(t_{2k+1}^*))$. Observing the identity $F^{-1}F = I_{2k+1}$ (here I_{2k+1} is the identity matrix) it follows

$$e_i^\top F^{-1} f(t_j^*) = \delta_{ij} \quad (i, j = 1, \ldots, 2k+1).$$

As these equations characterize the ith basis Lagrange interpolation polynomial with knots $t_1^*, \ldots, t_{2k+1}^*$ we have for any point $z \in \mathbb{R}$

$$e_i^\top F^{-1} f(z) = \bar{L}_i(z) = a_i^\top f(z)\ , \quad i = 1, \ldots, 2k+1,$$

where

$$a_i = (F^{-1})^\top e_i = (a_{i,1}, \ldots, a_{i,2k+1})^\top \tag{3.12}$$

is the vector of coefficients of the ith basis Lagrange interpolation polynomial $(i = 1, \ldots, 2k+1)$. Therefore we obtain for the solution of (3.11)

$$h\beta = F^{-1} e_p = (a_{1,p}, \ldots, a_{2k+1,p})^T$$

or equivalently (since $\beta_i = \omega_i T_{2k+1}(t_i^*)$)

$$h\beta_i = h\omega_i T_{2k+1}(t_i^*) = \frac{1}{p!} \frac{d^p}{d^p z} \bar{L}_i(z)\Big|_{z=0} = a_{i,p}\ , \quad i = 1, \ldots, 2k+1. \tag{3.13}$$

Therefore the representation (3.6) follows if $T_{2k+1}(t_1^*)a_{1,p}, \ldots, T_{2k+1}(t_{2k+1}^*)a_{2k+1,p}$ have the same sign. In this case part (3) of Theorem 2.1 is also satisfied (as we can solve (3.11) with positive weights) and the part (c) of Theorem 3.1 proved. For a proof of this property we now consider the different cases in Theorem 3.1 separately.

First consider the case $p = 1$ and let $t_1^*, \ldots, t_{2k+1}^*$ be either $x_1, \ldots, x_{2k+1}$ or $x_2, \ldots, x_{2k+2}$. Note that in this case either the smallest point -1 or the largest point 1 has been deleted from the whole set of the extremal points of the Chebyshev polynomial $T_{2k+1}(x)$. A direct calculation by Vieta' formulas gives for the ith coefficient of the polynomial (3.5)

$$a_{i,1} = \frac{\prod_{j=1}^{2k+1} t_j^*}{(t_i^*)^2 \prod_{j \neq i}(t_i^* - t_j^*)}\ , \quad i = 1, \ldots, 2k+1,$$

(note that the polynomial $\bar{L}_i(z) = a_i^T f(z)$ in (3.5) has the roots $t_1^*, \ldots, t_{2k+1}^*$ and 0). As the sign of the denominator is alternating with i and the sign of $T_{2k+1}(t_i^*)$ is also alternating with i it follows that all products $T_{2k+1}(t_i^*)a_{i,1}$ have the same sign, $i = 1, 2, \ldots, 2k+1$ (note

that the numerator does not depend on i).
In the case where $p = 2l+1 > 1$ is odd the argument is very similar. Here let $t_1^*, \dots, t_{2k+1}^*$ be either $x_1, x_2, \dots, x_k, x_{k+2}, \dots, x_{2k+2}$ or $x_1, x_2 \dots, x_{k+1}, x_{k+3}, \dots, x_{2k+2}$. This means that in this case one of the two points with minimal distance to 0 has been deleted from the set of the extremal points of $T_{2k+1}(x)$. By the Vieta' formulas we obtain for the ith coefficient of the polynomial $\bar{L}_{2l+1}(z)$ in (3.5) the representation

$$a_{i,2l+1} = -\frac{\sum_{\substack{1\le j_1<j_2<\ldots<j_{2l}\le 2k+1 \\ j_1,\ldots,j_{2l}\neq i}} \prod_{s=1}^{2l} t_{j_s}^*}{t_i^* \prod_{j\neq i}(t_i^* - t_j^*)} \ , \quad i = 1, \dots, 2k+1$$

