Limit Theorems and Statistical Inference for Solutions of Some Stochastic (Partial) Differential Equations

Dissertation
zur Erlangung des akademischen Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)

Der Fakultät für Mathematik
der Technischen Universität Dortmund
vorgelegt von

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im Oktober 2019

Dissertation

Limit Theorems and Statistical Inference for Solutions of Some Stochastic (Partial) Differential Equations

Fakultät für Mathematik

Technische Universität Dortmund

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Tag der mündlichen Prüfung: 8. Januar 2020

Acknowledgements

First of all, I thank my supervisor Prof. Jeannette Woerner for her guidance throughout the last three years, in particular, for suggesting a very interesting research topic, for our fruitful discussions, for sharing her thoughts and ideas, for teaching me how to work independently and how to cooperate, and for giving me many wonderful opportunities such as attending great conferences and spending time abroad. But above all I thank her for her unwavering support and encouragement in research, in teaching and in academic life in general as well as for her patience and kindness at all times.

I am also very grateful to Prof. Markus Reiß who has kindly agreed to be the second reviewer of this thesis for his inspiring questions and remarks concerning the manuscript. I thank him, moreover, for igniting my interest in probability back when I was a relatively clueless student jumping from one subject to another. I am also grateful to Prof. Michael Voit for agreeing to be my third examiner, but also for all the questions and discussions during the Oberseminar.

I am deeply thankful to Prof. Ciprian Tudor who has welcomed me in Lille for the possibility to work with him and learn from him, for creating a very productive atmosphere as well as for his patience and support. Working with him has been a privilege.

I express my sincere gratitude to everyone else who has guided and inspired me scientifically, particularly to Prof. Peter Imkeller for introducing me to Malliavin calculus, to Jan Gairing for generously and readily sharing his ideas and perspective, to Prof. Valentine Genon-Catalot whose notes have kickstarted Chapter 4 of this thesis, to Prof. Antoine Ayache for an insightful discussion and to Randolf Altmeyer for being my role model in research and in teaching.

Moreover, I am grateful to the RTG 2131 "High-Dimensional Phenomena in Probability - Fluctuations and Discontinuity" and to the SFB 823 "Statistical Modelling of Nonlinear Dynamic Processes" funded by the DFG for their financial support at different stages of my work, their excellent scientific programmes and the possibility to connect with great people and inspiring researchers.

Special thanks go out to my colleagues from Lehrstuhl IV in Dortmund and the équipe des probas in Lille as well as to everyone else in the open neighbourhood for making my time in academia enjoyable and insightful, for our passionate and fun discussions about maths (variétés stochastiques! and other mindblowing things), life, the universe and everything, 42 and 52. Among these I particularly thank Meryem who has become not only a great friend but also a great collaborator and Viktor for the careful proofreading of this thesis.

Outside of work I thank my friends and family for their love and support. Thank you for being in my life, you make it worth living. I love you very deeply. Daniel und Justus, ich weiß nicht, wo ich ohne euch wäre. Alexandre, merci d'avoir illuminé ma vie. Most importantly, I thank my parents, Nadiya and Oleksandr. Я вас дуже люблю.

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Chapter 1

Introduction

One of the defining areas in the modern stochastic analysis is the study of solutions of stochastic (partial) differential equations (S(P)DE in the sequel). Driven by the wish of defining a process that changes according to some random force, equations containing known random processes are being considered and analysed. In order to do this, one needs to find a "natural" definition and establish properties of stochastic integrals, i.e. integrals with respect to stochastic processes.

From the practical point of view an important question in this context is the statistical inference for such processes, in particular, estimation of parameters and functions involved in a given model. If a certain process is assumed to be described by an equation an estimator provides a precise quantification of the process's behaviour, leading to a better understanding of the dynamics. Conversely, a practically motivated study often helps advance the theory in order to accommodate some particular properties of the setup.

Possibly the simplest example of a stochastic differential equation is the so called Ornstein-Uhlenbeck equation. It combines a very basic ordinary differential equation (ODE) $x'(t) = \alpha x(t)$, $x(0) = x_0 \in \mathbb{R}$ with a parameter $\alpha \in \mathbb{R}$, with an additive white noise component driving the dynamics. The obtained equation has the form

$$X_t = X_0 + \int_0^t \alpha X_s ds + B_t, \quad t \geq 0,$$
$$X_0 = x_0 \in \mathbb{R},$$

where B is a Brownian motion and $\alpha \in \mathbb{R}$ is a parameter. From the ODE perspective, if B were a deterministic function, the solution of such an equation would have been a function given by

$$X_t = e^{\alpha t} x_0 + e^{\alpha t} \int_0^t e^{-\alpha s} dB_s$$

whenever the right-hand side is well defined. With the development of the pathwise as well as of the Itō stochastic calculus in the first half of the 20th century it became possible to assign a meaningful definition to this solution as an integral with respect to a Brownian motion and to consider integrals of and with respect to the solution process X needed for further analysis.

An important element for the definition of Itō integrals is a certain structural property of the Brownian motion, namely the fact that it is a semimartingale. With the statistical background in mind the estimation of the parameter α is a natural question, and a lot of literature has been devoted to it. In the framework of continuous observations on an interval $[0, T]$ classical ideas such as a maximum likelihood estimator obtained by the Girsanov theorem as well as the least squares ansatz lead to the estimator

$$\frac{\int_0^T X_s dX_s}{\int_0^T X_s^2 ds}$$

that is strongly consistent and asymptotically normal for $\alpha < 0$ and has a Cauchy limiting distribution for $\alpha > 0$. These results are described in [42] together with an optimality study of the estimator: For $\alpha < 0$ the local asymptotic normality (LAN) of the model is demonstrated and for $\alpha > 0$ the local asymptotic mixed normality (LAMN) property, and the estimator achieves the Hajek-LeCam bound in both cases (implying optimality in a certain sense). The applications for modelling with the Ornstein-Uhlenbeck process X are numerous and reach from describing the structure of interest rates in finance (see [81]) and analysing the particle dynamics in physics ([34]) to quantifying the phenotypic evolution in biology ([47]).

The simplicity of the Ornstein-Uhlenbeck model makes it a very versatile tool, but it also offers a lot of space for modification and adaptation for certain specific setups. For instance, one can consider a situation where the driving random process is non-Markovian and the increments exhibit a long-term correlation. A simple example of such a process is a fractional Brownian motion $(B_t^H)_{t\geq 0}$, which is a Gaussian process defined via a covariance formula depending on a parameter $H \in (0, 1)$. For $H > \frac{1}{2}$ the increment process $(B_{n+1}^H - B_n^H)_{n\in\mathbb{N}}$ is strongly correlated even over longer periods of time (this property is known as long memory or long range dependence and will be made precise later), allowing the modelling of processes with Gaussian marginal but having a more complex dependency structure than in case of the Brownian motion, which emerges as a special case for $H = \frac{1}{2}$. An application for this kind of processes related to studying turbulence in physics is described in [87]. The solution of the equation

$$X_t = X_0 + \int_0^t \alpha X_s ds + B_t^H, \quad t \geq 0,$$
$$X_0 = x_0 \in \mathbb{R},$$

can still be simply derived in the pathwise sense, however, the estimation study for α proves to be more challenging than for the Brownian motion. To start with, B^H is not a semimartingale for $H \neq \frac{1}{2}$, so one cannot directly rely on the classical Girsanov theorem for obtaining a maximum likelihood estimator. A way to overcome this problem is presented in [38]: The authors define semimartingales that can be associated with the fractional Brownian motion B^H as well as with the solution process X and subsequently make use of the Girsanov theorem. The estimator defined in this way thus has a more complicated representation compared to the Brownian case. It is strongly consistent, asymptotically normal and attains the Hajek-LeCam bound related to the LAN property of the model for $\alpha < 0$. This property has been shown only recently in [44]. Following the least squares approach, on the other hand, still yields the estimator

$$\frac{\int_0^T X_s dX_s}{\int_0^T X_s^2 ds},$$

however, the pathwise definition of the stochastic integral only yields a consistent estimator for $\alpha > 0$ (see [9]), and in order to obtain an estimator converging to the true parameter α for $\alpha < 0$ a different notion has to be considered, namely the so called divergence integral from the Malliavin calculus, an area of stochastic analysis whose foundation was laid in 1970-80s and which has gained popularity in recent years. These and other results concerning SDEs driven by a fractional Brownian motion have been proved starting from the late 2000s (in particular, strong consistency and asymptotic normality of the above estimator for $\alpha < 0$ has been proved in [30] in 2009) and still constitute a fruitful basis for further investigations despite the apparent simplicity of the objects involved. Nothing is known, for example, about the optimal convergence rates of estimators for $\alpha > 0$.

One instance of research providing a generalisation of the fractional Ornstein-Uhlenbeck model is a paper by H. Dehling, B. Franke and J.H.C. Woerner ([24]), where the equation

$$\begin{aligned} X_t &= X_0 + \int_0^t \left(\sum_{i=1}^p \mu_i \varphi_i(s) + \alpha X_s \right) ds + B_t^H, \quad t \geq 0, \\ X_0 &= x_0 \in \mathbb{R}, \end{aligned}$$

with periodic functions φ_i having the same period, $\mu_i \in \mathbb{R}$ and $\alpha < 0$ is considered. A driving idea for this model from the point of view of applications is time-continuous data with Gaussian marginals depending on a long-range dependent process and carrying some periodicities in its underlying structure. An example is the earth temperature data derived from the ice core analysis: It is known to have a long memory and to depend on solar cycles (see [85]).

Assuming continuous observations, the authors have proposed a least squares type estimator for the vector $(\mu_1, \dots, \mu_p, \alpha)$ and shown its consistency and asymptotic normality by means of Malliavin calculus. As of yet, no results concerning the optimality of this estimator have been shown.

The above paper has been the starting point for this thesis, and indeed, for the most part this manuscript should be considered to be a cloud of results and observations surrounding it. The thesis is far from being comprehensive (nor does it have the intention to be), and some of the results lead to many more questions worth investigating. Rather than that it should be considered as a kaleidoscope of answers to some questions leading in several major directions inspired by [24] and possibly widening the pavement for a better structural understanding of related problems.

Chapter 3, following the preliminary part of the thesis, deals with the least squares type estimator for the vector $(\mu_1, \dots, \mu_p, \alpha)$ in the case $\alpha > 0$ and investigates its asymptotic properties. One of the most curious results from this chapter (along with the non-Gaussian limiting distribution of the estimator) is the change in the convergence rate for some specific φ_i and the form of the limiting variance obtained in this case. Additionally, this result is transferred to the case $\alpha < 0$, where it also leads to improved rates of convergence. The principal part of this chapter is presented in the preprint

- R. Shevchenko, J. H. C. Woerner - Inference for fractional Ornstein-Uhlenbeck type processes with periodic mean in the non-ergodic case, 2019, arXiv:1903.08033.

In Chapter 4 the setting

$$
\begin{aligned}
X_t &= X_0 + \int_0^t (L(s) + \alpha X_s)\, ds + B_t^H,\ t \geq 0, \\
X_0 &= x_0 \in \mathbb{R},
\end{aligned}
$$

is considered for $\alpha < 0$. However, here we do not assume that L is a linear combination of known functions. Instead, we assume periodicity and estimate the function L nonparametrically, relying on such classical techniques as orthogonal projections and appropriate truncation. We define an estimator related to the one constructed in [24], show that it converges in L^2 and derive its rate of convergence using Malliavin calculus.

The extension considered in Chapter 5 concerns the random part of the equation studied in [24]. The solutions of the equations mentioned until now are Gaussian processes. However, a consideration of hydrological data (for which models with long memory processes are often considered and which gave the initial motivation to the definition of fractional Brownian motion in [46]) shows the presence of skewness in the observations (see [43]), requiring a non-Gaussian driving process. One class of long range dependent processes having this property are the non-Gaussian Hermite processes, prominent (and best studied) among them the Rosenblatt process. Its marginals do indeed have a non-symmetric Lebesgue density (see [82] for plots and theoretical results), making it an appropriate candidate for such a model. We consider thus the process

$$\begin{aligned} X_t &= X_0 + \int_0^t \left(\sum_{i=1}^{p} \mu_i \varphi_i(s) + \alpha X_s \right) ds + Z_t^H,\ t \geq 0, \\ X_0 &= 0, \end{aligned}$$

where Z^H is the Rosenblatt process, and estimate the vector $(\mu_1, \ldots, \mu_p, \alpha)$ for $\alpha < 0$ following the construction from [24]. For the proofs of asymptotic properties we rely this time on specific results from stochastic analysis with respect to the Rosenblatt process and further explore the structure of the estimator from the theoretical point of view. An additional value of this chapter is the definition of estimators using pathwise rather than divergence type integrals and the proof of their asymptotic properties. The results are presented in the paper

- R. Shevchenko, C. A. Tudor - Parameter estimation for the Rosenblatt Ornstein–Uhlenbeck process with periodic mean, 2019, Statistical Inference for Stochastic Processes.

So far there has been one parameter in the considered equations that has been assumed to be fixed, namely the so called Hurst parameter H responsible for the long range dependence structure of the driving process. In fact, for continuous observations this parameter is directly accessible and does not need to be estimated. For discrete observations there is a vast amount of literature dedicated to the estimation of H in a multitude of settings by studying the so called variations of observed processes: The monograph [77], for example, is dedicated entirely to this topic. In the last chapter of the thesis we also consider one such question, however, this time our main object is significantly more involved. We are dealing with the solution u of a stochastic wave equation, an SPDE of the form

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t,x) &= \frac{\partial^2 u}{\partial x^2} u(t,x) + \dot{W}^H(t,x),\ t \geq 0,\ x \in \mathbb{R}, \\ u(0,x) &= 0, \quad x \in \mathbb{R}, \\ \frac{\partial u}{\partial t}(0,x) &= 0, \quad x \in \mathbb{R}, \end{cases}$$

where W^H is a noise white in space and fractional in time. The starting point for the investigation of variations of u is, as opposed to the previous results, the paper [35] by M. Khalil and C. A. Tudor considering this equation. However, in the course of our study we encounter familiar objects such as fractional Brownian motion (as well as a Gaussian process with a similar covariance structure), but also a distribution related to a Rosenblatt marginal. Moreover, this part is methodically connected to the others by ideas from Malliavin calculus. The main results are limit theorems, most importantly a non-central limit theorem, where the distribution mentioned above emerges, as well as the study of several different estimators of the Hurst parameter H. The content of this chapter can be found in the preprint

- R. Shevchenko, M. Slaoui, C. A. Tudor - Generalized k-variations and Hurst parameter estimation for the fractional wave equation via Malliavin calculus, 2019, arXiv:1903.02369, accepted for publication in Journal of Statistical Planning and Inference.

Chapter 6 is the only part of the thesis that contains simulations illustrating the results. While it is methodically challenging to include simulations in Chapters 4 and 5 (given the fact that divergence integrals cannot in general be approximated by an appropriate discretisation), a simulation study concerning Chapter 3 is, in principle, possible. However, due to very high values emerging in the simulation, the numerical error is high, such that many observations are needed to adequately approximate the integrals involved. In Chapter 6, on the other hand, we do not encounter such difficulties and the simulation study seems to align with the theoretical results.

In total, the contents of this thesis demonstrate an interplay of theoretical and practical ideas that have largely motivated each other either through mathematical curiosity or following a concrete wish related to a (possible) application.

Chapter 2

Preliminaries and background

This chapter provides the reader with a toolkit of basic definitions as well as techniques that will be used throughout the thesis. At the same time it sets up the scene for the main chapters by making the reader familiar with reoccurring notations and machinery.

2.1 Preliminaries

2.1.1 Properties of the fractional Brownian motion

We begin the preliminaries with a short overview over one of the central objects of this thesis, namely the fractional Brownian motion. Its basic properties are well known and there are numerous sources explaining them in detail (for example, the monographs [50] and [11]). In this chapter our references are [48] and [57].

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space.

The (two-sided) fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$ is a centred Brownian process $B^H = (B_t^H)_{t\in\mathbb{R}}$ on $(\Omega, \mathcal{F}, P)$ with the properties $B_0^H = 0$ and

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \qquad t, s \in \mathbb{R}.$$

It follows from the definition that

$$\mathbb{E}[(B_t^H - B_s^H)(B_u^H - B_v^H)] = \frac{1}{2}(|s-u|^{2H} + |t-v|^{2H} - |t-u|^{2H} - |s-v|^{2H}).$$

This implies that the process B^H has stationary increments. Moreover, it follows that $\mathbb{E}[(B_t^H - B_s^H)^2] = |t-s|^{2H}$, which, combined with Gaussianity, yields that there is a continuous modification (with $(H-\varepsilon)$-Hölder continuous paths for any $\varepsilon > 0$) of B^H by Kolmogorov's continuity criterion. From now on we assume that the fBm we consider is a continuous modification.

Regarding the stationarity of increments one can compute the autocovariance function

$$r(n) := \mathbb{E}[B_1^H(B_{n+1}^H - B_n^H)] = (n+1)^{2H} - 2n^{2H} + (n-1)^{2H} \overset{n\to\infty}{\sim} n^{2H-2},$$

where the last step is due to a Taylor approximation. Depending on the value of H one can consider three cases regarding the asymptotics of this function:

- if $H \in \left(0, \frac{1}{2}\right)$, then $\sum_{n\in\mathbb{N}} |r(n)| \sim \sum_{n\in\mathbb{N}} n^{2H-2} < \infty$. In this case we say that B^H is short range dependent,
- if $H = \frac{1}{2}$, then $r(n) = 0$ for all $n \in \mathbb{N}$. More generally, in this case the increments of B^H are independent and $B^{\frac{1}{2}}$ is the usual Brownian motion,
- if $H \in \left(\frac{1}{2}, 1\right)$, then $\sum_{n\in\mathbb{N}} |r(n)| \sim \sum_{n\in\mathbb{N}} n^{2H-2} = \infty$, and we say that B^H has the property of long range dependence.

In Figures 2.1, 2.2, 2.3 examples of sample paths in these three cases are demonstrated.

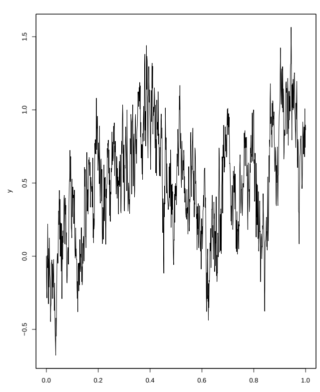

Figure 2.1: $H = 0.25$

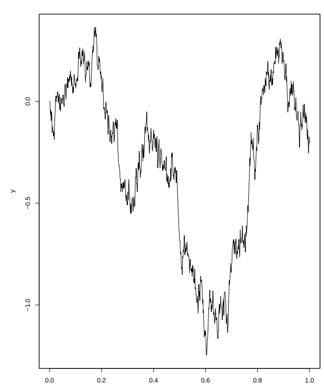

Figure 2.2: $H = 0.5$

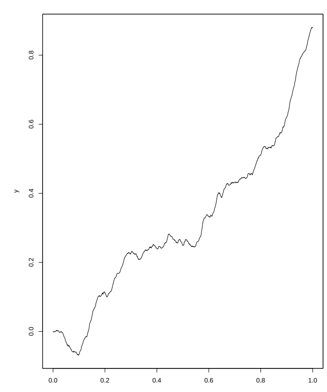

Figure 2.3: $H = 0.9$

Another important property of the fBm related to the covariance function is the fact that it is a self similar process. Indeed,

$$\{B_{at}^H,\, t \in \mathbb{R}\} \overset{d}{=} \{a^H B_t^H,\, t \in \mathbb{R}\}$$

in the sense of finite dimensional distributions.

In the course of this thesis we will be concerned with integrals with respect to fBm. It is well known that for $H = \frac{1}{2}$ the process B^H is a martingale and thus the Itō calculus is a good option. However, for $H \neq \frac{1}{2}$ this option is not available, as we will see in the upcoming part of this chapter.

2.1.1 Definition. For any $k > 0$, a natural number $n \geq 1$ and for any stochastic process $Z = (Z_t)_{t\geq 0}$ we call

$$V_k^n(Z) := \sum_{i=1}^{n} \left| Z_{\frac{i}{n}} - Z_{\frac{i-1}{n}} \right|^k$$

the realised k-variation of Z.

We call the limit of $V_n^k(Z)$ in probability its k-variation.

Recall that for establishing Itō's calculus with respect to a process it is necessary for this process to be a semimartingale, that is, a sum of a finite variation process and a local martingale (having a finite quadratic variation).

2.1.2 Proposition. *For $H \neq \frac{1}{2}$ the process B^H is not a semimartingale.*

Proof. For $k > 0$ consider the process

$$Y_{n,\,k} = n^{kH} \sum_{i=1}^{n} \left| B^H_{\frac{i}{n}} - B^H_{\frac{i-1}{n}} \right|^k$$

and note that by the self similarity the sequence $(Y_{n,\,k})_{n\geq 1}$ has the same distribution as a process $(\tilde{Y}_{n,\,k})_{n\geq 1}$ defined by

$$\tilde{Y}_{n,\,k} = \frac{1}{n} \sum_{i=1}^{n} \left| B^H_i - B^H_{i-1} \right|^k .$$

As discussed above, the sequence of the increments $\{B^H_i - B^H_{i-1},\, i \geq 1\}$ is stationary. By Gaussianity and the convergence of the covariance function it follows that this sequence is even ergodic. Thus, $\tilde{Y}_{n,\,k}$ converges almost surely to $\mathbb{E}[|B^H_1|^k]$ as n tends to infinity. Therefore, $V_n^k(B^H)$ converges in probability to zero if $kH > 1$ and to infinity if $kH < 1$.

- For $H < \frac{1}{2}$ the number $k > 2$ can be chosen such that $kH < 1$ and thus, the quadratic variation of B^H is infinite.
- For $H > \frac{1}{2}$ first consider k such that $\frac{1}{H} < k < 2$. It follows that the k-variation of B^H is zero, and thus also its quadratic variation. However, now we can choose k between 1 and $\frac{1}{H}$ and deduce that the 1-variation must be infinite.

Hence, in both cases B^H cannot be a semimartingale. □

We conclude this section by indicating a property connecting the fBm with a Brownian motion. Consider the square integrable kernel

$$K_H(t,\, s) := c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

where $c_H = \left(\frac{H(2H-1)}{\beta(2-2H,\, H-\frac{1}{2})}\right)^{\frac{1}{2}}$ and $t > s$. Then for a Wiener process $(W_t)_{t\in\mathbb{R}^+}$ the process defined by the Itō integral

$$\int_0^t K_H(t,\, s) dW_s$$

is a fractional Brownian motion.

A generalisation of this construction using the concept of multiple Wiener-Itō integrals creates a collection of processes having the same covariance structure as the fBm. They are known as Hermite processes and we will introduce them later.

2.1.2 Stochastic Integration

As mentioned earlier, integration with respect to the fractional Brownian motion poses a particular challenge in its own right. One of the main tools to define such integrals is the Malliavin calculus which is particularly well suited for Gaussian processes. The theory as well as the application to the fractional Brownian motion are described very extensively in [57]. Another possibility is to consider Young integrals which are defined as Riemann-Stieltjes type integrals under certain conditions introduced and studied in [86]. A comparable Riemann-type construction is also proposed in [22].

A comprehensive overview over different integration techniques is presented in the book [48].

Malliavin Calculus

In this section we will talk about the main definitions of Malliavin calculus following mainly [57] as well as [51] and quote some results that will be needed later, focussing in particular on the example of the fractional Brownian motion. If it is not specifically stated otherwise, the proofs of the results in this section can be found in [57] and are not given here. Throughout this section let $T \in \mathbb{R}^+ \cup \{\infty\}$.

For a real separable Hilbert space $(\mathscr{H},\, \langle \cdot\, ,\, \cdot \rangle_{\mathscr{H}})$ we call a stochastic process $W = \{W(h),\, h \in \mathscr{H}\}$ in a complete probability space $(\Omega,\, \mathscr{F},\, \mathbb{P})$ an isonormal Gaussian process on $\mathscr{H}$ if W is a centred Gaussian family of random variables such that for all $f,\, g \in \mathscr{H}$

$$\mathbb{E}[W(f)W(g)] = \langle f,\, g \rangle_{\mathscr{H}}.$$

For a fractional Brownian motion $(B_t^H)_{t\in[0,\,T)}$ with Hurst parameter $H \in (0,\, 1)$ let $\mathscr{H}^H$ be the closure of the set of indicator functions with respect to the inner product

$$\langle \mathbf{1}_{[0,\,s]},\, \mathbf{1}_{[0,\,t]} \rangle_{\mathscr{H}^H} := \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \quad t,\, s \in [0,\, T).$$

An isonormal process B^H corresponding to this inner product is defined via $B^H(\mathbf{1}_{[0,\,s]}) = B^H_s$ for intervals and is extended linearly to simple functions and by taking L^2-limits to all elements of $\mathscr{H}^H$. This is described in detail in Example 2.1.5 in [51]. Thus, we have embedded $(B^H_t)_{t\in[0,\,T)}$ into an isonormal Gaussian process on $\mathscr{H}^H$. The notation $\mathscr{H}^H$ will persist throughout the thesis.

For $H > \frac{1}{2}$ there exists an explicit characterisation of a large subset of $\mathscr{H}^H$ (see [61]). We consider the following definition:

$$|\mathscr{H}^H| := \left\{ f : [0,\,T) \to \mathbb{R} \text{ meas. s.th. } \int_0^T \int_0^T |f(v)||f(u)||u-v|^{2H-2} dv du < \infty \right\}.$$

It is shown in [60] that $|\mathscr{H}^H|$ is a subspace of $\mathscr{H}^H$, which yields a sufficient condition for functions to belong to the space $\mathscr{H}^H$.

The following result provides us with a useful representation of the inner product of two functions from the space $|\mathscr{H}^H|$.

2.1.3 Proposition. *Let $H > \frac{1}{2}$. Then we have for $f,\, g \in |\mathscr{H}^H|$*

$$\langle f,\, g\rangle_{\mathscr{H}^H} = \underbrace{H(2H-1)}_{=:\alpha_H} \int_0^T \int_0^T f(v)g(u)|u-v|^{2H-2} du dv.$$

2.1.4 Remark. Similarly to the above proposition one can show that for $H = \frac{1}{2}$ (that is, if W is the usual Brownian motion) the space $\mathscr{H}$ is identical with the space $L^2([0,\,T))$ endowed with the usual inner product.

For a Gaussian process $(W_t)_{t\in[0,\,T)}$ on $\mathscr{H}$ let $\mathscr{S}$ denote the set of smooth random variables of the form

$$\mathscr{S} := \left\{ g(W(h_1), \ldots, W(h_n)) | g \in C_p^\infty(\mathbb{R}^n),\, h_1, \ldots, h_n \in \mathscr{H},\, n \geq 1 \right\},$$

where $C_p^\infty(\mathbb{R}^n)$ denotes the space of infinitely continuously differentiable functions $g : \mathbb{R}^n \to \mathbb{R}$ such that g and all its partial derivatives grow at most polynomially. Then we can define the pth Malliavin derivative D^p (for $p \geq 1$) as an operator which acts on functions $F = g(W(h_1), \ldots, W(h_n)) \in \mathscr{S}$ by

$$D^p F = \sum_{i_1,\ldots,i_p=1}^{n} \frac{\partial^p g}{\partial x_{i_1} \ldots \partial x_{i_p}} (W(h_1), \ldots, W(h_n)) h_{i_1} \otimes \cdots \otimes h_{i_p},$$

which means that $D^p F$ is a random variable with values in $\mathscr{H}^{\otimes p}$. It does not depend on the choice of the function g and of $h_1, \ldots, h_n \in \mathscr{H}$.
D is clearly a linear operator. Moreover, $D^p : \mathscr{S} \to L^q(\Omega;\, \mathscr{H}^{\otimes p})$ is closable (this

is shown in [57]) and we will denote by $\mathbb{D}^{p,q}$ the closure of $\mathscr{S}$ with respect to the norm

$$\|F\|_{p,q} := \left(\mathbb{E}[|F|^q] + \mathbb{E}[\|DF\|^q_{\mathscr{H}}] + \cdots + \mathbb{E}[\|D^pF\|^q_{\mathscr{H}^{\otimes p}}]\right)^{\frac{1}{q}}.$$

Since Malliavin derivatives are $\mathscr{H}^{\otimes p}$-valued random variables, one can identify them with stochastic processes if those values in $\mathscr{H}^{\otimes p}$ are functions. In this case we will write $D^p_s X$ for $D^p X(s)$.

For Malliavin derivatives the chain rule holds, as described in the following proposition.

2.1.5 Proposition. *Let $f \in C(\mathbb{R}^m;\mathbb{R})$ be a function with $\|\partial_i f\|_\infty < M_i < \infty$ for some $M_i > 0$ ($i = 1,\dots,m$) and let $X_1,\dots X_m \in \mathbb{D}^{1,2}$. Then $f(X_1,\dots,X_m) \in \mathbb{D}^{1,2}$ and*

$$Df(X_1,\dots,X_m) = \sum_{i=1}^m \partial_i f(X_1,\dots,X_m) DX_i.$$

The product rule

$$D(X_1X_2) = X_1D(X_2) + X_2D(X_1) \tag{2.1.1}$$

is an immediate consequence.

For a fixed integer $p \geq 1$ the divergence operator, denoted by δ^p, is the adjoint of the derivative operator D. Its domain $\mathrm{Dom}(\delta^p)$ is the set of random variables $u \in L^2(\Omega;\, \mathscr{H}^{\otimes p})$ such that

$$|\,\mathbb{E}[\langle D^pF,\, u\rangle_{\mathscr{H}^{\otimes p}}]| \leq c_u \|F\|_2$$

for all $F \in \mathbb{D}^{p,2}$, where c_u is a constant.
For $u \in \mathrm{Dom}(\delta^p)$ the image $\delta^p(u)$ is the unique L^2-random variable satisfying the following characterising equation

$$\mathbb{E}[F\delta^p(u)] = \mathbb{E}[\langle D^pF,\, u\rangle_{\mathscr{H}^{\otimes p}}]$$

for all $F \in \mathbb{D}^{p,2}$. We call $\delta(u)(= \delta^1(u))$ the divergence integral (also known as Skorokhod integral) and denote it alternatively by $\int_0^T u_t \delta W_t$ to indicate the underlying process.

Now some classical results (also mentioned in [57]) will be presented which we will require for further proofs.

2.1.6 Lemma. *Let $F \in \mathbb{D}^{1,2}$, $u \in \mathrm{Dom}(\delta)$ such that $Fu \in L^2(\Omega;\mathscr{H})$. Then $Fu \in \mathrm{Dom}(\delta)$ and*

$$\delta(Fu) = F\delta(u) - \langle DF,\, u\rangle_{\mathscr{H}}$$

if the right-hand side is in L^2.

2.1.7 Lemma. *The inclusion* $\mathbb{D}^{1,2}(\mathscr{H}) \subseteq \operatorname{Dom}\delta$ *holds and for each* $u, v \in \mathbb{D}^{1,2}(\mathscr{H})$ *we have*

$$\mathbb{E}[\delta(u)\delta(v)] = \mathbb{E}[\langle u, v\rangle_{\mathscr{H}}] + \mathbb{E}[\operatorname{Tr}(Du \circ Dv)].$$

In the calculations related to Malliavin calculus explicit representations of integrals and derivatives in certain particular cases can be of use. In the following we will describe the definitions and techniques allowing such representations.

Let $H_n(x)$ denote the nth Hermite polynomial defined by

$$H_n(x) := \frac{(-1)^n}{n!} e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2})$$

and $H_0 := 1$. The main reason these polynomials are considered in the context of Gaussian families is the following orthogonality property.

2.1.8 Lemma. *Let* X, Y *be two jointly normally distributed centred random variables satisfying* $\mathbb{E}[X^2] = \mathbb{E}[Y^2] = 1$. *Then for all* $m, n \geq 0$ *we have*

$$\mathbb{E}[H_n(X)H_m(Y)] = \begin{cases} \frac{1}{n!}\mathbb{E}[XY]^n & \textit{if } m = n, \\ 0 & \textit{otherwise.} \end{cases}$$

If we now denote by $\mathcal{H}_n$ the closed linear subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the set $\{H_n(W(h)),\, h \in \mathscr{H}, \|h\|_{\mathscr{H}} = 1\}$, then such spaces will form an orthogonal family due to the previous lemma. We call the space $\mathcal{H}_n$ the nth Wiener chaos.

We will denote by $I_q(\cdot) := \delta^q(\cdot)$ restricted to the space $\mathscr{H}^{\otimes q}$ the qth multiple stochastic integral with respect to W (known as the Wiener-Itō multiple stochastic integral).

For functions $h \in \mathscr{H}$ of norm one the relation $m! H_m(W(h)) = I_m(h^{\otimes m})$ holds for all $m \in \mathbb{N}$, and as a consequence the following isometry of multiple integrals is obtained: for $p, q \geq 1$, $f \in \mathscr{H}^{\otimes p}$ and $g \in \mathscr{H}^{\otimes q}$

$$\mathbb{E}\Big(I_p(f)I_q(g)\Big) = \begin{cases} q!\langle \tilde{f}, \tilde{g}\rangle_{\mathscr{H}^{\otimes q}} & \text{if } p = q, \\ 0 & \text{otherwise,} \end{cases} \tag{2.1.2}$$

where $\tilde{f}$ denotes the canonical symmetrisation of f and is defined by

$$\tilde{f}(x_1, \ldots, x_q) := \frac{1}{q!} \sum_{\sigma \in \mathcal{S}_q} f(x_{\sigma(1)}, \ldots, x_{\sigma(q)}),$$

where the sum runs over all permutations σ of $\{1, \ldots, q\}$. We can therefore say that the multiple stochastic integral is an isometry between the Hilbert space $\mathscr{H}^{\odot q}$ (subspace of symmetrised functions of $\mathscr{H}^{\otimes q}$) equipped with the scaled norm $\frac{1}{\sqrt{q!}}\|\cdot\|_{\mathscr{H}^{\otimes q}}$ and the Wiener chaos of order q. Moreover, we have

$$I_q(f) = I_q\big(\tilde{f}\big).$$

The next important definition is the contraction of two functions. Consider $\{e_k, k \geq 1\}$ an orthonormal basis of $\mathscr{H}$ and let $f \in \mathscr{H}^{\odot p}$ and $g \in \mathscr{H}^{\odot q}$.

For $r = 1, \dots, p \wedge q$, the rth contraction $f \otimes_r g$ is an element of $\mathscr{H}^{\otimes(p+q-2r)}$, which is defined by:

$$f \otimes_r g := \sum_{j_1,\dots,j_p=1}^{\infty} \langle f, e_{k_1} \otimes e_{k_2} \otimes \dots e_{k_r} \rangle_{\mathscr{H}^{\otimes r}} \otimes \langle g, e_{k_1} \otimes e_{k_2} \otimes \dots e_{k_r} \rangle_{\mathscr{H}^{\otimes r}}. \quad (2.1.3)$$

In the particular case when $\mathscr{H} = L^2(T)$, the r-th contraction $f \otimes_r g$ is the element of $\mathscr{H}^{\otimes(p+q-2r)}$ which is defined by

$$\begin{aligned} &(f \otimes_r g)(s_1, \dots, s_{p-r}, t_1, \dots, t_{q-r}) \\ &= \int_{T^r} \mathrm{d}u_1 \dots \mathrm{d}u_r f(s_1, \dots, s_{p-r}, u_1, \dots, u_r) g(t_1, \dots, t_{q-r}, u_1, \dots, u_r) \end{aligned} \quad (2.1.4)$$

for every $f \in L^2(T^p)$, $g \in L^2(T^q)$ and $r = 1, \dots, p \wedge q$.

With this notation we can now formulate the following product rule: if $f \in \mathscr{H}^{\odot p}$ and $g \in \mathscr{H}^{\odot q}$, then

$$I_p(f) I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r} (f \otimes_r g).$$

An important property of Wiener chaoses is the fact that the space of $\sigma(W)$-measurable random variables can be decomposed into orthogonal spaces, which is known as the Wiener chaos decomposition. This can be written in terms of multiple integrals.

2.1.9 Lemma. *Let $\mathcal{G}$ denote the σ-field generated by W. Then any random variable $F \in L^2(\Omega, \mathcal{G}, P)$ can be written as*

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where $f_0 = \mathbb{E}[F]$, I_0 is the identity mapping and the functions f_n are up to symmetrisation uniquely determined by F.

It is clear that by definition $\delta(I_{n-1}(f)) = I_n(f)$ for a function $f \in \mathscr{H}^{\odot n}$. Moreover, for all $r \geq 1$,

$$D^r I_n(f) = \begin{cases} \frac{n!}{(n-r)!} I_{n-r}(f) & \text{if } r \leq n, \\ 0 & \text{otherwise,} \end{cases} \quad (2.1.5)$$

which is proved in [51].

An important property of finite sums of multiple integrals is the hypercontractivity. Namely, if $F = \sum_{k=0}^n I_k(f_k)$ with $f_k \in \mathscr{H}^{\otimes k}$ then

$$\mathbb{E}[|F|^p] \leq C_p \left(\mathbb{E}[F^2]\right)^{\frac{p}{2}}. \tag{2.1.6}$$

for every $p \geq 2$. This is proved in [51].

The last result that we will cover in this section is the interchangeability of the operators in Malliavin calculus and the Lebesgue integral. The following result (in a more general form) is shown in Proposition 6.5 of [40].

2.1.10 Proposition. *Consider an isonormal Gaussian process B^H on Ω generated by an fBm. Let λ be the Lebesgue measure on $\mathbb{R}$. Let $u : \mathbb{R} \times [0, T] \times \Omega \to \mathbb{R}$ be a measurable random field with the following properties:*

(i) $u(x, \cdot) \in \mathrm{Dom}(\delta)$ *for every* $x \in \mathbb{R}$,

(ii) $\mathbb{E}[\int_{\mathbb{R}} \int_{\mathbb{R}} \langle |u(x_1, \cdot)|, |u(x_2, \cdot)| \rangle_{\mathscr{H}^H} dx_1 dx_2] < \infty$,

(iii) there is a measurable version in $\Omega \times \mathbb{R}$ *of the random field* $\left(\int_0^T u(x, t)\delta B_t^H\right)_{x \in \mathbb{R}}$,

(iv) it holds that

$$\int_{\mathbb{R}} \mathbb{E}\left[\left(\int_0^T u(x, t)\delta B_t^H\right)^2\right] dx < \infty.$$

Then $\int_{\mathbb{R}} u(x, \cdot)dx \in \mathrm{Dom}(\delta)$ *and*

$$\int_0^T \int_{\mathbb{R}} u(x, t)dx\delta B_t^H = \int_{\mathbb{R}} \int_0^T u(x, t)\delta B_t^H dx.$$

2.1.11 Remark. If u is a multiple stochastic integral satisfying the above assumptions then it follows by (2.1.5) that also the Malliavin derivative and the Lebesgue integral of multiple stochastic integrals are interchangeable.

Further specific results around Malliavin calculus will be quoted whenever they are needed in the thesis.

Young integrals

Anther possibility to define stochastic integrals with respect to the fractional Brownian motion and derived processes is a pathwise approach. It uses the smoothness of the paths of a fractional Brownian motion. This section is based upon the book [28].

2.1.12 Definition. Given two continuous functions x, y on an interval $[0, T]$ having bounded p- and q-variation respectively, we call a continuous function z the (indefinite) Young integral of y against x if there exists a sequence (x_n, y_n) of continuous functions of bounded 1-variation which converges uniformly with uniform variation bounds in the sense

$$\|x_n - x\|_\infty \to 0 \text{ and } \sup_n \|x_n\|_{p-var} < \infty,$$
$$\|y_n - y\|_\infty \to 0 \text{ and } \sup_n \|y_n\|_{q-var} < \infty$$

and

$$\int_0^\cdot y_n dx_n \to z \text{ uniformly on } [0, T] \text{ as } n \to \infty.$$

If z is unique we write $\int_0^\cdot ydx$ instead of z and set $\int_s^t ydx := \int_0^t ydx - \int_0^s ydx$.

The following theorem is central for the definition of Young-type integrals.

2.1.13 Theorem (Young-Lóeve)**.** *Given x, y as in Definition 2.1.12, if $1/p + 1/q > 1$, there exists a unique (indefinite) Young integral of y against x and it has finite p-variation.*

Note that this integral then coincides with the Riemann-Stieltjes integral (which is shown in [86]).

We have already shown that paths of a fractional Brownian motion are almost surely $(H - \varepsilon)$-Hölder continuous for each $\varepsilon > 0$. Therefore, the paths have finite $\frac{1}{H-\varepsilon}$-variation. In view of the above theorem it means that Young integrals of functions of bounded variations are well defined with respect to the paths of a fractional Brownian motion, but also that such integrals, i.e. processes with finite $\frac{1}{H-\varepsilon}$-variation, can be integrated with respect to processes with finite $\frac{1}{H-\varepsilon}$-variation in Young's sense, in particular, with respect to a fractional Brownian motion itself (assuming that $H > \frac{1}{2}$).

The following result connecting the two notions of stochastic integrals is proved in [2].

2.1.14 Remark. Let $H > \frac{1}{2}$ and let $u_t \in \mathbb{D}^{1,2}$ (for all $t \in [0, T]$) be such that the Young integral $\int_0^T u_s dB_s^H$ is well-defined. Suppose, moreover, that

$$P\left(\int_0^T \int_0^T |D_s u_t||t-s|^{2H-2} dsdt < \infty\right) = 1.$$

Then $u \in \operatorname{Dom} \delta$ and for every $t \in [0, T]$

$$\int_0^t u_s dB_s^H = \int_0^t u_s \delta B_s^H + H(2H-1) \int_0^t \int_0^t D_s u_r |s-r|^{2H-2} drds.$$

In particular, if u is a non-random Hölder continuous function of order $\alpha > 1-H$, then the Young and the Skorokhod integrals over $[0, T]$ coincide.

The last conclusion has another useful consequence.

2.1.15 Remark. Let $H > \frac{1}{2}$ and let u be a non-random Hölder continuous function of order $\alpha > 1 - H$. Then for $t \in [0, T]$

$$\int_0^t u_s \delta B_s^H = \int_0^T u_s \mathbf{1}_{\{s \le t\}} \delta B_s^H,$$

which allows us to use a shorter notation $\delta(u.\mathbf{1}_{\{. \le t\}})$ for this integral without specifying the domain of integration.

2.1.3 Solutions of the Ornstein-Uhlenbeck type equations

Some of the main objects in our work are the Ornstein-Uhlenbeck type equations of the form

$$\begin{aligned} X_t &= X_0 + \int_0^t (L(s) - \alpha X_s)\, ds + \sigma B_t^H, \quad t \ge 0, \\ X_0 &= x_0 \in \mathbb{R}, \end{aligned} \tag{2.1.7}$$

We assume in the following that L is a 1-periodic function and $\alpha, \sigma \in \mathbb{R} \backslash \{0\}$ as well as $x_0 \in \mathbb{R}$. (Note, however, that all the subsequent statements can be generalised to functions with a known period $\nu \in \mathbb{R}^+$.) The equation does not contain stochastic integrals with respect to B^H, therefore, it can be solved pathwise using the methods from the ordinary differential equations theory. As explained in detail in [14], the equaton

$$f(t) = \int_0^t g(s) f(s) ds + h(t), \quad t \ge 0,$$

for almost surely continuous functions $g,\ h : \mathbb{R}^+ \to \mathbb{R}$ with $\sup_{0 \le s \le t}(|g(s)| + |h(s)|)$ for all $t \ge 0$ has a unique solution which can be written as

$$f(t) = e^{\int_0^t g(u)du} \left(h(0) + \int_0^t e^{-\int_0^s g(u)du} dh(s) \right), \quad t \ge 0,$$

the integral being defined in the Riemann-Stieltjes sense. With

$$g(t) = -\alpha, \qquad h(t) = \sigma B_t^H + \int_0^t L(s) ds + x_0, \qquad t \ge 0,$$

satisfying all the above conditions we can conclude that

$$X_t = e^{-\alpha t} \left(x_0 + \sigma \int_0^t e^{\alpha s} dB_s^H + \int_0^t e^{\alpha s} L(s) ds \right)$$

solves the equation 2.1.7. Note, moreover, that by Remark 2.1.14 the stochastic integral $\int_0^t e^{-\alpha s} dB_s^H$ can be replaced by a Skorokhod integral $\int_0^t e^{-\alpha s} \delta B_s^H$ if the latter is well defined. This is the case for $H > \frac{1}{2}$. Consequently, the solution is Malliavin differentiable with respect to the fractional Brownian motion.

Skorokhod integrals with respect to X (which is in general not a centred process) can be defined as

$$\int_0^T Y_t \delta X_t := \int_0^T Y_t d\left(\int_0^t L(s) - \alpha X_s ds\right) + \int_0^T Y_t \sigma \delta B_t^H$$

for processes Y for which the two summands are well defined. This is the definition that will be used in the thesis.

2.1.4 Solutions of the wave equation

This section is based on the books [20] as well as [77] which offer many more general statements and concepts concerning stochastic partial differential equations (SPDEs).

The fractional wave equation, one of the subjects of our study, is defined as follows:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t,x) &= \Delta u(t,x) + \dot{W}^H(t,x),\ t \in (0,T],\ T>0,\ x \in \mathbb{R}^d,\ d \geq 1, \\ u(0,x) &= 0, \quad x \in \mathbb{R}^d, \\ \frac{\partial u}{\partial t}(0,x) &= 0, \quad x \in \mathbb{R}^d, \end{cases} \tag{2.1.8}$$

where Δ is the Laplacian on $\mathbb{R}^d$, $d \geqslant 1$, and $\dot{W}^H$ is a fractional-white Gaussian noise which is defined in reference to a real valued centred Gaussian field $W^H = \{W_t^H(A); t \in [0,T], A \in B_b(\mathbb{R}^d)\}$ ($B_b(\mathbb{R}^d)$ being the class of bounded Borel subsets of $\mathbb{R}^d$) with covariance function given by

$$\mathbb{E}\left(W_t^H(A) W_s^H(B)\right) = R_H(t,s)\lambda(A \cap B), \quad A, B \in \mathcal{B}_b(\mathbb{R}^d), \tag{2.1.9}$$

where R_H is the covariance of the fractional Brownian motion

$$R_H(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad s,t \geq 0.$$

Due to the presence of the noise term it is impossible to find a differentiable strong solution. Moreover, a precise meaning needs to be given to the equation (2.1.8). Both will be achieved with the following notion of a solution.

2.1.16 Definition. We call a field $u = \{u(t,\, x);\, t \in [0,\, T],\, x \in \mathbb{R}\}$ the mild solution of the wave equation if

$$u(t,\, x) = \int_0^t \int_{\mathbb{R}^d} G_1(t-s,\, x-y) W^H(ds,\, dy), \tag{2.1.10}$$

where G_1 are the Green's functions of the homogeneous wave equation $\frac{\partial^2 u}{\partial t^2} = \Delta u$.

In dimension 1 we have $G_1(t, x) = \frac{1}{2}\mathbf{1}_{\{|x|<t\}}$, in dimension 2 $G_1(t, x)$ is equal to $\frac{1}{2\pi\sqrt{t^2-|x|^2}}\mathbf{1}_{\{|x|<t\}}$. In dimensions 3 or higher G_1 is a distribution, and the solution to the homogeneous equation is understood as a solution of a Fourier transform of the equation (see [66] for details).

This definition is motivated by the fact that

$$u(t, x) = \int_0^t \int_{\mathbb{R}^d} G_1(t-s, x-y)\varphi(s, y)dsdy$$

solves the deterministic equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t, x) &= \Delta u(t, x) + \varphi(t, x), \ t \in (0, T], \, T > 0, \ x \in \mathbb{R}^d, \, d \geq 1, \\ u(0, x) &= 0, \quad x \in \mathbb{R}^d, \\ \frac{\partial u}{\partial t}(0, x) &= 0, \quad x \in \mathbb{R}^d \end{cases}$$

in its mild formulation (see [76]).

In the stochastic case there is one more necessary addition to the definition: it is important to make sense of the integral with respect to W^H.

As described in [35], we can define it as a Wiener integral on the Hilbert space $\mathscr{H}^W$ defined as the closure of the space of simple functions $\{\mathbf{1}_{[0,\,t]\times A}, \, t \in [0, T], \, A \in \mathcal{B}_b(\mathbb{R}^d)\}$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,\,t]\times A}, \, \mathbf{1}_{[0,\,t]\times A} \rangle_{\mathscr{H}^W} := \mathbb{E}\left(W_t^H(A) W_s^H(B)\right) = \alpha_H \lambda(A \cap B) \int_0^t \int_0^s |u-v|^{2H-2} du dv,$$

where $\alpha_H = H(2H-1)$. For the existence of a solution it thus remains to check when the above integral is well defined, i.e. when $G_1(t-\cdot\,, x-\cdot\,)$ is an element of $\mathscr{H}^W$. This question is treated in the following proposition proved in [77].

2.1.17 Proposition. *The stochastic wave equation* (2.1.8) *admits a unique mild solution* $u(t, x)_{t \in [0,\,T],\, x \in \mathbb{R}^d}$ *if and only if*

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|x|^2} \right)^{H+\frac{1}{2}} \lambda(dx) < \infty.$$

Therefore, we retrieve the following specific result.

2.1.18 Proposition. *The wave solution process* $u = \{u(t, x), \, t \geq 0, \, x \in \mathbb{R}^d\}$ *defined in 2.1.10 exists if and only if* $d < 2H + 1$, *i.e. for* $H = \frac{1}{2}$ *in dimension one and for* $H > \frac{1}{2}$ *in dimensions one and two.*

2.2 Background and existing results

In this section we will focus on the estimation questions and existing results for Ornstein-Uhlenbeck type equations. Let us talk separately about the volatility and drift estimation.

2.2.1 Volatility estimation in OU-type equations based on quadratic variations

Since we are concerned with parameter estimation in the setting of the Ornstein-Uhlenbeck type equations, it is important to discuss results related to the estimation of the volatility coefficient $\sigma > 0$ and/or the Hurst parameter H based on observations of the solution X in the setting

$$\begin{aligned} X_t &= X_0 + \int_0^t (L(s) - \alpha X_s)\, ds + \sigma B_t^H, \quad t \geq 0, \\ X_0 &= x_0 \in \mathbb{R}, \end{aligned} \tag{2.2.1}$$

for $\alpha \in \mathbb{R}$, $\sigma \in \mathbb{R}\backslash\{0\}$ and a periodic bounded deterministic function L.

A well-known approach is to consider power variations as they have been introduced in Section 2.1.1. The realised k-variation of a stochastic process Z is given by

$$V_k^n(Z) := \sum_{i=1}^n \left| Z_{\frac{i}{n}} - Z_{\frac{i-1}{n}} \right|^k .$$

A starting point for the use of power variation in parameter estimation is a result concerning fractional Brownian motion stated in [69], which has also been used in Section 2.1.1: For a fractional Brownian motion $(B_t^H)_{t\geq 0}$ and some $\sigma > 0$

$$n^{kH-1} V_k^n(\sigma B^H) \overset{n\to\infty}{\to} c_k \sigma^k$$

holds in probability for an explicitly known constant c_k. The proof is based on the self-similarity property of the fractional Brownian motion, and the fact that it cannot be a semimartingale unless $H = 0.5$ follows directly from this result for different k (as seen in Section 2.1.1).

There are many extensions to this proposition; in particular, in [77] almost sure convergence is demonstrated for a normalised version of realised power variations

$$\sum_{i=1}^n \left(\frac{|Z_{\frac{i}{n}} - Z_{\frac{i-1}{n}}|^k}{\mathbb{E}\left[|Z_{\frac{i}{n}} - Z_{\frac{i-1}{n}}|^k \right]} - 1 \right)$$

(appropriately scaled) and its second order asymptotics is studied, thus providing a strongly consistent estimator for both σ and H (refer also to [16] for the definition of a joint estimator). Note in particular that if a continuous observation of a path of σB^H is available, it is sufficient to retrieve the exact value of σ in case H is known. An important generalisation from [19] allows us to extend this statement to the solution X of an Ornstein-Uhlenbeck SDE. This generalisation comprises two following results (Theorem 1 and Corollary 2 in [19] respectively).

2.2.1 Theorem. *Suppose that $u = \{u_t,\, t \in [0,\, 1]\}$ is a stochastic process of finite q-variation where $q < \frac{1}{1-H}$. Set $Z_t := \int_0^t u_s dB_s^H$. Then*

$$n^{kH-1} V_k^n(Z) \to c_k \int_0^1 |u_s|^k ds$$

in probability as n tends to infinity.

2.2.2 Proposition. *Assume the same conditions as in Theorem 2.2.1. Consider a stochastic process $Y = \{Y_t,\, t \in [0,\, 1]\}$ such that*

$$n^{kH-1} V_k^n(Y) \to 0$$

in probability as n tends to infinity. Then

$$n^{kH-1} V_k^n(Z+Y) \to c_k \int_0^1 |u_s|^k ds$$

in probability as n tends to infinity. In particular, the above condition is satisfied for processes whose trajectories are γ-Hölder for some $\gamma \in (H,\, 1]$.

The processes that are considered in our case are

$$Z_t := \int_0^t \sigma dB_s^H \text{ and}$$
$$Y_t := \int_0^t (\alpha X_s + L(s))\, ds.$$

As a deterministic integral of the function X with $(H-\varepsilon)$-Hölder trajectories ($\varepsilon > 0$) and the function L that is bounded by assumption the process Y has Lipschitz continuous trajectories. Thus, both Z and Y satisfy the assumptions of the above Theorem 2.2.1 and Proposition 2.2.2. We can conclude that an observation of a trajectory of X over the unit interval suffices to access the constant σ.

2.2.2 Drift estimation for continuous observations

There are several approaches to parameter estimation in case $L \equiv 0$, most of which are derived from corresponding ideas for classical diffusions (in other words, for diffusions driven by the Brownian motion). For the ergodic case (i.e. $\alpha > 0$) there is the maximum-likelihood-approach relying on Girsanov-type formulas in [38] as well as its discretised versions ([67]). In [30], moreover, a least-squares-type estimator defined using divergence integrals is presented. The non-ergodic case is treated in [9], where Young integrals are considered.

The case of the non-zero mean L is a by far less studied setting. In [24] periodic functions $L \equiv \sum_{i=1}^{p} \mu_i \varphi_i$ in equation (2.2.1) are considered, in which the number p and the functions φ_i are known and the parameters μ_i are estimated jointly with the multiplicative coefficient $\alpha > 0$. This construction is similar to the least-squares type estimator for the classical Ornstein-Uhlenbeck process with the same mean structure studied in [23], but the proofs rely on the properties of divergence integrals used in the definition of the estimators.

In this thesis we will consider drift estimation only in the setting of continuous observations. In this case, as explained in Section 2.2.1, the value of the volatility parameter σ can be obtained directly from observations on any compact interval. This allows us to consider settings in which we assume σ to be known and equal to one without loss of generality. This can be done whenever the fBm is the driving process, in particular, in Chapters 3 and 4.

Asymptotic behaviour: Ergodic and non-ergodic case

For the Ornstein-Uhlenbeck type equations that are considered in this thesis (2.1.7) three cases should be addressed separately depending on the value of the parameter α. In order to percieve the differences between these cases let us note that for the solution

$$X_t = e^{-\alpha t}\left(x_0 + \sigma \int_0^t e^{\alpha s} dB_s^H + \int_0^t e^{\alpha s} L(s) ds\right)$$

the following statement holds.

2.2.3 Proposition. *Let $H \in (0, 1)$. The random variable X_t has Gaussian distribution $N\left(e^{-\alpha t}\left(x_0 + \int_0^t e^{\alpha s} L(s) ds\right), v(\alpha, t)\right)$ with variance*

$$v(\alpha, t) = H\sigma^2 \int_0^t z^{2H-1}(e^{-\alpha z} + e^{\alpha(z-2t)}) dz.$$

The Gaussianity and the expectation can be read directly from the solution formula and the variance is computed in [41]. In the same source the following lemma is demonstrated.

2.2.4 Lemma. *For $v(t, \alpha)$ from the previous proposition we have*

(i) if $\alpha < 0$ then $v(\alpha, t) \sim \frac{\sigma^2 H \Gamma(2H)}{(-\alpha)^{2H}} e^{-2\alpha t}$ as $t \to \infty$,

(ii) if $\alpha > 0$ then $v(\alpha, t) \to \frac{\sigma^2 H\Gamma(2H)}{\alpha^{2H}}$ as $t \to \infty$,

(iii) $v(0, t) = \sigma^2 t^{2H}$ for all $t \geq 0$.

We see from these observations that for $\alpha < 0$ the mean as well as the variance of X explode as t goes to infinity, whereas for $\alpha > 0$ both quantities are bounded. For $\alpha > 0$ the process is mean-reverting: it drifts towards its long-term mean which inherits its periodicity from the function L.

A property of a similar kind is ergodicity. Indeed, it is shown in [41] that the solution of (2.1.7) with $L \equiv 0$, $x_0 = \sigma \int_{-\infty}^{0} e^{-\alpha(t-s)} dB_s^H$ (here we ignore the formal restriction $x_0 \in \mathbb{R}$) is a stationary, ergodic process for $\alpha > 0$. Solutions of (2.1.7) with a different initial condition approach this process with exponential speed, which means that many convergence results obtained by ergodicity can be translated to solutions with a general $x_0 \in \mathbb{R}$. Similar results can be obtained for processes where $L \not\equiv 0$ (see [24]). Due to this observation the case $\alpha > 0$ is often called the ergodic and the case $\alpha < 0$ the non-ergodic case.

In the borderline case $\alpha = 0$ the equation (2.1.7) reads

$$\begin{aligned} X_t &= X_0 + \int_0^t L(s)ds + B_t^H, \quad t \geq 0, \\ X_0 &= x_0 \in \mathbb{R}, \end{aligned}$$

which can be understood as a model of a periodic signal with an additive fractional Brownian component. In this case $X_t - x_0$ is a simple unbiased estimator of $\int_0^t L(s)ds$ (see [63]), but especially for the parametric version of the problem, e.g. for $L \equiv \sum_{i=1}^{p} \mu_i \varphi_i$ with known $\varphi_1, \ldots, \varphi_p$ more methods can be applied such as Girsanov's theorem or Bayesian estimation (see [4]). In this thesis we will only briefly consider this case.

For a visual impression of the differences between the three setups, consider a simulation of 10 sample paths of the solution of

$$\begin{aligned} X_t &= X_0 + \int_0^t (100 \sin(2\pi s) - \alpha X_s)\, ds + B_t^H,\ t \geq 0, \\ X_0 &= 0, \end{aligned}$$

for $H = 0.7$ and $\alpha = 1$, 0 and -1 depicted in Figures 2.4, 2.5 and 2.6. Note that although there are some results discussing the optimality of different drift estimators in the ergodic case (comparison of asymptotic variances in [31] and the proof of the LAN property in [44]), these results are restricted to the case $L \equiv 0$. More generally, to our knowledge no LAN or LAMN property has been established for fractional diffusions with a time dependent drift. While it is an interesting line of research, we will not follow it here and will restrict ourselves to presenting the relevant context and directly comparing our results to estimators with a similar structure.

In the following we will concentrate on the relevant results for the ergodic and the non-ergodic case separately.

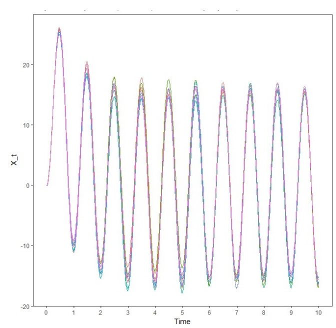

Figure 2.4: $\alpha = -1$

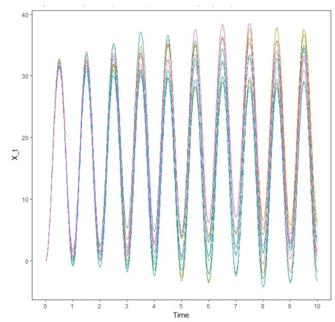

Figure 2.5: $\alpha = 0$

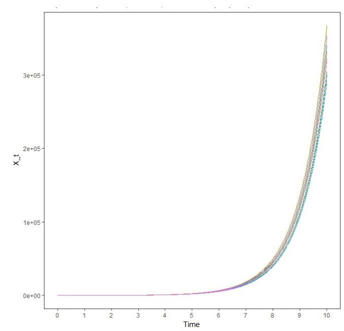

Figure 2.6: $\alpha = 1$

Results in the ergodic case

The first contribution to be mentioned here is the paper [30]. There the case $L \equiv 0$ is considered, i.e. the equation

$$X_t = \int_0^t \alpha X_s ds + B_t^H$$

for $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$. The construction of the estimator for α is based on (formally) minimising the term $\int_0^n (\dot{X}_t - \alpha X_t)^2 dt$, which yields the sequence

$$\frac{\int_0^n X_t dX_t}{\int_0^n X_t^2 dt}.$$

The stochastic integral in the numerator is considered to be the Skorokhod integral. It is shown in [30] that if the pathwise integral is considered instead, the estimator thus obtained is not consistent (it converges to zero for all negative α). The above sequence with the Skorokhod integral, however, is shown there to be strongly consistent and asymptotically normal with $\sqrt{n}$ as the speed of convergence.

Another result that is particularly relevant in Chapter 5 is a pathwise estimator of α that arises from some calculations within this paper and is also analysed there. It is defined as

$$\left(\frac{1}{H\Gamma(2H)n} \int_0^n X_t^2 dt\right)^{-\frac{1}{2H}},$$

is strongly consistent and asymptotically normal and has the practical advantage of involving no stochastic integrals and thus of having good simulation properties.

In the subsequent paper [31] a significant expansion of these results is achieved: The same two estimators are considered for $H \in \left(0, \frac{1}{2}\right]$ and for $H \in \left[\frac{3}{4}, 1\right)$. Once again, consistency and asymptotic normality are demonstrated.

In the paper [24] the idea of the least squares estimation is extended to the case of a periodic bounded function L having the form $L \equiv \sum_{i=1}^p \mu_i \varphi_i$ with functions φ_i that are assumed to be known, bounded and orthonormal.

For the construction of an estimator of $(\mu_1, \ldots, \mu_p,\, \alpha)$ a technique introduced in [27] is used where a least squares estimator is derived for a discretised version of a more general equation

$$dX_t = \langle \vartheta,\, f(t,\, X_t)\rangle dt + dB_t^H,$$

where $\vartheta := (\vartheta_1, \ldots, \vartheta_{p+1})$ is a parameter vector to be estimated and $f(t,\, x) := (f_1(t,\, x), \ldots, f_{p+1}(t,\, x))$ is a collection of known real-valued functions. For a time interval $[0,\, T]$ and a uniform mesh size $\Delta t := T/N$ the least squares approach for the equations

$$X_{(i+1)\Delta t} - X_{i\Delta t} = \sum_{j=1}^{p+1} f_j(i\Delta t,\, X_{i\Delta t})\vartheta_j \Delta t + (B^H_{(i+1)\Delta t} - B^H_{i\Delta t}),\ i \in \{0, \ldots, N\},$$

yields the estimator $\tilde{\vartheta}_{T,\,\Delta t} = Q^{-1}_{T,\,\Delta t} P_{T,\,\Delta t}$ with

$$Q_{T,\,\Delta t} = \left(\sum_{i=0}^{N} f_j(i\Delta t,\, X_{i\Delta t}) f_k(i\Delta t,\, X_{i\Delta t})\Delta t \right)_{j,\,k \in \{1, \ldots, p+1\}}$$

and

$$P_{T,\,\Delta t} = \Big(\sum_{i=0}^{N} f_1(i\Delta t,\, X_{i\Delta t})(X_{(i+1)\Delta t} - X_{i\Delta t}), \ldots, \\ \sum_{i=0}^{N} f_{p+1}(i\Delta t,\, X_{i\Delta t})(X_{(i+1)\Delta t} - X_{i\Delta t}) \Big)^T,$$

i.e., this estimator minimises the functional

$$(\vartheta_1, \ldots, \vartheta_{p+1}) \mapsto \sum_{i=0}^{N} \left(X_{(i+1)\Delta t} - X_{i\Delta t} - \sum_{j=1}^{p+1} f_j(i\Delta t,\, X_{i\Delta t})\vartheta_j \Delta t \right)^2 .$$

Plugging in $\vartheta := (\mu_1, \ldots, \mu_p,\, \alpha)$ and $f(t,\, x) := (\varphi_1(t), \ldots, \varphi_p(t),\, -x)^T$ and replacing the sums by their continuous counterparts an estimator $\hat{\vartheta}_n = Q_n^{-1} P_n$ is obtained, where

$$P_n := \left(\int_0^n \varphi_1(t) dX_t, \ldots, \int_0^n \varphi_p(t) dX_t, \int_0^n X_t dX_t \right)$$

and

$$Q_n := \begin{pmatrix} nE_p & a_n \\ a_n^T & b_n \end{pmatrix},$$

$$a_n^T := \left(\int_0^n \varphi_1(t) X_t dt \ldots, \int_0^n \varphi_p(t) X_t dt \right),$$

$$b_n := \int_0^n X_t^2 dt.$$

With Skorokhod stochastic integrals strong consistency as well as joint asymptotic normality is demonstrated for $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$ in this paper (strong consistency was later extended in [7] to $H \in \left(\frac{1}{2}, 1\right)$). The rate of convergence obtained is n^{1-H}.

In [56], moreover, an estimator for the case of the constant drift L is considered in the setting of a Hermite process driving the equation. A Hermite process of order q is a process in the qth Wiener chaos having the same covariance structure as the fractional Brownian motion, and for $q = 1$ the fBm is retrieved. The estimator in [56] is a generalisation of the pathwise estimator from [30].

Results in the non-ergodic case

The paper [9] deals with the estimation of α in the non-ergodic setting. There the classical Ornstein-Uhlenbeck process is considered (i.e. with $L \equiv 0$). The authors define the least-squares type estimator in the same way as it is done in [30] for the ergodic case, namely as

$$\frac{\int_0^n X_t dX_t}{\int_0^n X_t^2 dt},$$

assuming continuous observations of the process X. Interpreting the stochastic integrals as Young type integrals they demonstrate strong consistency and show a noncentral limit theorem as a second order convergence result: The error converges with exponential speed in distribution to a Cauchy random variable.

2.2.3 Realised quadratic variations of the wave equation solution

We finish the chapter by revising the results that serve as a starting point for Chapter 6.

In the paper [36] a thorough analysis of spatial quadratic variations for the wave equation with space-time white noise is conducted. These results are extended in [35] to the fractional-white noise with $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$. In particular, for the normalised realised quadratic variations

$$V_n := \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left(\frac{(u(t, \frac{j+1}{n}) - u(t, \frac{j}{n}))^2}{\mathbb{E}[(u(t, \frac{j+1}{n}) - u(t, \frac{j}{n}))^2]} - 1 \right)$$

(where u is the solution of (2.1.8)) it is shown for $t > 1$ (chosen in order to simplify the covariance structure, see Chapter 6) that $\frac{V_n}{\sqrt{\mathbb{E}[V_n^2]}}$ converges to the standard normal distribution in law and

$$d\left(\frac{V_n}{\sqrt{\mathbb{E}[V_n^2]}}, N(0,\,1)\right) \leq C \begin{cases} \frac{1}{\sqrt{n}} & \text{if } H \in (\frac{1}{2}, \frac{5}{8})\ , \\ \frac{\log(n)^{3/2}}{\sqrt{n}} & \text{if } H = \frac{5}{8}, \\ n^{4H-3} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}), \end{cases}$$

where d stands for either Wasserstein, Kolmogorov or the total variation distance and $C > 0$ is a constant. The main tool for the analysis is the calculation of the spatial covariance of the solution.
These results have allowed the authors to construct a consistent and asymptotically normal estimator for H.

Chapter 3

Parametric estimator for the fractional Ornstein-Uhlenbeck type processes

This chapter is dealing with parameter estimation in Ornstein-Uhlenbeck type equations with a periodic drift function $L \equiv \sum_{i=1}^{p} \mu_i \varphi_i$ using the least squares ansatz. The main result are the asymptotic properties for the estimator introduced in Section 2.2.2 in the non-ergodic case. Of particular interest is the asymptotics for the estimators of the parameters μ_i in the special case $\int_0^1 \varphi_i(s)ds = 0$: the estimator converges faster and the expression for the asymptotic variance is significantly more involved. At the end of the chapter this result is applied to the parameter estimation in the ergodic case.

The content of this chapter is partially contained in the preprint

- R. Shevchenko, J. H. C. Woerner - Inference for fractional Ornstein-Uhlenbeck type processes with periodic mean in the non-ergodic case, 2019, arXiv:1903.08033.

3.1 Parametric estimator (non-ergodic case)

We will consider the estimator $\hat{\vartheta}_n$ from Section 2.2.2 for the non-ergodic case and $H \in \left(\frac{1}{2}, 1\right)$ and investigate its asymptotic properties. In particular, we will show that the asymptotics is partly inherited from the ergodic case treated in [24] and partly follows the results for the non-ergodic case in [9].

3.1.1 Setting

Let us first recall the setting for this chapter in more detail.

Let $(B_t^H)_{t\geq 0}$ be a fractional Brownian motion with the Hurst index $H \in \left(\frac{1}{2}, 1\right)$. Consider a stochastic differential equation (or SDE) of the following form:

$$\begin{aligned} X_t &= X_0 + \int_0^t \left(\sum_{i=1}^p \mu_i \varphi_i(s) + \alpha X_s \right) ds + \sigma B_t^H, \quad t \geq 0, \\ X_0 &= x_0 \in \mathbb{R}. \end{aligned} \tag{3.1.1}$$

We assume to observe X continuously. L is assumed to be a bounded 1-periodic function which can be written as a linear combination of p known bounded 1-periodic $L^2([0,\, 1])$-orthonormal functions with unknown real coefficients, i.e.

$$L(s) = \sum_{i=1}^p \mu_i \varphi_i(s) \text{ for all } s \in [0,\, 1].$$

The factor $\alpha > 0$ is also assumed to be unknown. As argued in Section 2.2.1, σ can be estimated with probability one on any finite time interval, therefore it can be assumed to be known and equal to one without loss of generality.

Moreover, it is important to define stochastic integrals with respect to B^H. In this chapter we will consider them to be defined in Young's sense (cf. Section 2.1.2). Such integrals are well defined due to Hölder smoothness of paths of the fractional Brownian motion whenever the integrated process is sufficiently smooth. Note that for deterministic integrands stochastic integrals in Young's sense almost surely coincide with Skorokhod integrals (see Section 2.1.2).

As shown in Section 2.1.3, the equation (3.1.1) has a solution with almost surely continuous paths, which can be written as

$$X_t = e^{\alpha t} x_0 + e^{\alpha t} \int_0^t e^{-\alpha s} L(s) ds + e^{\alpha t} \int_0^t e^{-\alpha s} dB_s^H$$

for $\alpha > 0$. Let us fix the notation $\xi_t := \int_0^t e^{-\alpha s} dB_s^H$, $\tilde{\xi}_t := e^{-\alpha t} X_t$ as well as

$$\xi_\infty := \int_0^\infty e^{-\alpha s} dB_s^H$$

and

$$\tilde{\xi}_\infty := x_0 + \int_0^\infty e^{-\alpha s} L(s) ds + \int_0^\infty e^{-\alpha s} dB_s^H.$$

3.1.2 Construction of the estimator

The estimator that we are going to consider copies the structure of an estimator defined in [24] for the ergodic case.

In the general construction given in Section 2.2.2 we set $\vartheta = (\mu_1, \dots, \mu_p, \alpha)$, $f(t, x) = (\varphi_1(t), \dots, \varphi_p(t), x)$ as well as $T = n$ and, as for the ergodic case, consider the continuous counterparts of the components. By proceeding thus we obtain the estimator $\hat{\vartheta} := Q_n^{-1} P_n$ with

$$P_n = \left(\int_0^n \varphi_1(t) dX_t, \dots, \int_0^n \varphi_p(t) dX_t, \int_0^n X_t dX_t \right)$$

and

$$Q_n = \begin{pmatrix} nE_p & a_n \\ a_n^T & b_n \end{pmatrix},$$

where

$$a_n^T = \left(\int_0^n \varphi_1(t) X_t dt \dots, \int_0^n \varphi_p(t) X_t dt \right),$$

$$b_n = \int_0^n X_t^2 dt.$$

To make sure that the integrals in P_n are well-defined we need an additional assumption on φ_i: these functions must be at least $(1 - H)$-Hölder continuous.

The two following results are an immediate analogy to the calculations in [24].

3.1.1 Proposition. *We have* $\hat{\vartheta}_n = \vartheta + Q_n^{-1} R_n$, *where*

$$R_n = \left(\int_0^n \varphi_1(t) dB_t^H, \dots, \int_0^n \varphi_p(t) dB_t^H, \int_0^n X_t dB_t^H \right)^T.$$

Proof. Since

$$\int_0^n \varphi_i(t) dX_t = \sum_{j=1}^p \mu_j \int_0^n \varphi_i(t) \varphi_j(t) dt + \alpha \int_0^n \varphi_i(t) X_t dt + \int_0^n \varphi_i(t) dB_t^H$$

for $i \in \{1, \dots p\}$ and

$$\int_0^n X_t dX_t = \sum_{j=1}^p \mu_j \int_0^n X_t \varphi_j(t) dt + \alpha \int_0^n X_t^2 dt + \int_0^n X_t dB_t^H,$$

we have $P_n = Q_n \vartheta + R_n$, and the claim follows. □

3.1.2 Proposition. *We have an explicit representation for* Q_n^{-1}, *namely*

$$Q_n^{-1} = \frac{1}{n} \begin{pmatrix} E_p + \gamma_n \Lambda_n \Lambda_n^t & -\gamma_n \Lambda_n \\ -\gamma_n \Lambda_n^t & \gamma_n \end{pmatrix}$$

with

$$\Lambda_n = (\Lambda_{n,1}, \dots, \Lambda_{n,p})^t = \left(\frac{1}{n} \int_0^n \varphi_1(t) X_t dt, \dots, \frac{1}{n} \int_0^n \varphi_p(t) X_t dt \right)$$

and $\gamma_n = D_n^{-1} = \left(\frac{1}{n} \int_0^n X_t^2 dt - \sum_{i=1}^p \Lambda_{n,i}^2 \right)^{-1}$.

Proof. This is a consequence of the fact that

$$\begin{pmatrix} nE_p & -a_n \\ -a_n^T & b_n \end{pmatrix}^{-1} = \frac{1}{n} \begin{pmatrix} E_p + \gamma_n \Lambda_n \Lambda_n^t & \gamma_n \Lambda_n \\ \gamma_n \Lambda_n^t & \gamma_n \end{pmatrix},$$

which is proved in [24]. □

3.1.3 Auxiliary results

First let us present two elementary results that will be used in this chapter.

3.1.3 Lemma. *For a centred normal sequence $(X_n)_{n\in\mathbb{N}}$ of random variables we have: If the squared L^2 norms of X_n are of order at most $\frac{1}{n^\beta}$ for $\beta > 0$, then the sequence converges to zero almost surely.*

Proof. First note that the squared L^2 norm of a centred normal random variable is its variance. For $k \in \mathbb{N}$ the $2k$-th moment is completely determined by it; we have

$$\mathbb{E}[X_n^{2k}] = C_k \, \mathbb{E}[X_n^2]^k \lesssim \frac{1}{n^{\beta k}}$$

by assumption. If we now check the summability criterion (implied by the Borel-Cantelli lemma), this consideration allows us to get the result by Markov inequality for $f(x) = x^{2k}$ and k such that $\beta k > 1$:

$$\sum_{n=1}^{\infty} P(|X_n| > \varepsilon) \le \sum_{n=1}^{\infty} \frac{\mathbb{E}[X_n^{2k}]}{\varepsilon^{2k}} = \frac{1}{\varepsilon^{2k}} C_k \sum_{n=1}^{\infty} \mathbb{E}[X_n^2]^k \lesssim \sum_{n=1}^{\infty} \frac{1}{n^{\beta k}} < \infty.$$

□

3.1.4 Proposition. *For $\alpha > 0$ there exists a constant $C > 0$ such that $\int_0^t e^{\alpha u} u^{2H-2} du \le C t^{2H-2} e^{\alpha t}$ for any $t > 0$ and $H \in \left(0, \frac{1}{2}\right)$.*

Proof. It is a result from [1] that the left-hand side is bounded by a constant times the right-hand side for large $t > 0$. For smaller t, that is, for $t \le t_0$ for some t_0, note that the left side is continuous while the right side has one discontinuity at 0, where it tends to infinity. Therefore, it is also possible to find a constant for which the bound holds on the compact interval $[0, t_0]$. By taking the maximum of the two we obtain the result. □

The next lemma provides some necessary convergence results. This lemma as well as its proof are motivated by analogous results in [9].

3.1.5 Lemma. *With the above notation we have $e^{-\alpha t} X_t \to \tilde{\xi}_\infty$ as well as $e^{-2\alpha t} \int_0^t X_s^2 ds \to \frac{\tilde{\xi}_\infty^2}{2\alpha}$ almost surely as t tends to infinity.*

Proof. The first statement follows directly from the fact that $\xi_t \to \xi_\infty$ a.s. (shown in Lemma 2, [9]):

$$e^{-\alpha t} X_t = x_0 + \int_0^t e^{-\alpha s} L(s) ds + \xi_t \to x_0 + \int_0^\infty e^{-\alpha s} L(s) ds + \xi_\infty \text{ a.s.}$$

For the second statement we start by noticing that $\tilde{\xi}_t$ is a process with a.s. continuous paths. We have for each $t \geq 0$:

$$\int_0^t X_s^2 ds \geq \int_{t/2}^t e^{2\alpha s} \tilde{\xi}_s^2 ds \geq \frac{t}{2} e^{\alpha t} \inf_{\frac{t}{2} \leq s \leq t} \tilde{\xi}_s^2.$$

Since $\tilde{\xi}_t \to \tilde{\xi}_\infty$ a.s., it follows that

$$\lim_{t\to\infty} \inf_{\frac{t}{2} \leq s \leq t} \tilde{\xi}_s^2 = \tilde{\xi}_\infty^2 \text{ a.s.}$$

From the fact that $\xi_\infty \sim N(0, \frac{H\Gamma(2H)}{\alpha^{2H}})$ (shown in [9]) we can conclude that $\tilde{\xi}_\infty$ also follows a (non-degenerate) normal distribution, and hence, $\lim_{t\to\infty} \int_0^t X_s^2 ds = \infty$ a.s. Therefre, we get by l'Hopital's rule

$$\lim_{t\to\infty} \frac{\int_0^t e^{2\alpha s} \tilde{\xi}_s^2 ds}{e^{2\alpha t}} = \lim_{t\to\infty} \frac{\tilde{\xi}_t^2}{2\alpha} = \frac{\tilde{\xi}_\infty^2}{2\alpha}.$$

□

3.1.6 Lemma. *For $i \in \{1, \ldots, p\}$ the following hold almost surely:*

(1) $\frac{1}{n} \int_0^n \varphi_i(t) dB_t^H \to 0$,

(2) $e^{-\alpha n} \Lambda_{ni} \sqrt{n} \to 0$,

(3) $n D_n e^{-2\alpha n} \to \frac{\tilde{\xi}_\infty^2}{2\alpha}$,

(4) $e^{-\alpha n} \frac{1}{\sqrt{n}} \int_0^n X_t dB_t^H \to 0$.

Proof. (1) This is an application of Lemma 3.1.3: We have

$$\mathbb{E}[(\frac{1}{n} \int_0^n \varphi_i(t) dB_t^H)^2] = \frac{1}{n^2} \int_0^n \int_0^n \varphi_i(u)\varphi_i(v) |u-v|^{2H-2} du dv \lesssim n^{2H-2},$$

and the result follows for $k = 2$.

(2) We write Λ_{ni} as a sum of a deterministic and of a centred Gaussian part and show convergence separately:

$$e^{-\alpha n} \Lambda_{ni} = \frac{1}{n} e^{-\alpha n} \int_0^n \varphi_i(t)(e^{\alpha t} x_0 + e^{\alpha t} \int_0^t e^{-\alpha s} L(s) ds) dt + \frac{1}{n} e^{-\alpha n} \int_0^n \varphi_i(t) e^{\alpha t} \xi_t dt =: A + B,$$

where $\xi_t = \int_0^t e^{-\alpha r} dB_r^H$. Note that the summand B is indeed centred Gaussian: It is an almost sure limit of Riemann sums which are centred Gaussian random variables.

For the deterministic part we write

$$\sqrt{n}A = \frac{1}{\sqrt{n}} e^{-\alpha n} \int_0^n \varphi_i(t) e^{\alpha t} x_0 dt + \frac{1}{\sqrt{n}} e^{-\alpha n} \int_0^n e^{\alpha t} \int_0^t e^{-\alpha s} L(s) ds dt$$
$$=: A_1 + A_2,$$

and we can bound the two summands as follows:

$$|A_1| \lesssim \frac{1}{\sqrt{n}} e^{-\alpha n} \int_0^n e^{\alpha t} dt = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n}} e^{-\alpha n} \to 0$$

as well as

$$|A_2| \lesssim \frac{1}{\sqrt{n}} e^{-\alpha n} \int_0^n e^{\alpha t} \int_0^t e^{-\alpha s} ds dt = \frac{1}{\sqrt{n}} e^{-\alpha n} \int_0^n e^{\alpha t} dt - \frac{1}{\sqrt{n}} e^{-\alpha n} \to 0.$$

We have shown convergence for the deterministic part and now we will calculate the second moment of the Gaussian part in order to apply Lemma 3.1.3. We have

$$\begin{aligned}\mathbb{E}[(\sqrt{n}B)^2] =& \mathbb{E}[(\frac{1}{\sqrt{n}} e^{-\alpha n} \int_0^n \varphi_i(t) e^{\alpha t} \xi_t dt)^2] \\ =& \frac{1}{n} e^{-2\alpha n} \int_0^n \int_0^n \varphi_i(t) \varphi_i(s) e^{\alpha t} e^{\alpha s} \, \mathbb{E}[\xi_t \xi_s] ds dt\end{aligned}$$

and we get by treating the stochastic integrals as Skorokhod integrals

$$\mathbb{E}[\xi_t \xi_s] = \int_0^t \int_0^s e^{-\alpha r} e^{-\alpha v} |r - v|^{2H-2} dv dr.$$

In total, we obtain

$$\begin{aligned}&\mathbb{E}[(\sqrt{n}B)^2] \\ &= \frac{1}{n} e^{-2\alpha n} \int_0^n \int_0^n \varphi_i(t) \varphi_i(s) \int_0^t \int_0^s e^{\alpha s - \alpha r} e^{\alpha t - \alpha v} |r - v|^{2H-2} dv dr ds dt \\ &= \frac{1}{n} e^{-2\alpha n} \int_0^n \int_0^n |r - v|^{2H-2} \int_v^n \int_r^n \varphi_i(t) \varphi_i(s) e^{\alpha s - \alpha r} e^{\alpha t - \alpha v} ds dt dv dr \\ &\lesssim \frac{1}{\alpha^2} \frac{1}{n} e^{-2\alpha n} \int_0^n \int_0^n |r - v|^{2H-2} (e^{\alpha n - \alpha v} - 1)(e^{\alpha n - \alpha r} - 1) dr dv \\ &\sim \frac{1}{n} \int_0^n \int_0^n |r - v|^{2H-2} (e^{-\alpha v} - e^{-\alpha n})(e^{-\alpha r} - e^{-\alpha n}) dr dv \\ &\leq \frac{1}{n} \int_0^n \int_0^n |r - v|^{2H-2} e^{-\alpha v} e^{-\alpha r} dr dv \lesssim \frac{1}{n},\end{aligned}$$

because the last integral is bounded (this is shown in [30]). Lemma 3.1.3 yields almost sure convergence to zero and hence the desired result.

(3) This follows from the previous result and Lemma 3.1.5:

$$Dne^{-2\alpha n} = e^{-2\alpha n}\int_0^n X_t^2 dt - \underbrace{\sum_{i=1}^p (\sqrt{n}\Lambda_{ni}e^{-\alpha n})^2}_{\to 0 \text{ by } (2)} \overset{n\to\infty}{\to} \frac{\tilde{\xi}_\infty^2}{2\alpha} \text{ a.s.}$$

(4) We plug in the expression X_t and get

$$\begin{aligned} e^{-\alpha n}\frac{1}{\sqrt{n}}\int_0^n X_t dB_t^H &= e^{-\alpha n}\frac{1}{\sqrt{n}}\int_0^n e^{\alpha t}x_0 dB_t^H \\ &+ e^{-\alpha n}\frac{1}{\sqrt{n}}\int_0^n e^{\alpha t}\int_0^t e^{-\alpha s}L(s)ds dB_t^H \\ &+ e^{-\alpha n}\frac{1}{\sqrt{n}}\int_0^n e^{\alpha t}\int_0^t e^{-\alpha s}dB_s^H dB_t^H =: A + B + C. \end{aligned}$$

The integral in A can again be interpreted as a Skorokhod integral (yielding a centred Gaussian random variable) which allows us the computation of its L^2 norm:

$$\begin{aligned} \mathbb{E}[A^2] &= x_0^2\frac{1}{n}e^{-2\alpha n}\int_0^n\int_0^n e^{\alpha u}e^{\alpha v}|u-v|^{2H-2}dudv \\ &= x_0^2\frac{1}{n}\underbrace{\int_0^n\int_0^n e^{-\alpha(n-u)}e^{-\alpha(n-v)}|u-v|^{2H-2}dudv}_{=:I_n} \lesssim \frac{1}{n}, \end{aligned}$$

because I_n is bounded as shown in Lemma 5.1 in [30]. Lemma 3.1.3 implies almost sure convergence. For B, which is also a centred Gaussian sequence, the calculation is similar:

$$\begin{aligned} &\mathbb{E}[B^2] \\ &= \frac{1}{n}e^{-2\alpha n}\int_0^n\int_0^n e^{\alpha u}\int_0^u e^{-\alpha s}L(s)ds e^{\alpha v}\int_0^v e^{-\alpha r}L(r)dr|u-v|^{2H-2}dudv \\ &\lesssim \frac{1}{n}e^{-2\alpha n}\int_0^n\int_0^n e^{\alpha u}(1-e^{-\alpha u})e^{\alpha v}(1-e^{-\alpha v})|u-v|^{2H-2}dudv \\ &= \frac{1}{n}e^{-2\alpha n}\int_0^n\int_0^n (e^{\alpha u}-1)(e^{\alpha v}-1)|u-v|^{2H-2}dudv \le \frac{1}{n}I_n \lesssim \frac{1}{n}, \end{aligned}$$

and the almost sure convergence follows. For C we use Lemma 4 from [9] to decompose the double integral:

$$C = e^{-\alpha n}\frac{1}{\sqrt{n}}(\int_0^n e^{\alpha s} dB_s^H \int_0^t e^{-\alpha r} dB_r^H - \int_0^n e^{-\alpha s}\int_0^s e^{\alpha r}\delta B_r^H \delta B_s^H$$
$$- H(2H-1)\int_0^n e^{-\alpha s}\int_0^s e^{\alpha r}|s-r|^{2H-2} drds) =: C_1 - C_2 - C_3,$$

where δ stands for the Skorokhod integral. We show almost sure convergence for the three summands:

$$C_1 = e^{-\alpha n}\frac{1}{\sqrt{n}}\int_0^n e^{\alpha s} dB_s^H \xi_t,$$

and since we know from [9] that $\xi_t \to \xi_\infty$ a.s. as t tends to infinity (where $\xi_\infty \sim N(0, \frac{H\Gamma(2H)}{\alpha^{2H}})$), it is enough to show that $e^{-\alpha n}\frac{1}{\sqrt{n}}\int_0^n e^{\alpha s} dB_s^H \to 0$ almost surely for $n \to \infty$. Because it is a centred Gaussian sequence, we can again rely on Lemma 3.1.3 and compute the respective variances:

$$\mathbb{E}[(e^{-\alpha n}\frac{1}{\sqrt{n}}\int_0^n e^{\alpha s} dB_s^H)^2]$$
$$\sim e^{-2\alpha n}\frac{1}{n}\int_0^n\int_0^n e^{\alpha s}e^{\alpha r}|s-r|^{2H-2} dsdr = \frac{1}{n} I_n \lesssim \frac{1}{n}.$$

In order to treat C_2 note that by Lemma 7 in [9]

$$Y_n := e^{-\frac{\alpha n}{2}}\int_0^n e^{-\alpha s}\int_0^s e^{\alpha r}\delta B_r^H \delta B_s^H \xrightarrow{L^2} 0,$$

and consequently $\mathbb{E}[Y_n^2]$ is bounded. Since, moreover, Y_n is centred (as it is a Skorokhod integral), Markov inequality helps achieve the summability of tails:

$$\sum_{n=1}^\infty P(|C_2| \geq \varepsilon) = \sum_{n=1}^\infty P(|\frac{1}{\sqrt{n}} e^{-\frac{\alpha n}{2}} Y_n| \geq \varepsilon)$$
$$\leq \sum_{n=1}^\infty \frac{\mathbb{E}[Y_n^2]}{\varepsilon^2 n e^{\frac{\alpha n}{2}}} \lesssim \sum_{n=1}^\infty \frac{1}{n e^{\frac{\alpha n}{2}}} < \infty,$$

and almost sure convergence to zero follows. Finally, Lemma 7 in [9] ensures that $C_3 e^{\frac{\alpha n}{2}}\sqrt{n}$ converges to zero, which implies that also C_3 itself goes to zero as n tends to infinity. This completes the proof of the initial claim.

□

3.1.7 Corollary. *For $\beta < \frac{1}{2}$ we have $n^\beta e^{-\alpha n}\Lambda_{ni}\sqrt{n} \to 0$ as well as $n^\beta e^{-\alpha n}\frac{1}{\sqrt{n}}\int_0^n X_t dB_t^H \to 0$ almost surely as n tends to infinity.*

Proof. The deterministic part of the sequence $n^\beta e^{-\alpha n}\Lambda_{ni}\sqrt{n}$ (i.e. $n^\beta\sqrt{n}A$, cf. the notation from the proof of (2) in 3.1.6) is bounded up to a constant by $n^{\beta-0.5}$ and the variance of the random part by $n^{2\beta-1}$. This yields polynomial rates of convergence, thus Lemma 3.1.3 still can be applied and we obtain almost sure convergence. The same argument holds for the second convergence result. Lemma 3.1.3 can still be applied for A, B and C_1 (from the proof of (4) in 3.1.6), and for C_2 and C_3 the additional factor n^β changes nothing in the structure of the arguments, so the proofs can be followed verbatim. □

3.1.4 Asymptotic properties of the estimator

In this section we will establish strong consistency and asymptotic normality of the estimator $\hat{\vartheta}$ defined in Section 3.1.2. The study of the asymptotic normality is the highlight of the whole chapter; in particular, the search for a palatable form of the asymptotic variance in some special cases and even the proof of its positivity require lengthy calculations.

3.1.8 Theorem. *The estimator $\hat{\vartheta} = (\hat{\mu}_1, \dots, \hat{\mu}_p,\ \hat{\alpha})$ is strongly consistent, i.e.*

(1) for $i \in \{1, \dots, p\}$

$$\hat{\mu_i} - \mu_i = \frac{1}{n}(\int_0^n \varphi_i(t)dB_t^H + \frac{1}{D_n}\sum_{j=1}^p \Lambda_{ni}\Lambda_{nj}\int_0^n \varphi_j(t)dB_t^H - \frac{1}{D_n}\Lambda_{ni}\int_0^n X_t dB_t^H) \to 0,$$

(2) $\hat{\alpha} - \alpha = -\frac{1}{nD_n}(\sum_{i=1}^p \Lambda_{ni}\int_0^n \varphi_i(t)dB_t^H - \int_0^n X_t dB_t^H) \to 0,$

hold almost surely as n tends to infinity.

Proof. We treat each summand separately and exploit Lemma 3.1.6.

(1) Let us denote $M_1 := \frac{1}{n}\int_0^n \varphi_i(t)dB_t^H$, $M_{2j} := \frac{1}{n}\frac{1}{D_n}\Lambda_{ni}\Lambda_{nj}\int_0^n \varphi_j(t)dB_t^H$, $M_3 := \frac{1}{n}\frac{1}{D_n}\Lambda_{ni}\int_0^n X_t dB_t^H$. In order to prove the claim we have to show that each of these summands converges to zero almost surely. For M_1 this is shown in Lemma 3.1.6 (1). To see this for M_{2j} we rewrite it as follows:

$$\begin{aligned} M_{2j} &= \frac{1}{n}\frac{1}{D_n}\Lambda_{ni}\Lambda_{nj}\int_0^n \varphi_j(t)dB_t^H \\ &= \underbrace{\frac{1}{nD_n e^{-2\alpha n}}}_{\to\frac{2\alpha}{\xi_\infty^2}\text{ by 3.1.6(3)}}\underbrace{(e^{-\alpha n}\Lambda_{ni}\sqrt{n})}_{\to 0\text{ by 3.1.6(2)}}\underbrace{(e^{-\alpha n}\Lambda_{nj}\sqrt{n})}_{\to 0\text{ by 3.1.6(2)}}\underbrace{\frac{1}{n}\int_0^n \varphi_j(t)dB_t^H}_{\to 0\text{ by 3.1.6(1)}}, \end{aligned}$$

and since $\tilde{\xi}_\infty$ is almost surely nonzero, the whole expression converges a.s. to zero. M_3 can also be rewritten in a way that makes the convergence statement obvious:

$$\begin{aligned} M_3 &= \frac{1}{n}\frac{1}{D_n}\Lambda_{ni}\int_0^n X_t dB_t^H \\ &= \underbrace{\frac{1}{nD_n e^{-2\alpha n}}}_{\to\frac{2\alpha}{\tilde{\xi}_\infty^2}\text{ by 3.1.6(3)}} \underbrace{(e^{-\alpha n}\Lambda_{ni}\sqrt{n})}_{\to 0\text{ by 3.1.6(2)}} \underbrace{(e^{-\alpha n}\frac{1}{\sqrt{n}}\int_0^n X_t dB_t^H)}_{\to 0\text{ by 3.1.6(4)}}, \end{aligned}$$

the claim follows with the same argument as above and completes the proof of the theorem.

(2) In this case we also start by introducing a notation for each type of summands. Let us denote $A_{1i} := \frac{1}{nD_n}\Lambda_{ni}\int_0^n \varphi_i(t)dB_t^H$ and $A_2 := \frac{1}{nD_n}\int_0^n X_t dB_t^H$. For the first type of summands we write

$$\begin{aligned} A_{1i} &= \frac{1}{nD_n}\Lambda_{ni}\int_0^n \varphi_i(t)dB_t^H \\ &= \underbrace{\frac{1}{nD_n e^{-2\alpha n}}}_{\to\frac{2\alpha}{\tilde{\xi}_\infty^2}\text{ by 3.1.6(3)}} \underbrace{(e^{-\alpha n}\Lambda_{ni}\sqrt{n})}_{\to 0\text{ by 3.1.6(2)}} \underbrace{\sqrt{n}e^{-\alpha n}}_{\to 0} \underbrace{\frac{1}{n}\int_0^n \varphi_i(t)dB_t^H}_{\to 0\text{ by 3.1.6(1)}} \end{aligned}$$

and for the second kind we obtain

$$\begin{aligned} A_2 &= \frac{1}{nD_n}\int_0^n X_t dB_t^H \\ &= \underbrace{\frac{1}{nD_n e^{-2\alpha n}}}_{\to\frac{2\alpha}{\tilde{\xi}_\infty^2}\text{ by 3.1.6(3)}} \underbrace{\sqrt{n}e^{-\alpha n}}_{\to 0} \underbrace{(e^{-\alpha n}\frac{1}{\sqrt{n}}\int_0^n X_t dB_t^H)}_{\to 0\text{ by 3.1.6(4)}}. \end{aligned}$$

Both calculations yield almost sure convergence of the summands (again, using the argument given in (1)) and thus provide the proof for the initial claim.

□

The next lemma is an auxiliary result for a limit theorem that will be proved later.

3.1.9 Lemma. *Let F be any $\sigma(B^H)$-measurable random variable such that $P(F < \infty) = 1$. Then, as $n \to \infty$,*

$$(n^{-H}\delta_n(\varphi_1), \ldots, n^{-H}\delta_n(\varphi_p),\, F,\, e^{-\alpha n}\delta_n(e^{\alpha\cdot})) \xrightarrow{d} (Z_1, \ldots, Z_p,\, F,\, Z),$$

where δ_n is the integral over $[0, n]$ with respect to B^H, $Z_1, \dots, Z_p$ are centred and jointly normally distributed with the covariance matrix $(\int_0^1 \varphi_i(x)dx \int_0^1 \varphi_j(x)dx)_{i,\, j=1,\dots,p}$ and $((Z_1, \dots, Z_p),\, F,\, Z)$ are independent. Moreover, $\mathrm{Var}(Z) = \frac{H\Gamma(2H)}{\alpha^{2H}}$.

Proof. Due to an approximation argument rigorously explained in [25] it is enough to show that for any $d \geq 1$, $s_1, \dots, s_d \in [0, \infty)$

$$(n^{-H}\delta_n(\varphi_1), \dots, n^{-H}\delta_n(\varphi_p), B^H_{s_1}, \dots, B^H_{s_d},\, e^{-\alpha n}\delta_n(e^{\alpha \cdot})) \xrightarrow{d} (Z_1, \dots, Z_p,\, B^H_{s_1}, \dots, B^H_{s_d},\, Z)$$

as $n \to \infty$. The left hand side is a Gaussian vector, and hence it suffices to determine the limits of the covariances. It is shown in [9] that the limits of $\mathrm{Cov}(B^H_s,\, e^{-\alpha n}\delta_n(e^{\alpha \cdot}))$ and $\mathrm{Var}(e^{-\alpha n}\delta_n(e^{\alpha \cdot}))$ are as claimed. Moreover, in [7] the joint limiting distribution of $(n^{-H}\delta_n(\varphi_1), \dots, n^{-H}\delta_n(\varphi_p))$ is established. Therefore, we only have to show that $\mathrm{Cov}(n^{-H}\delta_n(\varphi_i),\, B^H_s)$ and $\mathrm{Cov}(n^{-H}\delta_n(\varphi_i),\, e^{-\alpha n}\delta_n(e^{\alpha \cdot}))$ converge to zero. For the first statement recall that $B^H_s = \int_0^n \mathbf{1}_{[0,\, s]} dB^H_t$ for any $n \geq s$. Then we can write (for n large enough) due to the isometry property of the integrals:

$$\begin{aligned}
\mathbb{E}[n^{-H}\delta_n(\varphi_i)B^H_s] &\lesssim n^{-H} \int_0^n \int_0^s |u-v|^{2H-2} du dv \\
&= n^{-H} \int_0^s \int_{-v}^{n-v} |z|^{2H-2} dz dv = n^{-H} \int_0^s \int_0^v z^{2H-2} dz + \int_0^{n-v} z^{2H-2} dz dv \\
&= \underbrace{n^{-H} \int_0^s v^{2H-1} dv}_{\to 0} + n^{-H} \int_0^s (n-v)^{2H-1} dv \lesssim n^{-H} \int_{n-s}^n z^{2H-1} dz \\
&= n^{-H}(n^{2H} - (n-s)^{2H}) \overset{\text{binom. series}}{=} n^{-H} O(n^{2H-1}) = O(n^{H-1}),
\end{aligned}$$

which goes to zero as n tends to infinity.

For the second convergence refer to Proposition 3.1.4 for the estimation $\int_0^t e^{\alpha u} u^{2H-2} du \lesssim t^{2H-2} e^{\alpha t}$. We use this for our calculation:

$$\begin{aligned}
\mathbb{E}[n^{-H}\delta_n(\varphi_i) e^{-\alpha n}\delta_n(e^{\alpha \cdot})] &\lesssim n^{-H} e^{-\alpha n} \int_0^n \int_0^n e^{\alpha v} |u-v|^{2H-2} du dv \\
&= n^{-H} e^{-\alpha n} \int_0^n e^{\alpha u} \int_0^n e^{\alpha(v-u)} |v-u|^{2H-2} dv du \\
&= n^{-H} e^{-\alpha n} \int_0^n \left(\underbrace{\int_0^u e^{-\alpha z} z^{2H-2} dz}_{\text{bounded}} + \int_0^{n-u} e^{\alpha z} z^{2H-2} dz \right) du \\
&\lesssim n^{-H} e^{-\alpha n} \int_0^n e^{\alpha u} e^{\alpha(n-u)} (n-u)^{2H-2} du = n^{-H} n^{2H-1} \to 0.
\end{aligned}$$

□

3.1.10 Theorem. *As n tends to infinity we obtain for the estimator $\hat{\vartheta}$*

$$(n^{1-H}(\hat{\mu}_1 - \mu_1, \dots, \hat{\mu}_p - \mu_p),\, e^{\alpha n}(\hat{\alpha} - \alpha)) \xrightarrow{d} (Z_1, \dots,\, Z_p,\, Z_{p+1})$$

with $Z_1, \dots, Z_p$ as above and $Z_{p+1} = 2\alpha N/M$ with $N \sim \mathrm{N}(0,\,1)$ and

$$M \sim \mathrm{N}\left(\frac{\alpha^H}{\sqrt{H\Gamma(2H)}}\left(x_0 + \int_0^\infty e^{-\alpha s} L(s)ds\right),\, 1\right)$$

independent of N. Moreover, $(Z_1, \dots, Z_p)$ and Z_{p+1} also are independent.

This result reflects the structure of the estimator: In the first p components the additive term $\frac{1}{n}\int_0^n \varphi_i(t)dB_t^H$ is the slowest summand (note that it does not include the solution process X and is, therefore, not influenced by its exponential growth), which yields the same rates of convergence as in the ergodic case. The estimator for α, however, does not contain such a term; it converges with the same exponential rate as the estimator in [9]. The limiting distribution is also structurally similar to the case $L \equiv 0$. As mentioned in [49], if the estimator from [9] is applied for an equation with a nonzero starting value, the limiting distribution will also contain this value as an additional additive term in the denominator. Moreover, due to the possibility of considering Young integrals and exploiting different techniques in the proofs our results are valid for $H \in \left(\frac{1}{2},\, 1\right)$ in contrast to $H \in \left(\frac{1}{2},\, \frac{3}{4}\right)$ for the ergodic case in [24].

Proof of Theorem 3.1.10. First of all we divide the error into parts that contribute to the limit and the rest. We use the notation from the previous theorem and write:

$$\begin{aligned} n^{1-H}(\hat{\mu}_1 - \mu_1) &= (n^{1-H}M_1 + n^{1-H}(\sum_{j=1}^p M_{2j} + M_3)), \\ e^{\alpha n}(\hat{\alpha} - \alpha) &= (-e^{\alpha n}\sum_{j=1}^p A_{1j} + e^{\alpha n}A_2). \end{aligned}$$

Now we will identify the rest terms by showing: $n^{1-H}(\sum_{j=1}^p M_{2j} + M_3)$ and $e^{\alpha n}\sum_{j=1}^p A_{1j}$ converge to zero almost surely. For M_{2j} and M_3 this follows from the fact that they contain the factor $(e^{-\alpha n}\Lambda_{nj}\sqrt{n})$ which would still converge to zero if multiplied by n^{1-H}, since $1 - H < 0.5$.

Each summand A_{1j} contains the factor

$$(e^{-\alpha n}\Lambda_{ni}\sqrt{n})\sqrt{n}e^{-\alpha n}\frac{1}{n}\int_0^n \varphi_j(t)dB_t^H$$

which converges to zero almost surely. The remainder $\frac{1}{nD_n e^{-2\alpha n}}$ tends almost surely to a random variable. We write

$$\begin{aligned} e^{\alpha n}(e^{-\alpha n}\Lambda_{ni}\sqrt{n})\sqrt{n}e^{-\alpha n}\frac{1}{n}\int_0^n \varphi_j(t)dB_t^H &= (e^{-\alpha n}\Lambda_{ni}\sqrt{n})\sqrt{n}\frac{1}{n}\int_0^n \varphi_j(t)dB_t^H \\ &= (e^{-\alpha n}\Lambda_{ni}\sqrt{n}n^{H-0.5})\left(n^{1-H}\frac{1}{n}\int_0^n \varphi_j(t)dB_t^H\right). \end{aligned}$$

The factor $e^{-\alpha n}\Lambda_{ni}\sqrt{n}n^{H-0.5}$ converges to zero almost surely, because $H-0.5 < 0.5$ and the factor $n^{1-H}\frac{1}{n}\int_0^n \varphi_j(t)dB_t^H$ converges in distribution to a normal random variable (this being a consequence of the previous lemma). In total we conclude that the above expression converges to zero in distribution and therefore in probability. Thus, also the whole term $e^{\alpha n}A_{1j}$ converges to zero in probability.

The next step is to consider and rewrite A_2. For this we apply the change of variables formula for Young integrals (see [9]) to the functions $e^{-\alpha n}X_n$ and $\int_0^n e^{\alpha t}dB_t^H$. We obtain the following formula:

$$\begin{aligned} \int_0^n X_s dB_s^H = \int_0^n e^{\alpha t}dB_t^H \tilde{\xi}_n - \int_0^n e^{-\alpha t}L(t)\int_0^t e^{\alpha s}dB_s^H dt \\ - \int_0^n e^{-\alpha t}\int_0^t e^{\alpha s}dB_s^H dB_t^H =: S_1 + S_2 + S_3, \end{aligned}$$

with which we can substitute the term $\int_0^n X_s dB_s^H$ in A_2. We will now show that only S_1 contributes to the convergence statement. Since

$$e^{\alpha n}A_2 = \frac{1}{nD_n e^{-2\alpha n}}e^{-\alpha n}\int_0^n X_t dB_t^H$$

and the denominator converges almost surely, it is enough to show that $e^{-\alpha n}(S_2 + S_3)$ tend to zero in probability. For S_3 this is shown in [9], so we only show this for S_2. As a Lebesgue integral of a Gaussian process $e^{-\alpha n}S_2$ is again centred Gaussian, showing its second moment's convergence will suffice:

$$\begin{aligned} &\mathbb{E}\left[\left(e^{-\alpha n}\int_0^n e^{-\alpha t}L(t)\int_0^t e^{\alpha s}dB_s^H dt\right)^2\right] \\ &\lesssim e^{-2\alpha n}\int_0^n\int_0^n e^{-\alpha u}L(u)e^{-\alpha v}L(v)\int_0^u\int_0^v e^{\alpha s}e^{\alpha r}|s-r|^{2H-2}dsdrdudv \\ &\lesssim e^{-2\alpha n}\int_0^n\int_0^n\int_0^u\int_0^v |s-r|^{2H-2}dsdrdudv \lesssim e^{-2\alpha n}n^{2H+2} \to 0 \end{aligned}$$

as n tends to infinity.