(note that one of the roots is equal to 0) and the symmetry of the roots yields

$$a_{i,2l+1} = -\frac{\sum_{\substack{1\le j_1<j_2<\ldots,j_l\le k+1 \\ j_1,\ldots,j_l\notin\{i,2k+2-i\}}} \prod_{s=1}^{l} (t_{j_s}^*)^2}{t_i^* \prod_{j\neq i}(t_i^* - t_j^*)} \ , \ i = 1, \dots, 2k+1.$$

Now it can be easily checked that $T_{2k+1}(t_1^*)a_{1,2l+1}, \dots T_{2k+1}(t_{2k+1}^*)a_{2k+1,2p+1}$ have the same sign. These arguments complete the proof of part (c) of Theorem 3.1.

Finally, it remains to show the representation (3.6) for the weights in the case (a) and (b). We omitt the details here as this can be done in a similar way as in the proof of part (c) of Theorem 3.1. □

Example 3.1 We determine the optimal designs for estimating the individual coefficients in a cubic regression with no intercept. For this purpose let $P(x)$ be an extremal polynomial from Elfving's theorem.

(a) If $p = 1$ we can use part (c) of Theorem 3.1. The extremal polynomial is given by $P(x) = x^3 - \frac{3}{4}x$ with extremal points -1, $-\frac{1}{2}$, $\frac{1}{2}$ and 1. There exist two 3-point e_1-optimal designs. One with masses $\frac{1}{9}$, $\frac{2}{3}$ and $\frac{2}{9}$ at the points -1, $-\frac{1}{2}$, and $\frac{1}{2}$ and the other one with masses $\frac{2}{9}$, $\frac{2}{3}$ and $\frac{1}{9}$ at the points $-\frac{1}{2}$, $\frac{1}{2}$ and 1.

(b) If $p = 2$ we can use part (a) of Theorem 3.1. Consequently, there exists a unique e_2-optimal design supported at 2 points, that is

$$\begin{pmatrix} -1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

In this case the corresponding extremal polynomial is not unique and given by $P(x) = x^2 - qx + qx^3$, where $q \in [-1, 1]$.

(c) If $p = 3$ we can again use part (c) of Theorem 3.1. The extremal polynomial is given by $P(x) = x^3 - \frac{3}{4}x$ with extremal points -1, $-\frac{1}{2}$, $\frac{1}{2}$ and 1. There exist two 3-point e_3-optimal designs. One with masses $\frac{1}{12}$, $\frac{2}{3}$ and $\frac{1}{4}$ at the points -1, $\frac{1}{2}$, and 1 and the other one with masses $\frac{1}{4}$, $\frac{2}{3}$ and $\frac{1}{12}$ at the points -1, $-\frac{1}{2}$ and 1.

Example 3.2 We determine the optimal designs for estimating the individual coefficients in a polynomial regression model of degree four with no intercept. Note that in this case Theorem 3.1(a) for $p = 2, 4$ and Theorem 3.1(b) for $p = 1, 3$ are applicable. Consequently the e_p-optimal designs are always unique

(a1) If $p = 2$, the extremal polynomial is given by $P(x) = x^4 - 2(\sqrt{2} - 1)x^2$ and the unique 4-point optimal design for estimating the coefficient of x^2 is given by

$$\begin{pmatrix} -1 & -\sqrt{\sqrt{2}-1} & \sqrt{\sqrt{2}-1} & 1 \\ \frac{\sqrt{2}}{8\sqrt{2}+8} & \frac{3\sqrt{2}+4}{8\sqrt{2}+8} & \frac{3\sqrt{2}+4}{8\sqrt{2}+8} & \frac{\sqrt{2}}{8\sqrt{2}+8} \end{pmatrix}.$$