For the last step of the proof we apply Lemma 3.1.9 to $F = \tilde{\xi}_\infty$ and obtain

$$(n^{-H}\delta_n(\varphi_1), \dots, n^{-H}\delta_n(\varphi_p), \tilde{\xi}_\infty, e^{-\alpha n}\delta_n(e^{\alpha \cdot})) \xrightarrow{d} (Z_1, \dots, Z_p, \tilde{\xi}_\infty, Z),$$

and consequently

$$\left(n^{-H}\delta_n(\varphi_1),\ldots,n^{-H}\delta_n(\varphi_p),\,\frac{e^{-\alpha n}\int_0^n e^{\alpha t}dB_t^H}{\tilde{\xi}_\infty}\right)\overset{d}{\to}\left(Z_1,\ldots,Z_p,\,\frac{Z}{\tilde{\xi}_\infty}\right),$$

where $Z \sim \sqrt{\frac{H\Gamma(2H)}{\alpha^{2H}}}\,\mathrm{N}(0,\,1)$ and

$$\tilde{\xi}_\infty \sim \sqrt{\frac{H\Gamma(2H)}{\alpha^{2H}}}\,\mathrm{N}\left(\frac{\alpha^H}{\sqrt{H\Gamma(2H)}}\left(x_0+\int_0^\infty e^{-\alpha s}L(s)ds\right),\,1\right).$$

Now note additionally that

$$\left(1,\ldots,1,\,\frac{\tilde{\xi}_n\tilde{\xi}_\infty}{nD_ne^{-2\alpha n}}\right)\overset{n\to\infty}{\to}(1,\ldots,1,\,2\alpha)\text{ a.s.}$$

Multiplying both vectors elementwise using Slutsky's lemma yields

$$\left(n^{-H}\delta_n(\varphi_1),\ldots,n^{-H}\delta_n(\varphi_p),\,\frac{e^{-\alpha n}S_1}{nD_ne^{-2\alpha n}}\right)\overset{d}{\to}\left(Z_1,\ldots,Z_p,\,2\alpha\frac{Z}{\tilde{\xi}_\infty}\right),$$

which is all that we need to show, since all the other summands converge to zero in probability. Note that we inherit the independence statement directly from Lemma 3.1.9. □

3.1.11 Remark. Recall that the covariance matrix of the limiting vector $(Z_1,\ldots,Z_p)$ has the form $\left(\int_0^1\varphi_i(t)dt\int_0^1\varphi_j(t)dt\right)_{i,\,j=1,\ldots,p}$. This matrix is singular of rank one; the limiting vector can be written as

$$\left(\int_0^1\varphi_1(t)dt,\ldots,\int_0^1\varphi_p(t)dt\right)^T Z',$$

where Z' is a standard normal random variable. This kind of limiting distributions does not often appear in the literature, one example being the simultaneous estimation of the parameters $\sigma>0$ and $H\in(0,\,1)$ from discrete observations of the fractional Gaussian noise $\left(\sigma B^H_{\Delta i}-\sigma B^H_{\Delta(i-1)}\right)_{i=1,\ldots,n}$ in the high frequency setting (considered in [13]). The authors proceed to show the LAN property for the model with a non-diagonal rate matrix and derive from it the efficient rates of convergence in case where both σ and H are unknown. These are worse than the efficient rates in case where just one of the two parameters is unknown. While we do not make such an analysis here and we do not know whether the rate n^{1-H} is efficient (although it is certainly an interesting question for future research), it is easy to see in our case that the knowledge of one of the parameters improves the speed of convergence: By transforming the parameter vector we obtain

$$n^{1-H}\begin{pmatrix} \hat{\mu}_1-\mu_1 \\ \hat{\mu}_2-\mu_2-\frac{\int_0^1\varphi_2(t)dt}{\int_0^1\varphi_1(t)dt}\left(\hat{\mu}_1-\mu_1\right) \\ \vdots \\ \hat{\mu}_p-\mu_p-\frac{\int_0^1\varphi_p(t)dt}{\int_0^1\varphi_1(t)dt}\left(\hat{\mu}_1-\mu_1\right) \end{pmatrix} \xrightarrow{d} \left(\int_0^1\varphi_1(t)dt,\, 0,\dots,0\right)^T Z'$$

as n tends to infinity (assuming that $\int_0^1\varphi_1(t)dt\neq 0$). We will see in the next step that the changed components converge with the speed $\sqrt{n}$, which will consequently be the speed of convergence of the appropriately changed parameter vector once μ_1 is known. Note that if $\int_0^1\varphi_1(t)dt=0$ the first component will already converge with a faster speed (this will also be shown subsequently).

Consider the special case of a basis element φ_k, $k\in\{1,\dots,p\}$, which integrates to zero on $[0,1]$. The results of our theorems continue to hold, but the limiting vector $(Z_1,\dots,Z_p)$ will have a zero entry at Z_k. This suggests that the convergence of the kth component of the estimator might be of a better order than n^{H-1}. The same observation applies to the "transformed estimator" from Remark 3.1.11. Indeed, one obtains the following facts.

3.1.12 Proposition. *If φ_k for $k\in\{1,\dots,p\}$ is such that $\int_0^1\varphi_k(t)dt=0$, then, as n tends to infinity,*

$$\sqrt{n}(\hat{\mu}_k-\mu_k)\xrightarrow{d} H(2H-1)\bar{Z}_k,$$

where $\bar{Z}_k$ is a zero mean Gaussian random variable with variance

$$\int_0^1\int_0^1\varphi_k(t)\varphi_k(s)|t-s|^{2H-2}dtds \\ +\sum_{l=1}^{\infty}2\binom{2H-2}{2l}\zeta(2l+2-2H)\int_0^1\int_0^1\varphi_k(t)\varphi_k(s)(t-s)^{2l}dtds,$$

where ζ denotes the Riemann zeta function.

Proof. Recall that

$$\sqrt{n}(\hat{\mu}_1-\mu_1)=\left(\sqrt{n}M_1+\sqrt{n}\left(\sum_{j=1}^{p}M_{2j}+M_3\right)\right)$$

with the notation from Theorem 3.1.8. As in Theorem 3.1.10, Corollary 3.1.7 ensures that $\sqrt{n}M_{2j}$ and $\sqrt{n}M_3$ converge to zero almost surely. Given that

$$\sqrt{n}M_1=\frac{1}{\sqrt{n}}\int_0^n\varphi_k(t)dB_t^H,$$

it is enough for our claim to investigate the term $\mathbb{E}\left[\left(\frac{1}{\sqrt{n}}\int_0^n \varphi_k(t)dB_t^H\right)^2\right]$.
With $\alpha_H = H(2H-1)$ we have by isometry and periodicity:

$$\begin{aligned}
&\frac{1}{\alpha_H}\mathbb{E}\left[\left(\frac{1}{\sqrt{n}}\int_0^n \varphi_k(t)dB_t^H\right)^2\right] \\
&\quad= \frac{1}{n}\int_0^n\int_0^n \varphi_k(t)\varphi_k(s)|t-s|^{2H-2}dtds \\
&\quad= \frac{1}{n}\sum_{i,j=0}^{n-1}\int_0^1\int_0^1 \varphi_k(t)\varphi_k(s)|t+i-s-j|^{2H-2}dtds \\
&\quad= \frac{1}{n}\int_0^1\int_0^1 \varphi_k(t)\varphi_k(s)n|t-s|^{2H-2}dtds \\
&\qquad+ \frac{1}{n}\int_0^1\int_0^1 \varphi_k(t)\varphi_k(s)\sum_{i>j}|t-s+i-j|^{2H-2}dtds \\
&\qquad+ \frac{1}{n}\int_0^1\int_0^1 \varphi_k(t)\varphi_k(s)\sum_{j>i}|s-t+j-i|^{2H-2}dtds. \qquad (3.1.2)
\end{aligned}$$

The first summand is independent of n, hence, it remains to consider the second and the third one (which are equal for symmetry reasons). By rearranging the sum in the second summand, we obtain the following:

$$\begin{aligned}
&\frac{1}{n}\int_0^1\int_0^1 \varphi_k(t)\varphi_k(s)\sum_{i>j}|t-s+i-j|^{2H-2}dtds \\
&\quad= \frac{1}{n}\int_0^1\int_0^1 \varphi_k(t)\varphi_k(s)\sum_{m=1}^{n-1}(n-m)|t-s+m|^{2H-2}dtds \\
&\quad= \frac{1}{n}\int_0^1\int_0^1 \varphi_k(t)\varphi_k(s)\sum_{m=1}^{n-1}(n-m)m^{2H-2}\left(\frac{t-s}{m}+1\right)^{2H-2}dtds \\
&\quad= \frac{1}{n}\int_0^1\int_0^1 \varphi_k(t)\varphi_k(s)\sum_{m=1}^{n-1}nm^{2H-2}\left(\frac{t-s}{m}+1\right)^{2H-2}dtds \\
&\qquad- \frac{1}{n}\int_0^1\int_0^1 \varphi_k(t)\varphi_k(s)\sum_{m=1}^{n-1}m\cdot m^{2H-2}\left(\frac{t-s}{m}+1\right)^{2H-2}dtds.
\end{aligned}$$

Now we can use the binomial series expansion to get

$$\left(\frac{t-s}{m}+1\right)^{2H-2} = \sum_{l=0}^{\infty}\binom{2H-2}{l}(t-s)^l m^{-l}$$

and use the zero integral assumption in order to evaluate the above expression. We conclude:

$$
\begin{aligned}
&\frac{1}{n}\int_0^1\int_0^1 \varphi_k(t)\varphi_k(s)\sum_{m=1}^{n-1} n m^{2H-2}\left(\frac{t-s}{m}+1\right)^{2H-2} dtds \\
&= \int_0^1\int_0^1 \varphi_k(t)\varphi_k(s)\sum_{m=1}^{n-1} m^{2H-2}\sum_{l=2}^{\infty}\binom{2H-2}{l}(t-s)^l m^{-l} dtds \\
&= \int_0^1\int_0^1 \varphi_k(t)\varphi_k(s)\sum_{l=2}^{\infty}\binom{2H-2}{l}(t-s)^l \sum_{m=1}^{n-1} m^{2H-2-l} dtds.
\end{aligned}
$$

By dominated convergence we now obtain

$$
\begin{aligned}
&\lim_{n\to\infty}\int_0^1\int_0^1 \varphi_k(t)\varphi_k(s)\sum_{l=2}^{\infty}\binom{2H-2}{l}(t-s)^l \sum_{m=1}^{n-1} m^{2H-2-l} dtds \\
&= \int_0^1\int_0^1 \varphi_k(t)\varphi_k(s)\sum_{l=2}^{\infty}\binom{2H-2}{l}(t-s)^l \sum_{m=1}^{\infty} m^{2H-2-l} dtds \\
&= \int_0^1\int_0^1 \varphi_k(t)\varphi_k(s)\sum_{l=2}^{\infty}\binom{2H-2}{l}(t-s)^l \zeta(l+2-2H) dtds,
\end{aligned}
$$

since the m^{2H-2-l} are summable for $l \geq 1$.
In a similar manner, we get

$$
\begin{aligned}
&\frac{1}{n}\int_0^1\int_0^1 \varphi_k(t)\varphi_k(s)\sum_{m=1}^{n-1} m\cdot m^{2H-2}\left(\frac{t-s}{m}+1\right)^{2H-2} dtds \\
&= \int_0^1\int_0^1 \varphi_k(t)\varphi_k(s)\sum_{l=2}^{\infty}\binom{2H-2}{l}(t-s)^l \frac{1}{n}\sum_{m=1}^{n-1} m^{2H-1-l} dtds,
\end{aligned}
$$

which converges to zero, again, due to summability of m^{2H-1-l}.

In total, we conclude that the second summand in (3.1.2) converges to

$$
\int_0^1\int_0^1 \varphi_k(t)\varphi_k(s)\sum_{l=2}^{\infty}\binom{2H-2}{l}(t-s)^l \zeta(l+2-2H) dtds,
$$

and thus, with a symmetric calculation, the third summand tends to

$$
\int_0^1\int_0^1 \varphi_k(t)\varphi_k(s)\sum_{l=2}^{\infty}\binom{2H-2}{l}(s-t)^l \zeta(l+2-2H) dtds.
$$

Adding up the two yields the desired result. □

3.1.13 Remark. Assume that $k_1,\ k_2 \in \{1,\ldots,p\}$ are both such that $\int_0^1 \varphi_{k_1}(t)dt = \int_0^1 \varphi_{k_2}(t)dt = 0$. Following the same calculations as above we obtain the term

$$\int_0^1 \int_0^1 \varphi_{k_1}(t)\varphi_{k_2}(s)|t-s|^{2H-2}dtds \\ + \sum_{l=1}^{\infty} 2\binom{2H-2}{2l}\zeta(2l+2-2H)\int_0^1\int_0^1 \varphi_{k_1}(t)\varphi_{k_2}(s)(t-s)^{2l}dtds$$

as limiting covariance of $\sqrt{n}(\hat{\mu}_{k_1}-\mu_{k_1})$ and $\sqrt{n}(\hat{\mu}_{k_2}-\mu_{k_2})$.

Now let us show an auxiliary result that will help us analyse the variance expression further.

3.1.14 Proposition. *Let $(f_n)_{n\in\mathbb{Z}\setminus\{0\}}$ be the real $L^2([0,\,1])$-Fourier basis without the constant element, i.e. $f_n(x)=\sqrt{2}\sin(2\pi nx)$ and $f_{-n}(x)=\sqrt{2}\cos(2\pi nx)$ for $n\in\mathbb{N}$. Then for any $u>0$ the integral*

$$\int_0^1\int_0^1 f_n(t)f_m(s)(e^{u(1-|t-s|)}+e^{u|t-s|}-2)dtds$$

is strictly positive and equal to $\frac{2(e^u-1)u}{(2\pi n)^2+u^2}$ if $m=n$ and zero otherwise.

Proof. Let us wirte $z=e^u$ and calculate for $m,\,n\in\mathbb{Z}\setminus\{0\}$:

$$\begin{aligned}\int_0^1\int_0^1 & f_n(t)f_m(s)(z^{(1-|t-s|)}+z^{|t-s|}-2)dtds \\ &= \int_0^1\int_{t-1}^{t} f_n(t)f_m(t-v)(z^{1-|v|}+z^{|v|})dvdt \\ &= \int_0^1 f_n(t)\int_{t-1}^{t} f_m(t-v)(z^{1-|v|}+z^{|v|})dvdt.\end{aligned}$$

By classical trigonometric identities we can decompose $f_m(t-v)$ as

$$\sqrt{2}f_m(t-v)=f_m(t)f_{-m}(v)-f_{-m}(t)f_m(v)$$

if m is positive and

$$\sqrt{2}f_m(t-v)=f_m(t)f_m(v)+f_{-m}(t)f_{-m}(v)$$

if m is negative. Thus, for the second part of the statement it suffices to show that the integral

$$\int_{t-1}^{t} f_m(v)(z^{1-|v|}+z^{|v|})dv$$

is independent of t for all $m \in \mathbb{Z}\backslash\{0\}$ and equal to zero for $m > 0$. This is indeed the case, because

$$\int_{t-1}^{0} f_m(v)(z^{1+v} + z^{-v})dv = \int_{t}^{1} f_m(v)(z^{1-v} + z^{v}),$$

and therefore,

$$\int_{t-1}^{t} f_m(v)(z^{1-|v|} + z^{|v|})dv = \int_{0}^{1} f_m(v)(z^{1-v} + z^{v})dv$$

is indeed independent of t. For symmetry reasons the integral vanishes for $m > 0$.

If $n = m$, the same trigonometric identities can be used to show that

$$\int_0^1 \int_0^1 f_n(t)f_n(s)(e^{u(1-|t-s|)} + e^{u|t-s|} - 2)dtds = \frac{1}{\sqrt{2}} \int_0^1 f_{-n}(v)(z^{1-v} + z^{v})dv$$

if n is positive and

$$\int_0^1 \int_0^1 f_n(t)f_n(s)(e^{u(1-|t-s|)} + e^{u|t-s|} - 2)dtds = \frac{1}{\sqrt{2}} \int_0^1 f_{n}(v)(z^{1-v} + z^{v})dv$$

if n is negative. Since

$$\int_0^1 \cos(2\pi nv)(z^{1-v} + z^{v})dv = \frac{2(z-1)\log(z)}{(2\pi n)^2 + (\log(z))^2} = \frac{2(e^u - 1)u}{(2\pi n)^2 + u^2}$$

is positive for all $u > 0$, the first part of the claim is proved. □

Now we can provide additional information about $\bar{Z}_k$ and find a more concise form for its variance.

3.1.15 Proposition. *The variance of $\bar{Z}_k$ from the Proposition 3.1.12 can be simplified to*

$$\frac{1}{\Gamma(2-2H)} \int_0^1 \int_0^1 \varphi_k(t)\varphi_k(s) \int_0^\infty \frac{u^{1-2H}}{e^u - 1}(e^{u(1-|t-s|)} + e^{u|t-s|} - 2)dudtds.$$

This expression is positive for all bounded nonzero L^2-functions φ_k with zero integrals.

Proof. Our goal is to show that

$$\int_0^1 \int_0^1 \varphi_k(t)\varphi_k(s)|t-s|^{2H-2}dtds \tag{3.1.3}$$

$$+ \sum_{l=1}^{\infty} 2\binom{2H-2}{2l}\zeta(2l+2-2H) \int_0^1 \int_0^1 \varphi_k(t)\varphi_k(s)(t-s)^{2l}dtds \tag{3.1.4}$$

can be rewritten in the above integral form. For the first summand the definition of Gamma function provides the representation

$$|t-s|^{2H-2} = \frac{1}{\Gamma(2-2H)} \int_0^\infty u^{1-2H} e^{-u|s-t|} du.$$

For the other summands we make use of the formula $\Gamma(z)\zeta(z) = \int_0^\infty \frac{u^{z-1}}{e^u-1} du$ for $z > 1$ (see [3], p. 251) and rewrite them as follows:

$$\begin{aligned}
&2\sum_{l=1}^\infty \binom{2H-2}{2l} \zeta(2l+2-2H) \int_0^1 \int_0^1 \varphi_k(t)\varphi_k(s)(t-s)^{2l} dtds \\
&= 2\int_0^1 \int_0^1 \varphi_k(t)\varphi_k(s) \sum_{l=1}^\infty \frac{(2H-2)_{2l}}{(2l)!} \frac{1}{\Gamma(2l+2-2H)} \int_0^\infty \frac{u^{2l+1-2H}}{e^u-1} du (t-s)^{2l} dsdt \\
&= 2\int_0^1 \int_0^1 \varphi_k(t)\varphi_k(s) \int_0^\infty \sum_{l=1}^\infty \frac{(2H-2)_{2l}}{(2l)!\Gamma(2-2H)(2-2H)^{(2l)}} \frac{u^{2l+1-2H}(t-s)^{2l}}{e^u-1} dudsdt \\
&= \frac{1}{\Gamma(2-2H)} 2\int_0^1 \int_0^1 \varphi_k(t)\varphi_k(s) \int_0^\infty \frac{u^{1-2H}}{e^u-1} \sum_{l=1}^\infty \frac{(u(t-s))^{2l}}{(2l)!} dudsdt \\
&= \frac{1}{\Gamma(2-2H)} 2\int_0^1 \int_0^1 \varphi_k(t)\varphi_k(s) \int_0^\infty \frac{u^{1-2H}}{e^u-1} (\cosh(u(t-s))-1) dudsdt,
\end{aligned}$$

where $(z)_k$ and $(z)^{(k)}$ denote the falling and rising factorials respectively: $(z)_n := z(z-1)\dots(z-n+1)$, $(z)^{(n)} := z(z+1)\dots(z+n-1)$. For even k it follows from the definition that $(-z)_k = (z)^{(k)}$.

Recall that

$$\cosh(u(t-s)) - 1 = \frac{e^{u(t-s)} + e^{u(s-t)} - 2}{2} = \frac{e^{u|t-s|} + e^{-u|t-s|} - 2}{2}$$

for any t, s and add up the summands of the variance expression (3.1.3) in order to obtain

$$\begin{aligned}
&\int_0^1 \int_0^1 \varphi_k(t)\varphi_k(s) \frac{1}{\Gamma(2-2H)} \\
&\qquad \times \int_0^\infty u^{1-2H} \left(e^{-u|s-t|} + \frac{2}{e^u-1} (\cosh(u(t-s))-1) \right) dudsdt \\
&= \int_0^1 \int_0^1 \varphi_k(t)\varphi_k(s) \frac{1}{\Gamma(2-2H)} \\
&\qquad \times \int_0^\infty u^{1-2H} \left(e^{-u|s-t|} + \frac{e^{u|t-s|} + e^{-u|t-s|} - 2}{e^u-1} \right) dudsdt \\
&= \int_0^1 \int_0^1 \varphi_k(t)\varphi_k(s) \frac{1}{\Gamma(2-2H)} \int_0^\infty \frac{u^{1-2H}}{e^u-1} (e^{u(1-|t-s|)} + e^{u|t-s|} - 2) dudsdt,
\end{aligned}$$

which is our claim.

Now let us prove that the obtained variance is indeed positive, thus confirming the rate of convergence suggested above. For elements of the real $L^2([0, 1])$-Fourier basis this claim is shown (up to an application of Fubini's Theorem) in Proposition 3.1.14. We also obtain from this proposition that in this particular case the variance (3.1.3) simplifies to

$$\frac{1}{\Gamma(2-2H)} \int_0^\infty \frac{u^{2-2H}}{(2\pi n)^2 + u^2} du$$

for $\varphi_k(x) = \sqrt{2}\sin(2\pi n)$ or $\varphi_k(x) = \sqrt{2}\cos(2\pi n)$.

An arbitrary L^2-function φ_k with zero integral can be written as $\sum_{n \in \mathbb{Z}\setminus\{0\}} c_n f_n$, where f_n are elements of the Fourier basis without the constant component, $c_n \in \mathbb{R}$, and we have for such a decomposition:

$$\begin{aligned}
&\int_0^1 \int_0^1 \varphi_k(t)\varphi_k(s) \frac{1}{\Gamma(2-2H)} \int_0^\infty \frac{u^{1-2H}}{e^u - 1} (e^{u(1-|t-s|)} + e^{u|t-s|} - 2) du ds dt \\
&= \int_0^1 \int_0^1 \sum_{m,\, n \in \mathbb{Z}\setminus\{0\}} c_n f_n(t) c_m f_m(s) \frac{1}{\Gamma(2-2H)} \\
&\quad \times \int_0^\infty \frac{u^{1-2H}}{e^u - 1} (e^{u(1-|t-s|)} + e^{u|t-s|} - 2) du ds dt \\
&= \sum_{m,\, n \in \mathbb{Z}\setminus\{0\}} c_m c_n \frac{1}{\Gamma(2-2H)} \\
&\quad \times \int_0^\infty \frac{u^{1-2H}}{e^u - 1} \int_0^1 \int_0^1 f_m(t) f_n(s) (e^{u(1-|t-s|)} + e^{u|t-s|} - 2) ds dt du \\
&= \sum_{n \in \mathbb{Z}\setminus\{0\}} c_n^2 \frac{1}{\Gamma(2-2H)} \\
&\quad \times \int_0^\infty \frac{u^{1-2H}}{e^u - 1} \int_0^1 \int_0^1 f_n(t) f_n(s) (e^{u(1-|t-s|)} + e^{u|t-s|} - 2) ds dt du,
\end{aligned}$$

since all the off-diagonal terms disappear, as demonstrated in Proposition 3.1.14. We can now use the result for the Fourier basis and complete the calculations:

$$\begin{aligned}
&\int_0^1 \int_0^1 \varphi_k(t)\varphi_k(s) \frac{1}{\Gamma(2-2H)} \int_0^\infty \frac{u^{1-2H}}{e^u - 1} (e^{u(1-|t-s|)} + e^{u|t-s|} - 2) du ds dt \\
&= \frac{1}{\Gamma(2-2H)} \sum_{n \in \mathbb{Z}\setminus\{0\}} c_n^2 \int_0^\infty \frac{u^{2-2H}}{(2\pi n)^2 + u^2} du,
\end{aligned}$$

which is clearly positive if φ_k is nonzero. □

3.1.16 Remark. Following the above calculations we can see that the same simplified form can be achieved for the covariance expression of two error terms each converging with the rate $\sqrt{n}$.

In the context of a different scaling for some of the components a natural question concerning "mixed" covariances arises. This question completes the analysis of the limiting covariance. It is answered in the following proposition.

3.1.17 Proposition. *Let $k_1,\ k_2 \in \{1,\dots,p\}$ be such that $\int_0^1 \varphi_{k_1}(t)dt = 0$ and $\int_0^1 \varphi_{k_2}(t)dt \neq 0$. Then the asymptotic covariance of $\sqrt{n}(\hat{\mu}_{k_1}-\mu_{k_1})$ and $n^{1-H}(\hat{\mu}_{k_2}-\mu_{k_2})$ is equal to zero.*

Proof. As in Proposition 3.1.12, we are left with investigating the term

$$\mathbb{E}\left[\left(n^{-\frac{1}{2}}\int_0^n \varphi_{k_1}(t)dB_t^H\right)\left(n^{-H}\int_0^n \varphi_{k_2}(t)dB_t^H\right)\right].$$

We obtain, again, with $\alpha_H = H(2H-1)$

$$\begin{aligned}
&\frac{1}{\alpha_H}\mathbb{E}\left[\left(n^{-\frac{1}{2}}\int_0^n \varphi_{k_1}(t)dB_t^H\right)\left(n^{-H}\int_0^n \varphi_{k_2}(t)dB_t^H\right)\right]\\
&= n^{-H-\frac{1}{2}}\int_0^1\int_0^1 \varphi_{k_1}(t)\varphi_{k_2}(s)n|t-s|^{2H-2}dtds\\
&\quad + n^{-H-\frac{1}{2}}\int_0^1\int_0^1 \varphi_{k_1}(t)\varphi_{k_2}(s)\sum_{i>j}|t-s+i-j|^{2H-2}dtds\\
&\quad + n^{-H-\frac{1}{2}}\int_0^1\int_0^1 \varphi_{k_1}(t)\varphi_{k_2}(s)\sum_{j>i}|s-t+j-i|^{2H-2}dtds.
\end{aligned}$$

In this case the first summand converges to zero. For the second summand we obtain similarly to Proposition 3.1.12 and using that φ_{k_1} integrates to zero

$$\begin{aligned}
&n^{-H-\frac{1}{2}}\int_0^1\int_0^1 \varphi_{k_1}(t)\varphi_{k_2}(s)\sum_{i>j}|t-s+i-j|^{2H-2}dtds\\
&= \frac{1}{n^{H-\frac{1}{2}}}\int_0^1\int_0^1 \varphi_{k_1}(t)\varphi_{k_2}\sum_{m=1}^{n-1} m^{2H-2}\sum_{l=1}^{\infty}\binom{2H-2}{l}(t-s)^l m^{-l}dtds\\
&\quad - \frac{1}{n^{H+\frac{1}{2}}}\int_0^1\int_0^1 \varphi_{k_1}(t)\varphi_{k_2}\sum_{m=1}^{n-1} m\cdot m^{2H-2}\sum_{l=1}^{\infty}\binom{2H-2}{l}(t-s)^l m^{-l}dtds\\
&= \frac{1}{n^{H-\frac{1}{2}}}\int_0^1\int_0^1 \varphi_{k_1}(t)\varphi_{k_2}\sum_{l=1}^{\infty}\binom{2H-2}{l}(t-s)^l\sum_{m=1}^{n-1} m^{2H-2-l}dtds\\
&\quad - \frac{1}{n^{H+\frac{1}{2}}}\int_0^1\int_0^1 \varphi_{k_1}(t)\varphi_{k_2}\sum_{l=1}^{\infty}\binom{2H-2}{l}(t-s)^l\frac{1}{n}\sum_{m=1}^{n-1} m^{2H-1-l}dtds,
\end{aligned}$$

which converges to zero as n tends to infinity. By an analogous calculation this follows also for the third summand. □

3.2 Remarks on parametric estimation in the ergodic case and for known α

This section contains several remarks that concern the parametric estimator in the ergodic case considered in [24] as well as a study of asymptotic properties of the least squares estimator if α is known.

3.2.1 Rates of convergence: Ergodic case

Consider the estimator $\hat{\vartheta}_n = (\hat{\mu}_1, \ldots, \hat{\mu}_p, \hat{\alpha})$ defined in Section 2.2.2 with Skorokhod integrals for $\alpha > 0$. Similarly to the non-ergodic case, the rate of convergence of $\hat{\vartheta}_n - \vartheta$ is determined by the behaviour of integrals of periodic deterministic functions with respect to B^H. More precisely, we have the following result demonstrated in [24] and [7].

3.2.1 Theorem. *For $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$ and with the decomposition $\hat{\vartheta}_n - \vartheta = \sigma Q_n^{-1} R_n$ we have*

$$n^{-H} R_n = \left(n^{-H} \int_0^n \varphi_1(t)dt, \ldots, n^{-H} \int_0^n \varphi_p(t)dt, \, -n^{-H} \int_0^n X_t dB_t^H\right)^T \xrightarrow{d} N(0, \Sigma),$$

where $\Sigma_{ij} = \int_0^1 \int_0^1 \varphi_i(u)\varphi_j(v)dudv$ with the notation

$$\varphi_{p+1}(t) := \tilde{h}(t) = e^{-\alpha t} \int_{-\infty}^t e^{\alpha s} L(s) ds$$

and

$$nQ_n^{-1} \xrightarrow{a.s.} \begin{pmatrix} E_p + \gamma \Lambda \Lambda^T & \gamma \Lambda \\ \gamma \Lambda^T & \gamma \end{pmatrix} =: C,$$

where

$$\Lambda = (\Lambda_1, \ldots, \Lambda_p)^T = \left(\int_0^1 \varphi_1(t)\tilde{h}(t)dt, \ldots, \int_0^1 \varphi_p(t)\tilde{h}(t)dt\right)^T$$

and

$$\gamma = \left(\int_0^1 \tilde{h}^2(t)dt + \alpha^{-2H} H\Gamma(2H - \sum_{i=1}^p \Lambda_i^2)\right)^{-1}$$

such that

$$n^{1-H}(\hat{\vartheta}_n - \vartheta) \xrightarrow{d} N(0, \sigma^2 C^T \Sigma C).$$

Therefore, if for some $i \in \{1, \ldots, p\}$ the integral $\int_0^1 \varphi_i(s)ds$ is equal to zero, the ith component of the vector $\hat{\vartheta} - \vartheta$ will by construction converge with a faster rate, as demonstrated in Proposition 3.1.12. In the following we will consider the last component of this vector, analyse under which conditions the speed of convergence might change and find a limiting distribution in this case. We start with the following lemma.

3.2.2 Lemma. *For $\tilde{h} = e^{-\alpha t} \int_{-\infty}^{t} e^{\alpha s} L(s) ds$ the integral $\int_0^1 \tilde{h}(t) dt$ is equal to zero if and only if $\int_0^1 L(s) ds = 0$.*

Proof. The proof is a straightforward calculation:

$$\begin{aligned}
\int_0^1 \tilde{h}(t)dt &= \int_0^1 e^{-\alpha t} \int_{-\infty}^{t} e^{\alpha s} L(s) ds dt = \int_{\mathbb{R}} e^{\alpha s} L(s) \int_0^1 e^{-\alpha t} \mathbf{1}_{\{s \leq t\}} dt ds \\
&= \int_{-\infty}^{0} e^{\alpha s} L(s) \int_0^1 e^{-\alpha t} dt ds + \int_0^1 e^{\alpha s} L(s) \int_s^1 e^{-\alpha t} dt ds \\
&= \frac{1}{\alpha} \Big(\int_{-\infty}^{0} e^{\alpha s} L(s) ds - \int_{-\infty}^{1} e^{\alpha(s-1)} L(s) ds + \int_0^1 L(s) ds \Big) \\
&= \frac{1}{\alpha} \int_0^1 L(s) ds.
\end{aligned}$$

□

In the next proposition we will analyse second order asymptotics for the case $\int_0^1 \tilde{h}(t)dt = 0$.

3.2.3 Proposition. *Let L be such that $\int_0^1 L(s)ds = 0$. Moreover, let $q \in \{0, \ldots, p\}$ be such that $\int_0^1 \varphi_i(t)dt \neq 0$ for $i \in \{1, \ldots, q\}$ and $\int_0^1 \varphi(t)dt = 0$ for $i \in \{q+1, \ldots, p\}$. Then the vector*

$$\big(n^{1-H}(\hat{\mu}_1 - \mu_1), \ldots, n^{1-H}(\hat{\mu}_q - \mu_q), \sqrt{n}(\hat{\mu}_{q+1} - \mu_{q+1}), \ldots, \sqrt{n}(\hat{\mu}_p - \mu_p), \sqrt{n}(\hat{\alpha}_{q+1} - \alpha)\big)$$

converges in law to $N(0, C^T \tilde{\Sigma} C)$, where C is defined in Theorem 3.2.1 and $\tilde{\Sigma}$ is a nondegenerate matrix which will be explicitly determined in the proof.

Proof. Since $nQ_n^{-1} \to C$ almost surely, it is enough to investigate the convergence of the appropriately weighted vector R_n, i.e.

$$\begin{aligned}
&\big(n^{-H} \int_0^n \varphi_1(t) dB_t^H, \ldots, n^{-H} \int_0^n \varphi_q(t) dB_t^H, \\
&\quad n^{-1/2} \int_0^n \varphi_{q+1}(t) dB_t^H, \ldots, n^{-1/2} \int_0^n \varphi_p(t) dB_t^H, -n^{-1/2} \int_0^n X_t dB_t^H\big).
\end{aligned}$$

As shown in [24], moreover, replacing $X_t = Z_t + h(t)$ by $Z_t + \tilde{h}_t$ changes nothing for the asymptotics, and in order to analyse the limiting distribution we consider the vector

$$\begin{aligned}
&\big(n^{-H} \int_0^n \varphi_1(t) dB_t^H, \ldots, n^{-H} \int_0^n \varphi_q(t) dB_t^H, n^{-1/2} \int_0^n \varphi_{q+1}(t) dB_t^H, \\
&\quad \ldots, n^{-1/2} \int_0^n \varphi_p(t) dB_t^H, -n^{-1/2} \int_0^n Z_t dB_t^H, -n^{-1/2} \int_0^n \tilde{h}(t) dB_t^H\big).
\end{aligned}$$

We know from Section 3.1.4 that in the above vector all the integrals with deterministic integrands converge to normally distributed random variables with positive variances. We also know the limits of covariances between them as well as the covariances between $-n^{-1/2}\int_0^n Z_t dB_t^H$ and other components (those are equal to zero by properties of the product of the different order Wiener integrals, see Section 2.1.2). Now note that it is shown in [30] that $-n^{-1/2}\int_0^n Z_t dB_t^H$ converges to $N(0, \sigma^2\alpha^{1-4H}\delta_H)$ in distribution, where

$$\delta_H = H^2(4H-1)\left(\Gamma(2H)^2 + \frac{\Gamma(2H)\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2-2H)}\right).$$

This allows us to use a multivariate version of the fourth moment theorem (see [59] and [58] for proofs), which states that for a sequence of vectors of multiple Wiener–Itō integrals componentwise convergence to Gaussian always implies joint convergence (as formulated in [51]). We therefore conclude that the vector converges jointly to a multivariate centred normal distribution whose covariance matrix contains the limiting covariances of the components. It follows that also the vector

$$\Big(n^{-H}\int_0^n \varphi_1(t)dB_t^H, \dots, n^{-H}\int_0^n \varphi_q(t)dB_t^H,$$
$$n^{-1/2}\int_0^n \varphi_{q+1}(t)dB_t^H, \dots, n^{-1/2}\int_0^n \varphi_p(t)dB_t^H,\ -n^{-1/2}\int_0^n X_t dB_t^H\Big)$$

converges to a centred normal random vector. The variance of the last component is the sum of the limiting variances of $-n^{-1/2}\int_0^n Z_t dB_t^H$ and $-n^{-1/2}\int_0^n \tilde{h}(t)dB_t^H$ and thus positive. □

Note that for $L \equiv 0$ we retrieve exactly the same rate of convergence and the same limiting distribution for α as in [30].

3.2.4 Remark. 1. As we have seen, the object determining the convergence of the estimators $\hat{\mu}_i$ is the integral $\int_0^n \varphi_i(t)dB_t^H$. In the ergodic case its behaviour is the same as that of $\int_0^n \varphi_i(t)dX_t$, which can be seen from the relationship

$$\int_0^n \varphi_i(t)dX_t = n\mu_i - \alpha\int_0^n \varphi_i(t)X_t dt + \int_0^n \varphi_i(t)dB_t^H$$

because the two remaining summands converge faster (see [24] for proofs). In other words, this term also converges faster if $\int_0^1 \varphi_i(t)dt = 0$. The difference between the two kinds of asymptotics of the integral $\int_0^n \varphi_i(t)dB_t^H$ can also be perceived on another level: The covariances of the increments of the process $\left(\int_0^n \varphi_i(t)dB_t^H\right)_{n\in\mathbb{N}}$ become summable if φ_i integrates to zero, transporting the setting from long to short range dependence. With this background in mind the different rate emerging in this special case does indeed seem less surprising (see the overview [70] for details on this connection).

2. Although the latter analysis shows a rate improvement in many cases, the least squares estimator is not optimal in general: In [56] an estimator in the case $p = 1$, $\varphi_1 \equiv 1$ is constructed, whose α-component converges with the speed $\sqrt{n}$. A simple intuition behind this might be that more information can be used in this special case. For the least squares type estimator the observations only on the intervals $[0, n]$, $n \in \mathbb{N}$, are considered in order to make use of ergodic type properties emerging due to periodicity (see [24] for details). For the constant function this restriction need not appear.

3.2.2 LSE for known α

In this section we will (merely for completeness reasons) briefly discuss the behaviour of the least squares type estimator in case where α is known.

In this case the SDE

$$X_t = X_0 + \int_0^t (L(s) - \alpha X_s)\, ds + B_t^H, \quad X_0 = x_0 \in \mathbb{R},$$

can be written as

$$X_t = \int_0^t L(s) ds + B_t^H$$

without loss of generality, assuming that the process X is observed continuously: The integral $\int_0^t \alpha X_s ds$ is known and can be brought to the other side. If in this setting we follow the construction used in [23] and [24] for the least squares estimator, we obtain the vector

$$\frac{1}{n}\left(\int_0^n \varphi_1(t) dX_t, \ldots, \int_0^n \varphi_p(t) dX_t\right)$$

as an estimator for $(\mu_1, \ldots, \mu_p)$. Since

$$\frac{1}{n}\int_0^n \varphi_i(t) dX_t = \mu_i + \frac{1}{n}\int_0^n \varphi_i(t) dB_t^H$$

for $i = 1, \ldots, p$, it follows that this estimator is strongly consistent (recall that $\frac{1}{n}\int_0^n \varphi_i(t) dB_t^H$ is Gaussian and converges to zero in L^2 with polynomial speed). From Section 3.1.4 we can also establish the asymptotic behaviour of the second order: The components are jointly asymptotically normal with the rate of convergence n^{1-H} for the components where $\int_0^1 \varphi_i(t) dt \neq 0$ and $\sqrt{n}$ for others.

Chapter 4

Nonparametric estimator for the fractional Ornstein-Uhlenbeck type processes (ergodic case)

The structure of the problem posed in [24] allows a generalisation possibility for the results obtained in this paper. Namely, one can construct a nonparametric estimator for the drift function L in the Ornstein-Uhlenbeck SDE by approximating it with finite linear combinations of known functions. The idea is similar to the construction of projection density estimators, see for example [17]. This chapter deals with this generalisation.

4.1 Setting

As before, we start with the SDE

$$\begin{aligned} X_t &= X_0 + \int_0^t L(s) - \alpha X_s ds + \sigma B_t^H, \quad t \geq 0 \\ X_0 &= x_0 \in \mathbb{R} \end{aligned} \tag{4.1.1}$$

with an fBm B^H, $H > \frac{1}{2}$, but now we assume L to be an unknown bounded 1-periodic function. Moreover, this time we consider the ergodic case and the factor $\alpha > 0$ is also assumed to be unknown (for convenience of this notation we have changed the sign in front of α in the equation in this chapter). We assume to observe X continuously on the whole positive real line. Therefore, by considerations in Section 2.2.1, the parameter $\sigma \neq 0$ can be assumed to be known and equal to one without loss of generality.

Recall that the unique almost surely continuous solution of the SDE (4.1.1), also known (for a zero mean) as the fractional Ornstein-Uhlenbeck process, has the following form:

$$X_t = e^{-\alpha t} x_0 + e^{-\alpha t} \int_0^t L(s) e^{\alpha s} ds + e^{-\alpha t} \int_0^t e^{\alpha s} dB_s^H.$$

Let us assume $x_0 = 0$ for simplicity of calculations. This assumption is not restrictive since the solution for $x_0 \neq 0$ approaches X_t for $x_0 = 0$ with exponential speed. As we will see later, this is enough for the statements to remain valid for all $x_0 \in \mathbb{R}$.

The integrals with respect to B^H in this chapter are understood as Skorokhod integrals on the space $\mathscr{H}^H$ corresponding to the driving fBm as described in Section 2.1.2. To avoid notation overload, let us simply write $\mathscr{H}$ for $\mathscr{H}^H$ in this chapter. For an integral $\int_0^t e^{\alpha s} \delta B_s^H$ we will sometimes write $\delta(e^{\alpha \cdot} \mathbf{1}_{\{\cdot \leq t\}})$, by which we mean the divergence integral from 0 to infinity as defined in Section 2.1.2. By Remark 2.1.15 it coincides with the divergence integral on $[0, t]$.

For the sake of clarity we will henceforth denote the solution of the SDE (4.1.1) by X^L, where L is the underlying mean function of the SDE.

As mentioned above, the main idea needed for the construction is similar to the construction of the projection density estimators. We pick an orthonormal basis $(\varphi_i)_{i \in \mathbb{N}}$ of $L^2([0, 1])$ and write an unknown function L as $L \equiv \sum_{i=1}^{\infty} \langle L, \varphi_i \rangle_{L^2([0,1])} \varphi_i$. Since an estimator for the coefficients of finite linear combinations has already been constructed in [24], we can use it to build an estimator of L.

Let us introduce a construction which will serve as a foundation and a starting point in the definition of our estimators.

4.1.1 Definition. For the SDE (4.1.1) with the mean function L, a given $L^2([0, 1])$-orthonormal basis (ONB in the sequel) $(\varphi_i)_{i \in \mathbb{N}}$ and some $p \in \mathbb{N}$ we define the p-cutoff estimator as follows:

$$\hat{\vartheta}_n^{L,p} := (\hat{\mu}_n^{Lp1}, \ldots, \hat{\mu}_n^{Lpp}, \alpha_n^{Lp}) := Q_n^{-1} P_n,$$

where

$$P_n := \left(\int_0^n \varphi_1(t) \delta X_t^L, \ldots, \int_0^n \varphi_p(t) \delta X_t^L, - \int_0^n X_t^L \delta X_t^L \right)^T$$

and

$$Q_n := \begin{pmatrix} nE_p & -a_n \\ -a_n^T & b_n \end{pmatrix}$$

with

$$a_n^T := \left(\int_0^n \varphi_1(t) X_t^L dt \ldots, \int_0^n \varphi_p(t) X_t^L dt \right)$$

and

$$b_n := \int_0^n (X_t^L)^2 dt.$$

The integrals in this definition are Skorokhod integrals in the sense of the definition in Section 2.1.3. Note that since X^L consists of a continuous deterministic function and an integral with respect to B^H over such a function, it will be integrable with respect to B^H.

4.1.2 Remark. If we can write L as $\sum_{i=1}^p \mu_i\varphi_i$ for some unknown μ_i's, then this is exactly the estimator given in [24]. It is proved there that the p-cutoff estimator is a weakly consistent and asymptotically normal estimator of $\vartheta = (\mu_1, \dots, \mu_p,\, \alpha)$ in this case.

Before we present other constructions let us cite (and adapt to our definition) some more results that are shown in [24].

4.1.3 Remark. Note that we have

$$Q_n^{-1} = \frac{1}{n}\begin{pmatrix} E_p + \gamma_n\Lambda_n\Lambda_n^T & -\gamma_n\Lambda_n \\ -\gamma_n\Lambda_n^T & \gamma_n \end{pmatrix}$$

with

$$\Lambda_n (= \Lambda_n^{Lp}) = (\Lambda_{n,1}, \dots, \Lambda_{n,p})^T = \left(\frac{1}{n}\int_0^n \varphi_1(t)X_t^L dt, \dots, \frac{1}{n}\int_0^n \varphi_p(t)X_t^L dt\right)$$

and $\gamma_n = \left(\frac{1}{n}\int_0^n (X_t^L)^2 dt - \sum_{i=1}^p \Lambda_{n,i}^2\right)^{-1}$. We will denote γ_n^{-1} by D_n^{Lp}.

Now let us cite a useful representation result for $\hat\vartheta_n^{L,p}$ which is proved in the same paper:

4.1.4 Proposition. *If L is of the form $L(s) = \sum_{i=1}^p \mu_i\varphi_i(s)$ for some real $\mu_1, \dots, \mu_p$, then the p-cutoff estimator has the following representation:*

$$\hat\vartheta_n^{L,p} = \vartheta + Q_n^{-1}R_n, \qquad \textit{where}$$

$$R_n = \left(\int_0^n \varphi_1(t)\delta B_t^H, \dots, \int_0^n \varphi_p(t)\delta B_t^H, -\int_0^n X_t^L \delta B_t^H\right).$$

Let us also state and prove a comparable representation for the general case:

4.1.5 Proposition. *For a given 1-periodic bounded function L and some $p \in \mathbb{N}$ the p-cutoff estimator can be written as follows:*

$$\hat\vartheta_n^{L,p} = \vartheta + Q_n^{-1}R_n + Q_n^{-1}N_n$$

with Q_n and R_n as above and

$$N_n = \left(0, \dots, 0, -\sum_{i=p+1}^\infty \mu_i \int_0^n \varphi_i(t)X_t^L dt\right)^T.$$

Proof. Exactly as in the special case, we replace δX_t^L in each component of P_n using the representation in the SDE (4.1.1) and obtain

$$\begin{aligned}\int_0^n \varphi_i(t)\delta X_t^L &= \int_0^n \varphi_i(t)L(t)dt - \alpha\int_0^n \varphi_i(t)X_t^L dt + \int_0^n \varphi_i(t)\delta B_t^H \\ &= \sum_{j=1}^\infty \mu_j \int_0^n \varphi_i(t)\varphi_j(t)dt - \alpha\int_0^n \varphi_i(t)X_t^L dt + \int_0^n \varphi_i(t)\delta B_t^H \\ &= \mu_i n - \alpha\int_0^n \varphi_i(t)X_t^L dt + \int_0^n \varphi_i(t)\delta B_t^H\end{aligned}$$

for all $i \in \{1, \ldots, p\}$ as well as

$$\begin{aligned}&-\int_0^n X_t^L \delta X_t^L = -\int_0^n X_t^L L(t)dt + \alpha\int_0^n (X_t^L)^2 dt - \int_0^n X_t^L \delta B_t^H \\ &= -\sum_{i=1}^p \mu_i \int_0^n \varphi_i(t)X_t^L dt + \alpha\int_0^n (X_t^L)^2 dt - \int_0^n X_t^L\delta B_t^H - \sum_{i=p+1}^\infty \mu_i \int_0^n \varphi_i(t)X_t^L dt.\end{aligned}$$

Since, moreover,

$$Q_n\vartheta = \begin{pmatrix} \mu_1 n - \alpha\int_0^n \varphi_1(t)X_t^L dt \\ \vdots \\ \mu_p n - \alpha\int_0^n \varphi_p(t)X_t^L dt \\ -\sum_{i=1}^p \mu_i \int_0^n \varphi_i(t)X_t^L dt + \alpha\int_0^n (X_t^L)^2 dt \end{pmatrix},$$

we can write

$$P_n = Q_n\vartheta + R_n + N_n$$

and obtain the result. □

Now let us look at a convergence result from [24] which we state here already in a slightly generalised form.

4.1.6 Proposition. *For a given 1-periodic bounded function L and some $p \in \mathbb{N}$ the following convergence statements hold for n going to infinity:*

$$\Lambda_{ni}^{Lp} \overset{a.s.}{\to} \int_0^1 \varphi_i(t)\tilde{h}(t)dt =: \Lambda_i^{Lp} \textit{ for all } i \in \{1, \ldots, p\},$$

$$D_n^{Lp} \overset{a.s.}{\to} \int_0^1 \tilde{h}^2(t)dt + \alpha^{-2H}H\Gamma(2H) - \sum_{i=1}^p (\Lambda_i^{Lp})^2 =: D^{Lp},$$

where $\tilde{h}(t) = e^{-\alpha t}\int_{-\infty}^t e^{\alpha s}L(s)ds$.

The proof is identical to the one given in [24].

Note that we can find a positive lower bound for the limits of D_n^{Lp} that is independent of p by observing that every limit is greater or equal than $\psi := \alpha^{-2H} H\Gamma(2H)$. However, it is not clear that the bound will be retained if n and p simultaneously tend to infinity. We will show this for a certain choice of p at the beginning of the next chapter.

Let us now pick a positive real number k that is below ψ. For this an additional assumption on α is needed: We assume that α is bounded from above by a known constant. This enables us to define a truncated version of our estimator:

4.1.7 Definition. For a given 1-periodic bounded function L and some $p \in \mathbb{N}$ we define the truncated p-cutoff estimator as

$$\hat{\vartheta}^{(T)} := \hat{\vartheta}_n^{(T),\,L,\,p} := \hat{\vartheta}_n^{L,\,p} \mathbf{1}_{\{D_n^{Lp} \geq k\}}.$$

4.2 Auxiliary results

The aim of this section is to prove some convergence and boundedness results for the objects introduced above that will be of use in later proofs.

Before we start, let us fix a notation. We will denote

$$h(t) := e^{-\alpha t} \int_0^t e^{\alpha s} L(s) ds$$

such that we can write

$$X_t^L = h(t) + e^{-\alpha t} \delta(e^{\alpha \cdot} \mathbf{1}_{\{\cdot \leq t\}}).$$

Note that the function $[t \mapsto h(t)]$ is bounded. Moreover, we write $\tilde{\Lambda}_{ni}^{Lp}$ for $\frac{1}{n} \int_0^n \tilde{X}_t^L \varphi_i(t) dt$, where

$$\tilde{X}_t^L = e^{-\alpha t} \int_{-\infty}^t L(s) e^{\alpha s} ds + e^{-\alpha t} \int_{-\infty}^t e^{\alpha s} dB_s^H = \tilde{h}(t) + \tilde{Z}_t$$

with

$$\tilde{Z}_t = e^{-\alpha t} \int_{-\infty}^t e^{\alpha s} dB_s^H.$$

4.2.1 Almost sure convergence of D_n^{Lp}

First of all, we will show almost sure convergence of D_n^{Lp} defined in Remark 4.1.3, as anticipated in the previous chapter. We will accomplish this in several steps.

4.2.1 Proposition. *For all $n \in \mathbb{N}$ we have $\mathbb{E}[(\tilde{\Lambda}_{ni}^{Lp} - \int_0^1 \tilde{h}(t) \varphi_i(t) dt)^2] \leq C n^{2H-2}$ with C independent of i.*

Proof. Firstly, we remark that the deterministic part of the integral is cancelled out, because

$$\tilde{\Lambda}^{Lp}_{ni} = \frac{1}{n}\int_0^n \tilde{h}(t)\varphi_i(t)dt + \frac{1}{n}\int_0^n \tilde{Z}_t\varphi_i(t)dt$$

and $\tilde{h}$ is 1-periodic, as stated in [24], hence, $\frac{1}{n}\int_0^n \tilde{h}(t)\varphi_i(t)dt = \int_0^1 \tilde{h}(t)\varphi_i(t)dt$. Therefore, we are left with evaluating

$$\mathbb{E}[(\frac{1}{n}\int_0^n \tilde{Z}_t\varphi_i(t)dt)^2] = \frac{1}{n^2}\int_0^n\int_0^n \varphi_i(t)\varphi_i(s)\,\mathbb{E}[\tilde{Z}_t\tilde{Z}_s]dtds,$$

which means considering $\mathbb{E}[\tilde{Z}_t\tilde{Z}_s]$. We rewrite it using the isometry property (Proposition 2.1.3):

$$\begin{aligned}
\mathbb{E}[\tilde{Z}_t\tilde{Z}_s] &= \alpha_H\int_{-\infty}^t\int_{-\infty}^s e^{-\alpha t+\alpha u}e^{-\alpha s+\alpha v}|u-v|^{2H-2}dudv\\
&=\alpha_H e^{-\alpha t}e^{-\alpha s}\big(\underbrace{\int_{-\infty}^0\int_{-\infty}^0 e^{\alpha u}e^{\alpha v}|u-v|^{2H-2}dudv}_{=:S_1} + \underbrace{\int_0^t\int_{-\infty}^0 e^{\alpha u}e^{\alpha v}|u-v|^{2H-2}dudv}_{=:S_{2.1}}\\
&\quad+\underbrace{\int_{-\infty}^0\int_0^s e^{\alpha u}e^{\alpha v}|u-v|^{2H-2}dudv}_{=:S_{2.2}} + \underbrace{\int_0^t\int_0^s e^{\alpha u}e^{\alpha v}|u-v|^{2H-2}dudv}_{=:S_3}\big).
\end{aligned}$$

We know due to [30] that S_1 is finite. Hence, we have

$$|\frac{1}{n^2}\int_0^n\int_0^n \varphi_i(t)\varphi_i(s)\alpha_H e^{-\alpha t}e^{-\alpha s}S_1dtds| \lesssim \frac{1}{n^2}\int_0^n\int_0^n e^{-\alpha t}e^{-\alpha s}dtds \lesssim n^{-2}.$$

For $S_{2.1}$ we observe that, by change of variables,

$$\begin{aligned}
S_{2.1} =&\alpha_H\int_0^t\int_v^\infty e^{-\alpha z}e^{2\alpha v}z^{2H-2}dzdv\\
=&\alpha_H\left(-\frac{1}{2\alpha}\int_0^\infty e^{-\alpha z}z^{2H-2}dz + \frac{1}{2\alpha}\left(\int_0^t e^{\alpha z}z^{2H-2}dz + \int_t^\infty e^{-\alpha z}e^{2\alpha t}z^{2H-2}dz\right)\right).
\end{aligned}$$

The first integral is finite, and for the second one we get

$$\begin{aligned}
e^{-\alpha t}e^{-\alpha s}\int_0^t e^{\alpha z}z^{2H-2}dz &= e^{-\alpha s}\int_0^t e^{-\alpha(t-z)}z^{2H-2}dz\\
=&e^{-\alpha s}\int_0^t e^{-\alpha z}(t-z)^{2H-2}dz = e^{-\alpha s}t^{2H-1}\int_0^1 e^{-\alpha tu}(1-u)^{2H-2}du,
\end{aligned}$$

which converges to zero as t goes to ∞ (and s is considered to be fixed), as shown in [9]. Therefore, the whole term is bounded by a constant times $e^{-\alpha s}$. For the last integral in $S_{2.1}$ we obtain

$$\begin{aligned} e^{-\alpha t}e^{-\alpha s}\int_t^\infty e^{-\alpha z}e^{2\alpha t}z^{2H-2}dz &= e^{-\alpha s}\int_t^\infty e^{-\alpha(z-t)}z^{2H-2}dz \\ \leq & e^{-\alpha s}\int_t^\infty e^{-\alpha(z-t)}(z-t)^{2H-2}dz = e^{-\alpha s}\int_0^\infty e^{-\alpha(z)}z^{2H-2}dz \lesssim e^{-\alpha s}. \end{aligned}$$

Therefore, in total $e^{-\alpha t}e^{-\alpha s}S_{2.1}$ is bounded by $e^{-\alpha s}$, and by symmetry $e^{-\alpha t}e^{-\alpha s}S_{2.2}$ is bounded by $e^{-\alpha t}$. We obtain thus

$$|\frac{1}{n^2}\int_0^n\int_0^n \varphi_i(t)\varphi_i(s)\alpha_H e^{-\alpha t}e^{-\alpha s}S_{2.2}dtds| \lesssim \frac{1}{n^2}\int_0^n\int_0^n e^{-\alpha s}dtds \lesssim n^{-1},$$

and the same result holds for $S_{2.2}$ by symmetry.

The methods used for S_3 are similar. For symmetry reasons we have $S_3\mathbf{1}_{\{t\geq s\}} = S_3\mathbf{1}_{\{s\geq t\}}$, and therefore,

$$S_3 = 2\int_0^s e^{\alpha u}\left(\int_0^u e^{\alpha v}|u-v|^{2H-2}dv + \int_u^t e^{\alpha v}|u-v|^{2H-2}dv\right)du.$$

The first of the two summands (let us denote it by $S_{3.1}$) is shown in the proof of Lemma 6 in [9] to be at most of the order $e^{2\alpha s}$, and therefore we get

$$\left|\frac{1}{n^2}\int_0^n\int_0^n \varphi_i(t)\varphi_i(s)\alpha_H e^{-\alpha t}e^{-\alpha s}S_{3.1}dtds\right| \lesssim n^{-1}$$

with the same calculation as for $S_{2.1}$. For the second summand, $S_{3.2}$, we first obtain by change of variables

$$\int_0^s e^{\alpha u}\int_u^t e^{\alpha v}|u-v|^{2H-2}dvdu = \int_0^s e^{2\alpha u}\int_0^{t-u} e^{\alpha z}z^{2H-2}dzdu.$$

From the proof of Lemma 6 in [9] we know that

$$e^{-\alpha t}\int_0^t e^{\alpha u}u^{2H-2}du = t^{2H-1}\int_0^1 e^{-\alpha tu}(1-u)^{2H-2}du$$

with the integral being a multiple of the confluent hypergeometric function ${}_1F_1$ with parameters 1, $2H$ and $-\alpha t$:

$$\int_0^1 e^{-\alpha tu}(1-u)^{2H-2}du = \frac{\Gamma(1)\Gamma(2H-1)}{\Gamma(2H)}{}_1F_1(1,\,2H,\,-\alpha t),$$

where ${}_1F_1$ is defined as

$${}_1F_1(a,\,b,\,z) = \sum_{n=0}^\infty \frac{(a)_n}{(b)_n}\frac{z^n}{n!}$$

(see [1], 13.2.1). The asymptotics of this function for large t is known (again, from [1], 13.1.5), with t^{-1} being an upper bound, and allows us the following estimation:

$$e^{-\alpha t}\int_0^t e^{\alpha u}u^{2H-2}du \lesssim t^{2H-1}t^{-1} = t^{2H-2}.$$

Therefore, we can continue on our calculations for the second summand as follows:

$$\begin{aligned}
&\int_0^s e^{2\alpha u}\int_0^{t-u} e^{\alpha z}z^{2H-2}dzdu = \int_0^s e^{2\alpha u}e^{\alpha(t-u)}e^{-\alpha(t-u)}\int_0^{t-u} e^{\alpha z}z^{2H-2}dzdu\\
&\lesssim \int_0^s e^{\alpha u}e^{\alpha t}(t-u)^{2H-2}du = e^{\alpha t}\int_{t-s}^t e^{\alpha(t-z)}z^{2H-2}dz\\
&\lesssim e^{2\alpha t}\int_{\alpha(t-s)}^{\alpha t} e^{-z}z^{2H-2}dz = e^{2\alpha t}(\Gamma(2H-1,\,\alpha(t-s)) - \Gamma(2H-1,\,\alpha t))\\
&\quad =: S_{3.2.1} + S_{3.2.2},
\end{aligned}$$

where $\Gamma(s,\,x)$ denotes the incomplete Gamma function defined via $\Gamma(s,\,x) := \int_x^\infty e^{-z}z^{s-1}dz$. Note that $\Gamma(s,\,x)e^x$ is asymptotically (for large x) of the order x^{s-1}, which allows for more practical estimates. We obtain thus

$$\begin{aligned}
&\left|\frac{1}{n^2}\int_0^n\int_0^n \varphi_i(t)\varphi_i(s)\alpha_H e^{-\alpha t}e^{-\alpha s}S_{3.2.1}dtds\right| \lesssim \frac{1}{n^2}\int_0^n\int_s^n e^{\alpha(t-s)}\Gamma(2H-1,\,\alpha(t-s))dtds\\
&\quad\lesssim \frac{1}{n^2}\int_0^n\int_s^n (t-s)^{2H-2}dtds = \frac{1}{n^2}\int_0^n\int_0^{n-s} z^{2H-2}dzds \sim n^{2H-2}
\end{aligned}$$

for the first and

$$\begin{aligned}
&\left|\frac{1}{n^2}\int_0^n\int_0^n \varphi_i(t)\varphi_i(s)\alpha_H e^{-\alpha t}e^{-\alpha s}S_{3.2.2}dtds\right| \lesssim \frac{1}{n^2}\int_0^n e^{-\alpha s}\int_s^n e^{\alpha t}\Gamma(2H-1,\,\alpha t)dtds\\
&\quad\lesssim \frac{1}{n^2}\int_0^n e^{-\alpha s}\int_s^n t^{2H-2}dtds = n^{2H-3}\int_0^n e^{-\alpha s}ds - \frac{1}{n^2}\int_0^n e^{-\alpha s}s^{2H-1}ds \lesssim n^{2H-3}
\end{aligned}$$

for the second summand. This completes the proof. □

Now we can show that $\sum_{i=1}^{p(n)}(\tilde{\Lambda}_{ni}^{Lp})^2 \to \sum_{i=1}^{\infty}(\int_0^1 \tilde{h}(t)\varphi_i(t)dt)^2$ almost surely as n tends to infinity.

4.2.2 Proposition. *For $p(n) = n^\vartheta$, $\vartheta < 1 - H$, we have*

$$\sum_{i=1}^{p(n)}\left(\tilde{\Lambda}_{ni}^{Lp}\right)^2 - \left(\int_0^1 \tilde{h}(t)\varphi_i(t)dt\right)^2 \overset{n\to\infty}{\to} 0$$

almost surely.

Proof. Since

$$\tilde{\Lambda}_{ni}^{Lp} = \int_0^1 \tilde{h}(t)\varphi_i(t)dt + \frac{1}{n}\int_0^n \tilde{Z}_t\varphi_i(t)dt,$$

it is enough to show that $\sum_{i=1}^{p(n)} \left(\frac{1}{n}\int_0^n \tilde{Z}_t\varphi_i(t)dt\right)^2$ and

$$\sum_{i=1}^{p(n)} \frac{1}{n}\int_0^n \tilde{Z}_t\varphi_i(t)dt \int_0^1 \tilde{h}(t)\varphi_i(t)dt$$

tend to zero almost surely. Both sequences are elements of a finite sum of Wiener chaoses, therefore, the hypercontractivity property (2.1.6) holds for them and it suffices to show that any of their moments are bounded by n to some negative power (up to a constant). Then the proof will follow by a Borel-Cantelli type argument similarly to Lemma 3.1.3 from the previous chapter. Note that we have established above that $\mathbb{E}\left[\left(\frac{1}{n}\int_0^n \tilde{Z}_t\varphi_i(t)dt\right)^2\right]$ can be bounded by $\sup \|\varphi_i\|_\infty n^{2H-2}$. For the L^2 norm of the first summand we calculate

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=1}^{p(n)} \left(\frac{1}{n}\int_0^n \tilde{Z}_t\varphi_i(t)dt\right)^2\right)^2\right] \\
&\lesssim \sum_{i,\,j=1}^{p(n)} \sqrt{\mathbb{E}\left[\left(\frac{1}{n}\int_0^n \tilde{Z}_t\varphi_i(t)dt\right)^4\right]}\sqrt{\mathbb{E}\left[\left(\frac{1}{n}\int_0^n \tilde{Z}_t\varphi_j(t)dt\right)^4\right]} \\
&\lesssim \sum_{i,\,j=1}^{p(n)} \mathbb{E}\left[\left(\frac{1}{n}\int_0^n \tilde{Z}_t\varphi_i(t)dt\right)^2\right]\mathbb{E}\left[\left(\frac{1}{n}\int_0^n \tilde{Z}_t\varphi_j(t)dt\right)^2\right] \\
&\lesssim p(n)^2 n^{4H-4}
\end{aligned}$$

by Gaussianity of $\frac{1}{n}\int_0^n \tilde{Z}_t\varphi_i(t)dt$. For the L^2 norm of the second summand we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=1}^{p(n)} \frac{1}{n}\int_0^n \tilde{Z}_t\varphi_i(t)dt \int_0^1 \tilde{h}(t)\varphi_i(t)dt\right)^2\right] \\
&\leq \sup \|\varphi_i\|_\infty^2 \|\tilde{h}\|_{L^1([0,\,1])}^2 \sum_{i,\,j=1}^{p(n)} \mathbb{E}\left[\left|\frac{1}{n}\int_0^n \tilde{Z}_t\varphi_i(t)dt\right|\left|\frac{1}{n}\int_0^n \tilde{Z}_t\varphi_j(t)dt\right|\right] \\
&\lesssim \sum_{i,\,j=1}^{p(n)} \sqrt{\mathbb{E}\left[\left(\frac{1}{n}\int_0^n \tilde{Z}_t\varphi_i(t)dt\right)^2\right]}\sqrt{\mathbb{E}\left[\left(\frac{1}{n}\int_0^n \tilde{Z}_t\varphi_j(t)dt\right)^2\right]} \\
&\lesssim p(n)^2 n^{2H-2}.
\end{aligned}$$

Hence, for $p(n) = n^\vartheta$ both summands converge almost surely, and the proof is complete. $\square$

Now we are prepared for the final result in this section.

4.2.3 Proposition. *For $p(n) = n^\beta$, $\beta < 1 - H$, we have $D_n^{Lp} \to \psi$ (with $\psi = \alpha^{-2H} H\Gamma(2H)$) almost surely as n tends to infinity.*

Proof. We know from [24] that $\frac{1}{n}\int_0^n X_t^2 dt$ converges to $\psi + \int_0^1 \tilde{h}^2(t)dt$, so with the above proposition it remains to show that $\sum_{i=1}^{p(n)} (\tilde{\Lambda}_{ni}^{Lp})^2 - (\Lambda_{ni}^{Lp})^2 \to 0$ almost surely. We use the facts from [24] that $\limsup_{n\to\infty} |\frac{1}{n}\int_0^n X_t dt|$ and $\limsup_{n\to\infty} |\frac{1}{n}\int_0^n \tilde{X}_t dt|$ are finite and that $|X_t - \tilde{X}_t|$ is almost surely bounded by $e^{-\alpha t} Z$, where Z is a random variable independent of t and calculate:

$$\begin{aligned}
&\sum_{i=1}^{p(n)} (\tilde{\Lambda}_{ni}^{Lp})^2 - (\Lambda_{ni}^{Lp})^2 = \sum_{i=1}^{p(n)} (\tilde{\Lambda}_{ni}^{Lp} - \Lambda_{ni}^{Lp})(\tilde{\Lambda}_{ni}^{Lp} + \Lambda_{ni}^{Lp}) \\
&\quad = \sum_{i=1}^{p(n)} \left(\frac{1}{n}\int_0^n (\tilde{X}_t - X_t)\varphi_i(t)dt \right) \left(\frac{1}{n}\int_0^n (\tilde{X}_t + X_t)\varphi_i(t)dt \right) \\
&\quad \lesssim \sum_{i=1}^{p(n)} \frac{1}{n}\int_0^n e^{-\alpha t} Z dt \frac{1}{n}\int_0^n |\tilde{X}_t + X_t| dt \lesssim Z p(n) \frac{1}{n}\int_0^n e^{-\alpha t} dt,
\end{aligned}$$

which goes to zero pointwise for the above choice of $p(n)$. $\square$

This result extends Proposition 4.1.6 to the case where p is not fixed but tends to infinity in a particular manner. Note that this result is not true for every choice of p: If we fix n and let p tend to infinity, then by Parseval's identity D_n^{Lp} will converge to zero almost surely, thus there can be no unique joint limit. For the above choice of p, however, the limit is pointwise and allows us to bound the denominator in the definition of the estimator (Definition 4.1.7) by a constant and thus eliminate it in further calculations.

4.2.2 Other results

We have already seen that $(\Lambda_{ni}^{Lp})^2$ converge almost surely for n going to infinity, now let us demonstrate that this statement also holds in L^2.

4.2.4 Proposition. *The sequences $(\Lambda_{ni}^{Lp})^2$ converge in L^2 to $(\Lambda_n^{Lp})^2$ for $i = 1, \ldots, p$ as n tends to infinity.*

Proof. We have $e^{\alpha \cdot}\mathbf{1}_{\{\cdot \le t\}} \in |\mathscr{H}|$, and consequently $\delta(e^{\alpha \cdot}\mathbf{1}_{\{\cdot \le t\}})$ and also X_t^L itself are Gaussian. Moreover, it has almost surely continuous paths. Therefore, also $\varphi_i(t)X_t^L$ is an almost surely continuous Gaussian stochastic process. We conclude that also its pathwise Lebesgue integral, $\int_0^n \varphi_i(t)X_t^L dt$, is a Gaussian random variable. Moreover, $\delta(e^{\alpha\cdot}\mathbf{1}_{\{\cdot\leq t\}})$ is centred, hence, also $\frac{1}{n}\int_0^n \varphi_i(t)\delta(e^{\alpha\cdot}\mathbf{1}_{\{\cdot\leq t\}})dt$ is centred. Therefore, we can write for every fixed i:

$$\Lambda_{ni}^{Lp} = Det_n + CG_n,$$

where Det_n is deterministic and CG_n is centred Gaussian.
We know due to the almost sure convergence result (and also due to Gaussianity) that Det_n has a (finite) limit. In order to show that $(\Lambda_{ni}^{Lp})^2$ converges in L^2 to $\lim_{n\to\infty} Det_n^2$, it is enough to verify that

$$4Det_n^2\,\mathbb{E}[CG_n^2] + \mathbb{E}[CG_n^4] \to 0,$$

since everything else either is cancelled out or contains an odd moment of CG_n as a factor. Due to Gaussianity we know that $\mathbb{E}[CG_n^4] = 3\,\mathbb{E}[CG_n^2]^2$, and therefore, it now suffices to prove that $\mathbb{E}[CG_n^2] \to 0$ holds for $n \to \infty$. We calculate:

$$\begin{aligned}
&\mathbb{E}[(\frac{1}{n}\int_0^n \varphi_i(t)\delta(e^{\alpha\cdot}\mathbf{1}_{\{\cdot\leq t\}}))^2] \\
=&\frac{1}{n^2}\int_0^n\int_0^n \varphi_i(t)\varphi_i(s)e^{-\alpha t}e^{-\alpha s}\,\mathbb{E}[\delta(e^{\alpha\cdot}\mathbf{1}_{\{\cdot\leq t\}})\delta(e^{\alpha\cdot}\mathbf{1}_{\{\cdot\leq s\}})]dsdt \\
\lesssim&\frac{1}{n^2}\int_0^n\int_0^n e^{\alpha v}e^{\alpha u}|u-v|^{2H-2}\int_v^n\int_u^n \varphi_i(t)\varphi_i(s)e^{-\alpha t}e^{-\alpha s}dtdsdudv.
\end{aligned}$$

Since the φ_i are uniformly bounded, we can write

$$\begin{aligned}
&\int_v^n\int_u^n \varphi_i(t)\varphi_i(s)e^{-\alpha t}e^{-\alpha s}dtds \\
&\qquad\lesssim \int_v^n e^{-\alpha t}dt\int_u^n e^{-\alpha s}ds = \frac{1}{\alpha^2}(e^{-\alpha v}-e^{-\alpha n})(e^{-\alpha u}-e^{-\alpha n}),
\end{aligned}$$

which means that

$$\begin{aligned}
\mathbb{E}[CG_n^2] &\lesssim \frac{1}{n^2}\int_0^n\int_0^n \underbrace{(1-e^{-\alpha(n-v)})}_{\text{bounded}}\underbrace{(1-e^{-\alpha(n-u)})}_{\text{bounded}}|u-v|^{2H-2}dudv \\
&\lesssim \frac{1}{n^2}n^{2H} = n^{2H-2} \overset{n\to\infty}{\to} 0.
\end{aligned}$$

□

Next we will demonstrate a boundedness result for the solution process X^L.

4.2.5 Proposition. *The sequence* $\mathbb{E}[(X_t^L)^2]$ *is uniformly bounded for* $t \geq 0$.

Proof. As mentioned above, we have:

$$X_t^L = h(t) + e^{-\alpha t}\delta(e^{\alpha\cdot}\mathbf{1}_{\{\cdot\leq t\}}).$$

The deterministic terms of $\mathbb{E}[(X_t^L)^2]$ are uniformly bounded, the mixed terms containing δ equal zero, and for the last term we get

$$\mathbb{E}[(e^{-\alpha t}\delta(e^{\alpha \cdot}\mathbf{1}_{\{\cdot \leq t\}}))^2] = e^{-2\alpha t}\int_0^t \int_0^t e^{\alpha(s+r)}|s-r|^{2H-2}dsdr,$$

which is bounded by a constant by Lemma 5.1 in [30].

□

4.2.6 Remark. $\mathbb{E}[(X_t^L)^4]$ is also uniformly bounded: For the third and fourth power of $e^{-\alpha t}\delta(e^{\alpha \cdot}\mathbf{1}_{\{\cdot \leq t\}})$ recall that it is an element of $\{B^H(h)|h \in \mathscr{H}\}$, hence, a Gaussian random variable with zero mean and a variance which is bounded by a constant, as proved in Proposition 4.2.5.

As a consequence, we obtain another boundedness result:

4.2.7 Proposition. $\mathbb{E}[(D_n^{Lp})^2]$ *is bounded. The bound is uniform with respect to* n *and to* p.

Proof. It follows from the Bessel inequality applied pointwise that

$$0 \leq \frac{1}{n}\int_0^n (X_t^L)^2 dt - \sum_{i=1}^p (\Lambda_{ni}^{Lp})^2 \leq \frac{1}{n}\int_0^n (X_t^L)^2 dt.$$

Hence,

$$\mathbb{E}[(D_n^{Lp})^2] = \mathbb{E}\left[\left(\frac{1}{n}\int_0^n (X_t^L)^2 dt - \sum_{i=1}^p (\Lambda_{ni}^{Lp})^2\right)^2\right] \leq \mathbb{E}\left[\left(\frac{1}{n}\int_0^n (X_t^L)^2 dt\right)^2\right].$$

To show that the right-hand side is bounded, we calculate:

$$\begin{aligned}\mathbb{E}[(\frac{1}{n}\int_0^n (X_t^L)^2 dt)^2] &= \frac{1}{n^2}\mathbb{E}[\int_0^n (X_t^L)^2 dt \int_0^n (X_s^L)^2 ds] \\ &= \frac{1}{n^2}\int_0^n \int_0^n E[(X_t^L)^2 (X_s^L)^2] dtds \lesssim \frac{1}{n^2} n^2 = 1,\end{aligned}$$

since we can find a uniform bound for $E[(X_t^L)^2(X_s^L)^2]$ due to the Remark 4.2.6 in combination with Cauchy-Schwarz. □

4.3 Error bounds

This section contains the crucial part of this chapter. Our goal is to find a bound on the L^2-error for the first p components of the truncated p-cutoff estimator

$\hat{\vartheta}^{(T)}$. Rigorously phrased, for a given number $p \in \mathbb{N}$, an $L^2([0, 1])$-orthonormal basis $(\varphi_i)_{i\in\mathbb{N}}$ and a 1-periodic bounded function $L = \sum_{i=1}^{\infty} \mu_i \varphi_i$ we write

$$\hat{\vartheta}^{(T)} = (\hat{\mu}_1^{(T)}, \dots, \hat{\mu}_p^{(T)}, \hat{\alpha}^{(T)})$$

and determine a bound on $\mathbb{E}[(\hat{\mu}_i^{(T)} - \mu_i)^2]$ for all $i \in \{1, \dots, p\}$. This bound might depend on n as well as on p.

Let us first make use of Proposition 4.1.5 and replace $\hat{\mu}_i^{(T)}$ for an $i \in \{1, \dots, p\}$ by the representation given in that proposition:

$$\begin{aligned}
\mathbb{E}[(\hat{\mu}_i^{(T)} - \mu_i)^2] &= \mathbb{E}\left[\left(\left(\vartheta + Q_n^{-1}R_n + Q_n^{-1}N_n\right)_i \mathbf{1}_{\{D_n^{Lp} \geq k\}} - \mu_i\right)^2\right] \\
&= \mathbb{E}\left[\left(\mu_i \mathbf{1}_{\{D_n^{Lp} \geq k\}} + (Q_n^{-1}R_n)_i \mathbf{1}_{\{D_n^{Lp} \geq k\}} + (Q_n^{-1}N_n)_i \mathbf{1}_{\{D_n^{Lp} \geq k\}} - \mu_i\right)^2\right] \\
&\leq 3\left(\mathbb{E}\left[(Q_n^{-1}R_n)_i^2 \mathbf{1}_{\{D_n^{Lp} \geq k\}}\right] + \mathbb{E}\left[(Q_n^{-1}N_n)_i^2 \mathbf{1}_{\{D_n^{Lp} \geq k\}}\right] + \mathbb{E}\left[\left(\mu_i \mathbf{1}_{\{D_n^{Lp} \geq k\}} - \mu_i\right)^2\right]\right).
\end{aligned}$$

We will find separate bounds for

$$\begin{aligned}
&\mathbb{E}\left[(Q_n^{-1}R_n)_j^2 \mathbf{1}_{\{D_n^{Lp} \geq k\}}\right], \\
&\mathbb{E}\left[(Q_n^{-1}N_n)_j^2 \mathbf{1}_{\{D_n^{Lp} \geq k\}}\right] \text{ and for} \\
&\mathbb{E}\left[\left(\mu_j \mathbf{1}_{\{D_n^{Lp} \geq k\}} - \mu_j\right)^2\right] = \mu_j^2 P(D_n^{Lp} < k)
\end{aligned}$$

for $j \in \{1, \dots, p\}$.

Let us proceed.

4.3.1 Bound of $(Q_n^{-1}R_n)_j \mathbf{1}_{\{D_n^{Lp} \geq k\}}$

In this subsection the following proposition will be demonstrated:

4.3.1 Proposition. *With the definitions given in Section 4.1 we have*

$$\mathbb{E}\left[(Q_n^{-1}R_n)_j^2 \mathbf{1}_{\{D_n^{Lp} \geq k\}}\right] \lesssim p^2 n^{2H-2}$$

for every $j \in \{1, \dots, p\}$, and the corresponding constant does not depend on j.

Proof. The proof consists of several parts.

1. We compute the jth entry in $Q_n^{-1}R_n$ explicitly and make use of the truncation in our estimator in order to eliminate the denominator (see Definition 4.1.7):

$$
\begin{aligned}
&\mathbb{E}\left[(Q_n^{-1}R_n)_j^2\mathbf{1}_{\{D_n^{Lp}\geq k\}}\right]\\
&\quad=\mathbb{E}\left[\frac{1}{n^2}\mathbf{1}_{\{D_n^{Lp}\geq k\}}\frac{1}{(D_n^{Lp})^2}\left(\sum_{i=1}^{p}\delta(\varphi_i)\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp}+\delta(\varphi_j)D_n^{Lp}-\delta(X^L)\Lambda_{nj}^{Lp}\right)^2\right]\\
&\quad\lesssim\frac{1}{n^2}\left(\mathbb{E}[\delta(\varphi_j)^2]+\mathbb{E}\left[\left(\sum_{i=1}^{p}\delta(\varphi_i)\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp}-\delta(X^L)\Lambda_{nj}^{Lp}\right)^2\right]\right).
\end{aligned}
$$

2. The isometry property of divergence integrals provides us with the asymptotic bound for the first summand:

$$
\frac{1}{n^2}\,\mathbb{E}[\delta(\varphi_j)^2]\lesssim\frac{1}{n^2}n^{2H}=n^{2H-2}.
$$

3. For each of the summands of

$$
\frac{1}{n^2}\,\mathbb{E}\left[\left(\sum_{i=1}^{p}\delta(\varphi_i)\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp}-\delta(X^L)\Lambda_{nj}^{Lp}\right)^2\right]
$$

we apply the formula $F\delta(u)=\delta(Fu)-\langle DF,u\rangle_{\mathscr{H}}$ (see Proposition 2.1.6), expand the expression, pull apart the summands and finally apply the Cauchy-Schwarz inequality (pointwise, i.e. $|2ab|\leq a^2+b^2$ for $a,\,b\in\mathbb{R}$) to the mixed terms (the latter action gives us a factor p in front of the sum):

$$
\begin{aligned}
&\frac{1}{n^2}\,\mathbb{E}\left[\left(\sum_{i=1}^{p}\delta(\varphi_i)\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp}-\delta(X^L)\Lambda_{nj}^{Lp}\right)^2\right]\\
&\quad\lesssim\frac{p}{n^2}\Bigg(\sum_{i=1}^{p}\mathbb{E}\left[\delta(\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp}\varphi_i)^2\right]+\sum_{i=1}^{p}\mathbb{E}\left[\langle D_\cdot(\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp}),\,\varphi_i(\cdot)\rangle_{\mathscr{H}}^2\right]\\
&\qquad+\mathbb{E}[\delta(X^L\Lambda_{nj}^{Lp})^2]+\mathbb{E}[\langle D_\cdot\Lambda_{nj}^{Lp},\,X_\cdot^L\rangle_{\mathscr{H}}^2]\Bigg)\\
&\quad=:\frac{p}{n^2}\left(\sum_{i=1}^{p}E_{1i}+\sum_{i=1}^{p}E_{2i}+E_3+E_4\right),
\end{aligned}
$$

We will now treat all the summands separately in order of their appearance in the sum.

3.1. Let us consider E_{1i}. Use $\mathbb{E}[\delta(u)\delta(v)] = \mathbb{E}[\langle u, v\rangle_{\mathscr{H}}] + \mathbb{E}[\langle D_{\cdot}u_*, D_*v_{\cdot}\rangle_{\mathscr{H}}]$ from Proposition 2.1.7 and write

$$\begin{aligned} E_{1i} &= \mathbb{E}\left[\delta(\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp}\varphi_i)^2\right] \\ &= \mathbb{E}[\langle \varphi_i\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp}, \varphi_i\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp}\rangle_{\mathscr{H}}] + \mathbb{E}[\langle D_{\cdot}(\varphi_i(*)\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp}), D_*(\varphi_i(\cdot)\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp})\rangle_{\mathscr{H}}] \\ &=: \epsilon_{11} + \epsilon_{12}. \end{aligned}$$

We have:

$$\begin{aligned} \epsilon_{11} = \mathbb{E}[\langle \varphi_i\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp}, \varphi_i\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp}\rangle_{\mathscr{H}}] \\ =\langle \varphi_i, \varphi_i\rangle_{\mathscr{H}}\, \mathbb{E}[(\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp})^2]. \end{aligned}$$

Because of the boundedness results from Section 4.2.2 we can find a constant uniform bound on the expectation term. And since the basis elements are bounded by assumption, we have $|\langle\varphi_i, \varphi_i\rangle_{\mathscr{H}}| \lesssim n^{2H}$, and hence, $|\epsilon_{11}| \lesssim n^{2H}$.

For ϵ_{12} we have to calculate and analyse the Malliavin derivatives of the term $\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp}$. We use the product rule for Malliavin derivatives (2.1.1) and write first

$$D_s(\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp}) = D_s\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp} + \Lambda_{ni}^{Lp}D_s\Lambda_{nj}^{Lp}.$$

Observe that, if it is not marked otherwise by the parentheses, the derivative operator applies only to the variable written directly behind it, i.e. $D_s\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp}$ is to be read as $D_s(\Lambda_{ni}^{Lp})\Lambda_{nj}^{Lp}$. We will use this notation throughout the thesis.

In total, we obtain

$$|\epsilon_{12}| = \mathbb{E}[\langle \varphi_i(*)(D_{\cdot}\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp} + \Lambda_{ni}^{Lp}D_{\cdot}\Lambda_{nj}^{Lp}), \varphi_i(\cdot)(D_*\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp} + \Lambda_{ni}^{Lp}D_*\Lambda_{nj}^{Lp})\rangle_{\mathscr{H}}]$$

These summands can be pulled apart by the triangle inequality applied to $|\epsilon_{12}|$. This will give us a total of four summands of the form $|\,\mathbb{E}[\langle f, g\rangle_{\mathscr{H}}]|$ with $f = \varphi_i(*)D_{\cdot}\Lambda_{nk}^{Lp}\Lambda_{nl}^{Lp}$ and $g = \varphi_i(\cdot)D_*\Lambda_{nm}^{Lp}\Lambda_{no}^{Lp}$ with $(k, l), (m, o) \in \{(i, j), (j, i)\}$. We will denote a summand of this form by $|\epsilon_{12}^{(1)}|$). Our next step is, of course, to find a bound for $|\epsilon_{12}^{(1)}|$.

First note that we can explicitly calculate the Malliavin derivative of X_t^L,

$$D_sX_t^L = e^{\alpha(s-t)}\mathbf{1}_{\{s\le t\}},$$

it is deterministic and uniformly bounded. We exchange the Malliavin derivative and the Lebesgue integral as reasoned in Remark 2.1.11 and conclude that also $D_s\Lambda_{ni}^{Lp} = \frac{1}{n}\int_0^n \varphi_i(t)D_sX_t^L dt$ are bounded and deterministic for $i \in \{1, \dots, p\}$, and, hence, we have

$$|\langle \varphi_i(*)D_{\cdot}\Lambda_{nk}^{Lp}, \varphi_i(\cdot)D_*\Lambda_{nm}^{Lp}\rangle_{\mathscr{H}}| \lesssim n^{2H},$$

which implies

$$|\epsilon_{12}^{(1)}| = |\,\mathbb{E}[\Lambda_{nl}^{Lp}\Lambda_{no}^{Lp}]\langle\varphi_i(*)D.\Lambda_{nk}^{Lp},\,\varphi_i(\cdot\,)D_*\Lambda_{nm}^{Lp}\rangle_{\mathscr{H}}| \lesssim n^{2H}$$

due to boundedness of $\mathbb{E}[(\Lambda_{nl}^{Lp})^2]$ and of $\mathbb{E}[(\Lambda_{no}^{Lp})^2]$. Note that the bound is independent of p: recall that in the notation Λ_{ni}^{Lp} the parameter p is an index and not a factor.

In total, taking into account the number of summands, we obtain $|\sum_{i=1}^{p} E_{1i}| \lesssim pn^{2H}$.

3.2. Now let us look at E_{2i}. Recall that

$$D_s(\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp}) = D_s\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp} + \Lambda_{ni}^{Lp}D_s\Lambda_{nj}^{Lp}$$

and paste this expression into the scalar product

$$E_{2i} = \mathbb{E}\left[\langle D.(\Lambda_{ni}^{Lp}\Lambda_{nj}^{Lp}),\,\varphi_i(\cdot\,)\rangle_{\mathscr{H}}^2\right].$$

Now the sum can be pulled apart again and, as above, we will be analysing every summand separately. Due to a comparable structure in this case we will again have four summands, this time of the following kind:

$$|E_2^{(1)}| = |\,\mathbb{E}[\langle D.\Lambda_{nk}^{Lp}\Lambda_{nl}^{Lp}, \varphi_i(\cdot\,)\rangle_{\mathscr{H}}\langle D.\Lambda_{nm}^{Lp}\Lambda_{no}^{Lp}, \varphi_i(\cdot\,)\rangle_{\mathscr{H}}]|$$

with $(k,\,l),\,(m,\,o) \in \{(i,\,j),\,(j,\,i)\}$.

Now note that we can find a bound for $D_s\Lambda_{ni}^{Lp}$, $s \le n$, $i \in \{1,\dots,p\}$, which is better than a constant:

$$\begin{aligned}|D_s\Lambda_{ni}^{Lp}| &= \left|\frac{1}{n}\int_0^n \varphi_i(t)e^{\alpha(s-t)}\mathbf{1}_{\{s\le t\}}dt\right|\\ &\lesssim e^{\alpha s}\frac{1}{n}\int_s^n e^{-\alpha t}dt \sim \frac{1}{n}(1-e^{-\alpha(n-s)}) \lesssim \frac{1}{n}.\end{aligned}$$

Using the boundedness result for Λ_{ni}^{Lp} (following from 4.2.4) as well as the above calculation, we get a bound for $|E_2^{(1)}|$:

$$|E_2^{(1)}| \lesssim |\langle D.\Lambda_{nk}^{Lp}, \varphi_i(\cdot\,)\rangle_{\mathscr{H}}\langle D.\Lambda_{nm}^{Lp}, \varphi_i(\cdot\,)\rangle_{\mathscr{H}}| \lesssim \frac{1}{n^2}n^{4H} \lesssim n^{2H}.$$

Again, considering the number of summands, we arrive at the bound $\sum_{i=1}^{p} E_{2i} \lesssim pn^{2H}$.

3.3 For E_3 we use the same formula as for E_1:

$$\mathbb{E}[\delta(X^L\Lambda_{nj}^{Lp})^2] = \underbrace{\mathbb{E}[\langle X^L_\cdot\Lambda_{nj}^{Lp},\,X^L_\cdot\Lambda_{nj}^{Lp}\rangle_{\mathscr{H}}]}_{=:E_{31}} + \underbrace{\mathbb{E}[\langle D.(X^L_*\Lambda_{nj}^{Lp}),\,D_*(X^L_\cdot\Lambda_{nj}^{Lp})\rangle_{\mathscr{H}}]}_{=:E_{32}}.$$

First, we have a look at E_{31}:

$$|E_{31}| = \left|\mathbb{E}\left[\int_0^n \int_0^n X_s^L \Lambda_{nj}^{Lp} X_t^L \Lambda_{nj}^{Lp} |s-t|^{2H-2} ds dt\right]\right|.$$

Since $\mathbb{E}[|X_s^L (\Lambda_{nj}^{Lp})^2 X_t|]$ is bounded by a constant, we have $|E_{31}| \lesssim n^{2H}$.

Let us now consider the second term:

$$|E_{32}| = \left|\mathbb{E}\left[\int_0^n \int_0^n D_s(X_t^L \Lambda_{nj}^{Lp}) D_t(X_s^L \Lambda_{nj}^{Lp}) |s-t|^{2H-2} ds dt\right]\right|.$$

The product rule (2.1.1) yields

$$\begin{aligned}
&|D_s(X_t^L \Lambda_{nj}^{Lp}) D_t(X_s^L \Lambda_{nj}^{Lp})| = |(D_s X_t^L \Lambda_{nj}^{Lp} + D_s \Lambda_{nj}^{Lp} X_t^L)(D_t X_s^L \Lambda_{nj}^{Lp} + D_t \Lambda_{nj}^{Lp} X_s^L)| \\
&\quad \leq \underbrace{|D_s X_t^L D_t X_s^L (\Lambda_{nj}^{Lp})^2|}_{=0 \text{ for } s \neq t} + \underbrace{|D_s \Lambda_{nj}^{Lp} X_t^L D_t X_s^L \Lambda_{nj}^{Lp}|}_{\lesssim |X_t^L \Lambda_{nj}^{Lp}|} \\
&\qquad + \underbrace{|D_t \Lambda_{nj}^{Lp} X_s^L D_s X_t^L \Lambda_{nj}^{Lp}|}_{\lesssim |X_s^L \Lambda_{nj}^{Lp}|} + \underbrace{|D_s \Lambda_{nj}^{Lp} D_t \Lambda_{nj}^{Lp} X_s^L X_t^L|}_{\lesssim |X_s^L X_t^L|},
\end{aligned}$$

which can be obtained from the above calculations for the derivatives. Due to boundedness results from Section 4.2.2 we now get a constant bound for

$$|D_s(X_t^L \Lambda_{nj}^{Lp}) D_t(X_s^L \Lambda_{nj}^{Lp})|$$

and hence a total bound of n^{2H} for the term $|E_{32}|$.

3.4 Finally, let us analyse E_4. We have

$$E_4 = \mathbb{E}[\langle D.\Lambda_{nj}^{Lp}, X_\cdot^L \rangle_{\mathscr{H}}^2] \lesssim \mathbb{E}[\langle \frac{1}{n}(1 - e^{-\alpha(n-\cdot)}), X_\cdot^L \rangle_{\mathscr{H}}^2] \lesssim \frac{1}{n^2} n^{4H} \lesssim n^{2H},$$

because $E[|X_t^L X_s^L|]$ is bounded.

Let us briefly summarise the results. In part 3 we have shown: $\sum_{i=1}^p E_{1i} \lesssim pn^{2H}$, $\sum_{i=1}^p E_{2i} \lesssim pn^{2H}$ as well as $E_3 \lesssim n^{2H}$ and $E_4 \lesssim n^{2H}$. In total, we obtain:

$$\begin{aligned}
&\frac{1}{n^2} \mathbb{E}\left[\left(\sum_{i=1}^p \delta(\varphi_i) \Lambda_{ni}^{Lp} \Lambda_{nj}^{Lp} - \delta(X^L) \Lambda_{nj}^{Lp}\right)^2\right] \\
&\lesssim \frac{p}{n^2} \left(\sum_{i=1}^p E_{1i} + \sum_{i=1}^p E_{2i} + E_3 + E_4\right) \\
&\lesssim \frac{p}{n^2} pn^{2H} = p^2 n^{2H-2}.
\end{aligned}$$

Combined with part 2 we get

$$\mathbb{E}\left[(Q_n^{-1} R_n)_j^2 \mathbf{1}_{\{D_n^{Lp} \geq k\}}\right] \lesssim p^2 n^{2H-2},$$

which is what we wanted to show. □

4.3.2 Bound of $(Q_n^{-1}N_n)_j \mathbf{1}_{\{D_n^{Lp}\geq k\}}$

First, let us introduce a notation that will also be of use later on:

4.3.2 Definition. For a 1-periodic L^2-function L and a given L^2-orthonormal basis $(\varphi_i)_{i\in\mathbb{N}}$ we will denote its projection on the space spanned by $\varphi_1, \ldots, \varphi_p$ (for a given $p \in \mathbb{N}$) by L_p, i.e., we will write

$$L_p = \sum_{i=1}^{p} \langle L, \varphi_i \rangle_{L^2([0,1])} \varphi_i.$$

Let us now consider the part of the L^2-error that we are going to analyse in this section:

$$(Q_n^{-1}N_n)_j \mathbf{1}_{\{D_n^{Lp}\geq k\}} = \left(\frac{1}{n} \frac{1}{D_n^{Lp}} \Lambda_{nj}^{Lp} \sum_{i=p+1}^{\infty} \mu_i \int_0^n X_t^L \varphi_i(t) dt \right) \mathbf{1}_{\{D_n^{Lp}\geq k\}}.$$

Now let us calculate the second moment of this term using Fubini's theorem:

$$\mathbb{E}\left[(Q_n^{-1}N_n)_j^2 \mathbf{1}_{\{D_n^{Lp}\geq k\}} \right] = \mathbb{E}\left[\frac{1}{n^2} \frac{1}{(D_n^{Lp})^2} (\Lambda_{nj}^{Lp})^2 \left(\sum_{i=p+1}^{\infty} \mu_i \int_0^n X_t^L \varphi_i(t) dt \right)^2 \mathbf{1}_{\{D_n^{Lp}\geq k\}} \right]$$

$$= \frac{1}{n^2} \int_0^n \int_0^n \left(\sum_{i=p+1}^{\infty} \mu_i \varphi_i(t) \right) \left(\sum_{i=p+1}^{\infty} \mu_i \varphi_i(s) \right) \underbrace{\mathbb{E}\left[\frac{1}{(D_n^{Lp})^2} (\Lambda_{nj}^{Lp})^2 X_t^L X_s^L \mathbf{1}_{\{D_n^{Lp}\geq k\}} \right]}_{|\ldots| \lesssim 1} dsdt$$

$$\lesssim \frac{1}{n^2} \left(\int_0^n |L(t) - L_p(t)| dt \right)^2 = \left(\int_0^1 |L(t) - L_p(t)| dt \right)^2.$$

The bound on the expectation is obtained by applying the Cauchy-Schwarz inequality and using the boundedness results for the fourth moments of the factors Λ_{nj}^{Lp} and X_t^L, which have been shown in the auxiliary subsection 4.2. The last step is due to periodicity of both L and L_p.

The bound we have found is nonrandom, it only depends on the class of functions L belongs to and on the ONB $(\varphi_i)_{i\in\mathbb{N}}$. A more useful bound on this expression with respect to p will be given as soon as we have made these choices.

4.3.3 Bound of $\mu_j^2 P(D_n^{Lp} < k)$

Let us write $\mu_j^2 b(n, p)$ for a bound on this term. We proceed as in the proof of convergence of D_n^{Lp} in order to find a suitable $b(n, p)$.

4.3.3 Proposition. *Let $\tilde{X}(= \tilde{X}^L)$ be the modified solution defined in the beginning of Section 4.2. We have*

$$\mathbb{E}\left[\left(\frac{1}{n} \int_0^n \tilde{X}_t^2 dt - \left(\psi + \sum_{i=1}^{\infty} \left(\int_0^1 \tilde{h}(t) \varphi_i(t) dt \right)^2 \right) \right)^2 \right] = O(n^{2H-2}).$$

Proof. We have $\tilde{X} = \tilde{h}(t) + \tilde{Z}_t$ (see beginning of Section 4.2), thus we can write

$$\frac{1}{n}\int_0^n \tilde{X}_t^2 dt = \frac{1}{n}\int_0^n \tilde{h}^2(t)dt + 2\frac{1}{n}\int_0^n \tilde{h}(t)\tilde{Z}_t dt + \frac{1}{n}\int_0^n \tilde{Z}_t^2 dt$$
$$\overset{\tilde{h}\text{ periodic}}{=} \int_0^1 \tilde{h}^2(t)dt + 2\frac{1}{n}\int_0^n \tilde{h}(t)\tilde{Z}_t dt + \frac{1}{n}\int_0^n \tilde{Z}_t^2 dt,$$

and therefore,

$$\mathbb{E}\left[\left(\frac{1}{n}\int_0^n \tilde{X}_t^2 dt - \left(\psi + \sum_{i=1}^\infty \left(\int_0^1 \tilde{h}(t)\varphi_i(t)dt\right)^2\right)\right)^2\right]$$
$$= \mathbb{E}\left[\left(\frac{1}{n}\int_0^n \tilde{Z}_t^2 dt - \psi + 2\frac{1}{n}\int_0^n \tilde{h}(t)\tilde{Z}_t dt\right)^2\right]$$
$$\lesssim \underbrace{\mathbb{E}\left[\left(\frac{1}{n}\int_0^n \tilde{Z}_t^2 dt - \psi\right)^2\right]}_{=:S_1} + 4\underbrace{\frac{1}{n^2}\mathbb{E}\left[\left(\int_0^n \tilde{h}(t)\tilde{Z}_t dt\right)^2\right]}_{=:S_2}.$$

Let us deal with the summands separately, as usual. For S_1, let us first observe that for two jointly normally distributed, centred random variables X, Y we have

$$\mathbb{E}[X^2Y^2] = \mathbb{E}[X^2]\,\mathbb{E}[Y^2] + 2\,\mathbb{E}[XY]^2.$$

With this in mind we can write

$$S_1 = \mathbb{E}\left[\left(\frac{1}{n}\int_0^n \tilde{Z}_t^2 dt\right)^2\right] - 2\psi\,\mathbb{E}\left[\frac{1}{n}\int_0^n \tilde{Z}_t^2 dt\right] + \psi^2$$
$$= \frac{1}{n^2}\int_0^n\int_0^n \mathbb{E}[\tilde{Z}_t^2\tilde{Z}_s^2]dtds - 2\psi\frac{1}{n}\int_0^n \mathbb{E}[\tilde{Z}_t^2]dt + \psi^2$$
$$= \left(\frac{1}{n}\int_0^n \mathbb{E}[\tilde{Z}_t^2]dt - \psi\right)^2 + 2\frac{1}{n^2}\int_0^n\int_0^n \mathbb{E}[\tilde{Z}_t\tilde{Z}_s]^2 dtds.$$

The second summand here can be bounded by n^{4H-4} ($\lesssim n^{2H-2}$) similarly to Proposition 4.2.1: we integrate this time over squares of S_1, $S_{2.1}$, $S_{2.2}$ and S_3 from Proposition 4.2.1 and using the same techniques we arrive at the desired bound. For the first summand let us observe that ψ equals $\mathbb{E}[\tilde{Z}_0^2]$ (as shown in [30]). As $\tilde{Z}$ is a stationary process, its second moments do not vary over time, and hence, pulling ψ into the integral yields $\mathbb{E}[\tilde{Z}_t^2] - \mathbb{E}[\tilde{Z}_0^2] = 0$ under the integral sign. Therefore, the first summand vanishes.

Now we turn to the second summand and rewrite it thus:

$$S_2 = \frac{1}{n^2}\int_0^n\int_0^n \tilde{h}(t)\tilde{h}(s)\,\mathbb{E}[\tilde{Z}_t\tilde{Z}_s]dtds \lesssim n^{2H-2},$$

as $\tilde{h}$ is bounded and thus the same arguments as in Proposition 4.2.1 can be applied. □

Next we will need another auxiliary convergence result.

4.3.4 Lemma. *Let $\tilde{\Lambda}_{ni}^{Lp}$ and $\tilde{h}$ be defined as in the beginning of Section 4.2. We have the following bound, using the Cauchy-Schwarz inequality and the boundedness results from Section 4.2.2:*

$$\mathbb{E}\left[\left(\sum_{i=1}^{p(n)}\left(\tilde{\Lambda}_{ni}^{Lp}\right)^2 - \left(\int_0^1 \tilde{h}(t)\varphi_i(t)dt\right)^2\right)^2\right] \lesssim p(n)^2 n^{2H-2}.$$

Proof. We calculate as follows:

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=1}^{p(n)}\left(\tilde{\Lambda}_{ni}^{Lp}\right)^2 - \left(\int_0^1 \tilde{h}(t)\varphi_i(t)dt\right)^2\right)^2\right] \\
&\lesssim p(n)\sum_{i=1}^{p(n)}\mathbb{E}\left[\left(\left(\tilde{\Lambda}_{ni}^{Lp}\right)^2 - \left(\int_0^1 \tilde{h}(t)\varphi_i(t)dt\right)^2\right)^2\right] \\
&= p(n)\sum_{i=1}^{p(n)}\mathbb{E}\left[\left(\left(\tilde{\Lambda}_{ni}^{Lp}\right) - \int_0^1 \tilde{h}(t)\varphi_i(t)dt\right)^2\left(\left(\tilde{\Lambda}_{ni}^{Lp}\right) + \int_0^1 \tilde{h}(t)\varphi_i(t)dt\right)^2\right] \\
&\lesssim p(n)\sum_{i=1}^{p(n)}\sqrt{\mathbb{E}\left[\left(\left(\tilde{\Lambda}_{ni}^{Lp}\right) - \int_0^1 \tilde{h}(t)\varphi_i(t)dt\right)^4\right]} \\
&\sim p(n)\sum_{i=1}^{p(n)}\mathbb{E}\left[\left(\left(\tilde{\Lambda}_{ni}^{Lp}\right) - \int_0^1 \tilde{h}(t)\varphi_i(t)dt\right)^2\right] \overset{4.2.1}{\lesssim} p(n)^2 n^{2H-2},
\end{aligned}$$

the step before last being true due to properties of the moments of centred normal random variables. □

Now we can derive the actual result:

4.3.5 Proposition. *For $k < \varphi$ we have*

$$P(D_n^{Lp} < k) = P\left(\frac{1}{n}\int_0^n X_t^2 dt - \sum_{i=1}^{p(n)}\left(\Lambda_{ni}^{Lp}\right)^2 < k\right) \lesssim p(n)^2 n^{2H-2}.$$

Proof. Pick ε_1, $\varepsilon_2 > 0$ such that $\varphi - \varepsilon_1 - \varepsilon_2 > k$. Then we can split up the set in question in the following way:

$$
\begin{aligned}
&P\left(\frac{1}{n}\int_0^n X_t^2 dt - \sum_{i=1}^{p(n)} \left(\Lambda_{ni}^{Lp}\right)^2 < k\right)\\
&\quad = P\Big(D_n^{Lp} < k, \left|\frac{1}{n}\int_0^n X_t^2 dt - \left(\psi + \int_0^1 \tilde{h}^2(t)dt\right)\right| < \varepsilon_1,\\
&\qquad\qquad \left|\sum_{i=1}^{p(n)} (\Lambda_{ni}^{Lp})^2 - \int_0^1 \tilde{h}^2(t)dt\right| < \varepsilon_2\Big)\\
&\quad + P\Big(D_n^{Lp} < k,\\
&\qquad\qquad \Big(\left|\frac{1}{n}\int_0^n X_t^2 dt - \left(\psi + \int_0^1 \tilde{h}^2(t)dt\right)\right| \geq \varepsilon_1 \text{ or } \left|\sum_{i=1}^{p(n)} (\Lambda_{ni}^{Lp})^2 - \int_0^1 \tilde{h}^2(t)dt\right| \geq \varepsilon_2\Big)\Big).
\end{aligned}
$$

The first summand is then zero by choice of ε_1, ε_2, because D_n^{Lp} must be enclosed in the interval $(\psi - \varepsilon_1 - \varepsilon_2,\ \psi + \varepsilon_1 + \varepsilon_2)$ due to the last two conditions. The second summand is bounded by the sum

$$
\underbrace{P\left(\left|\frac{1}{n}\int_0^n X_t^2 dt - \left(\psi + \int_0^1 \tilde{h}^2(t)dt\right)\right| \geq \varepsilon_1\right)}_{=:S_1} + \underbrace{P\left(\left|\sum_{i=1}^{p(n)} (\Lambda_{ni}^{Lp})^2 - \int_0^1 \tilde{h}^2(t)dt\right| \geq \varepsilon_2\right)}_{=:S_2}.
$$

Let us consider S_1. First note that, as mentioned in Proposition 4.2.3, X_t and $\tilde{X}_t (= \tilde{X}_t^L)$ approach each other as t tends to infinity. Therefore, we exchange X_t for $\tilde{X}_t$ by picking $0 < \tilde{\varepsilon}_1 < \varepsilon_1$ such that we have

$$
\begin{aligned}
S_1 =& P\left(\left|\frac{1}{n}\int_0^n X_t^2 dt - \left(\psi + \int_0^1 \tilde{h}^2(t)dt\right)\right| \geq \varepsilon_1, \left|\frac{1}{n}\int_0^n X_t^2 dt - \frac{1}{n}\int_0^n \tilde{X}_t^2 dt\right| \geq \tilde{\varepsilon}_1\right)\\
&+ P\left(\left|\frac{1}{n}\int_0^n X_t^2 dt - \left(\psi + \int_0^1 \tilde{h}^2(t)dt\right)\right| \geq \varepsilon_1, \left|\frac{1}{n}\int_0^n X_t^2 dt - \frac{1}{n}\int_0^n \tilde{X}_t^2 dt\right| < \tilde{\varepsilon}_1\right)\\
&\leq P\left(\left|\frac{1}{n}\int_0^n X_t^2 dt - \frac{1}{n}\int_0^n \tilde{X}_t^2 dt\right| \geq \tilde{\varepsilon}_1\right)\\
&\qquad + P\left(\left|\frac{1}{n}\int_0^n X_t^2 dt - \left(\psi + \int_0^1 \tilde{h}^2(t)dt\right)\right| \geq \varepsilon_1 - \tilde{\varepsilon}_1\right).
\end{aligned}
$$

Then we apply Markov inequality to each of the summands, obtaining for the first one

$$
\begin{aligned}
&P\left(\left|\frac{1}{n}\int_0^n X_t^2 dt - \frac{1}{n}\int_0^n \tilde{X}_t^2 dt\right| \geq \tilde{\varepsilon}_1\right) \\
&\lesssim \mathbb{E}\left[\frac{1}{n^2}\int_0^n \int_0^n (X_t^2 - \tilde{X}_t^2)(X_s^2 - \tilde{X}_s^2) dtds\right] \\
&\leq \frac{1}{n^2}\int_0^n \int_0^n \mathbb{E}[|(X_t + \tilde{X}_t)(X_s + \tilde{X}_s)| \underbrace{|X_t - \tilde{X}_t|}_{\leq e^{-\alpha t} Z} \underbrace{|X_s - \tilde{X}_s|}_{\leq e^{-\alpha s} Z}] dtds \\
&\lesssim \frac{1}{n^2}\int_0^n \int_0^n e^{-\alpha t} e^{-\alpha s}\, \mathbb{E}[|(X_t + \tilde{X}_t)(X_s + \tilde{X}_s)| Z^2] dtds \lesssim \frac{1}{n^2},
\end{aligned}
$$

as the random variables involved are jointly Gaussian with uniformly bounded variances (cf. Section 4.2.2), Z being a bound on $e^{\alpha t}|X_t - \tilde{X}_t|$ defined in [24] and mentioned in Proposition 4.2.3.

The second summand is bounded by n^{2H-2}, as shown in Proposition 4.3.3.