(a2) If $p = 4$, the extremal polynomial is given by $P(x) = x^4 - 2(\sqrt{2} - 1)x^2$ and the unique 4-point optimal design for estimating the coefficient of x^4 is given by

$$\begin{pmatrix} -1 & -\sqrt{\sqrt{2}-1} & \sqrt{\sqrt{2}-1} & 1 \\ \frac{\sqrt{2}}{4\sqrt{2}+4} & \frac{\sqrt{2}+2}{4\sqrt{2}+4} & \frac{\sqrt{2}+2}{4\sqrt{2}+4} & \frac{\sqrt{2}}{4\sqrt{2}+4} \end{pmatrix}.$$

(b1) If $p = 1$, the extremal polynomial is given by $P(x) = x^3 - \frac{3}{4}x$ and the unique 4-point optimal design for estimating the coefficient of x^1 is given by

$$\begin{pmatrix} -1 & -\frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{18} & \frac{4}{9} & \frac{4}{9} & \frac{1}{18} \end{pmatrix}.$$

(b2) If $p = 3$, the extremal polynomial is given by $P(x) = x^3 - \frac{3}{4}x$ and the unique 4-point optimal design for estimating the coefficient of x^3 is given by

$$\begin{pmatrix} -1 & -\frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}.$$

Note that this design is also optimal for estimating the coefficient of x^3 and in a cubic regression with intercept [see Dette (1990)].

Acknowledgements This work has been supported in part by the Collaborative Research Center "Statistical modeling of nonlinear dynamic processes" (SFB 823, Teilprojekt C2) of the German Research Foundation (DFG). The work of Viatcheslav Melas and Petr Shpilev was partly supported by Russian Foundation for Basic Research (project no. 17-01-00161).

References

Chang, F.-C. (1999). Exact D-optimal designs for polynomial regression with intercept. *Statistics & Probability Letters*, 44:131–136.

Chang, F.-C. and Heiligers, B. (1996). E-optimal designs for polynomial regression without intercept. *Journal of Statistical Planning and Inference.*, 55(3):371–387.

Dette, H. (1990). A generalization of D- and D_1-optimal designs in polynomial regression. *Annals of Statistics*, 18:1784–1805.

Dette, H., Melas, V. B., and Pepelyshev, A. (2004). Optimal designs for estimating individual coefficients in polynomial regressiona functional approach. *Journal of Statistical Planning and Inference*, 118(1):201 – 219.

Elfving, G. (1952). Optimal allocation in linear regression theory. *The Annals of Mathematical Statistics*, 23:255–262.

Fang, Z. (2002). D-optimal designs for polynomial regression models through origin. *Statistics & Probability Letters*, 57:343–351.

Huang, M.-N. L., Chang, F.-C., and K., W. W. (1995). D-optimal designs for polynomial regression without an intercept. *Statistica Sinica*, 5(2):441–458.

Kiefer, J. (1974). General equivalence theory for optimum designs (approximate theory). *Annals of Statistics*, 2:849–879.

Li, K.-H., Lau, T.-S., and Zhang, C. (2005). A note on D-optimal designs for models with and without an intercept. *Statistical Papers.*, 46(3):451–458.

Ortiz, I. and Rodríguez, C. (1998). D-optimal designs for weighted polynomial regression without any initial terms. In Atkinson, A. C., Pronzato, L., and Wynn, H. P., editors, *MODA 5 - Advances in Model-Oriented Design Analysis and Experimental Design.*, pages 67–74. Physica-Verlag, Heidelberg.

Pukelsheim, F. (2006). *Optimal Design of Experiments.* SIAM, Philadelphia.

Silvey, S. (1980). *Optimal Design.* Chapman and Hall, London.

Studden, W. J. (1968). Optimal designs on Tchebycheff points. *Annals of Mathematical Statistics*, 39(5):1435–1447.

Szegö, G. (1975). *Orthogonal Polynomials.* American Mathematical Society, Providence, R.I.