Now we resume the investigation of S_2. Pick an $0 < \bar{\varepsilon}_2 < \varepsilon_2$ and note that since $\sum_{i=1}^\infty \left(\int_0^1 \tilde{h}(t)\varphi_i(t)dt\right)^2$ is a convergent sum, there exists an $n \in \mathbb{N}$ such that $\sum_{i=p(n)}^\infty \left(\int_0^1 \tilde{h}(t)\varphi_i(t)dt\right)^2 < \bar{\varepsilon}_2$. We can, therefore, write

$$
S_2 \leq P\left(\left|\sum_{i=1}^{p(n)} \left((\Lambda_{ni}^{Lp})^2 - \left(\int_0^1 \tilde{h}(t)\varphi_i(t)dt\right)^2\right)\right| \geq \varepsilon_2 - \bar{\varepsilon}_2\right).
$$

We proceed as with the first summand and replace Λ_{ni}^{Lp} by $\tilde{\Lambda}_{ni}^{Lp}$:

$$
\begin{aligned}
S_2 \leq &P\left(\left|\sum_{i=1}^{p(n)} \left((\Lambda_{ni}^{Lp})^2 - \left(\int_0^1 \tilde{h}(t)\varphi_i(t)dt\right)^2\right)\right| \geq \varepsilon_2 - \bar{\varepsilon}_2,\ \left|\sum_{i=1}^{p(n)} \left((\Lambda_{ni}^{Lp})^2 - (\tilde{\Lambda}_{ni}^{Lp})^2\right)\right| > \tilde{\varepsilon}_2\right) \\
&+ P\left(\left|\sum_{i=1}^{p(n)} \left((\Lambda_{ni}^{Lp})^2 - \left(\int_0^1 \tilde{h}(t)\varphi_i(t)dt\right)^2\right)\right| \geq \varepsilon_2 - \bar{\varepsilon}_2,\ \left|\sum_{i=1}^{p(n)} \left((\Lambda_{ni}^{Lp})^2 - (\tilde{\Lambda}_{ni}^{Lp})^2\right)\right| \geq \tilde{\varepsilon}_2\right) \\
\leq &P\left(\left|\sum_{i=1}^{p(n)} \left((\Lambda_{ni}^{Lp})^2 - (\tilde{\Lambda}_{ni}^{Lp})^2\right)\right| \geq \tilde{\varepsilon}_2\right) \\
&+ P\left(\left|\sum_{i=1}^{p(n)} \left((\Lambda_{ni}^{Lp})^2 - \left(\int_0^1 \tilde{h}(t)\varphi_i(t)dt\right)^2\right)\right| \geq \varepsilon_2 - \bar{\varepsilon}_2 - \tilde{\varepsilon}_2\right),
\end{aligned}
$$

where $\tilde{\varepsilon}_2 > 0$ is picked such that $\varepsilon_2 - \bar{\varepsilon}_2 - \tilde{\varepsilon}_2 > 0$. For the first summand, Markov's inequality together with moment boundedness of X_t, $\tilde{X}_t$ yields the bound $p(n)^2 \frac{1}{n^2}$, since (by calculations from Proposition 4.2.3)

$$P\left(\left|\sum_{i=1}^{p(n)}\left((\Lambda_{ni}^{Lp})^2-(\tilde{\Lambda}_{ni}^{Lp})^2\right)\right|\geq\tilde{\varepsilon}_2\right)$$
$$\lesssim\mathbb{E}\left[\left(\sum_{i=1}^{p(n)}\frac{1}{n}\int_0^n e^{-\alpha t}Zdt\frac{1}{n}\int_0^n|\tilde{X}_t+X_t|dt\right)^2\right]$$
$$\lesssim p(n)^2\left(\frac{1}{n}\int_0^n e^{-\alpha t}dt\right)^2,$$

using Markov inequality.

Finally, for the second summand the bound $p(n)^2 n^{2H-2}$ has been proved in Lemma 4.3.4. □

Note that this bound is always faster than the bound derived in Section 4.3.1, hence we do not have to take this summand into consideration while determining the optimal $p(n)$.

4.4 Construction of the estimator

Let us now define an estimator for the function L.

4.4.1 Definition. Given the SDE (4.1.1) with the 1-periodic and bounded mean function L, an ONB $\{\varphi_i\}_{i\in\mathbb{N}}$ of $L^2([0, 1])$, define for a nondecreasing function $p:\mathbb{N}\to\mathbb{N}$, $p(n)\overset{n\to\infty}{\to}\infty$,

$$\hat{L}_{p(n)}:=\sum_{i=1}^{p(n)}\varphi_i\hat{\mu}_i^{(T)},$$

where $\hat{\mu}_i^{(T)}=(\hat{\vartheta}_n^{(T,\,L,\,p(n))})_i$ (see Definition 4.1.7).

In order to establish bounds on the rates of convergence which depend on n we are free to make two choices. We can pick a suitable orthonormal basis (note that in our previous calculations we have not specified one) and a class of functions where L should belong to, thus imposing new conditions on our problem. This will be done in the main proposition. Before that let us present a lemma from [68].

4.4.2 Lemma. *If a function f is continuous with the modulus of continuity $\omega(\delta)$, then it holds for the partial sum of its Fourier series $S_N(x)=\sum_{j=-N}^{N}\hat{f}(j)e^{2\pi ijx}$ (where $\hat{f}(n)$ are the corresponding Fourier coefficients):*

$$|f(x)-S_N(x)|\leq A\omega\left(\frac{1}{2\pi N}\right)\log(N)$$

for some constant A.

Therefore, it seems reasonable to consider the Fourier orthonormal basis $(e^{2\pi ijx})_{j\in\mathbb{Z}}$. However, as we need a basis of real functions, we consider the family $(1, \sqrt{2}\sin(2\pi x), \sqrt{2}\cos(2\pi x), \sqrt{2}\sin(2\pi\cdot 2x), \sqrt{2}\cos(2\pi\cdot 2x), \dots)$ as a choice for φ_i, which changes nothing for the uniform bound, since

$$S_N(x) = a_0 + \sum_{j=1}^{N} a_j\sqrt{2}\sin(2\pi jx) + b_j\sqrt{2}\cos(2\pi jx)$$

with a_i, b_i being defined as scalar products of f with the respective basis elements. As to suitable classes for L, let us consider the class H^γ of γ-Hölder continuous functions (for which we have $\omega(\delta) = k\delta^\gamma$ for some constant k). The above result will then mean

$$|L_{2p+1}(x) - L(x)| \lesssim \left(\frac{1}{2\pi p}\right)^\gamma \log(p) \lesssim \frac{\log(p)}{p^\gamma}.$$

It is enough to consider odd indexes, since for the even ones the order of the bounds remains asymptotically the same (this can be shown with the triangle inequality because the bound on the coefficients is known).

4.4.3 Proposition. *Consider the SDE* (4.1.1) *with the* 1*-periodic, γ-Hölder continuous (for $\gamma \geq \frac{1}{2}$) and bounded mean function L as well as the trigonometric orthonormal basis $\{\varphi_i\}_{i\in\mathbb{N}}$. Then $\hat{L}_{p(n)}$ is an L^2-consistent estimator of L for $p(n) \sim n^{\frac{2-2H}{2\gamma+2}}$.*

Proof. First, note that for $p(n) \sim n^{\frac{2-2H}{2\gamma+2}}$ Propositions 4.2.2 and 4.2.3 are satisfied.

We consider the mean integrated squared error between the estimator and the true function L and provide a tradeoff bound by applying the Cauchy-Schwarz inequality:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^1 (\hat{L}_{p(n)}(t) - L(t))^2 dt\right] \\
&\quad\leq 2\left(\mathbb{E}\left[\int_0^1 (\hat{L}_{p(n)}(t) - L_{p(n)}(t))^2 dt\right] + \int_0^1 (L_{p(n)}(t) - L(t))^2 dt\right).
\end{aligned}$$

Note that

$$\begin{aligned}
\mathbb{E}\left[\int_0^1 (\hat{L}_{p(n)}(t) - L_{p(n)}(t))^2 dt\right] &= \mathbb{E}\left[\int_0^1 \left(\sum_{i=1}^{p(n)} \varphi_i(t)\hat{\mu}_i - \sum_{i=1}^{p(n)} \varphi_i(t)\mu_i\right)^2 dt\right] \\
= \mathbb{E}\left[\int_0^1 \left(\sum_{i=1}^{p(n)} \varphi_i(t)(\hat{\mu}_i - \mu_i)\right)^2 dt\right] &= \sum_{i=1}^{p(n)} \mathbb{E}\left[(\hat{\mu}_i - \mu_i)^2\right],
\end{aligned}$$

and therefore,

$$\mathbb{E}\left[\int_0^1 (\hat{L}_{p(n)}(t) - L(t))^2 dt\right] \lesssim \sum_{i=1}^{p(n)} \mathbb{E}\left[(\hat{\mu}_i - \mu_i)^2\right] + \int_0^1 (L_{p(n)}(t) - L(t))^2 dt.$$

We know from Section 4.3 that the first term is asymptotically bounded by the slowest term out of $p(n)\cdot p(n)^2 n^{2H-2}$, $p(n)\cdot \left(\int_0^1 |L(t) - L_p(t)|dt\right)^2$ (the additional factor $p(n)$ emerging from the summation) and $\sum_{i=1}^{p(n)} \mu_i^2 b(n, p(n))$. As we can see from Lemma 4.4.2,

$$p(n)\cdot \left(\int_0^1 |L(t) - L_{p(n)}(t)|dt\right)^2 \lesssim p(n)\frac{\log(p(n))^2}{p(n)^{2\gamma}}$$

as well as

$$\int_0^1 (L_{p(n)}(t) - L(t))^2 dt \lesssim \frac{\log(p(n))^2}{p(n)^{2\gamma}}.$$

For our choice of $p(n)$ and γ each of the terms goes to zero as n tends to infinity. In particular, we get

$$\sum_{i=1}^{p(n)} \mu_i^2 b(n, p(n)) = b(n, p(n)) \sum_{i=1}^{p(n)} \mu_i^2 \lesssim b(n, p(n)) \to 0$$

by virtue of the Bessel's inequality, since $p(n) \to \infty$ for $n \to \infty$.

Moreover, the $p(n)$ chosen in the proposition minimises the bounds. □

This speed of convergence can be improved by assuming that the solution for every cutoff L_p can be observed. In this case the bound $p(n)\cdot \left(\int_0^1 |L(t) - L_p(t)|dt\right)^2$ disappears, such that convergence becomes faster.

Chapter 5

Parametric estimators for the Ornstein-Uhlenbeck type equations driven by a Rosenblatt process

As mentioned in Chapter 2, Hermite processes with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$ are stochastic processes in Wiener chaoses with respect to a fractional Brownian motion that have the same covariance structure, namely that of the fractional Brownian motion with this parameter. As we will see in this chapter, they have a representation in terms of multiple integrals, and thus calculations including them can be made my means of Malliavin calculus. For this reason it is a natural idea to analyse the behaviour of the least-squares type estimator in the Ornstein-Uhlenbeck setting (analysed in [24] with Malliavin calculus for fBm) for Hermite processes in higher order chaoses. In this chapter we have made this analysis for the Hermite process in the second chaos, also known as Rosenblatt process. Moreover, we have constructed other parameter estimators for this setting and analysed their asymptotic behaviour. The mathematical reason for the restriction to chaos two is that, although Skorokhod-type integrals are defined for general Hermite processes, the calculus for them is less studied than for the Rosenblatt processes.

Concerning the practical motivation we have briefly seen in the introduction that in some applications it makes sense to consider a driving process with skewed marginals, which exhibits self-similarity. The Rosenblatt process is well-adapted for this role since the choice of the parameter H defining the covariance allows to regulate the skewness of the process: In the limit $H \to 1$ the χ^2-marginals emerge and in the limit $H \to \frac{1}{2}$ normal marginals occur (see [82]). Moreover, Rosenblatt processes, just like their Gaussian counterparts, exhibit self-similarity (see [78]). From the practical point of view this offers a simple but versatile idea for modelling processes with a skewness.

Although the definition of the least-squares estimator is analogous to the fBm case and relies on calculus for Skorokhod type integrals, the proof techniques are quite different from those used in [24]. The proofs in the latter setting rely heavily upon the Gaussianity of the driving process, requiring a different approach for the former case. The techniques found for demonstrating the asymptotic properties for the Rosenblatt processes are more general, some of them allowing extensions to self-similar processes. These techniques, moreover, lay the foundation for considering general Ornstein-Uhlenbeck type equations driven by a general Hermite process. For this some additional properties of stochastic integrals with respect to Hermite processes need to be established; moreover, parts of the proofs will require a different treatment, however, this chapter provides an outline for future work in this direction which would even further generalise the results. The same is valid for the new estimators defined in this chapter and the techniques developed to analyse them.

We consider the following model:

$$\begin{aligned} X_t &= X_0 + \int_0^t (L(t) - \alpha X_t)\, dt + Z_t^H, \quad t \geq 0, \\ X_0 &= 0, \end{aligned} \tag{5.0.1}$$

where the random noise $(Z_t^H)_{t\geq 0}$ is a Rosenblatt process with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$, L is a periodic function and α is assumed to be positive (here, again, the sign in front of α in the equation is changed to accommodate this). We will assume that L can be written as $L(t) = \sum_{i=1}^{p} \mu_i \varphi_i(t)$ with some suitable known periodic functions φ, $i = 1, .., p$. The purpose is to estimate jointly the parameters $\mu_1, .., \mu_p$ and α based on a continuous-time observation of the solution to (5.0.1).

The contents of this Chapter can for the most part be found in the paper

- R. Shevchenko, C. A. Tudor - Parameter estimation for the Rosenblatt Ornstein–Uhlenbeck process with periodic mean, 2019, Statistical Inference for Stochastic Processes.

5.1 Preliminaries: The Rosenblatt process and the stochastic integral with respect to it

Let us start by recalling the definition and the basic properties of the Rosenblatt process as well as the construction of the stochastic integral with respect to this process, which is neither Gaussian nor a semimartingale. For a more complete exposition, we refer to the monographs [62], [77] or to the reference [78]. The properties mentioned in this section are demonstrated in these references. Note that there are several possible definitions of the Rosenblatt process. Here, we chose to work with the so-called finite interval representation of it. Let $H > \frac{1}{2}$ and $(B_t)_{t\geq 0}$ a Brownian motion. Consider the kernel

$$K^H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du \tag{5.1.1}$$

with $t > s$ and c_H a deterministic constant and recall that a fBm could be defined as a single integral of this kernel with respect to a Brownian motion (see Section 2.1.1). The Rosenblatt process with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$ is defined as

$$Z_t^H = d(H) \int_0^t \int_0^t \left(\int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du \right) \delta B_{y_1} \delta B_{y_2}, \quad t \geq 0, \tag{5.1.2}$$

with

$$H' = \frac{H+1}{2}$$

and $d(H)$ a deterministic constant that ensures $\mathbb{E}(Z_t^H)^2 = t^{2H}$ for every $t \geq 0$. The stochastic integral in (5.1.2) is a multiple integral of order 2 with respect to the Wiener process B (see Section 2.1.2). Similarly, Hermite processes of order q with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$ are defined as multiple integrals of order q of the same structure but with a different normalising constant, and the kernels involved are K^{H_q} with $H_q = 1 + \frac{H-1}{q}$.

The process $\left(Z_t^H\right)_{t\geq 0}$ is a self-similar stochastic process (with the self-similarity index H). As for the fBm (see Section 2.1.1), this means that

$$\{Z_{at}^H, \, t \in \mathbb{R}\} \stackrel{d}{=} \{a^H Z_t^H, \, t \in \mathbb{R}\}.$$

This scaling property is important for the analysis of integrals with respect to Z^H, since it allows a rescaling of the integrands.

Due to the fact that it has the same covariance structure as the fBm, it has stationary increments and a version with Hölder continuous paths of order $\delta \in (0, H)$. By definition it is clear that this process lives in the second Wiener chaos.

Let us denote by $\mathscr{H}^H$ the canonical Hilbert space associated to the fractional Brownian motion with parameter H, i.e. $\mathscr{H}^H$ is the closure of the linear space generated by the indicator functions $\{\mathbf{1}_{[0,t]}, t \geq 0\}$ with respect to the inner product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathscr{H}^H} = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad t, s \geq 0,$$

as explained in Section 2.1.2. It is also possible to define Skorokhod integrals of random integrands with respect to the Rosenblatt process on the (possibly infinite) interval $[0, T]$. For a square integrable stochastic process $(g_t)_{t\geq 0}$ we set

$$\int_0^T g_s \delta Z_s^H := \int_0^t \int_0^t I(g)(y_1, y_2) \delta B_{y_1} \delta B_{y_2} \tag{5.1.3}$$

with the transfer operator

$$I(g)(y_1, y_2) = \int_{y_1 \vee y_2}^T g_u \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du. \tag{5.1.4}$$

The notation δB in (5.1.3) indicates the Skorokhod integral with respect to the Wiener process $(B_y)_{y\geq 0}$. From Lemma 1 in [78], the Skorokhod integral (5.2.7) is well-defined if

$$\mathbb{E} \int_0^T \int_0^T \|D_{x_1,x_2} g\|^2_{\mathscr{H}^H} dx_1 dx_2 < \infty. \tag{5.1.5}$$

Moreover, if $g \in \mathbb{L}^{2,p} := L^2([0, T], \mathbb{D}^{2,p})$ $(p \geq 2)$, then for every $t \geq 0$

$$\mathbb{E}\left|\int_0^t g_s \delta Z_s^H\right|^p \leq c(p, H) \sup_{r\in[0,t]} \left[\mathbb{E}\,|g_r|^p + \mathbb{E}\int_0^t \int_0^t \|D^{(2)}_{x_1,x_2} g_r\|^p_{L^2([0,\,t])} dx_1 dx_2\right] t^{pH} \tag{5.1.6}$$

for some constant $c(p,\, H)$ (this is also proved in [78]).

If $g \in \mathscr{H}^H$ is deterministic, then the integral (5.1.3) is a Wiener integral with respect to the Rosenblatt process (also called Wiener-Rosenblatt integral) and it satisfies the following isometry

$$\mathbb{E}\left(\int_0^t g_s \delta Z_s^H \int_0^t h_s \delta Z_s^H\right) = H(2H-1)\int_0^t \int_0^t g(u)h(v)|u-v|^{2H-2} du dv = \langle g, h\rangle_{\mathscr{H}^H}$$

for any functions g, h such that $\int_0^t \int_0^t |g(u)h(v)||u-v|^{2H-2} du dv < \infty$ and for any $t \geq 0$ (see [78] for proof).

5.2 The Rosenblatt Ornstein-Uhlenbeck process with periodic mean

The Rosenblatt Ornstein-Uhlenbeck (ROU in the sequel) process is defined as the solution of the Ornstein-Uhlenbeck equation driven by a Rosenblatt noise, see e.g. [45] or [74]. Possible applications of such a model are mentioned at the beginning of this chapter. The ROU process with periodic mean is defined as the solution to the Ornstein-Uhlenbeck type equation whose drift is a periodic function. More precisely, we will consider the stochastic differential equation

$$X_t = \int_0^t (L(s) - \alpha X_s)\, ds + Z_t^H, \quad t \geq 0, \tag{5.2.1}$$

with vanishing initial condition, where Z^H is the Rosenblatt process with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$. As in Chapter 4, L is assumed to be a deterministic function that can be expressed as a linear combination of known bounded 1-periodic functions (assumed to be orthonormal in $L^2([0,1])$, without loss of generality), i.e., for $p \geq 1$,

$$L(s) = \sum_{i=1}^{p} \mu_i \varphi_i(s), \quad s \geq 0. \tag{5.2.2}$$

Let us focus on the basic properties of the solution to (5.2.1). As in the case when the noise is a fractional Brownian motion, it follows with the arguments given in Section 2.1.3 that (5.2.1) admits a unique strong solution which can be written as

$$X_t = e^{-\alpha t}\left(\int_0^t e^{\alpha s} L(s) ds + \int_0^t e^{\alpha s} \delta Z_s^H\right) =: h(t) + Y_t, \quad t \geq 0, \tag{5.2.3}$$

where we use the notation

$$h(t) = e^{-\alpha t}\int_0^t e^{\alpha s} L(s) ds \quad \text{and} \quad Y_t = e^{-\alpha t}\int_0^t e^{\alpha s} \delta Z_s^H \tag{5.2.4}$$

for every $t \geq 0$. The stochastic integral δZ_s^H in (5.2.3) is considered a Wiener integral with respect to the Rosenblatt process Z^H (coinciding, as in the fBm case, with the pathwise integral for deterministic integrands) and we will call the process $(X_t)_{t \geq 0}$ the Rosenblatt Ornstein-Uhlenbeck process with periodic mean. We can also define the so-called stationary Rosenblatt Ornstein-Uhlenbeck process with periodic mean by putting

$$\tilde{X}_t = e^{-\alpha t}\left(\int_{-\infty}^t e^{\alpha s} L(s) ds + \int_{-\infty}^t e^{\alpha s} \delta Z_s^H\right) =: \tilde{h}(t) + \tilde{Y}_t, \quad t \geq 0, \tag{5.2.5}$$

with

$$\tilde{h}(t) = e^{-\alpha t}\int_{-\infty}^t e^{\alpha s} L(s) ds \text{ and } \tilde{Y}_t = e^{-\alpha t}\int_{-\infty}^t e^{\alpha s} \delta Z_s^H. \tag{5.2.6}$$

The existence of the stochastic integrals in (5.2.3) and (5.2.5) is shown in e.g. [14] or [45]. We also recall the correlation structure of the process $\tilde{Y}$ (see [14] or [45]): for every $t \geq 0$ and for $s \to \infty$ we have with $c_H \in \mathbb{R}$

$$\mathbb{E}\, \tilde{Y}_t \tilde{Y}_{t+s} = c_H s^{2H-2} + O(s^{2H-4}). \tag{5.2.7}$$

We will start by proving some ergodic type properties of the process X. These properties will be needed in order to analyze the asymptotic properties of our estimators in the sequel.

5.2.1 Proposition. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a bounded 1-periodic function and let $(\tilde{Y}_t)_{t\geq 0}$ be given by (5.2.6). Then*

$$\frac{1}{n}\int_0^n \varphi(t)\tilde{Y}_t dt \overset{n\to\infty}{\to} 0 \text{ almost surely.}$$

Proof. We have for every $n \geq 1$

$$\mathbb{E}\left[\left(\frac{1}{n}\int_0^n \varphi(t)\tilde{Y}_t dt\right)^2\right] = \frac{1}{n^2}\int_0^n\int_0^n \varphi(t)\varphi(s)\,\mathbb{E}[\tilde{Y}_t\tilde{Y}_s]dtds.$$

First notice that for every integer $n_0 < n$ we have

$$\frac{1}{n^2}\int_{[0,n]^2\setminus[n_0,n]^2} \varphi(t)\varphi(s)\,\mathbb{E}[\tilde{Y}_t\tilde{Y}_s]dtds \overset{n\to\infty}{\to} 0. \tag{5.2.8}$$

Indeed, we can write

$$\begin{aligned} & \frac{1}{n^2}\int_{[0,n]^2\setminus[n_0,n]^2} \varphi(t)\varphi(s)\,\mathbb{E}[\tilde{Y}_t\tilde{Y}_s]dtds \\ = & \frac{1}{n^2}\int_0^{n_0}\int_0^{n_0} \varphi(t)\varphi(s)\,\mathbb{E}[\tilde{Y}_t\tilde{Y}_s]dtds + 2\frac{1}{n^2}\int_0^{n_0}\int_{n_0}^{n} \varphi(t)\varphi(s)\,\mathbb{E}[\tilde{Y}_t\tilde{Y}_s]dtds \\ \lesssim & \frac{1}{n^2}\int_0^{n_0}\int_{n_0}^{n} |\varphi(t)\varphi(s)|(\tilde{Y}_t^2+\tilde{Y}_s^2)dtds \leq cn^{-1}, \end{aligned}$$

where we used $\mathbb{E}\,Y_t^2 \leq c$ for every $t \geq 0$ (see relation (2.16) in [56]). We obtain by (5.2.8) and the periodicity of φ

$$\begin{aligned} \mathbb{E}\left[\left(\frac{1}{n}\int_0^n \varphi(t)\tilde{Y}_t dt\right)^2\right] & \lesssim \frac{1}{n^2}\int_{n_0}^n\int_{n_0}^n |\varphi(t)\varphi(s)||t-s|^{2H-2}dtds \\ & \lesssim \frac{1}{n^2}\int_0^n\int_0^n |\varphi(t)\varphi(s)||t-s|^{2H-2}dtds \\ & \lesssim \frac{1}{n^2}\sum_{i,j=0}^{n-1}\int_0^1\int_0^1 |\varphi(t)\varphi(s)||t-s-(i-j)|^{2H-2}dtds \\ & \lesssim \frac{1}{n^2}\sum_{i,j=0;|i-j|<2}^{n-1}\int_0^1\int_0^1 |\varphi(t)\varphi(s)||(i-j)-(t-s)|^{2H-2}dtds \\ & \quad +2\frac{1}{n^2}\sum_{i,j=0;i-j\geq 2}^{n-1}\int_0^1\int_0^1 |\varphi(t)\varphi(s)|((i-j)-(t-s))^{2H-2}dtds. \end{aligned}$$

Moreover,

$$\begin{aligned}
&\frac{1}{n^2}\sum_{i,\,j=0;|i-j|<2}^{n-1}\int_0^1\int_0^1|\varphi(t)\varphi(s)||(i-j)-(t-s)|^{2H-2}dtds\\
\lesssim\; &\frac{1}{n^2}n\max\left(\int_0^1\int_0^1|\varphi(t)\varphi(s)||t-s|^{2H-2}dtds,\,\int_0^1\int_0^1|\varphi(t)\varphi(s)||1-(t-s)|^{2H-2}dtds\right).
\end{aligned}$$

Because φ is bounded and $H>\frac{1}{2}$, the two integrals above are finite and then the summand converges to zero as $n\to\infty$.

For the second summand note that

$$((i-j)-(t-s))^{2H-2}=\left(1-\frac{t-s}{i-j}\right)^{2H-2}(i-j)^{2H-2},$$

and since for $i-j\geq 2$ we have $1-\frac{t-s}{i-j}\geq\frac{1}{2}$, we deduce that this summand is bounded by

$$\frac{1}{n^2}\sum_{i,\,j=0;|i-j|\geq 2}^{n-1}\int_0^1\int_0^1|\varphi(t)\varphi(s)||i-j|^{2H-2}dtds$$

up to a constant. In total, we have

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{1}{n}\int_0^n\varphi(t)\tilde{Y}_tdt\right)^2\right] &\lesssim \frac{1}{n^2}\sum_{i,\,j=0;|i-j|\geq 2}^{n-1}\int_0^1\int_0^1|\varphi(t)\varphi(s)||i-j|^{2H-2}dtds\\
&\lesssim \|\varphi\|^2_{L^2([0,\,1])}\frac{1}{n^2}\sum_{i,\,j=0}^{n-1}|i-j|^{2H-2}\lesssim n^{2H-2}.
\end{aligned}$$

Since $\tilde{Y}_t$ is a second Wiener chaos element, then so is the integral $\int_0^n\varphi(t)\tilde{Y}_tdt$, because it is an element of the L^2-closure of $(\varphi(t)\tilde{Y}_t)_{t\in\mathbb{R}^+}$: It follows by Fubini's theorem that it is orthogonal to every Z in the orthogonal complement of this closure. Therefore, due to the hypercontractivity property (2.1.6) we obtain the bound

$$\mathbb{E}\left[\left(\frac{1}{n}\int_0^n\varphi(t)\tilde{Y}_tdt\right)^{2m}\right]\lesssim n^{m(2H-2)}.$$

We can choose an $m\in\mathbb{N}$ big enough, depending on H, such that the statement follows by the usual Borel-Cantelli argument. □

As a consequence of Proposition 5.2.1, we can deduce a discrete ergodic property for the shifted process $\tilde{X}$.

5.2.2 Corollary. *For every $n\geq 1$, define the process $\mathbf{Y}_n:=\{\tilde{Y}_{n+s},\,s\in[0,\,1]\}$. Then $\mathbf{Y}$ satisfies the following discrete ergodic property*

$$\frac{1}{n}\sum_{i=0}^{n-1}\int_0^1\varphi(t)\mathbf{Y}_i(t)dt\overset{n\to\infty}{\to}0\ \textit{almost surely.}$$

Moreover, the process $\mathbf{X}_n := \{\tilde{X}_{n+s},\ s \in [0,1]\}$ *(*$n \in \mathbb{N}$*) also satisfies the discrete ergodic property, i.e.,*

$$\frac{1}{n}\sum_{i=0}^{n-1}\int_0^1 \varphi(t)\mathbf{X}_i(t)dt \overset{n\to\infty}{\to} \int_0^1 \varphi(t)\tilde{h}(t)dt \text{ almost surely.}$$

Proof. For $\mathbf{Y}_n$ the conclusion follows since

$$\frac{1}{n}\sum_{i=0}^{n-1}\int_0^1 \varphi(t)\mathbf{Y}_i(t)dt = \frac{1}{n}\sum_{i=0}^{n-1}\int_i^{i+1} \varphi(t)\tilde{Y}_t dt = \frac{1}{n}\int_0^n \varphi(t)\tilde{Y}_t dt,$$

while for $\mathbf{X}_n$ we simply use the fact that $\tilde{h}$ is 1-periodic. □

5.3 The least squares estimator

We will analyze the least squares estimator for the parameters of the model (5.2.1), inspired by the construction in [27] and [24]. In the first part we adapt the definition given in Section 2.2.2. and derive some of its basic properties and in the second part we study its consistency and its limit behavior in distribution.

5.3.1 Definition and basic properties

Our purpose is to estimate the $(p+1)$-dimensional parameter

$$\vartheta = (\mu_1, ..., \mu_p, \alpha) \tag{5.3.1}$$

where $\mu_i,\ i = 1, .., p$, are the coefficients that appear in the definition of the periodic function L (see formula (5.2.2)) while α is the drift parameter of the ROU process (5.2.1). We will construct a least squares estimator (LSE) to estimate ϑ. Following the construction explained in Section 2.2.2, we are led to the following estimator

$$\hat{\vartheta}_n := (\hat{\mu}_n^1, \dots, \hat{\mu}_n^p,\ \hat{\alpha}_n) := Q_n^{-1}P_n, \tag{5.3.2}$$

with the $(p+1)$- dimensional random vector P_n given by

$$P_n := \left(\int_0^n \varphi_1(t)\delta X_t, \dots, \int_0^n \varphi_p(t)\delta X_t, -\int_0^n X_t \delta X_t\right)^T \tag{5.3.3}$$

and with the matrix $Q_n \in M_{p+1}(\mathbb{R})$

$$Q_n := \begin{pmatrix} nId_p & -a_n \\ -a_n^T & b_n \end{pmatrix} \tag{5.3.4}$$

where

$$a_n^T := \left(\int_0^n \varphi_1(t)X_t dt \dots, \int_0^n \varphi_p(t)X_t dt\right)$$

and

$$b_n := \int_0^n X_t^2 dt.$$

Note that in the definition of the estimator $\hat{\vartheta}_n$ (5.3.2) stochastic integrals with respect to X appear. This integral is understood in the following sense (similarly to the integrals in the fBm setting defined in Section 2.1.2)

$$\int_0^t g_s \delta X_s := \int_0^t g_s \left(L(s) - \alpha X_s\right) ds + \int_0^t g_s \delta Z_s^H \tag{5.3.5}$$

for every $t \geq 0$, where the second integral is a Skorokhod integral with respect to the Rosenblatt process (see Section 5.1), provided that the integrals above exist. We need to chose a Skorokhod and not a pathwise integral with respect to the Rosenblatt process because, similarly to the explanation for the fBm given in e.g. [30], the choice of the pathwise integrals (which can be easily defined for the Rosenblatt process since it has Hölder continuous paths or every order $\delta \in (0, H)$, cf. Section 2.1.2) does not lead to a consistent estimator.

First, we need to argue that the stochastic integrals that appear in (5.3.3) and (5.3.4) are well-defined. The Wiener integrals $\int_0^t \varphi_i(s)\delta Z_s^H$ are obviously well-defined since $\varphi_i, i = 1, .., p$ are bounded and periodic. In the next result we show that the Skorokhod integral in (5.3.3) is also well-defined.

5.3.1 Proposition. *Let $(X_t)_{t\geq 0}$ be the solution to (5.2.1). Then for every $t \geq 0$ the Skorohod integral $\int_0^t X_s \delta Z_s^H$ is well-defined.*

Proof. From relation (5.1.5) in Section 5.1 we need to show that

$$\mathbb{E}\int_0^T \int_0^T \|D_{x_1,x_2} X\|^2_{\mathscr{H}^H} dx_1 dx_2 < \infty.$$

By taking the Malliavin derivative in (5.2.3), we get for every $x_1, x_2 > 0$

$$\begin{aligned} D_{x_1 x_2} X_u &= 2d(H)\mathbf{1}_{[0,\,u]^2}(x_1,\, x_2) I(e^{\alpha(\cdot - u)})(x_1,\, x_2) \\ &= 2d(H)\mathbf{1}_{[0,\,u]^2}(x_1,\, x_2) \int_{x_1\vee x_2}^u e^{\alpha(u'-u)} \frac{\partial K^{H'}}{\partial u'}(u',\, x_1)\frac{\partial K^{H'}}{\partial u'}(u',\, x_2) du', \end{aligned}$$

where I is the transfer operator (5.1.4). Hence,

$$\begin{aligned} \|D_{x_1 x_2} X\|^2_{\mathscr{H}^H} &= \int_{x_1\vee x_2}^n \int_{x_1\vee x_2}^n \int_{x_1\vee x_2}^u e^{\alpha(u'-u)} \frac{\partial K^{H'}}{\partial u'}(u',\, x_1)\frac{\partial K^{H'}}{\partial u'}(u',\, x_2) du' \\ &\times \int_{x_1\vee x_2}^v e^{\alpha(v'-v)} \frac{\partial K^{H'}}{\partial v'}(v',\, x_1)\frac{\partial K^{H'}}{\partial v'}(v',\, x_2) dv' |u-v|^{2H-2} du dv \\ &\leq \int_{x_1\vee x_2}^n \int_{x_1\vee x_2}^n |u-v|^{2H-2} \\ &\times \int_{x_1\vee x_2}^u \frac{\partial K^{H'}}{\partial u'}(u',\, x_1)\frac{\partial K^{H'}}{\partial u'}(u',\, x_2) du' \int_{x_1\vee x_2}^v \frac{\partial K^{H'}}{\partial v'}(v',\, x_1)\frac{\partial K^{H'}}{\partial v'}(v',\, x_2) dv' du dv \\ &= \|D_{x_1 x_2} Z^H\|^2_{\mathscr{H}^H}, \end{aligned}$$

since $e^{\alpha(u'-u)} \leq 1$ and the other integrands are nonnegative. From Example 1 in [78] we know that $\mathbb{E}[\int_0^n \int_0^n \|D_{x_1x_2}Z\|^2_{\mathscr{H}^H} dx_1 dx_2] < \infty$, and the result follows. □

In the sequel, we will need a more convenient expression of the estimator (5.3.2). Note that the inverse of the matrix Q_n can be expressed as (see [24])

$$Q_n^{-1} = \frac{1}{n} \begin{pmatrix} Id_p + \gamma_n \Lambda_n \Lambda_n^t & -\gamma_n \Lambda_n \\ -\gamma_n \Lambda_n^t & \gamma_n \end{pmatrix} \tag{5.3.6}$$

with

$$\Lambda_n = (\Lambda_{n,1}, \ldots, \Lambda_{n,p})^T = \left(\frac{1}{n} \int_0^n \varphi_1(t) X_t dt, \ldots, \frac{1}{n} \int_0^n \varphi_p(t) X_t dt \right) \tag{5.3.7}$$

and

$$\gamma_n = \left(\frac{1}{n} \int_0^n X_t^2 dt - \sum_{i=1}^p \Lambda_{n,i}^2 \right)^{-1}. \tag{5.3.8}$$

Another useful fact is that we can (also in a similar way to [24]) deduce a different expression for $\hat{\vartheta}_n$ which allows to access the error $\hat{\vartheta}_n - \vartheta$ directly.

5.3.2 Proposition. *The estimator $\hat{\vartheta}_n$ (5.3.2) has the following representation:*

$$\hat{\vartheta}_n = \vartheta + Q_n^{-1} R_n \tag{5.3.9}$$

with Q_n given by (5.3.4) and

$$R_n = \left(\int_0^n \varphi_1(t) \delta Z_t^H, \ldots, \int_0^n \varphi_p(t) \delta Z_t^H, -\int_0^n X_t \delta Z_t^H \right). \tag{5.3.10}$$

Proof. This follows easily if the relation $X_t = \int_0^t (L(s) - \alpha X_s)\, ds + Z_t^H$ is plugged as the integrator in each component of P_n (5.3.3). □

The relation (5.3.9) will be used in order to study the asymptotic behaviour of the LSE.

5.3.2 Strong consistency

We study the asymptotic properties of the LSE (5.3.2). In this part we prove that $\hat{\vartheta}_n$ is strongly consistent, i.e. it converges almost surely to the parameter ϑ (5.3.1) as $n \to \infty$. In order to prove the estimator's consistency we will need several auxiliary results. First, we quote a technical lemma from [39].

5.3.3 Lemma. *Let $\gamma > 0$ and $p_0 \in \mathbb{N}$. Moreover, let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of random variables. If for every $p \geq p_0$ there exists a constant $c_p > 0$ such that for all $n \in \mathbb{N}$*

$$(\mathbb{E}[|Z_n|^p])^{1/p} \leq c_p n^{-\gamma},$$

then for all $\varepsilon > 0$ there exists a random variable η_ε such that

$$|Z_n| \leq \eta_\varepsilon n^{-\gamma+\varepsilon} \ a.s.$$

for all $n \in \mathbb{N}$. Moreover, $\mathbb{E}[|\eta_\varepsilon|^p] < \infty$ for all $p \geq 1$.

To show strong consistency of the estimator (5.3.2), we will treat the quantities $\frac{1}{n}R_n$ and nQ_n^{-1} separately, as in [24] and [7].

5.3.4 Proposition. *Let R_n be given by (5.3.10). Then, as n tends to infinity, $\frac{1}{n}R_n \to 0$ almost surely.*

Proof. Due to (5.1.6) it suffices to demonstrate that

$$\sup_n \sup_{r\in[0,\,n]} (\mathbb{E}[|g_r|^{p^*}] + \mathbb{E}[\|D^{(2)}g_r\|^{p^*}_{L^2([0,\,n]^2)}]) < \infty \tag{5.3.11}$$

for $g = \varphi_i$ $(i = 1, \ldots, p)$ and for $g = X$ for all $p^* \in \mathbb{N}$. Then the result will follow by taking $\gamma = 1 - H$ in Lemma 5.3.3. Since by assumption all φ_i are bounded, the statement for $g = \varphi_i$ $(i = 1, \ldots, p)$ is immediate. For $g = X$ recall that

$$X_t = \int_0^t e^{\alpha(s-t)}L(s)ds + \int_0^t e^{\alpha(s-t)}\delta Z_s^H.$$

Using the fact that L is bounded, we clearly have

$$\int_0^t e^{\alpha(s-t)}L(s)ds \leq \|L\|_\infty \int_0^t e^{\alpha(s-t)}ds = \frac{1}{\alpha}\|L\|_\infty e^{-\alpha t}(e^{\alpha t} - 1) \leq \frac{1}{\alpha}\|L\|_\infty,$$

and by the triangle inequality it is enough to prove the inequality (5.1.6) for the random part of X, i.e. for $g_t = \int_0^t e^{\alpha(s-t)}dZ_s^H = Y_t$ (see (5.2.4)). We write for every $r > 0$

$$\mathbb{E}[|Y_r|^{p^*}] + \mathbb{E}[\|D^{(2)}Y_r\|^{p^*}_{L^2([0,\,n]^2)}] =: N_{1,r} + N_{2,r}.$$

For the term $N_{1,r}$ we note that since Y_r is a multiple Wiener-Itō integral of order two with respect to a Brownian motion, the hypercontractivity property (2.1.6) is applicable, yielding the inequality

$$\mathbb{E}[|Y_r|^{p^*}]^{1/p^*} \leq (p^* - 1)\,\mathbb{E}[|Y_r|^2]^{1/2}.$$

Therefore, since the above constant does not depend on the underlying space, it suffices to show boundedness of the L^2-norm. Due to isometry property of Wiener-Rosenblatt integrals (5.1) we have

$$\begin{aligned}\mathbb{E}[|Y_r|^2] &= \int_0^r \int_0^r e^{-2\alpha r}e^{\alpha u}e^{\alpha v}|u - v|^{2H-2}dudv \\ &= \int_0^r \int_0^r e^{-\alpha u}e^{-\alpha v}|u - v|^{2H-2}dudv\end{aligned}$$

and clearly $\sup_{r\in[0,t]} \mathbb{E}[|Y_r|^2] < C$ for every $t \geq 0$ with some $C > 0$. Concerning the summand $N_{2,r}$ we recall that

$$D_{x_1x_2} Y_r = 2d(H)\mathbf{1}_{[0,\,r]^2}(x_1,\, x_2) I(e^{\alpha(\cdot - r)})(x_1,\, x_2).$$

Since it is nonrandom, it is enough to prove the boundedness of $\|D^{(2)}Y_r\|^2_{L^2([0,\,n]^2)}$. We have, with I given by (5.1.4),

$$\begin{aligned}
\|D^{(2)}Y_r\|^2_{L^2([0,\,n]^2)} &= \int_0^n \int_0^n (2d(H)\mathbf{1}_{[0,\,r]^2}(x_1,\, x_2) I(e^{\alpha(\cdot - r)})(x_1,\, x_2))^2 dx_1 dx_2 \\
&= 4d(H)^2 \int_0^r \int_0^r (I(e^{\alpha(\cdot - r)})(x_1,\, x_2))^2 dx_1 dx_2 \\
&= 4d(H)^2 \|I(e^{\alpha(\cdot - r)})(x_1,\, x_2)\|^2_{L^2([0,\,r]^2)} \\
&= d(H)^2\, \mathbb{E}[I_2(I(e^{\alpha(\cdot - r)})(x_1,\, x_2))^2] = d(H)^2\, \mathbb{E}[Y_r^2]
\end{aligned}$$

due to isometry of the Wiener-Itō integrals (2.1.2). As shown above, the obtained expression is bounded by a constant independent of r and of n. Thus, our claim (5.3.11) is proved. □

The next step is the almost sure convergence of the matrix nQ_n^{-1}. The proof is similar to the one given in [24] for the case of the fractional Brownian motion.

5.3.5 Proposition. *Let Q_n be defined by (5.3.4). As n tends to infinity, nQ_n^{-1} tends almost surely to the deterministic matrix*

$$Q := \begin{pmatrix} Id_p + \gamma\Lambda\Lambda^T & -\gamma\Lambda \\ -\gamma\Lambda^T & \gamma \end{pmatrix}, \tag{5.3.12}$$

where

$$\Lambda = (\Lambda_1, ..., \Lambda_p) \text{ and } \Lambda_i := \langle \varphi_i, \tilde{h}(t)\rangle_{L^2[0,1]}, i = 1, .., p, \tag{5.3.13}$$

with $\tilde{h}$ from (5.2.6) and

$$\gamma := \left(\int_0^1 \tilde{h}^2(t)dt + \alpha^{-2H} H\Gamma(2H) - \sum_{i=1}^p \Lambda_i^2 \right)^{-1}. \tag{5.3.14}$$

Proof. We will use the expression (5.3.6) of the matrix Q_n^{-1}. From this formula it suffices to prove almost sure convergence of the quantities $\Lambda_{n,\,i}$ from (5.3.7) to the constant Λ_i given by (5.3.13) for every $i \in \{1, \ldots, p\}$ as well as almost sure convergence of γ_n^{-1} to the nonzero real number γ^{-1} from (5.3.14). Concerning $\Lambda_{n,\,i}$ using the fact that the difference

$$|Y_t - \tilde{Y}_t| = e^{-\alpha t} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{-\infty}^0 e^{\alpha u} \frac{\partial K^{H'}}{\partial u}(u,\, x_1) \frac{\partial K^{H'}}{\partial u}(u,\, x_2) du \delta B(x_1) \delta B(x_2) \right|$$

converges to zero almost surely as $t \to \infty$ (and the same holds true for $|X_t - \tilde{X}_t|$), we obtain almost surely via Corollary 5.2.2

$$\begin{aligned}\lim_{n\to\infty} \Lambda_{n,\,i} &= \lim_{n\to\infty} \frac{1}{n}\int_0^n \varphi_i(t) X_t dt = \lim_{n\to\infty} \frac{1}{n}\int_0^n \varphi_i(t)\tilde{X}_t dt \\ &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1}\int_i^{i+1} \varphi_i(t)\tilde{X}_t dt = \int_0^1 \varphi_i(t)\,\mathbb{E}[\tilde{X}_t]dt = \int_0^1 \varphi_i(t)\tilde{h}(t)dt = \Lambda_i\end{aligned}$$

for every $i = 1, .., p$. Concerning γ_n^{-1} we have from (5.3.8)

$$\frac{1}{n}\int_0^n X_t^2 dt \;=\; \frac{1}{n}\int_0^n h(t)^2 dt + \frac{2}{n}\int_0^n h(s) Y_s ds + \frac{1}{n}\int_0^n Y_s^2 ds.$$

Since $|h(t) - \tilde{h}(t)| = e^{-\alpha t}|\int_{-\infty}^0 e^{\alpha s} L(s) ds|$, we conclude that the first integral converges to $\int_0^1 \tilde{h}^2(t)dt$. For the second integral note that due to boundedness of $\frac{1}{n}\int_0^n Y_s ds$ (shown in [56]) and of $|\frac{1}{n}\int_0^n h(t)dt|$ we obtain almost surely

$$\lim_{n\to\infty} \frac{2}{n}\int_0^n h(s) Y_s ds = \lim_{n\to\infty} \frac{2}{n}\int_0^n \tilde{h}(s)\tilde{Y}_s ds = 0$$

by applying Proposition 5.2.1. The almost sure limit of the third integral equals $\alpha^{-2H} H\Gamma(2H)$, as demonstrated in [56]. So almost surely

$$\gamma_n^{-1} = \frac{1}{n}\int_0^n X_t^2 dt - \sum_{i=1}^p \Lambda_{n,\,i}^2 \overset{n\to\infty}{\to} \|\tilde{h}\|^2_{L^2([0,\,1])} - \sum_{i=1}^p \langle \tilde{h},\, \varphi_i\rangle^2_{L^2([0,\,1])} + \alpha^{-2H} H\Gamma(2H)$$

and by Bessel's inequality we can see as in [24] that the above limit is indeed a positive real number. □

As a consequence of Propositions 5.3.2, 5.3.4 and 5.3.5 we obtain the strong consistency of the least squares estimator.

5.3.6 Theorem. *As $n \to \infty$, the LSE (5.3.2) converges almost surely to the parameter $\vartheta = (\mu_1, ..., \mu_p, \alpha)$.*

5.4 Limit distribution of the least squares estimator

We will analyse the asymptotic behaviour in distribution of the LSE. We use the decomposition of $\hat{\vartheta}_n$ given in Proposition 5.3.2. It follows from this result, since the random matrix nQ_n^{-1} given by (5.3.4) converges almost surely to the deterministic matrix Q from Proposition 5.3.5, then it is enough to consider the asymptotics of the vector R_n in (5.3.10).

We start with a result concerning the first p components of the vector (5.3.10). In the sequel, by a Rosenblatt random variable we mean a random variable with the same law as Z_1^H from (5.1.2).

5.4.1 Proposition. *For every $n \geq 1$, consider $U_n := n^{-H} \int_0^n f(s)\delta Z_s^H$ for a bounded 1-periodic function f. As n tends to infinity, this sequence converges in distribution to $U = \left(\int_0^1 f(t)dt\right) V$, where V is a Rosenblatt random variable.*

Proof. It follows by the scaling property of the Rosenblatt process (see [78]) that $U_n \stackrel{d}{\equiv} \int_0^1 f(ns)\delta Z_s^H$, where $\stackrel{d}{\equiv}$ stands for the equivalence of finite dimensional distributions. We will show that this sequence converges in L^2 to the random variable $\left(\int_0^1 f(t)dt\right) Z_1^H$. We can write

$$\begin{aligned}
&\mathbb{E}\left[\int_0^1 f(ns)\delta Z_s^H - \left(\int_0^1 f(s)ds\right) Z_1^H\right]^2 = \mathbb{E}\left[\int_0^1 \left(f(ns) - \int_0^1 f(r)dr\right) \delta Z_s^H\right]^2 \\
= \;& H(2H-1)\int_0^1\int_0^1 f(nu)f(nv)|u-v|^{2H-2}dudv + \left(\int_0^1 f(s)ds\right)^2 \\
& -2H(2H-1)\int_0^1\int_0^1 f(nu)|u-v|^{2H-2}dudv \int_0^1 f(s)ds.
\end{aligned}$$

First,

$$\begin{aligned}
& H(2H-1)\int_0^1\int_0^1 f(nu)f(nv)|u-v|^{2H-2}dudv \\
= \;& H(2H-1)n^{-2H}\int_0^n\int_0^n f(nu)f(nv)|u-v|^{2H-2}dudv \\
= \;& n^{-2H}H(2H-1)\sum_{i,j=0}^{n-1}\int_0^1\int_0^1 f(u)f(v)|u-v+i-j|^{2H-2}dudv \\
\sim \;& n^{-2H}H(2H-1)\sum_{i,j=0,\, i\neq j}^{n-1}\int_0^1\int_0^1 f(u)f(v)|i-j|^{2H-2}\left|1+\frac{u-v}{i-j}\right|^{2H-2} dudv \\
\sim \;& n^{-2H}H(2H-1)\sum_{i,j=0; i\neq j}^{n-1} |i-j|^{2H-2}\left(\int_0^1 f(s)ds\right)^2 \stackrel{n\to\infty}{\to} \left(\int_0^1 f(s)ds\right)^2.
\end{aligned}$$

The equivalence is obtained by considering the binomial expansion of $\left|1+\frac{u-v}{i-j}\right|^{2H-2}$. On the other hand,

$$\begin{aligned}
& H(2H-1)\int_0^1\int_0^1 f(nu)|u-v|^{2H-2}dudv \\
= \;& H(2H-1)\left(\int_0^1 f(nu)du\int_0^u (u-v)^{2H-2}dv + \int_0^1\int_v^1 f(nu)du(v-u)^{2H-2}dv\right) \\
= \;& H\int_0^1 f(nu)u^{2H-1}du + H\int_0^1 f(nu)(1-u)^{2H-1}du.
\end{aligned}$$

Now, again by the binomial expansion,

$$
\begin{aligned}
& H\int_0^1 f(nu)u^{2H-1}du = Hn^{-2H}\int_0^n f(u)u^{2H-1}du \\
= \;& Hn^{-2H}\sum_{i=0}^{n-1}\int_0^1 f(u)(u+i)^{2H-1}du \sim Hn^{-2H}\int_0^1 f(u)\sum_{i=0}^{n-1}(u+i)^{2H-1}du \\
\sim \;& Hn^{-2H}\int_0^1 f(u)\sum_{i=1}^{n-1} i^{2H-1}\left(1+\frac{u}{i}\right)^{2H-1}du \sim Hn^{-2H}\int_0^1 f(u)du\frac{n^{2H}}{2H} \\
= \;& \frac{1}{2}\int_0^1 f(u)du.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
& H\int_0^1 f(nu)(1-u)^{2H-1}du = H\int_0^1 f(n(1-u))u^{2H-1}du \\
= \;& H\int_0^1 f(-nu)u^{2H-1}du \overset{n\to\infty}{\to} \frac{1}{2}\int_0^1 f(-u)du = \frac{1}{2}\int_0^1 f(u)du
\end{aligned}
$$

with the same argument as above. This gives the desired L^2-convergence. □

Now let us consider the last component of the vector R_n in (5.3.10). First we show that the stochastic integral part does not contribute to the limit.

5.4.2 Proposition. *Let $(Y_t)_{t\geq 0}$ be given by (5.2.4). Then, as n tends to infinity,*

$$
\mathbb{E}\left(n^{-H}\int_0^n Y_t\delta Z_t^H\right)^2 \to 0.
$$

Proof. Let us estimate the L^2-norm of the random variable $n^{-H}\int_0^n Y_t dZ_t^H$ with Y from (5.2.4). In [78] the following bound is given:

$$
\begin{aligned}
\mathbb{E}\;\left(\int_0^n Y_t\delta Z_t^H\right)^2 \leq C\Bigg(&\mathbb{E}\left[\int_0^n\int_0^n Y_uY_v|u-v|^{2H-2}dudv\right] \\
&+\mathbb{E}\left[\int_0^n\int_0^n\int_0^n\int_0^n D_{x_1,x_2}Y_uD_{x_1,x_2}Y_v|u-v|^{2H-2}dudvdx_1dx_2\right]\Bigg).
\end{aligned}
$$

Since Y_u is a double integral, it is easy to note that the two summands above only differ by a constant, so it is enough to consider one of them. We obtain using the isometry for the Rosenblatt process

$$
\begin{aligned}
& \mathbb{E}\left[\int_0^n\int_0^n Y_uY_v|u-v|^{2H-2}dudv\right] \\
= \;& \int_0^n\int_0^n\int_0^u\int_0^v e^{\alpha(s-u)}e^{\alpha(r-v)}|r-s|^{2H-2}drds|u-v|^{2H-2}dudv \\
\leq \;& \int_0^n\int_0^n\int_0^n\int_0^n e^{-\alpha|s-u|}e^{-\alpha|r-v|}|r-s|^{2H-2}|u-v|^{2H-2}drdsdudv,
\end{aligned}
$$

and it is demonstrated in [30] and [31] that this bound multiplied by $\frac{1}{n}$, $\frac{1}{n\log(n)}$ or n^{2-4H} in cases $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$, $H = \frac{3}{4}$ and $H > \frac{3}{4}$ respectively converges to a constant. Thus, the statement follows. □

The next proposition concludes the asymptotic analysis.

5.4.3 Proposition. *Let $(Y_t)_{t\geq 0}$ be given by (5.2.3). The sequence $n^{-H}\int_0^n X_t \delta Z_t^H$ converges in distribution to $U = \left(\int_0^1 \tilde{h}(t)dt\right) V$, where V is a Rosenblatt random variable.*

Proof. Recall that for every $t \geq 0$, $X_t = Y_t + h(t)$, see (5.2.4), so we need to analyse the limit of $n^{-H}\int_0^n h(t)\delta Z_t^H$. Since $\tilde{h}$ from (5.2.6) is a periodic function, it suffices to demonstrate that $n^{-H}\int_0^n (h(t) - \tilde{h}(t))\delta Z_t^H$ converges to zero in L^2 and then to apply Proposition 5.4.1. Since $|h(t) - \tilde{h}(t)|$ is bounded by $e^{-\alpha t}$ times a constant, we get by the isometry property (2.1.2)

$$\mathbb{E}\left[\left(\int_0^n (h(t) - \tilde{h}(t))\delta Z_t^H\right)^2\right] \leq c\int_0^n \int_0^n e^{-\alpha u} e^{-\alpha v}|u - v|^{2H-2} dudv,$$

for some positive constant c. The right hand side is bounded uniformly in n, and the desired convergence follows. □

By putting together the above results we state and prove the main result of this section.

5.4.4 Theorem. *Let $\hat{\vartheta}_n$ be given by (5.3.2). Then the sequence $n^{1-H}\left(\hat{\vartheta}_n - \vartheta\right)$ converges in distribution as $n \to \infty$ to QR where the matrix Q is given by (5.3.12) and R is the following random vector*

$$R = \left(\int_0^1 \varphi_1(s)ds, ..., \int_0^1 \varphi_p(s)ds, -\int_0^1 \tilde{h}_s ds\right)^T V,$$

where V is a Rosenblatt random variable (i.e. $V \stackrel{d}{=} Z_1^H$) and $\tilde{h}$ is defined by (5.2.6).

Proof. The almost sure convergence of nQ_n^{-1} to the matrix Q follows from Proposition 5.3.5 an we need to prove the asymptotic behaviour in distribution of the vector $\frac{1}{n}R_n$ (5.3.10). For any $a_1, \ldots, a_{p+1} \in \mathbb{R}$ and for 1-periodic functions $f_1, \ldots, f_{p+1}$ we have

$$\sum_{i=1}^{p+1} a_i n^{-H}\int_0^n f_i(t)\delta Z_t^H = n^{-H}\int_0^n \sum_{i=1}^{p+1} a_i f_i(t)\delta Z_t^H,$$

and by Proposition 5.4.1 this converges in distribution as $n \to \infty$ to $U = (\int_0^1 \sum_{i=1}^{p+1} a_i f_i(t)dt)V$ (where V is a Rosenblatt random variable), because

$\sum_{i=1}^{p+1} a_i f_i$ is again a 1-periodic function. By applying the results to $f_i = \varphi_i, i = 1, .., p$ and $f_{p+1} = -\tilde{h}$ and by using the L^2 convergence from Proposition 5.4.2, we obtain the conclusion. □

Note that for functions φ_i, $i = 1, \ldots, p$, whose integrals are equal to zero one might obtain an improvement in the speed of convergence (similarly to the results obtained in Chapter 3). This case is, however, not treated here. Also, similarly to the fBm-case, the limit is of dimension one, however, here it is an element of the second Wiener chaos. From the point of view of statistical applications the limit is not as well-studied as the normal distribution, in particular, there is no known closed expression for the density of Z_1^H. However, in [82] an algorithm for approximating quantiles is given which makes statistical evaluations such as the construction of confidence intervals accessible.

5.5 Alternative estimators

The estimator ϑ_n (5.3.2), although consistent and with explicit limit distribution, involves a Skorokhod integral. It is well-known that it is difficult to simulate such a stochastic object. Therefore, we will define some alternative estimators that can be expressed only in terms of Wiener and Lebesgue integrals and consequently they can be simulated. One of these new estimators represents an extended version of the estimators proposed in [30] or [56] (mentioned in Section 2.2.2) as it reduces to them when the periodic drift L reduces to a constant.

Recall that the the functions φ_i from (5.2.2) are assumed to be orthogonal in $L^2([0, 1])$. We will consider the following assumptions, or cases (the function $\tilde{h}$ is defined in (5.2.6)):

(A1) $\tilde{h}$ does not belong to $\mathrm{span}(\varphi_1, \ldots, \varphi_p)$. In this case there exists a bounded function φ_{p+1} orthogonal to all φ_i ($i \in \{1, p\}$), but not orthogonal to $\tilde{h}$.

(A1*) $\tilde{h} \in \mathrm{span}(\varphi_1, \ldots, \varphi_p)$. Then there is no L^2 function satisfying the above orthogonality conditions.

We will show below in Remark 5.5.5 that in the case when $\varphi_i, i = 1, .., p$ are elements of the trigonometric basis of $L^2([0, 1])$ it is easy to check which one of these assumptions is satisfied and to determine the function φ_{p+1} without the knowledge of $\tilde{h}$ in case of (A1).

5.5.1 Proposition. *Assume that (A1) is satisfied. Define for every $n \geq 1$*

$$\bar{\alpha}_n := -\frac{\int_0^n \varphi_{p+1}(t) dX_t}{\int_0^n \varphi_{p+1}(t) X_t dt}$$

and for $i = 1, .., p$

$$\bar{\mu}_{i,n} := \frac{1}{n}\left(\int_0^n \varphi_i(t)dX_t + \bar{\alpha}_n \int_0^n \varphi_i(t)X_t dt\right).$$

Then $(\bar{\alpha}_n, \bar{\mu}_{1,n}, ..., \bar{\mu}_{p,n})$ *is a consistent estimator of the parameter* $(\alpha, \mu_1, .., \mu_p)$ *of the model (5.2.1).*

Proof. From (5.3.5) and (A1) we have

$$\frac{1}{n}\int_0^n \varphi_{p+1}(t)dX_t = -\alpha\frac{1}{n}\int_0^n \varphi_{p+1}(t)X_t dt + \frac{1}{n}\int_0^n \varphi_{p+1}(t)dZ_t^H$$

so we can write

$$\bar{\alpha}_n - \alpha = \frac{n^{-1}\int_0^n \varphi_{p+1}(t)dZ_t^H}{n^{-1}\int_0^n \varphi_{p+1}(t)X_t dt}. \tag{5.5.1}$$

As demonstrated in Proposition 5.3.4, the numerator of (5.5.1) converges to zero almost surely as $n \to \infty$. Moreover, we can conclude using Proposition 5.2.1 that

$$\Lambda_{n,\,p+1} := \frac{1}{n}\int_0^n \varphi_{p+1}(t)X_t dt \overset{n\to\infty}{\to} \langle \tilde{h},\, \varphi_{p+1}\rangle_{L^2([0,\,1])}$$

almost surely. Since this is nonzero by the assumption (A1), strong consistency of $\bar{\alpha}_n$ follows. Consistency of $\bar{\mu}_i$ follows by observing that

$$\frac{1}{n}\int_0^n \varphi_i(t)dX_t = \mu_i - \alpha\frac{1}{n}\int_0^n \varphi_i(t)X_t dt + \frac{1}{n}\int_0^n \varphi_i(t)dZ_t^H,$$

and this implies, for every $i = 1, .., p$

$$\bar{\mu}_{i,n} - \mu_i = \frac{1}{n}(\bar{\alpha}_n - \alpha)\int_0^n \varphi_i(t)X_t dt + \frac{1}{n}\int_0^n \varphi_i(t)dZ_t^H \tag{5.5.2}$$

and the last summand again converges to zero almost surely as $n \to \infty$ while $\frac{1}{n}\int_0^n \varphi_i(t)X_t dt$ tends to a constant. □

The asymptotic behaviour in distribution of the above estimators can be easily obtained from the proofs in Section 5.4.

5.5.2 Proposition. *As* n *tends to infinity the vector* $n^{1-H}(\bar{\alpha}_n - \alpha, \bar{\mu}_{1,n} - \mu_1, \ldots, \bar{\mu}_{p,n} - \mu_p)^T$ *converges in distribution to the vector*

$$\begin{pmatrix} \int_0^1 \varphi_{p+1}(t)dt\frac{1}{\langle \varphi_{p+1}, \tilde{h}\rangle_{L^2([0,\,1])}} \\ \int_0^1 \varphi_{p+1}(t)dt\frac{\langle \varphi_1, \tilde{h}\rangle_{L^2([0,\,1])}}{\langle \varphi_{p+1}, \tilde{h}\rangle_{L^2([0,\,1])}} + \int_0^1 \varphi_1(t)dt \\ \vdots \\ \int_0^1 \varphi_{p+1}(t)dt\frac{\langle \varphi_p, \tilde{h}\rangle_{L^2([0,\,1])}}{\langle \varphi_{p+1}, \tilde{h}\rangle_{L^2([0,\,1])}} + \int_0^1 \varphi_p(t)dt \end{pmatrix} V,$$

where V *is a Rosenblatt random variable.*

Proof. This follows by construction from relations (5.5.1), (5.5.2), Proposition 5.2.1 and the non-central limit theorem in Proposition 5.4.1. □

When the assumption (A1*) is satisfied, we can also define consistent estimators for the parameters of the model (5.2.1) which involve only Wiener and deterministic integrals.

5.5.3 Proposition. *Assume that (A1*) is satisfied. Consider the following estimators*

$$\bar{\alpha}_n^{(1)} := \left(\frac{1}{H\Gamma(2H)}\gamma_n^{-1}\right)^{-\frac{1}{2H}}$$

and for $i = 1, .., p$,

$$\bar{\mu}_{n,i}^{(1)} := \frac{1}{n}\left(\int_0^n \varphi_i(t)dX_t + \bar{\alpha}_n^{(1)} \int_0^n \varphi_i(t)X_t dt\right)$$

Then $\left(\bar{\alpha}_n^{(1)}, \bar{\mu}_{1,n}^{(1)}, ..., \bar{\mu}_{p,n}^{(1)}\right)$ is a strongly consistent estimator of the parameter (5.3.1).

Proof. It is shown in Proposition 5.3.5 that with γ_n defined in (5.3.8)

$$\gamma_n^{-1} \overset{n\to\infty}{\to} \|\tilde{h}\|_{L^2([0,1])} - \sum_{i=1}^{p} \langle \tilde{h}, \varphi_i \rangle^2_{L^2([0,1])} + \alpha^{-2H} H\Gamma(2H)$$

almost surely. Because (A1*) is satisfied, we obtain the equality $\|\tilde{h}\|_{L^2([0,1])} = \sum_{i=1}^{p} \langle \tilde{h}, \varphi_i \rangle^2_{L^2([0,1])}$, and thus consistency follows by the continuous mapping theorem. Consistency of the estimators of the μ_i is a direct consequence and can be shown similarly to the strong consistency in Proposition 5.5.1. □

Concerning the limit in law of $\left(\bar{\alpha}_n^{(1)}, \bar{\mu}_{1,n}^{(1)}, ..., \bar{\mu}_{p,n}^{(1)}\right)$, we have the following result.

5.5.4 Proposition. *As n tends to infinity the vector $n^{1-H}(\bar{\alpha}_n^{(1)} - \alpha, \bar{\mu}_{1,n}^{(1)} - \mu_1, \ldots, \bar{\mu}_{p,n}^{(1)} - \mu_p)^T$ converges in distribution to the vector*

$$C_\alpha G_\infty \begin{pmatrix} 1 \\ \langle \tilde{h}, \varphi_1 \rangle_{L^2([0,1])} \\ \vdots \\ \langle \tilde{h}, \varphi_p \rangle_{L^2([0,1])} \end{pmatrix} + Z_1^H \begin{pmatrix} 0 \\ \int_0^1 \varphi_1(t)dt \\ \vdots \\ \int_0^1 \varphi_p(t)dt \end{pmatrix},$$

where $C_\alpha = \frac{\alpha^H}{2H^2\Gamma(2H)}$ and $G_\infty = B_H \times R$ with R being $\sigma(Z^H)$-measurable and having a Rosenblatt distribution and B_H being defined as follows:

$$B_H = \frac{(2H-1)\Gamma(H+1)}{\sqrt{\frac{H}{2}(2H-1)}}.$$

Proof. Using a Taylor expansion we obtain for large n

$$\begin{aligned}\bar{\alpha}_n^{(1)} - \alpha &= \alpha\left(\left(1+\frac{\alpha^{2H}(\gamma_n^{-1}-\alpha^{-2H}H\Gamma(2H))}{H\Gamma(2H)}\right)^{-\frac{1}{2H}}-1\right)\\ &= \frac{\alpha^{2H+1}}{2H^2\Gamma(2H)}(\gamma_n^{-1}-\alpha^{-2H}H\Gamma(2H))+o(1).\end{aligned}$$

Therefore, it suffices to calculate the asymptotics of the quantity

$$\gamma_n^{-1}-\alpha^{-2H}H\Gamma(2H) = \frac{1}{n}\int_0^n X_t^2dt - \sum_{i=1}^p\left(\frac{1}{n}\int_0^n X_t\varphi_i(t)dt\right)^2 - \alpha^{-2H}H\Gamma(2H).$$

As in the previous computations (e.g. in the proof of Proposition 5.3.2), the above expression has the same limit in distribution, as $n\to\infty$, as

$$\begin{aligned}&\left(\frac{1}{n}\int_0^n \tilde{X}_t^2dt - \sum_{i=1}^p\left(\frac{1}{n}\int_0^n \tilde{X}_t\varphi_i(t)dt\right)^2 - \alpha^{-2H}H\Gamma(2H)\right)\\ =\ & \frac{1}{n}\int_0^n \tilde{Y}_t^2dt - \frac{2}{n}\int_0^n \tilde{Y}_t\tilde{h}(t)dt + \frac{1}{n}\int_0^n \tilde{h}(t)^2dt - \alpha^{-2H}H\Gamma(2H) - \sum_{i=1}^p\left(\frac{1}{n}\int_0^n \tilde{Y}_t\varphi_i(t)dt\right)^2\\ &+2\sum_{i=1}^p\left(\frac{1}{n}\int_0^n \tilde{Y}_t\varphi_i(t)dt\right)\left(\frac{1}{n}\int_0^n \tilde{h}(t)\varphi_i(t)dt\right) - \sum_{i=1}^p\langle\tilde{h},\,\varphi_i\rangle^2_{L^2([0,\,1])}\\ =\ & \frac{1}{n}\int_0^n \tilde{Y}_t^2dt - \frac{2}{n}\int_0^n \tilde{Y}_t\tilde{h}(t)dt - \alpha^{-2H}H\Gamma(2H)\\ &-\sum_{i=1}^p\left(\frac{1}{n}\int_0^n \tilde{Y}_t\varphi_i(t)dt\right)^2 + 2\sum_{i=1}^p\left(\frac{1}{n}\int_0^n \tilde{Y}_t\varphi_i(t)dt\right)\left(\frac{1}{n}\int_0^n \tilde{h}(t)\varphi_i(t)dt\right). \quad (5.5.3)\end{aligned}$$

Note that $\frac{1}{n}\int_0^n \tilde{h}(t)^2dt$ and $\sum_{i=1}^p\langle\tilde{h},\,\varphi_i\rangle^2_{L^2([0,\,1])}$ cancel each other out by Parseval's identity due to (A1*). If we consider the space of square integrable functions on $[0,\,n]$ with the scalar product

$$\langle f,\,g\rangle_n := \frac{1}{n}\int_0^n f(x)g(x)dx,$$

the orthonormality assumption of φ_i, as well as (A1*), will still hold for the periodic extensions on $[0,n]$ of φ_i and $\tilde{h}$ under the scalar product $\langle\cdot\,,\cdot\rangle_n$, and by the assumption (A1*) we obtain

$$2\sum_{i=1}^p\left(\frac{1}{n}\int_0^n \tilde{Y}_t\varphi_i(t)dt\right)\left(\frac{1}{n}\int_0^n \tilde{h}(t)\varphi_i(t)dt\right) = 2\langle\tilde{h},\,\tilde{Y}\rangle_n = \frac{2}{n}\int_0^n \tilde{Y}_t\tilde{h}(t)dt.$$

Therefore, (5.5.3) reduces to the term

$$\frac{1}{n}\int_0^n \tilde{Y}_t^2 dt - \alpha^{-2H} H\Gamma(2H) - \sum_{i=1}^{p}\left(\frac{1}{n}\int_0^n \tilde{Y}_t\varphi_i(t)dt\right)^2$$
$$= \frac{1}{n}\int_0^n \left(\tilde{Y}_t^2 - \mathbb{E}[\tilde{Y}_t^2]\right)dt + \frac{1}{n}\int_0^n \left(\mathbb{E}[\tilde{Y}_t^2] - \alpha^{-2H}H\Gamma(2H)\right)dt$$
$$-\sum_{i=1}^{p}\left(\frac{1}{n}\int_0^n \tilde{Y}_t\varphi_i(t)dt\right)^2.$$

It follows from Proposition 5.2.1 that $n^{1-H}\sum_{i=1}^{p}\left(\frac{1}{n}\int_0^n \tilde{Y}_t\varphi_i(t)dt\right)^2$ converges to zero in L^2 an $n\to\infty$. As to the first two summands, by replacing once again $\tilde{Y}$ by Y, the quantity (5.5.3) will become asymptotically equivalent to

$$\frac{1}{n}\int_0^n \left(Y_t^2 - \mathbb{E}[Y_t^2]\right)dt + \frac{1}{n}\int_0^n \left(\mathbb{E}[Y_t^2] - \alpha^{-2H}H\Gamma(2H)\right)dt.$$

It is shown in [56] that $n^{1-H}\frac{1}{n}\int_0^n \mathbb{E}\left([Y_t^2] - \alpha^{-2H}H\Gamma(2H)\right)dt$ goes to zero in L^2 when $n\to\infty$. Another result from [56] by rescaling of Z^H by the factor n^{-H} is that

$$n^{1-H}\frac{1}{n}\int_0^n \left(Y_t^2 - \mathbb{E}[Y_t^2]\right)dt \overset{d}{\equiv} \alpha^{-H-1}G_{\alpha n}$$

where G_T are explicitly defined random variables converging in L^2 as $T\to\infty$ to a limit denoted by G_∞, whose distribution and properties are as claimed in the statement of the proposition. Thus, as $n\to\infty$

$$n^{1-H}\frac{\alpha^{2H+1}}{2H^2\Gamma(2H)}(\bar{\alpha}_n^{(1)} - \alpha) \xrightarrow{d} \alpha^{-H-1}G_\infty.$$

By the definition of $\bar{\mu}_i^{(1)}$, we can write for every $i = 1, .., p$

$$\bar{\mu}_{i,n}^{(1)} - \mu_i = (\bar{\alpha}_n^{(1)} - \alpha)\frac{1}{n}\int_0^n \varphi_i(t)X_t dt + \frac{1}{n}\int_0^n \varphi_i(t)dZ^H(t).$$

Since the sequence $\frac{1}{n}\int_0^n \varphi_i(t)X_t dt$ converges almost surely as $n\to\infty$ to $\langle \tilde{h}, \varphi_i\rangle_{L^2([0,1])}$, it now suffices to investigate joint convergence of

$$\left(\frac{1}{n}\int_0^n \left(Y_t^2 - \mathbb{E}[Y_t^2]\right)dt, \frac{1}{n}\int_0^n f(s)dZ_s^H\right)$$

for a periodic function f. First we rescale the Rosenblatt process involved in both elements by n^{-H} and obtain

$$\left(n^{1-H}\frac{1}{n}\int_0^n \left(Y_t^2 - \mathbb{E}[Y_t^2]\right)dt,\, n^{-H}\int_0^n f(s)dZ_s^H\right) \overset{d}{\equiv} \left(\alpha^{-H-1}G_{\alpha n}, \int_0^1 f(ns)dZ_s^H\right). \tag{5.5.4}$$

We know from Proposition 5.4.1 that $\int_0^1 f(ns)dZ_s^H$ converges in L^2 to $(\int_0^1 f(s)ds)Z_1^H$, and the first component also converges in L^2, as mentioned above. Consequently, we get the joint convergence in distribution of the vector (5.5.4) to $(\alpha^{-H-1}G_\infty, (\int_0^1 f(s)ds)Z_1^H)$. This fact combined with Slutsky's lemma for vectors yields the desired result. □

The random vector (G_∞, Z_1^H) whose components appear in the statement of the above result can be understood as a two dimensional Rosenblatt vector. Its marginals are Rosenblatt distributed and it is well-defined as a limit in L^2 of the sequence (5.5.4). From the practical point of view dealing with this estimator is even more cumbersome than with the least squares type estimator considered above: The limiting distribution is more difficult to handle than Z_1^H obtained before. Moreover, for both pathwise estimators the knowledge of $\tilde{h}$ is necessary for identifying the limiting distribution, while in the least squares case it is needed only for the last component. However, we will see that in certain cases one has to consider the setting (A1*), for which this is, to the best of our knowledge, the only construction of a consistent pathwise estimator made until now.

Let us end this chapter with a discussion concerning the hypotheses (A1) and (A1*) in the case of the trigonometric basis of $L^2([0,1])$.

5.5.5 Remark.

- Consider the orthonormal basis of $L^2([0,1])$ formed by $\{1, \sqrt{2}\sin(2\pi n\cdot\,), \sqrt{2}\cos(2\pi n\cdot\,), n \in \mathbb{N}\}$. Recall that

$$\tilde{h}(t) = \sum_{i=1}^{p} \mu_i \int_0^t e^{-\alpha(t-s)}\varphi_i(s)ds.$$

By direct calculation, we obtain

$$\int_{-\infty}^t e^{\alpha(s-t)}\sin(2\pi ns)ds = \frac{\alpha}{(2\pi n)^2+\alpha^2}\sin(2\pi nt) - \frac{2\pi n}{(2\pi n)^2+\alpha^2}\cos(2\pi nt),$$
$$\int_{-\infty}^t e^{\alpha(s-t)}\cos(2\pi ns)ds = \frac{\alpha}{(2\pi n)^2+\alpha^2}\cos(2\pi nt) + \frac{2\pi n}{(2\pi n)^2+\alpha^2}\sin(2\pi nt).$$

This implies a simple rule in the non-degenerate setting (i.e., if all μ_i, $i \in \{1,\dots,p\}$, are nonzero): If $\{\varphi_1,\dots,\varphi_p\}$ are elements of the trigonometric basis and if this set is "symmetric" (i.e., $\sin(2\pi n\cdot\,) \in \{\varphi_1,\dots,\varphi_p\} \Leftrightarrow \cos(2\pi n\cdot\,) \in \{\varphi_1,\dots,\varphi_p\}$), then the assumption (A1*) is satisfied; otherwise, (A1) is verified and φ_{p+1} can be chosen from the missing counterparts.

- The pathwise estimators of α considered in [30] and [56] are special cases of the estimator defined in Proposition 5.5.3. Indeed, for a constant mean function the assumption (A1*) is satisfied.

5.5.6 Remark. A natural question in this context would be the application of the ideas on pathwise estimators to the fractional Brownian case or to other Hermite processes. The transition is far from immediate because the speed of convergence of the estimator for α will change (see [56] for a special case) and under this different scaling the summands considered to establish the second order asymptotics will behave differently, making a careful separate analysis necessary. We do not include this analysis here, however, it remains an interesting and, we believe, worthwhile direction for future research.

Chapter 6

Power variations of the wave equation solution

In this chapter we go back to the variational methods mentioned in the preliminaries part of the thesis. The object we consider is the solution of the fractional wave equation described in Section 2.1.4. This model has a physical interpretation, namely, it describes the vibration of a string depending on time and space under the influence of a random force which has white noise properties in space and fractional structure in the time component. This can be used to model an influence which exhibits long memory in time, for example if there are intrinsic or experimental reasons to assume strong correlations.

In [35] an estimator for the Hurst parameter H is derived for $H < \frac{3}{4}$ (see Section 2.2.3 for more details). In this chapter we will complement this result and show by calculating the limiting distribution of the realised quadratic variation of the solution that for $H > \frac{3}{4}$ this estimator is not asymptotically Gaussian. This is inconvenient for statistical applications. In order to avoid this restriction and to get an estimator which is asymptotically Gaussian for every $H \in [\frac{1}{2}, 1)$, we will use the generalized k-variations, which means that the usual increment of the process is replaced by a higher order increment and consider higher powers of the increments. The idea was introduced in the reference [33] and since it has been used by many authors (see e.g. [16] or [15]). In particular, before constructing the new estimators we will prove several central limit theorems and derive bounds on the speed of convergence in terms of the Wasserstein distance.

The results of this chapter are presented in the preprint

- R. Shevchenko, M. Slaoui, C. A. Tudor - Generalized k-variations and Hurst parameter estimation for the fractional wave equation via Malliavin calculus, 2019, arXiv:1903.02369, accepted for publication in Journal of Statistical Planning and Inference.

6.1 Preliminaries

We recall here the fractional-white wave equation and its solution and we present the basic definitions and the notation concerning the filters used in our work.

6.1.1 The solution to the wave equation with fractional-white noise

We study the solution u of the one-dimensional equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t,x) &= \frac{\partial^2 u}{\partial x^2}u(t,x) + \dot{W}^H(t,x),\ t \geq 0,\ x \in \mathbb{R}, \\ u(0,x) &= 0, \quad x \in \mathbb{R}, \\ \frac{\partial u}{\partial t}(0,x) &= 0, \quad x \in \mathbb{R}, \end{cases} \tag{6.1.1}$$

i.e. a square-integrable centred field $u = (u(t,x);\ t \in [0,T], x \in \mathbb{R})$ for $T \in \mathbb{R}^+$ defined as

$$u(t,x) = \int_0^t \int_{\mathbb{R}} \frac{1}{2} \mathbb{1}_{\{|x-y|<t-s\}} W^H(\mathrm{d}s, \mathrm{d}y), \quad t \geq 0, x \in \mathbb{R}, \tag{6.1.2}$$

see Section 2.1.4 for details on these definitions.

It is shown in [8] that the solution (6.1.2) is self-similar in time and stationary in space. An important tool in this chapter is the use of the exact spatial covariance structure, which is calculated in [35] for $H > \frac{1}{2}$ and in [36] for $H = \frac{1}{2}$. Namely, the covariance can be expressed as follows:

$$\begin{aligned} \mathbb{E}\left(u(t,x)u(t,y)\right) &= \frac{1}{2}\left(c_H|y-x|^{2H+1} - t\frac{|y-x|^{2H}}{2} + \frac{t^{2H+1}}{2H+1}\right)1_{\{|y-x|<t\}} \\ &\quad + \frac{(2t-|y-x|)^{2H+1}}{8(2H+1)} 1_{\{t \leq |y-x| < 2t\}} \end{aligned} \tag{6.1.3}$$

with $c_H = \frac{4H-1}{4(2H+1)}$. When $t > 1$ and $x, y \in [0,1]$, this expression reduces to

$$\begin{aligned} &\mathbb{E}\left(u(t,x)u(t,y)\right) \\ &= \frac{1}{2}\left(c_H|y-x|^{2H+1} - t\frac{|y-x|^{2H}}{2} + \frac{t^{2H+1}}{2H+1}\right). \end{aligned} \tag{6.1.4}$$

To be able to work with the reduced expression we will fix for the rest of the chapter $t > 1$. Moreover, we will associate to the process $(u(t,x), x \in [0,1])$ its canonical Hilbert space $\mathscr{H} (=: \mathscr{H}^u)$ which is defined as the closure of the linear space generated by the indicator functions $\{\mathbf{1}_{[0,x]}, x \in [0,1]\}$ with respect to the inner product

$$\langle \mathbf{1}_{[0,x]}, \mathbf{1}_{[0,y]} \rangle_{\mathscr{H}} \;=\; \mathbb{E}\left(u(t,x)u(t,y)\right).$$

We will denote by I_q the multiple stochastic integral of order $q \geq 1$ with respect to the Gaussian process $(u(t,x), x \in [0,1])$ and by D the Malliavin derivative with respect to this process. We refer to Section 2.1.2 for the basic elements of the Malliavin calculus.

In Section 6.3 we will also use multiple stochastic integrals with respect to the fractional-white noise W^H with covariance (2.1.9) (defined analogously to the one-dimensional multiple integrals described in Section 2.1.2). We use the notation I_q^W to indicate the multiple integral of order $q \geq 1$ with respect to W^H.

6.1.2 Filters

In this section we will define filters and the increments of the solution to (6.1.1) along filters. We start with several definitions and some notation needed along this chapter.

6.1.1 Definition. Given $l, p \in \mathbb{N}^*(= \mathbb{N}\backslash\{0\})$, a vector $\alpha = (\alpha_0, ..., \alpha_l)$ is called a filter of length $l+1$ and order (or power) p if

$$\begin{cases} \sum_{q=0}^{l} \alpha_q q^r & = 0, \quad 0 \leq r \leq p-1, \\ \sum_{q=0}^{l} \alpha_q q^p & \neq 0 \end{cases}$$

with the convention $0^0 = 1$.

For instance, $\alpha = (1, -1)$ is a filter of length 2 and of order $p = 1$ while $\alpha = (1, -2, 1)$ is a filter of length 3 and of power $p = 2$.

For a filter $\alpha = (a_0, a_1, .., a_l)$ of length $l + 1 \geq 1$ and of order $p \geq 1$ we define the space-filtered process (or the spatial increment of the process u along the filter α) as

$$U^\alpha\left(\frac{i}{N}\right) = \sum_{r=0}^{l} a_r u\left(t, \frac{i-r}{N}\right) \text{ for } i = l, .., N, \tag{6.1.5}$$

where N is a natural number corresponding (from the statistical point of view) to the number of observations of the solution process for a fixed time t.

In the case of the filter $\alpha = (1, -1)$ of order one $U^\alpha\left(\frac{i}{N}\right) = u\left(t, \frac{i}{N}\right) - u\left(t, \frac{i-1}{N}\right)$ is the usual spatial increment of the solution while for $\alpha = (1, -2, 1)$ we have $U^\alpha\left(\frac{i}{N}\right) = u\left(t, \frac{i}{N}\right) - 2u\left(t, \frac{i-1}{N}\right) + u\left(t, \frac{i-2}{N}\right)$ which represents the rectangular spatial increment.

We denote for $j \geq 1$

$$\pi_H^{\alpha,N}(j) := \mathbb{E}\left[U^\alpha\left(\frac{i}{N}\right)U^\alpha\left(\frac{i+j}{N}\right)\right].$$

From the covariance formula (6.1.3) we can write

$$\begin{aligned}\pi_H^{\alpha,N}(j) &= \sum_{r_1,r_2=0}^{l} a_{r_1}a_{r_2}\,\mathbb{E}\left[u\left(t,\frac{i-r_1}{N}\right)u\left(t,\frac{i+j-r_2}{N}\right)\right]\\ &= k_1\frac{1}{N^{2H}}\Phi_{H,\alpha}(j)+k_2\frac{1}{N^{2H+1}}\Phi_{H+\frac{1}{2},\alpha}(j),\end{aligned} \tag{6.1.6}$$

with

$$\Phi_{H,\alpha}(j)=\sum_{r_1,r_2=0}^{l} a_{r_1}a_{r_2}|j+r_1-r_2|^{2H}, \quad j\geq 0,$$

and $k_1=-\frac{t}{4}$ and $k_2=\frac{c_H}{2}=\frac{4H-1}{8(2H+1)}$. We write for further use

$$c_1(H):=\frac{-t}{4}\sum_{q,r=0}^{l}\alpha_q\alpha_r|q-r|^{2H} \text{ and } c_2(H):=\frac{c_H}{2}\sum_{q,r=0}^{l}\alpha_q\alpha_r|q-r|^{2H+1}. \tag{6.1.7}$$

In particular, from (6.1.6) we obtain

$$\begin{aligned}\pi_H^{\alpha,N}(0) &= \mathbb{E}\left[U^\alpha\left(\frac{i}{N}\right)\right]^2 = k_1\frac{1}{N^{2H}}\Phi_{H,\alpha}(0)+k_2\frac{1}{N^{2H+1}}\Phi_{H+\frac{1}{2},\alpha}(0)\\ &= c_1(H)\frac{1}{N^{2H}}+c_2(H)\frac{1}{N^{2H+1}}.\end{aligned}$$

We will need the technical lemma below to establish the asymptotic equivalent of $\Phi_{H,\alpha}$ and similar expressions. The proof of the lemma is based on a Taylor expansion, similarly to the corresponding results in [16] or [33].

6.1.2 Lemma. *Let $l_1, l_2, p_1, p_2 \in \mathbb{N}^*$, $H \in \mathbb{R}^+\backslash\mathbb{N}$ and $\alpha^{(1)}$, $\alpha^{(2)}$ be filters of lengths l_1+1, l_2+2 and of orders p_1, p_2 respectively. Then*

$$\sum_{q=0}^{l_1}\sum_{r=0}^{l_2}\alpha_q^{(1)}\alpha_r^{(2)}|q-r+k|^{2H} \overset{k\to\infty}{\sim} \kappa_H k^{2H-2p},$$

with $\kappa_H=\sum_{q=0}^{l_1}\sum_{r=0}^{l_2}\alpha_q^{(1)}\alpha_r^{(2)}\frac{2H(2H-1)\ldots(2H-2p+1)}{2p!}(q-r)^{2p}$, where $p=\min(p_1,\,p_2)$.

Proof. We have

$$\sum_{q=0}^{l_1}\sum_{r=0}^{l_2}\alpha_q^{(1)}\alpha_r^{(2)}|q-r+k|^{2H} = \sum_{q=0}^{l_1}\sum_{r=0}^{l_2}\alpha_q^{(1)}\alpha_r^{(2)}k^{2H}\left|\frac{q-r}{k}+1\right|^{2H}$$

$$\overset{k\to\infty}{\sim}\sum_{q=0}^{l_1}\sum_{r=0}^{l_2}\alpha_q^{(1)}\alpha_r^{(2)}k^{2H}\left(\frac{q-r}{k}+1\right)^{2H} = \sum_{m=0}^{\infty}\binom{2H}{m}k^{2H-m}\sum_{q=0}^{l_1}\sum_{r=0}^{l_2}\alpha_q^{(1)}\alpha_r^{(2)}(q-r)^m$$

$$= \sum_{m=2p}^{\infty}\binom{2H}{m}k^{2H-m}\sum_{q=0}^{l_1}\sum_{r=0}^{l_2}\alpha_q^{(1)}\alpha_r^{(2)}(q-r)^m,$$

since all summands below $2p$ will disappear because the order of both filters is higher or equal to p. The asymptotically dominating summand is

$$\binom{2H}{2p}k^{2H-2p}\sum_{q=0}^{l_1}\sum_{r=0}^{l_2}\alpha_q^{(1)}\alpha_r^{(2)}(q-r)^{2p},$$

which is what we wanted to show. □

6.2 Central limit theorems for the spatial k-variations

In this section we focus on the asymptotic behaviour in distribution of the realised k-variations in space of the solution to the fractional-white wave equation, defined via a filter of power $p \geq 1$. In the first step we show the k-variation satisfies a central limit theorem (CLT) when $p > H + \frac{1}{4}$. Next, by taking k to be an even integer, we derive a Berry-Esséen type bound for this convergence in distribution via the Stein-Malliavin calculus. Restricting ourselves in addition to $k = 2$, we prove a multidimensional CLT, which is needed for the estimation of the Hurst parameter.

6.2.1 Central limit theorem

Fix $t > 1$ and $l, p \in \mathbb{N}^*$. Let α be a filter of length $l + 1$ and of power p as in Definition 6.1.1. Let u be given by (6.1.2). For any integer $k \geq 1$ we define the spatial k-variations of the process $(u(t, x), x \in \mathbb{R})$ by

$$V_N(k,\alpha) = \frac{1}{N-l}\sum_{i=l}^{N}\left[\frac{\left|U^\alpha\left(\frac{i}{N}\right)\right|^k}{\mathbb{E}\left|U^\alpha\left(\frac{i}{N}\right)\right|^k} - 1\right] \tag{6.2.1}$$

with $U^\alpha\left(\frac{i}{N}\right)$ given by (6.1.5). Note that these objects are often called realised or empirical variations, but for brevity reasons this adjective will be omitted here.

We will show that the sequence (6.2.1) satisfies a CLT. In order to do this we will use a criterion based on Malliavin calculus.

Chaos expansion

The first step is to derive a Wiener chaos expansion (see Section 2.1.2) of the k-variation sequence $V_N(k,\alpha)$ with respect to the Gaussian process $(u(t,\, x))_{x\in[0,\,1]}$. Noticing that the filtered process U^α as a linear combination of centred Gaussian random variables is a centered Gaussian process, we get

$$\mathbb{E}\left(U^\alpha\left(\frac{i}{N}\right)^k\right) = E_k\,\mathbb{E}\left(U^\alpha\left(\frac{i}{N}\right)^2\right)^{\frac{k}{2}}, \tag{6.2.2}$$

where E_k denotes the k-th absolute moment of a standard Gaussian variable given by $E_k = \frac{2^{\frac{k}{2}}\Gamma(\frac{k+1}{2})}{\Gamma(\frac{1}{2})}$. We introduce the variable

$$Z^\alpha\left(\frac{i}{N}\right) = \frac{U^\alpha\left(\frac{i}{N}\right)}{(\pi_H^{\alpha,N}(0))^{1/2}}. \tag{6.2.3}$$

It is clear that $Z^\alpha\left(\frac{i}{N}\right)$ is a standard Gaussian variable and $\mathrm{Corr}\left(Z^\alpha\left(\frac{i}{N}\right), Z^\alpha\left(\frac{j}{N}\right)\right) = \mathrm{Corr}\left(U^\alpha\left(\frac{i}{N}\right), U^\alpha\left(\frac{j}{N}\right)\right)$, where Corr denotes the correlation coefficient. Using (6.2.2) and (6.2.3) we can write V_N as follows:

$$V_N(k,\alpha) \;=\; \frac{1}{N-l}\sum_{i=l}^{N}\left[\frac{|U^\alpha(\frac{i}{N})|^k}{\mathbb{E}\,|U^\alpha(\frac{i}{N})|^k} - 1\right] = \frac{1}{N-l}\sum_{i=l}^{N}\left[\frac{|Z^\alpha(\frac{i}{N})|^k}{E_k} - 1\right].$$

In Lemma 2 of [16] the expansion in Hermite polynomials of the function $H^k(t) = \frac{|t|^k}{E_k} - 1$ is given:

$$H^k(t) = \sum_{j=1}^{\infty} c_j^k H_j(t),$$

where $c_{2j+1}^k = 0$ for $j \geqslant 0$, $c_{2j}^k = \frac{1}{(2j)!}\prod_{i=0}^{j-1}(k-2i)$ for $j \geqslant 1$ and $H_j(t)$ denotes the j-th Hermite polynomial defined in Section 2.1.2.

Observing that for

$$C_{i,\,\alpha} := \sum_{q=0}^{l} \alpha_q \mathbf{1}_{[0,\,\frac{i-q}{N}]}$$

we have from (6.1.6) that $\left\|\frac{C_{i,\,\alpha}}{(\pi_H^{\alpha,N}(0))^{1/2}}\right\|_{\mathscr{H}} = 1$ we can express $Z^\alpha\left(\frac{i}{N}\right)$ as an integral with respect to the process $(u(t,x), x \in [0,\,1])$ since the increment $u(t,y) - u(t,x)$

can be expressed as $I_1(\mathbf{1}_{[x,y]})$ (recall that I_1 represents the multiple integral of order 1 with respect to the Gaussian process $(u(t,x), x \in [0,1])$) for every $x < y$:

$$Z^{\alpha}\left(\frac{i}{N}\right) \quad = \quad I_1\left(\frac{C_{i,\alpha}}{(\pi_H^{\alpha,N}(0))^{1/2}}\right).$$

Since we have $H_q(I_1(h)) = \frac{1}{q!} I_q(h^{\otimes q})$ for $\|h\|_{\mathscr{H}} = 1$ we get

$$\begin{aligned}
V_N(k,\alpha) &= \frac{1}{N-l}\sum_{i=l}^{N} H^k\left(Z^{\alpha}\left(\frac{i}{N}\right)\right) = \frac{1}{N-l}\sum_{q\geqslant 1} c_{2q}^k \sum_{i=l}^{N} H_{2q}\left(Z^{\alpha}\left(\frac{i}{N}\right)\right) \\
&= \frac{1}{N-l}\sum_{q\geqslant 1} c_{2q}^k \sum_{i=l}^{N} H_{2q}\left(I_1\left(\frac{C_{i,\alpha}}{(\pi_H^{\alpha,N}(0))^{1/2}}\right)\right) \\
&= \frac{1}{N-l}\sum_{q\geqslant 1} \frac{c_{2q}^k}{(2q)!} \sum_{i=l}^{N} I_{2q}\left(\left(\frac{C_{i,\alpha}}{(\pi_H^{\alpha,N}(0))^{1/2}}\right)^{\otimes 2q}\right).
\end{aligned}$$

Hence, we obtain the following chaos expansion of the k-variation sequence:

$$V_N(k,\alpha) = \frac{1}{N-l}\sum_{i=l}^{N}\sum_{q=1}^{\infty}\frac{c_{2q}^k}{(2q)!} I_{2q}\left(\frac{C_{i,\alpha}^{\otimes 2q}}{(\pi_H^{\alpha,N}(0))^{q}}\right) = \sum_{q\geq 1} I_{2q}(f_{N,2q}) \tag{6.2.4}$$

with

$$f_{N,2q} = \frac{c_{2q}^k}{(2q)!}\frac{1}{N-l}\sum_{i=l}^{N}\frac{C_{i,\alpha}^{\otimes 2q}}{(\pi_H^{\alpha,N}(0))^{q}}. \tag{6.2.5}$$

Relation (6.2.4) shows that the random variable $V_N(k,\alpha)$ admits an infinite chaos expansion, which contains the chaoses of all orders from $q = 2$ to infinity. We will study the behaviour of each chaos component of $V_N(k,\alpha)$. Let us start by analysing the asymptotic behaviour of the mean square of each kernel $f_{N,2q}$ that appears in the chaos expansion of $V_N(k,\alpha)$. This will be needed for the proof of the CLT later in the chapter.

6.2.1 Lemma. *For $N, q \geq 1$, let $f_{N,2q}$ be given by (6.2.5). Then*

$$(N-l)(2q)!\|f_{N,2q}\|^2_{\mathscr{H}^{\otimes 2q}} \overset{N\to\infty}{\to} \frac{(c_{2q}^k)^2}{(2q)!}\sum_{v\in\mathbb{Z}}(\varphi_{H,\alpha}(v))^{2q} := \sigma_{2q}^2,$$

for $H < p - \frac{1}{4q}$ (i.e. $H < 1 - \frac{1}{4q}$ for $p = 1$ and $H \in \left[\frac{1}{2}, 1\right)$ for $p \geq 2$), where we use the notation

$$\varphi_{H,\alpha}(v) = \frac{\Phi_{H,\alpha}(v)}{\Phi_{H,\alpha}(0)}. \tag{6.2.6}$$

Moreover, $\sigma^2 := \sum_{q\geq 1}\sigma_{2q}^2 < \infty$. *For* $p = q = 1,\ H = 3/4$,

$$\frac{N-l}{\log(N-l)}2!\|f_{N,2}\|^2_{\mathscr{H}^{\otimes 2}} \overset{N\to\infty}{\to} c^2 := \frac{(c_2^k)^2}{2}\lim_{N\to\infty}\log(N)\sum_{|v|\leq N}(\rho_{H,\alpha}(v))^2 < \infty. \tag{6.2.7}$$

Proof. From (6.2.5), we get

$$\begin{aligned}(2q)!\|f_{N,2q}\|^2_{\mathscr{H}^{\otimes 2q}} &= \frac{(c_{2q}^k)^2}{(2q)!}\frac{1}{(N-l)^2}\sum_{i,j=l}^{N}\frac{\langle C_{i,\alpha}, C_{j,\alpha}\rangle^{2q}_{\mathscr{H}}}{(\pi_H^{\alpha,N}(0))^{2q}}\\ &= \frac{1}{(N-l)^2}\frac{(c_{2q}^k)^2}{(2q)!}\sum_{i,j=l}^{N}\left(\rho_H^{\alpha,N}(j-i)\right)^{2q},\end{aligned}$$

where we used the notation

$$\rho_H^{\alpha,N}(v) = \frac{\pi_H^{\alpha,N}(v)}{\pi_H^{\alpha,N}(0)} \text{ for } v\in\mathbb{Z}. \tag{6.2.8}$$

Next, we write

$$\frac{1}{N-l}\sum_{i,j=l}^{N}\left(\rho_H^{\alpha,N}(j-i)\right)^{2q} = \sum_{v\in\mathbb{Z}}\left(\rho_H^{\alpha,N}(v)\right)^{2q}\mathbf{1}_{\{|v|\leq N-l\}}\frac{N-|v|-l}{N-l},$$

and thus

$$(N-l)(2q)!\|f_{N,2q}\|^2_{\mathscr{H}^{\otimes 2q}} = \frac{(c_{2q}^k)^2}{(2q)!}\sum_{v\in\mathbb{Z}}\left(\rho_H^{\alpha,N}(v)\right)^{2q}\mathbf{1}_{\{|v|\leq N-l\}}\frac{N-|v|-l}{N-l}. \tag{6.2.9}$$

Using the expression

$$\rho_H^{\alpha,N}(v) = \frac{k_1\Phi_{H,\alpha}(v)N^{-2H} + k_2\Phi_{H+\frac{1}{2},\alpha}(v)N^{-2H-1}}{k_1\Phi_{H,\alpha}(0)N^{-2H} + k_2\Phi_{H+\frac{1}{2},\alpha}(0)N^{-2H-1}} = \frac{\Phi_{H,\alpha}(v)+a_N(v)}{\Phi_{H,\alpha}(0)+a_N(0)}$$

with

$$a_N(v) = \frac{k_2}{k_1 N}\Phi_{H+\frac{1}{2},\alpha}(v) \tag{6.2.10}$$

we can write, with $\varphi_{H,\alpha}$ and $\rho_H^{\alpha,N}$ given by (6.2.6) and (6.2.8) respectively,

$$b_{N,H}(v) := \rho_H^{\alpha,N}(v) - \varphi_{H,\alpha}(v) \tag{6.2.11}$$

and remark that due to Lemma 6.1.2 for v large enough

$$|b_{N,H}(v)| \overset{|v|\to\infty}{\sim} \left|a_N(v)\frac{1}{\Phi_{H,\alpha}(0)+a_N(0)}\right| \leq C\frac{1}{N}v^{2H+1-2p}, \tag{6.2.12}$$

where $C > 0$ does not depend on N, v. With this notation we can write

$$\begin{aligned}
(N-l)(2q)! \quad & \|f_{N,2q}\|^2_{\mathscr{H}^{\otimes 2q}} \\
= \quad & \frac{(c^k_{2q})^2}{(2q)!} \sum_{v\in\mathbb{Z}} \left(\varphi_{H,\alpha}(v) + b_{N,H}(v)\right)^{2q} \mathbf{1}_{\{|v|\le N-l\}} \frac{N-|v|-l}{N-l} \\
= \quad & \frac{(c^k_{2q})^2}{(2q)!} \sum_{v\in\mathbb{Z}} \sum_{m=0}^{2q} \binom{2q}{m} \varphi_{H,\alpha}(v)^m (b_{N,H}(v))^{2q-m} \mathbf{1}_{\{|v|\le N-l\}} \frac{N-|v|-l}{N-l} \\
= \quad & \frac{(c^k_{2q})^2}{(2q)!} \sum_{v\in\mathbb{Z}} \varphi_{H,\alpha}(v)^{2q} \mathbf{1}_{\{|v|\le N-l\}} \frac{N-|v|-l}{N-l} + r_{N,q,1},
\end{aligned}$$

with

$$r_{N,q,1} = \frac{(c^k_{2q})^2}{(2q)!} \sum_{v\in\mathbb{Z}} \sum_{m=0}^{2q-1} \binom{2q}{m} \varphi_{H,\alpha}(v)^m (b_{N,H}(v))^{2q-m} \mathbf{1}_{\{|v|\le N-l\}} \frac{N-|v|-l}{N-l}. \tag{6.2.13}$$

By the dominated convergence theorem we obtain

$$\frac{(c^k_{2q})^2}{(2q)!} \sum_{v\in\mathbb{Z}} \varphi_{H,\alpha}(v)^{2q} \mathbf{1}_{\{|v|\le N-l\}} \frac{N-|v|-l}{N-l} \overset{N\to\infty}{\to} \sigma^2_{2q},$$

which by Lemma 6.1.2 is finite if $p = 1$, $H < 1 - \frac{1}{4q}$, and for all $H \in [1/2, 1)$ if $p > 1$.

For $q = p = 1$, $H = 3/4$,

$$\frac{1}{\log(N-l)} \sum_{v\in\mathbb{Z}} \varphi_{H,\alpha}(v)^2 \mathbf{1}_{\{|v|\le N-l\}} \frac{N-|v|-l}{N-l}$$

converges to a positive constant and thus (6.2.7) is obtained.

In order to conclude it remains to show that the rest term $r_{N,q,1}$ (6.2.13) converges to 0 as $N \to \infty$ for every $q \ge 1$. From (6.2.13), using the bound (6.2.12) and Lemma 6.1.2, we have the estimate

$$|r_{N,q,1}| \;\le\; C \sum_{m=0}^{2q-1} \binom{2q}{m} \frac{1}{N^{2q-m}} \sum_{1\le v\le N-l} |v|^{(2H-2p)m} |v|^{(2H+1-2p)(2q-m)} := \sum_{m=0}^{2q-1} r_{N,q,1,m},$$

and for each $m = 0, .., 2q-1$,

$$r_{N,q,1,m} \le \frac{C}{N^{2q-m}} \sum_{1\le v\le N-l} |v|^{(2H-2p)2q+2q-m}.$$

If the series $\sum_{v\in\mathbb{Z}} |v|^{(2H-2p)2q+2q-m}$ converges we get

$$r_{N,q,1,m} \le C\frac{1}{N^{2q-m}} \le C\frac{1}{N} \overset{N\to\infty}{\to} 0,$$

and when the series diverges,

$$r_{N,q,1,m} \le C\frac{1}{N^{2q-m}}N^{(2H-2p)2q+2q-m+1} \le CN^{(2H-2p)2q+1} \overset{N\to\infty}{\to} 0$$

(up to an additional $\log N$ factor appearing whenever the exponent in the sum adds up to minus one) if $p = 1, H \in (\frac{1}{2}, 1 - \frac{1}{4q})$ or $p \ge 2$ and $H \in [\frac{1}{2}, 1)$. If $p = 1$ and $H = \frac{1}{2}$ we obtain for $m \neq 1$

$$r_{N,q,1,m} \le CN^{-2q+1} \overset{N\to\infty}{\to} 0$$

and for $m = 1$

$$r_{N,q,1,m} \le CN^{-2q+1}\log N \overset{N\to\infty}{\to} 0.$$

If $p = q = 1$, $H = 3/4$, the quantity

$$\frac{1}{\log(N-l)} r_{N,q,1}$$

will also converge to zero which can be seen using again (6.2.12) and Lemma 6.1.2.

The fact that the series $\sigma^2 = \sum_{q\ge1}\sigma_{2q}^2$ is finite for $H < p - \frac{1}{4q}$ follows from the study of the k-variations of the fractional Brownian motion, see [16] or [51]. □

Asymptotic normality for the renormalized k-variation

We will consider the renormalized k-variation sequence

$$G_N(k,\alpha) = \sqrt{N-l}V_N(k,\alpha). \tag{6.2.14}$$

From the above Lemma 6.2.1 it follows that

$$\mathbb{E}\left[G_N(k,\alpha)\right]^2 \overset{N\to\infty}{\to} \sigma^2,$$

with σ^2 given in the statement of Lemma 6.2.1. We will now show that the sequence (6.2.14) satisfies a central limit theorem, which is the main result of this section.

6.2.2 Theorem. *Let $l, p \in \mathbb{N}^*$. For a filter α of order p and of length $l+1$, with $p > H + \frac{1}{4}$, let $G_N(k,\alpha)$ be given by (6.2.14). Then the sequence $(G_N(k,\alpha))_{N\ge1}$ converges in distribution, as $N \to \infty$, to the Gaussian law $N(0,\sigma^2)$. Moreover, for $p = 1$, $H = 3/4$, the sequence $\left(\frac{1}{\sqrt{\log(N-l)}}G_N(k,\alpha)\right)_{N\ge1}$ converges in distribution to $N(0, c^2)$. The constants σ^2, c^2 are those appearing in Lemma 6.2.1.*

Proof. Notice that from (6.2.4), we can write

$$G_N(k,\alpha) = \sum_{q\geq 1} I_{2q}(g_{N,2q}) \quad \text{with} \quad g_{N,2q} = \sqrt{N-l} f_{N,2q} \tag{6.2.15}$$

with $f_{N,2q}$ given by (6.2.5). Our main tool to prove the asymptotic normality of (6.2.15) is Theorem 6.3.1 from [51]. According to it, for $p > H + 1/4$ it suffices to show that

1. $(2q)!\|g_{N,2q}\|^2_{\mathscr{H}^{\otimes 2q}} \overset{N\to\infty}{\to} \sigma^2_{2q}$ and $\sigma^2 := \sum_{q\geq 1} \sigma^2_{2q} < \infty$,
2. for every $q \geq 1$ and $r = 1,..,2q-1$, $\|g_{N,2q} \otimes_r g_{N,2q}\|_{\mathscr{H}^{\otimes 4q-2r}} \overset{N\to\infty}{\to} 0$ (where $\otimes_r$ denotes the contraction introduced ib Section 2.1.2),
3. $\lim_{M\to\infty} \sup_{N\geq 1} \sum_{q\geq M+1} (2q)!\|g_{N,2q}\|^2_{\mathscr{H}^{\otimes 2q}} = 0$

and for $p = 1$, $H = 3/4$,

1. $\frac{1}{\log(N-l)}(2q)!\|g_{N,2q}\|^2_{\mathscr{H}^{\otimes 2q}} \overset{N\to\infty}{\to} \mathbf{1}_{\{q=1\}} c^2$,
2. for every $q \geq 1$ and $r = 1,..,2q-1$, $\frac{1}{\log(N-l)}\|g_{N,2q} \otimes_r g_{N,2q}\|_{\mathscr{H}^{\otimes 4q-2r}} \overset{N\to\infty}{\to} 0$,
3. $\lim_{M\to\infty} \sup_{N\geq 1} \sum_{q\geq M+1} \frac{1}{\log(N-l)}(2q)!\|g_{N,2q}\|^2_{\mathscr{H}^{\otimes 2q}} = 0$.

Point 1 in both cases follows from Lemma 6.2.1. Let us investigate what happens for point 2. By definition of contraction (see (2.1.3)), we have for $q \geq 1$ and $r = 1,..,2q-1$

$$g_{N,2q} \otimes_r g_{N,2q} = \frac{1}{N-l} \frac{(c^k_{2q})^2}{(2q)!} \sum_{i,j=l}^{N} \frac{\langle C_{i,\alpha}, C_{j,\alpha}\rangle^r_{\mathscr{H}}}{\pi^{\alpha,N}_H(0)^{2q}} C^{\otimes 2q-r}_{i,\alpha} \otimes C^{\otimes 2q-r}_{j,\alpha}$$

and

$$\begin{aligned}
&\|g_{N,2q} \otimes_r g_{N,2q}\|^2_{\mathscr{H}^{\otimes 4q-2r}} \\
&= \left(\frac{(c^k_{2q})^2}{(2q)!}\right)^2 \frac{1}{(N-l)^2} \\
&\times \sum_{i_1,i_2,i_3,i_4=l}^{N} \frac{\langle C_{i_1,\alpha}, C_{i_2,\alpha}\rangle^{2q-r}_{\mathscr{H}} \langle C_{i_2,\alpha}, C_{i_3,\alpha}\rangle^r_{\mathscr{H}} \langle C_{i_3,\alpha}, C_{i_4,\alpha}\rangle^{2q-r}_{\mathscr{H}} \langle C_{i_4,\alpha}, C_{i_1,\alpha}\rangle^r_{\mathscr{H}}}{\pi^{\alpha,N}_H(0)^{2q}} \\
&= \left(\frac{(c^k_{2q})^2}{(2q)!}\right)^2 \frac{1}{(N-l)^2} \\
&\times \sum_{i_1,i_2,i_3,i_4=l}^{N} \rho^{\alpha,N}_H(i_1-i_2)^{2q-r} \rho^{\alpha,N}_H(i_2-i_3)^r \rho^{\alpha,N}_H(i_3-i_4)^{2q-r} \rho^{\alpha,N}_H(i_4-i_1)^r
\end{aligned}$$

with $\rho_H^{\alpha,N}$ given by (6.2.8). We use the fact that

$$\sum_{i_1,i_2,i_3,i_4=l}^{N} \rho_H^{\alpha,N}(i_1-i_2)^{2q-r}\rho_H^{\alpha,N}(i_2-i_3)^{r}\rho_H^{\alpha,N}(i_3-i_4)^{2q-r}\rho_H^{\alpha,N}(i_4-i_1)^{r}$$
$$\leq \sum_{n,\,m=l}^{N}\left(\left(\rho_H^{\alpha,N}\mathbf{1}_{\{|\cdot|\leq N-l\}}\right)^{2q-r} * \left(\rho_H^{\alpha,N}\mathbf{1}_{\{|\cdot|\leq N-l\}}\right)^{r}\right)^2(n-m),$$

(where $*$ denotes convolution of sequences on $\mathbb{Z}$) and we obtain

$$\begin{aligned}
&\|g_{N,2q}\otimes_r g_{N,2q}\|^2_{\mathscr{H}^{\otimes 4q-2r}}\\
\leq\ & C\frac{1}{N-l}\sum_{v=l}^{N}\left(\left(\rho_H^{\alpha,N}\mathbf{1}_{\{|\cdot|\leq N-l\}}\right)^{2q-r} * \left(\rho_H^{\alpha,N}\mathbf{1}_{\{|\cdot|\leq N-l\}}\right)^{r}\right)^2(v)\\
\leq\ & C\frac{1}{N-l}\left\|\left(\rho_H^{\alpha,N}\mathbf{1}_{\{|\cdot|\leq N-l\}}\right)^{2q-r}\right\|^2_{l^{4/3}(\mathbb{Z})}\left\|\left(\rho_H^{\alpha,N}\mathbf{1}_{\{|\cdot|\leq N-l\}}\right)^{r}\right\|^2_{l^{4/3}(\mathbb{Z})}\\
=\ & C\frac{1}{N-l}\left(\sum_{|v|\leq N-l}\left(\rho_H^{\alpha,N}(v)\right)^{(2q-r)\frac{4}{3}}\right)^{3/2}\left(\sum_{|v|\leq N-l}\left(\rho_H^{\alpha,N}(v)\right)^{r\frac{4}{3}}\right)^{3/2}
\end{aligned}$$

by virtue of the Young's inequality similarly to the calculations in [35] (i.e., $\|u*v\|_{l^s(\mathbb{Z})}\leq\|u\|_{l^p(\mathbb{Z})}\|v\|_{l^q(\mathbb{Z})}$ for $s,\,p,\,q\geq 1$ if $\frac{1}{s}+1=\frac{1}{p}+\frac{1}{q}$). Note that for v large enough we have by virtue of (6.2.12)

$$b_{N,\,H}(v)\mathbf{1}_{\{|v|\leq N-l\}}\leq C\frac{1}{N}v^{2H+1-2p}\mathbf{1}_{\{|v|\leq N-l\}}\leq Cv^{2H-2p}\leq C\varphi_H(v)\mathbf{1}_{\{|v|\leq N-l\}},$$

and since all the powers involved above are positive, this allows us to replace $\rho_H^{\alpha,N}$ with φ_H. Thus, for large N the norm $\|g_{N,2q}\otimes_r g_{N,2q}\|^2_{\mathscr{H}^{\otimes 4q-2r}}$ is bounded by

$$C\frac{1}{N-l}\left(\sum_{|v|\leq N-l}|v|^{(2H-2p)(2q-r)\frac{4}{3}}\right)^{3/2}\left(\sum_{|v|\leq N-l}|v|^{(2H-2p)r\frac{4}{3}}\right)^{3/2}.$$

For $p\geq 2$ all these series converge. For $p=1$ and $H\leq\frac{3}{4}$ the only cases in which some of the series do not converge are $r=2q-1$ and $r=1$. However, the observation

$$\frac{1}{N-l}\sum_{|v|\leq N-l}|v|^{(2H-2)\frac{4}{3}}\sum_{|v|\leq N-l}|v|^{(2H-2)\frac{4}{3}}\leq CN^{-1}N^{\frac{8}{3}H-\frac{5}{3}}N^{\frac{8}{3}H-\frac{5}{3}}\overset{N\to\infty}{\to}0$$

ensures that even in those cases the term $\|g_{N,2q}\otimes_r g_{N,2q}\|^2_{\mathscr{H}^{\otimes 4q-2r}}$ converges to zero.

Concerning point 3, fix $M \geq 1$ and recall that from (6.2.9)

$$(2q)!\|g_{N,2q}\|^2_{\mathscr{H}^{\otimes 2q}} = \frac{(c^k_{2q})^2}{(2q)!} \sum_{v\in\mathbb{Z}} \left(\rho_H^{\alpha,N}(v)\right)^{2q} \mathbf{1}_{\{|v|\leq N-l\}} \frac{N-|v|-l}{N-l},$$

and therefore, since $|\rho_H^{\alpha,N}(v)| \leq 1$ for $|v|$ large enough,

$$\begin{aligned}
\sup_{N\geq 1} & \sum_{q\geq M+1} (2q)!\|g_{N,2q}\|^2_{\mathscr{H}^{\otimes 2q}} \\
\leq & \sup_{N\geq 1} \sum_{q\geq M+1} \frac{(c^k_{2q})^2}{(2q)!} \sum_{v\in\mathbb{Z}} \left(\rho_H^{\alpha,N}(v)\right)^2 \mathbf{1}_{\{|v|\leq N-l\}} \frac{N-|v|-l}{N-l} \\
\leq & \; C \sum_{q\geq M+1} \frac{(c^k_{2q})^2}{(2q)!} \sum_{v\in\mathbb{Z}} \varphi_{H,\alpha}(v)^2 \\
& + C \sup_{N\geq 1} \sum_{q\geq M+1} \frac{(c^k_{2q})^2}{(2q)!} \sum_{v\in\mathbb{Z}} b_{N,H}(v)^2 \mathbf{1}_{\{|v|\leq N-l\}} \frac{N-|v|-l}{N-l}.
\end{aligned}$$

From (6.2.12)

$$b_{N,H}(v)^2 \leq C\frac{1}{N^2} \text{ if } p \geq 2,$$

and

$$\sum_{|v|\leq N-l} b_{N,H}(v)^2 \frac{N-|v|-l}{N-l} \leq C\frac{1}{N^2} \sum_{|v|\leq N-l} v^{(2H-1)2} \leq CN^{4H-3} \quad \text{ if } p=1, H<\frac{3}{4}.$$

Consequently,

$$\sup_{N\geq 1} \sum_{q\geq M+1} (2q)!\|g_{N,2q}\|^2_{\mathscr{H}^{\otimes 2q}} \leq C \sum_{q\geq M+1} \frac{(c^k_{2q})^2}{(2q)!} \sum_{v\in\mathbb{Z}} \varphi_{H,\alpha}(v)^2,$$

and this tends to zero as $M \to \infty$ due to the convergence of the series $\sum_{q\geq 1} \frac{(c^k_{2q})^2}{(2q)!}$.

For $\frac{1}{\sqrt{\log(N-l)}} G_N(k,\alpha)$ there is nothing to show since the case $q=1$ does not contribute to the limit. □

6.2.2 Rate of convergence for even power variations

In this section we will further quantify the CLT proved above (Theorem 6.2.2) by deriving a rate of convergence in Wasserstein distance for even power variations. The choice of even powers enables us to obtain a finite sum in the Hermite expansion and treat a finite number of summands later on. This constraint is particularly important because there is no dominating chaos component (in the L^2 sense), and thus the question of the interplay of an infinite number of summands becomes difficult to treat. While there is a general CLT for such a case (and it is used in the proof of Theorem 6.2.2), there are no known results concerning the convergence rates. However, deriving such results still might be possible and the constraint is not an intrinsically motivated one.

Let $k \geq 2$ be an even integer. Consider the sequence $G_N(k, \alpha)$ defined by (6.2.14). From (6.2.4), since the coefficients c_{2j}^k vanish if $2j > k$, we get

$$G_N(k, \alpha) = \frac{1}{\sqrt{N-l}} \sum_{i=l}^{N} \sum_{q=1}^{\frac{k}{2}} \frac{c_{2q}^k}{(2q)!} I_{2q} \left(\frac{C_{i,\alpha}^{\otimes 2q}}{(\pi_H^{\alpha,N}(0))^q} \right). \tag{6.2.16}$$

Denote for every $q = 1, 2, .., \frac{k}{2}$ the $2q$-th chaos component of $G_N(k, \alpha)$ by

$$G_N^{(2q)}(k, \alpha) := I_{2q}(g_{N,2q}), \tag{6.2.17}$$

with $g_{N,2q}$ from (6.2.15). Let us consider the $\frac{k}{2}$-dimensional random vector

$$\mathbf{G}_N(k,\, \alpha) := \left(G_N^{(2)}(k, \alpha), G_N^{(4)}(k, \alpha), ..., G_N^{(k)}(k, \alpha) \right).$$

Notice that for every $q_1, q_2 = 1, .., \frac{k}{2}$ with $q_1 \neq q_2$

$$\mathbb{E}\left(G_N^{(2q_1)}(k, \alpha) G_N^{(2q_2)}(k, \alpha) \right) = 0,$$

while for $q_1 = q_2 = q$

$$\mathbb{E}\left[G_N^{(2q)}(k, \alpha) \right]^2 = \frac{(c_{2q}^k)^2}{(2q)!} \sum_{v \in \mathbb{Z}} \rho_H^{\alpha,\, N}(v)^{2q} \mathbf{1}_{\{|v| \leq N-l\}} \left(1 - \frac{|v|}{N-l} \right).$$

Let us introduce the matrix $C = (C_{q_1,q_2})_{q_1,q_2=1,..,\frac{k}{2}}$ with components $C_{q_1,q_2} = 0$ if $q_1 \neq q_2$ and

$$C_{q,q} = \frac{(c_{2q}^k)^2}{(2q)!} \sum_{v \in \mathbb{Z}} \varphi_{H,\alpha}(v)^{2q}. \tag{6.2.18}$$

The objective in this section is to calculate the rate of convergence of $V_N(k,\, \alpha)$ in the CLT proved in Section 6.2.1. In order to obtain this rate in terms of the Wasserstein distance we will use Corollary 3.6 from [54] to show that the vector $\mathbf{G}_N(k,\, \alpha)$ converges to a normal distribution with the covariance matrix C and determine its convergence rate. This will provide corresponding results for the k-variation statistics $V_N(k, \alpha)$. For the sake of completeness we cite this corollary here.

6.2.3 Corollary. *Fix $d \geq 2$ and $1 \leq q_1 \leq \cdots \leq q_d$. Consider a vector $F := (F_1, \ldots F_d) = (I_{q_1}(f_1) \ldots I_{q_d}(f_d))$ with $f_i \in \mathscr{H}^{\odot q_i}$ (where $\mathscr{H}$ is the underlying Hilbert space) for any $i = 1, \ldots, d$. Let $Z \sim N_d(0, C)$ with C positive definite. Then*

$$d_W(F,\, Z) \leq c \sqrt{\sum_{1 \leq i,\, j \leq d} \mathbb{E}\left[\left(C_{ij} - \frac{1}{q_j} \langle DF_i,\, DF_j \rangle_{\mathscr{H}}\right)^2\right]}$$

for some constant strictly positive c.

(In the one-dimensional case for a standard normal Z this result is also true and can be found in [51]. For $k = 2$ the required norming condition is satisfied, and the corollary is applicable.)

Before we begin with the proof of the main result of this chapter, let us briefly recall the definition of the Wasserstein distance. The Wasserstein distance between the laws of two $\mathbb{R}^d$-valued random variables F and G is defined as

$$d_W(F, G) = \sup_{h \in \mathcal{A}} \left| \mathbb{E}\, h(F) - \mathbb{E}\, h(G) \right|, \tag{6.2.19}$$

where $\mathcal{A}$ is the class of Lipschitz continuous functions $h : \mathbb{R}^d \to \mathbb{R}$ such that $\|h\|_{Lip} \leq 1$, where

$$\|h\|_{Lip} = \sup_{x, y \in \mathbb{R}^d, x \neq y} \frac{|h(x) - h(y)|}{\|x - y\|_{\mathbb{R}^d}}.$$

In order to apply the corollary for $F_i = G_N^{(2i)}, i = 1, .., \frac{k}{2}$, we will write each summand as

$$\begin{aligned}
&\mathbb{E}\left[\left(C_{ij} - \frac{1}{q_j} \langle DF_i,\, DF_j \rangle_{\mathscr{H}}\right)^2\right] \\
&\quad \leq 2 \left(C_{ij} - \frac{1}{q_j}\, \mathbb{E}[\langle DF_i,\, DF_j \rangle_{\mathscr{H}}]\right)^2 \\
&\qquad + 2\, \mathbb{E}\left[\left(\frac{1}{q_j}\, \mathbb{E}[\langle DF_i,\, DF_j \rangle_{\mathscr{H}}] - \frac{1}{q_j} \langle DF_i,\, DF_j \rangle_{\mathscr{H}}\right)^2\right]
\end{aligned} \tag{6.2.20}$$

and conduct separate calculations for both parts. We start with a lemma for the deterministic part.

6.2.4 Lemma. *Let $G_N^{(2q)}, C_{q,q}$ be given by (6.2.17), (6.2.18) respectively and assume $p \geq 2$. For N large enough and for every $q = 1, .., \frac{k}{2}$ we have for every $H \in \left[\frac{1}{2}, 1\right)$,*

$$\left| \mathbb{E}\left[G_N^{(2q)}(k, \alpha)^2\right] - C_{q,q} \right| \leq C \frac{1}{N}.$$

For $p = 1$ we have for $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$

$$\left| \mathbb{E}\left[G_N^{(2q)}(k, \alpha)^2\right] - C_{q,q} \right| \leq C N^{4H-3}$$

and for $p = 1,\ H = \frac{1}{2}$,

$$\left| \mathbb{E}\left[G_N^{(2q)}(k,\alpha)^2 \right] - C_{q,q} \right| \leq C \frac{\log N}{N}.$$

Proof. As in the proof of Lemma 6.2.1, we have

$$\mathbb{E}\left[G_N^{(2q)}(k,\alpha)^2 \right] = \frac{(c_{2q}^k)^2}{(2q)!} \sum_{v\in\mathbb{Z}} \left(\rho_H^{\alpha,N}(v) \right)^{2q} \mathbf{1}_{\{|v|\leq N-l\}} \frac{N-|v|-l}{N-l}. \quad (6.2.21)$$

Recall the representation $\rho_H^{\alpha,\,N}(v) = \varphi_H(v) + b_{N,\,H}(v)$ introduced in Lemma 6.2.1. We obtain by the binomial formula

$$\begin{aligned} \mathbb{E} & \left[G_N^{(2q)}(k,\alpha)^2 \right] \\ = & \frac{(c_{2q}^k)^2}{(2q)!} \sum_{m=0}^{2q} \binom{2q}{m} \sum_{v\in\mathbb{Z}} \varphi_{H,\alpha}(v)^m b_{N,\,H}(v)^{2q-m} \mathbf{1}_{\{|v|\leq N-l\}} \frac{N-|v|-l}{N-l} \\ = & \frac{(c_{2q}^k)^2}{(2q)!} \sum_{v\in\mathbb{Z}} \varphi_{H,\alpha}(v)^{2q} \mathbf{1}_{\{|v|\leq N-l\}} \frac{N-|v|-l}{N-l} + r_{N,\,q,\,1}, \end{aligned}$$

where we separated the summand with $m = 2q$ above and we used the notation (6.2.13). Consequently,

$$\begin{aligned} \mathbb{E}\left[G_N^{(2q)}(k,\alpha)^2 \right] = & \frac{(c_{2q}^k)^2}{(2q)!} \sum_{v\in\mathbb{Z}} \varphi_{H,\alpha}(v)^{2q} - \frac{(c_{2q}^k)^2}{(2q)!} \sum_{|v|\geq N-l+1} \varphi_{H,\alpha}(v)^{2q} \\ & + \frac{(c_{2q}^k)^2}{(2q)!} \sum_{|v|\leq N-l} \varphi_{H,\alpha}(v)^{2q} \left(\frac{N-|v|-l}{N-l} - 1 \right) + r_{N,\,q,1} \\ = & \frac{(c_{2q}^k)^2}{(2q)!} \sum_{v\in\mathbb{Z}} \varphi_{H,\alpha}(v)^{2q} + r_{N,\,q,3} + r_{N,\,q,2} + r_{N,\,q,1} \end{aligned}$$

with

$$r_{N,\,q,2} = \frac{(c_{2q}^k)^2}{(2q)!} \sum_{|v|\leq N-l} \varphi_{H,\alpha}(v)^{2q} \left(\frac{N-|v|-l}{N-l} - 1 \right),$$

$$r_{N,\,q,3} = -\frac{(c_{2q}^k)^2}{(2q)!} \sum_{|v|\geq N-l+1} \varphi_{H,\alpha}(v)^{2q}.$$

The asymptotics for $r_{N,\,q,\,1}$ has been studied in Lemma 6.2.1:

$$|r_{N,\,q,\,1}| \leq C \begin{cases} \frac{1}{N} & \text{if } p \geq 2 \text{ or } p = 1, q \geq 2, H \in \left[\frac{1}{2},\, 1 - \frac{1}{2q}\right), \\ N^{4H-3} & \text{if } p = 1, q = 1, H \in \left(\frac{1}{2},\, \frac{3}{4}\right), \\ \frac{\log N}{N} & \text{if } p = 1, q = 1, H = \frac{1}{2} \end{cases}$$

for some $C > 0$.

For $r_{N,q,2}$ we calculate

$$|r_{N,q,2}| \leq C \left| \sum_{|v|\leq N-l} \varphi_{H,\alpha}(v)^{2q} \left(\frac{|v|}{N-l}\right) \right| \leq C\frac{1}{N} \sum_{|v|\leq N-l} |v|^{2q(2H-2p)+1}.$$

Note that the above series is convergent for $p \geq 2$ or for $p = 1$ and $q \geq 2$ if $H < 1 - \frac{1}{2q}$ (which is satisfied for $H < \frac{3}{4}$). In these cases, we will find the estimate

$$|r_{N,q,2}| \leq C\frac{1}{N}.$$

For $p = 1$ and $q = 1$, the sequence $\sum_{1\leq v\leq N-l} |v|^{(2H-2p)2q+1} = \sum_{1\leq |v|\leq N-l} |v|^{4H-3}$ behaves as N^{4H-2} and we get

$$|r_{N,q,2}| \leq CN^{4H-3},$$

so here we obtain the bounds

$$|r_{N,q,2}| \leq C \begin{cases} \frac{1}{N} & \text{if } p \geq 2 \text{ or } p = 1, q \geq 2, H \in \left[\frac{1}{2}, 1 - \frac{1}{2q}\right), \\ N^{4H-3} & \text{if } p = 1, q = 1, H \in \left(\frac{1}{2}, \frac{3}{4}\right), \\ \frac{\log N}{N} & \text{if } p = 1, q = 1, H = \frac{1}{2}. \end{cases}$$

Finally, for $r_{N,q,3}$ the same bounds can be established. An application of Lemma 6.1.2 yields

$$|r_{N,q,3}| \leq C \sum_{|v|\geq N-l} \varphi_{H,\alpha}(v)^{2q} \leq CN^{(2H-2p)2q+1},$$

and consequently,

$$|r_{N,q,3}| \leq C \begin{cases} \frac{1}{N} & \text{if } p \geq 2 \text{ or } p = 1, q \geq 2, H < 1 - \frac{1}{2q}, \\ N^{4H-3} & \text{if } p = 1, q = 1, H < \frac{3}{4}. \end{cases}$$

Since $\frac{1}{N} < N^{4H-3}$ for H between $\frac{1}{2}$ and $\frac{3}{4}$, the result for $p = 1$, $q \geq 2$ follows. □

The following proposition provides a bound for the random part in (6.2.20).

6.2.5 Proposition. *Let G_N be given by (6.2.14). For $q_1, q_2 \in \{1, \ldots, \frac{k}{2}\}$, $p \geq 2$ and $H \in \left[\frac{1}{2}, 1\right)$,*

$$\mathrm{Var}(\langle DG_N^{(2q_1)}(k,\alpha),\, DG_N^{(2q_2)}(k,\alpha)\rangle_{\mathscr{H}}) \leq C\frac{1}{N}$$

with some positive constant C. For $p = 1$ and $H < 3/4$

$$\mathrm{Var}(\langle DG_N^{(2q_1)}(k,\alpha),\, DG_N^{(2q_2)}(k,\alpha)\rangle_{\mathscr{H}}) \leq C \begin{cases} \frac{1}{N} & \text{if } H \in [\frac{1}{2}, \frac{5}{8})\ , \\ \frac{\log(N)^3}{N} & \text{if } H = \frac{5}{8}, \\ N^{8H-6} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}) \end{cases}$$

also with a positive constant C.

Proof. We can explicitly compute the Malliavin derivatives in the statement. For $q \in \{1, \ldots, \frac{k}{2}\}$

$$D_{\cdot}G_N^{(2q)}(k,\alpha) = \frac{1}{\sqrt{N-l}} \sum_{i=l}^{N} \frac{c_{2q}^k}{(2q-1)!} I_{2q-1}\left(\frac{C_{i,\alpha}^{\otimes(2q-1)}}{(\pi_H^{\alpha,N}(0))^q}\right) C_{i,\alpha}(\cdot).$$

Assume without loss of generality $q_1 \leq q_2$. We have

$$\begin{aligned} & \langle DG_N^{(2q_1)}(k,\alpha),\, DG_N^{(2q_2)}(k,\alpha)\rangle_{\mathscr{H}} \\ = \; & \frac{1}{(N-l)(\pi_H^{\alpha,N}(0))^{q_1+q_2}} \frac{c_{2q_1}^k}{(2q_1-1)!} \frac{c_{2q_2}^k}{(2q_2-1)!} \\ & \times \sum_{i,\,j=l}^{N} I_{2q_1-1}\left(C_{i,\alpha}^{\otimes(2q_1-1)}\right) I_{2q_2-1}\left(C_{j,\alpha}^{\otimes(2q_2-1)}\right) \langle C_{i,\,\alpha},\, C_{j,\,\alpha}\rangle_{\mathscr{H}} \\ = \; & \frac{1}{(N-l)(\pi_H^{\alpha,N}(0))^{q_1+q_2}} \frac{c_{2q_1}^k}{(2q_1-1)!} \frac{c_{2q_2}^k}{(2q_2-1)!} \sum_{i,\,j=l}^{N} \langle C_{i,\,\alpha},\, C_{j,\,\alpha}\rangle_{\mathscr{H}} \\ & \times \left(\sum_{r=0}^{2q_1-1} r! \binom{2q_1-1}{r} \binom{2q_2-1}{r} I_{2q_1+2q_2-2-2r}\left(C_{i,\alpha}^{\otimes(2q_1-1)} \otimes_r C_{j,\alpha}^{\otimes(2q_2-1)}\right)\right), \end{aligned}$$

and $\mathbb{E}[\langle DG_N^{(2q_1)}(k,\alpha),\, DG_N^{(2q_2)}(k,\alpha)\rangle_{\mathscr{H}}]$ is the term containing I_0. It follows that

$$\begin{aligned} & \langle DG_N^{(2q_1)}(k,\alpha),\, DG_N^{(2q_2)}(k,\alpha)\rangle_{\mathscr{H}} - \mathbb{E}[\langle DG_N^{(2q_1)}(k,\alpha),\, DG_N^{(2q_2)}(k,\alpha)\rangle_{\mathscr{H}}] \\ = \; & \frac{1}{(N-l)(\pi_H^{\alpha,N}(0))^{q_1+q_2}} \frac{c_{2q_1}^k}{(2q_1-1)!} \frac{c_{2q_2}^k}{(2q_2-1)!} \sum_{i,\,j=l}^{N} \langle C_{i,\,\alpha},\, C_{j,\,\alpha}\rangle_{\mathscr{H}} \\ & \times \left(\sum_{r=0}^{2q_1-1-w} r! \binom{2q_1-1}{r} \binom{2q_2-1}{r} I_{2q_1+2q_2-2-2r}\left(C_{i,\alpha}^{\otimes(2q_1-1)} \otimes_r C_{j,\alpha}^{\otimes(2q_2-1)}\right)\right), \end{aligned}$$

where $w = 1$ if $l_1 \neq l_2$ and $w = 2$ otherwise.

Due to the fact that products of integrals of different orders have zero expectation we obtain

$$
\begin{aligned}
P \ &:= \mathbb{E}[(\langle DG_N^{(2q_1)}(k,\alpha),\, DG_N^{(2q_2)}(k,\alpha)\rangle_{\mathscr{H}} - \mathbb{E}[\langle DG_N^{(2q_1)}(k,\alpha),\, DG_N^{(2q_2)}(k,\alpha)\rangle_{\mathscr{H}}]])^2] \\
&\overset{N\to\infty}{\sim} \frac{C}{(N-l)^2(\pi_H^{\alpha,N}(0))^{2(q_1+q_2)}} \\
&\times \sum_{r=0}^{2q_1-w} \mathbb{E}\Big[\Big(\sum_{i,j=l}^{N} r!\binom{2q_1-1}{r}\binom{2q_2-1}{r} I_{2(q_1+q_2-1-r)} \\
&\qquad \times (C_{i,\alpha}^{\otimes(2q_1-1)} \otimes_r C_{j,\alpha}^{\otimes(2q_2-1)})\langle C_{i,\alpha},\, C_{j,\alpha}\rangle_{\mathscr{H}}\Big)^2\Big] \\
= \ & \frac{C}{(N-l)^2(\pi_H^{\alpha,N}(0))^{2(q_1+q_2)}} \sum_{r=0}^{2q_1-w} \sum_{i,j,k,m=l}^{N} r!^2 \binom{2q_1-1}{r}^2 \binom{2q_2-1}{r}^2 \langle C_{i,\alpha},\, C_{j,\alpha}\rangle_{\mathscr{H}} \\
&\times \langle C_{k,\alpha},\, C_{m,\alpha}\rangle_{\mathscr{H}} \langle C_{i,\alpha}^{\otimes(2q_1-1)} \widetilde{\otimes_r} C_{j,\alpha}^{\otimes(2q_2-1)},\, C_{k,\alpha}^{\otimes(2q_1-1)} \widetilde{\otimes_r} C_{m,\alpha}^{\otimes(2q_2-1)}\rangle_{\mathscr{H}^{\otimes(2q_1+2q_2-2-2r)}} \\
&=: \sum_{r=0}^{2q_1-w} P_r,
\end{aligned}
$$

where the tildas denote the symmetrisation of functions as explained in Section 2.1.2. We can compute the contractions involved and get via (2.1.3)

$$
C_{i,\alpha}^{\otimes(2q_1-1)} \otimes_r C_{j,\alpha}^{\otimes(2q_2-1)} = C_{i,\alpha}^{\otimes(2q_1-r-1)} \otimes C_{j,\alpha}^{\otimes(2q_2-r-1)} \left(\langle C_{i,\alpha},\, C_{j,\alpha}\rangle_{\mathscr{H}}\right)^r .
$$

Consequently, one can write for $r \geq 0$

$$
\begin{aligned}
&\left| \langle C_{i,\alpha}^{\otimes(2q_1-1)} \widetilde{\otimes_r} C_{j,\alpha}^{\otimes(2q_2-1)},\, C_{k,\alpha}^{\otimes(2q_1-1)} \widetilde{\otimes_r} C_{m,\alpha}^{\otimes(2q_2-1)}\rangle_{\mathscr{H}^{\otimes(2q_1+2q_2-2-2r)}} \right| \\
\lesssim \ & |(\langle C_{i,\alpha},\, C_{j,\alpha}\rangle_{\mathscr{H}} \langle C_{k,\alpha},\, C_{m,\alpha}\rangle_{\mathscr{H}})^r| \\
\times \ & \max_a \left| \langle C_{i,\alpha},\, C_{k,\alpha}\rangle_{\mathscr{H}}^{l_1-r-1-a} \langle C_{j,\alpha},\, C_{m,\alpha}\rangle_{\mathscr{H}}^{l_2-r-1-a} \langle C_{i,\alpha},\, C_{m,\alpha}\rangle_{\mathscr{H}}^{a} \langle C_{j,\alpha},\, C_{k,\alpha}\rangle_{\mathscr{H}}^{a} \right|
\end{aligned}
$$

due to symmetrisation: the maximum (with a going from 0 to $l_1 - r - 1$) is taken over all outcomes of different permutations of the first and second component of the inner product, the number a signifying the number of C_i in the first component that are appearing in the same places as C_m in the second component in a given permutation.

In total, we obtain for a fixed $r \in \{0, \ldots, 2q_1 - w\}$

$$\begin{aligned} |P_r| &\le C \frac{1}{(N-l)^2 (\pi_H^{\alpha, N}(0))^{2(q_1+q_2)}} \sum_{i,j,k,m=l}^{N} \left| \langle C_{i,\alpha},\, C_{j,\alpha} \rangle_{\mathscr{H}}^{r+1} \langle C_{k,\alpha},\, C_{m,\alpha} \rangle_{\mathscr{H}}^{r+1} \right| \\ &\times \max_a \left| \langle C_{i,\alpha},\, C_{k,\alpha} \rangle_{\mathscr{H}}^{l_1-r-1-a} \langle C_{j,\alpha},\, C_{m,\alpha} \rangle_{\mathscr{H}}^{l_2-r-1-a} \langle C_{i,\alpha},\, C_{m,\alpha} \rangle_{\mathscr{H}}^{a} \langle C_{j,\alpha},\, C_{k,\alpha} \rangle_{\mathscr{H}}^{a} \right| \\ &= C \frac{1}{(N-l)^2} \sum_{i,j,k,m=l}^{N} \left| \rho_H^{\alpha, N}(i-j)^{r+1} \rho_H^{\alpha, N}(k-m)^{r+1} \right| \\ &\quad \times \max_a \left| \rho_H^{\alpha, N}(i-k)^{l_1-r-1-a} \rho_H^{\alpha, N}(j-m)^{l_2-r-1-a} \rho_H^{\alpha, N}(i-m)^{a} \rho_H^{\alpha, N}(j-k)^{a} \right|, \end{aligned}$$

with $\rho_H^{\alpha, N}$ defined in (6.2.8) and a ranging over $0, \ldots, l_1 - r - 1$ as above. Due to boundedness of $\rho_H^{\alpha, N}$ we can without loss of generality reduce the number of factors. In particular,

$$\begin{aligned} \max_{a=0,\ldots,l_1-r-1} & \left| \rho_H^{\alpha, N}(i-k)^{l_1-r-1-a} \rho_H^{\alpha, N}(j-m)^{l_2-r-1-a} \rho_H^{\alpha, N}(i-m)^{a} \rho_H^{\alpha, N}(j-k)^{a} \right| \\ &\le C |\rho_H^{\alpha, N}(i-k) \rho_H^{\alpha, N}(j-m)|, \end{aligned}$$

since either the factor $|\rho_H^{\alpha, N}(i-k)\rho_H^{\alpha, N}(j-m)|$ or $|\rho_H^{\alpha, N}(i-m)\rho_H^{\alpha, N}(j-k)|$ is contained in the product and for symmetry reasons there is no need to distinguish between these cases. Using this inequality and bounding the first two factors in the same way we arrive at a bound

$$\begin{aligned} |P_r| &\le C \frac{1}{(N-l)^2} \sum_{i,j,k,m=l}^{N} |\rho_H^{\alpha, N}(i-j) \rho_H^{\alpha, N}(k-m) \rho_H^{\alpha, N}(i-k) \rho_H^{\alpha, N}(j-m)| \\ &\le C \frac{1}{(N-l)^2} N \left(\sum_{v=1}^{N} |\rho_H^{\alpha, N}(v)|^{4/3} \right)^3, \end{aligned}$$

where the last step follows via Young's inequality (as in Theorem 6.2.2). The representation $\rho_H^{\alpha, N}(v) = \varphi_H(v) + b_{N,H}(v)$ together with the fact that for $|v| \le N$ we have $b_{N,H}(v) \le C\varphi_H(v)$ for some constant C allows us to replace $\rho_H^{\alpha, N}$ with $\varphi_H(v)$ in the last bound, since the powers involved are positive. Finally, by Lemma 6.1.2

$$\sum_{v=1}^{N} |\varphi_H(v)|^{4/3} \lesssim \begin{cases} 1 & \text{if } H \in (0, \frac{5}{8}) \ , \\ \log(N) & \text{if } H = \frac{5}{8}, \\ N^{\frac{8H}{3} - \frac{5}{3}} & \text{if } H \in (\frac{5}{8}, 1) \end{cases}$$

for $p = 1$ and $\sum_{v=1}^{N} |\varphi_H(v)|^{4/3} = \mathrm{O}(1)$ for $p > 1$, and thus the result follows. $\square$

Let us now state and prove the main result of this section.

6.2.6 Theorem. *Let σ^2, c^2 be constants as in Lemma 6.2.1. Let $p \geq 2$ and consider the sequence (6.2.16). Let $Z \sim N(0, \sigma^2)$. Then there exists a constant C such that*

$$d_W(G_N(k, \alpha), Z) \leq C \frac{1}{\sqrt{N}}.$$

For $p = 1$ and $H < 3/4$ let $Z \sim N(0, c^2)$. Then there exists a constant C such that

$$d_W(G_N(k, \alpha), Z) \leq C \begin{cases} \frac{1}{\sqrt{N}} & \text{if } H \in [\frac{1}{2}, \frac{5}{8}) \ , \\ \frac{\log(N)^{3/2}}{\sqrt{N}} & \text{if } H = \frac{5}{8}, \\ N^{4H-3} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}). \end{cases}$$

Proof. Consider the function $f : \mathbb{R}^{\frac{k}{2}} \to \mathbb{R}$, $f(x) = \frac{2}{k}(x_1 + \dots + x_{\frac{k}{2}})$. Note that f is a Lipschitz continuous function with $\|f\|_{Lip} \leq 1$. From Lemma 6.2.4 and Proposition 6.2.5 it is easy to see that by Corollary 6.2.3

$$d_W(\frac{k}{2}\mathbf{G}_N(k, \alpha), \frac{k}{2}\mathbf{Z}) = d_W\left(\frac{k}{2}(G_N^{(2)}(k, \alpha), \dots, G_N^{(k)}(k, \alpha)), \frac{k}{2}\mathbf{Z}\right) \leq C \frac{1}{\sqrt{N}},$$

where $\mathbf{Z} \sim N(0, C)$ if $p \geq 2$ and

$$d_W(\frac{k}{2}\mathbf{G}_N(k, \alpha), \frac{k}{2}\mathbf{Z}) \leq C \begin{cases} \frac{1}{\sqrt{N}} & \text{if } H \in [\frac{1}{2}, \frac{5}{8}) \ , \\ \frac{\log(N)^{3/2}}{\sqrt{N}} & \text{if } H = \frac{5}{8}, \\ N^{4H-3} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}). \end{cases}$$

if $p = 1$ and $H < 3/4$. Now,

$$\begin{aligned} d_W(G_N(k, \alpha), Z) &= \sup_{\|g\|_{Lip} \leq 1} |\mathbb{E}\, g(G_N(k, \alpha)) - \mathbb{E}\, g(Z)| \\ &= \sup_{\|g\|_{Lip} \leq 1} \left| \mathbf{E}(g \circ f)\left(\frac{k}{2}\mathbf{G}_N(k,\, \alpha)\right) - \mathbf{E}(g \circ f)\left(\frac{k}{2}Z\right) \right| \\ &\leq \sup_{\|h\|_{Lip} \leq 1} \left| \mathbb{E}\, h\left(\frac{k}{2}\mathbf{G}_N(k,\, \alpha)\right) - \mathbb{E}\, h(\mathbf{Z}) \right| \\ &= d_W\left(\frac{k}{2}\mathbf{G}_N(k,\, \alpha), \mathbf{Z}\right). \end{aligned}$$

□

6.2.7 Remark. For $p = 1$ and $k = 2$ we retrieve the bounds obtained in [35] (and in [36] for $H = \frac{1}{2}$), which also coincide with the speed of convergence for the quadratic variations of the fBm (see [51]) under the Wasserstein distance. The part of the covariance structure depending on the parameter $H + \frac{1}{2}$ becomes insignificant in the limit due to the fact that it contains a faster converging factor $\frac{1}{N^{2H+1}}$. For $k = 2$ it might also be possible to get optimal rates under the total variation distance based on the criteria in [52] by following the outline of a similar proof in [35].

6.2.3 Multivariate central limit theorem

In this part we restrict ourselves to the case of quadratic variations (i.e. $k = 2$) and we derive a multidimensional CLT. This result will be needed in Section 6.4 which deals with the estimation of the Hurst parameter of the solution to (6.1.1).

To establish multidimensional convergence, we will use Theorem 6.2.3 in [59], which is a version of the multivariate fourth moment theorem. Let us recall its statement.

6.2.8 Theorem. *Let $d \geqslant 2$ and $q_1, \ldots, q_d \geqslant 1$ be some fixed integers. Consider vectors*

$$F_n = (F_{1,n}, \ldots, F_{d,n}) = (I_{q_1}(f_{1,n}), \ldots, I_{q_d}(f_{d,n}))$$

with $f_{i,n} \in \mathscr{H}^{\odot q_i}$ (with $\mathscr{H}$ being the underlying Hilbert space). Let C be a real-valued symmetric non negative definite matrix and let $N \sim N_d(0, C)$. Assume that

$$\lim_{n\to\infty} \mathbb{E}\left(F_{i,n}F_{j,n}\right) = C_{ij} \text{ for } i,\, j \in \{1, \ldots, d\}. \tag{6.2.22}$$

Then, as n tends to ∞, the following two conditions are equivalent:

- *F_n converges in law to N,*
- *for every $1 \leqslant i \leqslant d$ $F_{i,n}$ converges in law to $N(0, C_{ij})$.*

We now state and prove the multivariate CLT for the renormalized sequence (6.2.1) with $k = 2$.

6.2.9 Theorem. *Let $P \geqslant 1$ be an integer and $\alpha^1, \ldots, \alpha^P$ be filters of orders $p_1, \ldots p_P$ and lengths $l_1 + 1, \ldots, l_P + 1$ respectively, where $l_i, p_i \in \mathbb{N}^*, i = 1, .., P$. Let $V_N(2, \alpha)$ be given by (6.2.1). If $p_1, \ldots, p_P > H + \frac{1}{4}$, we have*

$$(\sqrt{N}V_N(2, \alpha^1), \ldots, \sqrt{N}V_N(2, \alpha^P)) \to N\left(0, \Theta\right),$$

where $(\Theta)_{i,j=1\ldots,P}$ denotes a $P \times P$ matrix with entries given by

$$\Theta_{n,m} = \frac{t^2}{8c_1(H)^2} \sum_{k=l}^{\infty} \left(\sum_{q_1=0}^{l_1} \sum_{q_2=0}^{l_2} \alpha^n_{q_1} \alpha^m_{q_2} |k + q_1 - q_2|^{2H} \right)^2. \tag{6.2.23}$$

Proof. By (6.2.4) with $k = 2$, with $c_1(H), c_2(H)$ from (6.1.7),

$$\begin{aligned}
\mathbb{E} \quad & \left(V_N\left(k, \alpha^n\right) V_N\left(k, \alpha^m\right)\right) \\
= \quad & \frac{N^{4H+2}}{(N-l)^2\left(c_1(H)N + c_2(H)\right)^2} \sum_{i,j=l}^{N} \mathbb{E}\left(I_2\left({C_{i,\alpha^n}}^{\otimes 2}\right) I_2\left({C_{j,\alpha^m}}^{\otimes 2}\right)\right) \\
= \quad & \frac{2N^{4H+2}}{(N-l)^2\left(c_1(H)N + c_2(H)\right)^2} \sum_{i,j=l}^{N} \langle C_{i,\alpha^n}, C_{j,\alpha^m}\rangle^2_{\mathscr{H}}.
\end{aligned}$$

By (6.1.6), we have for $i, j = l, .., N$

$$\langle C_{i,\alpha^n}, C_{j,\alpha^m}\rangle_{\mathscr{H}} = \mathbb{E}\left(U^{\alpha^n}\left(\frac{i}{N}\right)U^{\alpha^m}\left(\frac{j}{N}\right)\right)$$

$$= \sum_{q_1=0}^{l_1}\sum_{q_2=0}^{l_2}\alpha_{q_1}^n\alpha_{q_2}^m\,\mathbb{E}\left(u\left(t,\frac{i-q_1}{N}\right)u\left(t,\frac{j-q_2}{N}\right)\right)$$

$$= \sum_{q_1=0}^{l_1}\sum_{q_2=0}^{l_2}\alpha_{q_1}^n\alpha_{q_2}^m\left(\frac{N^{-2H-1}}{2}c_H|j-i+q_1-q_2|^{2H+1}-\frac{tN^{-2H}}{4}|j-i+q_1-q_2|^{2H}\right).$$

Plugging this into the covariance expression and using similar computations as in [35], we get

$$\mathbb{E}\left(V_N\left(k,\alpha^n\right)V_N\left(k,\alpha^m\right)\right) \overset{N\to\infty}{\sim} \frac{2N^{4H+3}}{(N-l)^2\left(c_1(H)N+c_2(H)\right)^2}$$
$$\times\sum_{k=l}^{N}\Big(\frac{N^{-2H-1}}{2}c_H\sum_{q_1=0}^{l_1}\sum_{q_2=0}^{l_2}\alpha_{q_1}^n\alpha_{q_2}^m|k+q_1-q_2|^{2H+1}$$
$$-\frac{tN^{-2H}}{4}\sum_{q_1=0}^{l_1}\sum_{q_2=0}^{l_2}\alpha_{q_1}^n\alpha_{q_2}^m|k+q_1-q_2|^{2H}\Big)^2$$
$$=: P_1+P_2+P_3.$$

Using Lemma 6.1.2, we get with $p := \min(p_n,\, p_m)$

$$P_1 \overset{N\to\infty}{\sim} \frac{c_1(H)}{N}\sum_{v=l}^{N}v^{4H-4p},$$

$$P_2 \overset{N\to\infty}{\sim} \frac{c_2(H)}{N^2}\sum_{v=l}^{N}v^{4H-4p+1},$$

$$P_3 \overset{N\to\infty}{\sim} \frac{c_3(H)}{N^3}\sum_{v=l}^{N}v^{4H-4p+2}.$$

This shows that P_1 is the dominant term and it converges for $H < p+\frac{1}{4}$, while the other terms are negligible. We thus obtain the claimed limit:

$$\mathbb{E}\left(\sqrt{N}V_N\left(k,\alpha^n\right)\sqrt{N}V_N\left(k,\alpha^m\right)\right) \overset{N\to\infty}{\to} \Theta_{n,m},$$

where $\Theta_{n,m}$ are given by (6.2.23). The second part of the equivalence in Theorem 6.2.8 has been proved as a particular case of the CLT for higher powers (see Theorem 6.2.2), and thus the statement of the proposition follows. □

6.3 Noncentral limit theorem

The asymptotic normality obtained in the previous section holds for any filter of order $p \geq 2$ or for any filter of order $p = 1$ and $H \leq \frac{3}{4}$. It remains to understand what happens in the case $p = 1$ and $H > \frac{3}{4}$. In this section we consider the filter $\alpha = (1, -1)$ (which has order $p = 1$) and we will show that, after a proper normalization, the quadratic variation associated to this filter converges in distribution to a non-Gaussian limit. Let us start by studying the behaviour of the mean square of the quadratic variation in order to determine a suitable normalisation.

6.3.1 Lemma. *Let $V_N(2, (1, -1))$ be given by (6.2.1). If $v_N := \mathbb{E}[V_N(2,\, (1,\, -1))^2]$ and $H > \frac{3}{4}$ we have*

$$N^{4-4H} v_N \overset{N\to\infty}{\to} \frac{4K_0}{k_1^2},$$

where K_0 will be given in the proof (see (6.3.1)) and k_1 appears in (6.1.6).

Proof. As in Lemma 2 in [35], we have by (6.1.6) with $\Phi_H (= \Phi_{H,(1,-1)})$ defined in Section 6.1.2

$$\begin{aligned}
v_N &= \frac{2N^{4H}}{(k_1 N + k_2)^2} \\
&\quad \times \sum_{i,j=0}^{N-1} \left[\mathbb{E}\left(\left(u\left(t, \frac{i+1}{N}\right) - u\left(t, \frac{i}{N}\right) \right) \left(u\left(t, \frac{j+1}{N}\right) - u\left(t, \frac{j}{N}\right) \right) \right) \right]^2 \\
&= \frac{2N^{4H}}{(k_1 N + k_2)^2} \sum_{i,j=1}^{N} \left[k_1 \frac{\Phi_H(i-j)}{N^{2H}} + k_2 \frac{\Phi_{H+\frac{1}{2}}(i-j)}{N^{2H+1}} \right]^2 \\
&= \frac{4N^{4H}}{(k_1 N + k_2)^2} \sum_{j=1}^{N} \sum_{i=j+1}^{N-1} \left[k_1 \frac{\Phi_H(i-j)}{N^{2H}} + k_2 \frac{\Phi_{H+\frac{1}{2}}(i-j)}{N^{2H+1}} \right]^2 \\
&\quad + \frac{2N^{4H}}{(k_1 N + k_2)^2} \sum_{i=1}^{N} \left[k_1 \frac{1}{N^{2H}} + k_2 \frac{1}{N^{2H+1}} \right]^2 \\
&= \frac{4N^{4H}}{(k_1 N + k_2)^2} \sum_{l=1}^{N} \left[k_1 \frac{\Phi_H(l)}{N^{2H}} + k_2 \frac{\Phi_{H+\frac{1}{2}}(l)}{N^{2H+1}} \right]^2 (N - l) \\
&\quad + \frac{2N^{4H}}{(k_1 N + k_2)^2} \sum_{i=1}^{N} \left[k_1 \frac{1}{N^{2H}} + k_2 \frac{1}{N^{2H+1}} \right]^2.
\end{aligned}$$

The last summand satisfies

$$\frac{2N^{4H}}{(k_1 N + k_2)^2} \sum_{i=1}^{N} \left[k_1 \frac{1}{N^{2H}} + k_2 \frac{1}{N^{2H+1}} \right]^2 \leq C$$

for N large enough while the first summand converges to infinity, see below. Using the asymptotic behaviour of Φ_H and $\Phi_{H+\frac{1}{2}}$, namely

$$\Phi_H(l) = H(2H-1)l^{2H-2} + o(l^{2H-2})$$

and

$$\Phi_{H+\frac{1}{2}}(l) = H(2H+1)l^{2H-1} + o(l^{2H-1})$$

for l large, we obtain

$$\begin{aligned} v_N \quad & \overset{N\to\infty}{\sim} \quad \frac{4}{k_1^2} N^{4H-2} \sum_{l=1}^{N} \left[k_1 H(2H-1) \frac{l^{2H-2}}{N^{2H}} + k_2 H(2H+1) \frac{l^{2H-1}}{N^{2H+1}} \right]^2 (N-l) \\ & = \quad \frac{4}{k_1^2} N^{4H-4} \frac{1}{N} \\ & \quad \times \sum_{l=1}^{N} \left[k_1 H(2H-1) \left(\frac{l}{N}\right)^{2H-2} + k_2 H(2H+1) \left(\frac{l}{N}\right)^{2H-1} \right]^2 \left(\frac{N-l}{N}\right), \end{aligned}$$

and therefore,

$$N^{4-4H} \frac{k_1^2}{4K_0} v_N \overset{N\to\infty}{\to} 1$$

with

$$\begin{aligned} K_0 &= \int_0^1 \left(k_1 H(2H-1)x^{2H-2} + k_2 H(2H+1)x^{2H-1}\right)^2 (1-x)dx \\ &= k_1^2 \frac{H^2(2H-1)}{2(4H-3)} + 2k_1k_2 \frac{H^2(2H+1)}{2(4H-1)} + k_2^2 \frac{H(2H+1)^2}{4(4H-1)}. \qquad (6.3.1) \end{aligned}$$

□

Recall that the solution to the wave equation with fractional-white noise can be written as

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G_1(t-s, x-y) W^H(\mathrm{d}s, \mathrm{d}y). \qquad (6.3.2)$$

Let $x_i = \frac{i}{N}$, $i = 0, 1, .., N$ be a partition of the unit interval $[0,1]$. Denote

$$g_{t,i}(s,\, x) = G_1(t-s, x_{i+1}-x) - G_1(t-s, x_i - x)$$

for $i = 0, 1, .., N-1$ and for $t \geq 0, x \in \mathbb{R}$, with G_1 given by

$$G_1(t,\, x) = \frac{1}{2} \mathbf{1}_{\{|x|<t\}}$$

(see Section 2.1.4). We can write

$$u(t,x_{i+1})-u(t,x_i)=I_1^W(g_{t,i}),$$

where I_1^W represents the multiple integral of order 1 with respect to the fractional-white Gaussian noise W^H. Then we have

$$V_N(2,(1,-1)):=V_N=\frac{1}{N}\sum_{i=1}^{N}\frac{I_2^W(g_{t,i}^{\otimes 2})}{\mathbb{E}\left(u(t,x_{i+1})-u(t,x_i)\right)^2},$$

and so

$$F_N:=\frac{V_N}{\sqrt{v_N}}=I_2(f_N)\quad \text{with}\quad f_N(x_1,x_2)=\frac{1}{\sqrt{Nv_N}}\frac{N^{2H+\frac{1}{2}}}{k_1N+k_2}\sum_{i=1}^{N}g_{t,i}^{\otimes 2}(x_1,x_2). \tag{6.3.3}$$

Since in this part we will use the multiple stochastic integrals with respect to the Gaussian noise W^H with covariance (2.1.9), let us recall some facts about them. Designate by ξ the set of linear combinations of the simple functions $\mathbf{1}_{\{[0,t]\times A\}},\ t\in[0,T],\ A\in\mathcal{B}_b(\mathbb{R}^d)$. The canonical Hilbert space $\mathscr{H}^W$ associated to the field W^H, when $H>\frac{1}{2}$, is defined as the closure of the linear space generated by ξ with respect to the inner product $\langle .,.\rangle_{\mathscr{H}^W}$ which is expressed by

$$\begin{aligned}\langle \mathbf{1}_{\{[0,t]\times A\}},\mathbf{1}_{\{[0,s]\times B\}}\rangle_{\mathscr{H}^W} &:= \mathbb{E}(W_t^H(A)W_s^H(B))\\ &= \alpha_H\lambda(A\cap B)\int_0^t\int_0^s\mid u-v\mid^{2H-2}\mathrm{d}u\mathrm{d}v.\end{aligned}$$

The scalar product in $\mathscr{H}^W$ is given by

$$\langle f,g\rangle_{\mathscr{H}^W}=\mathbb{E}(W^H(f)W^H(g))=\alpha_H\int_0^T\int_0^T\int_{\mathbb{R}^d}f(u,x)g(v,x)\mid u-v\mid^{2H-2}\mathrm{d}x\mathrm{d}u\mathrm{d}v. \tag{6.3.4}$$

for every $f,g\in\mathscr{H}^W$ such that

$$\int_0^T\int_0^T\int_{\mathbb{R}^d}|f(u,x)g(v,x)|\mid u-v\mid^{2H-2}\mathrm{d}x\mathrm{d}u\mathrm{d}v<\infty.$$

It is possible to represent the Wiener integral with respect to W^H as an integral with respect to a white noise field with space-time white noise W via a transfer formula given by

$$\int_0^T\int_{\mathbb{R}}f(s,y)dW^H(s,y)=\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\mathbb{1}_{\{[0,t]\}}(u)f(u,x)(u-s)_+^{H-\frac{3}{2}}du\right)dW(s,y) \tag{6.3.5}$$

(see [77] for details).

Let us introduce a definition which will be useful for further investigation.

6.3.2 Definition. For a random variable F having all moments we define its mth cumulant as

$$k_m(F) = (-i)^n \frac{\partial^n}{\partial t^n} \ln \mathbb{E}(e^{itF})|_{t=0}.$$

We have the following link between the moments and the cumulants of F (see [51]): for every $m \geq 1$,

$$k_m(F) = \sum_{\sigma=(a_1,..,a_r)\in\mathcal{P}(\{1,..,n\})} (-1)^{r-1}(r-1)!\, \mathbb{E}\, X^{|a_1|} \dots \mathbb{E}\, X^{|a_r|} \tag{6.3.6}$$

if $F \in L^m$, where $\mathcal{P}(b)$ is the set of all partitions of b. In particular, for centred random variables F we have $k_1(F) = \mathbb{E}\, F, k_2(F) = \mathbb{E}\, F^2, k_3(F) = \mathbb{E}\, F^3, k_4 = \mathbb{E}\, F^4 - (\mathbb{E}\, F^2)^2$. As stated in [51], the law of the second Wiener chaos elements is completely determined by its cumulants (or equivalently, by its moments). That is, if F, G are elements of the second Wiener chaos then F and G have the same law if and only if they have the same cumulants. Moreover, the convergence of the cumulants to cumulants of an element of the second Wiener chaos implies convergence in distribution. Hence, we can analyse the asymptotic behaviour of the cumulants of the sequence F_N in order to prove a limit theorem for it and characterise its limiting distribution.

In the particular situation when $F = I_2(f)$ (which is satisfied for the random variables F_N) its cumulants can be computed as (see e.g. [50], Proposition 7.2 or [77])

$$k_m(F) = 2^{m-1}(m-1)! \int_{\mathbb{R}^m} f(u_1,u_2)f(u_2,u_3)\dots f(u_{m-1},u_m)f(u_m,u_1)du_1\dots du_m, \tag{6.3.7}$$

with $u_1, \dots, u_m$ possibly being multidimensional.

Based on the formula (6.3.7) we obtain the limit in distribution of (6.3.3).

6.3.3 Theorem. *Let F_N be given by (6.3.3) with $H > \frac{3}{4}$. Then the sequence $(F_N)_{N\geq 1}$ converges in distribution to a random variable F whose law is given by the cumulants explicitly determined in the proof (see (6.3.9) and (6.3.10)).*

Proof. Note first that by the transfer formula (6.3.5) $W^H(g_{t,i})$ has a representation as $W(\tilde{g}_{t,i})$ for some (explicitly known) function $\tilde{g}_{t,i}$, where W is a two-dimensional Gaussian noise. Therefore, $k_1(F_N) = 0, k_2(F_N) = 1$, the above formula for cumulants (6.3.7) can be applied and we obtain for $m \geq 3$

$$k_m(F_N) = 2^{m-1}(m-1)! \left(\frac{1}{\sqrt{N v_N}} \frac{N^{2H+\frac{1}{2}}}{k_1 N + k_2} \right)^m \int_{\mathbb{R}^m} \left(\sum_{j_1=1}^{N} \tilde{g}_{t,j_1}^{\otimes 2}(x_1, x_2) \right)$$

$$\times \left(\sum_{j_2=1}^{N} \tilde{g}_{t,j_2}^{\otimes 2}(x_2, x_3) \right) \dots \left(\sum_{j_m=1}^{N} \tilde{g}_{t,j_m}^{\otimes 2}(x_m, x_1) \right) dx_1 \dots dx_m$$

$$= 2^{m-1}(m-1)! \left(\frac{1}{\sqrt{N v_N}} \frac{N^{2H+\frac{1}{2}}}{k_1 N + k_2} \right)^m \sum_{j_1,\dots,j_m=1}^{N} \left(\int_{\mathbb{R}} \tilde{g}_{t,j_1}(x) \tilde{g}_{t,j_2}(x) dx \right)$$

$$\times \left(\int_{\mathbb{R}} \tilde{g}_{t,j_2}(x) \tilde{g}_{t,j_3}(x) dx \right) \dots \left(\int_{\mathbb{R}} \tilde{g}_{t,j_m}(x) \tilde{g}_{t,j_1}(x) dx \right).$$

We use the isometry formula for multiple integrals with respect to W (6.3.4) as well as the transfer formula (6.3.5) in order to get

$$\begin{aligned} \int_{\mathbb{R}} \tilde{g}_{t,j_1}(x) \tilde{g}_{t,j_2}(x) dx &= \mathbb{E} \left(u(t, x_{i+1}) - u(t, x_i) \right) \left(u(t, x_{j+1}) - u(t, x_j) \right) \\ &= k_1 \Phi_H \left(\frac{i-j}{N} \right) + k_2 \Phi_{H+\frac{1}{2}} \left(\frac{i-j}{N} \right), \end{aligned}$$

where

$$\Phi_H(k) = \frac{1}{2} \left(|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H} \right), \quad k \in \mathbb{R}, \qquad (6.3.8)$$

(see Section 6.1.2) and we obtain

$$k_m(F_N) = 2^{m-1}(m-1)! \left(\tfrac{1}{\sqrt{N v_N}} \tfrac{N^{2H+\frac{1}{2}}}{k_1 N + k_2} \right)^m$$
$$\times \sum_{j_1,\dots,j_m=1}^{N} \left[k_1 \Phi_H \left(\tfrac{j_1 - j_2}{N} \right) + k_2 \Phi_{H+\frac{1}{2}} \left(\tfrac{j_1 - j_2}{N} \right) \right] \dots \left[k_1 \Phi_H \left(\tfrac{j_m - j_1}{N} \right) + k_2 \Phi_{H+\frac{1}{2}} \left(\tfrac{j_m - j_1}{N} \right) \right].$$

By Lemma 6.3.1

$$k_m(F_N) \overset{N \to \infty}{\sim} 2^{m-1}(m-1)! (4K_0)^{-\frac{m}{2}} N^m$$
$$\times \sum_{j_1,..,j_m=1}^{N} \left[k_1 \Phi_H \left(\tfrac{j_1 - j_2}{N} \right) + k_2 \Phi_{H+\frac{1}{2}} \left(\tfrac{j_1 - j_2}{N} \right) \right] \dots \left[k_1 \Phi_H \left(\tfrac{j_m - j_1}{N} \right) + k_2 \Phi_{H+\frac{1}{2}} \left(\tfrac{j_m - j_1}{N} \right) \right].$$

By writing

$$\Phi_H \left(\frac{i-j}{N} \right) = H(2H-1) \int_{\frac{i}{N}}^{\frac{i+1}{N}} \int_{\frac{j}{N}}^{\frac{j+1}{N}} |u-v|^{2H-2} du dv,$$

and similarly

$$\Phi_{H+\frac{1}{2}} \left(\frac{i-j}{N} \right) = H(2H+1) \int_{\frac{i}{N}}^{\frac{i+1}{N}} \int_{\frac{j}{N}}^{\frac{j+1}{N}} |u-v|^{2H-1} du dv,$$

we get, for any $m \geq 3$,

$$k_m(F_N) \overset{N\to\infty}{\sim} 2^{m-1}(m-1)!(4K_0)^{-\frac{m}{2}} N^m \sum_{j_1,\dots,j_m=1}^{N}$$
$$\int_0^1 \int_0^1 (k_1 H(2H-1) N^{-2H} |u-v+j_1-j_2|^{2H-2}$$
$$+k_2 H(2H+1) N^{-2H-1} |u-v+j_1-j_2|^{2H-1}) dudv$$
$$\dots$$
$$\dots$$
$$\times \int_0^1 \int_0^1 (k_1 H(2H-1) N^{-2H} |u-v+j_m-j_1|^{2H-2}$$
$$+k_2 H(2H+1) N^{-2H-1} |u-v+j_m-j_1|^{2H-1}) dudv.$$

Next, we write

$$N^{-2H}|u-v+j_m-j_1|^{2H-2} = N^{-2} \left| \frac{j_1-j_2}{N} \right|^{2H-2} \left| 1 + \frac{u-v}{j_1-j_2} \right|^{2H-2}$$

and

$$N^{-2H-1}|u-v+j_1-j_2|^{2H-1} = N^{-2} \left| \frac{j_1-j_2}{N} \right|^{2H-1} \left| 1 + \frac{u-v}{j_1-j_2} \right|^{2H-1}$$

and obtain

$$k_m(F_N) \overset{N\to\infty}{\sim} 2^{m-1}(m-1)!(4K_0)^{-\frac{m}{2}} N^{-m} \sum_{j_1,..,j_m=1}^{N}$$
$$\int_0^1 \int_0^1 \left[k_1 H(2H-1) \left| \frac{j_1-j_2}{N} \right|^{2H-2} \left| 1 + \frac{u-v}{j_1-j_2} \right|^{2H-2} \right.$$
$$\left. +k_2 H(2H+1) \left| \frac{j_1-j_2}{N} \right|^{2H-1} \left| 1 + \frac{u-v}{j_1-j_2} \right|^{2H-1} \right] dudv$$
$$\dots$$
$$\dots$$
$$\int_0^1 \int_0^1 \left[k_1 H(2H-1) \left| \frac{j_m-j_1}{N} \right|^{2H-2} \left| 1 + \frac{u-v}{j_m-j_1} \right|^{2H-2} \right.$$
$$\left. +k_2 H(2H+1) \left| \frac{j_m-j_1}{N} \right|^{2H-1} \left| 1 + \frac{u-v}{j_m-j_1} \right|^{2H-1} \right] dudv.$$

We claim that

$$k_m(F_N) \overset{N\to\infty}{\sim} 2^{m-1}(m-1)!(4K_0)^{-\frac{m}{2}} N^{-m} \sum_{j_1,..,j_m=1}^{N}$$
$$\left[k_1 H(2H-1)\left|\frac{j_1-j_2}{N}\right|^{2H-2} + k_2 H(2H+1)\left|\frac{j_1-j_2}{N}\right|^{2H-1}\right]$$
$$\dots$$
$$\dots$$
$$\times\left[k_1 H(2H-1)\left|\frac{j_m-j_1}{N}\right|^{2H-2} + k_2 H(2H+1)\left|\frac{j_m-j_1}{N}\right|^{2H-1}\right].$$

This follows by a standard procedure (see [50] or [77]) from the Taylor expansion in the vicinity of $x=0$ of the functions

$$1-(1+x)^{2H-2} \quad \text{and} \quad 1-(1+x)^{2H-1}$$

and by the dominated convergence theorem. Therefore, for $m \geq 3$

$$k_m(F_N) \overset{N\to\infty}{\to} 2^{m-1}(m-1)!(4K_0)^{-\frac{m}{2}} \int_{[0,1]^m}$$
$$\left(k_1 H(2H-1)|x_1-x_2|^{2H-2} + k_2 H(2H+1)|x_1-x_2|^{2H-1}\right)$$
$$\dots$$
$$\left(k_1 H(2H-1)|x_m-x_2|^{2H-2} + k_2 H(2H+1)|x_m-x_1|^{2H-1}\right) dx_1 \dots dx_m.$$

Note that the above integral is finite by Lemma 3.3 in [6]. Also, clearly

$$k_1(F_N)=0 \quad \text{and} \quad k_2(F_N)=1.$$

Since F_N belongs to the second Wiener chaos, the convergence of cumulants determines the convergence of F_N in law to a random variable F with cumulants given by

$$k_m(F) = 2^{m-1}(m-1)!(4K_0)^{-\frac{m}{2}} \int_{[0,1]^m}$$
$$\left(k_1 H(2H-1)|x_1-x_2|^{2H-2} + k_2 H(2H+1)|x_1-x_2|^{2H-1}\right)$$
$$\dots$$
$$\times\left(k_1 H(2H-1)|x_m-x_1|^{2H-2} + k_2 H(2H+1)|x_m-x_1|^{2H-1}\right)$$
$$dx_1 \dots dx_m \qquad (6.3.9)$$

for $m \geq 3$ and

$$k_1(F)=0 \quad \text{and} \quad k_2(F)=1. \qquad (6.3.10)$$

The existence of such a limit is ensured by the Fréchet-Shohat theorem (see [37]): It follows from the convergence of cumulants that also all the moments of F_N converge to some real numbers M_m, $m \in \mathbb{N}$, as N tends to infinity. Moreover, by hypercontractivity (2.1.6) the mth absolute moments of F_N are bounded by $(m-1)^m$. Therefore, also the limits of the moment sequences will be bounded by $(m-1)^m$, which means that the growth condition $\limsup_{m\to\infty}(\frac{1}{m!}|M_m|)^{1/m} < \infty$ is satisfied, thus yielding the existence of a limiting distribution with the cumulants obtained above. □

Note that the limit law with cumulants (6.3.9) and (6.3.10) is related to the Rosenblatt distribution but is more complex. For instance, if the constant k_2 vanished in (6.3.9), then we would have obtained a Rosenblatt distribution in the limit. If the constant k_1 vanished the cumulants would have described a Rosenblatt distribution associated with the parameter $H + \frac{1}{2}$ given the existence of such an object. In total, the obtained limit reflects the covariance structure of the solution and in contrast to the CLTs proved in this chapter also includes the part resembling an fBm-type process with the parameter $H + \frac{1}{2}$.

6.4 Estimation of the Hurst parameter H

We will apply the theoretical results from Section 6.2 in order to construct and analyse several estimators for the Hurst index of the mild solution (6.1.2) to the wave equation (6.1.1). It is worth to emphasize that the estimators are based on the observations of the process u at a fixed time and at discrete points in space.

We will define two kinds of estimators for the Hurst parameter. For the first kind we will consider the observation time t of our equation to be known, and the estimators obtained will be asymptotically normal with the rate of convergence of order $\sqrt{N}\log(N)$ for $H < p - \frac{1}{4}$. In the second case we develop an estimator for H if the time $t > 1$ is not known. This estimator will also be asymptotically normal, but with a slower rate of convergence, namely $\sqrt{N}$. Both kinds of estimators are strongly consistent.

6.4.1 Estimators for known t

We follow the standard procedure from [16] or [15] to construct our estimators. First, let us define an auxiliary object, namely the k-th empirical absolute moment of discrete variations of the mild solution $u(t, x)$ defined in (6.1.2) for a fixed time $t > 1$ and a filter α as follows:

$$S_N(k, \alpha) = \frac{1}{N-l}\sum_{i=l}^{N-1}\left|U^\alpha\left(\frac{i}{N}\right)\right|^k \qquad (6.4.1)$$

with $U^\alpha\left(\frac{i}{N}\right)$ defined in (6.1.5). Since $U^\alpha\left(\frac{i}{N}\right)$ is Gaussian, we have $\mathbb{E}\left[\left|U^\alpha\left(\frac{i}{N}\right)\right|^k\right] = \left(\pi_H^{\alpha,N}(0)\right)^{\frac{k}{2}} E_k$, where E_k denotes the k-th absolute moment of a standard Gaussian random variable, and therefore we obtain

$$\mathbb{E}[S_N(k,\,\alpha)] = \left(\pi_H^{\alpha,N}(0)\right)^{\frac{k}{2}} E_k.$$

Thus, for a given k, by replacing $\mathbb{E}[S_N(k,\,\alpha)]$ by $S_N(k,\,\alpha)$ we obtain an estimator for H that is a pointwise solution to the equation

$$S_N(k,\,\alpha)^{\frac{2}{k}} - E_k^{\frac{2}{k}} \pi_x^{\alpha,N}(0) = 0$$

with respect to x. Recall that (see (6.1.6))

$$\pi_x^{\alpha,N}(0) = \frac{t}{2N^{2x}} \Phi_{x,\alpha}(0) - \frac{c_x}{N^{2x+1}} \Phi_{x+\frac{1}{2},\alpha}(0),$$

and we denote

$$c_1(x) := \Phi_{x,\alpha}(0) = -\frac{1}{2} \sum_{q,r=0}^{l} \alpha_q \alpha_r |q-r|^{2x}, \qquad c_2(x) = c_x \Phi_{x+\frac{1}{2},\alpha}(0).$$

Note that for large N the function $g(x) := \pi_x^{\alpha,N}(0)$ is invertible. In order to see this we consider the derivative

$$g'(x) = \frac{t}{2}\left(\frac{c_1'(x)}{N^{2x}} - \frac{2\log(N)c_1(x)}{N^{2x}}\right) - \left(\frac{c_2'(x)}{N^{2x+1}} - \frac{2\log(N)c_2(x)}{N^{2x+1}}\right).$$

As shown in [16], the expression in the first parentheses becomes negative for large N, and since it is the asymptotically dominating term, also the whole function will become negative for N large enough. Therefore, for such N the function g is strictly decreasing and we can define estimators by inverting it:

$$\widehat{H}_{N,k} := \left(\pi_\cdot^{\alpha,N}(0)\right)^{-1}\left(\left(\frac{S_N(k,\,\alpha)}{E_k}\right)^{\frac{2}{k}}\right). \qquad (6.4.2)$$

Another estimator can be obtained by inverting only the dominant part of the function g. Notice that asymptotically $\pi_x^{\alpha,N}(0)$ is equal to $\frac{t}{2N^{2x}}\Phi_{x,\alpha}(0) =: \bar{g}(x)$, which is easier to invert than its exact counterpart. This motivates the definition of another class of estimators,

$$\bar{H}_{N,k} := \bar{g}^{-1}\left(\left(\frac{S_N(k,\,\alpha)}{E_k}\right)^{\frac{2}{k}}\right). \qquad (6.4.3)$$

We show that the two estimators constructed above are consistent and we give their limiting behavior in distribution.

6.4.1 Proposition. *The estimators $\widehat{H}_{N,k}$ and $\bar{H}_{N,k}$ given by (6.4.2) and (6.4.3) of the Hurst parameter $H \geq \frac{1}{2}$ are strongly consistent. Moreover, with $v_N^{(k)} := \mathbb{E}[V_N(k,\alpha)^2]$, for $H \leq p - \frac{1}{4}$ we have as N tends to infinity*

$$\frac{k\log(N)}{\sqrt{v_N^{(k)}}}\left(H - \widehat{H}_{N,k}\right) \xrightarrow{d} N(0,1)$$

and for $H > \frac{3}{4}$, $\alpha = (1, -1)$, $k = 2$

$$\frac{2\log(N)}{\sqrt{v_N^{(2)}}}\left(H - \widehat{H}_{N,2}\right) \xrightarrow{d} F,$$

where F is the random variable from Proposition 6.3.3. The same statements hold for $\bar{H}_{N,k}$.

Proof. Since for every $k \geq 2$

$$v_N^{(k)} \lesssim \begin{cases} 1/N & \text{if } H < p - \frac{1}{4}\ , \\ \frac{\log(N)}{N} & \text{if } H = \frac{3}{4},\ p = 1, \\ \frac{1}{N^{2-2H}} & \text{if } H > \frac{3}{4},\ \alpha = (1,\,-1),\ k = 2, \end{cases} \tag{6.4.4}$$

the almost sure convergence to zero of $V_N(k,\alpha)$ follows by hypercontractivity with a Borel-Cantelli argument, see e.g. [75]. Due to the fact that the functions g and $\bar{g}$ are asymptotically equal we obtain the asymptotic equality of $\bar{H}_{N,k}$ and $\widehat{H}_{N,k}$ and thus also strong consistency of $\bar{H}_{N,k}$.

For the asymptotic behaviour we can refer to the calculations from [16] and obtain

$$V_N(k,\alpha) = k\log(N)(H - \widehat{H}_{N,k})(1 + o(1)),$$

which means that by Slutsky's lemma we will get

$$\frac{k\log(N)}{\sqrt{v_N^{(k)}}}\left(H - \widehat{H}_{N,k}\right) \xrightarrow{d} N(0,1)$$

for $H \leq p - \frac{1}{4}$ as N tends to infinity. For $H < p - \frac{1}{4}$ this implies in particular that

$$k\log(N)\sqrt{N}\left(H - \widehat{H}_{N,k}\right) \xrightarrow{d} N(0,\sigma^2)$$

for $N \to \infty$ with σ^2 defined in Lemma 6.2.1. For $H > \frac{3}{4}$, $\alpha = (1,\,-1)$, $k = 2$ the relation yields

$$\frac{2\log(N)}{\sqrt{v_N^{(2)}}}\left(H - \widehat{H}_{N,2}\right) \xrightarrow{d} F$$

for F given above when N goes to infinity. The same results follow for $\bar{H}$ due to its asymptotic equality to $\widehat{H}_{N,k}$. □

6.4.2 Remark. Note that this result provides the following speeds of convergence (see (6.4.4)): $\sqrt{N}\log(N)$ for $H < p - \frac{1}{4}$, $\sqrt{N}\sqrt{\log(N)}$ for $H = \frac{3}{4}$, $p = 1$ and $N^{2-2H}\log(N)$ for $H > \frac{3}{4}$, $\alpha = (1, -1)$, $k = 2$.

6.4.2 An estimator for unknown t

Assume that the time $t > 1$ at which the solution (6.1.2) is observed is not known. Similarly to [33], if two sequences $(a_i^{(1)})_{i\in\{0,\dots,p\}}$ and $(a_i^{(2)})_{i\in\{0,\dots,2p\}}$ are considered, where $a^{(2)}$ is obtained by "thinning" the sequence $a^{(1)}$ (i.e., $a_{2k}^{(2)} := a_k^{(1)}$ for $k \in \{0,\dots,p\}$ and zero otherwise), then it follows that

$$\Phi_{H,\,a^{(2)}}(0) = 2^{2H}\Phi_{H,\,a^{(1)}}(0) \text{ and } \Phi_{H+\frac{1}{2},\,a^{(2)}}(0) = 2^{2H+1}\Phi_{H+\frac{1}{2},\,a^{(1)}}(0),$$

which implies that for large N we have $\pi_H^{a^{(2)},N}(0) \sim 2^{2H}\pi_H^{a^{(1)},N}(0)$. This, in turn, can be transferred to S_N:

$$\mathbb{E}[S_N(k,\,a^{(2)})] = \left(\pi_H^{a^{(2)},N}(0)\right)^{\frac{k}{2}} E_k \sim 2^{Hk}\left(\pi_H^{a^{(1)},N}(0)\right)^{\frac{k}{2}} E_k = 2^{Hk}\,\mathbb{E}[S_N(k,\,a^{(1)})].$$

This motivates another estimator for H defined by

$$\tilde{H}_N := \frac{1}{k}\log_2\left(\frac{S_N(k,\,a^{(2)})}{S_N(k,\,a^{(1)})}\right). \tag{6.4.5}$$

Its limit behavior is given below.

6.4.3 Proposition. *The estimator $\tilde{H}_N$ (6.4.5) is strongly consistent for all $H \geq \frac{1}{2}$. Moreover for $H < p - \frac{1}{4}$, we have*

$$\sqrt{N}(\tilde{H}_N - H) \xrightarrow{d} N(0,\,\sigma^2)$$

with $\sigma > 0$.

Proof. It follows from the fact that $V_N(k,\alpha) \overset{N\to\infty}{\to} 0$ almost surely that S_N converges almost surely to its expectation. Thus, strong consistency is clear by construction of $\tilde{H}_N$. The multivariate convergence statement yields asymptotic normality by the delta method, similarly to [16]. □

6.4.3 Numerical computations and simulation experiments

In this section we conduct simulations of the solution process and compare numerical performances of different estimators introduced in the previous section. More specifically, we are going to analyse the behaviour of $\bar{H}_{N,2}$ for filters $(1, -1)$ as well as $(1, -2, 1)$, that of its exact counterpart $\hat{H}_{N,2}$ for the second filter as well as that of $\tilde{H}_N$ for different values of H. Methodically the simulation scheme is simple: We simulate a vector of discrete observations of the solution as a multivariate normally distributed vector with a given covariance matrix using the function `mvrnorm` in the programming language `R` (the matrix is decomposed using the eigendecomposition). The construction of estimators from this vector is straightforward and the source code for both can be found in the Appendix. Possible errors can occur when the covariance matrix is decomposed or when numerical operations are carried out (such as calculating the inverse of a function at a given point). The obtained results, however, do not seem to be strongly jeopardised by those as they are in accordance with the theory.

For $N = 1000$ and $t = 3$ we get the following results for the mean squared errors (MSE) computed from 100 iterations:

	$H = 0.51$	$H = 0.7$	$H = 0.95$
$\bar{H}_{N,2}(1, -1)$	$1.02 \cdot 10^{-5}$	$1.61 \cdot 10^{-5}$	0.001
$\bar{H}_{N,2}(1, -2, 1)$	$1.2 \cdot 10^{-5}$	$9.626 \cdot 10^{-6}$	$1.98 \cdot 10^{-6}$
$\hat{H}_{N,2}(1, -2, 1)$	$1.2 \cdot 10^{-5}$	$9.634 \cdot 10^{-6}$	$1.99 \cdot 10^{-6}$
$\tilde{H}_N$	0.002	0.001	0.001

.

The estimator $\tilde{H}_N$ performs the worst. This can be explained heuristically by the fact that it contains two sources of error instead of one, this being the practical trade-off in the case where time t is not available. Another interesting observation is that the exact estimator $\hat{H}_{N,2}$ is not performing better than the estimator $\bar{H}_{N,2}$ which uses the inverse of an approximation of the actual function. This encourages the use of the simpler version in applications.

True value H	Mean $\bar{H}_{N,2}(1, -1)$	Mean $\bar{H}_{N,2}(1, -2, 1)$	Mean $\hat{H}_{N,2}(1, -2, 1)$	Mean$\tilde{H}_N$
0.51	0.5107118	0.5138851	0.5110081	0.5110082
0.55	0.5499827	0.5362797	0.549677	0.549678
0.60	0.5997487	0.6007376	0.5999698	0.5999722
0.65	0.6498786	0.6510065	0.6502865	0.6502909
0.70	0.7005558	0.6925	0.7003125	0.7003196
0.75	0.7500486	0.7482407	0.7499587	0.74997
0.80	0.8005769	0.7966326	0.7998019	0.7998186
0.85	0.8512704	0.8517664	0.8500505	0.8500754
0.90	0.9042009	0.8927607	0.8997257	0.8997638
0.95	0.9587621	0.9540507	0.9498974	0.9499602
0.99	1.01826	0.9959974	0.9898137	0.9899168

Table 6.1: Mean of the estimated values for 100 simulations

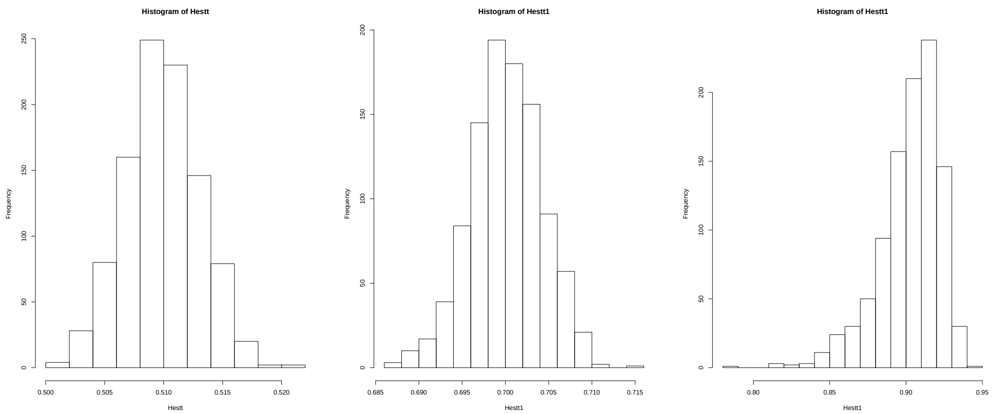

Figure 6.1: Histograms for $H = 0.51$, 0.7 and 0.9 respectively.

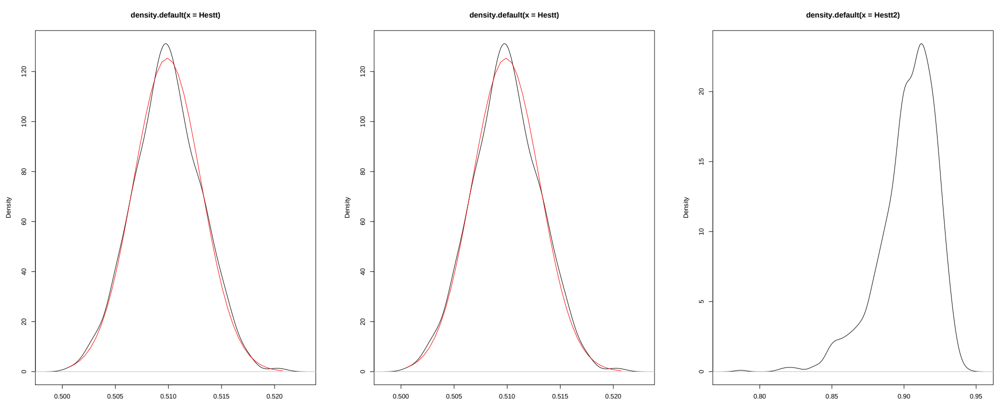

Figure 6.2: Normal fits of empirical densities for $H = 0.51$, 0.7, empirical density plot for $H = 0.9$.

The Figures 6.1 and 6.2 show the change in the limiting distribution (illustrated by histograms and density fits for the simple estimator $\bar{H}_{N,2}(1, -1)$ over 100 simulations): For $H = 0.51$, $H = 0.7$ the limiting distribution is normal and for $H > \frac{3}{4}$ it is not. Additionally, the boxplots in Figure 6.3 illustrate the changes in the speed of convergence indicated in the discussion for $\bar{H}_{N,2}(1, -1)$ and provide a comparison to the rates of convergence for the other three estimators (see Remark 6.4.2).

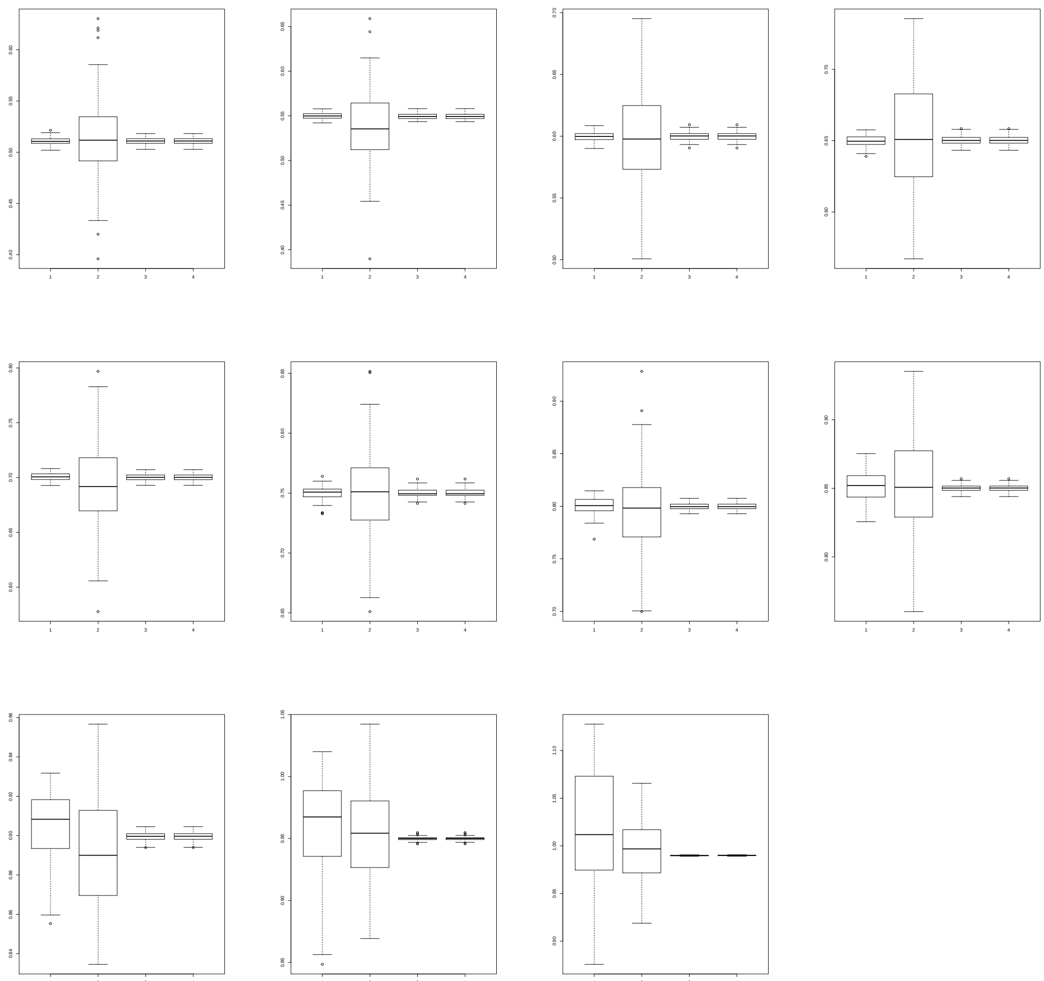

Figure 6.3: Boxplots of $\bar{H}_{N,2}(1,\,-1)$, $\tilde{H}_N$, $\bar{H}_{N,2}(1,\,-2,\,1)$, $\hat{H}_{N,2}(1,\,-2,\,1)$ from left to right for the values of H listed above.

List of Symbols

$\lesssim$	$a_n \lesssim b_n :\Leftrightarrow a_n \leq cb_n$ for n large
$\sim$	$a_n \overset{n\to\infty}{\sim} b_n :\Leftrightarrow a_n \lesssim b_n$ and $b_n \lesssim a_n$
β	Beta function
Γ	Gamma function
ζ	Riemann zeta function
$(\cdot)_n$	falling factorial, $(z)_n := z(z-1)\dots(z-n+1)$
$(\cdot)^{(n)}$	rising factorial, $(z)^{(n)} := z(z+1)\dots(z+n-1)$
$\mathscr{H}$	separable Hilbert space with scalar product $\langle\cdot,\cdot\rangle_{\mathscr{H}}$, Section 2.1.2
$\mathscr{H}^H$	separable Hilbert space induced by an fBm with scalar product $\langle\cdot,\cdot\rangle_{\mathscr{H}^H}$, Section 2.1.2
$\mathscr{H}^W$	separable Hilbert space induced by a fractional-Brownian field with scalar product $\langle\cdot,\cdot\rangle_{\mathscr{H}^W}$, Section 2.1.4
$\mathscr{H}^{\otimes p}$	p-fold tensor product of $\mathscr{H}$
$\mathscr{H}^{\odot p}$	p-fold symmetric tensor product of $\mathscr{H}$, Section 2.1.2
$\mathscr{S}$	space of smooth random variables, Section 2.1.2
D^p	(p-fold) derivative operator, Section 2.1.2
$\\|\cdot\\|_{p,q}$	norm on $\mathscr{S}$, Section 2.1.2
$\mathbb{D}^{p,q}$	closure of the derivative operator w.r.t. $\\|\cdot\\|_{p,q}$, Section 2.1.2
δ^p	(p-fold) divergence operator, Section 2.1.2
$\mathrm{Dom}(\delta^p)$	domain of the divergence operator
$\mathrm{Tr}(\cdot)$	trace of an operator on a Hilbert space
$\\|\cdot\\|_{Lip}$	Lipschitz norm, Section 6.2.2

$\|\cdot\|_{p-var}$	$\|x\|_{p-var} =: \|x\|_{p-var,\,[0,\,T]}$ $=: \left(\sup_{\Pi\in\Delta[0,\,T]} \sum_{k=0}^{n-1} \lvert x(t_{k+1}) - x(t_k)\rvert^p\right)^{\frac{1}{p}}$, where the supremum is taken over partitions of the interval $[0,\,T]$
$\mathcal{B}_b(\mathbb{R}^d)$	bounded Borel subsets of $\mathbb{R}^d$
$l^p(\mathbb{Z})$	l^p space of sequences indexed over $\mathbb{Z}$
$M_p(\mathbb{R})$	real $p \times p$ matrices
$\stackrel{d}{=}$	equality in law for random variables
$\stackrel{d}{\equiv}$	equality of all finite-dimensional distributions for processes
$\stackrel{d}{\to}$	convergence in distribution
d_W	Wasserstein distance, Section 6.2.2

Appendix

Source code for the simulation of the wave equation solution, implementation of four estimators from Section 6.4 and calculation of the respective mean squared errors written in the programming language R. The source code for the plots is not included.

```
library(kergp)
#define the covar-fctn
kernFun<-function(x1,x2,par){
h<-abs(x1-x2)
cH<-(4*par[1]-1)/(4*(2*par[1]+1))
K<-0.5*(cH*h^(2*par[1]+1)-0.5*par[2]*h^(2*par[1])
+(par[2]^(2*par[1]+1))/(2*par[1]+1))
return(K)
}

#set parameters, build a matrix, simulate and plot
H<-0.51
t<-3
covar<-covMan(kernFun, d=1, parNames = c("Hurst", "time"), par = c(H, t))
covar
nGrid=1000
xGrid=seq(from=0,to=1,length=nGrid)
Kmat<-covMat(object=covar, X=as.matrix(xGrid))
library(MASS)
yGrid<-mvrnorm(mu=rep(0, nGrid), Sigma=Kmat)
plot(xGrid, yGrid, type = "l")

#k=2, est for (1,-1)
incr<-numeric(nGrid-1)
for (i in 1:(nGrid-1)) {
incr[i]=yGrid[i+1]-yGrid[i]
}
SN=sum(incr^2)/nGrid
Hest=(-log(SN)+log(t/2))/(2*log(nGrid))
```

```
Hest

#k=2, est for a quotient
U1 = 0
for (i in 3:nGrid)
{
U1=U1+(yGrid[i-2]-2*yGrid[i-1]+yGrid[i])^2
}
U2 = 0
for (i in 5:nGrid)
{
U2=U2+(yGrid[i]-2*yGrid[i-2]+yGrid[i-4])^2
}
Hest1=(1/(2*log (2)))*log(U2/U1)
Hest1

#inverting a fctn
inverse = function (f, lower = -100, upper = 100) {
function (y) uniroot((function (x) f(x) - y),
lower = lower, upper = upper)[1]
}

#approx and exact inverse + est for k=2, (1,-2,1)
#(calculate U1 first!, U1/nGrid=SN)
invp2 = inverse(function (x) (t*(2-2^(2*x-1)))/(nGrid^(2*x)), 0.05, 10)
invp2(U1/nGrid)
invp2exact = inverse(function (x) (t*(2-2^(2*x-1)))/(nGrid^(2*x))-
(4*x-1)*(2-2^(2*x))/(2*(2*x+1)*(nGrid^(2*x+1))), 0.05, 10)
invp2exact(U1/nGrid)

#MSE for diff est, takes around 10-20 mins
Hest0<-numeric(100)#simple est
Hest1<-numeric(100)#quotient
Hest2<-numeric(100)#approx
Hest3<-numeric(100)#exact
for (k in 1:100)
{
yGrid<-mvrnorm(mu=rep(0, nGrid), Sigma=Kmat)
for (i in 1:(nGrid-1)) {
incr[i]=yGrid[i+1]-yGrid[i]
}
SN=sum(incr^2)/nGrid
Hest0[k]=(-log(SN)+log(t/2))/(2*log(nGrid))
```

```
U1 = 0
for (i in 3:nGrid)
{
U1=U1+(yGrid[i-2]-2*yGrid[i-1]+yGrid[i])^2
}
U2 = 0
for (i in 5:nGrid)
{
U2=U2+(yGrid[i]-2*yGrid[i-2]+yGrid[i-4])^2
}
Hest1[k]=(1/(2*log (2)))*log(U2/U1)

Hest2[k]=invp2(U1/nGrid)
Hest3[k]=invp2exact(U1/nGrid)
}
Htrue<-rep(H, 100)
library(Metrics)
mean(Hest0)
mse(Hest0, Htrue)
mse(Hest1, Htrue)
mse(as.numeric(Hest2), Htrue)
mse(as.numeric(Hest3), Htrue)
```

Bibliography

[1] M. Abramowitz, I. Stegun - Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. U.S. Government Printing Office, Washington (1972).

[2] E. Alòs, D. Nualart - Stochastic integration with respect to the fractional Brownian motion.
Stochastics and Stochastics Reports, 75(3), 129-152 (2003).

[3] T. M. Apostol - Introduction to Analytic Number Theory. Springer, New York Heidelberg Berlin (1976).

[4] A. V. Artemov, E. V. Burnaev - Optimal estimation of a signal perturbed by a fractional Brownian noise.
Theory of Probability and Its Applications, 60(1), 126-134 (2016).

[5] E. Azmoodeh, L. Viitasaari - Parameter estimation based on discrete observations of fractional Ornstein-Uhlenbeck process of the second kind.
Statistical Inference for Stochastic Processes, 18(3), 205-227 (2015).

[6] S. Bai, M. Taqqu - Behavior of the generalized Rosenblatt process at extremes critical exponent values.
The Annals of Probability, 45(2), 1278-1324 (2017).

[7] S. Bajja, K. Es-Sebaiy, L. Viitasaari - Least squares estimator of fractional Ornstein Uhlenbeck processes with periodic mean.
Journal of Korean Statistical Society 46(4), 608-622 (2017).

[8] R. M. Balan, C. A. Tudor - The stochastic wave equation with fractional colored noise: a random field approach.
Stochastic Processes and their Applications, 120, 2468-2494 (2010).

[9] R. Belfadli, K. Es-Sebaiy, Y. Ouknine - Parameter estimation for fractional Ornstein-Uhlenbeck processes: non-ergodic case.
Frontiers of Environmental Science & Engineering in China, 1(1), 1-16 (2011).

[10] F. E. Benth - On arbitrage-free pricing of weather derivatives based on fractional Brownian motion.
Applied Mathematical Finance, 10, 303-324 (2003).

[11] F. Biagini, Y. Hu, B. Øksendal, T. Zhang - Stochastic Calculus for Fractional Brownian Motion and Applications. Springer, London (2008).

[12] P. Breuer, P. Major - Central limit theorems for non-linear functionals of Gaussian fields.
Journal of Multivariate Analysis, 3, 425-441 (1983).

[13] A. Brouste, M. Fukasawa - Local asymptotic normality property for fractional Gaussian noise under high-frequency observations.
The Annals of Statistics, 46(5), 2045-2061 (2018).

[14] P. Cheridito, H. Kawaguchi, M. Maejima - Fractional Ornstein-Uhlenbeck processes.
Electronic Journal of Probability, 8, 1-14 (2003).

[15] A. Chronopoulou, C. A. Tudor, F. G. Viens - Variations and Hurst index estimation for a Rosenblatt process using longer filters.
Electronic Journal of Statistics 3, 1393-1435 (2009).

[16] J.-F. Coeurjolly - Estimating the parameters of a fractional Brownian motion by discrete variations of its sample paths.
Statistical Inference for Stochastic Processes 30, 199-227 (2001).

[17] F. Comte - Nonparametric Estimation. Spartacus-idh, Paris (2017).

[18] F. Comte, V. Genon-Catalot - Nonparametric adaptive estimation for pure jump Lévy processes.
Annales de l'I.H.P. Probabilités et statistiques, 46, 595-617 (2010).

[19] M. Corcuera, D. Nualart, J. H. C. Woerner - Power variation of some integral fractional processes.
Bernoulli, 12, 713-735 (2006).

[20] R. Dalang, D. Khoshnevisan, C. Mueller, D. Nualart, Y. Xiao - A Minicourse on Stochastic Partial Differential Equations. Springer Science & Business Media (2009).

[21] J. Clarke De la Cerda, C.A. Tudor - Hitting probabilities for the stochastic wave equation with fractional colored noise.
Revista Matemática Iberoamericana, 30 (2), 685-709 (2014).

[22] L. Decreusefond, A. S. Üstünel - Stochastic analysis of the fractional Brownian motion.
Potential Analysis, 10, 177-214 (1999).

[23] H. Dehling, B. Franke, T. Kott - Drift estimation for a periodic mean reversion process.
Statistical Inference for Stochastic Processes, 13, 175-192 (2010).

[24] H. Dehling, B. Franke, J. H. C. Woerner - Estimating drift parameters in a fractional Ornstein Uhlenbeck process with periodic mean.
Statistical Inference for Stochastic Processes, 20, 1-14 (2017).

[25] K. Es-Sebaiy, I. Nourdin - Parameter estimation for α-fractional bridges.
Malliavin Calculus and Stochastic Analysis (2011).

[26] R. Fox, M. S. Taqqu - Multiple stochastic integrals with dependent integrators.
Journal of Multivariate Analysis 21, 105-127 (1987).

[27] B. Franke, T. Kott - Parameter estimation for the drift of a time-inhomogenous jump diffusion process.
Statistica Neerlandica, 13, 175-192 (2013).

[28] P. K. Friz, N. B. Victoir - Multidimensional Stochastic Processes as Rough Paths: Theory and Applications. Cambridge University Press (2010).

[29] M. Gubinelli, A. Lejay, S. Tindel - Young integrals and SPDE's.
Potential Analysis 25, 307-326 (2006).

[30] Y. Hu, D. Nualart - Parameter estimation for fractional Ornstein Uhlenbeck processes.
Statistics and Probability Letters, 80, 1030-1038 (2010).

[31] Y. Hu, D. Nualart, H. Zhou - Parameter estimation for fractional Ornstein-Uhlenbeck processes of general Hurst parameter.
Statistical Inference for Stochastic Processes, 22, 111-124 (2019).

[32] H. E. Hurst - Long-term storage capacity of reservoirs.
Transactions of the American Society of Civil Engineers, 116, 770-799 (1951).

[33] J. Istas, G. Lang - Quadratic variations and estimation of the local Hölder index of a Gaussian process.
Annales de l'I.H.P. Probabilités et statistiques, 33(4), 407-436 (1997).

[34] J. Keizer - Statistical Thermodynamics of Nonequilibrium Processes. springer, New York (1987).

[35] M. Khalil, C. A. Tudor - Correlation structure, quadratic variations and parameter estimation for the solution to the wave equation with fractional noise.
Electronic Journal of Statistics, 12 (2), 3639-3672 (2018).

[36] M. Khalil, C.A. Tudor, M. Zili - Spatial variation for the solution to the stochastic linear wave equation driven by additive space-time white noise.
Stochastics and Dynamics, 18 (5) (2018).

[37] A. Klenke - Probability Theory A Comprehensive Course. Springer, London (2014).

[38] M. L. Kleptsyna, A. Le Breton - Statistical analysis of the fractional Ornstein-Uhlenbeck type process.
Statistical Inference for Stochastic Processes, 5, 229-248 (2002).

[39] P. Kloeden, A. Neuenkirch - The pathwise convergence of approximation schemes for stochastic differential equations.
LMS Journal of Computation and Mathematics, 10, 235-253 (2007).

[40] I. Kruk, F. Russo, C. A. Tudor - Wiener integrals, Malliavin calculus and covariance measure structure.
Journal of Functional Analysis, 249 (1), 92-142 (2007).

[41] K. Kubilius, Y. Mishura, K. Ralchenko - Parameter Estimation in Fractional Diffusion Models. Springer, Cham (2017).

[42] Y. A. Kutoyants - Statistical Inference for Ergodic Diffusion Processes. Springer, London (2004).

[43] A. J. Lawrance, N. T. Kottegoda - Stochastic modelling of riverflow time series.
Journal of the Royal Statistical Society. Series A (General), 140(1), 1-47 (1977).

[44] Y. Liu, E. Nualart, S. Tindel - LAN property for stochastic differential equations with additive fractional noise and continuous time observation.
Stochastic Processes and their Applications, 129, 2880-2902 (2019).

[45] M. Maejima, C. A. Tudor - Wiener Integrals with respect to the Hermite process and a Non-Central Limit Theorem.
Stochastic Analysis and its Applications 25(5), 1043-1056 (2007).

[46] B. Mandelbrot, J. W. van Ness - Fractional Brownian motions, fractional noises and applications.
SIAM Review, 10, 422-437 (1968).

[47] E. Martins - Estimating the rate of phenotypic evolution from comparative data.
American Naturalist, 144, 2, 193-209 (1994).

[48] Y. Mishura - Stochastic Calculus for Fractional Brownian Motion and Related Processes. Springer, Berlin-Heidelberg (2008).

[49] M. Moers - Hypothesis testing in a fractional Ornstein-Uhlenbeck model.
International Journal of Stochastic Analysis (2012).

[50] I. Nourdin - Selected Aspects of Fractional Brownian Motion. Springer, Mailand (2012).

[51] I. Nourdin, G. Peccati - Normal Approximations With Malliavin Calculus. Cambridge University Press (2012).

[52] I. Nourdin, G. Peccati - The optimal fourth moment theorem.
Proc. Amer. Math. Soc., 143(7), 3123-3133 (2015).

[53] I. Nourdin, G. Peccati, M. Podolskij - Quantitative Breuer-Major theorems.
Stochastic Process. Appl. 121(4), 793–812 (2011).

[54] I. Nourdin, G. Peccati, A. Réveillac - Multivariate normal approximation using Stein's method and Malliavin calculus.
Ann. Inst. Henri Poincaré Probab. Stat. 46 (1), 45–58 (2010).

[55] I. Nourdin, G. Poly - Convergence in law in the second Wiener/Wigner chaos.
Electronic Communications in Probability 17 (36), 1-12 (2012).

[56] I. Nourdin, T. Tran - Statistical inference for Vasicek-type model driven by Hermite processes.
Preprint, to appear in *Stochastic Processes and their Applications* (2017).

[57] D. Nualart - The Malliavin Calculus and Related Topics. Springer, Heidelberg (2006).

[58] D. Nualart, S. Ortiz-Latorre - Central limit theorem for multiple stochastic integrals and Malliavin calculus.
Stochastic Processes and theis Applications, 118, 614-628 (2008).

[59] G. Peccati, C. A. Tudor - Gaussian limits for vector-valued multiple stochastic integrals.
Séminaire de Probabilités, XXXVIII, 247-262 (2005).

[60] V. Pipiras, M. Taqqu - Integration questions related to fractional Brownian motion.
Probability Theory and Related Fields, 118(2), 251-291 (2000).

[61] V. Pipiras, M. Taqqu - Are classes of deterministic integrands for fractional Brownian motion on an interval complete?
Bernoulli 7(6), 873-897 (2001).

[62] V. Pipiras, M. Taqqu - Long-Range Dependence and Self-Similarity. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press (2017).

[63] N. Privault, A. Réveillac - Stein estimation for the drift of Gaussian processes using the Malliavin calculus.
The Annals of Statistics, 36(5), 2531-2550 (2008).

[64] L. Quer-Sardanyons, S. Tindel - The 1-d stochastic wave equation driven by a fractional Brownian sheet.
Stochastic Processes and their Applications, 117, 1448-1472 (2007).

[65] B. L. S. Prakasa Rao - Statistical Inference for Fractional Diffusion Processes. Wiley Series in Probability and Statistics, Chichester, John Wiley & Sons (2010).

[66] M. Renardy, R. C. Rogers - An Introduction to Partial Differential Equations. Springer, New York (2004).

[67] L. R. Rifo, S. Torres, C. Tudor - Comparative estimation for discrete fractional Ornstein-Uhlenbeck process.
Stochastic Models, 29, 291-305 (2013).

[68] T. J. Rivlin - An Introduction to the Approximation of Functions. Dover (2003).

[69] L. C. G. Rogers - Arbitrage with fractional Brownian motion.
Mathematical Finance, 7 (1), 95-105 (1997).

[70] G. Samorodnitsky - Long range dependence.
Foundations and Trends in Stochastic Systems, 1 (3), 163–257 (2007).

[71] R. Shevchenko, M. Slaoui, C. A. Tudor - Generalised k-variations and Hurst parameter estimation for the fractional wave equation via Malliavin calculus.
To appear in *Journal of Statistical Planning anf Inference* (2019).

[72] R. Shevchenko, C. A. Tudor - Parameter estimation for the Rosenblatt Ornstein-Uhlenbeck process with periodic mean.
Statistical Inference for Stochastic Processes (2019).

[73] R. Shevchenko, J. H. C. Woerner - Inference for fractional Ornstein-Uhlenbeck type processes with periodic mean in the non-ergodic case.
Preprint, arXiv:1903.08033 (2019).

[74] M. Slaoui, C. A. Tudor - Limit behaviour of the Rosenblatt Ornstein-Uhlenbeck process with respect to the Hurst index.
Theory of Probability and its Applications, 98, 173-187 (2018).

[75] S. Torres, C. A. Tudor, F. Viens - Quadratic variations for the fractional-colored stochastic heat equation.
Electronic Journal of Probability, 19(76), 1-51 (2014).

[76] F. Treves - Basic Linear Partial Differential Equations. Academic Press (1975).

[77] C. A. Tudor - Analysis of Variations for Self-Similar Processes. A Stochastic Calculus Approach. Probability and its Applications (New York). Springer, Cham (2013).

[78] C. A. Tudor - Analysis of the Rosenblatt process.
ESAIM: Probability and Statistics, 12, 230–257 (2008).

[79] C. A. Tudor, F. G. Viens - Statistical aspects of the fractional stochastic calculus.
The Annals of Statistics 35 (3) 1183-1212 (2007).

[80] A. B. Tsybakov - Introduction to Nonparametric Estimation. Springer, New York (2009).

[81] O. Vasicek - An equilibrium characterization of the term structure.
Journal of Financial Economics, 5(2), 177-188 (1977).

[82] M. S. Veillette, M. S. Taqqu - Properties and numerical evaluation of the Rosenblatt distribution.
Bernoulli, 19(3), 982–1005 (2013).

[83] P. Walters - An Introduction to Ergodic Theory. Springer, New York (1982).

[84] W. Xiao, W. Zhang, W. Xu - Parameter estimation for fractional Ornstein-Uhlenbeck processes at discrete observation.
Applied Mathematical Modelling, 35, 4196-4207 (2011).

[85] Y. Yamamoto, N. Kitahara, M. Kano - Long memory effect of past climate change in Vostok ice core records.
Thermochimica Acta, 532, 41-44 (2012).

[86] L. C. Young - An inequality of the Hölder type connected with Stieltjes integration.
Acta Mathematica, 67, 251-282 (1936).

[87] C. Zeng, Y. Chen, Q. Yang - The fBm-driven Ornstein-Uhlenbeck process: Probability density and anomalous diffusion.
Fractional Calculus and Applied Analysis, 15(3) 479-492 (2012).